Some exercises before the test

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Reminder: most of exercises from the home works are suitable for the test difficulty-wise. Sometimes they were lengthy though so you can expect some specific parts of these exercises rather than the entire exercises.

Exercise 1 (Currying)**.** Show that there is a bijection between sets $A^{B\times C}$ and $(A^B)^C$.

Exercise 2. Show that the following diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle h} \\ D & \xrightarrow{j} & E & \xrightarrow{k} & F \end{array}$$

commutes if and only if its components

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle a} \\ D & \xrightarrow{j} & E \end{array} \qquad \begin{array}{ccc} B & \xrightarrow{g} & C \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle h} \\ E & \xrightarrow{k} & F \end{array}$$

commute.

Exercise 3. Determine initial and terminal objects in

a) SET with sets as objects and functions as arrows.

b) GROUP with groups as objects and homomorphisms as arrows.

Exercise 4. Show that every monoid $(M, \cdot)$ can be treated as a category with one object and vice versa - each single object category gives rise to a monoid.

Exercise 5. How to treat a partial order $(X, \preceq)$ as a category? Localize initial and terminal objects. What are products and co-products of objects of this category? Do products always exists for posets? What if $\preceq$ is a linear order?

Exercise 6. Consider the category where objects are sets and arrows are pairs of sets (A, B) such that $A \subseteq B$. Determine initial and terminal objects, products and co-products, if they exist.

Exercise 7. Consider the category $\mathcal{C}$ where Obj($\mathcal{C}$)= $\mathbb{N}^+$ and Arr($\mathcal{C}$) $= \{(k, n) : k|n\}$. Determine initial and terminal objects, products and co-products, if they exist.

Exercise 8. Show that if arrows $f : A \to B$ and $g : B \to C$ are monic then so is $g \circ f$. Conversely, if $g \circ f$ is monic then so is f.

Exercise 9. State and prove analogous results for epic arrows. Be careful with the second part.

Exercise 10. Show that for the SET category epimorphisms are surjections and monomorphisms are injections.

Exercise 11. Show that each isomorphism is also an epimorphism and a monomorphism, however (contrary to intuition from the previous exercise) not every arrow which is a monomorphism and epimorphism is an isomorphism.

Exercise 12. Consider a category derived from a partial order. Which arrows are monic/epic/isomorphisms?

Exercise 13. Show that $f : \mathbb{N} \to \mathbb{Z}$ given by $f(n) = n$ is epic as an arrow from the monoid $(\mathbb{N}, +, 0)$ to $(\mathbb{Z}, +, 0)$.

Exercise 14. Let $(M, \cdot, e_1)$ and $(N, \star, e_2)$ be monoids. Show that the product monoid $(M, \cdot, e_1) \otimes (N, \star, e_2) = (M \times N, \cdot \times \star, (e_1, e_2))$ is actually a monoid.

Exercise 15. Are $(\mathbb{Z}, +, 0)$ and $(\mathbb{Z}, \cdot, 1)$ isomorphic? Same question for $(\mathbb{Z}, +, 0)$ and $(\mathbb{Z}, +, 0) \otimes (\mathbb{Z}, +, 0)$.

Exercise 16. Let L be a list functor. Show that for every set X a list reversing function $\mathrm{reverse}_X$ is a natural transformation.

Exercise 17. Let $(\mathcal{P}, \preceq)$ be a preorder (partial order without antisymmetry), $\mathcal{C}$ any category and $F, G : \mathcal{C} \to \mathcal{P}$ functors. Show that there is exactly one natural transformation $\eta : F \dot{\to} G$ if and only if $F(A) \preceq G(A)$ for every $A \in \mathrm{Obj}(\mathcal{C})$.

Exercise 18. Show that equalizers and pullbacks are unique up to isomorphisms.

Exercise 19. Show that if (X, e) is an equalizer then e is a monic arrow.

Exercise 20. Show that if (X, e) is an equalizer and e is epic then e is an isomorphism.

Exercise 21. Show that if $f = g$ and (X, e) is the equalizer for arrows f, g then e is an isomorphism.

Exercise 22. Describe equalizers in the category SET, i.e. for functions $f, g : A \to B$ find the equalizer (X, e) of these functions.

Exercise 23. Complete the following diagram (and show it's correct)

$$\begin{tikzcd} & B \arrow[d, "\subseteq"] \\ A \arrow[r, "\subseteq"] & C \end{tikzcd}$$

to a pullback in the category with sets as objects and pairs of sets (X, Y) with $X \subseteq Y$ as arrows.

Exercise 24. Show that if the following diagram

$$\begin{tikzcd} P \arrow[r, "e"] \arrow[d, "e"] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

depicts a pullback then (P, e) is the equalizer of arrows f and g.

Exercise 25. Let T be a terminal object. Show that if the following diagram

$$\begin{tikzcd}
P \arrow[r, "f'"] \arrow[d, "g'"] & B \arrow[d, "g"] \\
A \arrow[r, "f"] & T
\end{tikzcd}$$

depicts a pullback then $P = A \times B$ and f' and g' are projections.

References

[1] Benjamin C. Pierce, Basic Category Theory for Computer Scientists, The MIT Press Cambridge, Massachusetts, London, England, 1991.

[2] Jacek Cichoń, List of exercises for functional programming: `https://cs.pwr.edu.pl/cichon/2018_19_b/Functional/Zadania/Functional_2019.pdf`