DUAL HARDY SPACES OF DIRICHLET SERIES, MULTIPLICATIVE TOEPLITZ OPERATORS AND THEIR DYNAMICS

RODRIGO CARDECCIA, SANTIAGO MURO, MATÍAS PALUMBO

Abstract. We study multiplicative Toeplitz operators on Hardy spaces of Dirichlet series $\mathcal{H}_p$. To this aim, we first study structural properties of $\mathcal{H}_p^*$ that are parallel to those of $\mathcal{H}_p$. Among other results, we prove a Cole-Gamelin type inequality and we find the abscissa of absolute convergence of $\mathcal{H}_p^*$. We also identify $\mathcal{H}_p^*$ with a space of co-analytic series on $\mathbb{D}_2^\infty$. We apply these results to describe the dynamics of adjoint multiplication operators and multiplicative Toeplitz operators, and show among other results that they are hypercyclic if and only if the image of the symbol intersects the unit circle $\mathbb{T}$.

1. Introduction

The study of Dirichlet series—functions of the form $\sum_{n=1}^\infty a_n n^{-s}$ where s is a complex variable—occupies a central position in complex analysis and analytic number theory. Since their introduction by Dirichlet in the 19th century to study arithmetic progressions, these series have revealed profound connections to diverse areas of mathematics. A key aspect of the study of Dirichlet series came with Harald Bohr's work, which established that the theory of Dirichlet series is intimately linked to function theory in infinitely many variables through what is now known as the Bohr transform. The importance of this correspondence, which identifies Dirichlet series with power series $\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha$ on the infinite-dimensional polydisc, was emphasized by the work of Hedenmalm, Lindqvist and Seip [29], who developed it using modern functional analysis language and applied it to the study of some spaces of Dirichlet series. This approach, showing the interplay between Dirichlet series and infinite dimensional function theory, has been fundamental for several advances in this field.

A significant development in this framework was the introduction of Hardy spaces of Dirichlet series, denoted $\mathcal{H}_p$, which serve as a natural generalization of classical Hardy spaces on the disc $H_p(\mathbb{D})$ to the setting of Dirichlet series. These spaces, first studied by Hedenmalm, Lindqvist, and Seip [29] (for $p = 2$) and later expanded by Bayart [5], exhibit rich interplay between harmonic analysis, operator theory, and number theory. The space $\mathcal{H}_p$ is isometrically isomorphic to a subspace of holomorphic functions on the half-plane $\mathbb{C}_{1/2} = \{s \in \mathbb{C} : \mathrm{Re}(s) > 1/2\}$, and its multiplier algebra $\mathcal{H}_\infty$ is linked to bounded analytic functions on $\mathbb{C}_0$. However, unlike the classical setting of the unit disk $\mathbb{D}$, where the dual space $H_p(\mathbb{D})^*$ is well understood, the structure of $\mathcal{H}_p^*$ for $p \neq 2$ remains elusive, a gap highlighted by Saksman and Seip [39, Problem 2.8] (see also [8, 38]).

This duality gap becomes particularly relevant when studying the behavior of some operators on $\mathcal{H}_p$. In the classical Hardy space $H_2(\mathbb{D})$, the dynamical properties of adjoint multiplication operators M_φ^* (or

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equivalently, co-analytic Toeplitz operators $T_{\overline{\varphi}}$) is an important line of research (see e.g. [13, 20, 21, 42]). Their hypercyclicity is fully characterized in a seminal article due to Godefroy and Shapiro [23]: M_φ^* is hypercyclic if and only if the image of the symbol φ intersects the unit circle $\mathbb{T}$. This result extends to more general domains $\Omega \subset \mathbb{C}^N$ under the key assumption that $\|M_\varphi\| \leq \sup_{z\in\Omega} |\varphi(z)|$. However, in the setting of Dirichlet series, this framework breaks down. The elements of Hardy spaces $\mathcal{H}_2$ are analytic on $\mathbb{C}_{1/2}$, while their multipliers are defined via boundedness on $\mathbb{C}_0$, violating the Godefroy-Shapiro's Theorem hypothesis. Moreover, for $p \neq 2$, the lack of a duality isomorphism between $\mathcal{H}_p^*$ and $\mathcal{H}_{p'}$ also breaks the analogous equivalence between adjoint multiplication and co-analytic (multiplicative) Toeplitz operators, raising fundamental questions about the nature of hypercyclicity in this context.

This leads us to the central questions of this work. First, we examine whether a version of Godefroy-Shapiro's theorem might hold despite these obstacles:

Question A. Let $D \in \mathcal{H}_\infty$. Can the hypercyclicity of $M_D^* : \mathcal{H}_p^* \to \mathcal{H}_p^*$ be characterized by the condition $D(\mathbb{C}_+) \cap \mathbb{T} \neq \emptyset$?

In the language of Toeplitz operators we may ask for the validity following non-equivalent formulation of a Godefroy-Shapiro Theorem (see Section 4 for precise definitions).

Question B. Let D be a Dirichlet series in $\mathcal{H}_\infty$ which induces a (multiplicative) co-analytic Toeplitz operator $T_{\jmath D} : \mathcal{H}_p \to \mathcal{H}_p$. Is the hypercyclicity of $T_{\jmath D}$ characterized by the condition $D(\mathbb{C}_+) \cap \mathbb{T} \neq \emptyset$?

In the latter case of multiplicative Toeplitz operators on $\mathcal{H}_p$, it is not even known which are the Dirichlet series D (over $\mathbb{Q}_+$, see Section 3.2 for a description) that are symbols of bounded Toeplitz operators on $\mathcal{H}_p$. One way to describe them is to consider the dual space $\mathcal{H}_p^*$ as a function space and determine the multiplication operators therein. A major difficulty is that, as previously mentioned, the spaces $\mathcal{H}_p^*$ are yet not well understood. So, in order to tackle Question B, it would be useful to investigate dual Hardy spaces of Dirichlet series. We are particularly interested in finding good structural properties of $\mathcal{H}_p^*$ parallel to those of $\mathcal{H}_p$.

It is an fundamental fact that, via the Bohr transform, $\mathcal{H}_p$ is isometrically isomorphic to a space of analytic functions of infinitely many variables, specifically on the open set of ℓ_2, $B_{c_0} \cap \ell_2$.

Question C. Can $\mathcal{H}_p^*$ be realized as a space of analytic functions on $\mathbb{C}_{1/2}$ or identified with a function space on a subset of ℓ_2? Does it inherit some of the key properties of $\mathcal{H}_p$?

Question D. Which Dirichlet series (over $\mathbb{Q}_+$) define bounded (multiplicative) Toeplitz operators on $\mathcal{H}_p$?

We provide answers to Questions A-C and some partial results concerning Question D. After a Preliminaries Section, in Section 3 we will show that $\mathcal{H}_p^*$ can be described as a space of co-analytic Dirichlet series on a left-half plane. We also prove there a Hilbert-type criterion and a Cole-Gamelin type inequality. To this end we need to investigate the image under the Bohr transform of $\mathcal{H}_p^*$ as a function space on the unit ball of c_0, and to determine its set of monomial convergence. In Section 4 we proceed to study multiplicative Toeplitz operators on $\mathcal{H}_p$ and relate them with the multipliers on $\mathcal{H}_p^*$. We show that when the symbol is a Dirichlet polynomial (over $\mathbb{Q}_+$), the associated Toeplitz operator is bounded on $\mathcal{H}_p$ and that the Bohr transform of symbols of bounded Toeplitz operators must be bounded on $\mathbb{T}^\infty$. However, we also show that there are bounded symbols (their Bohr transform may even be bounded on rB_{c_0} for some $r > 1$) that do not define bounded Toeplitz operators; so in contrast to the $\mathcal{H}_p$ case, the space of multipliers of $\mathcal{H}_p^*$ cannot be identified with $\mathcal{H}_\infty$. In Section 5 we tackle the questions on dynamics. To give a complete answer we need first to study the finite dimensional case and then the infinite dimensional case of adjoint multipliers. This is translated through the Bohr transform to adjoint multipliers of Dirichlet series and finally apply these results to answer Question B on the hypercyclicity (and also frequently hypercyclicity and chaos) of Toeplitz operators on $\mathcal{H}_p$. We also prove more precise results about the dynamic behavior of adjoint multipliers (we call them Trichotomy results following the one dimensional case in [10]): if a (non-constant) adjoint multiplier is not hypercyclic then either all orbits tend to 0, or all tend to infinity. Finally, in the last section we give some remarks and present some related open problems.

2. Preliminaries

We provide a comprehensive overview of both Hardy spaces of Dirichlet series and Linear Dynamics, to make the article as self-contained as possible

2.1. **Hardy spaces of Dirichlet series.** A Dirichlet series is a (formal) series of the form

$$D = \sum_{n=1}^{\infty} a_n n^{-s},$$

where the a_n are complex numbers and s denotes the complex variable. The Dirichlet series with $a_n = 1$ for all n is the classical Riemann zeta function ζ.

A (formal) power series in infinitely many variables is a series of the form

$$\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_{\alpha_1,\dots,\alpha_k} z_1^{\alpha_1} \cdot z_2^{\alpha_2} \cdots z_k^{\alpha_k},$$

where $z = (z_1, z_2, \dots)$ is a sequence of complex numbers and, given a set X, we denote by $X^{(\mathbb{N})}$ to the set of all finite sequences $(x_1, \dots, x_k) \in X^k$ and $k \in \mathbb{N}$. Dirichlet series and power series can be identified via the Bohr transform. Let $\mathfrak{p} = (\mathfrak{p}_k)_k$ be the sequence of prime numbers. Given $n \in \mathbb{N}$, the Fundamental Theorem of Arithmetic implies that there is a unique $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$, $\alpha = (\alpha_1, \dots, \alpha_k)$ such that $n = \mathfrak{p}^\alpha := \mathfrak{p}_1^{\alpha_1} \dots \mathfrak{p}_k^{\alpha_k}$. Thus, the Bohr transform $\mathfrak{B}$ is formally defined by

$$\begin{array}{cccc} & & (a_n := c_{p^\alpha}) & \\ \mathfrak{B}: & \textbf{Power Series} & \longrightarrow & \textbf{Dirichlet Series} \\ & \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha & & \sum_{n=1}^{\infty} a_n n^{-s}. \end{array}$$

This transform is a Banach space isomorphism between the corresponding Hardy function spaces, which we describe next.

Let $\mathbb{T}^N$ the N-dimensional torus endowed with the normalized Lebesgue measure. For $1 \le p \le \infty$ the Hardy space $H_p(\mathbb{T}^N)$ is the subspace of $L_p(\mathbb{T}^N)$ of functions with vanishing Fourier coefficients in the complement of the non-negative cone $\mathbb{N}_0^N$, i.e.,

$$H_p(\mathbb{T}^N) := \{f \in L_p(\mathbb{T}^N) : \int_{\mathbb{T}^N} f(w) w^{-\alpha} dw = 0 \ \forall \alpha \in \mathbb{Z}^N \setminus \mathbb{N}_0^N\}.$$

Analogously, $\mathbb{T}^\infty$ denotes the infinite dimensional polytorus endowed with the product measure and

$$H_p(\mathbb{T}^\infty) = \{f \in L_p(\mathbb{T}^\infty) : \int_{\mathbb{T}^\infty} f(w)w^{-\alpha}dw = 0 \ \forall \alpha \in \mathbb{Z}^{(\mathbb{N})} \setminus \mathbb{N}_0^{(\mathbb{N})}\}.$$

For $1 < p < \infty$, the Fourier basis $(w^\alpha)_{\alpha \in \mathbb{N}^N} = (w_1^{\alpha_1} \cdots w_N^{\alpha_k})_{\alpha \in \mathbb{N}_0^N}$ in the case of $H_p(\mathbb{T}^N)$ and $(w^\alpha)_{\alpha \in \mathbb{N}^{(\mathbb{N})}} = (w_1^{\alpha_1} \cdots w_k^{\alpha_k})_{\alpha \in \mathbb{N}^{(\mathbb{N})}}$ in the case of $H_p(\mathbb{T}^\infty)$ is a Schauder basis, see [1].

Functions in $H_p(\mathbb{T}^N)$ extend analytically to $\mathbb{D}^N$ via the N-dimensional Poisson operator

$$P_N : \sum_{\alpha \in \mathbb{N}_0^N} c_\alpha w^\alpha \mapsto \sum_{\alpha \in \mathbb{N}_0^N} c_\alpha z^\alpha.$$

The Hardy space $H_p(\mathbb{D}^N)$ is the image of $H_p(\mathbb{T}^N)$ via the Poisson operator P_N. This space has an intrinsic norm given by $\|f\|_{H_p(\mathbb{D}^N)} = \sup_{0<r_1,\dots r_N \leq 1} \|f(rz)\|_{H_p(\mathbb{T}^N)}$. For $p = \infty$ the norm just becomes $\|f\|_{H_\infty(\mathbb{D}^N)} = \sup_{z \in \mathbb{D}^N} |f(z)|$. Reciprocally, if a function f belongs to $H_p(\mathbb{D}^N)$ then its radial limit $\lim_{r \to 1} f(rz) \in H_p(\mathbb{T}^N)$.

In a similar way, functions in $H_p(\mathbb{T}^\infty)$ have an analytic extension to $H_p(\mathbb{D}_2^\infty)$ if $p < \infty$ and to $H_\infty(B_{c_0})$ when $p = \infty$; here we denote $\mathbb{D}_2^\infty := B_{c_0} \cap \ell_2$, which is an open subset of ℓ_2. The space $H_\infty(B_{c_0})$ is the space of holomorphic and bounded functions in B_{c_0} and the norm is given by $\|f\|_{H_\infty(B_{c_0})} = \sup_{z \in B_{c_0}} |f(z)|$.

Given a formal power series $f = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha$, we consider for each $N \in \mathbb{N}$ its restriction to N variables via

$$f_N(z_1, \dots, z_N) := f(z_1, \dots, z_N, 0 \dots) = \sum_{\alpha \in \mathbb{N}^N} c_\alpha z^\alpha.$$

The space $H_p(\mathbb{D}_2^\infty)$ consists on all the analytic functions in $\mathbb{D}_2^\infty$ such that

(1) for each $N \in \mathbb{N}$, $f_N(z) = f(z_1, \dots, z_N, 0, \dots) \in H_p(\mathbb{D}^N)$ and
(2) $\sup_{N \in \mathbb{N}} \|f_N\|_{H_p(\mathbb{D}^N)} < \infty$.

The norm is given by

$$\|f\|_{H_p(\mathbb{D}_2^\infty)} = \sup_{N \in \mathbb{N}} \|f_N\|_{H_p(\mathbb{D}^N)} = \sup_{N \in \mathbb{N}} \sup_{0<r<1} \left(\int_{\mathbb{T}^N} |f(rw)|^p dw \right)^{\frac{1}{p}}.$$

For each $N \in \mathbb{N}$, we denote by $i_N : H_p(\mathbb{D}^N) \to H_p(\mathbb{D}_2^\infty)$ and $\pi_N : H_p(\mathbb{D}_2^\infty) \to H_p(\mathbb{D}^N)$ the canonical inclusion and projection, respectively.

The infinite Poisson operator $P_\infty : \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha w^\alpha \to \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha$ is a surjective isometry between $H_p(\mathbb{T}^\infty)$ and $H_p(\mathbb{D}_2^\infty)$ and between $H_\infty(\mathbb{T}^\infty)$ and $H_\infty(B_{c_0})$.

As mentioned before, power series can be identified formally with Dirichlet series via the Bohr transform $\mathfrak{B}$. For $1 \leq p < \infty$ the space $\mathcal{H}_p$ of Dirichlet series is the image of the Hardy space $H_p(\mathbb{D}_2^\infty)$ and for $p = \infty$ $\mathcal{H}_\infty$ is the image of $H_\infty(B_{c_0})$ via the Bohr transform $\mathfrak{B}$. They also have an intrinsic norm. When $p = \infty$ we obtain $\mathcal{H}_\infty$ the space of holomorphic and bounded Dirichlet series in $\mathbb{C}_+$, with norm

$\|D(s)\|_{\mathcal{H}_\infty} = \sup_{Re(s)>0} |D(s)|$. When $p < \infty$ the norm is given by the Besicovitch norm

$$\lim_{k\to\infty} \lim_{R\to\infty} \left(\frac{1}{2R} \int_{-R}^{R} \left| \sum_{n=1}^{k} a_n n^{-it} \right|^p dt) \right)^{\frac{1}{p}}.$$

Alternatively, $\mathcal{H}_p$ is the completion of the Dirichlet polynomials $\sum_{n=1}^{k} a_n n^{-s}$ under the Besicovitch norm. It is worth noting that $(n^{-s})_n$ was proved to be a Schauder basis for $\mathcal{H}_p$ with $p > 1$ [1].

For $1 \leq p < \infty$ the Dirichlet series in $\mathcal{H}_p$ are holomorphic (at least) on the half-plane $Re(s) > \frac{1}{2}$ while every Dirichlet series in $\mathcal{H}_\infty$ is holomorphic on $Re(s) > 0$.

Given $N \in \mathbb{N}$, $\mathcal{H}_{p,N}$ denotes the space of Dirichlet series depending on the first N primes and given $D \in \mathcal{H}_p$, $D_N := \sum_{n=\mathfrak{p}^\alpha, \alpha\in\mathbb{N}^N} a_n n^{-s}$ denotes its restriction to $\mathcal{H}_{p,N}$. The operator $\pi : \mathcal{H}_p \to \mathcal{H}_{p,N}$, $D \mapsto D_N$ is a norm one operator with $\|D\|_{\mathcal{H}_p} = \sup_{N\in\mathbb{N}} \|D_N\|_{\mathcal{H}_{p,N}}$.

We will also consider another version of the Bohr transform, that we denote by $\mathscr{B}$ whose domain are analytic trigonometric series $\mathbb{T}^\infty$, with only nonzero coefficients on the nonnegative cone $\mathbb{N}_0^{(\mathbb{N})} \subset \mathbb{Z}^{(\mathbb{N})}$:

$$\mathscr{B}: \begin{array}{ccc} \textbf{Analytic Trigonometric Series} & \xrightarrow{(a_n := c_{p^\alpha})} & \textbf{Dirichlet Series} \\ \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} c_\alpha w^\alpha & & \sum_{n=1}^{\infty} a_n n^{-s}. \end{array}$$

For Hardy spaces, $\mathscr{B}$ is a surjective isometry from $H_p(\mathbb{T}^\infty)$ to $\mathcal{H}_p$ and of course we can link both versions of the Bohr transform through the Poisson operator via the identification

$$\mathscr{B} = \mathfrak{B} P_\infty.$$

Given a Banach space of analytic functions X, the multipliers of X is the subspace of all the analytic functions φ for which $M_\varphi f = \varphi f$ defines a bounded operator,

$$\mathcal{M}(X) := \{\varphi \in X : \varphi \cdot f \in X \text{ for every } f \in X\}.$$

It is known from [5, 29] that for every $1 \leq p < \infty$, $\mathcal{M}(\mathcal{H}_p) = \mathcal{H}_\infty$. Applying the Bohr transform, $\mathcal{M}(H_p(\mathbb{D}_2^\infty)) = H_\infty(B_{c_0})$.

For more on Hardy spaces of Dirichlet series we refer to the books [19, 36].

2.2. **Linear Dynamics.** Given an infinite dimensional and separable Banach space X, a linear operator $T : X \to X$ is said to be hypercyclic provided that there exist $x \in X$ such that

$$Orb_T(x) := \{T^n(x) : n \in \mathbb{N}\}$$

is dense in X. Equivalently, Birkhoff's transitivity Theorem states that an operator is hypercyclic if and only if it is transitive, which means that for every non empty open sets U, V there is $n \in \mathbb{N}$ such that

$$T^n(U) \cap V \neq \emptyset.$$

A vector $x \in X$ is said to be recurrent if for every open $U \ni x$ we have that $N(x, U) := \{n \in \mathbb{N} : T^n(x) \in U\}$ is not empty (and hence infinite). An operator is said to be recurrent if it has a dense set of recurrent vectors.

An operator $T : X \to X$ is said to be chaotic if it is hypercyclic and it has a dense set of periodic vectors, i.e. $\{x \in X : \exists n \in \mathbb{N} \text{ with } T^n(x) = x\}$ is dense in X.

A set $A \subseteq \mathbb{N}_0$ is said to have positive lower density whenever

$$\underline{d}(A) := \liminf_{n\to\infty} \frac{\#A \cap [0, n]}{n + 1}$$

and an operator is said to be frequently hypercyclic if there is a vector $x \in X$ such that for every nonempty open set U,

$$N(x, U) \text{ has positive lower density.}$$

All these properties are preserved via quasi-conjugacy. An operator $S : Y \to Y$ is said to be *quasi-conjugated* to $T : X \to X$ provided that there exist a dense and continuous map $L : X \to Y$ such that $SL = LX$. If L is invertible we say that S is *conjugated* to T.

$$\begin{array}{ccc} X & \xrightarrow{T} & X \\ \big\downarrow{\scriptstyle L} & & \big\downarrow{\scriptstyle L} \\ Y & \xrightarrow{S} & Y \end{array}$$

Quasi-conjugacy preserves the usual dynamical properties, i.e., if T is hypercyclic (or chaotic, etc) then so is S. In the case that L is not a dense map, we still have regular pointwise preservation, if x is a recurrent vector or a periodic vector for T, then so is $L(x)$ for S.

We conclude the preliminaries by recalling the seminal Godefroy-Shapiro Theorem [23] for the Hardy space case (in fact, it is proved there for a wide class of Hilbert spaces of analytic functions in several variables; see also [11]).

Theorem 2.1 (Godefroy-Shapiro)**.** *Let $\phi \in H_\infty(\mathbb{D}^N)$. Then, $M_\phi : H_2(\mathbb{D}^N)^* \to H_2(\mathbb{D}^N)^*$ is hypercyclic (and chaotic) if and only if $\phi(\mathbb{D}^N) \cap \mathbb{T} \neq \emptyset$.*

This theorem was extended in [37] to some reflexive spaces of one variable analytic functions, including the case of the Hardy spaces of the disc $H_p(\mathbb{D})$, $1 < p < \infty$ (see also [27, Exercise 4.4.5]).

3. The dual of $\mathcal{H}_p$

To answer the problems of linear dynamics (Questions A and B), we need first to investigate properties of $\mathcal{H}_p^*$ parallel to those of $\mathcal{H}_p$.

3.1. **Elementary definitions and the abscissa of convergence.** Since for $1 < p < \infty$, $\mathcal{H}_p$ is reflexive and $(n^{-s})_{n\in\mathbb{N}}$ is a Schauder basis for $\mathcal{H}_p$ [1] (although it is unconditional only for $p = 2$ [16, 17]), a first approach is to identify the dual of $\mathcal{H}_p$ with a space of (formal) Dirichlet series

$$\{D : \langle D, E\rangle_{\mathcal{H}_2} \text{ converges for every } E \in \mathcal{H}_p\}, \tag{1}$$

endowed with the dual norm, i.e.

$$\|D\| = \sup_{E\in\mathcal{H}_p, \|E\|_p\leq 1} \langle D, E\rangle_{\mathcal{H}_2}.$$

Formally speaking, this space is not the linear dual Banach space of $\mathcal{H}_p$. Indeed, we obtain conjugate-linear functionals. Another equivalent procedure, which we will adopt, is to consider Dirichlet series in dual Hardy spaces as *co-analytic* series. A formal co-analytic Dirichlet series is a series of the form

$$\sum_{n=1}^{\infty} a_n n^s.$$

Thus co-analytic series converge on left half-planes $\{Re(s) < c\}$. Associated with co-analytic Dirichlet series we will consider a "conjugation" map $\jmath$,

$$\jmath : \sum_{n=1}^{\infty} a_n n^s \to \sum_{n=1}^{\infty} \overline{a}_n n^{-s}.$$

The motivation to consider dual Hardy Dirichlet series as co-analytic series comes from harmonic analysis, where we have natural duality relations. Indeed, if $f = \sum_{\alpha\in\mathbb{N}^{(\mathbb{N})}} c_\alpha \overline{w}^\alpha$ is a co-analytic polynomial and $g = \sum_{\alpha\in\mathbb{N}^{(\mathbb{N})}} d_\alpha w^\alpha$ is an analytic polynomial, then

$$\int_{\mathbb{T}^N} fg = \sum_{\alpha\in\mathbb{N}^{(\mathbb{N})}} c_\alpha d_\alpha.$$

Similarly in the Dirichlet series setting, co-analytic Dirichlet series satisfy a duality relation by integrating in the imaginary axis.

Proposition 3.1. *Let $D = \sum_{n=1}^{k} d_n n^s$ be a co-analytic Dirichlet polynomial and $E = \sum_{n=1}^{k} e_n n^{-s}$ be an analytic Dirichlet polynomial. Then*

$$\lim_{R\to\infty} \frac{1}{2R}\int_{-R}^{R} D(it)E(it)dt = \sum_{n=1}^{k} d_n e_n.$$

Proof. By linearity it suffices to check that

$$\lim_{R\to\infty} \frac{1}{2R}\int_{-R}^{R} n^{it} m^{-it} dt = \delta_{n,m}$$

for every $n, m \in \mathbb{N}$. If $n = m$ we obtain that $\frac{1}{2R}\int_{-R}^{R} n^{it}m^{-it}dt = \frac{1}{2R}\int_{-R}^{R} 1dt = 1$. On the other hand, if $n \neq m$ we have

$$\frac{1}{2R}\int_{-R}^{R} n^{it}m^{-it}dt = \frac{\sin(R\log(n/m))}{R\log(n/m)},$$

which tends to zero as $R \to \infty$. $\square$

We will hence identify $\mathcal{H}_p^*$ with the Banach space of formal co-analytic series $D = \sum_{n=1}^{\infty} d_n n^s$ such that $\sum_{n=1}^{\infty} d_n e_n$ converges for every $E = \sum_{n=1}^{\infty} e_n n^{-s} \in \mathcal{H}_p$. The norm is given by

$$\|D\|_{\mathcal{H}_p^*} = \sup_{\|E\|_{\mathcal{H}_p}\leq 1} |\langle D, E\rangle| := \sup_{\|E\|_{\mathcal{H}_p}\leq 1} \left|\sum_{n=1}^{\infty} d_n e_n\right|.$$

With this identification, $(n^s)_{n\in\mathbb{N}}$ is the dual basis of $(n^{-s})_{n\in\mathbb{N}}$, and the duality can also be given by

$$\langle n^s, E\rangle = \lim_{k\to\infty}\lim_{R\to\infty} \frac{1}{2R}\int_{-R}^{R} n^s \cdot \sum_{j=1}^{k} e_j j^{-it} dt = e_n.$$

Note also that, if we denote by $\overline{\mathcal{H}_p^*}$, the image of the conjugate-linear mapping $\jmath : \mathcal{H}_p^* \to \overline{\mathcal{H}_p^*}$, then $\overline{\mathcal{H}_p^*}$ is the space given by (1).

We will study now properties of dual Hardy spaces parallel to those of $\mathcal{H}_p$. So far we have worked with formal co-analytic Dirichlet series and we wonder which is the largest half-plane where all series in $\mathcal{H}_p^*$ converge and whether they are holomorphic in the half-plane $Re(s) < -\frac{1}{2}$.

Given a Banach space of co-analytic Dirichlet series X, we consider its abscissa of convergence which is $\sigma_c(X) = \sup\{c \in \mathbb{R} : D(w) \text{ converges } \forall\, Re(w) < c, D \in X\}$. Co-analytic Dirichlet series are holomorphic in $Re(w) < \sigma_c(X)$ [19, Theorem 1.1].

Theorem 3.2. *Let $1 < p < \infty$. Then $\sigma_c(\mathcal{H}_p^*) = -\frac{1}{2}$.*

Proof. Given a (formal) co-analytic Dirichlet series $\sum_{n=1}^\infty a_n n^s$, we have that $D(w) = \langle D, \zeta(\cdot - w)\rangle = \sum_{n=1}^\infty a_n n^w$. Hence, $D(w)$ converges for every $D \in \mathcal{H}_p^*$ if and only if $\zeta(\cdot - w) \in \mathcal{H}_p$.

For an arbitrary Dirichlet series E, let $\sigma_{\mathcal{H}_p}(E) := \inf\{\sigma \in \mathbb{R} : \sum_{n=1}^\infty a_n n^{-s-\sigma} \in \mathcal{H}_p\}$. By [19, Theorem 12.7], we have that for every Dirichlet series E, $\sigma_{\mathcal{H}_p}(E) = \sigma_{\mathcal{H}_2}(E)$. Therefore, $\zeta(\cdot - w) \in \mathcal{H}_p$ if and only if $Re(-w) > \sigma_{\mathcal{H}_p}(\zeta) = \sigma_{\mathcal{H}_2}(\zeta) = \frac{1}{2}$. $\square$

Another natural question is which is the abscissa of absolute convergence of $\mathcal{H}_p^*$,

$$\sigma_a(\mathcal{H}_p^*) := \sup\{\sigma \in \mathbb{R} : D(w) \text{ converges absolutely for every } Re(w) < \sigma \text{ and every } D \in \mathcal{H}_p^*\}.$$

Clearly, $\sigma_a(\mathcal{H}_p^*) \le \sigma_c(\mathcal{H}_p^*)$. For $p < 2$ we have an easy solution, since $\mathcal{H}_p^* \hookrightarrow \mathcal{H}_2^*$ (because $\mathcal{H}_2 \hookrightarrow \mathcal{H}_p$) and $\sigma_a(\mathcal{H}_2^*) = -\frac{1}{2}$, we conclude that $\sigma_a(\mathcal{H}_p^*) \ge -\frac{1}{2} = \sigma_c(\mathcal{H}_p^*)$. Thus, $\sigma_a(\mathcal{H}_p^*) = -\frac{1}{2}$.

In Theorem 3.11 we will prove that for $p > 2$ we also have that $\sigma_a(\mathcal{H}_p^*) = -\frac{1}{2}$. To this aim we will study $H_p(\mathbb{D}_2^\infty)^*$ as a space of co-analytic power series.

3.2. **Properties of $H_p(\mathbb{D}_2^\infty)^*$ parallel to $H_p(\mathbb{D}_2^\infty)$.** The aim of this section is to describe $H_p(\mathbb{D}_2^\infty)^*$ and $H_p(\mathbb{T}^\infty)^*$ as spaces of co-analytic series, in such a way that we recover analogues of some of the main properties of $H_p(\mathbb{D}_2^\infty)$ and $H_p(\mathbb{T}^\infty)$. We are particularly interested in the set of monomial convergence, the Khinchin-Steinhaus inequality for homogeneous polynomials, and a Hilbert type criterion. In the next subsection we will address Cole-Gamelin type inequalities.

We start by defining $H_p(\mathbb{D}_2^\infty)^*$ and $H_p(\mathbb{T}^\infty)^*$ as formal co-analytic series. The Bohr transform $n^{-s} \mapsto z^\alpha$, shows that $H_p(\mathbb{D}_2^\infty)$ is a Banach space with basis $(z^\alpha)_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}}$. Here, and in the following, we consider the order given by $\alpha < \beta$ if $\mathfrak{p}^\alpha < \mathfrak{p}^\beta$. Since $H_p(\mathbb{D}_2^\infty)$ is reflexive, its dual basis, which we will identify with $(z^{-\alpha})_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}}$, is a Schauder basis of $H_p(\mathbb{D}_2^\infty)^*$.

Hence, a formal co-analytic series $\varphi = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha \overline{z}^\alpha$ belongs to $H_p(\mathbb{D}_2^\infty)^*$ provided that for every $f = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} d_\alpha z^\alpha \in H_p(\mathbb{D}_2^\infty)$ we have that

$$\langle \varphi, f \rangle = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha d_\alpha = \lim_{N \to \infty} \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})} : \mathfrak{p}^\alpha \le N} c_\alpha d_\alpha$$

converges. In this case,

$$\|\varphi\|_{H_p(\mathbb{D}_2^\infty)^*} = \sup_{\|f\|_{H_p(\mathbb{D}_2^\infty)} \le 1} |\langle \varphi, f \rangle|.$$

We define $H_p(\mathbb{T}^\infty)^*$ in a similar way, they are just the co-analytic trigonometric series $\varphi = \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} c_\alpha w^{-\alpha}$ for which $\langle \varphi, f\rangle$ converges for every $f \in H_p(\mathbb{T}^\infty)$. The duality relation is given by

$$\langle \varphi, f\rangle = \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} c_\alpha d_\alpha,$$

where $f = \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} d_\alpha w^\alpha \in H_p(\mathbb{T}^\infty)$.

If φ is a co-analytic polynomial then the duality can also be written as

$$\langle \varphi, f\rangle = \int_{\mathbb{T}^\infty} \varphi(w) f(w) dw.$$

Note that, identifying $H_p(\mathbb{T}^\infty)$ with $H_p(\mathbb{D}_2^\infty)$ through the Poisson operator P_∞, we also have a surjective isometry between the dual spaces $H_p(\mathbb{D}_2^\infty)^*$ and $H_p(\mathbb{T}^\infty)^*$.

The spaces $H_p(\mathbb{T}^\infty)$ and $\mathcal{H}_p$ are isometrically isomorphic via the Bohr transform $\mathscr{B}$. We define an extended Bohr transform which sends formal trigonometric series $\sum_{\alpha\in\mathbb{N}^{(\mathbb{Z})}} c_\alpha w^\alpha$ on $\mathbb{T}^\infty$ to formal Dirichlet series with indices in $\mathbb{Q}_+$, $\sum_{q\in\mathbb{Q}_+} a_q q^{-s}$. Note that each rational number can be written in a unique (irreducible) way as $q = \mathfrak{p}_1^{\alpha_1}\dots\mathfrak{p}_k^{\alpha_k}$, where $\alpha_1,\dots,\alpha_k \in \mathbb{Z}$. Thus, we have

$$\mathscr{B}: \begin{array}{c}\textbf{Trigonometric Series}\\ \sum_{\alpha\in\mathbb{Z}^{(\mathbb{N})}} c_\alpha w^\alpha\end{array} \xrightarrow{(a_q := c_{p^\alpha})} \begin{array}{c}\textbf{Dirichlet Series}\\ \sum_{q\in\mathbb{Q}_+} a_n n^{-s}.\end{array}$$

The extended Bohr transform maps co-analytic trigonometric series to co-analytic Dirichlet series and it is a surjective isometry between $H_p(\mathbb{T}^\infty)^*$ and $\mathcal{H}_p^*$. We will also consider an isometry between $H_p(\mathbb{T}^\infty)^*$ and $H_p(\mathbb{D}_2^\infty)^*$ given by an extended infinite Poisson operator (which we also denote by P_∞), $\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} c_\alpha w^{-\alpha} \mapsto \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} c_\alpha \overline{z}^\alpha$. Thus, $P_\infty\mathscr{B}^{-1}$ is a surjective isometry between $\mathcal{H}_p^*$ and $H_p(\mathbb{D}_2^\infty)^*$ given by

$$\sum_{n=1}^\infty a_n n^s \in \mathcal{H}_p^* \mapsto \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} a_{\mathfrak{p}^\alpha} \overline{z}^\alpha \in H_p(\mathbb{D}_2^\infty)^*. \tag{2}$$

Since $H_p(\mathbb{D}^N)$ and $H_p(\mathbb{T}^N)$ are reflexive spaces with bases (as subspaces of $H_p(\mathbb{D}_2^\infty)$ and $H_p(\mathbb{T}^\infty)$, respectively), we may describe $H_p(\mathbb{D}^N)^*$ and $H_p(\mathbb{T}^N)^*$ using the dual basis as follows. For each $N \in \mathbb{N}$, $H_p(\mathbb{T}^N)^*$ is the space of formal series $\varphi = \sum_{\alpha\in\mathbb{N}_0^N} c_\alpha w^{-\alpha}$ such that $\langle\varphi, f\rangle = \sum_{\alpha\in\mathbb{N}_0^N} c_\alpha d_\alpha$ converges for every $f = \sum_{\alpha\in\mathbb{N}_0^N} d_\alpha w^\alpha \in H_p(\mathbb{T}^N)$. Analogously $H_p(\mathbb{D}^N)^*$ is the space of co-analytic series $\varphi = \sum_{\alpha\in\mathbb{N}_0^N} c_\alpha \overline{z}^\alpha$ such that $\langle\varphi, f\rangle = \sum_{\alpha\in\mathbb{N}_0^N} c_\alpha d_\alpha$ converges for every $f = \sum_{\alpha\in\mathbb{N}_0^N} d_\alpha z^\alpha \in H_p(\mathbb{D}^N)$.

It is worth noting that here, although we are considering the order induced by $(n^{-s})_n$ through the Bohr transform for the trigonometric basis, it is well known that in the N-dimensional case the partial sums of degree $\leq n$ converge in the $L_p(\mathbb{T}^n)$ norm (see e.g. [24, Section 3.5]).

We will also consider here the conjugation map $\jmath(f)(z) = \overline{f(z)}$, which maps formal co-analytic series to formal analytic series and vice versa.

Remark 3.3. *Thanks to the Riesz projection and Hölder's inequality, we have, for $p' = \frac{p}{p-1}$, that $H_p(\mathbb{T}^N)^*$ and $H_{p'}(\mathbb{T}^N)$ are conjugate-linear isomorphic as Banach spaces. Furthermore, if $\varphi \in H_p(\mathbb{T}^N)^*$ then $\varphi \in L_{p'}(\mathbb{T}^N)$ and $\jmath\varphi \in H_{p'}(\mathbb{T}^N)$.*

While for the infinite torus, $H_p(\mathbb{T}^\infty)^*$ and $H_{p'}(\mathbb{T}^\infty)$ are not isomorphic, from Hölder's inequality we have $H_{p'}(\mathbb{T}^\infty) \subset \jmath(H_p(\mathbb{T}^\infty)^*)$.

Some further elementary properties of $\jmath$ are the following:

(1) *$\jmath : L_p(\mathbb{T}^\infty) \to L_p(\mathbb{T}^\infty)$ is a conjugate-linear isometry.*
(2) *$\jmath\mathscr{B} = \mathscr{B}\jmath$ and $\jmath\mathscr{B}^{-1} = \mathscr{B}^{-1}\jmath$.*
(3) *Let $1 < p < \infty$. Then, $\jmath H_{p'}(\mathbb{T}^\infty) \subseteq H_p(\mathbb{T}^\infty)^*$ and $\jmath\mathcal{H}_{p'} \subseteq \mathcal{H}_p^*$.*
(4) *$\jmath : H_{p'}(\mathbb{T}^N) \to H_p(\mathbb{T}^N)^*$ is a conjugate-linear isomorphism.*

Hilbert type criterion. The first property we want to recover is a Hilbert criterion for dual Hardy spaces. For each $N \in \mathbb{N}$ we have an isometric inclusion $i_N : H_p(\mathbb{D}^N) \to H_p(\mathbb{D}_2^\infty)$ and a norm one projection $\pi_N : H_p(\mathbb{D}_2^\infty) \to H_p(\mathbb{D}^N)$. Thus we have for every $N \in \mathbb{N}$ a dual norm one projection $i_N^* : H_p(\mathbb{D}_2^\infty)^* \to H_p(\mathbb{D}^N)^*$ and a dual isometric inclusion $\pi_N^* : H_p(\mathbb{D}^N)^* \to H_p(\mathbb{D}_2^\infty)^*$.

We prove now our first property parallel to $H_p(\mathbb{D}_2^\infty)$: a formal co-analytic series belongs to $H_p(\mathbb{D}_2^\infty)^*$ provided that its restrictions to finitely many coordinates belong to $H_p(\mathbb{D}^N)^*$ and have uniformly bounded norm.

Proposition 3.4 (Hilbert type criterion)**.** *Let $(c_\alpha)_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}}$ be coefficients. The following are equivalent:*

(1) *$\varphi = \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} c_\alpha \overline{z}^\alpha \in H_p(\mathbb{D}_2^\infty)^*$ and*
(2) *for every $N \in \mathbb{N}$, $\varphi_N := \sum_{\alpha\in\mathbb{N}_0^N} c_\alpha \overline{z}^\alpha \in H_p(\mathbb{D}^N)^*$ and $\sup_{N\in\mathbb{N}} \|\varphi_N\|_{H_p(\mathbb{D}^N)^*} < \infty$.*

In this case, $\|\varphi\| = \sup_{N\in\mathbb{N}} \|i_N^*(\varphi)\|_{H_p(\mathbb{D}^N)^*}$ and $\pi_N^*(\varphi_N) \to \varphi$ in $H_p(\mathbb{D}^N)^*$.

Proof. (1) $\Rightarrow$ (2). Since the functions depending on finitely many variables are dense in $H_p(\mathbb{D}_2^\infty)$, we have

$$\|\varphi\|_{H_p(\mathbb{D}_2^\infty)^*} = \sup_{\|f\|_{H_p(\mathbb{D}_2^\infty)}\leq 1} |\langle \varphi, f\rangle| = \sup_{N\in\mathbb{N}} \sup_{\|f\|_{H_p(\mathbb{D}^N)}\leq 1} |\langle \varphi, i_N(f)\rangle| = \sup_{N\in\mathbb{N}} \|i_N^*(\varphi)\|_{H_p(\mathbb{D}^N)}.$$

(2) $\Rightarrow$ (1). Consider for each $N \in \mathbb{N}$ the function $\pi_N^*(\varphi_N) \in H_p(\mathbb{D}_2^\infty)^*$. By assumption, $(\pi_N^*(\varphi_N))_{N\in\mathbb{N}}$ is a bounded sequence and thus has a weakly convergent subsequence, say $\pi_{N_k}^*(\varphi_{N_k}) \xrightarrow{w} \varphi \in H_p(\mathbb{D}_2^\infty)^*$, with $\|\varphi\|_{H_p(\mathbb{D}_2^\infty)^*} \leq \sup_{k\in\mathbb{N}} \|\varphi_{N_k}\|_{H_p(\mathbb{D}^{N_k})^*}$. By the uniqueness of coefficients it follows that φ equals $\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} c_\alpha \overline{z}^\alpha$.

The last assertion holds because $(\|\pi_N^*(\varphi_N)\|_{H_p(\mathbb{D}_2^\infty)^*})_N$ is nondecreasing. $\square$

As a consequence of the above proposition, we show that if $\varphi \in \jmath H_{p'}(\mathbb{T}^\infty)$ then we have a better duality relation.

Corollary 3.5. *Let $\varphi \in \jmath H_{p'}(\mathbb{T}^\infty)$ and $f \in H_p(\mathbb{T}^\infty)$. Then*

$$\langle \varphi, f\rangle_{(H_p(\mathbb{T}^\infty)^*, H_p(\mathbb{T}^\infty))} = \int_{\mathbb{T}^\infty} \varphi(w) f(w) dw = \lim_{N\to\infty} \int_{\mathbb{T}^N} \varphi_N(w) f_N(w) dw.$$

Proof. Since $i_N(f_N) \to f$ in $H_p(\mathbb{T}^\infty)$ and $\pi_N^*(\varphi_N) \to \varphi$ in $L_{p'}(\mathbb{T}^\infty)$-norm,

$$\begin{aligned}\int_{\mathbb{T}^\infty} \varphi(w) f(w) dw &= \lim_{N\to\infty} \int_{\mathbb{T}^\infty} (f - f_N)\varphi + \int_{\mathbb{T}^\infty} f_N(\varphi - \varphi_N) + \int_{\mathbb{T}^\infty} f_N \varphi_N = \lim_{N\to\infty} \int_{\mathbb{T}^N} \varphi_N f_N \\ &= \lim_{N\to\infty} \langle \varphi_N, f_N \rangle_{(H_p(\mathbb{T}^N)^*, H_p(\mathbb{T}^N))} = \lim_{N\to\infty} \langle \pi_N^*(\varphi_N), i_N(f_N) \rangle_{(H_p(\mathbb{T}^\infty)^*, H_p(\mathbb{T}^\infty))} \\ &= \langle \varphi, f \rangle_{(H_p(\mathbb{T}^\infty)^*, H_p(\mathbb{T}^\infty))},\end{aligned}$$

where the last equality holds since we also have $\pi_N^*(\varphi_N) \to \varphi$ in $H_p(\mathbb{T}^N)^*$ by the preceding proposition. $\square$

Khinchin-Steinhaus Inequality. The second property we want to recover is the polynomial Khinchin-Steinhaus inequality, see [19, Theorem 8.10]. To this aim we consider consider homogeneous parts. For each $m \in \mathbb{N}$, consider the space

$$H_p(\mathbb{D}_2^\infty)^{*m} = \{\varphi \in H_p(\mathbb{D}_2^\infty)^* : c_\alpha = 0 \text{ if } |\alpha| \neq m\}$$

of m-homogeneous functions in $H_p(\mathbb{D}_2^\infty)^*$, endowed with the subspace norm of $H_p(\mathbb{D}_2^\infty)^*$. The space $H_p(\mathbb{D}_2^\infty)^{*m}$ is 1-complemented in $H_p(\mathbb{D}_2^\infty)^*$ as the following proposition shows (see [19, Proposition 11.13] for the analogous result for $H_p(\mathbb{D}_2^\infty)$).

Proposition 3.6. *The space $H_p(\mathbb{D}_2^\infty)^{*m}$ is closed in $H_p(\mathbb{D}_2^\infty)^*$, and the projection*

$$\begin{aligned}\Pi_m : H_p(\mathbb{D}_2^\infty)^* &\longrightarrow H_p(\mathbb{D}_2^\infty)^{*m} \\ \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} c_\alpha \overline{z}^\alpha &\longmapsto \sum_{\substack{\alpha\in\mathbb{N}_0^{(\mathbb{N})} \\ |\alpha|=m}} c_\alpha \overline{z}^\alpha\end{aligned}$$

is a contraction.

Proof. The closeness of $H_p(\mathbb{D}_2^\infty)^{*m}$ follows from the fact that $(\overline{z}^\alpha)_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}}$ is a Schauder basis of $H_p(\mathbb{D}_2^\infty)^*$. The bound for the norm of the projection Π_m follows from the corresponding Cauchy estimate in $H_p(\mathbb{D}_2^\infty)$ (see [19, Proposition 11.13]):

$$\begin{aligned}\|\Pi_m(\varphi)\| &= \sup_{\|f\|_{H_p(\mathbb{D}_2^\infty)}\le 1} |\langle \Pi_m(\varphi), f\rangle| = \sup_{\substack{f\in H_p^m(\mathbb{D}_2^\infty) \\ \|f\|_{H_p^m(\mathbb{D}_2^\infty)}\le 1}} |\langle \Pi_m(\varphi), f\rangle| \\ &= \sup_{\substack{f\in H_p^m(\mathbb{D}_2^\infty) \\ \|f\|_{H_p^m(\mathbb{D}_2^\infty)}\le 1}} |\langle \varphi, f\rangle| \le \|\varphi\|_{H_p(\mathbb{D}_2^\infty)^*}.\end{aligned}$$

$\square$

It is well known that all the spaces $H_p^m(\mathbb{D}_2^\infty)$, $1 \le p < \infty$, are equal as linear spaces and isomorphic as Banach spaces [19, Proposition 11.12]. Hence, $(z^\alpha)_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}, |\alpha|=m}$ is a Schauder basis of $H_p^m(\mathbb{D}_2^\infty)$ for every $1 \le p < \infty$. We can thus identify $H_p^m(\mathbb{D}_2^\infty)^*$ with the analytic series $\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} c_\alpha \overline{z}^\alpha$ such that $\left|\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} c_\alpha a_\alpha\right| < \infty$ for every $f = \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}, |\alpha|=m} a_\alpha z^\alpha$. We will see next that this space is equal to $H_p(\mathbb{D}_2^\infty)^{*m}$.

Proposition 3.7. *Let $1 < p < \infty$. The spaces $H_p(\mathbb{D}_2^\infty)^{*m}$ and $H_p^m(\mathbb{D}_2^\infty)^*$ are the same space.*

Proof. Let $\varphi \in H_p(\mathbb{D}_2^\infty)^{*m} \cap H_p^m(\mathbb{D}_2^\infty)^*$. Then,

$$\begin{aligned}\|\varphi\|_{H_p(\mathbb{D}_2^\infty)^{*m}} &= \sup_{\substack{f\in H_p(\mathbb{D}_2^\infty)\\ \|f\|_{H_p(\mathbb{D}_2^\infty)}\le 1}} |\langle \varphi, f\rangle| = \sup_{\substack{f\in H_p^m(\mathbb{D}_2^\infty)\\ \|f\|_{H_p^m(\mathbb{D}_2^\infty)}\le 1}} |\langle \varphi, f\rangle| \\ &= \|\varphi\|_{H_p^m(\mathbb{D}_2^\infty)^*}.\end{aligned}$$

The fact that the m-homogeneous co-analytic polynomials are dense in both spaces implies that $H_p(\mathbb{D}_2^\infty)^{*m} = H_p^m(\mathbb{D}_2^\infty)^*$. □

As a consequence, we recover the Khinchin-Steinhaus inequality. See [14, Theorem 2] for an alternative proof of the case $m = 1$.

Corollary 3.8 (Khinchin-Steinhaus inequality)**.** *Let $m \in \mathbb{N}$. Then,*

(1) For every $1 < p, q < \infty$, we have that $H_p(\mathbb{D}_2^\infty)^{*m}$ and $H_q(\mathbb{D}_2^\infty)^{*m}$ are equal as linear sets and isomorphic as Banach spaces. Moreover, for each $1 < p \le q < \infty$ there exists a constant $C_{p,q}$ such that for all $\varphi \in H_p(\mathbb{D}_2^\infty)^{*m}$ we have

$$\|\varphi\|_{H_q(\mathbb{D}_2^\infty)^{*m}} \le \|\varphi\|_{H_p(\mathbb{D}_2^\infty)^{*m}} \le C_{p,q}^m \|\varphi\|_{H_q(\mathbb{D}_2^\infty)^{*m}}.$$

(2) For every $1 < p, q < \infty$ we have that $\jmath H_p(\mathbb{D}_2^\infty)^{*m} = H_q(\mathbb{D}_2^\infty)^m$ as linear sets, and $\jmath$ is a conjugate linear isomorphism.

Proof. All the spaces $H_q^m(\mathbb{D}_2^\infty)$ are equal as linear spaces and isomorphic as Banach spaces [19, Proposition 11.12]. This, together with the above result shows that the $H_p(\mathbb{D}_2^\infty)^{*m}$ are equal as linear spaces and isomorphic as Banach spaces. The proof concludes by observing that each $H_p(\mathbb{D}_2^\infty)^{*m}$ is isomorphic to $H_2^m(\mathbb{D}_2^\infty)^*$, $\jmath H_2^m(\mathbb{D}_2^\infty)^*$ and $H_2^m(\mathbb{D}_2^\infty)$ are equal as linear space and conjugate isomorphic as linear Banach spaces which in turn is isomorphic to each $H_q^m(\mathbb{D}_2^\infty)$. □

The set of monomial convergence of $H_p(\mathbb{D}_2^\infty)^*$. We consider now the problem of monomial convergence. Given a set of formal power series X, recall that the set of monomial convergence of X is

$$\operatorname{mon}(X) = \left\{ z \in \mathbb{C}^{\mathbb{N}} : \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} |c_\alpha z^\alpha| < \infty \text{ for every } f = \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha \in X \right\}.$$

We extend the definition to sets of co-analytic functions in the obvious way.

Applying the Khinchin-Steinhaus inequality and using the known properties of $H_p(\mathbb{D}_2^\infty)^m$ (namely [19, Proposition 11.4]), we immediately obtain that $\operatorname{mon}(H_p(\mathbb{D}_2^\infty)^{*m}) = \ell_2$.

Corollary 3.9. *For every $m \ge 1$ and $1 < p < \infty$ we have*

$$\operatorname{mon}(H_p(\mathbb{D}_2^\infty)^{*m}) = \ell_2.$$

We prove now the main result of the subsection which characterizes $\operatorname{mon}(H_p(\mathbb{D}_2^\infty)^*)$.

Theorem 3.10. *Let $1 < p < \infty$. Then* $\operatorname{mon}(H_p(\mathbb{D}_2^\infty)^*) = \mathbb{D}_2^\infty$.

Proof. Since $\jmath H_{p'}(\mathbb{D}_2^\infty) \subseteq H_p(\mathbb{D}_2^\infty)^*$, we have that

$$\operatorname{mon}(H_p(\mathbb{D}_2^\infty)^*) \subseteq \operatorname{mon}(\jmath H_{p'}(\mathbb{D}_2^\infty)) = \operatorname{mon}(H_{p'}(\mathbb{D}_2^\infty)) = \mathbb{D}_2^\infty.$$

For the other inclusion we will first prove that there is $0 < r < 1$ such that $rB_{\ell_2} \subseteq \mathrm{mon}(H_p(\mathbb{D}_2^\infty)^*)$. By [19, Equation (11.4)] we have that for every $\varphi \in H_2^m(\mathbb{D}_2^\infty)$,

$$\sum_{|\alpha|=m} |c_\alpha(\varphi) z^\alpha| \leq \|\varphi\|_{H_2(\mathbb{D}_2^\infty)} \|z\|_2^m.$$

Since $H_p(\mathbb{D}_2^\infty)^{*m}$ and $H_2^m(\mathbb{D}_2^\infty)$ are isomorphic as Banach spaces, there is $C > 0$ such that for every $\varphi \in H_p(\mathbb{D}_2^\infty)^{*m}$ we have that

$$\sum_{|\alpha|=m} |c_\alpha(\varphi) z^\alpha| \leq C^m \|\varphi\|_{H_p(\mathbb{D}_2^\infty)^*} \|z\|_2^m.$$

Let now $r < \frac{1}{C}$, $z \in rB_{\ell_2}$ and $\varphi \in H_p(\mathbb{D}_2^\infty)^*$. Hence,

$$\begin{aligned}
\sum_{\alpha \in \mathbb{N}^{(\mathbb{N})}} |c_\alpha(\varphi) z^\alpha| &= \sum_{m=0}^\infty \sum_{|\alpha|=m} |c_\alpha(\varphi) z^\alpha| = \sum_{m=0}^\infty \sum_{|\alpha|=m} |c_\alpha(\varphi_m) z^\alpha| \\
&\leq \sum_{m=0}^\infty r^m C^m \|\varphi_m\|_{H_p^m(\mathbb{D}_2^\infty)^*} \leq \sum_{m=0}^\infty r^m C^m \|\varphi\|_{H_p(\mathbb{D}_2^\infty)^*} \\
&\leq \frac{1}{1-rC} \|\varphi\|_{H_p(\mathbb{D}_2^\infty)^*} \qquad (3)
\end{aligned}$$

Let $z \in \mathbb{D}_2^\infty$, $k \in \mathbb{N}$ such that $\sum_{n>k} |z_j|^2 < r^2$, and let $\varphi \in H_p(\mathbb{D}_2^\infty)^*$. For each $n_1, \ldots, n_k \in -\mathbb{N}$ we will consider $\varphi_{n_1,\ldots,n_k} \in H_p(\mathbb{D}_2^\infty)^*$ such that for every $\alpha \in -\mathbb{N}_0^{(\mathbb{N})}$ we have that

$$c_\alpha(\varphi_{n_1,\ldots,n_k}) = c_{n_1,\ldots,n_k,\alpha}(\varphi). \qquad (4)$$

To define $\varphi_{n_1,\ldots,n_k}$, we consider first $[\varphi_{n_1,\ldots,n_k}]_N \in H_p(\mathbb{D}^N)^*$ given by

$$[\varphi_{n_1,\ldots,n_k}]_N(u) = \int_{\mathbb{T}^k} \varphi(w,u,0) w_1^{-n_1} \cdots w_k^{-n_k} dw.$$

Since for finitely many variables, $H_p(\mathbb{D}^N)^*$ and $\jmath H_{p'}(\mathbb{D}^k)$ coincide as linear sets, we have that $\varphi(\cdot,u,0)$ is measurable in $\mathbb{T}^k$ for almost every $u \in \mathbb{T}^N$. We observe now that

$$\begin{aligned}
\|[\varphi_{n_1,\ldots,n_k}]_N\|_{H_p(\mathbb{D}^N)^*} &= \sup_{\|f\|_{H_p(\mathbb{D}^N)} \leq 1} \int_{\mathbb{T}^N} \int_{\mathbb{T}^k} \varphi(w,u,0) w_1^{-n_1} \cdots w_k^{-n_k} f(u) dw du \\
&\leq \sup_{\|f\|_{H_p(\mathbb{D}_2^\infty)} \leq 1} \|\varphi_{N+k}\|_{H_p(\mathbb{D}^{N+k})^*} \|w_1^{-n_1} \cdots w_k^{-n_k} f\|_{H_p(\mathbb{D}^{N+k})} \leq \|\varphi\|_{H_p(\mathbb{D}_2^\infty)^*}.
\end{aligned}$$

On the other hand, for every $\alpha \in -\mathbb{N}_0^{(\mathbb{N})}$ we have that

$$c_\alpha([\varphi_{n_1,\ldots,n_k}]_N) = \int_{\mathbb{T}^N} \int_{\mathbb{T}^k} \varphi(w,u,0) w_1^{-n_1} \cdots w_k^{-n_k} u^{-\alpha} dw du = c_{n_1,\ldots,n_k,\alpha}(\varphi_{N+k}) = c_{n_1,\ldots,n_k,\alpha}(\varphi).$$

By Proposition 3.4 we conclude that $\varphi_{n_1,\ldots,n_k} = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_{n_1,\ldots,n_k,\alpha}(\varphi) \overline{z}^\alpha \in H_p(\mathbb{D}_2^\infty)^*$ with

$$\|\varphi_{n_1,\ldots,n_k}\|_{H_p(\mathbb{D}_2^\infty)^*} \leq \|\varphi\|_{H_p(\mathbb{D}_2^\infty)^*}. \qquad (5)$$

Let $R<1$ such that $\|z\|_\infty < R$. By (5), (4) and (3) applied to $z-\sum_{j>k} z_j e_j$ and to each $\varphi_{n_1,\dots,n_k}$, we obtain that

$$\begin{aligned}\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} |c_\alpha(\varphi)z^\alpha| &= \sum_{n_1,\dots,n_k}\sum_{\beta\in\mathbb{N}_0^{(\mathbb{N})}} |c_{n_1,\dots,n_k,\beta}(\varphi)||z_1|^{n_1}\cdots|z_k|^{n_k}|z_{k+1}|^{\beta_1}|z_{k+2}|^{\beta_2}\cdots \\ &< \sum_{n_1,\dots,n_k} R^{n_1+\cdots+n_k}\sum_{\beta\in\mathbb{N}_0^{(\mathbb{N})}} |c_\beta(\varphi_{n_1,\dots,n_k})||z_{k+1}|^{\beta_1}|z_{k+2}|^{\beta_2}\cdots \\ &\le \sum_{n_1,\dots,n_k} R^{n_1+\cdots+n_k}\frac{1}{1-rC}\|\varphi_{n_1,\dots,n_k}\| \le \frac{1}{(1-R)^k}\frac{1}{1-rC}\|\varphi\|<\infty.\end{aligned}$$

□

An application of the Bohr transform gives us the analogous result for co-analytic Dirichlet series.

Theorem 3.11. *Let $1<p<\infty$. Then, $\sigma_a(\mathcal{H}_p^*)=-\frac{1}{2}$.*

Proof. We already know that $\sigma_a(\mathcal{H}_p^*)\le\sigma_c(\mathcal{H}_p^*)=-\frac{1}{2}$. So, it suffices to show that for every $s\in Re(s)<-\frac{1}{2}$, we have that $D(s)$ converges absolutely for every $D\in\mathcal{H}_p^*$. Let $D=\sum_{n=1}^\infty a_n n^s\in\mathcal{H}_p^*$ and $Re(s)<-\frac{1}{2}$. Then

$$\sum_{n=1}^\infty |a_n n^s| = \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} |a_{\mathfrak{p}^\alpha}|\left(\mathfrak{p}^{Re(s)}\right)^\alpha.$$

The right hand side is convergent because $P_\infty\mathscr{B}^{-1}(D)=\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} a_{\mathfrak{p}^\alpha}\overline{z}^\alpha\in H_p(\mathbb{D}_2^\infty)^*$ and $\mathfrak{p}^s\in\mathbb{D}_2^\infty=\operatorname{mon}(H_p(\mathbb{D}_2^\infty)^*)$. □

3.3. **Cole-Gamelin type inequalities: the norm of point evaluation functionals.** In Theorem 3.2 we proved that for every $1<p<\infty$, $\sigma_c(\mathcal{H}_p^*)=-\frac{1}{2}$. In particular, the linear evaluation functional, $D\mapsto k_s(D)=D(s)$, is continuous whenever $Re(s)<-\frac{1}{2}$ and analogously, $\varphi\mapsto\delta_z(\varphi):=\varphi(z)$ is continuous when $z\in\mathbb{D}_2^\infty$ and $\varphi\in H_p(\mathbb{D}_2^\infty)^*$. It is desirable to obtain good estimations of $\|k_s\|_{\mathcal{H}_p}$. Recall that, when we see k_s as an element of $\mathcal{H}_p=(\mathcal{H}_p^*)^*$, we have that $k_s=\zeta(\cdot-s)$. We mention that in [39, Problem 4.3] it was asked to find optimal estimations for $\mathcal{H}_p$-norm of the truncations of $\zeta(\cdot-s)$.

The following theorem summarizes our findings. Part i) is immediate, it is a consequence of the inclusion $\mathcal{H}_2\hookrightarrow\mathcal{H}_p$, for $p<2$. Part ii) also follows easily since, for $p>2$, $\|\zeta(\cdot+w)\|_{\mathcal{H}_p}\le\|\zeta(\cdot+w)\|_{\mathcal{H}_\infty}\le\sup_{Re(s)>0}\sum_{n=1}^\infty|n^{-(s+w)}|\le\zeta(Re(w))$.

Theorem 3.12. *Let $1<p<\infty$. Then,*

i) If $p\le 2$, and $\frac{1}{2}<Re(w)$, $\|\zeta(\cdot+w)\|_{\mathcal{H}_p}\le\|\zeta(\cdot+w)\|_{\mathcal{H}_2}=\zeta(2Re(w))^{\frac{1}{2}}$;
ii) if $p>2$, and $1\le Re(w)$ then $\|\zeta(\cdot+w)\|_{\mathcal{H}_p}\le\zeta(Re(w))$;
iii) if $2<p\le 4$ and $\frac{1}{2}<Re(w)$. Then,

$$\|\zeta(\cdot+w)\|_{\mathcal{H}_p}\le\frac{\zeta(2Re(w))^{\frac{2}{p'}-\frac{1}{2}}}{\zeta(4Re(w))^{\frac{1}{p'}-\frac{1}{2}}};$$

iv) if $4 < p$ *and* $\frac{1}{2} < Re(w) \leq 1$*. Then,*

$$\|\zeta(\cdot + w)\|_{\mathcal{H}_p} \leq \frac{\zeta(2Re(w))^{\frac{p}{4}}}{\zeta(4Re(w))^{\frac{p-2}{8}}} \exp(C(p-2)^{1-Re(w)}),$$

where $C > 0$ *is an absolute constant.*

For the proof we need some preparation. We will work again with $H_p(\mathbb{D}_2^\infty)^*$. For each p and each $z \in \mathbb{D}$ let $\gamma_p(z) := \left(\int_{\mathbb{T}} |\frac{1}{w-z}|^p dw\right)^{\frac{1}{p}}$.

Theorem 3.13. *Let* $\varphi \in H_p(\mathbb{D}^N)^*$ *and* $z \in \mathbb{D}^N$*, then*

$$\varphi(z) = \int_{\mathbb{T}^N} \prod_{j=1}^N \frac{1}{1 - w_j \overline{z}_j} P_N^{-1}(\varphi)(w_1, w_2, \ldots, w_N) d_{w_1} \ldots d_{w_N}. \tag{6}$$

Consequently $|\varphi(z)| \leq \prod_{j=1}^N \gamma_p(z_j) \|\varphi\|_{H_p(\mathbb{D}^N)^*}$.

Proof. We prove it first for co-analytic polynomials. We will prove it by induction. Let $Q = \sum_{n=0}^\infty c_n \overline{z}^n \in H_p(\mathbb{D})^*$ be a co-analytic polynomial in one variable and let $z \in \mathbb{D}$. For each $n \in \mathbb{Z}$ we have that $c_n(Q) = \int_{\mathbb{T}} Q(w) w^{-n} dw$. Since Q has only negative nonzero Fourier coefficients, we obtain that

$$Q(z) = \sum_{n=0}^\infty \int_{\mathbb{T}} Q(w) w^n dw \overline{z}^n = \int_{\mathbb{T}} Q(w) \sum_{n=0}^\infty w^n \overline{z}^n dw = \int_{\mathbb{T}} Q(w) \frac{1}{1 - w\overline{z}} dw.$$

Consider now a co-analytic polynomial Q depending on N-variables ($N > 1$) and let $z = (z_1, \ldots, z_N) \in \mathbb{D}^N$. Then, Q_{z_1} is a co-analytic polynomial in $N-1$ variables and hence

$$Q_{z_1}(z_2, \ldots, z_N) = \int_{\mathbb{T}^{N-1}} Q_{z_1}(w_2, \ldots, w_N) \prod_{j=2}^N \frac{1}{1 - w_j \overline{z}_j} d_{w_2} \ldots d_{w_N}.$$

By the inductive hypothesis applied to each $Q_{w_2,\ldots,w_N}$, which is a co-analytic polynomial in one variable, we get

$$\begin{aligned} Q(z) &= \int_{\mathbb{T}^{n-1}} \left(\int_{\mathbb{T}} Q_{w_2,\ldots,w_N}(w_1) \frac{1}{1 - w_1 \overline{z}_1} \right) d_{w_1} \prod_{j=2}^N \frac{1}{1 - w_j \overline{z}_j} d_{w_2} \ldots d_{w_N} \\ &= \int_{\mathbb{T}^N} Q(w_1, \ldots w_N) \prod_{j=1}^N \frac{1}{1 - w_j \overline{z}_j} d_{w_1} \ldots d_{w_N}. \end{aligned}$$

Therefore we obtain that $|Q(z)| \leq \| \prod_{j=1}^N \frac{1}{w_j - z_j} \|_{H_p(\mathbb{T}^N)} \|Q\|_{H_p(\mathbb{T}^N)^*} = \prod_{j=1}^N \gamma_p(z_j) \|Q\|_{H_p(\mathbb{D}^N)^*}$.

Let $\varphi \in H_p(\mathbb{D}^N)^*$ be arbitrary and let $z = (z_1, \ldots, z_N) \in \mathbb{D}^N$. Let $(Q_n)_n$ be co-analytic polynomials such that $Q_n \to \varphi$ in $H_p(\mathbb{D})^*$. The evaluations are continous, because $H_p(\mathbb{D}^N)^*$ and $H_{p'}(\mathbb{D}^N)$ are conjugate-linear isomorphic. Hence, $Q_n(z) \to \varphi(z)$. On the other hand, we have that for each $1 \leq j \leq N$, $\frac{1}{1 - w_j \overline{z}_j} \in H_p(\mathbb{T})$ and hence $\prod_{j=1}^N \frac{1}{1 - w_j z_j} \in H_p(\mathbb{T}^N)$. By the duality of $H_p(\mathbb{T}^N)$ we obtain that,

$$\int_{\mathbb{T}^N} Q_n(w_1, \ldots w_N) \prod_{j=1}^N \frac{1}{1 - w_j \overline{z}_j} d_{w_1} \ldots d_{w_N} \to \int_{\mathbb{T}^N} P_N^{-1}(\varphi)(w_1, \ldots w_N) \prod_{j=1}^N \frac{1}{1 - w_j \overline{z}_j} d_{w_1} \ldots d_{w_N}.$$

Altogether we finally obtain

$$\varphi(z) = \int_{\mathbb{T}^N} P_N^{-1}(\varphi)(w_1, \ldots w_N) \prod_{j=1}^{N} \frac{1}{1 - w_j \overline{z}_j} d_{w_1} \ldots d_{w_N}.$$

□

Corollary 3.14. *Let $z \in \mathbb{D}^{\mathbb{N}}$ such that $\delta_z(\varphi) = \varphi(z)$ is continuous on $H_p(\mathbb{D}_2^\infty)^*$. Then, $\|\delta_z\| \leq \prod_{j=1}^{\infty} \gamma_p(z_j)$.*

Proof. Let $\varphi \in H_p(\mathbb{D}_2^\infty)^*$. Since $\varphi_N = i_N^*(\varphi) \to \varphi$ and the evaluation at z is continuous, we have that for each $N \in \mathbb{N}$, $|\varphi_N(z)| \leq \prod_{j=1}^{N} \gamma_p(z_j)$. Therefore,

$$|\varphi(z)| = \lim_{N \to \infty} |\varphi_N(z)| \leq \prod_{j=1}^{\infty} \gamma_p(z_j).$$

□

We already know by Theorem 3.10 that δ_z is continuous for every $z \in \mathbb{D}_2^\infty$. We want to estimate the norm of these evaluations functionals. Our strategy will be to find estimations for $\prod_{j=1}^{\infty} \gamma_p(z_j)$.

Corollary 3.15. *Let $\varphi \in H_p(\mathbb{D}_2^\infty)^*$ and $z \in B_{c_0} \cap \ell_1$. Then, $\prod_{j=1}^{\infty} \gamma_p(z_j) \leq \prod_{j=1}^{\infty} \frac{1}{1-|z_j|}$ and $|\varphi(z)| \leq \prod_{j=1}^{\infty} \frac{1}{1-|z_j|} \|\varphi\|$.*

Proof. Note that for each $z_j \in \mathbb{D}$ we have by Hölder's inequality that $\|\frac{1}{w_j - z_j}\|_p \leq \|\frac{1}{w_j - z_j}\|_\infty = \frac{1}{1-|z_j|}$. Therefore, for $z \in B_{c_0} \cap \ell_1$, $\prod_{j=1}^{\infty} \gamma_p(z_j) \leq \prod_{j=1}^{\infty} \frac{1}{1-|z_j|}$. This product converges, because $\frac{1}{1-|z_j|} = 1 + \frac{|z_j|}{1-|z_j|} \leq e^{\frac{|z_j|}{1-|z_j|}}$. Hence, $\prod_{j=1}^{\infty} \frac{1}{1-|z_j|} \leq e^{\frac{\|z\|_1}{1-\|z\|_\infty}}$.

Let now $\varphi \in H_p(\mathbb{D}_2^\infty)^*$. Since $\varphi_N = i_N^*(\varphi) \to \varphi$ and the evaluation at $z \in B_{c_0} \cap \ell_1$ is continuous, we have by the above theorem applied to φ_N, that $|\varphi(z)| = \lim_{N \to \infty} |\varphi_N(z)| \leq \prod_{j=1}^{\infty} \frac{1}{1-|z_j|} \|\varphi\|_{H_p(\mathbb{D}_2^\infty)^*}$. □

We obtain now an estimation for $z \in \mathbb{D}_2^\infty$.

Theorem 3.16. *Let $z \in \mathbb{D}_2^\infty$ and $\varphi \in H_p(\mathbb{D}_2^\infty)^*$. Then,*

(i) if $2 < p \leq 4$,

$$|\varphi(z)| \leq \left(\exp \frac{\|z\|_2^2}{1 - \|z\|_\infty^2} \right)^{\frac{2}{p'} - \frac{1}{2}} \|\varphi\|,$$

(ii) and if $p > 4$,

$$|\varphi(z)| \leq \left(\exp \frac{\|z\|_2^2}{1 - \|z\|_\infty^2} \right)^{\frac{p}{4}} \left(\frac{1 + \|z\|_\infty}{1 - \|z\|_\infty} \right)^{(p-2)\left(\frac{1}{2} - \frac{2}{p}\right)} \|\varphi\|.$$

Proof. Consider first $z \in \mathbb{D}$ and let $r = |z|$. Then,

$$\begin{aligned}\gamma_p(z)^p &= \int_{\mathbb{T}} \frac{1}{|z-w|^p} dw = \int_{\mathbb{T}} \left(\frac{1}{|z-w|^2}\right)^{\frac{p}{2}} dw = \int_{\mathbb{T}} \left(\frac{1}{1-|z|^2}\frac{|w|^2-|z|^2}{|w-z|^2}\right)^{\frac{p}{2}} dw \\ &= \left(\frac{1}{1-|z|^2}\right)^{\frac{p}{2}} \int_{\mathbb{T}} \left(Re\left(\frac{w+z}{w-z}\right)\right)^{\frac{p}{2}} dw = \left(\frac{1}{1-r^2}\right)^{\frac{p}{2}} \int_0^{2\pi} \left(\frac{1-r^2}{1-2rcos(\theta)+r^2}\right)^{\frac{p}{2}} d\theta \\ &= \left(\frac{1}{1-r^2}\right)^{\frac{p}{2}} \|p_r\|_{\frac{p}{2}}^{p/2}, \end{aligned} \tag{7}$$

where p_r denotes the Poisson kernel. Recall that $p_r(\theta) = \sum_{k\in\mathbb{Z}} r^{|k|}e^{ik\theta} = \frac{1-r^2}{1-2rcos(\theta)+r^2}$.

Note that for $p = 4$, Plancherel's formula implies

$$\|p_r\|_{\frac{p}{2}} = \|p_r\|_2 = \left(\frac{1+r^2}{1-r^2}\right)^{\frac{1}{2}},$$

and for $p = 2$, $\|p_r\|_{\frac{p}{2}} = \|p_r\|_1 = 1$. Thus for $2 < p < 4$, we may apply complex interpolation to the mapping $\mathbb{C} \ni t \mapsto tp_r \in L_{\frac{p}{2}}(\mathbb{T})$, to obtain

$$\|p_r\|_{\frac{p}{2}} \leq \left(\frac{1+r^2}{1-r^2}\right)^{\frac{p-2}{p}}.$$

We conclude from Corollary 3.14 that for $2 < p \leq 4$, $z \in \mathbb{D}_2^\infty$,

$$\begin{aligned}|\varphi(z)| &\leq \prod_{j=1}^\infty \gamma_p(z_j) = \prod_{j=1}^\infty \left(\frac{1}{1-|z_j|^2}\right)^{\frac{1}{2}} \|p_{|z_j|}\|_{\frac{p}{2}}^{\frac{1}{2}} \leq \prod_{j=1}^\infty \left(\frac{1}{1-|z_j|^2}\right)^{\frac{1}{2}} \left(\frac{1+|z_j|^2}{1-|z_j|^2}\right)^{\frac{p-2}{2p}} \\ &\leq \left(\exp\frac{\|z\|_2^2}{1-\|z\|_\infty^2}\right)^{\frac{1}{2}} \left(\exp\frac{\|z\|_2^2}{1-\|z\|_\infty^2}\right)^{\frac{p-2}{p}} = \left(\exp\frac{\|z\|_2^2}{1-\|z\|_\infty^2}\right)^{\frac{2}{p'}-\frac{1}{2}}. \end{aligned} \tag{8}$$

For $p > 4$, we will use the hypercontractivity of the Poisson operator proved by Weissler in [41, Theorem 2]: let $1 < p \leq q < \infty$ and let $\mathbf{P}_r : L_p(\mathbb{T}) \to L_q(\mathbb{T})$ denote the Poisson operator, that is, the operator defined by

$$\mathbf{P}_r f(w) = \sum_{k\in\mathbb{Z}} a_k r^{|k|} w^k$$

for $f(w) = \sum_{k\in\mathbb{Z}} a_k w^k$. Then, $\|\mathbf{P}_r\| \leq 1$ if and only if $r^2 \leq \frac{p-1}{q-1}$. Note that for $r < t < 1$, $\mathbf{P}_t(p_{\frac{r}{t}}) = p_r$, thus by the hypercontractivity of $\mathbf{P}_t : L_2(\mathbb{T}) \to L_{\frac{p}{2}}(\mathbb{T})$ for any $r < t < \sqrt{\frac{2}{p-2}}$ we have

$$\|p_r\|_{\frac{p}{2}} = \|\mathbf{P}_t(p_{\frac{r}{t}})\|_{\frac{p}{2}} \leq \|p_{\frac{r}{t}}\|_2 = \left(\frac{1+(\frac{r}{t})^2}{1-(\frac{r}{t})^2}\right)^{\frac{1}{2}}.$$

Let $N_z = \{j \in \mathbb{N} : |z_j|^2 > \frac{1}{p-2}\}$. Using (7) and letting $t \to \sqrt{\frac{2}{p-2}}$ we obtain

$$\gamma_p(z_j) \leq \left(\frac{1}{1-|z_j|^2}\right)^{\frac{1}{2}} \left(\frac{1+\frac{p-2}{2}|z_j|^2}{1-\frac{p-2}{2}|z_j|^2}\right)^{\frac{1}{4}} \quad \text{if } j \notin N_z.$$

On the other hand, interpolating the norms $\|p_r\|_2 = \left(\frac{1+r^2}{1-r^2}\right)^{\frac{1}{2}}$ and $\|p_r\|_\infty = \frac{1+r}{1-r}$ we have for any $0<r<1$, $4<p<\infty$,

$$\|p_r\|_{\frac{p}{2}} \leq \left(\frac{1+r^2}{1-r^2}\right)^{\frac{2}{p}} \left(\frac{1+r}{1-r}\right)^{1-\frac{4}{p}},$$

thus we have for $j \in N_z$,

$$\gamma_p(z_j) \leq \left(\frac{1}{1-|z_j|^2}\right)^{\frac{1}{2}} \left(\frac{1+|z_j|^2}{1-|z_j|^2}\right)^{\frac{1}{p}} \left(\frac{1+|z_j|}{1-|z_j|}\right)^{\frac{1}{2}-\frac{2}{p}}.$$

Note that since $\|z\|_2^2 < 1$, we have, $\#N_z < p-2$. Hence, from Corollary 3.14 we obtain, for $p>4$, $z \in \mathbb{D}_2^\infty$,

$$\begin{aligned}
|\varphi(z)| &\leq \prod_{j=1}^{\infty} \gamma_p(z_j) = \prod_{j=1}^{\infty} \left(\frac{1}{1-|z_j|^2}\right)^{\frac{1}{2}} \|p_{|z_j|}\|_{\frac{p}{2}}^{\frac{1}{2}} \\
&\leq \Big(\prod_{j=1}^{\infty} \frac{1}{1-|z_j|^2}\Big)^{\frac{1}{2}} \left(\prod_{j\notin N_z} \frac{1+\frac{p-2}{2}|z_j|^2}{1-\frac{p-2}{2}|z_j|^2}\right)^{\frac{1}{4}} \prod_{j\in N_z} \left(\frac{1+|z_j|^2}{1-|z_j|^2}\right)^{\frac{1}{p}} \left(\frac{1+|z_j|}{1-|z_j|}\right)^{\frac{1}{2}-\frac{2}{p}} \\
&\leq \Big(\prod_{j=1}^{\infty} \frac{1}{1-|z_j|^2}\Big)^{\frac{1}{2}} \left(\prod_{j\notin N_z} \frac{1+|z_j|^2}{1-|z_j|^2}\right)^{\frac{p-2}{8}} \prod_{j\in N_z} \left(\frac{1+|z_j|^2}{1-|z_j|^2}\right)^{\frac{1}{p}} \left(\frac{1+|z_j|}{1-|z_j|}\right)^{\frac{1}{2}-\frac{2}{p}},
\end{aligned}$$

where for the last inequality we have used the inequality

$$\log \frac{1+cx}{1-cx} \leq c \log \frac{1+x}{1-x},$$

which can be proved through elementary calculus for any $c>1$ and any $0 \leq x \leq \frac{1}{2c}$. We thus have, since $\frac{p-2}{8} > \frac{1}{p}$,

$$\begin{aligned}
|\varphi(z)| &\leq \Big(\prod_{j=1}^{\infty} \frac{1}{1-|z_j|^2}\Big)^{\frac{1}{2}} \left(\prod_{j=1}^{\infty} \frac{1+|z_j|^2}{1-|z_j|^2}\right)^{\frac{p-2}{8}} \prod_{j\in N_z} \left(\frac{1+|z_j|}{1-|z_j|}\right)^{\frac{1}{2}-\frac{2}{p}} \\
&\leq \left(\exp \frac{\|z\|_2^2}{1-\|z\|_\infty^2}\right)^{\frac{p-2}{4}+\frac{1}{2}} \left(\frac{1+\|z\|_\infty}{1-\|z\|_\infty}\right)^{(p-2)\left(\frac{1}{2}-\frac{2}{p}\right)}.
\end{aligned} \tag{9}$$

□

Applying the Bohr lift we can obtain analogous results for Dirichlet series, which prove the remaining cases of Theorem 3.12.

Corollary 3.17. *Let $2<p\leq 4$ and $\frac{1}{2} < Re(w)$. Then,*

$$\|\zeta(\cdot + w)\|_{\mathcal{H}_p} \leq \frac{\zeta(2Re(w))^{\frac{2}{p'}-\frac{1}{2}}}{\zeta(4Re(w))^{\frac{1}{p'}-\frac{1}{2}}}.$$

Proof. If $D \in \mathcal{H}_p^*$ and $\frac{1}{2} < Re(w)$ we have by (2) that $D(-w) = P_\infty \mathscr{B}^{-1}(D)(\overline{\mathfrak{p}^{-w}})$, where $\mathfrak{p}^{-w} = (\mathfrak{p}_1^{-w}, \mathfrak{p}_2^{-w}, \dots)$. By the proof of above theorem (see equation (8)) together with some known properties of Euler products (see for example [32, Corollary 1.10]) we have,

$$|D(-w)| \leq \prod_{j=1}^{\infty} \frac{\left(1+\mathfrak{p}_j^{-2Re(w)}\right)^{\frac{p-2}{2p}}}{\left(1-\mathfrak{p}_j^{-2Re(w)}\right)^{\frac{p-1}{p}}} \|D\|_{\mathcal{H}_p^*} = \zeta(2Re(w))^{\frac{p-1}{p}} \left(\frac{\zeta(2Re(w))}{\zeta(4Re(w))}\right)^{\frac{p-2}{2p}} = \frac{\zeta(2Re(w))^{\frac{3p-4}{2p}}}{\zeta(4Re(w))^{\frac{p-2}{2p}}}.$$

□

Corollary 3.18. *Let* $4 < p$ *and* $\frac{1}{2} < Re(w) \leq 1$*. Then,*

$$\|\zeta(\cdot + w)\|_{\mathcal{H}_p} \leq |D(-w)| \leq \frac{\zeta(2Re(w))^{\frac{p}{4}}}{\zeta(4Re(w))^{\frac{p-2}{8}}} \exp\left(C(p-2)^{1-Re(w)}\right),$$

where C *is an absolute constant.*

Proof. As in the proof of the previous corollary we start noting that if $D \in \mathcal{H}_p^*$ and $\frac{1}{2} < Re(w)$ we have that $D(-w) = P_\infty \mathscr{B}^{-1} D(\overline{\mathfrak{p}^{-w}})|$. By Theorem 3.16 (see equation (9)) together with some known properties of Euler products we have,

$$|D(-w)| \leq \prod_{j=1}^{\infty} \frac{\left(1+\mathfrak{p}_j^{-2Re(w)}\right)^{\frac{p-2}{8}}}{\left(1-\mathfrak{p}_j^{-2Re(w)}\right)^{\frac{p+2}{8}}} \left(\prod_{j\in N_z} \frac{1+\mathfrak{p}_j^{-Re(w)}}{1-\mathfrak{p}_j^{-Re(w)}}\right)^{\frac{1}{2}-\frac{2}{p}} \|D\|_{\mathcal{H}_p^*}$$

$$= \zeta(2Re(w))^{\frac{p+2}{8}} \left(\frac{\zeta(2Re(w))}{\zeta(4Re(w))}\right)^{\frac{p-2}{8}} \left(\prod_{j\in N_z} \frac{1+\mathfrak{p}_j^{-Re(w)}}{1-\mathfrak{p}_j^{-Re(w)}}\right)^{\frac{1}{2}-\frac{2}{p}}.$$

Let $\pi(x)$ denote the number of prime numbers smaller than or equal to x, which behaves, by the Prime Number Theorem, asymptotically as $\frac{x}{ln(x)}$. Using Abel summation, we derive the following Mertens'-type inequality:

(10)
$$\sum_{\substack{\mathfrak{p} \text{ prime} \\ \mathfrak{p}\leq x}} \mathfrak{p}^{-Re(w)} = \pi(x)x^{-Re(w)} + Re(w)\int_2^x \pi(t)t^{-Re(w)-1}dt \leq C\left(\frac{x^{1-Re(w)}}{ln(x)} + \int_2^x \frac{1}{t^{Re(w)}}dt\right) \leq Cx^{1-Re(w)},$$

where C denotes an absolute constant and $Re(w) \leq 1$.

Since $\frac{1+x}{1-x} \leq e^{3x}$ for $x \geq 0$ and $N_z \leq p-2$ we have that

$$\prod_{j\in N_z} \frac{1+\mathfrak{p}_j^{-Re(w)}}{1-\mathfrak{p}_j^{-Re(w)}} \leq \prod_{j=1}^{p-2} \exp\left(3\mathfrak{p}_j^{-Re(w)}\right).$$

By (10), it follows that

$$\prod_{j=1}^{p-2} \exp\left(3\mathfrak{p}_j^{-Re(w)}\right) = \exp\left(3\sum_{j=1}^{p-2} \mathfrak{p}_j^{-Re(w)}\right) \leq \exp\left(3C(p-2)^{1-Re(w)}\right).$$

Altogether we find that,

$$|D(-w)| \leq \zeta(2Re(w))^{\frac{p+2}{8}} \left(\frac{\zeta(2Re(w))}{\zeta(4Re(w))}\right)^{\frac{p-2}{8}} \exp\left(C(p-2)^{1-Re(w)}\right),$$

where C is an absolute constant. □

4. Multiplicative Toeplitz operators on Hardy spaces

The classical Toeplitz matrices $T = (a_{ij})_{i,j}$ are infinite matrices of the form $\phi(i-j)$, where ϕ is a function over $\mathbb{Z}$. It is known by the work of Otto Toeplitz that T defines a bounded operator on ℓ_2 if and only if ϕ is a bounded symbol. In this case the operator is called a Toeplitz operator T_ϕ with symbol ϕ. Mapping the canonical basis of ℓ_2 to other bases in Hilbert spaces one can obtain other realizations of the Toeplitz matrix. For example, the Wiener-Hopf integral operators on $L_2(\mathbb{R}_+)$ have Toeplitz matrices with respect to the basis of Laguerre functions [35]. The concept of Toeplitz operator extends to Banach spaces with Schauder basis in the obvious way. In the particular case of the $H_p(\mathbb{T})$ spaces, a Toeplitz operator is bounded if and only if $\phi \in L_\infty(\mathbb{T})$ and moreover $T_\phi(f) = P_+(\phi f)$, where $P_+ : L_p(\mathbb{T}) \to H_p(\mathbb{T})$ is the Riesz projection. The extension to the N-dimensional torus is given by operators on $H_p(\mathbb{T}^N)$, $N \in \mathbb{N} \cup \{\infty\}$, of the form

$$f \in H_p(\mathbb{T}^N) \mapsto T_\phi(f) = P_+(\phi f),$$

for some bounded function ϕ defined on $\mathbb{T}^N$.

The main objective of this section is to study analogues on $\mathcal{H}_p$ spaces of Dirichlet series.

Let us first recall that an infinite matrix $(a_{ij})_{i,j\in\mathbb{N}}$ is a *muliplicative Toeplitz matrix* if $a_{ij} = a_{nm}$ for indices satisfying $\frac{i}{j} = \frac{n}{m}$. Given a Hilbert space H [35], a linear operator T is said to be a multiplicative Toeplitz operator if its matrix with respect to the canonical basis $(e_n)_{n\in\mathbb{N}}$ is a multiplicative Toeplitz matrix. Multiplicative Toeplitz operators arise as a natural generalization of classical Toeplitz operators $T_\varphi : H_2(\mathbb{T}) \to H_2(\mathbb{T})$ to Dirichlet series. Indeed, if we consider as a symbol a given formal Dirichlet series over the rational numbers, $D = \sum_{q\in\mathbb{Q}_+} a_q q^{-s}$, consider the linear operator $T_D : \mathcal{H}_2 \to \mathcal{H}_2$,

$$T_D(E) = P_{\mathbb{N}}(D \cdot E).$$

Here $P_{\mathbb{N}}$ denotes the projection over the natural numbers, $\sum_{q\in\mathbb{Q}_+} c_q q^{-s} \mapsto \sum_{n=1}^{\infty} c_n n^{-s}$ defined on the Hilbert space of general Dirichlet series spanned by $(q^{-s})_{q\in\mathbb{Q}_+}$ (sometimes called Besicovitch space) to the subspace $\mathcal{H}_2$. This projection is the Dirichlet series counterpart of the Riesz projection. Then, the coefficients of the matrix associated to T_D are, for $n, m \in \mathbb{N}$,

$$\langle T_D m^{-s}, n^{-s}\rangle = \langle P_{\mathbb{N}} \sum_{q\in\mathbb{Q}_+} a_q (qm)^{-s}, n^{-s}\rangle = a_{n/m},$$

so they define a multiplicative Toeplitz matrix.

When we consider the extended Bohr transform defined in Subsection 3.2 we have that

$$\mathscr{B}^{-1} P_{\mathbb{N}} \mathscr{B}$$

is the Riesz projection P_+ that sends $L_2(\mathbb{T}^\infty)$ to $H_2(\mathbb{T}^\infty)$. Thus, $T_D : \mathcal{H}_2 \to \mathcal{H}_2$ is bounded if and only if $\mathscr{B}^{-1} D \in L_\infty(\mathbb{T}^\infty)$. Moreover, $\mathscr{B}^{-1} T_D \mathscr{B} = T_{\mathscr{B}^{-1}D}$ is a Toeplitz operator on $H_2(\mathbb{T}^\infty)$.

When we want to generalize these operators to $\mathcal{H}_p$ spaces we face a serious obstacle: the Riesz projection P_+ sending $L_p(\mathbb{T}^\infty)$ to $H_p(\mathbb{T}^\infty)$ is bounded only for $p = 2$. Despite this restriction, we can still operate formally and consider the Dirichlet series D for which

$$T_D : E \mapsto P_{\mathbb{N}}(D \cdot E)$$

defines a bounded operator on $\mathcal{H}_p$. We will call such an operator *multiplicative Toeplitz operator* with symbol D and we denote by $\mathcal{T}_p$ to the set of all Dirichlet series over the rational numbers that are symbols of bounded multiplicative Toeplitz operators on $\mathcal{H}_p$. Again, a simple computation shows that multiplicative Toeplitz operators have a multiplicative Toeplitz matrix representation with respect to the canonical basis $\{n^{-s}\}$.

If D is an analytic Dirichlet series then T_D is just a multiplication operator on $\mathcal{H}_p$. It is well-known by [29, 5] that the multipliers $\mathcal{M}(\mathcal{H}_p)$ of $\mathcal{H}_p$, $1 < p < \infty$, are exactly $\mathcal{H}_\infty$. Hence $\mathcal{H}_\infty \subset \mathcal{T}_p$.

When D is a co-analytic Dirichlet series, we will say that T_D is a co-analytic multiplicative Toeplitz operator.

Remark 4.1. *Co-analytic multiplicative Toeplitz operators are in correspondence with the multipliers of $\mathcal{H}_p^*$. More precisely, $T_D = M_D^*$, where $M_D : \mathcal{H}_p^* \to \mathcal{H}_p^*$ is the multiplication operator by D.*

To see this, just note that if $D = \sum_{k=1}^\infty d_k k^s$, and $n, m \in \mathbb{N}$ then

$$\langle M_D^* n^{-s}, m^s \rangle_{\mathcal{H}_p, \mathcal{H}_p^*} = \langle n^{-s}, Dm^s \rangle_{\mathcal{H}_p, \mathcal{H}_p^*} = \langle n^{-s}, \sum_{k=1}^\infty d_k (k \cdot m)^s \rangle_{\mathcal{H}_p, \mathcal{H}_p^*} = \sum_{k=1}^\infty d_k \delta_{n, k\cdot m} = d_{\frac{n}{m}},$$

while

$$\langle T_D n^{-s}, m^s \rangle = \langle P_{\mathbb{N}} \left(\sum_{k=1}^\infty d_k \left(\frac{n}{k}\right)^{-s} \right), m^s \rangle = \langle \sum_{\substack{k\in\mathbb{N} \\ k|n}} d_k \left(\frac{n}{k}\right)^{-s}, m^s \rangle = \sum_{k=1}^\infty d_k \delta_{\frac{n}{k}, m} = d_{\frac{n}{m}}.$$

Of course, $d_{\frac{n}{m}} = 0$ if $\frac{n}{m} \notin \mathbb{N}$. Thus, the co-analytic Dirichlet series for which the multiplicative Toeplitz operators is bounded, coincides with those that define bounded multipliers on $H_p(\mathbb{D}_2^\infty)^*$.

Another initial observation is that in contrast to the case $p = 2$, $\mathcal{M}(\mathcal{H}_p^*) \neq \jmath\mathcal{H}_\infty$, at least isometrically. Indeed, if $p \neq 2$ then the Toeplitz operator with symbol $\overline{z}$ has norm strictly bigger than one on $H_p(\mathbb{T})$ (see [12, Theorem 7.7] and [15]). Therefore, since $T_{2^s} : \mathcal{H}_p \to \mathcal{H}_p$, satisfies $T_{2^s} = \mathscr{B} T_{\overline{z}_1} \mathscr{B}^{-1}$, then if it is bounded, it must satisfy $\|T_{2^s}\| \geq \|T_{\overline{z}_1}\| > 1 = \|\jmath(2^s)\|_{\mathcal{H}_\infty}$.

So far, the only examples of multiplicative Toeplitz operators on $\mathcal{H}_p$ that we have are the analytic ones (those with symbol in $\mathcal{H}_\infty$). Our first goal will be to prove that the Dirichlet polynomials over the rational numbers are in $\mathcal{T}_p$ for any $1 < p < \infty$. First we need the following.

Lemma 4.2. *For each prime number $\mathfrak{p}_k$, the projection $\mathcal{R}_{\mathfrak{p}_k} : \mathcal{H}_p \to \mathcal{H}_p$, defined by*

$$\sum_{n=1}^\infty a_n n^{-s} \mapsto \sum_{\substack{n=1 \\ \mathfrak{p}_k | n}}^\infty a_n n^{-s}$$

is a bounded operator with $\|\mathcal{R}_{\mathfrak{p}_k}\| \leq 2$.

Proof. Equivalently we will prove that for each $n \in \mathbb{N}$, $R_n : H_p(\mathbb{T}^\infty) \to H_p(\mathbb{T}^\infty)$, $\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha w^\alpha \mapsto \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})} : \alpha_n \neq 0} c_\alpha w^\alpha$ is bounded.

Given $f \in H_p(\mathbb{T}^\infty)$, $u \in \mathbb{T}^\infty$ we consider

$$\tilde{f}(u) = \int_{\mathbb{T}} f(u_1, \ldots, u_{n-1}, w, u_{n+1}, \ldots) dw.$$

It follows by the integral Minkowski inequality, Hölder's inequality and Fubini's Theorem that $\tilde{f} \in H_p(\mathbb{T}^\infty)$, with $\|\tilde{f}\|_p \leq \|f\|_p$. Indeed,

$$\|\tilde{f}\|_p = \left(\int_{\mathbb{T}^\infty}\left|\int_{\mathbb{T}} f(u_1,\ldots,u_{n-1},w,u_{n+1},\ldots)dw\right|^p du\right)^{\frac{1}{p}} \leq \int_{\mathbb{T}}\left(\int_{\mathbb{T}^\infty}|f(u_1,\ldots,u_{n-1},w,u_{n+1},\ldots|^p du\right)^{\frac{1}{p}} dw$$
$$\leq \left(\int_{\mathbb{T}}\int_{\mathbb{T}^\infty}|f(u_1,\ldots,u_{n-1},w,u_{n+1},\ldots|^p dudw\right)^{\frac{1}{p}} \cdot \left(\int_{\mathbb{T}} 1^q\right)^{\frac{1}{q}} = \|f\|_p.$$

It also satisfies that $c_\alpha(\tilde{f}) = c_\alpha(f)$ if $\alpha_n = 0$ and $c_\alpha(\tilde{f}) = 0$ if $\alpha_n \neq 0$. Indeed, if $\alpha_n \neq 0$ then $c_\alpha(\tilde{f}) = 0$ because $\tilde{f}$ does not depend on the n-th coordinate. On the other hand, if $\alpha_n = 0$, we have that

$$c_\alpha(\tilde{f}) = \int_{\mathbb{T}^\infty}\int_{\mathbb{T}} f(u_1,\ldots,u_{n-1},w,u_{n+1},\ldots)u^{-\alpha}dwdu = c_\alpha(f),$$

again Fubini's Theorem. This implies that $(I - R_n)(f) = \tilde{f}$, and hence $I - R_n$ is a contraction. In particular, $\|R_n\| \leq 2$.

Since $\mathscr{B}R_k\mathscr{B}^{-1} = \mathcal{R}_{\mathfrak{p}_k}$ we conclude that $\|\mathcal{R}_{\mathfrak{p}_k}\| \leq 2$.

□

Proposition 4.3. *For every $1 \leq p < \infty$ and every Dirichlet polynomial over the rationals, i.e. for every $Q = \sum_{n=1}^N a_{q_n} q_n^{-s}$ with $q_n \in \mathbb{Q}_+$, the Toeplitz operator $T_Q : \mathcal{H}_p \to \mathcal{H}_p$ is bounded.*

Proof. Let $D = \sum_{n=1}^\infty a_n n^{-s}$ be a norm 1 Dirichlet series in $\mathcal{H}_p$. Note that if $q = \frac{1}{\mathfrak{p}}$, with $\mathfrak{p}$ a prime number, then

$$T_{q^{-s}}(D) = P_{\mathbb{N}}(q^{-s}\cdot D) = P_{\mathbb{N}}\Big(\sum_{n=1}^\infty a_n\left(\frac{n}{\mathfrak{p}}\right)^{-s}\Big) = \sum_{\substack{n=1\\ \mathfrak{p}|n}}^\infty a_n\left(\frac{n}{\mathfrak{p}}\right)^{-s} = q^{-s}\cdot\mathcal{R}_{\mathfrak{p}}(D).$$

Since $|q^{it}| = 1$ whenever t is a real number we have by Lemma 4.2 that the Besicovitch norm is

$$\|T_{q^{-s}}(D)\| = \lim_{N\to\infty}\lim_{R\to\infty}\left(\int_{-R}^R\left|q^{-it}\cdot\sum_{\substack{n=1\\ \mathfrak{p}|n}}^N a_n n^{-it}\right|^p dt\right)^{\frac{1}{p}} = \|\mathcal{R}_{\mathfrak{p}}(D)\| \leq 2.$$

It follows that for any natural number $n = \mathfrak{p}^\alpha = \prod_{j=1}^l \mathfrak{p}_j^{\alpha_j}$ the Toeplitz operator

$$T_{\left(\frac{1}{n}\right)^{-s}} = T^{\alpha_1}_{\left(\frac{1}{\mathfrak{p}_1}\right)^{-s}} \circ\cdots\circ T^{\alpha_l}_{\left(\frac{1}{\mathfrak{p}_l}\right)^{-s}}$$

is bounded. Suppose now that $q \in \mathbb{Q}_+$, say $q = \frac{n_1}{n_2}$, for coprime natural numbers $n_1, n_2 \in \mathbb{N}$. Then, using the fact that $M_{n_2^{-s}}$ is a bounded multiplication operator on $\mathcal{H}_p$, we have for every Dirichlet series D that

$$T_{q^{-s}}(D) = P_{\mathbb{N}}\left(\left(\frac{n_1}{n_2}\right)^{-s}\cdot D\right) = P_{\mathbb{N}}\left(\left(\frac{1}{n_2}\right)^{-s}\cdot(M_{n_1^{-s}}D)\right) = T_{\left(\frac{1}{n_2}\right)^{-s}}\circ M_{n_1^{-s}}(D).$$

Hence $T_{q^{-s}}$ is a composition of bounded operators.

Finally if Q is now a polynomial, $Q = \sum_{n=1}^N a_{q_n} q_n^{-s}$, with $q_n \in \mathbb{Q}_+$, we have that $T_Q = \sum_{n=1}^N a_{q_n} T_{q_n^{-s}}$. □

It is well-known that, in the classical Hardy spaces $H_p(\mathbb{T}^N)$, the functions that are symbols of Toeplitz operators are exactly those in $L_\infty(\mathbb{T}^N)$. In the infinite dimensional case, Toeplitz operators on $H_2(\mathbb{T}^\infty)$ were recently studied in [28], where it is proved that again, the set of symbols that determine bounded Toeplitz operators on $H_2(\mathbb{T}^\infty)$ is $L_\infty(\mathbb{T}^\infty)$. This implies that the for the Dirichlet series case $\mathcal{H}_2$, the symbols satisfy $\mathscr{B}^{-1}\mathcal{T}_2 = L_\infty(\mathbb{T}^\infty)$.

We will show that $\mathscr{B}^{-1}\mathcal{T}_p \subset L_\infty(\mathbb{T}^\infty)$ for any $1 < p < \infty$, but that in contrast to the aforementioned cases, not every bounded function defines a bounded Toeplitz operator on $\mathcal{H}_p$ if $p \neq 2$.

The algebra of Dirichlet series $\mathcal{A}(\mathbb{C}_+)$ was introduced in [4] and consists of the closure of the Dirichlet polynomials under the supremum norm. Thus, $\mathcal{A}(\mathbb{C}_+) \subseteq \mathcal{H}_\infty$. Through the Bohr transform $\mathfrak{B}$, the space $\mathcal{A}(\mathbb{C}_+)$ is isometrically isomorphic to the subspace $\mathcal{A}_u(B_{c_0}) \subset H_\infty(B_{c_0})$ of uniformly continuous analytic functions on the unit ball of c_0, see [4, Theorem 2.4].

We will construct a co-analytic Dirichlet series $D \in \jmath\mathcal{A}(\mathbb{C}_+)$ which is not the symbol of a bounded Toeplitz operator. In particular, $L_\infty(\mathbb{T}^\infty) \not\subset \mathscr{B}^{-1}\mathcal{T}_p$. This also implies that there are Dirichlet co-analytic series in $\jmath\mathcal{A}(\mathbb{C}_+)$ which are not multipliers of $\mathcal{H}_p^*$.

Theorem 4.4. *For every $1 \leq p < \infty$, $p \neq 2$, there exists a co-analytic Dirichlet series $D \in \jmath\mathcal{A}(\mathbb{C}_+)$ for which the multiplicative Toeplitz operator $T_D : \mathcal{H}_p \to \mathcal{H}_p$ is not a bounded operator. In particular, for $1 < p < \infty$ and $p \neq 2$, $\mathcal{M}(\mathcal{H}_p^*) \neq \jmath\mathcal{H}_\infty$.*

Proof. Using the Bohr transform, it suffices to prove that there is $\phi \in \jmath\mathcal{A}_u(B_{c_0})$ for which $T_\phi : H_p(\mathbb{D}_2^\infty) \to H_p(\mathbb{D}_2^\infty)$ is not bounded.

Let $n_1 = 1$ and $n_k = n_{k-1} + k$ for $k \geq 2$. Let, for $z \in B_{c_0}$, $\phi_k(z) := \prod_{j=n_k-k+1}^{n_k} \overline{z}_j$. Each ϕ_k is a co-analytic polynomial of degree k with supremum norm equal to one. Moreover, there exists $\lambda > 1$ such that $\|T_{\phi_k}\| > \lambda^k$. Indeed, by [12, Theorem 7.7] there are $\lambda > 1$ and $f \in H_p(\mathbb{D})$ with $\|f\|_{H_p(\mathbb{D})} = 1$ such that $\|T_{\overline{z}}(f)\| > \lambda$. Hence, if we define $f_j \in H_p(\mathbb{D}_2^\infty)$ by $f_j(z) := f(z_j)$ we have that $\|f_j\|_{H_p(\mathbb{D}_2^\infty)} = 1$ and that $\|T_{\overline{z}_j}(f_j)\|_{H_p(\mathbb{D}_2^\infty)} = \|T_{\overline{z}} f\|_{H_p(\mathbb{D})} > \lambda$. Consider now $g_k \in H_p(\mathbb{D}_2^\infty)$ given by $g_k(z) := \prod_{j=n_k-k+1}^{n_k} f_j$. Since g_k is a product of functions which depend on different variables, we have that $\|g_k\|_{H_p(\mathbb{D}_2^\infty)} = \prod_{j=n_k-k+1}^{n_k} \|f_j\| = 1$. In a similar way, we obtain that

$$\|T_{\phi_k}(g_k)\| = \left\| \prod_{j=n_k-k+1}^{n_k} T_{\overline{z}_j} f_j \right\| > \lambda^k. \tag{11}$$

Consider now $(a_k)_k \in \ell_1$ such that $a_k \lambda^k \to \infty$. We define $\phi := \sum_{k=1}^\infty a_k \phi_k$. Since $(a_k)_k \in \ell_1$ and each $\|\jmath\phi_k\|_\infty = 1$, we have that $\jmath\phi \in \mathcal{A}_u(B_{c_0})$ with $\|\phi\|_\infty \leq \|(a_k)_k\|_1$.

On the other hand, $T_\phi(g_k) = \sum_{j=1}^\infty a_j T_{\phi_j}(g_k) = a_k T_{\phi_k}(g_k)$. We conclude from (11) that

$$\|T_\phi(g_k)\|_{H_p(\mathbb{D}_2^\infty)} \geq a_k \lambda^k \to \infty.$$

This implies that T_ϕ is not bounded on $H_p(\mathbb{D}_2^\infty)$. $\square$

Our next step is to show that $\mathscr{B}^{-1}\mathcal{T}_p \subset L_\infty(\mathbb{T}^\infty)$. We first need to prove that multiplicative Toeplitz operators behave well when we restrict to finitely many primes.

Recall that $\mathcal{H}_{p,N}$ is the subspace of $\mathcal{H}_p$ of Dirichlet series depending on the first N primes, i.e. series with zero n-th coefficient for every n not of the form $\mathfrak{p}_1^{\alpha_1}\dots\mathfrak{p}_N^{\alpha_N}$, for some $\alpha_j\in\mathbb{N}_0$. Also, given a Dirichlet series D, D_N denotes its restriction to the first N primes. Note that $\mathfrak{B}: H_p(\mathbb{D}^N)\to\mathcal{H}_{p,N}$ is a surjective isomorphism.

Theorem 4.5. *Let $1<p<\infty$ and let D be a formal co-analytic Dirichlet series. Then $T_D:\mathcal{H}_p\to\mathcal{H}_p$ defines a bounded operator if and only if*

$$\sup_{N\in\mathbb{N}}\|T_{D_N}\|_{\mathcal{H}_{p,N}\to\mathcal{H}_{p,N}}<\infty. \tag{12}$$

Moreover, in this case, $\|T_D\|$ equals the supremum in (12).

Proof. It is plain that for each $N\in\mathbb{N}$, $\|T_{D_N}\|_{\mathcal{H}_{p,N}\to\mathcal{H}_{p,N}}\le\|T_{D_N}\|_{\mathcal{H}_p\to\mathcal{H}_p}\le\|T_D\|_{\mathcal{H}_p\to\mathcal{H}_p}$, because if E depends only on the first N-primes then $T_DE=T_{D_N}E$.

Reciprocally, let D be a formal co-analytic Dirichlet series such that $\sup_{N\in\mathbb{N}}\|T_{D_N}\|_{\mathcal{H}_{p,N}\to\mathcal{H}_{p,N}}<\infty$. We will prove that $M_D:\mathcal{H}_p^*\to\mathcal{H}_p^*$, $M_DE=D\cdot E$ defines a bounded operator with $\|M_D\|\le\sup_{N\in\mathbb{N}}\|T_{D_N}\|$.

Let $E\in\mathcal{H}_p^*$ and consider the formal co-analytic series given by $D\cdot E$. Then, $[D\cdot E]_N=D_N\cdot E_N=M_{D_N}E_N\in\mathcal{H}_{p,N}^*$ with $\|[D\cdot E]_N\|\le\|M_{D_N}\|_{\mathcal{H}_{p,N}^*\to\mathcal{H}_{p,N}^*}\|E_N\|_{\mathcal{H}_{p,N}^*}\le\sup_{N\in\mathbb{N}}\|M_{D_N}\|_{\mathcal{H}_{p,N}^*\to\mathcal{H}_{p,N}^*}\|E\|_{\mathcal{H}_p^*}=\sup_{N\in\mathbb{N}}\|T_{D_N}\|_{\mathcal{H}_{p,N}\to\mathcal{H}_{p,N}}\|E\|_{\mathcal{H}_p^*}$, where in the second inequality we have used Proposition 3.4. Applying again Proposition 3.4, we obtain that $D\cdot E\in\mathcal{H}_p^*$, with $\|D\cdot E\|_{\mathcal{H}_p^*}\le\sup_{N\in\mathbb{N}}\|T_{D_N}\|_{\mathcal{H}_{p,N}\to\mathcal{H}_{p,N}}\|E\|_{\mathcal{H}_p^*}$. This implies that M_D is a well defined operator with $\|M_D\|_{\mathcal{H}_p^*\to\mathcal{H}_p^*}\le\sup_{N\in\mathbb{N}}\|T_{D_N}\|_{\mathcal{H}_{p,N}\to\mathcal{H}_{p,N}}$. Consequently, $T_D=M_D^*$ defines a bounded operator. ∎

Using Remark 4.1, the above theorem may be restated in terms of the multipliers of $H_p(\mathbb{D}_2^\infty)^*$.

Remark 4.6. *For each $N\in\mathbb{N}$, consider $\mathcal{T}_{p,N}:=\mathfrak{B}L_\infty(\mathbb{T}^N)$ endowed with the Toeplitz operator norm given by $\|D\|_{\mathcal{T}_p^N}:=\|T_D\|_{\mathcal{H}_{p,N}\to\mathcal{H}_{p,N}}$. Rephrasing Theorem 4.5, $\mathcal{M}(\mathcal{H}_p^*)$ is the space of co-analytic Dirichlet series for which $D_N\in\mathcal{T}_{p,N}$ and $\sup_{N\in\mathbb{N}}\|D_N\|_{\mathcal{T}_{p,N}}<\infty$.*

The next theorem shows that $\mathcal{T}_p$ is contained in $\mathfrak{B}L_\infty(\mathbb{T}^\infty)$. First, we state a likely well-known lemma.

Lemma 4.7. *Let $a\in L_\infty(\mathbb{T}^N)$, T_a the associated Toeplitz operator on $H_p(\mathbb{T}^N)$, $1\le p<\infty$. Then $r(T_a)\ge\|a\|_\infty$.*

Proof. Denote M_a and $S=M_{z_1\cdots z_N}$ the multiplication operators on $L_p(\mathbb{T}^N)$. Note that S is an isometry. Recall also that the approximate point spectrum of M_a, $\sigma_{ap}(M_a)$, coincides with the essential range of a.

If $\lambda\in\sigma_{ap}(M_a)$ there is a sequence of norm one trigonometrical polynomials $(q_k)_k$ in $L_p(\mathbb{T}^N)$ such that $(M_a-\lambda)q_k\to0$. Let $d_k\ge0$ be such that $S^{d_k}q_k\in H_p(\mathbb{T}^N)$ for each k. Thus $(S^{d_k}q_k)_k$ is a norm one sequence in $H_p(\mathbb{T}^N)$ such that

$$\begin{aligned}\|(T_a-\lambda)S^{d_k}q_k\|_p&=\|P_+(M_a-\lambda)S^{d_k}q_k\|_p=\|P_+S^{d_k}(M_a-\lambda)q_k\|_p\le\sin(\pi/p)^{-N}\|S^{d_k}(M_a-\lambda)q_k\|_p\\&=\sin(\pi/p)^{-N}\|(M_a-\lambda)q_k\|_p\to0.\end{aligned}$$

Hence, $\sigma_{ap}(M_a)\subseteq\sigma_{ap}(T_a)$ and therefore $r(T_a)\ge r(M_a)\ge\|a\|_\infty$. ∎

Theorem 4.8. *Let $D = \sum_{q\in\mathbb{Q}_+} a_q q^{-s}$ be a Dirichlet series in $\mathcal{T}_p$, i.e. such that $T_D : \mathcal{H}_p \to \mathcal{H}_p$ defines a bounded multiplicative Toeplitz operator. Then $\mathscr{B}^{-1}(D) \in L_\infty(\mathbb{T}^\infty)$ and $\|\mathscr{B}^{-1}(D)\|_{L_\infty(\mathbb{T}^\infty)} \leq \|T_D\|$.*

Proof. If $E \in \mathcal{H}_p$ depends on the first N primes, then $T_D(E) = T_{D_N}E$. Hence, $\|T_D\| \geq \sup_{N\in\mathbb{N}} \|T_{D_N}\| = \sup_{N\in\mathbb{N}} \|T_{\mathscr{B}^{-1}(D_N)}\|_{H_p(\mathbb{T}^N)\to H_p(\mathbb{T}^N)}$. By Lemma 4.7 we have that $\|T_{\mathscr{B}^{-1}(D_N)}\|_{H_p(\mathbb{T}^N)\to H_p(\mathbb{T}^N)} \geq r(T_{\mathscr{B}^{-1}(D_N)}) \geq \|\mathscr{B}^{-1}(D_N)\|_{L_\infty(\mathbb{T}^N)}$ and hence $\sup_{N\in\mathbb{N}} \|\mathscr{B}^{-1}(D_N)\|_{L_\infty(\mathbb{T}^N)} < \infty$. Since $L_1(\mathbb{T}^\infty)$ is separable, there is a convergent w^* subsequence of $(\mathscr{B}^{-1}(D_N))_N$ which has to converge, by the uniqueness of the Fourier coefficients, to $\mathscr{B}^{-1}(D)$. In particular, $\mathscr{B}^{-1}(D) \in L_\infty(\mathbb{T}^\infty)$ with $\|\mathscr{B}^{-1}(D)\|_\infty \leq \sup_{N\in\mathbb{N}} \|T_{D_N}\| \leq \|T_D\|$. ☐

Corollary 4.9. *Let $1 < p < \infty$. Then $\mathcal{M}(\mathcal{H}_p^*)$ is contained in $\jmath\mathcal{H}_\infty$.*

Proof. Let $D \in \mathcal{M}(\mathcal{H}_p^*)$. By the above theorem D is a co-analytic Dirichlet series for which $\mathscr{B}^{-1}(D) \in L_\infty(\mathbb{T}^\infty)$. Hence, $\mathscr{B}^{-1}(\jmath D) = \jmath\mathscr{B}^{-1}(D) \in H_\infty(\mathbb{T}^\infty)$. Thus, $\jmath D = \mathscr{B}\jmath\mathscr{B}^{-1}(D) \in \mathcal{H}_\infty$. ☐

5. Linear Dynamics on Hardy spaces of Dirichlet series

The main aim of this section is to describe the dynamics of multiplicative co-analytic Toeplitz operators on Hardy spaces of Dirichlet series. We will deduce their dynamical properties from the corresponding properties of adjoint of multiplication operators on dual Hardy spaces of Dirichlet series. First we will need to extend the Godefroy-Shapiro theorem 2.1 to Hardy spaces on the N-polydisc and on $\mathbb{D}_2^\infty$.

The main theorems of the section are Theorems 5.13 and 5.17 which answer both Questions A and B affirmatively: adjoint multiplication operators in $\mathcal{H}_p^*$ and multiplicative co-analytic Toeplitz operators in $\mathcal{H}_p$ are hypercyclic (and chaotic) whenever $D(\mathbb{C}_+) \cap \mathbb{T} \neq \emptyset$. We also study frequent hypercyclicity and analyze zero-one laws and trichotomy results.

The structure of the section is the following: we first study the linear dynamics of adjoint multiplication operators on $H_p(\mathbb{D}^N)^*$, then we extend this results to the infinite dimensional case of $H_p(\mathbb{D}_2^\infty)^*$. Next, via the Bohr transform, we derive properties for the dynamics on $\mathcal{H}_p^*$. Finally, applying these results, we study the dynamics of multiplicative co-analytic Toeplitz operators on $\mathcal{H}_p$.

5.1. Linear dynamics of adjoint of multiplication operators on $H_p(\mathbb{D}^N)^*$.

In this subsection we first prove an extension of Godefroy-Shapiro Theorem 2.1 for $p \neq 2$ and show that they are frequently hypercyclic whenever they are hypercyclic. These results are probably known by specialists, but we were not able to find them in the literature.

After this, we will prove a zero-one type law for adjoint of multipliers (as shown in [18] and [10] for the one dimensional Bergman and Hardy spaces respectively): if an orbit has a nonzero limit point then the operator must be hypercyclic. Moreover, if ϕ is analytic on a neighborhood of $\mathbb{D}^N$, we show that a trichotomy holds for M_ϕ^*: either every orbit converges to 0, or every orbit tends to ∞, or M_ϕ^* is hypercyclic.

We start with a result on frequent hypercyclicity, see [7, Example 3.10] for a proof of the case $N = 1$. Recall that an operator is said to have a perfectly spanning set of unimodular eigenvectors provided that there is a measure $(\sigma, \mathbb{T})$ such that for every set of zero measure $A \subseteq \mathbb{T}$, we have that the span of the eigenvectors with eigenvalue in $\mathbb{T} \setminus A$ is dense in X. Every operator supporting a perfectly spanning set of unimodular eigenvectors is ergodic with respect to a measure of full support μ and hence it is frequently hypercyclic, see [25, Theorem 1.4].

Lemma 5.1. *Let $\phi \in H_\infty(\mathbb{D}^N)$ be a non-constant function such that $\phi(\mathbb{D}^N) \cap \mathbb{T} \neq \emptyset$ and $A \subseteq \mathbb{T}$ such that $\mathbb{T} \setminus A$ is dense in $\mathbb{T}$. Then the span of the eigenvectors of M_ϕ^* with eigenvalue in $\mathbb{T} \setminus A$ is dense in $H_p(\mathbb{D}^N)^*$. In particular, M_ϕ^* supports a perfect spanning set of unimodular eigenvectors with respect to the normalized Lebesgue measure of $\mathbb{T}$.*

Proof. Recall that if δ_z denotes the evaluation at $z \in \mathbb{D}^N$, then δ_z is an eigenvector of M_ϕ^* of eigenvalue $\phi(z)$.

Since ϕ is holomorphic, there is an arc $I \subseteq \mathbb{T}$ such that $I \subseteq \phi(\mathbb{D}^N) \cap \mathbb{T}$. Assume that $\mathbb{T} \setminus A$ is dense in $\mathbb{T}$ and consider $\Gamma = \phi^{-1}(I \setminus A) = \{\lambda \in \mathbb{D}^N : \phi(\lambda) \in I \setminus A\}$.

Then there exists an accumulation point $\lambda \in \overline{\mathbb{D}^N}$ of Γ. Moreover, this point λ may be taken in $\mathbb{D}^N$. Indeed, consider first $\lambda_0 \in \Gamma$ and then a one-dimensional disk D centered at λ_0 with radius strictly smaller than $d(\lambda_0, \partial\mathbb{D}^N)$ such that $\phi(D) \cap \mathbb{T} \subseteq I$ and has positive measure. Hence any accumulation point of $\{z \in D : \phi(z) \in I \setminus A\}$ belongs to $\mathbb{D}^N$ and is an accumulation point of Γ.

Take a collection of lines L_k through λ such that ϕ restricted to each L_k is non-constant and $\bigcup_{k\in\mathbb{N}} L_k \cap \mathbb{D}^N$ is dense in $\mathbb{D}^N$.

Note that $\phi|_{L_k \cap \mathbb{D}^N}$ is a non-constant holomorphic map of one variable whose image intersects I. Thus, if $D \subset L_k \cap \mathbb{D}^N$ is a one-dimensional disc around λ, we have that $\phi|_{L_k \cap \mathbb{D}^N}(D)$ contains a sub-arc of I. Hence $\Gamma \cap L_k$ has an accumulation point in $L_k \cap \mathbb{D}^N$.

Let $f \in H_p(\mathbb{D}^N)$ such that $\langle \delta_z, f \rangle = f(z) = 0$ for every $z \in \Gamma$. In particular, f is identically zero in each $\Gamma \cap L_k$, which is a set that contains an accumulation point. This implies that f restricted to each $L_k \cap \mathbb{D}^N$ is 0. Since $\bigcup_{k\in\mathbb{N}} L_k \cap \mathbb{D}^N$ is dense in $\mathbb{D}^N$, we conclude that f is identically zero. This implies that *span*$\{\delta_\lambda : \lambda \in I \setminus A\}$ is dense in $H_p(\mathbb{D}^N)^*$ which proves the claim. ☐

Theorem 5.2. *Let $\phi \in H_\infty(\mathbb{D}^N)$ be a non-constant function. Then, the following are equivalent:*

(1) $M_\phi^* : H_p(\mathbb{D}^N)^* \to H_p(\mathbb{D}^N)^*$ is hypercyclic;
(2) $M_\phi^* : H_p(\mathbb{D}^N)^* \to H_p(\mathbb{D}^N)^*$ is chaotic and frequently hypercyclic and
(3) $\phi(\mathbb{D}^N) \cap \mathbb{T} \neq \emptyset$.

Proof. (1) $\Rightarrow$ (3). This implication follows as the one dimensional case: assume that $\phi(\mathbb{D}^N) \cap \mathbb{T} = \emptyset$. Since $\mathbb{D}^N$ is connected either $\phi(\mathbb{D}^N) \subset \mathbb{D}$ or $\phi(\mathbb{D}^N) \subset \mathbb{C} \setminus \bar{\mathbb{D}}$.

If $\phi(\mathbb{D}^N) \subset \mathbb{D}$, we have $\|M_\phi^*\| = \|M_\phi\| = \|\phi\|_\infty \leq 1$, and thus M_ϕ^* cannot be hypercyclic. If $\phi(\mathbb{D}^N) \subset \mathbb{C} \setminus \bar{\mathbb{D}}$, the same holds for $M_{\frac{1}{\phi}}^*$, and hence its inverse M_ϕ^* is not hypercyclic.

(3) $\Rightarrow$ (2). It follows from Lemma 5.1 that M_ϕ^* is frequently hypercyclic. Moreover, from the same lemma, we have that the span of the eigenvectors with rational eigenvalues are dense in $H_p(\mathbb{D}^N)^*$, and the set of the density of periodic vectors follows, see e.g. [27, Proposition 2.33]. ☐

We now proceed with the zero-one law on $H_p(\mathbb{D}^N)^*$. The proof is analogous to the one dimensional result [10, Theorem 4.1], we sketch it for the sake of completeness.

Theorem 5.3 (Zero-one law)**.** *Let $1 < p < \infty$ and let $\phi \in H_\infty(\mathbb{D}^N)$. If $M_\phi^* : H_p(\mathbb{D}^N)^* \to H_p(\mathbb{D}^N)^*$ has an orbit with a nonzero limit point, then M_ϕ^* is hypercyclic.*

Proof. If $\phi(\mathbb{D}^N) \cap \mathbb{T} \neq \emptyset$ then the operator is hypercyclic by the above theorem. So suppose first that $\phi(\mathbb{D}^N) \subseteq \mathbb{D}$. Then M_ϕ^* is supercyclic and $\|M_\phi^*\| \leq 1$. From a result by Ansari and Bourdon [3] every orbit tends to zero, which is a contradiction. If $\phi(\mathbb{D}^N) \subseteq \bar{\mathbb{D}}^c$ then $\psi = \frac{1}{\phi}$ and $\|M_\psi^*\| \leq 1$ because $\psi(\mathbb{D}) \subseteq \mathbb{D}$. Applying again [3], every orbit under M_ψ^* must tend to zero. Hence if $(M_\phi^*)^{n_k} x \to y \neq 0$, we have $\|y - (M_\phi^*)^{n_k} x\| \geq \|(M_\psi^*)^{n_k} y - x\| \to 0$, which is a contradiction. □

From the proof of the above result, it is natural to ask how wild the orbits of M_ϕ^* can be in the case $\phi(\mathbb{D}^N) \subseteq \overline{\mathbb{D}}^c$. The zero-one law shows that they cannot have limit points but, is it for example possible that they oscillate? We will prove that when the symbol ϕ can be extended to a neighborhood of $\overline{\mathbb{D}^N}$ then every orbit must tend to infinity. It is worth to mention that this does not imply that the orbits have an easy behavior, indeed, by a deep result due to Shkarin [40, Theorem 1.6], there are orbits which, although they tend to infinite, are dense in the weak topology.

We will need the following lemmas, the first one is taken from [10, 40].

Lemma 5.4. *Let $U \subset \mathbb{C}$ be an open set containing $\mathbb{D}$, and let $\phi : U \to \overline{\mathbb{D}}^c$ be analytic and non-constant. Then $\phi^{-1}(\mathbb{T}) \cap \mathbb{T}$ has zero measure in $\mathbb{T}$.*

Proof. We will prove that $\phi(\mathbb{T}) \cap \mathbb{T}$ is finite. Suppose otherwise, then $g(t) = |\phi(e^{it})|^2$ is real analytic and $g = 1$ in a set with an accumulation point. Thus, $g = 1$ in $\mathbb{T}$. This implies that $\phi(\mathbb{T}) \subseteq \mathbb{T}$. By the maximum modulus principle, it follows that $\phi(\overline{\mathbb{D}}) \subseteq \overline{\mathbb{D}}$ which is a contradiction. □

The second lemma is a well known result in harmonic analysis. The case $N = 1$ follows from the classical F. and M. Riesz Theorem, see e.g. [30, Corollary p.52]. For the $N > 1$ case, see [43, Theorem 4.24].

Lemma 5.5. *Let $\phi \in H_1(\mathbb{T}^N)$, $N \in \mathbb{N}$. If ϕ vanishes on a set $A \subset \mathbb{T}^N$ of positive measure, then ϕ is identically zero.*

We are ready to prove the trichotomy result for sufficiently smooth symbols.

Theorem 5.6. *Let $\phi \in H_\infty(\mathbb{D}^N)$ be non-constant, and suppose further that ϕ is analytic in a neighborhood of $\mathbb{D}^N$. The following trichotomy holds:*

either – M_ϕ^* *is hypercyclic,*
or – *for any $\varphi \in H_p(\mathbb{D}^N)^*$, $M_\phi^{*n} \varphi \to 0$,*
or – *for any non-zero $\varphi \in H^p(\mathbb{D}^N)^*$, $\|M_\phi^{*n} \varphi\| \to \infty$.*

Proof. Either $\phi(\mathbb{D}^N) \subseteq \mathbb{D}$, $\phi(\mathbb{D}^N) \cap \mathbb{T} \neq \emptyset$ or $\phi(\mathbb{D}^N) \subseteq \overline{\mathbb{D}}^c$. In the first case, proceeding as in the zero-one law we conclude that $M_\phi^{*n}(\varphi) \to 0$ for every $\varphi \in H_p(\mathbb{D}^N)^*$. If $\phi(\mathbb{D}^N) \cap \mathbb{T} \neq 0$, the operator is hypercyclic by Theorem 2.1.

It therefore suffices to address the case $\phi(\mathbb{D}^N) \subseteq \overline{\mathbb{D}}^c$. We claim that $\phi^{-1}(\mathbb{T}) \cap \mathbb{T}^N$ has zero measure in $\mathbb{T}^N$. To this aim, we choose $j \in \{1, \ldots, N\}$ such that ϕ depends on the j-th variable, and for each $\hat{w}_j := (w_1, \dots, w_{j-1}, w_{j+1}, \dots, w_N) \in \mathbb{T}^{N-1}$ we consider $\phi_{\hat{w}_j}(z) = \phi(w_1, \dots, w_{j-1}, z, w_{j+1}, \dots, w_N)$. Clearly $\phi_{\hat{w}_j}$ is analytic on a neighborhood of $\overline{\mathbb{D}}$, and $\phi_{\hat{w}_j}(\mathbb{D}) \subseteq \overline{\mathbb{D}}^c$.

If $A = \phi^{-1}(\mathbb{T}) \cap \mathbb{T}^N$, $A_1 = \{w \in A : \phi_{\hat{w}_j} \text{ is not constant}\}$ and $A_2 = \{w \in A : \phi_{\hat{w}_j} \text{ is constant}\}$, it follows that $A = A_1 \cup A_2$ and it suffices to show that A_1 and A_2 have zero measure. By Fubini's Theorem we know that

$$\begin{aligned} m(A_1) &= \int_{\mathbb{T}^N} \mathbb{1}_{A_1}(w) dw = \int_{\mathbb{T}^{N-1}} \int_{\mathbb{T}} \mathbb{1}_{A_{1,\hat{w}_j}}(w_j) dw_j d\hat{w}_j = \int_{\mathbb{T}^{N-1}} m(A_{1,\hat{w}_j}) d\hat{w}_j, \\ m(A_2) &= \int_{\mathbb{T}^N} \mathbb{1}_{A_2}(w) dw = \int_{\mathbb{T}} \int_{\mathbb{T}^{N-1}} \mathbb{1}_{A_{2,w_j}}(\hat{w}_j) d\hat{w}_j dw_j = \int_{\mathbb{T}} m(A_{2,w_j}) dw_j, \end{aligned}$$

where

$$\begin{aligned} A_{1,\hat{w}_j} &= \{\lambda \in \mathbb{T} : (w_1, \dots, w_{j-1}, \lambda, w_{j+1}, \dots, w_N) \in A_1\} \\ &= \{\lambda \in \mathbb{T} : (w_1, \dots, w_{j-1}, \lambda, w_{j+1}, \dots, w_N) \in A \text{ and } \phi_{\hat{w}_j} \text{ is not constant}\} \\ &= \{\lambda \in \phi_{\hat{w}_j}^{-1}(\mathbb{T}) \cap \mathbb{T} \text{ for } \hat{w}_j \text{ such that } \phi_{\hat{w}_j} \text{ is not constant}\}, \end{aligned}$$

$$A_{2,w_j} = \{\hat{w}_j \in \mathbb{T}^{N-1} : (w_1, \dots, w_{j-1}, w_j, w_{j+1}, \dots, w_N) \in A_2\} = \{\hat{w}_j \in \mathbb{T}^{N-1} : \phi_{\hat{w}_j} \text{ is constant}\}.$$

By Lemma 5.4, $A_{1,\hat{w}_j}$ has zero measure in $\mathbb{T}$. We show next that A_{2,w_j} has zero measure in $\mathbb{T}^{N-1}$. Indeed, if $\phi_{\hat{w}_j}$ is constant then $\phi'_{\hat{w}_j}$ vanishes on a neighborhood of $\overline{\mathbb{D}}$. Thus, if A_{2,w_j} has positive measure then $\frac{\partial \phi}{\partial z_j}$ vanishes in a set of positive measure composed by the product of A_{2,w_j} and a neighborhood of $\overline{\mathbb{D}}$. In particular, $\frac{\partial \phi}{\partial z_j}$ vanishes in a set of positive measure of $\mathbb{T}^N$, and since $\frac{\partial \phi}{\partial z_j}$ is analytic on a neighborhood of $\overline{\mathbb{D}^N}$ we can apply Lemma 5.5 to conclude that $\frac{\partial \phi}{\partial z_j}$ is zero everywhere. This contradicts the fact that ϕ depends on z_j, and therefore A_{2,w_j} has zero measure.

Now let $g \in H_p(\mathbb{T}^N)$ and $\psi = \frac{1}{\phi} \in H_\infty(\mathbb{D}^N)$. Thus, $P_N^{-1}(\psi)^n g \to 0$ almost everywhere in $\mathbb{T}^N$. By the dominated convergence theorem, it follows that $P_N^{-1}(\psi)^n g \to 0$ in $H_p(\mathbb{T}^N)$. We conclude that $\psi^n g \to 0$ for every $g \in H_p(\mathbb{D}^N)$.

Finally, if $\varphi \in H_p(\mathbb{D}^N)^*$, let $g \in H_p(\mathbb{D}^N)$ such that $\langle \varphi, g \rangle \neq 0$. Then $\langle M_\phi^{*n} \varphi, M_\psi^n g \rangle = \langle \varphi, g \rangle$. Since $M_\psi^n g \to 0$, we obtain that $M_\phi^{*n} \varphi \to \infty$. $\square$

5.2. **Linear dynamics of adjoint of multiplication operators on $H_p(\mathbb{D}_2^\infty)^*$.** We now analyze the dynamics on $H_p(\mathbb{D}_2^\infty)^*$, extending the previous results to the infinite dimensional setting. An important ingredient for the proof of Theorem 5.9 is the following lemma, which will help us to derive the dynamical properties of $M_\phi^* : H_p(\mathbb{D}_2^\infty)^* \to H_p(\mathbb{D}_2^\infty)^*$, from the dynamics of $M_{\phi_N} : H_p(\mathbb{D}^N)^* \to H_p(\mathbb{D}^N)^*$.

Lemma 5.7. *Let $N \in \mathbb{N}$ and $\phi \in H_\infty(B_{c_0})$. Then $\pi_N^* M_{\phi_N}^* = M_\phi^* \pi_N^*$, that is, the following diagram is commutative:*

$$\begin{tikzcd} H_p(\mathbb{D}^N)^* \arrow[r, "M_{\phi_N}^*"] \arrow[d, "\pi_N^*"] & H_p(\mathbb{D}^N)^* \arrow[d, "\pi_N^*"] \\ H_p(\mathbb{D}_2^\infty)^* \arrow[r, "M_\phi^*"] & H_p(\mathbb{D}_2^\infty)^* \end{tikzcd}$$

Proof. Let $\beta \in \mathbb{N}_0^{(\mathbb{N})}$. It suffices to show that $\pi_N(\phi z^\beta) = \phi_N \pi_N(z^\beta)$, since

$$\langle \pi_N^*(M_{\phi_N}^*(h)), z^\beta \rangle = \langle h, \phi_N \pi_N(z^\beta) \rangle \quad \text{and} \quad \langle M_\phi^*(\pi_N^*(h)), z^\beta \rangle = \langle h, \pi_N(\phi z^\beta) \rangle,$$

for each $h \in H_p(\mathbb{D}^n)$.

If $\phi = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha$ we have

$$\phi z^\beta = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c'_\alpha z^\alpha,$$

where $c'_{\gamma+\beta} = c_\gamma$ for $\gamma \in \mathbb{N}_0^{(\mathbb{N})}$ (and $c'_\alpha = 0$ if $\alpha \neq \gamma + \beta$ for some γ). Note in addition that

$$\pi_N(\phi z^\beta) = \sum_{\alpha \in \mathbb{N}_0^N} c'_\alpha z^\alpha.$$

Now, $\pi_N(z^\beta) = 0$ (and thus $\phi_N \pi_N(z^\beta) = 0$) if $\beta \notin \mathbb{N}_0^N$, and in this case $\pi_N(\phi z^\beta) = 0$ as well, since $c'_\alpha = 0$ for each $\alpha \in \mathbb{N}_0^N$. This proves the claim for $\beta \notin \mathbb{N}_0^N$.

If $\beta \in \mathbb{N}_0^N$ we have

$$\phi_N \pi_N(z^\beta) = \phi_N z^\beta = \sum_{\alpha \in \mathbb{N}_0^N} c'_\alpha z^\alpha,$$

which proves the claim.

□

Our first aim is to prove frequent hypercyclicity of adjoint multiplication operator. For the proof we will use the fact that the operators $\phi_N : H_p(\mathbb{D}^N)^* \to H_p(\mathbb{D}^N)^*$ support compatible perfectly spanning sets of unimodular eigenvectors.

Theorem 5.8. *Let $\phi \in H_\infty(B_{c_0})$. If $M_\phi^* : H_p(\mathbb{D}_2^\infty)^* \to H_p(\mathbb{D}_2^\infty)^*$ is hypercyclic then it supports a perfect spanning set of unimodular eigenvectors with respect to the normalized Lebesgue measure of $\mathbb{T}$. In particular M_ϕ^* is frequently hypercyclic.*

Proof. Since M_ϕ^* is hypercyclic there is $N_0 \in \mathbb{N}$ such that $\phi_N(\mathbb{D}^N) \cap \mathbb{T} \neq \emptyset$ for every $N \geq N_0$. By Lemma 5.1, we have that for each $N \geq N_0$ each $M_{\phi_N}^*$ supports a perfectly spanning set with respect to the normalized Lebesgue measure.

Let $A \subset \mathbb{T}$ be a set of zero measure. Let $U \subseteq H_p(\mathbb{D}_2^\infty)^*$ be a nonempty open set and consider $\varphi \in U$ to be a co-analytic polynomial, depending on $N \geq N_0$ variables, and let $\varepsilon > 0$ such that the ball $B(\varphi, \varepsilon) \subseteq U$. Since $M_{\phi_N}^*$ supports a perfectly spanning set of eigenvectors, there are eigenvectors $\eta_1, \ldots \eta_l$ with eigenvalues in $\mathbb{T} \setminus A$ and coefficients $a_1, \ldots a_l$ such that $\| i_N^* \varphi - \sum_{j=1}^l a_j \eta_j \|_{H_p(\mathbb{D}^N)^*} < \varepsilon$. Since the inclusion $\pi_N^* : H_p(\mathbb{D}^N)^* \to H_p(\mathbb{D}_2^\infty)^*$ is isometric, we conclude $\| \varphi - \pi_N^* \left(\sum_{j=1}^l a_j \eta_j \right) \|_{H_p(\mathbb{D}_2^\infty)^*} < \varepsilon$.

Finally observe that if η_j is an eigenvector of $M_{\phi_N}^*$, then $\pi_N^*(\eta_j)$ is an eigenvector of M_ϕ^* with the same eigenvalue, because by Lemma 5.7, $\pi_N^* M_{\phi_N}^* = M_\phi^* \pi_N^*$. □

Applying the previous lemma together with the Hilbert criterion proved in Section 3 we will obtain the following extension of Godefroy-Shapiro's theorem.

Theorem 5.9. *Let $\phi \in H_\infty(B_{c_0})$ be a non-constant function. Then, the following are equivalent:*

(1) $M_\phi^* : H_p(\mathbb{D}_2^\infty)^* \to H_p(\mathbb{D}_2^\infty)^*$ is hypercyclic;
(2) $M_\phi^* : H_p(\mathbb{D}_2^\infty)^* \to H_p(\mathbb{D}_2^\infty)^*$ is chaotic and frequently hypercyclic;
(3) $\phi(B_{c_0}) \cap \mathbb{T} \neq \emptyset$;
(4) There exists $N \in \mathbb{N}$ such that $M_{\phi_N}^* : H_p(\mathbb{D}^N)^* \to H_p(\mathbb{D}^N)^*$ is hypercyclic and
(5) There exists $N_0 \in \mathbb{N}$ such that for each $N \geq N_0$, $M_{\phi_N}^* : H_p(\mathbb{D}^N)^* \to H_p(\mathbb{D}^N)^*$ is hypercyclic.

Proof. (1) $\Rightarrow$ (3) follows as in the proof of Theorem 5.2.

(3) $\Rightarrow$ (4). The function ϕ is holomorphic and non-constant, and thus an open mapping. The hypothesis $\phi(B_{c_0}) \cap \mathbb{T} \neq \emptyset$ then implies that the image of ϕ intersects both $\mathbb{D}$ and $\mathbb{C} \setminus \bar{\mathbb{D}}$. Since eventually-zero sequences are dense in B_{c_0}, we can consider $b = (b_1, \dots, b_N, 0, \dots), c = (c_1, \dots, c_N, 0, \dots) \in B_{c_0}$ such that $|\phi(b)| = |\phi_N(b_1, \dots, b_N)| < 1$ and $|\phi(c)| = |\phi_N(c_1, \dots, c_N)| > 1$ for some $N \in \mathbb{N}$. This implies that $\phi_N(\mathbb{D}^N) \cap \mathbb{T} \neq \emptyset$, and we conclude by Theorem 5.2 that $M_{\phi_N}^*$ is hypercyclic.

(4) $\Rightarrow$ (5). Given $N \in \mathbb{N}$ such that $M_{\phi_N}^*$ is hypercyclic (and thus $\phi_N(\mathbb{D}^N) \cap \mathbb{T} \neq \emptyset$), it is clear that $\phi_M(\mathbb{D}^M) \cap \mathbb{T} \neq \emptyset$ for every $M \geq N$. The result then follows from Theorem 5.2.

(5) $\Rightarrow$ (1). Let $U, V \subset H_p(\mathbb{D}_2^\infty)^*$ be nonempty open sets, and let $N_0 \in \mathbb{N}$ such that $M_{\phi_N}^*$ is hypercyclic for every $N \geq N_0$. For any such N consider the inclusion $\pi_N^* : H_p(\mathbb{D}^N)^* \to H_p(\mathbb{D}_2^\infty)^*$ and note that the sets ${\pi_N^*}^{-1}(U), {\pi_N^*}^{-1}(V) \subset H_p(\mathbb{D}^N)^*$ are open. Since the polynomials are dense in $H_p(\mathbb{D}_2^\infty)^*$, there is $N_1 \geq N_0$ such that for every $N \geq N_1$ both ${\pi_N^*}^{-1}(U)$ and ${\pi_N^*}^{-1}(V)$ are nonempty.

Let $N \geq N_1$. Hence, $M_{\phi_N}^*$ is topologically transitive, and thus there exist $f \in {\pi_N^*}^{-1}(U)$ and $k \in \mathbb{N}$ such that ${M_{\phi_N}^*}^k(f) \in {\pi_N^*}^{-1}(V)$. Clearly $\pi_N^*(f) \in U$, and Lemma 5.7 yields

$$ {M_\phi^*}^k(\pi_N^*(f)) = \pi_N^*({M_{\phi_N}^*}^k(f)) \in V. $$

This means that M_ϕ^* is topologically transitive, and thus hypercyclic.

It remains to prove that under any of these conditions we have (2): M_ϕ^* is chaotic and frequently hypercyclic. The latter is Theorem 5.8.

To prove the density of periodic vectors assuming (5), fix a nonempty open set U and let $N \geq N_0$ such that ${\pi_N^*}^{-1}(U) \neq \emptyset$. We then have, by Theorem 5.2 that there exists a periodic point x of $M_{\phi_N^*}$ in ${\pi_N^*}^{-1}(U)$. It follows that $\pi_N^*(x) \in U$ is a periodic vector of M_ϕ^* and hence M_ϕ^* is chaotic.

$\square$

In [10], a zero-one law for adjoint of multipliers on $H_p(\mathbb{D})$ has been proven: if an orbit has a limit point different from zero then the operator must be hypercyclic. Moreover, if ϕ is analytic in a neighborhood of $\mathbb{D}$ then a stronger trichotomy holds: either every orbit tends to 0, every orbit tends to ∞ or M_ϕ^* is hypercyclic. We extend these results to infinitely many variables.

We now present the infinite dimensional version of the zero-one law. The proof is analogous to that of Theorem 5.3, so we omit it.

Theorem 5.10 (Zero-one law)**.** *Let $1 < p < \infty$ and let $\phi \in H_\infty(B_{c_0})$ (or $\phi \in H_\infty(\mathbb{D}^N)$). If $M_\phi^* : H_p(\mathbb{D}_2^\infty)^* \to H_p(\mathbb{D}_2^\infty)^*$ (or $M_\phi^* : H_p(\mathbb{D}^N)^* \to H_p(\mathbb{D}^N)^*$) has an orbit with a nonzero limit point, then M_ϕ^* is hypercyclic.*

Let us turn to the infinite dimensional version of the trichotomy result. While the proof follows the general strategy of the N-dimensional case, there are key differences and additional challenges that arise. In particular, proving that $P_\infty^{-1}(\phi)^{-1}(\mathbb{T}) \cap \mathbb{T}^\infty$ has zero measure in $\mathbb{T}^\infty$ is no longer simple.

The infinite dimensional case of Lemma 5.5 follows from the extension to $\mathbb{T}^\infty$ of the brothers Riesz Theorem (see [6] and [2, Corollary 2]).

Lemma 5.11. *Let $\phi \in H_1(\mathbb{T}^\infty)$. If ϕ vanishes in a set $A \subset \mathbb{T}^\infty$ of positive measure, then ϕ is identically zero.*

Theorem 5.12 (Trichotomy)**.** *Let $r > 1$ and let $\phi \in H_\infty(rB_{c_0})$ be non-constant. The following trichotomy holds:*

either – M_ϕ^* is hypercyclic,
or – for any $\varphi \in H^p(\mathbb{D}_2^\infty)^*$, $M_\phi^{*n}\varphi \to 0$,
or – for any non-zero $\varphi \in H^p(\mathbb{D}_2^\infty)^*$, $\|M_\phi^{*n}\varphi\| \to \infty$.

Proof. Either $\phi(B_{c_0}) \subseteq \mathbb{D}$, $\phi(B_{c_0}) \cap \mathbb{T} \neq 0$ or $\phi(B_{c_0}) \subseteq \overline{\mathbb{D}}^c$. In the first case we have by the proof of Theorem 5.10 that $M_\phi^{*n}\varphi \to 0$ for every $\varphi \in (H_p(\mathbb{D}_2^\infty))^*$ and if $\phi(B_{c_0}) \cap \mathbb{T} \neq 0$, then the operator is hypercyclic by Theorem 5.9.

For the case $\phi(B_{c_0}) \subseteq \overline{\mathbb{D}}^c$ we will show that $P_\infty^{-1}(\phi)^{-1}(\mathbb{T}) \cap \mathbb{T}^\infty$ has zero measure in $\mathbb{T}^\infty$. Choose $j \in \mathbb{N}$ such that ϕ depends on the j-th variable. For almost every $\hat{w}_j := (w_1, \ldots, w_{j-1}, w_{j+1}, \ldots) \in \mathbb{T}^\infty$, the function $P_\infty^{-1}(\phi)(w_1, \ldots, w_{j-1}, \cdot, w_{j+1}, \ldots)$ is in $H_\infty(\mathbb{T})$. Hence, for those $\hat{w}_j$ we may consider $t \mapsto \phi_{\hat{w}_j}(t) = P_1[P_\infty^{-1}(\phi)(w_1, \ldots, w_{j-1}, \cdot, w_{j+1}, \ldots)](t)$. We claim that $\phi_{\hat{w}_j} \in H_\infty(r\mathbb{D})$ for almost all $\hat{w}_j \in \mathbb{T}^\infty$. To see this, consider also the function $\tilde{\phi}(z) = \phi(z_1, \ldots, z_{j-1}, rz_j, z_{j+1}, \ldots)$. Since $\tilde{\phi} \in H_\infty(B_{c_0})$ we have $\tilde{\phi}_{\hat{w}_j} = P_1[P_\infty^{-1}(\tilde{\phi})(w_1, \ldots, w_{j-1}, \cdot, w_{j+1}, \ldots)] \in H_\infty(\mathbb{D})$ for almost all $\hat{w}_j \in \mathbb{T}^\infty$. Since the Poisson operators (P_1 and P_∞) preserve coefficients, we have for $t \in \mathbb{D}$,

$$\tilde{\phi}_{\hat{w}_j}(t) = \phi_{\hat{w}_j}(rt).$$

Since the left hand side function is bounded on $\mathbb{D}$, the claim that $\phi_{\hat{w}_j} \in H_\infty(r\mathbb{D})$ follows.

We claim now that $\phi_{\hat{w}_j}(\mathbb{D}) \subseteq \overline{\mathbb{D}}^c$ for almost all $\hat{w}_j \in \mathbb{T}^\infty$. Indeed, note that since the Poisson operators P_1 and P_∞ are isometric isomorphisms, the mapping $H_\infty(B_{c_0}) \ni \eta \mapsto \eta_{\hat{w}_j} \in H_\infty(\mathbb{D})$ is a contraction. Thus if $\psi = \phi^{-1}$, then $\psi \in H_\infty(B_{c_0})$ and $\|\psi\| \leq 1$. Hence $|\psi_{\hat{w}_j}(z)| \leq 1$. Moreover, since the Poisson operators are multiplicative, $\psi_{\hat{w}_j}(z) = \phi_{\hat{w}_j}(z)^{-1}$. We conclude that $\phi_{\hat{w}_j}(\mathbb{D}) \subseteq \mathbb{D}^c$, but since $\phi_{\hat{w}_j}(\mathbb{D})$ must be open, the claim follows.

If $A = P_\infty^{-1}(\phi)^{-1}(\mathbb{T}) \cap \mathbb{T}^\infty$, $A_1 = \{w \in A : \phi_{\hat{w}_j} \text{ is not constant}\}$ and $A_2 = \{w \in A : \phi_{\hat{w}_j} \text{ is constant}\}$, it follows that $m(A \setminus (A_1 \cup A_2)) = 0$ and it suffices to show that A_1 and A_2 have zero measure. As in the N-dimensional case, it follows from Fubini's Theorem that

$$m(A_1) = \int_{\mathbb{T}^\infty} m(A_{1,\hat{w}_j}) d\hat{w}_j, \qquad \text{and} \qquad m(A_2) = \int_{\mathbb{T}} m(A_{2,w_j}) dw_j,$$

where $A_{1,\hat{w}_j} = \{\lambda \in \phi_{\hat{w}_j}^{-1}(\mathbb{T}) \cap \mathbb{T}$ for $\hat{w}_j$ such that $\phi_{\hat{w}_j}$ is not constant$\}$, and $A_{2,w_j} = \{\hat{w}_j \in \mathbb{T}^\infty : \phi_{\hat{w}_j}$ is constant$\}$. By Lemma 5.4, $A_{1,\hat{w}_j}$ has zero measure in $\mathbb{T}$ and hence so does A_1. For A_2, note that if $\phi_{\hat{w}_j}$ is constant then $\phi'_{\hat{w}_j}$ is the zero function on $r\mathbb{D}$. Thus, if A_{2,w_j} had positive measure then $P_\infty^{-1}\frac{\partial\phi}{\partial z_j}$ would vanish on $A_{2,w_j} \times \mathbb{T}$, which has positive measure in $\mathbb{T}^\infty$.

Moreover, since $\phi \in H_\infty(rB_{c_0})$, we have that $\frac{\partial\phi}{\partial z_j} \in H_\infty(B_{c_0})$. Indeed, [34, Lemma 2.11] together with [33, Lemma 4.5] (using there $X = rB_{c_0}$ and $\mathfrak{A} = \mathcal{P}$ the current holomorphy type, wich means that in this case $H_{\mathfrak{A}b}(X)$ is just the classical space $H_b(rB_{c_0})$ of holomorphic functions of bounded type on rB_{c_0}) we have that $P_\infty^{-1}\frac{\partial\phi}{\partial z_j} \in H_b(rB_{c_0}) \subset H_\infty(B_{c_0})$.

Thus we can apply Lemma 5.11 to $P_\infty^{-1}\frac{\partial\phi}{\partial z_j}$ to conclude that $\frac{\partial\phi}{\partial z_j}$ vanishes everywhere on B_{c_0}. This contradicts the fact that ϕ depends on z_j, and therefore A_{2,w_j} has zero measure.

Now let $g \in H_p(\mathbb{D}_2^\infty)$ and $\psi = \frac{1}{\phi} \in H_\infty(B_{c_0})$. Thus, $P_\infty^{-1}(\psi^n g) \to 0$ almost everywhere in $\mathbb{T}^\infty$. By the dominated convergence theorem, it follows that $P_\infty^{-1}(\psi^n g) \to 0$ in $H_p(\mathbb{T}^\infty)$. We conclude that $\psi^n g \to 0$ in $H_p(\mathbb{D}_2^\infty)$, for every $g \in H_p(\mathbb{D}_2^\infty)$.

Finally, if $\varphi \in H_p(\mathbb{D}_2^\infty)^*$, let $g \in H_p(\mathbb{D}_2^\infty)$ such that $\langle\varphi, g\rangle \neq 0$. Then since $\langle M_\phi^{*n}\varphi, M_\psi^n g\rangle = \langle\varphi, g\rangle$ and $M_\psi^n g \to 0$, we obtain that $M_\phi^{*n}\varphi \to \infty$. $\square$

5.3. **Linear dynamics of adjoint of multiplication operators in $\mathcal{H}_p^*$.** Our aim is now to derive, applying the Bohr transform, similar results for adjoint of multipliers on Hardy spaces of Dirichlet series.

We note given a Dirichlet series D we have, via the extended Bohr transform, that $\mathfrak{B}^{-1}M_D^* = M_{\mathfrak{B}^{-1}D}^*\mathfrak{B}^{-1}$.

$$
\begin{tikzcd}
\mathcal{H}_p^* \arrow[r, "M_D^*"] \arrow[d, leftrightarrow, "\mathfrak{B}^{-1}"] & \mathcal{H}_p^* \arrow[d, leftrightarrow, "\mathfrak{B}^{-1}"] \\
H_p(\mathbb{D}_2^\infty)^* \arrow[r, "M_{\mathfrak{B}^{-1}D}^*"] & H_p(\mathbb{D}_2^\infty)^*
\end{tikzcd}
\tag{13}
$$

Recall that, given $D \in \mathcal{H}_\infty$ and $N \in \mathbb{N}$, we refer by D_N to the restriction of D to the first N primes. That is, if $D(s) = \sum_{n=1}^\infty a_n n^{-s}$ the restriction is as follows:

$$D_N(s) = \sum_{\substack{n=\mathfrak{p}^\alpha \\ \alpha\in\mathbb{N}_0^N}}^\infty a_n n^{-s}.$$

The restrictions D_N are elements of $\mathcal{H}_\infty$ themselves (and thus multipliers of $\mathcal{H}_p$) and $\mathfrak{B}^{-1}D_N = (\mathfrak{B}^{-1}D)_N$.

Theorem 5.13. *Let $D \in \mathcal{H}_\infty$ be non-constant. The following are equivalent:*

(1) *$M_D^* : \mathcal{H}_p^* \to \mathcal{H}_p^*$ is hypercyclic;*
(2) *$M_D^* : \mathcal{H}_p^* \to \mathcal{H}_p^*$ is chaotic and frequently hypercyclic;*
(3) *$D(\mathbb{C}_+) \cap \mathbb{T} \neq \emptyset$;*
(4) *There exists $N \in \mathbb{N}$ such that $D_N(\mathbb{C}_+) \cap \mathbb{T} \neq \emptyset$ and*

(5) There exists $N_0 \in \mathbb{N}$ such that for each $N \geq N_0$, $D_N(\mathbb{C}_+) \cap \mathbb{T} \neq \emptyset$.

Proof. The conjugation (13) together with Theorem 5.9 shows (1) $\Rightarrow$ (2), while (5) $\Rightarrow$ (4) and (2) $\Rightarrow$ (1) are immediate.

(1) $\Rightarrow$ (3). By the conjugation (13) we have that $M^*_{\mathfrak{B}^{-1}D}$ is hypercyclic, and thus $\|M^*_{\mathfrak{B}^{-1}D}\| = \|D\|_\infty > 1$. Assuming that $D(\mathbb{C}_+) \cap \mathbb{T} = \emptyset$, it necessarily follows that $D(\mathbb{C}_+) \subset \mathbb{C} \setminus \overline{\mathbb{D}}$. This implies that $\frac{1}{D} \in \mathcal{H}_\infty$, see e.g. [36, Theorem 6.2.1]. The contradiction arises from the fact that $\|M^*_{\frac{1}{D}}\| = \|\frac{1}{D}\|_\infty \leq 1$, but $M^*_{\frac{1}{D}} = (M^*_D)^{-1}$ must be hypercyclic.

(1) $\Rightarrow$ (5). Applying again (13) and Theorem 5.9, there exists $N_0 \in \mathbb{N}$ such that the operator $M^*_{\mathfrak{B}^{-1}D_N}$ is hypercyclic for each $N \geq N_0$. Thus

$$1 < \|M^*_{\mathfrak{B}^{-1}D_N}\| = \|\mathfrak{B}^{-1}D_N\|_\infty = \|D_N\|_\infty.$$

From an argument similar to the one used in implication (1) $\Rightarrow$ (3) we conclude that $D_N(\mathbb{C}_+) \cap \mathbb{T} \neq \emptyset$.

(3) $\Rightarrow$ (4). Consider the sequence $(D_N)_{N\in\mathbb{N}}$. This sequence is bounded in $\mathcal{H}_\infty$ because

$$\|D_N\|_{\mathcal{H}_\infty} = \|\mathfrak{B}^{-1}D_N\|_{H_\infty(B_{c_0})} \leq \|\mathfrak{B}^{-1}D\|_{H_\infty(B_{c_0})} = \|D\|_{\mathcal{H}_\infty}$$

for every $N \in \mathbb{N}$. From Bayart's Montel-type theorem [5], each subsequence has a sub-subsequence $(D_{N_k})_{k\in\mathbb{N}}$ which converges in the Fréchet space $\mathcal{H}^+_\infty$ to a Dirichlet series $E \in \mathcal{H}_\infty$ (see [9, Section 2]), that is, it converges uniformly to E on $\mathbb{C}_\varepsilon$ on each $\varepsilon > 0$. Since $(n^{-s})_n$ is a basis of $\mathcal{H}^+_\infty$ (see again [9]), it follows that $E = D$. This implies that $D_N \to D$ in $\mathcal{H}^+_\infty$.

On the other hand, since D is holomorphic and non-constant it is an open mapping. This, along with the hypothesis $D(\mathbb{C}_+) \cap \mathbb{T} \neq \emptyset$, means that there are nonempty open subsets $U, V \subset \mathbb{C}_+$ such that $D(U) \subset \mathbb{D}$ and $D(V) \subset \mathbb{C} \setminus \overline{\mathbb{D}}$. Since $(D_N)_{N\in\mathbb{N}}$ converges pointwise in $\mathbb{C}_+$, which implies that there exists $N_0 \in \mathbb{N}$ such that $D_N(\mathbb{C}_+) \cap \mathbb{T} \neq \emptyset$ for each $N \geq N_0$.

(4) $\Rightarrow$ (1). It suffices to show that $\mathfrak{B}^{-1}D_N(\mathbb{D}^N) \cap \mathbb{T} \neq \emptyset$, since in that case it follows from Theorems 5.2 and 5.9 that $M^*_{\mathfrak{B}^{-1}D}$ is hypercyclic, and hence our claim follows from the conjugation (13).

So assuming $D_N(\mathbb{C}_+) \cap \mathbb{T} \neq \emptyset$, let $s_0 \in \mathbb{C}_+$ be such that $|D_N(s_0)| = 1$. Note that $z_0 = (\mathfrak{p}_1^{-s_0}, \ldots, \mathfrak{p}_N^{-s_0}) \in \mathbb{D}^N$. If d_n denote the coefficients of D, we have

$$\mathfrak{B}^{-1}D_N(z_0) = \sum_{\alpha\in\mathbb{N}_0^N} d_\alpha z_0^\alpha = \sum_{\alpha\in\mathbb{N}_0^N} d_\alpha (\mathfrak{p}^\alpha)^{-s_0} = \sum_{\substack{n=\mathfrak{p}^\alpha \\ \alpha\in\mathbb{N}_0^N}} d_n n^{-s_0} = D_N(s_0),$$

from which we conclude that $\mathfrak{B}^{-1}D_N(\mathbb{D}^N) \cap \mathbb{T} \neq \emptyset$. □

The zero-one law for adjoint multiplication operators on $\mathcal{H}^*_p$ is an immediate consequence of (13) together with Theorem 5.10.

Theorem 5.14 (Zero-one law)**.** *Let $D \in \mathcal{H}_\infty$ and $p > 1$. If $M^*_D : \mathcal{H}^*_p \to \mathcal{H}^*_p$ has an orbit with a nonzero limit point, then M^*_D is hypercyclic.*

We finally derive a trichotomy result on $\mathcal{H}^*_p$ for adjoint of multipliers with negative abscissa of uniform convergence.

Theorem 5.15 (Trichotomy)**.** *Let $D \in \mathcal{H}_\infty$ such that $\sigma_u(D) < 0$ and let $1 < p < \infty$. The following trichotomy holds:*

either – M_D^* is hypercyclic,
or – for any $E \in \mathcal{H}_p^*$, $M_D^{*n} E \to 0$,
or – for any non-zero $E \in \mathcal{H}_p^*$, $\|M_D^{*n} E\| \to \infty$.

Proof. By the trichotomy on the infinite polydisc, Theorem 5.12 and (13), it suffices to show that there is $t > 1$ such that $\phi := \mathfrak{B}^{-1} D \in H_\infty(tB_{c_0})$. Let $\varepsilon > 0$ such that $\sigma_u(D) < -\varepsilon < 0$. Then if $\tilde{D}(s) = D(s - \varepsilon) = \sum_{n=1}^\infty d_n n^{-s+\varepsilon} = \sum_{n=1}^\infty d_n n^\varepsilon n^{-s}$ we have $\tilde{D} \in \mathcal{H}_\infty$. In particular, if we consider the translation operator $\tau_\varepsilon : \mathcal{H}_\infty \to \mathcal{H}_\infty,\ D \mapsto D(\cdot + \varepsilon)$ we have that $\tau_\varepsilon(\tilde{D}) = D$.

Let $r = (r_k)_k = (\mathfrak{p}_k^{-\varepsilon})_k$ and $T_r : H_\infty(B_{c_0}) \to H_\infty(r^{-1}B_{c_0})$ given by $T_r(\phi)(z) = \phi(rz) = \phi((r_k z_k)_k)$. Of course here, $r^{-1}B_{c_0}$ denotes $\{z \in c_0 : (r_k z_k)_k \in B_{c_0}\}$. We will show that $\tau_\varepsilon = \mathfrak{B} T_r \mathfrak{B}^{-1}$.

To this aim we consider a Dirichlet series $E = \sum_{n=1}^\infty a_n n^{-s}$. Hence,

$$\mathfrak{B} T_r \mathfrak{B}^{-1} E = \mathfrak{B} T_r \left(\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} a_{\mathfrak{p}^\alpha} z^\alpha \right) = \mathfrak{B} \left(\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} a_{\mathfrak{p}^\alpha} r^\alpha z^\alpha \right) = \mathfrak{B} \left(\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} a_{\mathfrak{p}^\alpha} (\mathfrak{p}^\alpha)^{-\varepsilon} z^\alpha \right) = \sum_{n=1}^\infty a_n n^{-\varepsilon} n^{-s}.$$

Let $\tilde{\phi} = \mathfrak{B}^{-1} \tilde{D}$. Hence, $\tilde{\phi} \in H_\infty(B_{c_0})$ and $\mathfrak{B} T_r(\tilde{\phi}) = \mathfrak{B} T_r \mathfrak{B}^{-1} \tilde{D} = \tau_\varepsilon \tilde{D} = D$. Hence, $T_r(\tilde{\phi}) = \phi$.

In particular, $\phi \in H_\infty(r^{-1}B_{c_0})$. Since $\mathfrak{p}_k^\varepsilon$ is increasing, we have that $2^\varepsilon B_{c_0} \subseteq r^{-1} B_{c_0}$ and hence $\phi \in H_\infty(2^\varepsilon B_{c_0})$. ☐

5.4. **Dynamics of multiplicative Toeplitz operators on Hardy spaces of Dirichlet series.** We will now study the dynamics of co-analytic Toeplitz operators on $\mathcal{H}_p$. The situation here is much subtler than for adjoint multipliers since, since as seen in Section 4, we do not have an explicit characterization of their boundedness. Nevertheless, assuming that we have a *bounded* multitplicative co-analytic Topelitz operator we will be able to describe its dynamical properties, hence answering Question B. The key to obtain this answer is the following simple lemma, which relates the dynamics of $T_D : \mathcal{H}_p \to \mathcal{H}_p$ with the dynamics of $M_{\jmath D}^* : \mathcal{H}_{p'}^* \to \mathcal{H}_{p'}^*$.

Lemma 5.16. *Let $1 < p < \infty$, $p' = \frac{p}{p-1}$ and let $D = \sum_{n=1}^\infty a_n n^s \in \mathcal{T}_p$ be the symbol of a co-analytic multiplicative Toeplitz operator. Then,*

(1) $M_{\jmath D}^* : \mathcal{H}_{p'}^* \to \mathcal{H}_{p'}^*$ is quasiconjugated to $T_D : \mathcal{H}_p \to \mathcal{H}_p$ through the conjugate-linear inclusion $\mathcal{H}_p \hookrightarrow \mathcal{H}_{p'}^*$ (see Remark 3.3). That is, the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{H}_p & \xrightarrow{T_D} & \mathcal{H}_p \\ \big\downarrow{\scriptstyle \jmath} & & \big\downarrow{\scriptstyle \jmath} \\ \mathcal{H}_{p'}^* & \xrightarrow{M_{\jmath D}^*} & \mathcal{H}_{p'}^* \end{array}$$

(2) For each $N \in \mathbb{N}$ we have that $M^*_{(\jmath D)_N} : \mathcal{H}^*_{p',N} \to \mathcal{H}^*_{p',N}$ is conjugated to $T_{D_N} : \mathcal{H}_{p,N} \to \mathcal{H}_{p,N}$ through the conjugate-linear isomorphism $\mathcal{H}_{p,N} \hookrightarrow \mathcal{H}^*_{p',N}$;

$$\begin{array}{ccc} \mathcal{H}_{p,N} & \xrightarrow{T_{D_N}} & \mathcal{H}_{p,N} \\ \jmath \updownarrow & & \jmath \updownarrow \\ \mathcal{H}^*_{p',N} & \xrightarrow{M^*_{(\jmath D)_N}} & \mathcal{H}^*_{p',N} \end{array}$$

Proof. We note that, by Corollary 4.9, $D \in \jmath H_\infty$ and hence $M_{\jmath D}$ defines a bounded operator.

Proof of (1). To prove quasiconjugation, by linearity, the density of the Dirichlet polynomials in $\mathcal{H}_{p'}$ and the density of co-analytic Dirichlet polynomials in $\mathcal{H}^*_{p'}$, it suffices to consider the pairings evaluated on monomials. Thus, given $n_1, n_2 \in \mathbb{N}$ we have

$$\langle M^*_{\jmath D}(\jmath(n_1^{-s})), n_2^{-s}\rangle_{(\mathcal{H}^*_{p'},\mathcal{H}_{p'})} = \langle n_1^s, \jmath D \cdot n_2^{-s}\rangle_{(\mathcal{H}^*_{p'},\mathcal{H}_{p'})} = \langle n_1^s, \sum_{k=1}^{\infty} \overline{a_k}(kn_2)^{-s}\rangle_{(\mathcal{H}^*_{p'},\mathcal{H}_{p'})} = \begin{cases} \overline{a_{\frac{n_1}{n_2}}} & \text{if } n_2 | n_1, \\ 0 & \text{else .} \end{cases}$$

Furthermore, from definition we have

$$T_D(n_1^{-s}) = P_{\mathbb{N}}\left(\sum_{n=1}^{\infty} a_n \left(\frac{n_1}{n}\right)^{-s}\right)$$

which yields

$$\langle \jmath T_D(n_1^{-s}), n_2^{-s}\rangle_{(\mathcal{H}^*_{p'},\mathcal{H}_{p'})} = \left\langle \jmath P_{\mathbb{N}}\left(\sum_{n=1}^{\infty} a_n \left(\frac{n_1}{n}\right)^{-s}\right), n_2^{-s}\right\rangle_{(\mathcal{H}^*_{p'},\mathcal{H}_{p'})} = \left\langle \jmath \sum_{n|n_1} a_n \left(\frac{n_1}{n}\right)^{-s}, n_2^{-s}\right\rangle_{(\mathcal{H}^*_{p'},\mathcal{H}_{p'})}$$
$$= \begin{cases} \overline{a_{\frac{n_1}{n_2}}} & \text{if } n_2 | n_1, \\ 0 & \text{else.} \end{cases}$$

(2) follows analogously to (1) together with the fact that $\jmath$ is now an isomorphism. □

Theorem 5.17. *Let $1 < p < \infty$ and let $D \in \mathcal{T}_p$ be the symbol of a co-analytic multiplicative Toeplitz operator. The following are equivalent:*

(1) $T_D : \mathcal{H}_p \to \mathcal{H}_p$ has an orbit with a non-zero limit point;
(2) $T_D : \mathcal{H}_p \to \mathcal{H}_p$ is chaotic and frequently hypercyclic;
(3) There exists $N_0 \in \mathbb{N}$ such that $T_{D_N} : \mathcal{H}_{p,N} \to \mathcal{H}_{p,N}$ is hypercyclic for all $N \geq N_0$ and
(4) $\jmath D(\mathbb{C}_+) \cap \mathbb{T} \neq \emptyset$.

Proof. (1) ⇒ (4). If $0 \neq E \in \mathcal{H}_p$ is a limit point of an orbit of T_D, then, it follows by quasiconjugation that $\jmath(E)$ is a limit point of an orbit of $M^*_{\jmath D} : \mathcal{H}^*_{p'} \to \mathcal{H}^*_{p'}$. Since $M^*_{\jmath D}$ has an orbit with a nonzero limit point, it follows from Theorem 5.14 that $M^*_{\jmath D}$ is hypercyclic and by Theorem 5.13 we can conclude that $\jmath D(\mathbb{C}_+) \cap \mathbb{T} \neq \emptyset$.

(4) ⇒ (3) follows from Theorems 5.13 and from the fact that $M^*_{(\jmath D)_N}$ and T_{D_N} are conjugated through a conjugate-linear isomorphism.

(2) ⇒ (1) is immediate.

(3) ⇒ (2). Note first that since each T_{D_N} is conjugated to $M^*_{(\jmath D)_N}$, we have by Theorem 5.13 that $M^*_{\jmath D}$ is hypercyclic, that $M^*_{(\jmath D)_N}$ is chaotic for all $N \geq N_0$, and that T_{D_N} is chaotic for all $N \geq N_0$.

Let us prove that T_D has a dense set of periodic vectors. Indeed, for every $N \in \mathbb{N}$, the operators $T_D : \mathcal{H}_p \to \mathcal{H}_p$ and $T_{D_N} : \mathcal{H}_{p,N} \to \mathcal{H}_{p,N}$ verify $T_D \circ i_N = i_N \circ T_{D_N}$, where i_N is the usual inclusion $\mathcal{H}_{p,N} \hookrightarrow \mathcal{H}_p$. Thus each periodic point of T_{D_N} translates into a periodic point of T_D. This means that T_D has a dense set of periodic points, since $\bigcup_{N\geq N_0} \mathcal{H}_{p,N}$ is dense in $\mathcal{H}_p$.

Applying the Bohr transform and Lemma 5.1 we have that each $M_{(\jmath D)_N} : \mathcal{H}^*_{p',N} \to \mathcal{H}^*_{p',N}$ has a perfectly spanning set of unimodular eigenvectors with respect to the normalized Lebesgue measure of $\mathbb{T}$. Again by conjugation between $M_{(\jmath D)_N} : \mathcal{H}^*_{p',N} \to \mathcal{H}^*_{p',N}$ and $T_{D_N} : \mathcal{H}_{p,N} \to \mathcal{H}_{p,N}$ this implies that T_{D_N} has a perfectly spanning set of unimodular eigenvectors with to respect the normalized Lebesgue measure of $\mathbb{T}$. The same argument of Theorem 5.8 implies now that T_D is frequently hypercyclic. ☐

6. Final remarks and questions

We would like to end the paper with some comments and presenting a few open problems. Most of the results in the paper hold for $1 < p < \infty$. Since our original motivation was to solve some problems on linear dynamics, which requires the space to be separable, this was a reasonable assumption. On the other hand, clearly the space $\mathcal{H}_1$ is more involved: it is not reflexive, its dual is not separable and $\{n^{-s}\}$ is not a Schauder basis. However, $\mathcal{H}_1$ is an important space, so it is natural to ask which of the results of Sections 3 and 4 hold for $p = 1$.

We only partially answered Question **D** from the Introduction: In Theorem 4.5 we gave a characterization of co-analytic Dirichlet series being in $\mathcal{T}_p$, as those Dirichlet series D for which $\sup_{n\in\mathbb{N}} \|D_N\|_{\mathcal{T}_{p,N}} < \infty$. We know that Dirichlet polynomials over $\mathbb{Q}_+$ are in $\mathcal{T}_p$ but that there exists a co-analytic Dirichlet series D such that $\jmath D$ is uniformly continuous on $\overline{\mathbb{C}_+}$ but $D \notin \mathcal{T}_p$.

Question 1. Is it possible to find an explicit description of $\mathcal{T}_p$? Assume that D is a co-analytic Dirichlet series with $\sigma_u(\jmath D) < 0$, does D induce a co-analytic multiplicative Toeplitz operator on $\mathcal{H}_p$? In view of Remark 4.1, we have the following related question: what are the multipliers of $\mathcal{H}^*_p$?

We remark that, taking $a_k = \mu^{-k}$ for $1 < \mu < \lambda < \|T_{\overline{z}}\|$ in the proof of Theorem 4.4, we obtain a function $\phi = \sum_{k=1}^{\infty} \mu^{-k}\phi_k$ such that $\frac{d^k}{k!}\phi = \mu^{-k}\phi_k$ has norm μ^{-k}. Thus $\jmath\phi \in H_b(\mu^{-1}B_{c_0}) \subset \mathcal{A}_u(rB_{c_0})$ for any $r < \mu^{-1}$, but T_ϕ is not a bounded operator on $H_p(\mathbb{D}^\infty_2)$. Note that this is not enough to assure that the corresponding Dirichlet series $D = \jmath\mathfrak{B}\jmath\phi$ satisfies $\sigma_u(\jmath D) < 0$.

Concerning the linear dynamics of Toeplitz operators, let us consider the simple example $D(s) = 2^s$. Then $M^*_{\jmath D} : \mathcal{H}^*_p \to \mathcal{H}^*_p$ is not hypercyclic because $\|M^*_{\jmath D}\| = \|\jmath D\|_\infty = \|2^{-s}\|_\infty = 1$. However, the norm of T_D is strictly bigger than one (it coincides with the norm of $T_{\overline{z}} : H_p(\mathbb{D}) \to H_p(\mathbb{D})$, whose exact norm is an open question [31, Open problem 5.4] or also [15, 22]). So, the same simple argument fails to prove that T_D is not hypercyclic or chaotic in $\mathcal{H}_p$. A closer look shows that if we denote by $(e_n)_n = (n^{-s})_n$ the basis of $\mathcal{H}_p$, then T_D acts as a "divide by two" backward shift: $T_D e_n = \begin{cases} e_{\frac{n}{2}} & \text{if } n \text{ is even;} \\ 0 & \text{else.} \end{cases}$. In other words, T_{2^s} is a pseudoshift, i.e. it satisfies $[T_{2^s}(a_n)_n]_k = a_{\varphi(k)}$, where $\varphi(n) = 2n$. From [26], every hypercyclic pseudoshift must satisfy that there is a subsequence $(n_k)_k$ such that for each j, $e_{\varphi^{n_k}(j)} \to 0$. But since in this case $\|e_{\varphi^{n_k}(j)}\|_{\mathcal{H}_p} = \|e_{2^{n_k}j}\|_{\mathcal{H}_p} = 1$ for every $j, k \in \mathbb{N}$ we conclude that T_{2^s} is not hypercyclic nor chaotic. A similar argument shows that every bounded *weighted* pseudoshift $(a_n)_n \mapsto (w_k a_{\varphi(k)})_k$, with $\varphi(n) = 2n$, is hypercyclic if and only if there exists $(n_k)_k$ such that $\prod_{j=1}^{n_k-1} w_j \to \infty$ as $k \to \infty$. However, as the basis is not unconditional, the same argument breaks to prove chaos. We thus have the following open problems:

Question 2. Which are the chaotic (or frequently hypercyclic) weighted "divide by two" operators in $\mathcal{H}_p$? Which are the chaotic (frequently hypercyclic) weighted backward shifts in $\mathcal{H}_p$, i.e. operators of the form $\sum_{n=1}^{\infty} a_n n^{-s} \mapsto \sum_{n=1}^{\infty} w_{n+1} a_{n+1} n^{-s}$? Up to our knowledge, even the following is an open problem: Which are the chaotic weighted backward shifts in $H_p(\mathbb{D})$?

The fact that $\|D\|_\infty$ is in general smaller than the norm of the Toeplitz operator T_D also makes our arguments fail to show the Trichotomy for the dynamics of multiplicative Toeplitz operators on $\mathcal{H}_p$. We know that if a non-constant co-analytic Dirichlet series $D \in \mathcal{T}_p$ satisfies $\jmath D(\mathbb{C}_+) \cap \mathbb{T} \neq \emptyset$ then T_D is hypercyclic. In the case $\jmath D(\mathbb{C}_+) \subset \overline{\mathbb{D}}^c$, assuming also $\sigma_u(\jmath D) < 0$, we deduce from Lemma 5.16 (1) and the Trichotomy for adjoint multipliers on $\mathcal{H}_p^*$ (Theorem 5.15), that every non-zero orbit of T_D goes to ∞. However, we are unable to deal with the case $\jmath D(\mathbb{C}_+) \subset \mathbb{D}$ because generally $\|T_D\| > 1$.

Question 3. Assume that $D \in \mathcal{T}_p$ defines a co-analytic multiplicative Toeplitz operator on $\mathcal{H}_p$ with $\jmath D(\mathbb{C}_+) \subset \mathbb{D}$. Is T_D a power-bounded operator? A positive answer to this question would solve the following: Assume that $\sigma_u(\jmath D) < 0$, does the trichotomy for the dynamics of T_D hold?

In case D depends only on finitely many primes the answer to these questions is affirmative: if $D = D_N$ for some $N \in \mathbb{N}$ with $\sigma_u(\jmath D) < 0$ and $\jmath D(\mathbb{C}_+) \subset \mathbb{D}$ then we can factorize T_D as $\mathcal{H}_p \xrightarrow{\tilde{\pi}_N} \mathcal{H}_{p,N} \xrightarrow{T_{D_N}} \mathcal{H}_{p,N} \xrightarrow{\tilde{\imath}_N} \mathcal{H}_p$ where $\tilde{\pi}_N = \mathfrak{B}\pi_N\mathfrak{B}^{-1}$ and $\tilde{\imath}_N = \mathfrak{B} i_N \mathfrak{B}^{-1}$ are norm one operators. Thus since $M_{\jmath D}^*$ is a contraction, we conclude by Lemma 5.16 (2) that T_D is power bounded. Hence we have the following.

Proposition 6.1. *Let D be a co-analytic Dirichlet polynomial. Then,*

either – *T_D is hypercyclic,*
or – *for any $E \in \mathcal{H}_p$, $T_D^n E \to 0$,*
or – *for any non-zero $E \in \mathcal{H}_p$, $\|T_D^n E\| \to \infty$.*

Our last question concerns the size of $\mathcal{H}_{p'}$ as a subset of $\mathcal{H}_p^*$.

Question 4. Let $D \in \mathcal{H}_\infty$. Take a typical hypercyclic vector for M_D^*. Does it belong to $\mathcal{H}_p^* \setminus \jmath\mathcal{H}_{p'}$ or to $\jmath\mathcal{H}_{p'}$? Are both sets $(\mathcal{H}_p^* \setminus \jmath\mathcal{H}_{p'}) \cap HC(M_D^*)$ and $\jmath\mathcal{H}_{p'} \cap HC(M_D^*)$ non-empty? Does there exist a hypercyclic vector whose whole orbit is contained in $\mathcal{H}_p^* \setminus \jmath\mathcal{H}_{p'}$?

We can also ask the analogous problem for frequent hypercyclity: suppose that μ is an ergodic (or invariant) measure of full support for M_D^*. Do we always have that $\mu(\mathcal{H}_p^* \setminus \jmath\mathcal{H}_{p'}) = 1$ (or that $\mu(\jmath\mathcal{H}_{p'}) = 1$)?

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Departamento de Matemática, Instituto Balseiro, Universidad Nacional de Cuyo – C.N.E.A. and CONICET, San Carlos de Bariloche, República Argentina

Email address: rodrigo.cardeccia@ib.edu.ar

FCEIA, Universidad Nacional de Rosario and CIFASIS, CONICET

Email address: smuro@fceia.unr.edu.ar

FCEIA, Universidad Nacional de Rosario

Email address: matiaspalumbo19@gmail.com