SOME EXERCICES

wtorek, 18 stycznia 2022 02:55

Task 1

REVISION

A triple $(\mathcal{M}, \cdot, e)$ is a monoid if $\mathcal{M}$ is a set, $\cdot : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is a binary operation on $\mathcal{M}$ and $e \in \mathcal{M}$ that satisfy:

- $(\forall x, y, z)[(x \cdot y) \cdot z = x \cdot (y \cdot z)]$ associativity
- $(\forall x)(x \cdot e = e \cdot x = x)$

Two mathematical structures are isomorphic if an isomorphism exists between them. A function $h: \mathcal{M}_1 \to \mathcal{M}_2$ between monoids $(\mathcal{M}_1, \cdot, e_1)$ and $(\mathcal{M}_2, *, e_2)$ is called homomorphism if:

- $\forall_{x, y \in \mathcal{M}_1} \; h(x \cdot y) = h(x) * h(y)$
- $h(e_1) = e_2$

A bijective monoid homomorphism is called a monoid isomorphism.

a) $\mathcal{M}_1 = (\mathbb{N}, +, 0)$, $(\mathbb{N}^+, \cdot, 1) = \mathcal{M}_2$

$h(x + y) = h(x) \cdot h(y)$

$h(0) = 1$ or $h(1) = 0$

Let's check what happend if we take $x = 1$, $y = 1$

$h(1 + 1) = h(2) = h(1) \cdot h(1) = [h(1)]^2$

Now let's take $x = y$

$h(2x) = h(x) \cdot h(x) = [h(x)]^2$

For $y = 2x$

$h(3x) = h(2x) \cdot h(x) = [h(x)]^2 \cdot h(x) = [h(x)]^3$

So we can notice that for any $a \in \mathbb{N}$, $y = (a-1)x$

$h(ax) = [h(x)]^a$ (Because $h(nx) = h((n-1) \cdot x) \cdot h(x) = \ldots = [h(x)]^{n-1} \cdot h(x)$ - recurrence)

Thanks to this observation we can notice that any natural number can be represented as ax for $x = 1$ and a equal this number, so

$h(x') = [h(1)]^{x'}$.

$$h(x+y) = [h(1)]^{x+y}$$

Because $h(1)$ have to be natural number (no matter if we go from $M_1 \to M_2$ or $M_2 \to M_1$), so we can write $h(1) = C \in \mathbb{N}$. Then for $z = x+y \in \mathbb{N}$, we got

$$h(z) = C^z$$

for $z = 0$, $h(z) = C^0 = 1$

So it's homomorphism. Of course it's injective but it's not surjective (not all elements can be represented as C^z, f.e. $C+1$)

It's equivalent to fact that there is not exist byective homomorphism between this two monoids, so M_1, M_2 are not isomorphic.

Now let's check $M_2 \to M_1$. Because if we want to obtain byection homomorphism then it must exist a inverse homomorphism. But we have just proved that this inverse homomorphism have to be in following form

$$h^{-1}(x) = C^x \Rightarrow h(x) = \ln_C(x) \Rightarrow h(x \cdot y) = \ln_C(x \cdot y) = \ln_C(x) + \ln_C(y) = $$
$$= h(x) + h(y)$$
$$h(e_2) = h(1) = \ln_C(1) = 0 = e_1$$

But again logarithm function isnot byection if arguments are natural numbers (not all elements can be represented in this way).

2) $M_1 = (\mathbb{N}, +, 0)$, $(\mathbb{N}^2, \oplus, (0,0)) = M_2$

$\oplus$ is just adding in $\mathbb{N}^2$
I changed sign for clarity

$$h(x \oplus y) \overset{M_2 \to M_1}{=} h(x) + h(y)$$
$$h(e_2) = e_1 \Rightarrow h((0,0)) = 0$$

or

$$h(x+y) \overset{M_1 \to M_2}{=} h(x) \oplus h(y)$$
$$h(e_1) = e_2 \Rightarrow h(0) = (0,0)$$

Let's us concider first case: $M_2 \to M_1$. We know that $h((0,0)) = 0$

Moreover $h((0,1)) = x$, where $x \in \mathbb{N}$
$h((1,0)) = y$, where $y \in \mathbb{N}$

We know that, any member of $\mathbb{N}^2$ can be represented as a linear combination of $(0,1)$ and $(1,0)$

$h((1,1)) = h((0,1) \oplus (1,0)) = h((0,1)) + h((1,0)) = x+y$

In general

$h((k,l)) = h((k,0) \oplus (0,l)) = h((k,0)) + h((0,l)) = h((k-1,0) + (1,0)) + h((0,l)) =$

$= \ldots = k \cdot h((1,0)) + l\, h((0,1)) = kx + ly$

Such construct is homomorphism. Now let's check if it's byjection.

$y \neq 0 \wedge x \neq 0$, $x \neq y \Rightarrow$ because otherwise $h(x_1) = h(x_2)$ for $x_1 \neq x_2$.

It's mean that only one can be equal 1. This mean that x or y have to be bigger than 1. Without loss of generality we can assume that it's $y > 1$.

First let's check injection.

$(k_1, l_1) \neq (k_2, l_2)$

$h((k_1, l_1)) = k_1 x + l_1 y \quad \wedge \quad h((k_2, l_2)) = k_2 x + l_2 y$

If h is injection then $h((k_1, l_1)) \neq h((k_2, l_2)) \Rightarrow k_1 x + l_1 y \neq k_2 x + l_2 y$

$\Rightarrow (k_1 - k_2)x + (l_1 - l_2)y \neq 0$ but it's not true for some

$k_1, k_2, l_1, l_2 \Rightarrow k_1 - k_2 = -y \wedge l_1 - l_2 = x$

So it's not byjection

Now let's concider second case: $M_1 \to M_2$. We know that

$h(0) = (0,0)$. $h(1) = (x,y) \in \mathbb{N}^2$ $\quad x, y \in \mathbb{N}$

$h(2) = h(1+1) = h(1) \oplus h(1) = (x,y) \oplus (x,y) = (2x, 2y)$

$\vdots$ next

$h(n) = h((n-1)+1) = h(n-1) \oplus h(1) = h(n-2) \oplus h(1) \oplus h(1) = \ldots = n\, h(1) = (nx, ny)$

This function isn't surjective because we cannot obtain $(x+1, y)$ as result.

So M_1 and M_2 also aren't isomorphic.

TASK 2

$X = \mathbb{N}$, $M = (\mathbb{N}, +, 0)$, $f: X \to M$ $\quad f(x) = x$

$h: X^* \to M$ such that $h \circ \eta = f$

First let's recall Universal mapping property (UMP) for monoids.

Let X be a set and let $(M, \cdot, e)$ be a monad. Say we have function ...

$f: X \to M$. then there exist exactly one homomorphism $h: X^*$ such that the following diagrams comutes:

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & X^* \arrow[dl, dashed, "h"] \\ M & \end{tikzcd}$$

$h \circ \eta_X = f$

Now let's find such h.

$\eta : X \to X^*$

$\eta(x) = [x]$

$h([\ ]) = 0$

$f(x) = h(\eta(x)) = h([x]) = x$

$h([x,y]) = h([x] ++ [y]) = h([x]) + h([y]) = f(x) + f(y) = x + y$

induction

$h([x_1, \dots, x_n]) = h([x_1, \dots, x_{n-1}] ++ [x_n]) = h([x_1, \dots, x_{n-1}]) + h([x_n]) = f(x_1) + \dots + f(x_n) = \sum_{i=1}^{n} x_i$ ■

Task 3

$C \to$ Category that consist one object looks following

$$\begin{tikzcd} A \arrow[loop above, "id_A"] \end{tikzcd}$$

So 1-object category have one object A. Now let us notice that operations on monoid M are exactly the same as the operations on $Morph(C)$ so we may regard the category C as a monoid M. It's of course reversible : given category $C = \{A\}$ we obtain a monoid M whose elements are the arrows of C. It's work's because the binary operations are the same for both structures. ■

Task 4

Let $(G, \cdot)$ be a group and $C = (Obj(C), Arr(C)) = (\{G\}, G)$ is the category with exactly one object $\{G\}$ and morphisms from $\{G\}$ to itself are given by G. Let h, g be a morphisms mentioned before. Then composition of morphisms look in the following way:

$\{G\} \xrightarrow{g} \{G\}$

$\{G\} \xrightarrow{h} \{G\}$

$$\begin{tikzcd} \{G\} \arrow[r, "g"] \arrow[dr, dashed, "hg"'] & \{G\} \arrow[d, "h"] \\ & \{G\} \end{tikzcd}$$

Every group have to have unit element e. Thanks to fact that G is a group then there is always exist an inverse morphism. These two facts let us tell that:

The unit elemement of a group is given by by the identity morphism on $\{G\}$ and vice versa - identity morphism (identity arrow) is given by unit element of the group $(G, \cdot)$

of the group $(G, \cdot)$.

Task 5

Reminder:

A dual (or opposite) category to $C = \{Obj(C), Arr(C)\}$ is the category $C^{OP} = (Obj(C^{OP}), Arr(C^{OP}))$ with

- $Obj(C^{OP}) = Obj(C)$
- $Arr(C^{OP}) = \{f^{OP} : f \in Arr(C)\}$ where $f^{OP} : B \to A$ is an arrow $f : A \to B$ with a flipped domain with codomain:
- there is a dual composition $\bullet$ such that for arrows f, g we have $f^{OP} \bullet g^{OP} = (g \circ f)^{OP}$

From the previous task we know that

- a group $(G, \cdot)$ forms a category $C = (\{G\}, G\}$ with exactly one object within which every morphism is reversible - is isomorphism

So by definition of dual category

$Obj(C^{OP}) = Obj(C) = \{G\}$

$Arr(C^{OP}) = \{f^{OP} : f \in Arr(C)\}$

but f^{OP} differ from f by place of domain and codomain. In our case domain and codomain are equal so $f \equiv f^{OP} \Rightarrow Arr(C^{OP}) = Arr(C)$

dual composition

$f^{OP} \bullet g^{OP} = (g \circ f)^{OP} \Rightarrow f \bullet g = g \circ f$

Thus categories $C^{OP} = (\{G\}, G\}$ and $C = (\{G\}, G\}$ are isomorphic, because there is exist a isomorphism between them (identity).

Task 6

REMINDER

Partially order is defined as $(X, \leq)$. It can be treated as a category in the following way:

$Obj(C) = X$; pair (x, y) is an arrow if $x \leq y$

An object T is terminal if for every object A there is exactly one arrow $f : A \to T$

An object I is initial if for every object A there is exactly one arrow $f : I \to T$

1) Let $C = (X, \subseteq)$ be a category where objects are sets.

Initial objects: exist iff X contain the smallest element $x \Rightarrow x$

Terminal objects: exist iff X contain the greatest element $x \Rightarrow x$

Isomorphisms: $x \subseteq y \wedge y \subseteq x \Rightarrow x = y$ so only identity arrows are isomorphism.

$C^{op} = (X^{op}, (\subseteq)^{op})$; By definition objects of duality category are the same as in C So $X^{op} = X$. Now arrows have swapped domain with codomain. It's implefied that '$(\subseteq)^{op} = \supseteq$'. So $C^{op} = (X, \supseteq)$

Initial object: exist iff X contain the greatest element: $x \Rightarrow X$

Terminal object: exist iff X contain the smallest element: $x \Rightarrow X$

Isomorphisms: $x \supseteq y \wedge y \supseteq x \Rightarrow x = y$. So only identity arrows are isomorphisms.

2) Let $C = (\mathbb{N}^+, |)$ be a category where objects are natural numbers greater than 0 and arrow is division. Arrow between x and y $x \to y$ exist iff $x | y$.

Initial object: Exist iff there is element which can divide without rest any other element

Terminal object: Exist iff there is element which is divisible by all other elements.

Isomorphisms: $x|y \wedge y|x \Rightarrow x = y$. So only identity arrows are isomorphisms.

$C^{op} = ((\mathbb{N}^+)^{op}, (|)^{op})$; By definition objects of duality category are the same as in C So $(\mathbb{N}^+)^{op} = \mathbb{N}^+$ Now arrows have swapped domain with codomain. It's implefied that arrow between x and y $x \to y$ exist iff $y | x$.

Initial object: Exist iff there is element which is divisible by all other elements

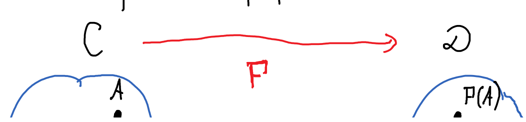
Terminal object: Exist iff there is element which can divide without rest all other elements

Isomorphisms: $y | x \wedge x | y \Rightarrow y = x$ So only identity arrows are isomorphisms.

3) $C = (\mathbb{Z}, \leq)$ is analogicall to the first subpoint.

Task 7 REMINDER

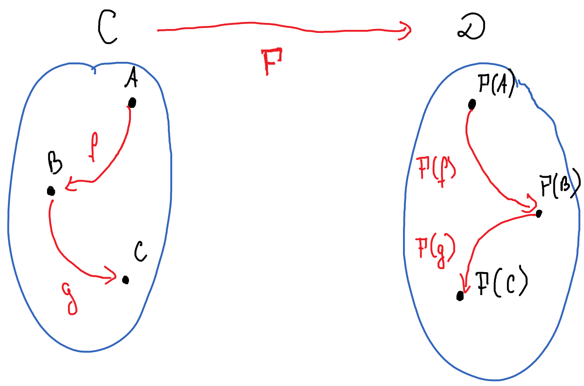
Let C, D be categories. We call $F: C \to D$ a functor if for all objects $A, B, C \in Ob(C)$ and arrows $f: A \to B$, $g: B \to C$



• $F(A) \in Ob(D)$

arrows $f: A \to B$, $g: B \to C$

- $F(A) \in Ob_j(\mathcal{D})$
- $F(f): F(A) \to F(B)$
- $F(id_A) = id_{F(A)}$
- $F(g \circ f) = F(g) \circ F(f)$



Let's introduce a counterexample:

Let C be a category with just one object X, and just one arrow: id_X

Let $\mathcal{D}$ be a category with just one object Y, and two arrows: id_Y and f, where composition is defined as follow: $f \circ f = f$.

Now let F be a "functor" which send X into Y and id_X to f. It's preserve composition

$$F(id_X \circ id_X) = F(id_X) = f = f \circ f = F(id_X) \circ F(id_X)$$

but it's not preserve identities $\Rightarrow$ Preservation of identity is necessary $\blacksquare$

Task 8

$F: C \to \mathcal{D}$, $G: \mathcal{D} \to \mathcal{E}$ } functors, $\quad C, \mathcal{D}, \mathcal{E}$ - categories

Let $X \in Ob_j(C)$. Then $F(X) \in Ob_j(\mathcal{D})$

$$(\forall Y \in Ob_j(\mathcal{D}))(\exists G(Y): G(Y) \in Ob_j(\mathcal{E}))$$

Thus:

$$(\forall X \in Ob_j(C))\ (G \circ F)(X) = G(F(X)) : (C \to \mathcal{D}) \to \mathcal{E} \equiv C \to \mathcal{E}$$

$$(\forall f \in Arr(C))\ (G \circ F)(f) = G(F(f)) \in Arr(\mathcal{E}) \quad \blacksquare$$

Task 9

1) By identity functor Id (C - category) we understand functor which works in following way: $\quad Id: C \to C$

$$(\forall x \in Ob_j(C))\ Id(x) = x$$

$$(\forall f \in Arr(f))\ \ Id(f) = f$$

Let's check if it's really a functor:

First condition is obvious thanks to above definition:

- $Id(id_X) = id_X = id_{Id(X)}$
- $(\forall f, g \in Arr(C): f: A \to B,\ g: B \to C)\ Id(g \circ f) = g \circ f = Id(g) \circ Id(f)$

2) By forgetful functor $\mathcal{U}$ we understand a functor which works as follow:

$\mathcal{U}: MON \to SET$

If $(M, \cdot, e)$ is a monoid then $\mathcal{U}((M, \cdot, e)) = M$

If h is monoid homomorphism then $\mathcal{U}(h) = h$

First condition is obvious thanks to above definition:

- $\mathcal{U}(id_X) = id_X = id_{\mathcal{U}(X)}$
- $(\forall f, g \in Arr(C) : f : A \to B \land g : B \to C)\ \mathcal{U}(g \circ f) = g \circ f = \mathcal{U}(g) \circ \mathcal{U}(f)$ ∎

Task 10

Let $F, G, H : C \to D$ be a functors and C, D - categories

$\eta : F \xrightarrow{\cdot} G \quad \land \quad \mu : G \xrightarrow{\cdot} H$

$$(\forall X \in Obj(C))\ (\mu \circ \eta)\, F(X) = \mu(\eta(F(X))) = \mu(G(X)) = H(X)$$

$$(\forall f \in Arr(C))\ (\mu \circ \eta)\, F(f) = \mu(\eta(F(f))) = \mu(G(f)) = H(f)$$

Thus

$$\mu \circ \eta : F \to G \to H \equiv \mu \circ \eta : F \to H$$ ∎

Task 11

$\eta : L \xrightarrow{\cdot} L$ defined as a list reversing transformation

To check if it's a natural transformation we have to check if the following diagram commute.

$$\begin{tikzcd} L(X) \arrow[r, "L(f)"] \arrow[d, "\mu_X"'] & L(Y) \arrow[d, "\mu_Y"] \\ L(X) \arrow[r, "L(f)"] & L(Y) \end{tikzcd}$$

$L = L(f) \circ \mu_X([x_1, \dots, x_n]) = L(f)([x_n, \dots, x_1]) =$

$= [f(x_n), \dots, f(x_1)] = [y_n, \dots, y_1] =$

$= \mu_Y([y_1, \dots, y_n]) = \mu_Y([f(x_1), \dots, f(x_n)]) =$

$= \mu_Y \circ L(f)\,[x_1, \dots, x_n] = R \Rightarrow$ it's commute ∎

Task 12

In HOMEWORK 3

Task 13

Let $F(X) = X^{\mathbb{N}}$. Let's check how it work for some $n \in \mathbb{N}$

$F(X) = \underbrace{(x_1, \dots, x)}_{n \text{ times}}$, $F(f) = \underbrace{(f_1, \dots, f)}_{n \text{ - times}}$

$F(f)(x) = (f(x_1), \dots, f(x_n))$

$\eta : F \dot{\to} F$, for example:

$\eta((x_1, \dots, x_n)) = (x_n, \dots, x_1)$

To check if it's a natural transformation we have to check if following diagram commute: $f : X \to Y$

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_x"'] & F(Y) \arrow[d, "\eta_y"] \\ F(X) \arrow[r, "F(f)"] & F(Y) \end{tikzcd}$$

$L = (F(f) \circ \eta_x)((x_1, \dots, x_n)) = F(f)(\eta_x((x_1, \dots, x_n))) =$

$= F(f)((x_n, \dots, x_1)) = (f(x_n), \dots, f(x_1)) = (y_n, \dots, y_1) =$

$= \mu_y((y_1, \dots, y_n)) = \mu_y((f(x_1), \dots, f(x_n))) =$

$= F(f)(\mu_y((x_1, \dots, x_n))) = (F(f) \circ \mu_y)((x_1, \dots, x_n)) = R$

Our function is in $Y_A = X^A$ form so we can use Yoneda lemma:

$$Nat(F(X), F(X)) = Nat(Y_n, F(X)) \approx F(n) = n^n$$

So we have n^n such natural transformations ■

Task 14

1) $V_n = X^n$

a) functor

$V_n(X) = (x_1, \dots, x_n)$ $\qquad V_n(f \circ g) = (f \circ g_1, \dots, f \circ g) = (f_1, \dots, f) \circ (g_1, \dots, g)$

$V_n(f) = \underbrace{(f_1, \dots, f)}_{n \text{ - times}}$ $\qquad V_n(id_X) = (id_{X1}, \dots, id_X) = id_{V_n(X)}$

b) natural transformations: $\mu : Id \dot{\to} V_n$; $\eta : V_n^2 \dot{\to} V_n$

Let $\mu(x) = \underbrace{(x_1, \dots, x)}_{n}$, $f : X \to Y$

Check if it's natural transformation:

$$Id(X) \xrightarrow{Id(f)} Id(Y)$$

$L = (V_n(f) \circ \mu_X)(x) = V_n(f)(\mu(x)) = V_n(f)(x_1, \dots, x) =$

$= (f(x), \dots, f(x)) = (y_1, \dots, y) = \mu_Y(y) =$

$$\begin{tikzcd}
Id(X) \arrow[r, "Id(f)"] \arrow[d, "\mu_X"'] & Id(Y) \arrow[d, "\mu_Y"] \\
V_n(X) \arrow[r, "V_n(f)"] & V_n(Y)
\end{tikzcd}$$

$= (f(x), \dots, f(x)) = (y, \dots, y) = \mu_Y(y) =$

$= \mu_Y\left(Id(f)(x)\right) = \mu_Y \circ Id(f)(x) = R$

It's commute so μ is natural transformation

Let $\eta(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n) = (x_1^{(1)}, x_2^{(2)}, \dots, x_n^{(n)})$ where $x_i^{(i)}$ means i'th element in i'th vector. Let's diagram: $f: X \to Y$

$$\begin{tikzcd}
V_n^2(X) \arrow[r, "V_n^2(f)"] \arrow[d, "\eta_X"'] & V_n^2(Y) \arrow[d, "\eta_Y"] \\
V_n(X) \arrow[r, "V_n(f)"] & V_n(Y)
\end{tikzcd}$$

$L = (V_n(f) \circ \eta_X)\left((\bar{x}_1, \dots, \bar{x}_n)\right) =$

$= V_n(f)\left(\eta_X((\bar{x}_1, \dots, \bar{x}_n))\right) = V_n(f)\left((x_1^{(1)}, \dots, x_n^{(n)})\right) =$

$= \left(f(x_1^{(1)}), \dots, f(x_n^{(n)})\right) = (y_1, \dots, y_n)$

$R = (\eta_Y \circ V_n^2(f))\left((\bar{x}_1, \dots, \bar{x}_n)\right) = \eta_Y\left(V_n^2(f)((\bar{x}_1, \dots, \bar{x}_n))\right)$

$= \eta_Y\left(\left(f(x_1^{(1)}), \dots, f(x_n^{(1)})\right), \dots, \left(f(x_1^{(n)}), \dots, f(x_n^{(n)})\right)\right) =$

$= \left(f(x_1^{(1)}), \dots, f(x_n^{(n)})\right) = (y_1, \dots, y_n) = L$

So η is also natural transformation.

c) monadic diagrams:

$\bar{x} = (x_1, \dots, x_n)$; $\bar{x}_i = (x_i, \dots, x_i)$

$$\begin{tikzcd}
V_n(X) \arrow[r, "V_n(\mu_X)"] \arrow[d, "\mu_{V_n(X)}"'] & V_n^2(X) \arrow[d, "\eta_X"] \\
V_n^2(X) \arrow[r, "\eta_X"] & V_n(X)
\end{tikzcd}$$

$L = (\eta_X \circ \mu_{V_n(X)})\left((x_1, \dots, x_n)\right) = \eta_X\left(\mu_{V_n(X)}(V_n(X))\right) =$

$= \eta_X(\bar{x}, \dots, \bar{x}) = (x_1^{(1)}, \dots, x_n^{(n)}) = (x_1, \dots, x_n)$

$R = (\eta_X \circ V_n(\mu_X))\left((x_1, \dots, x_n)\right) = \eta_X\left(V_n(\mu_X)(x_1, \dots, x_n)\right) =$

$= \eta_X(\bar{x}_1, \dots, \bar{x}_n) = (x_1^{(1)}, \dots, x_n^{(n)}) = (x_1, \dots, x_n)$

$$\eta_X \circ \mu_{V_n(X)} = \eta_X \circ V_n(\mu_X) = id_{V_n(X)}$$

$$\begin{tikzcd}
V_n^3(X) \arrow[r, "V_n(\eta_X)"] \arrow[d, "\eta_{V_n(X)}"'] & V_n^2(X) \arrow[d, "\eta_X"] \\
 &
\end{tikzcd}$$

$L = (\eta_X \circ \eta_{V_n(X)})\left(V_n^3(X)\right) = \eta_X\left(\eta_{V_n(X)}(V_n^3(X))\right) =$

$= \eta_X(\bar{x}_1^{(1)}, \dots, \bar{x}_n^{(n)}) = (x_1^{1,1}, \dots, x_n^{n,n})$

$$\begin{tikzcd} V_n^2(X) \arrow[r, "\eta_X"] & V_n(X) \end{tikzcd}$$

$\ldots \eta_X(x_1, \ldots, x_n) = (x_1', \ldots, x_n')$

$R = (\eta_X \circ V_n(\eta_X))(V_n^3(x)) = \eta_X(V_n(\eta_X)(V_n^3(x))) =$

$= \eta_X((x_1^{1,1}, \ldots, x_n^{1,n}), \ldots, (x_1^{n,1}, \ldots, x_n^{n,n})) =$

$= (x_1^{1,1}, \ldots, x_n^{n,n}) = L \quad \square$

2) Maybe M

a) functor

$M(X) = X \cup \{n_X\}$

$M(f) = f \cup \{(n_X, n_Y)\}$

$n_X \notin X, \; f: X \to Y, \; g: Y \to Z$

b) natural transformations $\eta: Id \to M$; $\mu: M^2 \to M$

$\eta_X(x) = x$

$$\begin{tikzcd} Id(X) \arrow[r, "Id(f)"] \arrow[d, "\eta_X"'] & Id(Y) \arrow[d, "\eta_Y"] \\ M(X) \arrow[r, "M(f)"] & M(Y) \end{tikzcd}$$

$R = \eta_Y \circ Id(f)(x) = \eta_Y(y) = y$

$L = M(f) \circ \eta_X(x) = M(f)(x) = y$

$\mu_X(x) = \begin{cases} n_X, & \text{for } x \in \{n_X, n_{X \cup \{n_X\}}\} \\ x, & \text{otherwise} \end{cases}$

$n_X \notin X$

$$\begin{tikzcd} M^2(X) \arrow[r, "M^2(f)"] \arrow[d, "\mu_X"'] & M^2(Y) \arrow[d, "\mu_Y"] \\ M(X) \arrow[r, "M(f)"] & M(Y) \end{tikzcd}$$

1. $L = \mu_Y\, M^2(f)(n_{X \cup \{n_X\}}) = \mu_Y(n_{Y \cup \{n_Y\}}) =$
$= n_Y$

2. $L = \mu_Y\, M^2(f)(x) = \mu_Y(y) = y$

1. $R = M(f) \circ \mu_X(n_{X \cup \{n_X\}}) = M(f)(n_{X \cup \{n_X\}}) =$
$= n_{Y \cup \{n_Y\}} = L$

2. $R = M(f) \circ \mu_X(x) = M(f)(x) = y$ ■

3) monoid diagrams:

$M(\eta_\cdot)$

$R = \mu \circ M(\eta_\cdot)(x) = \mu_\cdot(x) = x$

3) monoid diagrams:

$$\begin{tikzcd} \mathcal{M}(X) \arrow[r, "\mathcal{M}(\eta_X)"] \arrow[d, "\eta_{\mathcal{M}(X)}"'] & \mathcal{M}^2(X) \arrow[d, "\mu_X"] \\ \mathcal{M}^2(X) \arrow[r, "\mu_X"'] & \mathcal{M}(X) \end{tikzcd}$$

$R = \mu_X \circ \mathcal{M}(\eta_X)(x) = \mu_X(x) = x$

$L = \mu_X \circ \eta_{\mathcal{M}(X)}(x) = \mu_X(x) = x = R = id_X$

$$\begin{tikzcd} \mathcal{M}^3(X) \arrow[r, "\mathcal{M}(\mu_X)"] \arrow[d, "\mu_{\mathcal{M}(X)}"'] & \mathcal{M}^2(X) \arrow[d, "\mu_X"] \\ \mathcal{M}^2(X) \arrow[r, "\mu_X"'] & \mathcal{M}(X) \end{tikzcd}$$

$\emptyset$ $L = \mu_X \circ \mathcal{M}(\mu_X)\left(n_{x \cup \{n_x\} \cup \{n_{x \cup \{n_x\}}\}}\right) =$
$= \mu_X(n_x) = n_x$

$\emptyset$ $L = \mu_X \mathcal{M}(\mu_X)(x) = \mu_X(x) = x$

① $R = \mu_X \cdot \mu_{\mathcal{M}(X)}\left(n_{x \cup \{n_x, n_{x \cup \{n_x\}}\}}\right) =$
$= \mu_X(n_{x \cup \{n_x\}}) = n_x = L$

② $R = \mu_X \circ \mu_{\mathcal{M}(X)}(X) = \mu_X(x) = x = R$

3) Writer W_n

a) functor

$W_n(X) = X \times M$

$W_n(f) = (f, id_M)$

$id_M \circ id_M = id_M$

$W_n(f \circ g) = (f \circ g, id_M) = W_n(f) \circ W_n(g)$

$W_n(id_X) = (id_X, id_M) = id_{W_n}$

b) natural transformations: $\eta : Id \Rightarrow W_n$; $\mu : W_n^2 \Rightarrow W_n$

$\eta_X(x) = (x, e)$

$$\begin{tikzcd} Id(X) \arrow[r, "Id(f)"] \arrow[d, "\eta_X"'] \arrow[dr, phantom, "\circlearrowleft"] & Id(Y) \arrow[d, "\eta_Y"] \\ W_n(X) \arrow[r, "W_n(f)"'] & W_n(Y) \end{tikzcd}$$

$R = \eta_Y \circ Id(f)(x) = \eta_Y(y) = (y, e)$

$L = W_n(f) \circ \eta_X(x) = W_n(f)(x, e) = (y, e)$

$\mu_X((x, m_1), m_2) = (x, m_1 \cdot m_2)$

$$\begin{tikzcd} W_n^2(X) \arrow[r, "W_n^2(f)"'] \arrow[d, "\mu_X"'] & W_n^2(Y) \arrow[d, "\mu_Y"] \\ W_n(X) \arrow[r, "W(f)"'] & W_n(Y) \end{tikzcd}$$

$R = \mu_Y \circ W_n^2(f)((x, m_1), m_2) = \mu_Y((y, m_1), m_2) =$
$= (y, m_1 \cdot m_2)$

$L = W_n(f) \cdot \mu_X((x, m_1), m_2) = W_n(f)(x, m_1 \cdot m_2) =$
$= (y, m_1 \cdot m_2) = R$ □

$$\begin{tikzcd} W_n(X) \arrow[r, "W_n(f)"'] & W_n(Y) \end{tikzcd}$$

$$= (y, m_1 \cdot m_2) = R \quad \square$$

3) monad diagrams

$$\begin{tikzcd} W_n(X) \arrow[r, "W_n(\eta_X)"] \arrow[d, "\eta_{W_n(X)}"'] & W_n^2(X) \arrow[d, "\mu_X"] \\ W_n^2(X) \arrow[r, "\mu_X"'] & W_n(X) \end{tikzcd}$$

$$R = \mu_X \circ W_n(\eta_X)((x, m_1)) = \mu_X((x, m_1), e) =$$
$$= (x_1, m_1 \cdot e) = (x_1, m_1)$$

$$L = \mu_X \circ \eta_{W_n(X)}((x, m_1)) = \mu_X((x, m_1), e) =$$
$$= (x_1, m_1 \cdot e) = (x_1, m_1) = R$$

$$\mu_X \circ W_n(\eta_X) = \mu_X \circ \eta_{W_n(X)} = id_{W_n(X)}$$

$$\begin{tikzcd} W_n^3(X) \arrow[r, "W(\mu_X)"] \arrow[d, "\mu_{W(X)}"'] & W_n^2(X) \arrow[d, "\mu_X"] \\ W_n^2(X) \arrow[r, "\mu_X"'] & W(X) \end{tikzcd}$$

$$R = \mu_X \circ W(\mu_X)(((x, m_1), m_2), m_3) =$$
$$= \mu_X((x_1, m_1 \cdot m_2), m_3) = (x_1, m_1 \cdot m_2 \cdot m_3)$$

$$L = \mu_X \circ \mu_{W(X)}(((x_1, m_1), m_2), m_3) =$$
$$= \mu_X((x_1, m_1), m_2 \cdot m_3) = (x_1, m_1 \cdot m_2 \cdot m_3) = R$$

$\blacksquare$