HYPERCYCLIC OPERATORS ON HARDY SPACES OF DIRICHLET SERIES

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1. Introduction

The study of Dirichlet series, that is, series of the form $\sum_n a_n n^{-s}$ where s is a complex variable, has garnered interest over the past century due to its significant impact on analytic number theory. One key aspect of the class of Dirichlet series is its deep connection to the class of holomorphic functions in infinitely many variables. This connection, established by Bohr as one of his early contributions and referred to as the Bohr transform, harnesses the Fundamental Theorem of Arithmetic to identify Dirichlet series with the power series $\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha$ which represent holomorphic functions in infinitely many variables.

Within the theory of Dirichlet series, one crucial concept, introduced relatively recently by Hedenmalm, Lindqvist and Seip [8] and further developed by Bayart [7], are the Hardy spaces of Dirichlet series $\mathcal{H}_p$. These spaces are intimately related to Hardy spaces of holomorphic functions in infinitely many variables $H_p(B_{c_0} \cap \ell_2)$ through the Bohr transform, which we denote by $\mathfrak{B}$.

In this paper, we study the dynamics of certain linear operators in Hardy spaces of Dirichlet series and holomorphic functions, with a focus on multiplication and Toeplitz operators. As a recurrent method in the results that follow, we first prove a given property of a linear operator that acts on Hardy spaces of holomorphic functions, and then transfer the property to the realm of Dirichlet series by applying the Bohr transform, which is remarkably well-behaved for this purpose. We also generalize some well-known results to more general settings.

We begin in Section 2 by analyzing the orbits of multiplication operators acting on general spaces of holomorphic functions on a Banach space domain. In Section 3, we provide a generalization of the classical Godefroy-Shapiro theorem, building on an existing generalization proved by Rong [11].

In Section 4 we introduce the dual spaces of the Hardy spaces $H_p(B_{c_0} \cap \ell_2)$ of holomorphic functions, which are of interest on their own as they present some difficulties (for example, $H_p(B_{c_0} \cap \ell_2)^*$ is not isomorphic to $H_q(B_{c_0} \cap \ell_2)$, with $\frac{1}{p} + \frac{1}{q} = 1$, and the Riesz projection $L^p(\mathbb{T}^\infty) \to H_p(\mathbb{T}^\infty)$ is not bounded). We identify the elements of the dual spaces $H_p(B_{c_0} \cap \ell_2)^*$ as formal co-analytic series $\sum_\alpha c_\alpha z^{-\alpha}$, and prove that these spaces are indeed spaces of analytic functions.

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Section 5 is centered around two Godefroy and Shapiro-type characterizations of the adjoint multiplication operators on $H_p(B_{c_0} \cap \ell_2)$ and $\mathcal{H}_p$ that are hypercyclic (among other analyzed properties). The characterizations not only relate the image of the multipliper to the hypercyclicity of the operator (as the theorem by Godefroy and Shapiro does) but also include results on the hypercyclicity of the operator associated to certain restrictions of the multiplier. In addition to this, we prove a trichotomy that holds for the orbits of adjoint multiplication operators in the finite-variable Hardy spaces $H_p(\mathbb{D}^N)$.

In Section 6, we shift the focus to multiplication operators M_ϕ on the dual space $H_p(B_{c_0} \cap \ell_2)^*$, where now the adjoint operator M_ϕ^* acts on $H_p(B_{c_0} \cap \ell_2)$ and, if well-defined, coincides with the Toeplitz operator associated to $\bar{\phi}$. As opposed to the previous analysis of multiplication operators in $H_p(B_{c_0} \cap \ell_2)$, the difficulty in this setting lies in the fact that we possess information about the space and the elements the adjoint multiplication operator acts on, but no so much about how the operator acts. We study the space of functions for which the Toeplitz operators in $H_p(B_{c_0} \cap \ell_2)^*$ are well-defined, and then provide a characterization of the hypercyclic Toeplitz operators in a similar fashion to Section 5.

We conclude in Section 7 by analyzing the hypercyclicity of composition operators on $\mathcal{H}_p$. We generalize a result for the case $p = 2$ by Bayart [7] to the general case $1 \leq p < \infty$.

2. Multiplication operators

Proposition 2.1. *Let $\Omega \subset E$, a domain on a Banach space E and let X be a reflexive Banach space of holomorphic functions on Ω such that for each $\phi \in H_\infty(\Omega)$ such that M_ϕ defines a bounded multiplication operator, we have $\|M_\phi\| \leq C \sup_{z\in\Omega} |\phi(z)|$. Let ϕ be a non-constant bounded holomorphic function on Ω. Then the orbits of M_ϕ cannot have non-zero weak limit points. Moreover, either*

i) $\|M_\phi^n(f)\| \to \infty$ for every $0 \neq f \in X$, or

ii) $M_\phi^n(f) \xrightarrow{w} 0$ for every $f \in X$, and in particular $M_\phi^n(f)(z) \to 0$ for all $z \in \Omega$ and $f \in X$.

Proof. We prove the first assertion. If g is a non-zero function that is a weak limit point of $\{M_\phi^n(f)\} = \{\phi^n f\}$ then by the continuity of evaluations (and hence weak-weak continuity), for every $z \in \Omega$ such that $f(z), g(z) \neq 0$, we have that $0 \neq g(z)$ is a limit point of the set $\{\phi(z)^n f(z)\}$. This implies that $|\phi(z)| = 1$ for every $z \in \Omega$ such that $f(z), g(z) \neq 0$. Since non-constant holomorphic are open mappings, ϕ must be constant on Ω.

Note that for any non-constant ϕ, either $|\phi(z)| < 1$ for every $z \in \Omega$ or $|\phi(z)| > 1$ for all $z \in U$ for some nonempty open set $U \subset \Omega$.

Suppose first that $|\phi(z)| < 1$ for every $z \in \Omega$. Thus, the orbit of any function f, $\{M_\phi^n(f)\} = \{\phi^n f\}$ is a bounded set in X. By the last assertion just proved, all its weak limit points must be the zero vector. This implies that $M_\phi^n(f) \xrightarrow{w} 0$ for every f.

We assume now that $|\phi(z)| > 1$ for all $z \in U$ for some nonempty open set $U \subset \Omega$. Let f be a non-zero function and let $z_0 \in U$ such that $f(z_0) \neq 0$. We claim that $\|M_\phi^n(f)\| \to \infty$.

Indeed, suppose that $\{M_\phi^n(f)\}$ has a bounded subsequence. Then $\{M_\phi^n(f)\}$ must have a weak limit point, which by the first assertion must be the zero vector. Hence 0 is a limit point of $\{\phi(z_0)^n f(z_0)\}$, which implies that $|\phi(z_0)| < 1$. This contradiction proves our claim. $\square$

Remark 2.2. *Proposition 2.1 cannot be applied to the spaces $\mathcal{H}_p$ of Dirichlet series because the elements of $\mathcal{H}_p$ and the multipliers define holomorphic functions on different domains ($\mathbb{C}_{1/2}$ and $\mathbb{C}_+$ respectively). However, the proof of the first assertion also works in this case. We will show through the Bohr transform that the whole proposition holds.*

Corollary 2.3. *Let $\psi \in \mathcal{H}_\infty$ be a non-constant multiplier of $\mathcal{H}_p$. Then the orbits of M_ψ cannot have non-zero weak limit points. Moreover, either*

i) $\|M_\psi^n(D)\| \to \infty$ for every $0 \neq D \in \mathcal{H}_p$, or

ii) $M_\psi^n(D) \xrightarrow{w} 0$ for every $D \in \mathcal{H}_p$, and in particular $M_\psi^n(D)(s) \to 0$ for all $s \in \mathbb{C}_{1/2}$ and $D \in \mathcal{H}_p$.

Proof. Proposition 2.1 can be applied to multiplication operators in $H_p(B_{c_0} \cap \ell_2)$, considering $\Omega = B_{c_0} \cap \ell_2$ as an open subset of ℓ_2. The thesis follows from observing that $\mathfrak{B} M_\phi^n(f) = M_{\mathfrak{B}\phi}^n(\mathfrak{B} f)$ and recalling that the Bohr transform $\mathfrak{B}$ induces isometric isomorphisms $H_p(B_{c_0} \cap \ell_2) \mapsto \mathcal{H}_p$ and $H_\infty(B_{c_0}) \mapsto \mathcal{H}_\infty$ (which preserve both weak and strong convergence). $\square$

Corollary 2.4. *Let F be the forward shift on ℓ_p and let $\phi(z) = \sum_{j=0}^\infty a_j z^j$ be non-constant, with $(a_j)_j \in \ell_1$. Then $\phi(F)$ cannot have orbits with non-zero weak limit points.*

Proof. Consider $\Phi : \ell_p \to H_p(\mathbb{D})$, given by $(x_n)_n \to \sum_n x_n z^n$ is a weak-weak continuous mapping and hence that $\phi(F)$ on (ℓ_p, w) is quasiconjugated to M_ϕ on $(H_p(\mathbb{D}), w)$. Therefore, by Proposition 2.1, $\phi(F)$ cannot have orbits with non-zero weak limit points. $\square$

3. Adjoint multiplication operators

A result by Rong [11] characterizes hypercyclic adjoint multiplication operators in reflexive spaces of analytic functions on a domain $\Omega \subset \mathbb{C}$, where point evaluations are continuous. An analogous characterization is valid in similar spaces of analytic functions on a domain Ω of a Banach space.

Lemma 3.1. *Let Λ be a nonempty open subset of a domain Ω in a Banach space E. Let X be a reflexive space of analytic functions on Ω in which the evaluation at each $\lambda \in \Omega$, k_λ, is continuous. Then span$\{k_\lambda : \lambda \in \Lambda\}$ is dense in X^*.*

Proof. Let $f \in X$ be such that $\langle k_\lambda, f\rangle = 0$ for every $\lambda \in \Lambda$. Thus $f_{|\Lambda} = 0$, and the Identity Principle, see e.g. [10, Proposition 5.7], shows that f is the zero function on Ω. $\square$

Theorem 3.2. *Let E be a Banach space and $\Omega \subset E$ a domain. Let X be a separable and reflexive space of analytic functions on Ω in which point evaluations are continuous, and suppose further that for each $\phi \in H_\infty(\Omega)$ we have $\|M_\phi\| \leq C \sup_{z\in\Omega} |\phi(z)|$. If $\phi \in H_\infty(\Omega)$ is non-constant, then the following are equivalent:*

(1) $M_\phi^* : X^* \to X^*$ is hypercyclic,
(2) $M_\phi^* : X^* \to X^*$ is mixing,
(3) $M_\phi^* : X^* \to X^*$ is chaotic,
(4) $\phi(\Omega) \cap \mathbb{T} \neq \emptyset$.

Proof. (4) $\Rightarrow$ (2). Let $\lambda \in \Omega$ and $f \in X$. Note that

$$M_\phi^*(k_\lambda)(f) = k_\lambda(\phi f) = \phi(\lambda)k_\lambda(f).$$

Moreover, since ϕ is non-constant and $\phi(\Omega) \cap \mathbb{T} \neq \emptyset$ we have that both $\phi^{-1}(\mathbb{D}) = \{\lambda \in \Omega : |\phi(\lambda)| < 1\}$ and $\phi^{-1}(\mathbb{C} \setminus \mathbb{D}) = \{\lambda \in \Omega : |\phi(\lambda)| > 1\}$ are nonempty open sets. By Lemma 3.1 we have that these sets generate dense subspaces in X^*, and the classical Godefroy-Shapiro criterion lets us conclude that M_ϕ^* is mixing.

(2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1) are immediate.

(1) $\Rightarrow$ (4) is completely analogous to the argument shown in [11].

(4) $\Rightarrow$ (3). We show that the set $\Gamma = \{\lambda \in \Omega : \phi(\lambda) \in e^{2\pi i\mathbb{Q}}\}$ has an accumulation point. Let $\lambda \in \Omega$ such that $\phi(\lambda) \in e^{2\pi i\mathbb{Q}}$, let $\varepsilon > 0$ and let $U = B(\lambda, \varepsilon)$ be a ball of radius ε centered at λ. Clearly $\phi(U) \cap \mathbb{T} \neq \emptyset$, and moreover $\phi(U)$ intersects a nontrivial arc of $\mathbb{T}$. Choosing any $\lambda \neq \lambda' \in \phi^{-1}(\phi(U) \cap \mathbb{T})$ yields an element of Γ at distance less than ε from λ.

We now claim that there is a collection $(L_k)_{k\in\mathbb{N}} \subset E$ of complex lines through an accumulation point $\lambda \in \Gamma$ such that $\phi_{|\Omega\cap L_k}$ is non constant for each $k \in \mathbb{N}$ and $\bigcup_{k\in\mathbb{N}} L_k \cap \Omega$ is dense in Ω. Indeed, since ϕ is not identically zero in Ω, given a base $\{U_k\}_{k\in\mathbb{N}}$ of Ω we have from the Identity Principle that ϕ is not identically zero in U_k, $k \in \mathbb{N}$. Thus, for every such k there exists $\lambda_k \in \Omega$ such that $\phi(\lambda_k) \neq \phi(\lambda)$. We define L_k as the complex line through λ_k and λ.

Note in addition that λ is an accumulation point of $\Gamma \cap L_k$ for each $k \in \mathbb{N}$. Indeed, the non-constant, holomorphic function of one variable $\phi_{|\Omega\cap L_k}$ is open and and thus its image intersects a nontrivial arc of $\mathbb{T}$. A similar argument than before can be used to show that λ is an accumulation point of $\Gamma \cap L_k$.

Now let $f \in X$ such that $\langle k_z, f\rangle = f(z) = 0$ for each $z \in \Gamma$. In particular, $f = 0$ in $\Gamma \cap L_k$ (which contains an accumulation point) and from the Identity Principle for holomorphic functions of one variable we have that $f = 0$ in $\Omega \cap L_k$. Since $\bigcup_{k\in\mathbb{N}} L_k \cap \Omega$ is dense in Ω it follows that f is identically zero, and from the Godefroy-Shapiro criterion we conclude that M_ϕ^* is chaotic.

$\square$

4. The dual space of Hardy spaces as a space of analytic functions

While the dual of each $H_p(\mathbb{D}^N)$ may be identified with $H_q(\mathbb{D}^N)$, the identification collapses when we consider $H_p(B_{c_0} \cap \ell_2)$. Indeed, the isomorphism between $H_p(\mathbb{D}^N)^*$ and $H_q(\mathbb{D}^N)$ has norm $|sen(\pi/q)|^{-N}$, which diverges as $N \to \infty$. Another significant obstacle is that the Riesz projection $L^p(\mathbb{T}^\infty) \to H_p(\mathbb{T}^\infty)$ is no longer bounded. Although the space $H_p(B_{c_0} \cap \ell_2)^*$ is not well understood, we can give an elementary description of its elements in terms of formal series.

4.1. **The dual of $H_p(B_{c_0} \cap \ell_2)$.** Since the spaces $H_p(B_{c_0} \cap \ell_2)$ are reflexive and $(z^\alpha)_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}}$ is a basis of $H_p(B_{c_0} \cap \ell_2)$, it follows that its dual basis (which we will denote by $(z^{-\alpha})_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}}$) is a basis of $H_p(B_{c_0} \cap \ell_2)^*$. In this way, we can identify $H_p(B_{c_0} \cap \ell_2)^*$ with the formal (co-analytic) series $\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^{-\alpha}$, equipped with the norm induced by the duality with $H_p(B_{c_0} \cap \ell_2)$. We will denote by $\jmath$ the conjugation map, i.e. $\jmath(f)(z) = \overline{f(z)}$.

Definition 4.1. *A formal series $\varphi = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^{-\alpha}$ belongs to $H_p(B_{c_0} \cap \ell_2)^*$ if and only if for every $f = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} d_\alpha z^\alpha \in H_p(B_{c_0} \cap \ell_2)$ we have $|\langle \varphi, f \rangle| = \left| \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha d_\alpha \right| < \infty$. In this case,*

$$\|\varphi\|_{H_p(B_{c_0} \cap \ell_2)^*} = \sup_{\|f\|_{H_p(B_{c_0} \cap \ell_2)} \leq 1} \langle \varphi, f \rangle.$$

Remark 4.2. *In the same way, for each $N \in \mathbb{N}$ we identify $H_p(\mathbb{D}^N)^*$ with the space of formal series $\varphi = \sum_{\alpha \in \mathbb{N}_0^N} c_\alpha z^\alpha$ such that $|\langle \varphi, f \rangle| < \infty$ for every $f \in H_p(\mathbb{D}^N)$. In this case, thanks to the Riesz projection and the Hölder inequality we have that $H_p(\mathbb{D}^N)^*$ and $H_q(\mathbb{D}^N)$ are isomorphic as Banach spaces, for $q = \frac{p}{p-1}$. Furthermore, since the Poisson operator and taking radial limits produce an isometry between $H_q(\mathbb{D}^N)$ and $H_q(\mathbb{T}^N)$ (with $z^\alpha \mapsto w^\alpha$), the same operators produce an isometry between $H_p(\mathbb{D}^N)^*$ and $H_p(\mathbb{T}^N)^*$.*

Remark 4.3. *While $H_p(B_{c_0} \cap \ell_2)^*$ and $H_q(B_{c_0} \cap \ell_2)$ are not isomorphic ($\frac{1}{p} + \frac{1}{q} = 1$), from the Hahn-Banach theorem and the Hölder inequality we have $H_q(B_{c_0} \cap \ell_2) \subset \jmath(H_p(B_{c_0} \cap \ell_2)^*)$ upon identifying the elements of each space as formal series.*

Since for each N we have an inclusion $i_N : H_p(\mathbb{D}^N) \to H_p(B_{c_0} \cap \ell_2)$ and a projection $\pi_N : H_p(B_{c_0} \cap \ell_2) \to H_p(\mathbb{D}^N)$, we have a dual projection $i_N^* : H_p(B_{c_0} \cap \ell_2)^* \to H_p(\mathbb{D}^N)^*$ and a dual inclusion $\pi_N^* : H_p(\mathbb{D}^N)^* \to H_p(B_{c_0} \cap \ell_2)^*$, both operators having norm equal to one. We emphasise that we see $H_p(\mathbb{D}^N)^*$ as a space of analytic functions endowed with the topology of dual space given by $H_p(\mathbb{D}^N)$ and not the one given by $H_q(\mathbb{D}^N)$.

Proposition 4.4. *Let $\varphi \in H_p(B_{c_0} \cap \ell_2)^*$. Then $\|\varphi\| = \sup_{N \in \mathbb{N}} \|i_N^*(\varphi)\|_{H_p(\mathbb{D}^N)^*}$.*

Proof. Since the functions depending on finitely many variables are dense in $H_p(B_{c_0} \cap \ell_2)$, we have

$$\|\varphi\|_{H_p(B_{c_0} \cap \ell_2)^*} = \sup_{\|f\|_{H_p(B_{c_0} \cap \ell_2)} \leq 1} \langle \varphi, f \rangle = \sup_{N \in \mathbb{N}} \sup_{\|f\|_{H_p(\mathbb{D}^N)} \leq 1} \langle \varphi, i_N(f) \rangle = \sup_{N \in \mathbb{N}} \|i_N^*(\varphi)\|_{H_p(\mathbb{D}^N)}.$$

$\square$

Proposition 4.5. *Let $(c_\alpha)_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}}$ be coefficients. The following are equivalent:*

(1) $\varphi = \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^{-\alpha} \in H_p(B_{c_0}\cap\ell_2)^*$.
(2) For every $N\in\mathbb{N}$, $\varphi_N := \sum_{\alpha\in\mathbb{N}_0^N} c_\alpha z^{-\alpha} \in H_p(\mathbb{D}^N)^*$ *and* $\sup_{N\in\mathbb{N}} \|\varphi_N\|_{H_p(\mathbb{D}^N)^*} < \infty$.

Proof. (1) $\Rightarrow$ (2) is the above proposition.

(2) $\Rightarrow$ (1). Consider for each $N\in\mathbb{N}$ the function $\pi_N^*(\varphi_N) \in H_p(B_{c_0}\cap\ell_2)^*$. By assumption, $(\pi_N^*(\varphi_N))_{N\in\mathbb{N}}$ is a bounded sequence and thus has a w-convergent subsequence, say $\pi_{N_k}^*(\varphi_{N_k}) \xrightarrow{w} \varphi \in H_p(B_{c_0}\cap\ell_2)^*$, with $\|\varphi\|_{H_p(B_{c_0}\cap\ell_2)^*} \le \sup_{k\in\mathbb{N}} \|\varphi_{N_k}\|_{H_p(\mathbb{D}^{N_k})^*}$. By the uniqueness of coefficients it follows that $\varphi = \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^{-\alpha}$. □

We will need to consider homogeneous parts. For each $m\in\mathbb{N}$, consider the space

$$H_p(B_{c_0}\cap\ell_2)^{*m} = \{\varphi\in H_p(B_{c_0}\cap\ell_2)^* : c_\alpha = 0 \text{ if } |\alpha|\neq m\}$$

of m-homogeneous functions in $H_p(B_{c_0}\cap\ell_2)^*$, endowed with the subspace norm of $H_p(B_{c_0}\cap\ell_2)^*$.

Proposition 4.6. *The space $H_p(B_{c_0}\cap\ell_2)^{*m}$ is closed in $H_p(B_{c_0}\cap\ell_2)^*$, and the projection*

$$\begin{aligned}\pi_m : H_p(B_{c_0}\cap\ell_2)^* &\longrightarrow H_p(B_{c_0}\cap\ell_2)^{*m}\\ \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^{-\alpha} &\longmapsto \sum_{\substack{\alpha\in\mathbb{N}_0^{(\mathbb{N})}\\ |\alpha|=m}} c_\alpha z^{-\alpha}\end{aligned}$$

is a contraction.

Proof. The closeness of $H_p(B_{c_0}\cap\ell_2)^{*m}$ follows from the fact that $(z^{-\alpha})_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}}$ is a Schauder basis of $H_p(B_{c_0}\cap\ell_2)^*$. As for the projection π_m, we have

$$\begin{aligned}\|\pi_m(\varphi)\| &= \sup_{\|f\|_{H_p(B_{c_0}\cap\ell_2)}\le 1} \langle \pi_m(\varphi), f\rangle\\ &= \sup_{\substack{f\in H_p^m(B_{c_0}\cap\ell_2)\\ \|f\|_{H_p^m(B_{c_0}\cap\ell_2)}\le 1}} \langle \pi_m(\varphi), f\rangle\\ &= \sup_{\substack{f\in H_p^m(B_{c_0}\cap\ell_2)\\ \|f\|_{H_p^m(B_{c_0}\cap\ell_2)}\le 1}} \langle \varphi, f\rangle\\ &\le \|\varphi\|_{H_p(B_{c_0}\cap\ell_2)^*}.\end{aligned}$$

□

Remark 4.7. *See* [4, Lemma 4] *for an analogous result on* $\pi_m : H_p(B_{c_0}\cap\ell_2) \to H_p^m(B_{c_0}\cap\ell_2)$.

It is well known that all the spaces $H_p^m(B_{c_0}\cap\ell_2)$, $1\le p<\infty$, are equal as linear spaces and isomorphic as Banach spaces [6, Proposition 11.12]. Hence, $(z^\alpha)_{\alpha\in\mathbb{N}_0^{(\mathbb{N})},|\alpha|=m}$ is a Schauder basis of $H_p^m(B_{c_0}\cap\ell_2)$ for every $1\le p<\infty$. We can thus identify $H_p^m(B_{c_0}\cap\ell_2)^*$ with the analytic series $\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^{-\alpha}$ such that $\left|\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} c_\alpha a_\alpha\right| < \infty$ for every $f = \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})},|\alpha|=m} a_\alpha z^\alpha$. We will see next that this space is equal to $H_p(B_{c_0}\cap\ell_2)^{*m}$.

Proposition 4.8. *Let $1 < p < \infty$. The spaces $H_p(B_{c_0}\cap\ell_2)^{*m}$ and $H_p^m(B_{c_0}\cap\ell_2)^*$ are the same space.*

Proof. Let $\varphi \in H_p(B_{c_0}\cap\ell_2)^{*m} \cap H_p^m(B_{c_0}\cap\ell_2)^*$. Then,

$$\begin{aligned}\|\varphi\|_{H_p(B_{c_0}\cap\ell_2)^{*m}} &= \sup_{\substack{f\in H_p(B_{c_0}\cap\ell_2)\\ \|f\|_{H_p(B_{c_0}\cap\ell_2)}\leq 1}} \langle\varphi, f\rangle \\ &= \sup_{\substack{f\in H_p^m(B_{c_0}\cap\ell_2)\\ \|f\|_{H_p^m(B_{c_0}\cap\ell_2)}\leq 1}} \langle\varphi, f\rangle \\ &= \|\varphi\|_{H_p^m(B_{c_0}\cap\ell_2)^*}.\end{aligned}$$

The fact that the m-homogeneous polynomials are dense in both spaces implies that $H_p(B_{c_0}\cap\ell_2)^{*m} = H_p^m(B_{c_0}\cap\ell_2)^*$. ☐

The following extends a theorem of Brevig [4, Theorem 2] to m-homogeneous functions.

Corollary 4.9. *Let $1 < p < \infty$ and $1 \leq q < \infty$. The spaces $H_p(B_{c_0}\cap\ell_2)^{*m}$ and $H_q^m(B_{c_0}\cap\ell_2)$, are all equal as linear spaces and isomorphic as Banach spaces. Moreover, for each $1 < p_1 \leq p_2 < \infty$ there exists a constant C_{p_1,p_2} such that for all $\varphi \in H_{p_1}(B_{c_0}\cap\ell_2)^{*m}$ we have*

$$\|\varphi\|_{H_{p_2}(B_{c_0}\cap\ell_2)^{*m}} \leq \|\varphi\|_{H_{p_1}(B_{c_0}\cap\ell_2)^{*m}} \leq C_{p_1,p_2}^m \|\varphi\|_{H_{p_2}(B_{c_0}\cap\ell_2)^{*m}}.$$

Proof. All the spaces $H_q^m(B_{c_0}\cap\ell_2)$ are equal as linear spaces and isomorphic as Banach spaces [6, Proposition 11.12]. This, together with the above result shows that the $H_p(B_{c_0}\cap\ell_2)^{*m}$ are equal as linear spaces and isomorphic as Banach spaces. The proof concludes by observing that each $H_p(B_{c_0}\cap\ell_2)^{*m}$ is isomorphic to $H_2^m(B_{c_0}\cap\ell_2)^* = H_2^m(B_{c_0}\cap\ell_2)$ which in turn is isomorphic to each $H_q^m(B_{c_0}\cap\ell_2)$. ☐

As an elementary corollary, we obtain a result about the set of monomial convergence of $\jmath(H_p(B_{c_0}\cap\ell_2)^{*m})$ (denoted by $\mathrm{mon}(\jmath(H_p(B_{c_0}\cap\ell_2)^{*m}))$) due to the fact that $\mathrm{mon}(H_p(B_{c_0}\cap\ell_2)) = \ell_2$ [6, Proposition 12.22]. The conjugation map is needed to view $H_p(B_{c_0}\cap\ell_2)^{*m}$ as a space of analytic functions.

Corollary 4.10. *For every $m \geq 2$ and $1 < p < \infty$ we have*

$$\mathrm{mon}(\jmath(H_p(B_{c_0}\cap\ell_2)^{*m})) := \left\{ z \in \mathbb{N}_0^{(\mathbb{N})} : \sum_{\substack{\alpha\in\mathbb{N}_0^{(\mathbb{N})}\\ |\alpha|=m}} |c_\alpha z^\alpha| < \infty \text{ for every } \sum_{\substack{\alpha\in\mathbb{N}_0^{(\mathbb{N})}\\ |\alpha|=m}} \bar{c}_\alpha z^{-\alpha} \in H_p(B_{c_0}\cap\ell_2)^{*m} \right\} = \ell_2.$$

Theorem 4.11. *Let $1 < p < \infty$. Then $\mathrm{mon}(\jmath(H_p(B_{c_0}\cap\ell_2)^*)) = B_{c_0}\cap\ell_2$.*

Proof. Since $H_q(B_{c_0}\cap\ell_2) \subseteq \jmath(H_p(B_{c_0}\cap\ell_2)^*)$, we have that $\mathrm{mon}(\jmath(H_p(B_{c_0}\cap\ell_2)^*)) \subseteq \mathrm{mon}(H_q(B_{c_0}\cap\ell_2)) = B_{c_0}\cap\ell_2$.

For the other inclusion we will first prove that there is $0 < r < 1$ such that $rB_{\ell_2} \subseteq \operatorname{mon}(\jmath(H_p(B_{c_0} \cap \ell_2)^*))$. By [6, Proposition 11.4] we have that for every $\varphi \in H_2^m(B_{c_0} \cap \ell_2)$,

$$\sum_{|\alpha|=m} |c_\alpha(\varphi) z^\alpha| \leq \|\varphi\|_{H_2(B_{c_0}\cap\ell_2)} \|z\|_2^m.$$

Since $H_p(B_{c_0} \cap \ell_2)^{*m}$ and $H_2^m(B_{c_0} \cap \ell_2)$ are isomorphic as Banach spaces, there is $C > 0$ such that for every $\varphi \in H_p(B_{c_0} \cap \ell_2)^{*m}$ we have that

$$\sum_{|\alpha|=m} |c_\alpha(\varphi) z^\alpha| \leq C^m \|\varphi\|_{H_p(B_{c_0}\cap\ell_2)^*} \|z\|_2^m.$$

Let now $r < \frac{1}{C}$, $z \in rB_{\ell_2}$ and $\varphi \in H_p(B_{c_0} \cap \ell_2)^*$. Hence,

$$\begin{aligned}
\sum_{\alpha\in\mathbb{N}^{(\mathbb{N})}} |c_\alpha(\varphi) z^\alpha| &= \sum_{m=0}^{\infty} \sum_{|\alpha|=m} |c_\alpha(\varphi) z^\alpha| = \sum_{m=0}^{\infty} \sum_{|\alpha|=m} |c_\alpha(\varphi_m) z^\alpha| \\
&\leq \sum_{m=0}^{\infty} r^m C^m \|\varphi_m\|_{H_p^m(B_{c_0}\cap\ell_2)^*} \leq \sum_{m=0}^{\infty} r^m C^m \|\varphi\|_{H_p(B_{c_0}\cap\ell_2)^*} \\
&\leq \frac{1}{1-rC} \|\varphi\|_{H_p(B_{c_0}\cap\ell_2)^*} \qquad (1)
\end{aligned}$$

Let $z \in B_{c_0} \cap \ell_2$, $k \in \mathbb{N}$ such that $\sum_{n>k} |z_j|^2 < r^2$, and let $\varphi \in H_p(B_{c_0} \cap \ell_2)^*$. For each $n_1, \ldots, n_k \in \mathbb{N}$ we will consider $\varphi_{n_1,\ldots,n_k} \in H_p(B_{c_0} \cap \ell_2)^*$ such that for every $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ we have that

$$c_\alpha(\varphi_{n_1,\ldots,n_k}) = c_{n_1,\ldots,n_k,\alpha}(\varphi). \qquad (2)$$

To define $\varphi_{n_1,\ldots,n_k}$, we consider first $[\varphi_{n_1,\ldots,n_k}]_N \in H_p(\mathbb{D}^N)^*$ given by

$$[\varphi_{n_1,\ldots,n_k}]_N(u) = \int_{\mathbb{T}^k} \varphi(w, u, 0) w_1^{-n_1} \cdots w_k^{-n_k} dw.$$

Since for finitely many variables, $H_p(\mathbb{D}^N)^*$ and $H_q(\mathbb{D}^k)$ coincide as linear sets, we have that $\varphi(-, u, 0)$ is measurable in $\mathbb{T}^k$ for almost every $u \in \mathbb{T}^N$. We observe now that

$$\begin{aligned}
\|[\varphi_{n_1,\ldots,n_k}]_N(u)\|_{H_p(\mathbb{D}^N)^*} &= \sup_{\|f\|_{H_p(\mathbb{D}^N)}\leq 1} \int_{\mathbb{T}^N} \int_{\mathbb{T}^k} \varphi(w, u, 0) w_1^{-n_1} \cdots w_k^{-n_k} \overline{f}(u) dw du \\
&\leq \sup_{\|f\|_{H_p(B_{c_0}\cap\ell_2)}\leq 1} \|\varphi_{N+k}\|_{H_p(\mathbb{D}^{N+k})^*} \|w_1^{n_1} \cdots w_k^{n_k} f\|_{H_p(\mathbb{D}^{N+k})} \leq \|\varphi\|_{H_p(B_{c_0}\cap\ell_2)^*}.
\end{aligned}$$

On the other hand, for every $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ we have that

$$c_\alpha([\varphi_{n_1,\ldots,n_k}]_N) = \int_{\mathbb{T}^N} \int_{\mathbb{T}^k} \varphi(w, u, 0) w_1^{-n_1} \cdots w_k^{-n_k} u^{-\alpha} dw du = c_{n_1,\ldots,n_k,\alpha}(\varphi_{N+k}) = c_{n_1,\ldots,n_k,\alpha}(\varphi).$$

By Proposition 4.5 we conclude that $\varphi_{n_1,\ldots,n_k} = \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} c_{n_1,\ldots,n_k,\alpha}(\varphi) z^\alpha \in H_p(B_{c_0} \cap \ell_2)^*$ with

$$\|\varphi_{n_1,\ldots,n_k}\|_{H_p(B_{c_0}\cap\ell_2)^*} \leq \|\varphi\|_{H_p(B_{c_0}\cap\ell_2)^*}. \qquad (3)$$

Let $R<1$ such that $\|z\|_\infty<R$. By (3), (2) and (1) applied to $z-\sum_{j>k} z_j e_j$ and to each $\varphi_{n_1,\dots,n_k}$, we obtain that

$$\begin{aligned}\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} |c_\alpha(\varphi) z^\alpha| &= \sum_{n_1,\dots,n_k}\sum_{\beta\in\mathbb{N}_0^{(\mathbb{N})}} |c_{n_1,\dots,n_k,\beta}(\varphi)||z_1|^{n_1}\cdots|z_k|^{n_k}|z_{k+1}|^{\beta_1}|z_{k+2}|^{\beta_2}\cdots\\ &< \sum_{n_1,\dots,n_k} R^{n_1+\cdots+n_k}\sum_{\beta\in\mathbb{N}_0^{(\mathbb{N})}} |c_\beta(\varphi_{n_1,\dots,n_k})||z_{k+1}|^{\beta_1}|z_{k+2}|^{\beta_2}\cdots\\ &\le \sum_{n_1,\dots,n_k} R^{n_1+\cdots+n_k}\frac{1}{1-rC}\|\varphi_{n_1,\dots,n_k}\| \le \frac{1}{(1-R)^k}\frac{1}{1-rC}\|\varphi\|<\infty.\end{aligned}$$

□

5. Linear dynamics of adjoint of multiplication operators on Hardy spaces

Theorem 5.1. *Let $\phi\in H_\infty(B_{c_0})$. Then ϕ defines a bounded multiplication operator $M_\phi: H_p(B_{c_0}\cap\ell_2)\to H_p(B_{c_0}\cap\ell_2)$, and $\|M_\phi\|=\|\phi\|_\infty$.*

Proof. As shown in [7], $\mathfrak{B}\phi\in\mathcal{H}_\infty$ defines a multiplication operator in $\mathcal{H}_p$ whose norm equals $\|\mathfrak{B}\phi\|_\infty$. Given $f\in H_p(B_{c_0}\cap\ell_2)$, using the fact that $\mathfrak{B}$ is multiplicative and an isometric isomorphism between $H_p(B_{c_0}\cap\ell_2)$ and $\mathcal{H}_p$ we have

$$\|M_\phi(f)\|_{H_p(B_{c_0}\cap\ell_2)} = \|\phi f\|_{H_p(B_{c_0}\cap\ell_2)} = \|\mathfrak{B}\phi\mathfrak{B}f\|_{\mathcal{H}_p} = \|M_{\mathfrak{B}\phi}(\mathfrak{B}f)\|_{\mathcal{H}_p}<\infty.$$

Moreover

$$\|M_\varphi\| = \sup_{\substack{f\in H_p(B_{c_0}\cap\ell_2)\\ \|f\|\le 1}}\|M_\varphi(f)\| = \sup_{\substack{g\in\mathcal{H}_p\\ \|g\|\le 1}}\|M_{\mathfrak{B}\varphi}(g)\| = \|M_{\mathfrak{B}\varphi}\| = \|\mathfrak{B}\varphi\|_\infty = \|\varphi\|_\infty.$$

□

Lemma 5.2. *Let $N\in\mathbb{N}$ and $\phi\in H_\infty(B_{c_0})$. Then $\pi_N^*\circ M_{\phi_N}^* = M_\phi^*\circ\pi_N^*$, that is, the following diagram is commutative:*

$$\begin{array}{ccc} H_p(B_{c_0}\cap\ell_2)^* & \xrightarrow{M_\phi^*} & H_p(B_{c_0}\cap\ell_2)^* \\ {\scriptstyle\pi_N^*}\big\uparrow & & \big\uparrow{\scriptstyle\pi_N^*} \\ H_p(\mathbb{D}^n)^* & \xrightarrow{M_{\phi_N}^*} & H_p(\mathbb{D}^n)^* \end{array}$$

Proof. Let $\beta\in\mathbb{N}_0^{(\mathbb{N})}$. It suffices to show that $\pi_N(\phi z^\beta)=\phi_N\pi_N(z^\beta)$, since

$$\langle\pi_N^*(M_{\phi_N}^*(h)), z^\beta\rangle = \langle h, \phi_N\pi_N(z^\beta)\rangle,$$

$$\langle M_\phi^*(\pi_N^*(h)), z^\beta\rangle = \langle h, \pi_N(\phi z^\beta)\rangle$$

for each $h\in H_p(\mathbb{D}^n)$.

If $\phi = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha$ we have

$$\phi z^\beta = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c'_\alpha z^\alpha,$$

where $c'_{\gamma+\beta} = c_\gamma$ for $\gamma \in \mathbb{N}_0^{(\mathbb{N})}$ (and $c'_\alpha = 0$ if $\alpha \neq \gamma + \beta$ for some γ). Note in addition that

$$\pi_N(\phi z^\beta) = \sum_{\alpha \in \mathbb{N}_0^N} c'_\alpha z^\alpha.$$

Now, $\pi_N(z^\beta) = 0$ (and thus $\phi_N \pi_N(z^\beta) = 0$) if $\beta \notin \mathbb{N}_0^N$, and in this case $\pi_N(\phi z^\beta) = 0$ as well, since $c'_\alpha = 0$ for each $\alpha \in \mathbb{N}_0^N$. This proves the claim for $\beta \notin \mathbb{N}_0^N$.

If $\beta \in \mathbb{N}_0^N$ we have

$$\phi_N \pi_N(z^\beta) = \phi_N z^\beta = \sum_{\alpha \in \mathbb{N}_0^N} c'_\alpha z^\alpha,$$

which proves the claim.

□

Theorem 5.3. *Let $\phi \in H_\infty(B_{c_0})$ be a non-constant function. Then, the following are equivalent:*

(1) $M_\phi^* : H_p(B_{c_0} \cap \ell_2)^* \to H_p(B_{c_0} \cap \ell_2)^*$ is hypercyclic.
(2) $M_\phi^* : H_p(B_{c_0} \cap \ell_2)^* \to H_p(B_{c_0} \cap \ell_2)^*$ is chaotic.
(3) $\phi(B_{c_0}) \cap \mathbb{T} \neq \emptyset$.
(4) There exists $N \in \mathbb{N}$ such that $M_{\phi_N}^* : H_p(\mathbb{D}^N)^* \to H_p(\mathbb{D}^N)^*$ is hypercyclic.
(5) There exists $N_0 \in \mathbb{N}$ such that for each $N \geq N_0$, $M_{\phi_N}^* : H_p(\mathbb{D}^N)^* \to H_p(\mathbb{D}^N)^*$ is hypercyclic.

Proof. (1) ⇒ (3). Assuming that M_ϕ^* is hypercyclic and $\phi(B_{c_0}) \cap \mathbb{T} = \emptyset$, since B_{c_0} is connected either $\phi(B_{c_0}) \subset \mathbb{D}$ or $\phi(B_{c_0}) \subset \mathbb{C} \setminus \bar{\mathbb{D}}$. If $\phi(B_{c_0}) \subset \mathbb{D}$, from Theorem 5.1 we have $\|M_\phi^*\| = \|M_\phi\| = \|\phi\|_\infty \leq 1$, and the operator cannot be hypercyclic.

If $\phi(B_{c_0}) \subset \mathbb{C} \setminus \bar{\mathbb{D}}$, the function ϕ is nowhere zero and thus $\frac{1}{\phi}$ is well-defined and holomorphic in B_{c_0}. An analogous argument shows that $M_{\frac{1}{\phi}}^*$ cannot by hypercyclic, in contradiction to the fact that M_ϕ^* is its inverse operator.

(3) ⇒ (4). The function ϕ is holomorphic and non-constant, and thus an open mapping. The hypothesis $\phi(B_{c_0}) \cap \mathbb{T} \neq \emptyset$ then implies that the image of ϕ intersects both $\mathbb{D}$ and $\mathbb{C} \setminus \bar{\mathbb{D}}$. Since eventually-zero sequences are dense in B_{c_0}, we can consider $b = (b_1, \dots, b_N, 0, \dots), c = (c_1, \dots, c_N, 0, \dots) \in B_{c_0}$ such that $|\phi(b)| = |\phi_N(b_1, \dots, b_N)| < 1$ and $|\phi(c)| = |\phi_N(c_1, \dots, c_N)| > 1$ for some $N \in \mathbb{N}$. This and the connectedness of B_{c_0} clearly imply that $\phi_N(\mathbb{D}^N) \cap \mathbb{T} \neq \emptyset$, and from Theorem 3.2 we conclude that $M_{\phi_N}^*$ is hypercyclic.

(4) ⇒ (5). Given $N \in \mathbb{N}$ such that $M_{\phi_N}^*$ is hypercyclic (and thus $\phi_N(\mathbb{D}^N) \cap \mathbb{T} \neq \emptyset$), the result follows from completing to any larger number of variables with zeros.

(5) $\Rightarrow$ (1). Let $U, V \subset H_p(B_{c_0} \cap \ell_2)^*$ be nonempty open sets, and let $N_0 \in \mathbb{N}$ such that $M^*_{\phi_N}$ is hypercyclic for every $N \geq N_0$. For any such N consider the inclusion $\pi^*_N : H_p(\mathbb{D}^N)^* \to H_p(B_{c_0} \cap \ell_2)^*$ and note that the sets ${\pi^*_N}^{-1}(U), {\pi^*_N}^{-1}(V) \subset H_p(\mathbb{D}^N)^*$ are open.

Since $M^*_{\phi_N}$ is hypercyclic, it is topologically transitive, and thus there exist $f \in {\pi^*_N}^{-1}(U)$ and $k \in \mathbb{N}$ such that ${M^*_{\phi_N}}^k(f) \in {\pi^*_N}^{-1}(V)$. Clearly $\pi^*_N(f) \in U$, and Lemma 5.2 yields

$$ {M^*_\phi}^k(\pi^*_N(f)) = \pi^*_N({M^*_{\phi_N}}^k(f)) \in V. $$

This means that M^*_ϕ is topologically transitive, and thus hypercyclic.

(3) $\Rightarrow$ (2). If $\phi(B_{c_0}) \cap \mathbb{T} \neq \emptyset$, it follows by Theorem 3.2 that $M^*_{\phi_N}$ is chaotic for every N larger than some $N_0 \in \mathbb{N}$. Fix any such N and let $U \subset H_p(B_{c_0} \cap \ell_2)^*$ be a nonempty open set. There exist $f \in {\pi^*_N}^{-1}(U)$ and $k \in \mathbb{N}$ such that ${M^*_{\phi_N}}^k(f) = f$. Moreover, from Lemma 3.2 we have

$$ \pi^*_N(f) = \pi^*_N({M^*_{\phi_N}}^k(f)) = {M^*_\phi}^k(\pi^*_N(f)). $$

The density of $\bigcup_{N \geq N_0} \pi^*_N(H_p(\mathbb{D}^N)^*)$ in $H_p(B_{c_0} \cap \ell_2)^*$ implies that the set of periodic points of M^*_ϕ is dense, and thus M^*_ϕ is chaotic.

(2) $\Rightarrow$ (1) is immediate. $\square$

In [2], a zero-one law for adjoint of multipliers on $H_p(\mathbb{D})$ has been proven: if an orbit has a limit point different from zero then the operator must be hypercyclic. Moreover, if ϕ is analytic in a neighborhood of $\mathbb{D}$ then a stronger trichotomy holds: either every orbit tends to 0, every orbit tends to ∞ or M^*_ϕ is hypercyclic. We extend this result to the finite-variable case.

Theorem 5.4 (Zero-one law)**.** *Let $\phi \in H_\infty(B_{c_0})$ and $p > 1$. If $M^*_\phi : H_p(B_{c_0} \cap \ell_2)^* \to H_p(B_{c_0} \cap \ell_2)^*$ has an orbit with a nonzero limit point, then M^*_ϕ is hypercyclic.*

Proof. The proof is analogous to [2, Theorem 4.1]. We give a sketch for the sake of completeness. If $\phi(B_{c_0}) \cap \mathbb{T} \neq \emptyset$ then the operator is hypercyclic by the above theorem. So suppose first that $\phi(B_{c_0}) \subseteq \mathbb{D}$. Then M^*_ϕ is supercyclic and $\|M^*_\phi\| \leq 1$. From a result by Ansari and Bourdon [1] every orbit tends to zero, which is a contradiction. If $\phi(B_{c_0}) \subseteq \bar{\mathbb{D}}^c$, then $\psi = \frac{1}{\phi} \in H_\infty(B_{c_0})$, $\|M^*_\psi\| \leq 1, \psi(\mathbb{D}) \subseteq \mathbb{D}$ and M^*_ψ has an orbit with an accumulation point. Hence, every orbit must again tend to zero. $\square$

Theorem 5.4 invites us to study how wild orbits can be when $\phi(B_{c_0}) \subseteq \overline{\mathbb{D}}^c$. Intuitively, every orbit should tend to infinity, and this is indeed the case when ϕ has a smooth extension to a neighborhood of B_{c_0}. By a deep result of Shkarin [12, Theorem 1.6], there are orbits which, although they tend to infinite, are dense in the weak topology.

We will need the following lemma, which is taken from [2, 12].

Lemma 5.5. *Let U be an open set containing $\mathbb{D}$, and let $\phi : U \to \overline{\mathbb{D}}^c$ be analytic and not constant. Then $\phi^{-1}(\mathbb{T}) \cap \mathbb{T}$ has 0 measure in $\mathbb{T}$.*

Proof. We will prove that $\phi(\mathbb{T}) \cap \mathbb{T}$ is finite. Suppose otherwise, then $g(t) = |\phi(e^{it})|^2$ is real analytic and $g = 1$ in a set with an accumulation point. Thus, $g = 1$ in $\mathbb{T}$. This implies that $\phi(\mathbb{T}) \subseteq \mathbb{T}$. By the maximum modulus principle, it follows that $\phi(\overline{\mathbb{D}}) \subseteq \overline{\mathbb{D}}$ which is a contradiction. $\square$

Lemma 5.6. *Let $\phi \in H_1(\mathbb{D}^N)$. If $P_N^{-1}(\phi)$ vanishes in a set $A \subset \mathbb{T}^N$ of positive measure, then ϕ is identically zero.*

Proof. We prove the claim by induction on N. For $N = 1$, the result follows from Hoffman [9].

Denoting the Lebesgue measure by m and $w = (w_1, \dots, w_n) \in \mathbb{T}^N$, we recall that

$$m(A) = \int_{\mathbb{T}} \mathbb{1}_{A_{w_N}}(w_1, \dots, w_{N-1}) dw_N,$$

where $A_{w_N} = \{(w_1, \dots, w_{N-1}) \in \mathbb{T}^{N-1} : (w_1, \dots, w_{N-1}, w_N) \in A\}$. Since $m(A) > 0$, it follows that $m(A_{w_N}) > 0$ for all w_N in a certain set $B \subset \mathbb{T}$ with positive measure.

Fixing $w_N \in B$, we consider the function $\phi_{w_N}(z_1, \dots, z_{N-1}) = \phi(z_1, \dots, z_{N-1}, w_N) \in H_1(\mathbb{D}^{N-1})$. Since ϕ_{w_N} vanishes in A_{w_N}, from the inductive hypothesis it follows that ϕ_{w_N} is identically zero. Additionally, since $w_N \in B$ is arbitrary (and since ϕ_{w_N} is defined in $\mathbb{T}^{N-1}$ for every $w_N \in B$) we have that ϕ vanishes in $\mathbb{T}^{N-1} \times B$.

Given $(w_1, \dots, w_{N-1}) \in \mathbb{T}^{N-1}$, we now consider the analytic function

$$\phi_{w_1,\dots,w_{N-1}}(w) = \phi(w_1, \dots, w_{N-1}, w) \in H_1(\mathbb{D}).$$

Again, since $\phi_{w_1,\dots,w_{N-1}}$ vanishes in B it follows from the result for $N = 1$ that $\phi_{w_1,\dots,w_{N-1}}$ is identically zero. Combining this with the argument for ϕ_{w_N} we conclude that ϕ is identically zero. $\square$

Theorem 5.7. *Let $\phi \in H_\infty(\mathbb{D}^N)$ be non-constant, and suppose further that ϕ is analytic in a neighborhood of $\mathbb{D}^N$. The following trichotomy holds: either*

(1) $M_\phi^{*n}(f) \to 0$ for every $f \in H_p(\mathbb{D}^N)^*$;
(2) M_ϕ^* is hypercyclic or
(3) $M_\phi^{*n}(f) \to \infty$ for every $f \in H_p(\mathbb{D}^N)^*$.

Proof. Either $\phi(\mathbb{D}^N) \subseteq \mathbb{D}$, $\phi(\mathbb{D}^N) \cap \mathbb{T} \neq \emptyset$ or $\phi(\mathbb{D}^N) \subseteq \overline{\mathbb{D}}^c$. In the first case, proceeding as in [2, Theorem 4.1] we have $\|M_\phi^*\| \leq 1$, which means that M_ϕ^* is power bounded (i.e. the sequence $(M_\phi^{*n})_{n \in \mathbb{N}}$ is bounded). In addition, M_ϕ^* is supercyclic, since a multiple of $\phi(\mathbb{D}^N)$ intersects $\mathbb{T}$ and is thus hypercyclic by Theorem 3.2. We conclude from a result by Ansari and Bourdon [1] that $M_\phi^{*n}(f) \to 0$ for every $f \in H_p(\mathbb{D}^N)^*$.

If $\phi(\mathbb{D}^N) \cap \mathbb{T} \neq 0$, the operator is hypercyclic by Theorem 3.2.

It therefore suffices to address the case $\phi(\mathbb{D}^N) \subseteq \overline{\mathbb{D}}^c$. From We claim that $\phi^{-1}(\mathbb{T}) \cap \mathbb{T}^N$ has zero measure in $\mathbb{T}^N$. To this aim, we choose $j \in \{1, \dots, N\}$ such that ϕ depends on the j-th variable, and for each $\hat{w}_j := (w_1, \dots, w_{j-1}, w_{j+1}, \dots, w_N) \in \mathbb{T}^{N-1}$ we consider $\phi_{\hat{w}_j}(z) = \phi(w_1, \dots, w_{j-1}, z, w_{j+1}, \dots, w_N)$. Clearly $\phi_{\hat{w}_j}$ is analytic on a neighborhood of $\mathbb{D}$, and $\phi_{\hat{w}_j}(\mathbb{D}) \subseteq \overline{\mathbb{D}}^c$.

If $A = \phi^{-1}(\mathbb{T}) \cap \mathbb{T}^N$, $A_1 = \{w \in A : \phi_{\hat{w}_j} \text{ is not constant}\}$ and $A_2 = \{w \in A : \phi_{\hat{w}_j} \text{ is constant}\}$, it follows that $A = A_1 \cup A_2$ and it suffices to show that A_1 and A_2 have zero measure. By Fubini's Theorem we know that

$$m(A_1) = \int_{\mathbb{T}^N} \mathbb{1}_{A_1}(w)dw = \int_{\mathbb{T}^{N-1}} \int_{\mathbb{T}} \mathbb{1}_{A_{1,\hat{w}_j}}(w_j)dw_j d\hat{w}_j = \int_{\mathbb{T}^{N-1}} m(A_{1,\hat{w}_j})d\hat{w}_j,$$
$$m(A_2) = \int_{\mathbb{T}^N} \mathbb{1}_{A_2}(w)dw = \int_{\mathbb{T}} \int_{\mathbb{T}^{N-1}} \mathbb{1}_{A_{2,w_j}}(\hat{w}_j)d\hat{w}_j dw_j = \int_{\mathbb{T}} m(A_{2,w_j})dw_j,$$

where

$$\begin{aligned} A_{1,\hat{w}_j} &= \{\lambda \in \mathbb{T} : (w_1, \ldots, w_{j-1}, \lambda, w_{j+1}, \ldots, w_N) \in A_1\} \\ &= \{\lambda \in \mathbb{T} : (w_1, \ldots, w_{j-1}, \lambda, w_{j+1}, \ldots, w_N) \in A \text{ and } \phi_{\hat{w}_j} \text{ is not constant}\} \\ &= \{\lambda \in \phi_{\hat{w}_j}^{-1}(\mathbb{T}) \cap \mathbb{T} \text{ for } \hat{w}_j \text{ such that } \phi_{\hat{w}_j} \text{ is not constant}\}, \end{aligned}$$

$$A_{2,w_j} = \{\hat{w}_j \in \mathbb{T}^{N-1} : (w_1, \ldots, w_{j-1}, w_j, w_{j+1}, \ldots, w_N) \in A_2\} = \{\hat{w}_j \in \mathbb{T}^{N-1} : \phi_{\hat{w}_j} \text{ is constant}\}.$$

By Lemma 5.5, $A_{1,\hat{w}_j}$ has zero measure in $\mathbb{T}$. We show next that A_{2,w_j} has zero measure in $\mathbb{T}^{N-1}$. Indeed, if $\phi_{\hat{w}_j}$ is constant then $\phi'_{\hat{w}_j}$ vanishes in a neighborhood of $\overline{\mathbb{D}}$. Thus, if A_{2,w_j} has positive measure then $\frac{\partial \phi}{\partial z_j}$ vanishes in a set of positive measure composed by the product of A_{2,w_j} and a neighborhood of $\overline{\mathbb{D}}$. In particular, $\frac{\partial \phi}{\partial z_j}$ vanishes in a set of positive measure of $\mathbb{T}^N$, and by Lemma 5.6 we have that $\frac{\partial \phi}{\partial z_j}$ is zero everywhere. This contradicts the fact that ϕ depends on z_j, and therefore A_{2,w_j} has zero measure.

Now let $g \in H_p(\mathbb{T}^N)$ and $\psi = \frac{1}{\phi} \in H_\infty(\mathbb{D}^N)$. Thus, $P_N^{-1}(\psi)^n g \to 0$ almost everywhere in $\mathbb{T}^N$. By the dominated convergence theorem, it follows that $P_N^{-1}(\psi)^n g \to 0$ in $H_p(\mathbb{T}^N)$. We conclude that $\psi^n g \to 0$ for every $g \in H_p(\mathbb{D}^N)$.

Finally, if $f \in H_p(\mathbb{D}^N)^*$, let $g \in H_p(\mathbb{D}^N)$ such that $\langle f, g\rangle \neq 0$. Then $\langle M_\phi^{*n} f, M_\psi^n g\rangle = \langle f, g\rangle$. Since $M_\psi^n g \to 0$, we obtain that $M_\phi^{*n} f \to \infty$. $\square$

Our aim is now to derive similar results for adjoints of multiplication operators in $\mathcal{H}_p$. Note that, similarly to elements of $H_p(B_{c_0} \cap \ell_2)^*$, linear functionals in $\mathcal{H}_p^*$ admit a representation in terms of the dual Schauder basis $(n^{-s})_{n\in\mathbb{N}}$, where $D = \sum_{n=1}^\infty a_n n^{-s} \in \mathcal{H}_p^*$ if and only if for each $E = \sum_{n=1}^\infty b_n n^{-s} \in \mathcal{H}_p$ we have $|\langle D, E\rangle| = |\sum_{n=1}^\infty a_n b_n| < \infty$.

Given $\phi \in H_\infty(B_{c_0})$, we also observe that $\mathfrak{B}^*$ factors the operators $M_\phi^* : H_p(B_{c_0} \cap \ell_2)^* \to H_p(B_{c_0} \cap \ell_2)^*$ and $M_{\mathfrak{B}\phi}^* : \mathcal{H}_p^* \to \mathcal{H}_p^*$. Firstly, the operator $M_{\mathfrak{B}\phi}$ is well-defined; as shown in [7], the multipliers of $\mathcal{H}_p$ are exactly the elements of $\mathcal{H}_\infty$. Moreover, identifying the elements of each space as formal series we have $\mathcal{H}_p^* = \mathfrak{B} H_p(B_{c_0} \cap \ell_2)^*$. It is also clear that $M_{\mathfrak{B}\phi} \circ \mathfrak{B} = \mathfrak{B} \circ M_\phi$, from which it follows that $\mathfrak{B}^* \circ M_{\mathfrak{B}\phi}^* = M_\phi^* \circ \mathfrak{B}^*$.

The usefulness of M_ϕ^* and $M_{\mathfrak{B}\phi}^*$ being conjugated through $\mathfrak{B}^*$ lies in the following lemma:

Lemma 5.8. *Let $\phi \in H_\infty(B_{c_0})$. Then, $M_\phi^* : H_p(B_{c_0} \cap \ell_2)^* \to H_p(B_{c_0} \cap \ell_2)^*$ is hypercyclic (resp. chaotic) if and only if $M_{\mathfrak{B}\phi}^* : \mathcal{H}_p^* \to \mathcal{H}_p^*$ is hypercyclic (resp. chaotic).*

The conjugation of M_ϕ and $M^*_{\mathfrak{B}\phi}$ allows us to immediately derive a zero-one law for multiplication operators on $\mathcal{H}_p$, based to Theorem 5.4.

Theorem 5.9 (Zero-one law)**.** *Let $\psi \in \mathcal{H}_\infty$ and $p > 1$. If $M^*_\psi : \mathcal{H}^*_p \to \mathcal{H}^*_p$ has an orbit with a nonzero limit point, then M^*_ψ is hypercyclic.*

Given $\psi \in \mathcal{H}_\infty$ and $N \in \mathbb{N}$, we refer by ψ_N to the restriction of ψ to the first N primes. That is, if $\psi(s) = \sum_{n=1}^\infty a_n n^{-s}$ the restriction is as follows:

$$\psi_N(s) = \sum_{\substack{n=\mathfrak{p}^\alpha=1 \\ \alpha\in\mathbb{N}_0^N}}^{\infty} a_n n^{-s}.$$

The restrictions ψ_N are elements of $\mathcal{H}_\infty$ themselves (and thus multipliers in $\mathcal{H}_p$) and are well-behaved through the Bohr transform. Specifically, $\mathfrak{B}\psi_N = (\mathfrak{B}\psi)_N$, that is, $\mathfrak{B}\psi_N$ is the restriction to $\mathbb{D}^n$ of the holomorphic function $\mathfrak{B}\psi \in H_\infty(B_{c_0})$.

Theorem 5.10. *Let $\psi \in \mathcal{H}_\infty$ be non-constant. Then, the following are equivalent:*

(1) $M^*_\psi : \mathcal{H}^*_p \to \mathcal{H}^*_p$ is hypercyclic.
(2) $M^*_\psi : \mathcal{H}^*_p \to \mathcal{H}^*_p$ is chaotic.
(3) $\psi(\mathbb{C}_+) \cap \mathbb{T} \neq \emptyset$.
(4) There exists $N \in \mathbb{N}$ such that $\psi_N(\mathbb{C}_+) \cap \mathbb{T} \neq \emptyset$.
(5) There exists $N_0 \in \mathbb{N}$ such that for each $N \geq N_0$, $\psi_N(\mathbb{C}_+) \cap \mathbb{T} \neq \emptyset$.

Proof. Lemma 5.8 proves (1) ⇒ (2), while (5) ⇒ (4) and (2) ⇒ (1) are immediate.

(1) ⇒ (3). From Lemma 5.8 we have that $M^*_{\mathfrak{B}^{-1}\psi}$ is hypercyclic, and thus $\|M^*_{\mathfrak{B}^{-1}\psi}\| = \|\psi\|_\infty > 1$. Supposing that $\psi(\mathbb{C}_+) \cap \mathbb{T} = \emptyset$, it follows necessarily that $\psi(\mathbb{C}_+) \subset \mathbb{C} \setminus \bar{\mathbb{D}}$. On the other hand, if $\psi(s) = \sum_{n=1}^\infty a_n n^{-s}$ we have that $\psi(s) \longrightarrow a_1$ uniformly as $\Re s \longrightarrow \infty$. This means that $|a_1| \geq 1$ and therefore $\frac{1}{\psi} \in \mathcal{H}_\infty$. The contradiction arises from the fact that $\|M^*_{\frac{1}{\psi}}\| = \|\frac{1}{\psi}\|_\infty \leq 1$, but $M^*_{\frac{1}{\psi}}$ must be hypercyclic.

(1) ⇒ (5). From Lemma 5.8 and Theorem 5.3, there exists $N_0 \in \mathbb{N}$ such that $\mathfrak{B}^{-1}\psi_N(\mathbb{D}^n) \cap \mathbb{T} \neq \emptyset$ for each $N \geq N_0$. Fixed any such N, by Theorem 3.2 the operator $M^*_{\mathfrak{B}^{-1}\psi_N}$ is hypercyclic, and thus

$$1 < \|M^*_{\mathfrak{B}^{-1}\psi_N}\| = \|\mathfrak{B}^{-1}\psi_N\|_\infty = \|\psi_N\|_\infty.$$

From a similar argument to the one used in implication (1) ⇒ (3) we conclude that $\psi_N(\mathbb{C}_+) \cap \mathbb{T} \neq \emptyset$.

(3) ⇒ (5). Consider the sequence $(\psi_N)_{N\in\mathbb{N}}$ in $\mathcal{H}_\infty$, which is bounded. Indeed,

$$\|\psi_N\|_\infty = \|\mathfrak{B}^{-1}\psi_N\|_{H_\infty(B_{c_0})} \leq \|\mathfrak{B}^{-1}\psi\|_{H_\infty(B_{c_0})} = \|\psi\|_\infty$$

for each $N \in \mathbb{N}$. From Bayart's Montel-type theorem [7], there exists a subsequence $(\psi_{N_k})_{k\in\mathbb{N}}$ and a function $g \in \mathcal{H}_\infty$ such that $(\psi_{N_k})_{k\in\mathbb{N}}$ converges to g in $\mathcal{H}^+_\infty$, that is, the subsequence converges uniformly to g in each $\mathbb{C}_\varepsilon$, $\varepsilon > 0$. Since coordinate functionals are continuous in $\mathcal{H}^+_\infty$ it follows that $g = \psi$.

On the other hand, since ψ is holomorphic and non-constant it is an open mapping. This, along with the hypothesis $\psi(\mathbb{C}_+) \cap \mathbb{T} \neq \emptyset$, means that there exist open subsets $U, V \subset \mathbb{C}_+$ such that $\psi(U) \subset \mathbb{D}$ y $\psi(V) \subset \mathbb{C} \setminus \bar{\mathbb{D}}$. Clearly $(\psi_{N_k})_{k\in\mathbb{N}}$ converges pointwise in $\mathbb{C}_+$, which means that there exists $k_0 \in \mathbb{N}$ such that $\psi_{N_k}(\mathbb{C}_+) \cap \mathbb{T} \neq \emptyset$ for each $k \geq k_0$, and thus $\psi_N(\mathbb{C}_+) \cap \mathbb{T} \neq \emptyset$ for some $N \in \mathbb{N}$.

(4) $\Rightarrow$ (1). It suffices to show that $\mathfrak{B}^{-1}\psi_N(\mathbb{D}^n) \cap \mathbb{T} \neq \emptyset$ for some $N \in \mathbb{N}$, since in that case we can conclude from Lemma 5.8 that M_ψ^* is hypercyclic.

Let $N \in \mathbb{N}$ such that $\psi_N(\mathbb{C}_+) \cap \mathbb{T} \neq \emptyset$ and $s_0 \in \mathbb{C}_+$ such that $|\psi_N(s_0)| = 1$. Let $z_0 = (\mathfrak{p}_1^{-s_0}, \ldots, \mathfrak{p}_n^{-s_0})$, where $\mathfrak{p}_j$ denotes the j-th prime number. Given $\alpha \in \mathbb{N}_0^N$, clearly $z_0^\alpha = (\mathfrak{p}^\alpha)^{-s_0}$, where $\mathfrak{p} = (\mathfrak{p}_j)_{j\in\mathbb{N}}$. Additionally, let $\psi(s) = \sum_{n=1}^\infty a_n n^{-s}$, and note $a_\alpha = a_n$ if $n = \mathfrak{p}^\alpha$.

In view of the above we have

$$\mathfrak{B}^{-1}\psi_N(z_0) = \sum_{\alpha\in\mathbb{N}_0^N} a_\alpha z_0^\alpha = \sum_{\alpha\in\mathbb{N}_0^N} a_\alpha (\mathfrak{p}^\alpha)^{-s_0} = \sum_{\substack{n=\mathfrak{p}^\alpha \\ \alpha\in\mathbb{N}_0^N}} a_n n^{-s_0} = \psi_N(s_0),$$

from which we conclude that $\mathfrak{B}^{-1}\psi_N(\mathbb{D}^n) \cap \mathbb{T} \neq \emptyset$. $\square$

6. Toeplitz operators on Hardy spaces

In a similar fashion, we may study the adjoints of multiplication operators acting on $H_p(B_{c_0} \cap \ell_2)^*$. Namely, given $M_\phi : H_p(B_{c_0} \cap \ell_2)^* \to H_p(B_{c_0} \cap \ell_2)^*$ we study the linear dynamics of $M_\phi^* : H_p(B_{c_0} \cap \ell_2) \to H_p(B_{c_0} \cap \ell_2)$. In this setting, it is even unclear what the multipliers of $H_p(B_{c_0} \cap \ell_2)^*$ are. As an initial observation, the space of multipliers of$H_p(B_{c_0} \cap \ell_2)^*$ does not equal $H_\infty(B_{c_0})$. To see this, we observe that the backwardshift operator on $H_p(\mathbb{T})$, $B(f) = \frac{f-f(0)}{z}$, is the adjoint of $M_z : H_p(\mathbb{T})^* \to H_p(\mathbb{T})^*$ and has norm bigger than one if $p \neq 2$ (see [3, Theorem 7.7] and [5]). This implies that the operator $M_{z_1}^* : H_p(B_{c_0} \cap \ell_2) \to H_p(B_{c_0} \cap \ell_2)$, if bounded, must have norm bigger than one and hence $\|M_{z_1}^*\| > 1 = \|z_1\|_{H_\infty(B_{c_0})}$.

Lemma 6.1. *For each $n \in \mathbb{N}$, the projection*

$$\begin{aligned} P_n : H_p(B_{c_0} \cap \ell_2) &\to H_p(B_{c_0} \cap \ell_2) \\ \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha &\to \sum_{\substack{\alpha\in\mathbb{N}_0^{(\mathbb{N})} \\ \alpha_n\neq 0}} c_\alpha z^\alpha \end{aligned}$$

is bounded.

Proof. We will prove that $I - P_n$ is bounded. Given $f \in H_p(\mathbb{T}^\infty)$, we consider

$$\tilde{f}(u) = \int_{\mathbb{T}} f(u_1, \ldots, u_{n-1}, w, u_{n+1}, \ldots) dw.$$

It follows by the integral Minkowski inequality, the Hölder inequality and Fubini's Theorem that $\tilde{f} \in H_p(\mathbb{T}^\infty)$, with $\|\tilde{f}\| \leq \|f\|$ and that $c_\alpha(\tilde{f}) = c_\alpha(f)$ if $\alpha_n = 0$ and $c_\alpha(\tilde{f}) = 0$ if $\alpha_n \neq 0$. Indeed, if $\alpha_n \neq 0$

then $c_\alpha(\tilde{f}) = 0$ because $\tilde{f}$ does not depend on the n-th coordinate. On the other hand, if $\alpha_n = 0$, we have that

$$c_\alpha(\tilde{f}) = \int_{\mathbb{T}^\infty} \int_{\mathbb{T}} f(u_1, \dots, u_{n-1}, w, u_{n+1}, \dots) u^{-\alpha} dw du = c_\alpha(f),$$

again Fubini's Theorem. This implies that $I - P_n(f) = \tilde{f}$, and hence $I - P_n$ is a contraction.

□

Proposition 6.2. *For every $1 \leq p < \infty$ and every polynomial Q, $M_Q^* : H_p(B_{c_0} \cap \ell_2) \to H_p(B_{c_0} \cap \ell_2)$ is bounded.*

Proof. Note that $M_{z_n}^*(f) = \frac{f-(I-P_n)(f)}{z_n}$. Since z_n has constant modulus 1 in the infinite politorus and $I - P_n$ is a contraction, we conclude that $M_{z_n}^*(f)$ has norm smaller than 2. This implies that $M_{z^\alpha}^*$ is bounded, and by linearity we conclude that M_Q^* is bounded. □

Theorem 6.3. *There exists $\phi \in H_\infty(B_{c_0})$ such that the Toeplitz operator with symbol $\overline{\phi}$ is not well defined in $H_p(B_{c_0} \cap \ell_2)$.*

Proof. Let $(n_k)_k$ be a sequence of natural numbers such that $n_k > n_{k-1} + k$. Let $\phi_k(z) := \prod_{j=n_k}^{n_{k+1}-1} z_j$. It is plain that each $\phi_k \in H_\infty(B_{c_0})$ with infinite norm equal to one. Moreover, there exist $\lambda > 1$ such that $\|T_{\overline{\phi_k}}\| > \lambda^k$. Indeed, by [3, Theorem 7.7] there are $\lambda > 1$ and $f \in H_p(\mathbb{T})$ with $\|f\|_{H_p(\mathbb{T})} = 1$ such that $\|T_{\overline{z}}(f)\| > \lambda$. Hence, if we define $f_j \in H_p(B_{c_0} \cap \ell_2)$ by $f_j = f(z_j)$ we have that $\|f_j\|_{H_p(B_{c_0} \cap \ell_2)} = 1$ and $\|T_{\overline{z}_j}(f_j)\|_{H_p(B_{c_0} \cap \ell_2)} = \|T_{\overline{z}} f\|_{H_p(\mathbb{T})} > \lambda$.

Consider now $g_k \in H_p(B_{c_0} \cap \ell_2)$ given by $g_k(z) := \prod_{j=n_k}^{n_k-1} f_j$. Since g_k is a product of functions which depend on different variables, we have that $\|g_k\|_{H_p(B_{c_0} \cap \ell_2)} = \prod_{j=n_k}^{n_{k+1}-1} \|f_j\| = 1$. In a similar way, we obtain that

$$\|T_{\overline{\phi_k}}(g_k)\| = \left\| \prod_{j=n_k}^{n_k-1} T_{\overline{z}_j} f_j \right\| > \lambda^k. \tag{4}$$

Consider now $a_k \in \ell_1$ such that $a_k \lambda^k \to \infty$. We define $\phi := \sum_{k=1}^\infty a_k \phi_k$. Since $a_k \in \ell_1$ and each $\|\phi_k\|_\infty = 1$, we have that $\phi \in H_\infty(B_{c_0})$ with $\|\phi\|_\infty \leq \|a_k\|_1$.

On the other hand, $T_{\overline{\phi}}(g_k) = \sum_{j=1}^\infty a_j T_{\overline{\phi_j}}(g_k) = a_k T_{\overline{\phi_k}}(g_k)$. We conclude from (4) that

$$\|T_{\overline{\phi_k}}(g_k)\|_{H_p(B_{c_0} \cap \ell_2)} \geq a_k \lambda^k \to \infty.$$

This implies that $T_{\overline{\phi}}$ is not bounded on $H_p(B_{c_0} \cap \ell_2)$. □

Theorem 6.4. *Let $p > 1$ and $\phi = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha$ be a formal series. Then $T_{\overline{\phi}} : H_p(B_{c_0} \cap \ell_2) \to H_p(B_{c_0} \cap \ell_2)$ defines a bounded operator if and only if*

$$\sup_{N \in \mathbb{N}} \|T_{\overline{\phi_N}}\|_{H_p(\mathbb{D}^N) \to H_p(\mathbb{D}^N)} < \infty.$$

Proof. It is plain that for each $N \in \mathbb{N}$, $\|T_{\overline{\phi_N}}\|_{H_p(\mathbb{D}^N)\to H_p(\mathbb{D}^N)} \leq \|T_{\overline{\phi}}\|$, because if f depends on the first N-variables then $T_{\overline{\phi}}f = T_{\overline{\phi_N}}f$.

Reciprocally, let $\phi = \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha$ be a formal series such that $\sup_{N\in\mathbb{N}} \|T_{\overline{\phi_N}}\|_{H_p(\mathbb{D}^N)\to H_p(\mathbb{D}^N)} < \infty$.

We will first prove that $M_\phi : H_p(B_{c_0} \cap \ell_2)^* \to H_p(B_{c_0} \cap \ell_2)^*$, $M_\phi f = \phi f$ defines a bounded operator with $\|M_\phi\| \leq \sup_{N\in\mathbb{N}} \|T_{\overline{\phi_N}}\|$. Since for each $N \in \mathbb{N}$, $M^*_{\phi_N} = T_{\overline{\phi_N}}$, we have that M_{ϕ_N} defines a bounded operator with norm $\|M_{\phi_N}\| \leq \sup_{N\in\mathbb{N}} \|T_{\overline{\phi_N}}\|$.

Let $g \in H_p(B_{c_0} \cap \ell_2)^*$ and consider the formal series given by ϕg. Then, $[\phi g]_N = \phi_N g_N = M_{\phi_N} g_N \in H_p(\mathbb{D}^N)^*$ with $\|[\phi g]_N\| \leq \|M_{\phi_N}\|\|g_N\| \leq \sup_{N\in\mathbb{N}} \|T_{\overline{\phi_N}}\|\|g\|_{H_p(B_{c_0}\cap\ell_2)^*}$. By Proposition 4.5, we get that $\phi g \in H_p(B_{c_0} \cap \ell_2)^*$. We conclude that for every $g \in H_p(B_{c_0} \cap \ell_2)^*$, $\phi g \in H_p(B_{c_0} \cap \ell_2)^*$, with $\|\phi g\| \leq \sup_{N\in\mathbb{N}} \|T_{\overline{\phi_N}}\|\|g\|$. This implies that M_ϕ is a well defined operator with $\|M_\phi\| \leq \sup_{N\in\mathbb{N}} \|T_{\overline{\phi}}\|$. Consequently $T_{\overline{\phi}} = M^*_\phi$ defines a bounded operator. $\square$

The above theorem may be restated in terms of the multipliers of $H_p(B_{c_0} \cap \ell_2)^*$.

Remark 6.5. *For each $N \in \mathbb{N}$, consider $T_p^N := H_\infty(\mathbb{D}^N)$ endowed with the Toeplitz operator norm given by $\|\phi\|_{T_p^N} := \|T_{\overline{\phi}}\|_{H_p(\mathbb{D}^N)\to H_p(\mathbb{D}^N)}$. Rephrasing Theorem 6.4, $\mathcal{M}(H_p(B_{c_0} \cap \ell_2)^*)$ is the space of functions for which $\phi_N \in T_p^N$ and $\sup_{N\in\mathbb{N}} \|\phi_N\|_{T_p^N} < \infty$*

The next theorem shows that $\mathcal{M}(H_p(B_{c_0}\cap\ell_2)^*)$ is a space of analytic functions contained in $H_\infty(B_{c_0})$. First, we state a likely well-known lemma.

Lemma 6.6. *Let $a \in L^\infty(\mathbb{T}^N)$, T_a the associated Toeplitz operator on $H_p(\mathbb{T}^N)$, $1 \leq p < \infty$. Then $r(T_a) \geq \|a\|_\infty$.*

Proof. Denote M_a and $S = M_{z_1\cdots z_N}$ the multiplication operators on $L^p(\mathbb{T}^N)$. Note that S is an isometry. Recall also that the approximate point spectrum of M_a, $\sigma_{ap}(M_a)$ coincides with the essential range of a.

If $\lambda \in \sigma_{ap}(M_a)$ there is a sequence of norm one trigonometrical polynomials $(q_k)_k$ in $L^p(\mathbb{T}^N)$ such that $(M_a - \lambda)q_k \to 0$. Let $d_k \geq 0$ be such that $S^{d_k}q_k \in H_p(\mathbb{T}^N)$ for each k. Thus $(S^{d_k}q_k)_k$ is a norm one sequence in $H_p(\mathbb{T}^N)$ such that

$$\begin{aligned}\|(T_a - \lambda)S^{d_k}q_k\|_p &= \|P_N(M_a - \lambda)S^{d_k}q_k\|_p = \|P_N S^{d_k}(M_a - \lambda)q_k\|_p \leq \sin(\pi/p)^{-N}\|S^{d_k}(M_a - \lambda)q_k\|_p \\ &= \sin(\pi/p)^{-N}\|(M_a - \lambda)q_k\|_p \to 0.\end{aligned}$$

Hence, $\sigma_{ap}(M_a) \subseteq \sigma_{ap}(T_a)$ and therefore $r(T_a) \geq r(M_a) \geq \|a\|_\infty$. $\square$

Theorem 6.7. *Let $1 < p < \infty$. Then $\mathcal{M}(H_p(B_{c_0} \cap \ell_2)^*)$ is contained in $H_\infty(B_{c_0})$.*

Proof. Let ϕ be a multiplier of $H_p(B_{c_0} \cap \ell_2)^*$. Each ϕ_N induces a Toeplitz operator $T_{\overline{\phi_N}} : H_p(\mathbb{D}^N) \to H_p(\mathbb{D}^N)$. From Lemma 6.6,

$$\|M_\phi\| = \|T_{\overline{\phi}}\| \geq \|T_{\overline{\phi_N}}\| \geq r(T_{\overline{\phi_N}}) \geq \|\phi_N\|_\infty.$$

On the other hand, since $1 \in H_p(B_{c_0} \cap \ell_2)^*$, we obtain that $\phi = \phi \cdot 1 \in H_p(B_{c_0} \cap \ell_2)^*$. By Theorem 4.11 we know that $B_{c_0} \cap \ell_2 \subseteq \operatorname{mon}(\jmath(H_p(B_{c_0} \cap \ell_2)^*))$. In particular, we have that

$$\sum_{\alpha \in \mathbb{N}^N} |c_\alpha z^\alpha| < \infty$$

for every $z \in \mathbb{D}^N$, where $\phi = \sum_\alpha c_\alpha z^\alpha$. It follows by the Hilbert Criterion [6, Theorem 2.21] that $\phi \in H_\infty(B_{c_0})$. $\square$

We now study the dynamics of Toeplitz operators in $H_p(B_{c_0} \cap \ell_2)$. Given $\phi(z) = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha \in H_\infty(B_{c_0})$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$, the key in the following theorem is the fact that the operator $M_\phi^* : H_q(B_{c_0} \cap \ell_2)^* \to H_q(B_{c_0} \cap \ell_2)^*$ is quasiconjugated to $T_{\bar{\phi}} : H_p(B_{c_0} \cap \ell_2) \to H_p(B_{c_0} \cap \ell_2)$ through the conjugate-linear inclusion $H_p(B_{c_0} \cap \ell_2) \hookrightarrow H_q(B_{c_0} \cap \ell_2)^*$ (see Remark 4.3). That is, the following diagram is commutative:

$$\begin{tikzcd} H_q(B_{c_0} \cap \ell_2)^* \arrow[r, "M_\phi^*"] & H_q(B_{c_0} \cap \ell_2)^* \\ H_p(B_{c_0} \cap \ell_2) \arrow[r, "T_{\bar{\phi}}"] \arrow[u, hook] & H_p(B_{c_0} \cap \ell_2) \arrow[u, hook] \end{tikzcd}$$

To prove quasiconjugation, by linearity and density of polynomials it suffices to consider the pairings evaluated on monomials. Thus, given $\alpha, \beta \in \mathbb{N}_0^{(\mathbb{N})}$ we have

$$\langle M_\phi^*(i(z^\alpha)), z^\beta \rangle = \langle z^{-\alpha}, \phi z^\beta \rangle = \begin{cases} 0 & \text{if } \alpha - \beta \notin \mathbb{N}_0^{(\mathbb{N})}, \\ c_{\alpha-\beta} & \text{if } \alpha - \beta \in \mathbb{N}_0^{(\mathbb{N})}. \end{cases}$$

Furthermore, from definition we have

$$T_{\bar{\varphi}}(z^\alpha) = \sum_{\substack{\gamma \in \mathbb{N}_0^{(\mathbb{N})} \\ \alpha - \gamma \in \mathbb{N}_0^{(\mathbb{N})}}} \bar{c}_\gamma z^{\alpha - \gamma},$$

which yields

$$\langle i(T_{\bar{\varphi}}(z^\alpha)), z^\beta \rangle = \begin{cases} 0 & \text{if } \alpha - \beta \notin \mathbb{N}_0^{(\mathbb{N})}, \\ c_{\alpha-\beta} & \text{if } \alpha - \beta \in \mathbb{N}_0^{(\mathbb{N})}. \end{cases}$$

Theorem 6.8. *Let $\phi \in \mathcal{M}(H_p(B_{c_0} \cap \ell_2)^*)$. The following are equivalent:*

(1) $T_{\bar{\phi}} : H_p(B_{c_0} \cap \ell_2) \to H_p(B_{c_0} \cap \ell_2)$ has a recurrent vector.
(2) $T_{\bar{\phi}} : H_p(B_{c_0} \cap \ell_2) \to H_p(B_{c_0} \cap \ell_2)$ is chaotic.
(3) There exists $N_0 \in \mathbb{N}$ such that $T_{\bar{\phi}_N} : H_p(\mathbb{D}^N) \to H_p(\mathbb{D}^N)$ is chaotic for all $N \geq N_0$.
(4) $\phi(B_{c_0}) \cap \mathbb{T} \neq \emptyset$.

Proof. (1) $\Rightarrow$ (4). If $f \in H_p(B_{c_0} \cap \ell_2)$ is a recurrent vector of $T_{\bar{\phi}}$, there is an increasing sequence $(n_k)_{k \in \mathbb{N}} \subset \mathbb{N}$ such that $T_{\bar{\phi}}^{n_k}(f) \to f$ as $k \to \infty$. This means that $i(f)$ is a recurrent vector of $M_\phi^* : H_q(B_{c_0} \cap \ell_2)^* \to H_q(B_{c_0} \cap \ell_2)^*$, since $(M_\phi^*)^{n_k}(i(f)) = i(T_{\bar{\phi}}^{n_k}(f)) \to i(f)$ as $k \to \infty$. Because M_ϕ^* has an orbit with a nonzero limit point, it follows from Theorem 5.4 that it is hypercyclic, and by Theorem 5.3 we can conclude that $\phi(B_{c_0}) \cap \mathbb{T} \neq \emptyset$.

(4) $\Rightarrow$ (3) follows from Theorem 5.3 and from the fact that $M^*_{\phi_N}$ and $T_{\bar{\phi}_N}$ are conjugates through a conjugate-linear isomorphism.

(2) $\Rightarrow$ (1) is immediate.

(3) $\Rightarrow$ (2). For all $N \in \mathbb{N}$, the operators $T_{\bar{\phi}} : H_p(B_{c_0} \cap \ell_2) \to H_p(B_{c_0} \cap \ell_2)$ and $T_{\bar{\phi}_N} : H_p(\mathbb{D}^N) \to H_p(\mathbb{D}^N)$ verify $T_{\bar{\phi}} \circ i_N = i_N \circ T_{\bar{\phi}_N}$, where i_N is the usual inclusion $H_p(\mathbb{D}^N) \hookrightarrow H_p(B_{c_0} \cap \ell_2)$. Since $T_{\bar{\phi}_N}$ is chaotic for all $N \geq N_0$, each operator $T_{\bar{\phi}_N}$ has a dense set of periodic points. It follows from $T_{\bar{\phi}} \circ i_N = i_N \circ T_{\bar{\phi_N}}$ that each periodic point of $T_{\bar{\phi}_N}$ translates into a periodic point of $T_{\bar{\phi}}$. This means that $T_{\bar{\phi}}$ has a dense set of periodic points, since $\bigcup_{N \geq N_0} H_p(\mathbb{D}^N)$ is dense in $H_p(B_{c_0} \cap \ell_2)$.

It remains to prove that $T_{\bar{\phi}}$ is hypercyclic. Let $U, V \subset H_p(B_{c_0} \cap \ell_2)$ be nonempty open sets, and let $N_0 \in \mathbb{N}$ such that $T_{\bar{\phi}_N}$ is hypercyclic for every $N \geq N_0$. For any such N consider the inclusion $i_N : H_p(\mathbb{D}^N) \to H_p(B_{c_0} \cap \ell_2)$.

Since $T_{\bar{\phi}_N}$ is hypercyclic, it is topologically transitive, and thus there exist $f \in {i_N}^{-1}(U)$ and $k \in \mathbb{N}$ such that $T^k_{\bar{\phi}_N}(f) \in {i_N}^{-1}(V)$. Clearly $i_N(f) \in U$, and we have

$$T^k_{\bar{\phi}}(i_N(f)) = i_N(T^k_{\bar{\phi}_N}(f)) \in V.$$

This means that $T_{\bar{\phi}}$ is topologically transitive, and thus hypercyclic. $\square$

7. Composition operators

Let us recall that by [7, Theorem 11], each analytic function $\Phi : \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ inducing a composition operator on $\mathcal{H}_p$ can be extended analytically to $\mathbb{C}_+$ and is of the form $\Phi(s) = c_0 + \varphi(s)$, where $c_0 \in \mathbb{N}_0$ and φ admits representation as a convergent Dirichlet series in some half-plane. Furthermore, $\Phi(\mathbb{C}_+) \subset \mathbb{C}_+$ if $c_0 > 0$ (in which case C_Φ is a contraction) and $\Phi(\mathbb{C}_+) \subset \mathbb{C}_{1/2}$ if $c_0 = 0$. If $p \geq 2$, this representation characterizes composition operators on $\mathcal{H}_p$ completely, whereas in the case $1 \leq p < 2$ there exist functions of this form which do not induce composition operators on $\mathcal{H}_p$.

Proposition 7.1. *No composition operator on $\mathcal{H}_p$ is hypercyclic for $1 \leq p < \infty$.*

Proof. As noted above, $\Phi(s) = c_0 + \varphi(s)$ where $c_0 \in \mathbb{N}_0$ and φ can be represented as a convergent Dirichlet series in some half-plane. We suppose $c_0 = 0$, since otherwise C_Φ is a contraction and therefore not hypercyclic.

Given a composition operator $C_\Phi : \mathcal{H}_p \to \mathcal{H}_p$, we calculate

$$|C^n_\Phi(f)(s)|^p = |\delta_{\Phi^n(s)}(f)|^p \leq \|f\|^p \|\delta_{\Phi^n(s)}\|^p = \|f\|^p \zeta(2\Re(\Phi^n(s))).$$

We also have $\Phi(\mathbb{C}_+) \subset \mathbb{C}_{1/2}$, and moreover there exists $\varepsilon > 0$ such that $\Phi^n(\mathbb{C}_+) \subset \mathbb{C}_{1/2+\varepsilon}$ for each $n \geq 2$. Thus $\zeta(2\Re(\Phi^n(s))) \leq \zeta(1+2\varepsilon)$, and we reach the inequality

$$|C_\Phi^n(f)(s)|^p \leq \|f\|^p \max\{\zeta(2\Re(\Phi(s))), \zeta(1+2\varepsilon)\} := \|f\|^p C(s), \tag{5}$$

valid for all $n \in \mathbb{N}$. This, together with the continuity of evaluations, implies that the orbit

$$O(f, C_\Phi) = \{C_\Phi^n(f) : n \in \mathbb{N}\}$$

is not dense in $\mathcal{H}_p$. □

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