Ex03 — The Z-transform

Delivery deadline: **5 September** 2025.

We highly recommend you to collaborate and interact with each other when working on these tasks. Still, the exercises shall be solved independently and must be assessed in the weekly submission sessions. It is expected that everyone is familiar with the approval requirements for exercises. This is described on the course homepage: `http://www.uio.no/studier/emner/matnat/ifi/IN3190/h25/curriculum-related/`

Observe: The sum of points in total is 18, while the maximum score obtainable in one week is 10.

Goals and reflections:

- These geometric series-related sums might be useful:
  - A finite geometric series is $a\sum_{n=0}^{N} r^n = a + ar + a^2 + \ldots + ar^N$.
    For $r \neq 1$, this sum has the analytical value:
    $$a\sum_{n=0}^{N} r^n = a\left(\frac{1-r^{N+1}}{1-r}\right). \tag{1}$$
    This can, for example, be shown based on a "telescopic sum" approach, which you can look up in the literature if you want.
  - Some geometric series with infinitely many terms do also converge to a finite value.
    For this case, and under the condition that $|r| < 1$, we have that:
    $$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}. \tag{2}$$
    This expression can be used, for example, when determining the region of convergence for the Z-transform of a falling exponential.
- The Z-transform is a key transform in the analysis and processing of discrete signals and systems. It converts a discrete sequence of complex or real values into a representation in the z-domain, which is a complex frequency-domain representation of the sequence. The tasks this week will help you discern the nature of the Z-transform.
- The region of convergence (ROC) is the part of the complex plane where (all values of z) where the Z-transform of the signal $x(n)$ exists. This means that the Z-transform sum converges and hence that
  $$|X(z)| = |\sum_{-\infty}^{\infty} x(n)z^{-n}| < \infty.$$
  Using the triangle inequality, we saw on introduction slides 3 and 4 that this implies that the ROC is some band between some radii r_1 and r_2 outside of the origin of the complex plane: $r_1 < |z| < r_2$. (Note that r_1 can be zero and r_2 can be ∞.) There are several properties of a signal (or system) that can be discerned from the ROC of the signal (or system):
  - Causal system: the ROC is outside of some radius r in the complex plane.
  - Stable system: the unit circle is included in the ROC.
- *Note:* it is hence the Z-transform $X(z)$ *and* the ROC that *together* describe the z-domain representation of a signal. There can be two (or more) sequences $x_1(n)$ and $x_2(n)$ that have the same transform $X_1(z) = X_2(z)$ – but their ROCs will differ! Check for example the z-transform of $x_1(n) = u(n)$ and $x_2(n) = -u(-n-1)$ which are both the same. But their ROCs are not the same: $|z| > 1$, and $|z| < 1$, respectively.
- Analyzing the poles and zeroes of the system function $H(z) = Y(z)/X(z)$ involves finding the z values where $H(z)$ is zero, and the z values where it diverges (goes towards infinity).

- Although we have formally not yet treated the Fourier theory and the frequency-domain, we will also let you apply lowpass and bandpass filtering to some signals from the real world. This will demonstrate that a lowpass filter keeps only "slowly varying parts of a signal", while bandpass filtering keeps only "slowly enough and still fastly enough varying parts of a time-domain signal". We will use the Scipy wrapped methods of the widespread ObsPy seismology related data processing package. Here, you'll also have a first aquintance with a spectrogram – a concept that will be treated later in the course. The current week, we use readily designed frequency filtering methods and treat these more or less as black boxes. Later in the course, you'll learn much more about how to design various kinds of filters.

Exercise 1 — Z-transform, ROC, and implementation: 6 Points

An LTI system is defined by the following difference equation that connects the input $x(n)$ to the output $y(n)$:

$$y(n) = \frac{3}{4}y(n-1) - \frac{1}{8}y(n-2) + x(n). \tag{3}$$

a) Apply the Z-transform to Eq. (3) and collect all terms involving $Y(z)$ and $X(z)$. Use $X(z)$ to denote the Z-transform of $x(n)$ and $Y(z)$ to denote the Z-transform of $y(n)$. You can take advantage of the Z-transform property $\mathcal{Z}\{x(n-k)\} = z^{-k}\mathcal{Z}\{x(n)\}$.

$$\boxed{Y(z) = \frac{X(z)}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}}$$

b) We define the system function $H(z)$ as $H(z) = \dfrac{Y(z)}{X(z)}$.

Show that $H(z)$ can be written as Eq. (4) below. Where are the poles and zeroes of $H(z)$? What is the ROC of this system if the system is causal? Is the system stable for the case of it being causal?

$$H(z) = \frac{1}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{2}z^{-1})} = \frac{z^2}{(z - \frac{1}{4})(z - \frac{1}{2})} \tag{4}$$

c) What is the impulse response $h(n)$ of this system?

$h(n)$ is the inverse Z-transform of $H(z)$, taking the ROC of $H(z)$ into account. A convenient approach to solve this problem can be to perform a partial fraction decomposition of Eq. (4). Thereafter, you can identify the inverse Z transform of each of the fractions using a table of Z transforms, e.g., Table 3.2 in Rao & Swamy.

Exercise 2 — Z–Transform, from exam in 2014 3 points

a) Show that the Z–transform does not exist for the discrete signal $x(n) = 1$, for $-\infty \leq n \leq \infty$.

b) Prove the *differentiation* property of the Z–transform:

$$\frac{d}{dz}X(z) = -\frac{\mathcal{Z}\{n\,x(n)\}}{z},$$

where $X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)\,z^{-n}$. Here, $\mathcal{Z}$ is the Z–transform operator.

c) The Z–transform $X(z)$ of a causal system $x[n]$ is

$$X(z) = \frac{10z^3 + 40z}{z(z^2 - 2z - 3)}.$$

Find the poles and zeroes of $X(z)$ and tell what is their order.

Exercise 3 — Exam task in 2012: Z-transform and region of convergence (ROC) 3 points

a) Find the Z-transform and ROC to the data sequence

$$x(n) = \begin{cases} \frac{1}{n} & \text{for} \quad n \in [-2, 2], \quad n \in \mathbb{Z} \\ 0 & \text{for} \quad n = 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathbb{Z}$ represents the room of integer numbers.

b) Find the Z-transform and ROC to the function

$$x(n) = n2^{n-1}u(n-1)$$

Hint: You can apply some properties of the Z-transform to simplify the task – or you can go directly into the Z-transform definition an apply an appropriate variable substitution.

c) Consider two finite data sequences $x(n)$ and $h(n)$. Show that this rule for convolution holds:

$$x(n) * h(n) \overset{\mathcal{Z}}{\longleftrightarrow} X(z)H(z),$$

where $*$ denotes the convolution operator. Briefly explain why this property can be useful.

Exercise 4 — Parts of an exam task in 2017: Averaging filter 3 points

A system can be described by its impulse response, $h(n)$, such that the out signal is given by $y(n) = h(n) * x(n)$.

a) A system has $h(n) = \{1, \underline{2}, 3, 3\}$. Sketch $h(n)$ and express $h(n)$ using a sum of unit steps $u(n)$. Make sure to label the figure axes.

b) What is the energy of this system? Is the system causal?

c) What is $h(-n+1)$?

d) Find $y(n)$ when $x(n) = \{\underline{1}, 0, 1\}$ and $h(n) = \{\underline{1}, 2, 3, 4\}$.

e) We will design a system which is an averaging filter. For each n, $y(n)$ shall be the average of the input and the previous two samples of the input $x(n)$.

- Write the difference equation that connects $y(n)$ and $x(n)$.
- Is this system recursive?

f) Calculate and sketch the impulse response of the system.

g) Assume the transfer function $H(z) = Y(z)/X(z)$ is given as:

$$H(z) = \frac{1}{3z^2}(z^2 + z + 1)$$

What are the zeroes and poles of this system?

Exercise 5—Highpass, lowpass, and bandpass filtering example 3 Points

A Python code was run to download a small dataset from a seismic station in *Schwartzwald*. This script generated the plots in Figures 1 and 2. Figure 1 displays the original data and filtered versions these data, while Figure 2 shows the *spectrogram* of the original data.

This is the Python code that was run to download the data and to generate these plots. You may also access the file listed below directly from `https://www.uio.no/studier/emner/matnat/ifi/IN3190/h24/weekly-tasks/obspy_signal_example.py`.

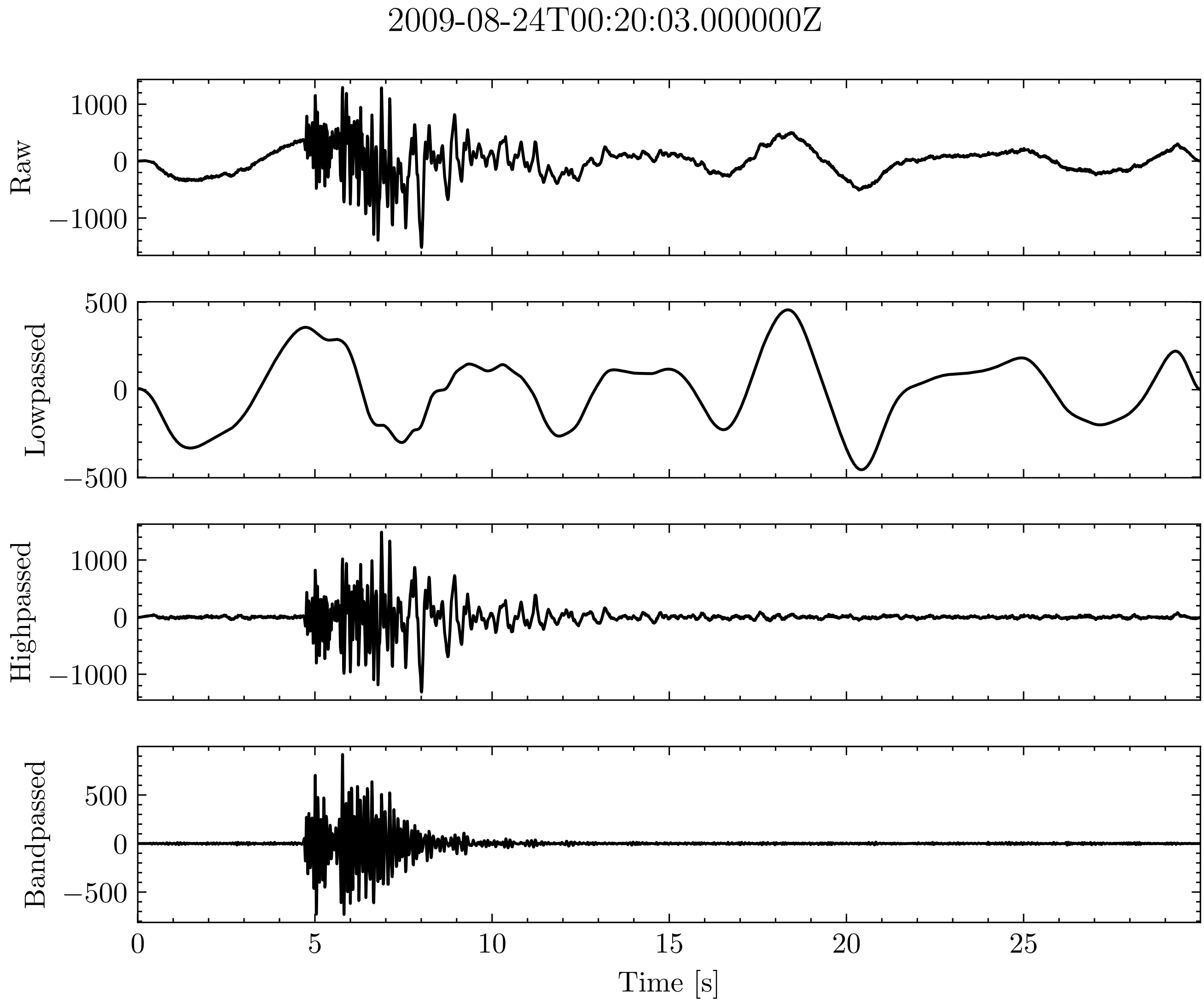

Figure 1: Geographical distribution of the 100 seismic events.

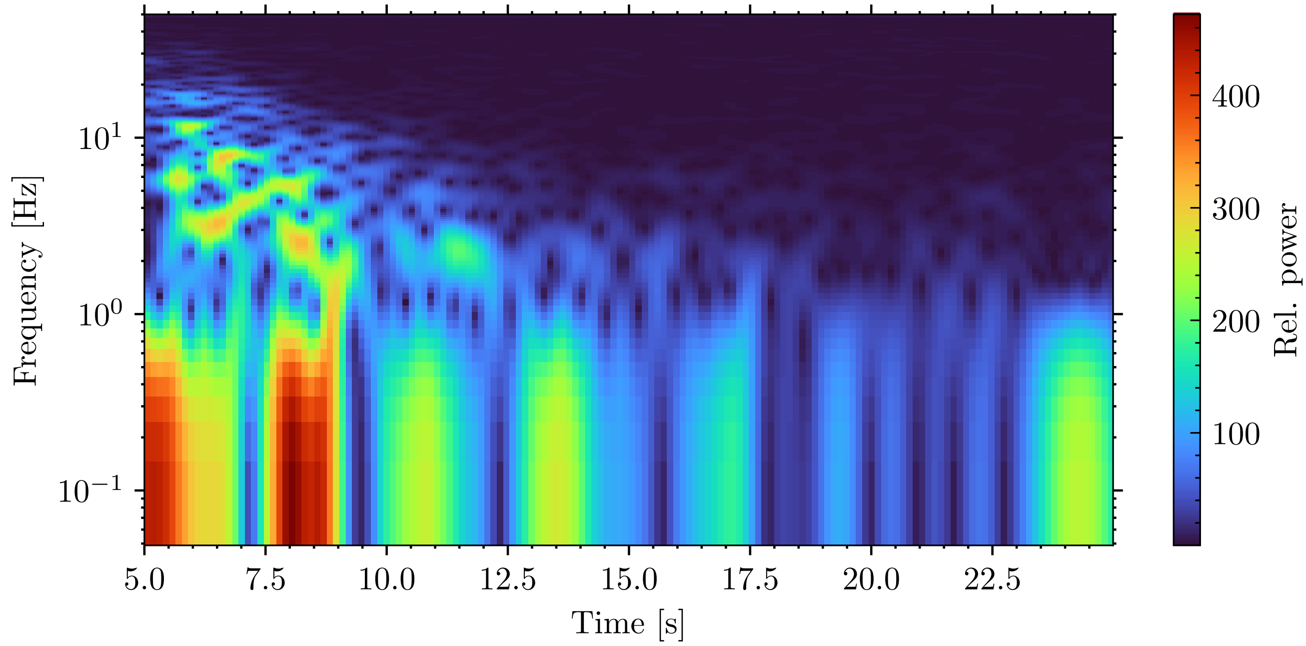

Figure 2: Geographical distribution of the 100 seismic events.

```python
# For interactive plotting in Ipython: remember to issue %matplotlib

import numpy as np
import matplotlib.pyplot as plt

import obspy

import scienceplots

plt.style.use('science') # Or you can use RCparams instead to set your plotting
    defaults


# Read the seismogram
st = obspy.read() # ObsPy default example. From a seismic station in Schwartzwald
    (Black Forest)


starttime = st[0].meta.starttime
endtime = st[0].meta.endtime
my_len = endtime-starttime
print(starttime)
print(endtime)

print(st[0].meta.starttime)

# There is only one trace in the Stream object, let's work on that trace...
tr = st[0]

# Time axis
t = np.arange(0, tr.stats.npts / tr.stats.sampling_rate, tr.stats.delta)

## Filtering with a lowpass on a copy of the original Trace
tr_filt = tr.copy()
tr_filt.filter('lowpass', freq=0.8, corners=2, zerophase=True)

## Highpass
tr_filt_bp1 = tr.copy()
tr_filt_bp1.filter('highpass', freq=0.8, corners=2, zerophase=True)

## Bandpass
tr_filt_bp2 = tr.copy()
tr_filt_bp2.filter('bandpass', freqmin=8.0, freqmax=20.0, corners=2, zerophase=
    True)


## --- Plot a spectrogram
fig_spect, axes_spect = plt.subplots(nrows=1, ncols=1,
                                      sharex=True,
                                      figsize=(6,3))

st.spectrogram(log=True, title='BW.RJOB ' + str(st[0].stats.starttime), axes=
    axes_spect, cmap='turbo')
plt.xlim(5, my_len-5)
axes_spect.set_xlabel('Time [s]')
axes_spect.set_ylabel('Frequency [Hz]')
axes_spect.tick_params(axis='both', direction='out', length=3, width=0.7, colors='
    black', grid_color='w', grid_alpha=0.5, which='major')
axes_spect.tick_params(axis='both', direction='out', length=1, width=0.7, colors='
    black', grid_color='w', grid_alpha=0.5, which='minor')
cbar = fig_spect.colorbar(axes_spect.collections[-1], ax=axes_spect, label='Rel.
    power')
```

```
plt.tight_layout()



## ---- Plotting the signal traces
fig_traces, axes_traces = plt.subplots(nrows=4, ncols=1,
                              sharex=True,
                              figsize=(6,5))

fig_traces.suptitle(tr.stats.starttime)

axes_traces[0].autoscale(enable=True, axis='x', tight=True)
axes_traces[0].plot(t, tr.data, 'k')
axes_traces[0].set_ylabel('Raw')

axes_traces[1].plot(t, tr_filt.data, 'k')
axes_traces[1].set_ylabel('Lowpassed')

axes_traces[2].plot(t, tr_filt_bp1.data, 'k')
axes_traces[2].set_ylabel('Highpassed')

axes_traces[3].plot(t, tr_filt_bp2.data, 'k')
axes_traces[3].set_ylabel('Bandpassed')

#fig_traces.tight_layout(rect=[0, 0.03, 1, 0.9])
axes_traces[-1].set_xlabel('Time [s]')
fig_traces.tight_layout()


plt.show(block=False)
```

Your tasks are to:

a) Download and run the `obspy_signal_example.py` script. You need Pyton installed on the machine where you run the code, plus the obspy and scienceplots packages (including their dependencies). We recommend installing this using Anaconda/Conda as described here: `https://github.com/obspy/obspy/wiki` and here: `https://github.com/garrettj403/SciencePlots/`.

b) Recreate the plots from Figures 1 and 2, but modified so that the bandpassed trace is filtered between 1.5 and 5 Hz. This is done on line 41 in the script.

c) Describe what aspects of the original signal that you see preserved in the lowpass, the highpass, and the bandpassed versions of the traces.