POWODZEINA!!!!!!!!!
1.

Exercise 1 (Currying). Show that there is a bijection between sets $A^{B\times C}$ and $(A^B)^C$.

(1) Let $f : C \times B \to A$ and $g : A^{B^C}$ such that

$$(g(x))(y) = f(x,y)$$

Then $\Phi : A^{B\times C} \longrightarrow A^{B^C}$ given by $((\phi(f))(x))(y) = f(x,y)$ is a bijection.

Exercise 2. Show that the following diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"] & B \arrow[r, "g"] \arrow[d, "a"] & C \arrow[d, "h"] \\
D \arrow[r, "j"] & E \arrow[r, "k"] & F
\end{tikzcd}$$

commutes if and only if its components

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"] & B \arrow[d, "a"] \\
D \arrow[r, "j"] & E
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, "a"] & C \arrow[d, "h"] \\
E \arrow[r, "k"] & F
\end{tikzcd}$$

commute.

②

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"] & B \arrow[r, "g"] \arrow[d, "a"] & C \arrow[d, "h"] \\
D \arrow[r, "j"'] & E \arrow[r, "k"'] & F
\end{tikzcd} \quad ***$$

$$k \circ j \circ i = k \circ a \circ f = h \circ g \circ f$$

$$* \quad \begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"] & B \arrow[d, "a"] \\
D \arrow[r, "j"'] & E
\end{tikzcd} \qquad \begin{tikzcd}
B \arrow[r, "g"] \arrow[d, "a"] & C \arrow[d, "h"] \\
E \arrow[r, "k"'] & F
\end{tikzcd} \quad **$$

if $j \circ i = a \circ f$ then $k \circ j \circ i = k \circ a \circ f$ | if $k \circ a = h \circ g$ then $k \circ a \circ f = h \circ g \circ f$

but if $*$ or $**$ don't commute then $***$ will not commute e.g. $j \circ i \neq a \circ f \Rightarrow k \circ j \circ i \neq k \circ a \circ f$

Exercise 3. Determine initial and terminal objects in

a) SET with sets as objects and functions as arrows.

b) GROUP with groups as objects and homomorphisms as arrows.

3.

③

For ~~sets~~ SET an initial object is an empty set $\emptyset$ and terminal object is ~~a~~ set with only one element $\{x\}$.

For GROUP an initial object is neutral element e and it is also a terminal object.

Exercise 4. Show that every monoid $(M, \cdot)$ can be treated as a category with one object and vice versa - each single object category gives rise to a monoid.

4.

(4)

Monoid $(M, \cdot, e)$:

1a) $\forall_{x,y} \in M \quad x \cdot y \in M$

2a) $\forall_{x,y,z} \in M \quad x \cdot (y \cdot z) = (x \cdot y) \cdot z$

3a) $\forall_{x} \in M \quad x \cdot e = e \circ x = x$

Category ($\mathcal{C} = (Obj(\mathcal{C}), Arr(\mathcal{C}))$

$A, B \in Obj(\mathcal{C})$

1b) $\exists\, f \in Arr(\mathcal{C}) \quad f: A \to B$

2b) $\begin{matrix} f: A \to B \\ g: B \to C \end{matrix} \quad \forall_{f,g} \quad g \circ f : A \to C$

3b) $\forall_{A \in Obj(\mathcal{C})}$ exists $id A \cdot \forall_{g: A \to B}$ $g \circ id A = g$

Monoid as Category:

ad 1b) There is only one object M and we know 1a so each $f \in Arr(\mathcal{C})$ $f: M \to M$

ad 2b) There is only one object and all arrows fulfil $f: M \to M$

ad 3b) There is only one object.

Exercise 5. How to treat a partial order $(X, \preceq)$ as a category? Localize initial and terminal objects. What are products and co-products of objects of this category? Do products always exists for posets? What if $\preceq$ is a linear order?

5.

5

If $(X, \leq)$ is a poset then $\mathcal{C} = (\mathrm{Obj}(\mathcal{C}) = X,$
$\mathrm{Arr}(\mathcal{C}) = \{(x,y) : x, y \in X \wedge x \leq y\})$

Terminal object is the greatest element of X,
initial object is the smallest element of X.

Product

Exercise 1. Let $(X, \preceq)$ be a poset. What can be said about initial and terminal objects? Do they always exist for any poset? Which arrows are isomorphisms? Illustrate these questions using some examples of known to you partial orders.

Solution. Recall that if $(X, \preceq)$ is a poset then we may treat it as a category $\mathcal{C}$ with $\mathrm{Obj}(\mathcal{C}) = X$ and $\mathrm{Arr}(\mathcal{C}) = \{(x, y) : x, y \in X \wedge x \preceq y\}$. The definition of, say, terminal objects says that T is a terminal object if

$$(\forall A \in \mathrm{Obj}(\mathcal{C}))(\exists! f \in \mathrm{Arr}(\mathcal{C}))(f : A \to T).$$

Since arrows are of the form (x, y) with $x \preceq y$, for our poset it means that

$$(\forall x \in X)(x \preceq T),$$

i.e. T is the greatest element. Similarly an initial object in a poset is the least element in the poset. Such elements not always exist, e.g. intervals $(0, 1), [0, 1), (0, 1], [0, 1]$ equipped with the usual order $\leqslant$ exhaust all possible combinations of possessing/not possessing the least/greatest elements.

Arrows that are isomorphisms are inversible. If we consider an arrow (x, y) whose inverse (y, x) is a legitimate arrow then $x \preceq y$ and $y \preceq x$, hence by antisymmetry $x = y$. Therefore the only arrows that are isomorphisms are of the form (x, x). $\square$

Common mistakes:

- Minimal and maximal instead of the least and the greatest.
- Not giving proper arguments for the claims.
- Confusing order isomorphisms (*between* posets) with isomorphisms in a poset as a category.

Exercise 6. Consider the category where objects are sets and arrows are pairs of sets (A, B) such that $A \subseteq B$. Determine initial and terminal objects, products and co-products, if they exist.

So, a product of two objects in a poset is actually the greatest object which is both smaller than both (also called the greatest lower bound). It is worth noting that in a total ordering, this corresponds to the function `min(a, b)`, since every object must be relateable to any other object (Wolfram call this trichotomy law).

Analog to the product definition, the coproduct corresponds to the smallest object greater than or equal to both `a` and `b`. In a total ordering, this corresponds to the maximum of both objects. You can work this one out for yourself.

6. Initial and terminal is empty set: terminal doesn't exist

In the category **Rel** of sets and relations, the empty set is the unique initial object, the unique terminal object, and hence the unique zero object.

produkt to czesc wspolna bo z czesci wspolnej fragment zawiera się i w A i w B
a co-produkt to suma bo A i B zawiera się w sumie A u B

Exercise 7. Consider the category $\mathcal{C}$ where Obj($\mathcal{C}$)= $\mathbb{N}^+$ and $\mathrm{Arr}(\mathcal{C}) = \{(k, n) : k|n\}$. Determine initial and terminal objects, products and co-products, if they exist.

7. initial 1? [(1,n) 1|n] dokładnie jedno takie I->A
terminal takie ze x->T jest dokladnie jedno to może zero bo wszystko jest dzielnikiem zera;
produkt to GCD (NWD - najwiekszy wspólny dzielnik)
co-produkt to NWW najmniejsza wspólna wielokrotność

Exercise 8. Show that if arrows $f : A \to B$ and $g : B \to C$ are monic then so is $g \circ f$. Conversely, if $g \circ f$ is monic then so is f.

8.

1.3.2 *Goal [1/2]:* $\forall f, g.\ f$ monic $\wedge\ g$ monic $\Rightarrow g \circ f$ monic

- Suffices to show: $\forall a, b.\ (g \circ f) \circ a = (g \circ f) \circ b \Rightarrow a = b$
(by definition $g \circ f$ monic)
- Assume $(g \circ f) \circ a = (g \circ f) \circ b$
- $g \circ (f \circ a) = g \circ (f \circ b)$
(by associativity)
- $f \circ a = f \circ b$
(by g monic)
- Conclude $a = b$
(by f monic) □

Goal [2/2]: $\forall f, g.\ g \circ f$ monic $\Rightarrow f$ monic

- Suffices to show: $\forall a, b.\ f \circ a = f \circ b \Rightarrow a = b$
- Assume $f \circ a = f \circ b$
- $g \circ (f \circ a) = g \circ (f \circ b)$
(because g is an arrow)
- $(g \circ f) \circ a = (g \circ f) \circ b$
(by associativity)
- Conclude $a = b$
(by $g \circ f$ monic) □

Exercise 9. State and prove analogous results for epic arrows. Be careful with the second part.

9.

1.3.3 *Goal [1/2]:* $\forall f, g.\ f$ epic $\wedge\ g$ epic $\Rightarrow g \circ f$ epic

- Suffices to show: $\forall a, b.\ a \circ (g \circ f) = b \circ (g \circ f) \Rightarrow a = b$
(by definition $g \circ f$ epic)
- Assume $a \circ (g \circ f) = b \circ (g \circ f)$
- $(a \circ g) \circ f = (b \circ g) \circ f$
(by associativity)
- $a \circ g = b \circ g$
(by f epic)

- Conclude $a = b$
  (by g epic) □

Goal [2/2]: $\forall f, g.\, g \circ f$ epic $\Rightarrow g$ epic

- Suffices to show: $\forall a, b.\, a \circ g = b \circ g \Rightarrow a = b$
- Assume $a \circ g = b \circ g$
- $(a \circ g) \circ f = (b \circ g) \circ f$
  (because $f = f$)
- $a \circ (g \circ f) = b \circ (g \circ f)$
  (by associativity)
- Conclude $a = b$
  (by $g \circ f$ epic) □

Exercise 10. Show that for the SET category epimorphisms are surjections and monomorphisms are injections.

10.

1.3.1 To show that the epimorphisms in **Set** are the surjective functions, let $f : A \to B$ be a surjective function and let $g : B \to C$ and $h : B \to C$ be other arrows in **Set**. First, assume both that g and h are equal and that $g \circ f \neq h \circ f$ to derive a contradiction. Let x be an element of A such that $g(f(x)) \neq h(f(x))$. If $f(x) = y$ we have that $g(y) \neq h(y)$, but this contradicts the assumption that g and h are equal.

For the converse, assume that $g \circ f = h \circ f$ but that $g \neq h$. The first equality states that $\forall x \in A \,.\, g(f(x)) = h(f(x)) \to g(y) = h(y)$ if $y = f(x)$. Because f is surjective, this derivation shows that g and h agree on all elements of their common domain, thus contradicting the assumption that $g \neq h$.

Sufficient Condition

Suppose that $f : X \twoheadrightarrow Y$ is an epimorphism.

By definition of surjection, it will suffice to show that:

$$\forall y \in Y : \exists x \in X : f \left({x}\right) = y$$

Let us reason by contradiction.

So suppose f were not surjective.

Then there would be an $y_0 \in Y$ such that:

$$\forall x \in X : f \left({x}\right) \ne y_0$$

Consider the mappings defined by:

$$g : Y \to Y \cup \left\{ {Y}\right\}, g \left({y}\right) := y$$
$$h : Y \to Y \cup \left\{ {Y}\right\}, h \left({y}\right) := \begin{cases} y & \text{if } y \ne y_0 \\ Y & \text{if } y = y_0 \end{cases}$$

The assumption on f yields that $g \circ f = h \circ f$.

Since $h \ne g$, it follows that f cannot be epic.

This contradiction shows that f is necessarily surjective.

$\blacksquare$

1.3.2 Proposition In **Set**, the monomorphisms are just the injective functions (the functions f such that $f(x) = f(y)$ implies $x = y$.)

Proof: Let $f : B \to C$ be an injective function, and let $g, h : A \to B$ be such that $f \circ g = f \circ h$ but $g \neq h$. Then there is some element $a \in A$ for which $g(a) \neq h(a)$. But since f is injective, $f(g(a)) \neq f(h(a))$, which

contradicts our assumption that $f \circ g = f \circ h$. This shows that f is a monomorphism.

Conversely, let $f : B \to C$ be a monomorphism. If f is not injective, then there are distinct elements $b, b' \in B$ for which $f(b) = f(b')$. Let A be the one-element set $\{a\}$, and let $g : A \to B$ map a to b while $h : A \to B$ maps a to b'. Then $f(g(a)) = f(h(a))$, contradicting the assumption that f is a monomorphism. *(End of Proof)*

Exercise 11. Show that each isomorphism is also an epimorphism and a monomorphism, however (contrary to intuition from the previous exercise) not every arrow which is a monomorphism and epimorphism is an isomorphism.

11.

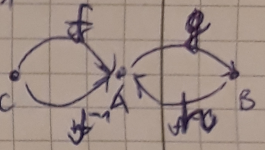
(11) iso ~~is~~ is monic and epic

Let $f: A \to B$ be a isomorphism

Then for any C, g, h:

epic:

$$g \circ f = h \circ f \quad / \circ f^{-1}$$



$$g \circ f \circ f^{-1} = h \circ f \circ f^{-1}$$

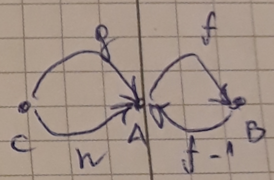
$$g \circ id = h \circ id$$

$$g = h \quad \boxtimes$$

monic:

$$f \circ g = f \circ h \quad / f^{-1} \circ$$

$$f^{-1} \circ f \circ g = f^{-1} \circ f \circ h$$

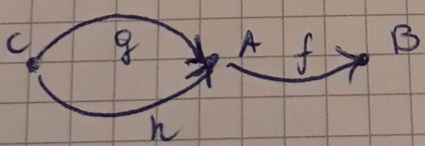


$$id \circ g = id \circ h$$

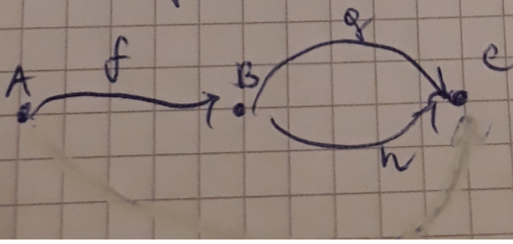
$$g = h \quad \boxtimes$$

$A \to B$ iso where $f: A \to B$ that $g: B \to A$ with $g \circ f = id_A$ $f \circ g = id_B$

monic, not isomorphism



epic, not isomorphism



W pierwszej czesci zakladamy ze f jest izomofrizmem ale wcale nie musi byc
ale przecież w zadaniu bylo, że trzeba wziąć izomorfizm
XD

Exercise 12. Consider a category derived from a partial order. Which arrows are monic/epic/isomorphisms?

12.

Proposition 1.18. *In a fixed poset category P, every arrow $p \leq q$ is both monic and epic.*

Proof. By construction, a poset category has at most one arrow between any two objects. Thus, any parallel arrows must be equal. This makes all arrows trivially monic and epic. □

id is isomorphism (x<=x)

Exercise 13. Show that $f : \mathbb{N} \to \mathbb{Z}$ given by $f(n) = n$ is epic as an arrow from the monoid $(\mathbb{N}, +, 0)$ to $(\mathbb{Z}, +, 0)$.

13.

1.3.6 Example Both $(\mathbf{Z}, +, 0)$, the monoid of integers under addition, and $(\mathbf{N}, +, 0)$, the monoid of nonnegative integers under addition, are objects of the category **Mon**. The inclusion function $i : (\mathbf{N}, +, 0) \to (\mathbf{Z}, +, 0)$ that maps each nonnegative integer z to the integer z is a monomorphism, as we would expect by analogy with **Set**. But i is also an epimorphism, although it is clearly *not* surjective. To see this, assume that $f \circ i = g \circ i$ for two homomorphisms f and g from $(\mathbf{Z}, +, 0)$ to some monoid $(M, *, E)$. Take any $z \in \mathbf{Z}$. If $z \geq 0$, then it is the image under i of the same z considered as an element of $\mathbf{N}$, so

$$f(z) = f(i(z)) = g(i(z)) = g(z).$$

If $z < 0$, then $-z \geq 0$ and $-z \in \mathbf{N}$; we reason as follows:

$$\begin{aligned}
f(z) &= f(z) * E \\
&= f(z) * g(0) \\
&= f(z) * g(-z + z) \\
&= f(z) * (g(-z) * g(z)) \\
&= (f(z) * g(-z)) * g(z) \\
&= (f(z) * g(i(-z))) * g(z) \\
&= (f(z) * f(i(-z))) * g(z) \\
&= (f(z) * f(-z)) * g(z) \\
&= f(z + -z) * g(z) \\
&= f(0) * g(z) \\
&= E * g(z) \\
&= g(z).
\end{aligned}$$

Since $f(z) = g(z)$ for all z, we have $f = g$; so i is an epimorphism.

Exercise 14. Let $(M, \cdot, e_1)$ and $(N, \star, e_2)$ be monoids. Show that the product monoid $(M, \cdot, e_1) \otimes (N, \star, e_2) = (M \times N, \cdot \times \star, (e_1, e_2))$ is actually a monoid.

14.

14. $X = (M \times N, \overbrace{\cdot \times \star}^{\circ}, (e_1, e_2))$

checking monoid axioms:

1) $\forall (m_1, m_2), (n_1, n_2) \in (m_1, m_2) \circ (n_1, n_2) = (m_1 \cdot n_1, m_2 \star n_2) \in X$ ✓

2) $\forall (m_1, m_2)\ (n_1, n_2)\ (a_1, a_2) \in X$

$(m_1, m_2) \circ ((n_1, n_2) \circ (a_1, a_2)) = \cancel{(m_1, m_2) \circ ((n_1 \cdot a_1}$

$= (m_1, m_2) \circ (n_1 \cdot a_1, n_2 \star a_2) =$

$= (m_1 \cdot (n_1 \cdot a_1), m_2 \star (n_2 \star a_2)) =$

$= ((m_1 \cdot n_1) \cdot a_1, (m_2 \star n_2) \star a_2) =$

$= (m_1 \cdot n_1, m_2 \star n_2) \circ (a_1, a_2) = ((m_1, m_2) \circ (n_1, n_2)) \circ (a_1, a_2)$ ✓

3) $\forall (m_1, m_2) \in X$

$(m_1, m_2) \circ (e_1, e_2) = (m_1 \cdot e_1, m_2 \star e_2) = (e_1 \cdot m_1, e_2 \star m_2) =$

$= (e_1, e_2) \circ (m_1, m_2) = (m_1, m_2)$ ✓

Exercise 15. Are $(\mathbb{Z},+,0)$ and $(\mathbb{Z},\cdot,1)$ isomorphic? Same question for $(\mathbb{Z},+,0)$ and $(\mathbb{Z},+,0)\otimes(\mathbb{Z},+,0)$.

15. z + tak bo jest takie coś w żeby wykonując działanie otrzymać id natomiast * nie bo jesteśmy w liczbach całkowitych, a nie wymiernych i nie ma takiego 1/n żeby * n dał id

a) nie są bo musiałaby istnieć homomorfizm (który jest bijekcją):
f(0)=1
f(a+b)=f(a)*f(b) => musi to być jakaś eksponenta. ale żadna nie jest bijekcją w Z, więc nie są.

b)to samo:f(a)=(f1(a),f2(a))
f(0)=(0,0)
f(a+b)=f(a)+f(b)=(f1(a),f2(a))+(f1(b),f2(b))=(f1(a)+f1(b),f2(a)+f2(b))
taką własność ma tylko liniowa funkcja:
f(a)=(c1*a,c2*a) (nie jest ona bijekcją)

Exercise 16. Let L be a list functor. Show that for every set X a list reversing function $\mathrm{reverse}_X$ is a natural transformation.

16.

$\boxed{\text{Example.}}$ Let L be a List constructor. Let X, Y be sets and let $f : X \to Y$. Let's find a nat. trans $\eta : L \to L$.

$$\begin{tikzcd}
L(X)=X^* \arrow[r, "L(f)"] \arrow[d, "\eta_X"'] \arrow[dr, phantom, "\circlearrowright"] & L(Y)=Y^* \arrow[d, "\eta_Y"] \\
X^*=L(X) \arrow[r, "L(f)"] & L(Y)=Y^*
\end{tikzcd}$$

$L(f)\circ\eta_X = \eta_Y\circ L(f)$. Boring: $\eta_X = id_{X^*}$. Better: $\eta_X = \mathrm{reverse}_{X^*}$

$\eta_Y \circ L(f)[x_1,\ldots,x_n] = \eta_Y[f(x_1), f(x_2),\ldots,f(x_n)] = [f(x_n), f(x_{n-1}),\ldots,f(x_1)]$

$L(f)\circ \eta_X [x_1,\ldots,x_n] = L(f)[x_n,x_{n-1},\ldots,x_1] = [f(x_n), f(x_{n-1}),\ldots,f(x_1)]$

$\eta_X = \mathrm{reverse}_{X^*}$

Exercise 17. Let $(\mathcal{P}, \preceq)$ be a preorder (partial order without antisymmetry), $\mathcal{C}$ any category and $F, G : \mathcal{C} \to \mathcal{P}$ functors. Show that there is exactly one natural transformation $\eta : F \dot{\to} G$ if and only if $F(A) \preceq G(A)$ for every $A \in \mathrm{Obj}(\mathcal{C})$.

17.
There is always max 1 arrow for preorder and it exists if F=<G. So natural transformation exists iff for every A: F(A) = < F(G).

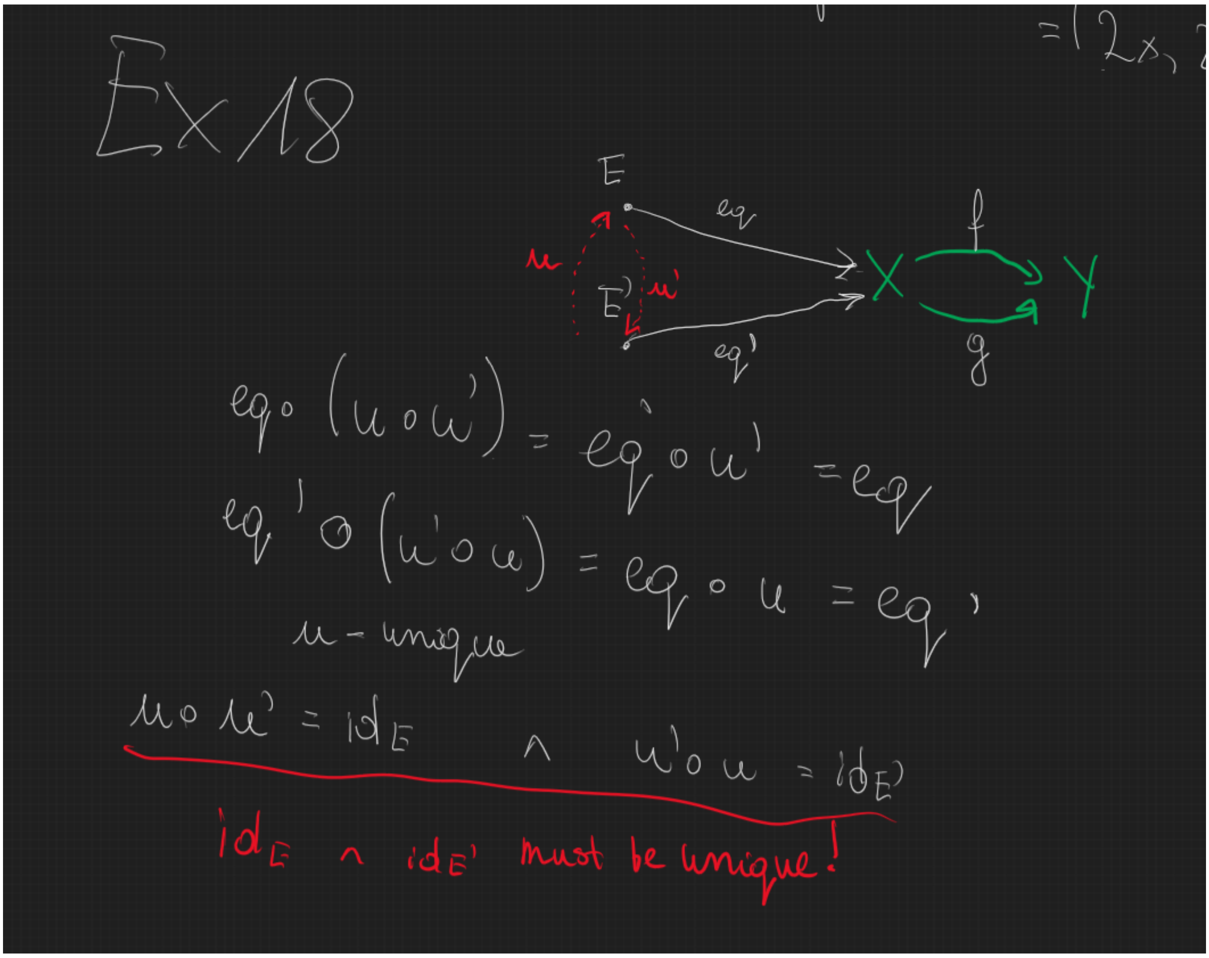
Exercise 18. Show that equalizers and pullbacks are unique up to isomorphisms.

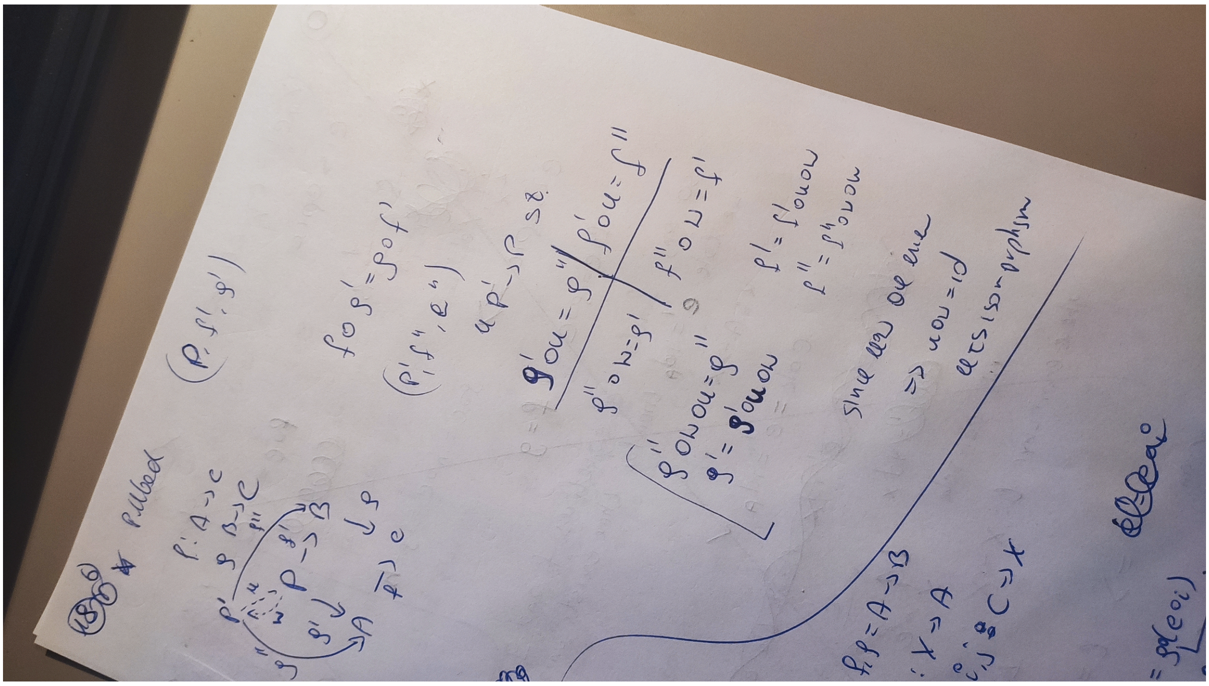
18.

This is a typical use of the universal property of the equaliser: if E and E' are both equalisers of $f, g : X \to Y$, with respective morphisms $eq : E \to X$ (resp. $eq' : E' \to X$), then (somewhat following the Wikipedia notation) there are unique morphisms $u : E' \to E$ and $u' : E \to E'$ such that $eq \circ u = eq'$ and $eq' \circ u' = eq$. In fact these are mutually inverse isomorphisms: we have $eq \circ (u \circ u') = eq' \circ u' = eq$ and $eq' \circ (u' \circ u) = eq \circ u = eq'$, and so by uniqueness we conclude that $u \circ u' = id_E$ and $u' \circ u = id_{E'}$.

The same argument shows more generally that objects defined by a universal property are unique up to a unique isomorphism in any category.

We do Pullback analogically!





Exercise 19. Show that if (X, e) is an equalizer then e is a monic arrow.

19.

Proposition 3.10. *Equalizer arrows are monic.*

Proof. Given the equalizer diagram $E \xrightarrow{e} A \overset{f}{\underset{g}{\rightrightarrows}} B$, suppose we have parallel arrows $z_1, z_2 : Z \to E$ such that $ez_1 = ez_2$. By precomposition, $fez_1 = gez_1 : Z \to A$. Thus, by the universal mapping property of equalizers, there exists a unique arrow $u : Z \to E$ such that $eu = ez_1$. Thus, $u = z_1$. But since $ez_1 = ez_2$, then $u = z_2$. Therefore, $z_1 = z_2$, so e is monic. $\square$

Exercise 20. Show that if (X, e) is an equalizer and e is epic then e is an isomorphism.

$$
\begin{tikzcd}
C \arrow[d, shift right, "v"'] \arrow[d, shift left, "u"] \arrow[dr, bend left, "e\circ u = e\circ v"] & & \\
K \arrow[r, "e"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B
\end{tikzcd}
$$

unique

Monic

$(e \circ u = e \circ v)$ implies $u = v$

20.
Zadanie wyżej dowód że jest monic jeżeli dodatkowo jest epic to jest iso:

As e equalises f and g, $f \circ e = g \circ e$.

Since e is epic, it follows that $f = g$.

Then in the equaliser diagram:

$$
\begin{tikzcd}
E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
A \arrow[u, dotted, "k"] \arrow[ur, "\mathrm{id}_A"'] & &
\end{tikzcd}
$$

We have that:

$$f \circ 1_A = g \circ 1_A$$

so there is a unique k with $e \circ k = \mathrm{id}_A$.

Then:

$$
\begin{aligned}
e \circ k \circ e &= \mathrm{id}_A \circ e \\
&= e \\
&= e
\end{aligned}
$$

e \circ \operatorname{id}_E

By Equalizer is Monomorphism, it follows that $k \circ e = \mathrm{id}_E$.

This gives k as an inverse to e.

Thus e is an isomorphism.

■

20/21

Monic from def

if epic $f=g$

$$\begin{tikzcd}
A \arrow[d, "u"'] \arrow[dr, "id_A"] & & \\
K \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[l, bend left, "u"] & B
\end{tikzcd}$$

unique

$e \circ u = id_A$

$e \circ u \circ e = id_A \circ e = e$

$u \circ e = id_E$

unique

20.

Zadanie wyżej dowód że jest monic jeżeli dodatkowo jest epic to jest iso:

As e equalises f and g, $f \circ e = g \circ e$.

Since e is epic, it follows that $f = g$.

Then in the equaliser diagram:

$$\begin{tikzcd}
E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
A \arrow[u, dotted, "k"] \arrow[ur, "\mathrm{id}_A"'] & &
\end{tikzcd}$$

We have that:

$f \circ 1_A = g \circ 1_A$

so there is a unique k with $e \circ k = \mathrm{id}_A$.

Then:

$$\begin{aligned} e \circ k \circ e &= \mathrm{id}_A \circ e \\ &= e \\ &= e \end{aligned}$$

e \circ \operatorname{id}_E

By Equalizer is Monomorphism, it follows that $k \circ e = \mathrm{id}_E$.

This gives k as an inverse to e.

Thus e is an isomorphism.

■

As $f=g$ we can put A into diagram

bcs $f \circ id_A = g \circ id_A$

$f \circ id = f$ $\quad g \circ id = g$ $\quad$ so $\quad g=f$

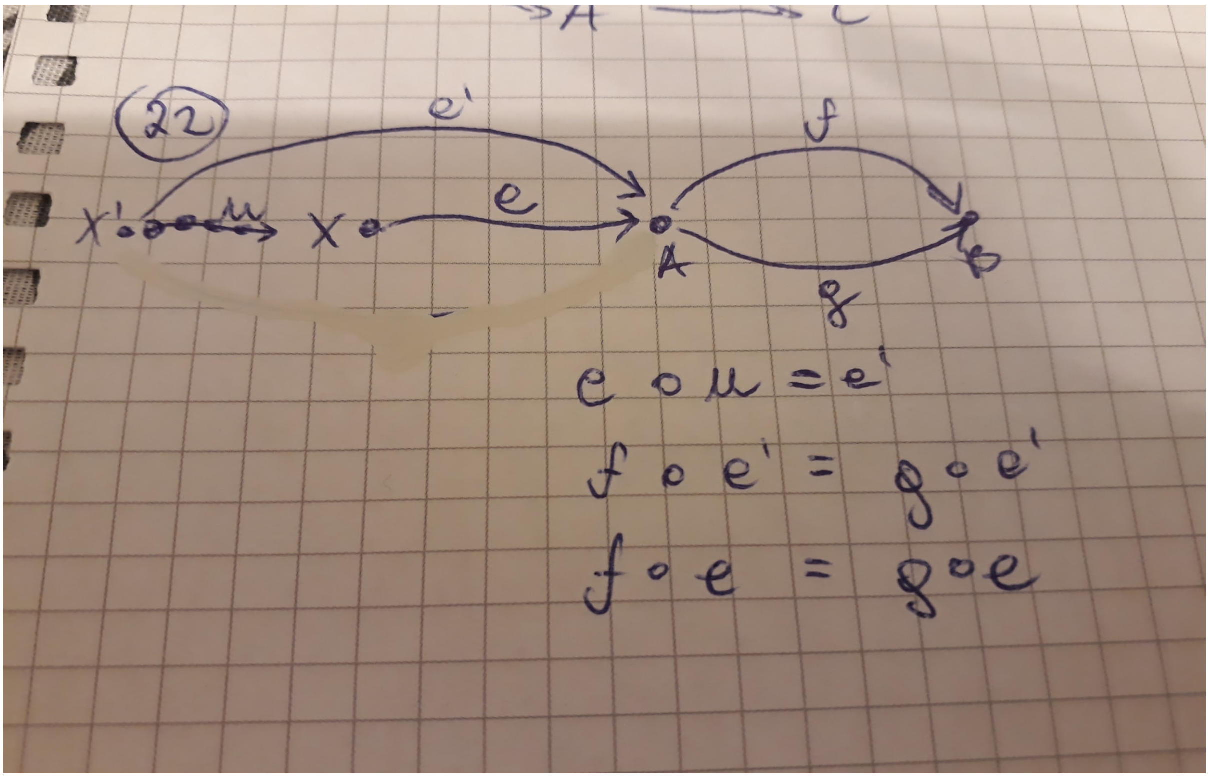
Exercise 21. Show that if $f = g$ and (X, e) is the equalizer for arrows f, g then e is an isomorphism.

21.

If f=g for equalizer (X,e) then e is epic. Then proof from ex 20.

Exercise 22. Describe equalizers in the category SET, i.e. for functions $f, g : A \to B$ find the equalizer (X, e) of these functions.

22.



In category theory [edit]

Equalisers can be defined by a universal property, which allows the notion to be generalised from the category of sets to arbitrary categories.

In the general context, X and Y are objects, while f and g are morphisms from X to Y. These objects and morphisms form a diagram in the category in question, and the equaliser is simply the limit of that diagram.

In more explicit terms, the equaliser consists of an object E and a morphism $eq : E \to X$ satisfying $f \circ eq = g \circ eq$, and such that, given any object O and morphism $m : O \to X$, if $f \circ m = g \circ m$, then there exists a unique morphism $u : O \to E$ such that $eq \circ u = m$.

$$\begin{tikzcd}
E \arrow[r, "eq"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\
O \arrow[u, dashed, "u"] \arrow[ur, "m"'] & &
\end{tikzcd}$$

A morphism $m : O \to X$ is said to **equalise** f and g if $f \circ m = g \circ m$.[1]

Example 40 In **Set**, the equaliser of f, g is given by the inclusion

$$\{x \in A \mid f(x) = g(x)\} \hookrightarrow A\,.$$

This allows *equationally defined subsets* to be defined as equalisers. For example, consider the pair of maps $\mathbb{R}^2 \overset{f}{\underset{g}{\rightrightarrows}} \mathbb{R}$, where

$$f : (x, y) \mapsto x^2 + y^2, \qquad g : (x, y) \mapsto 1\,.$$

Then, the equaliser is the unit circle as a subset of $\mathbb{R}^2$.

Exercise 22. Describe equalizers in the category SET, i.e. for functions $f, g : A \to B$ find the equalizer (X, e) of these functions.

$$\begin{tikzcd}
X' \arrow[d, "u"'] \arrow[dr, bend left, "e'"] & & \\
X \arrow[r, "e"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B
\end{tikzcd}$$

unique

Example 40 In **Set**, the equaliser of f, g is given by the inclusion

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$$f : (x, y) \mapsto x^2 + y^2, \qquad g : (x, y) \mapsto 1\,.$$

Then, the equaliser is the unit circle as a subset of $\mathbb{R}^2$.

to spełnia to

$$\boxed{\begin{aligned} &f \circ e = g \circ e \\ &f \circ e' = g \circ e' \\ &e' = e \circ u \\ \Rightarrow\ &f(x) = g(x) \end{aligned}} \quad X = \{x \in A \mid \underline{f(x) = g(x)}\} \hookrightarrow A$$

jeżeli

$\hookrightarrow$ equaliser

Equalizer $(X, \subseteq)$!

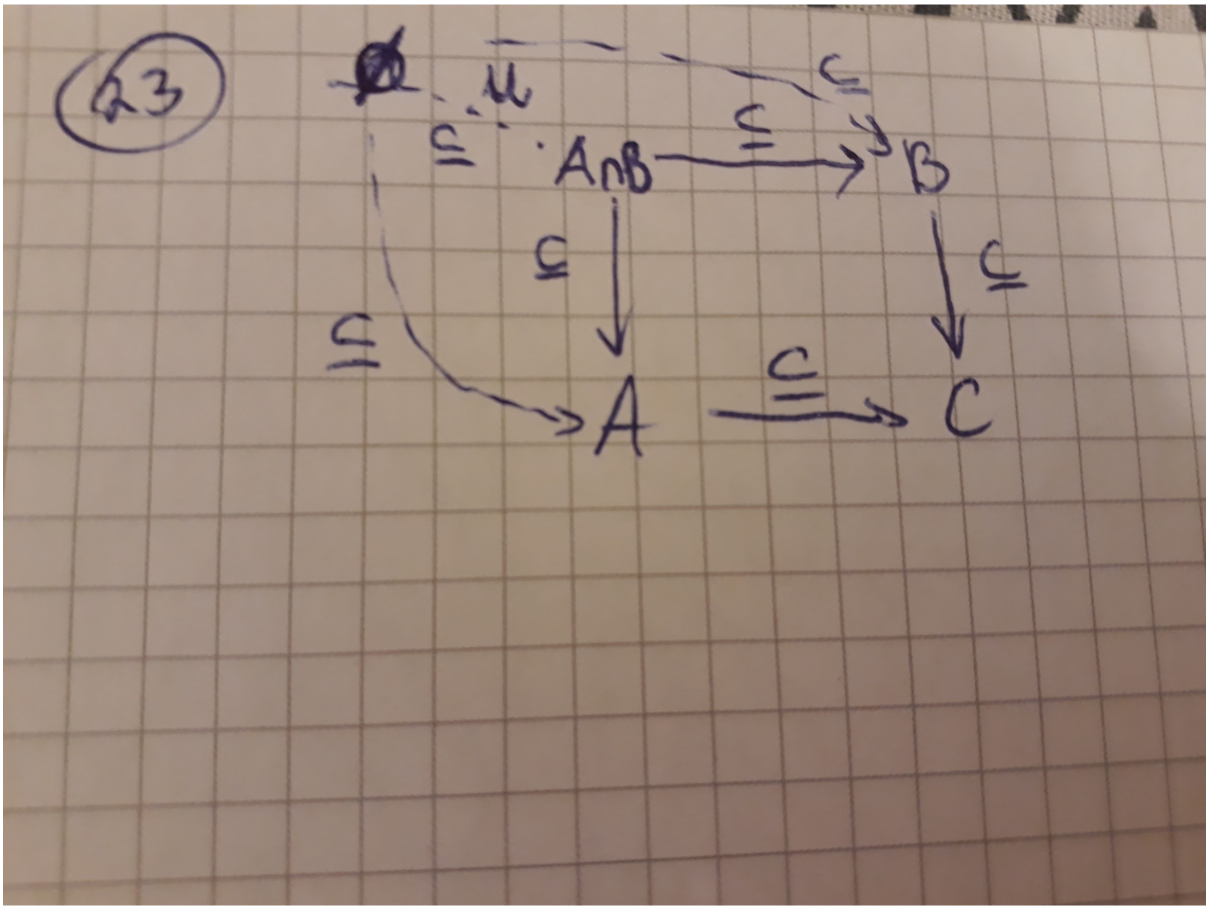
✓

Exercise 23. Complete the following diagram (and show it's correct)

$$\begin{tikzcd}
& B \arrow[d, "\subseteq"] \\
A \arrow[r, "\subseteq"] & C
\end{tikzcd}$$

to a pullback in the category with sets as objects and pairs of sets (X, Y) with $X \subseteq Y$ as arrows.

23.

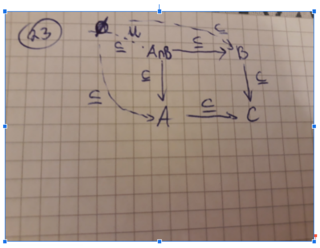


Exercise 23. Complete the following diagram (and show it's correct)

$$\begin{tikzcd} & B \arrow[d, "\subseteq"] \\ A \arrow[r, "\subseteq"] & C \end{tikzcd}$$

(D', D) $D' \subseteq D$

to a pullback in the category with sets as objects and pairs of sets (X, Y) with $X \subseteq Y$ as arrows.



$$\begin{tikzcd} D' \arrow[dr, dashed, "h = \subseteq"] \arrow[drr, bend left, "q' = \subseteq"] \arrow[ddr, bend right, "p' = \subseteq"'] & & \\ & D \arrow[r, "q"] \arrow[d, "p"] & B \arrow[d, "\subseteq = g"] \\ & A \arrow[r, "\subseteq = f"'] & C \end{tikzcd}$$

1) D: $A \xleftarrow{p} D \xrightarrow{q} B$

$f: A \to C \wedge g: B \to C$

$f \circ p = g \circ q$

– So we assume $p: \subseteq$ $q: \subseteq$ so $D = A \cap B$

if $A \subseteq C$ and $D \subseteq A$ then $D \subseteq C$

if $B \subseteq C$ and $D \subseteq B$ then $D \subseteq C$

and so $D = A \cap B$

2) $A \xleftarrow{p'} D' \xrightarrow{q'} B$

$f \circ p' = g \circ q'$

3) there exist unique $h: D' \to D$ such $p' = p \circ h$ and $q' = q \circ h$

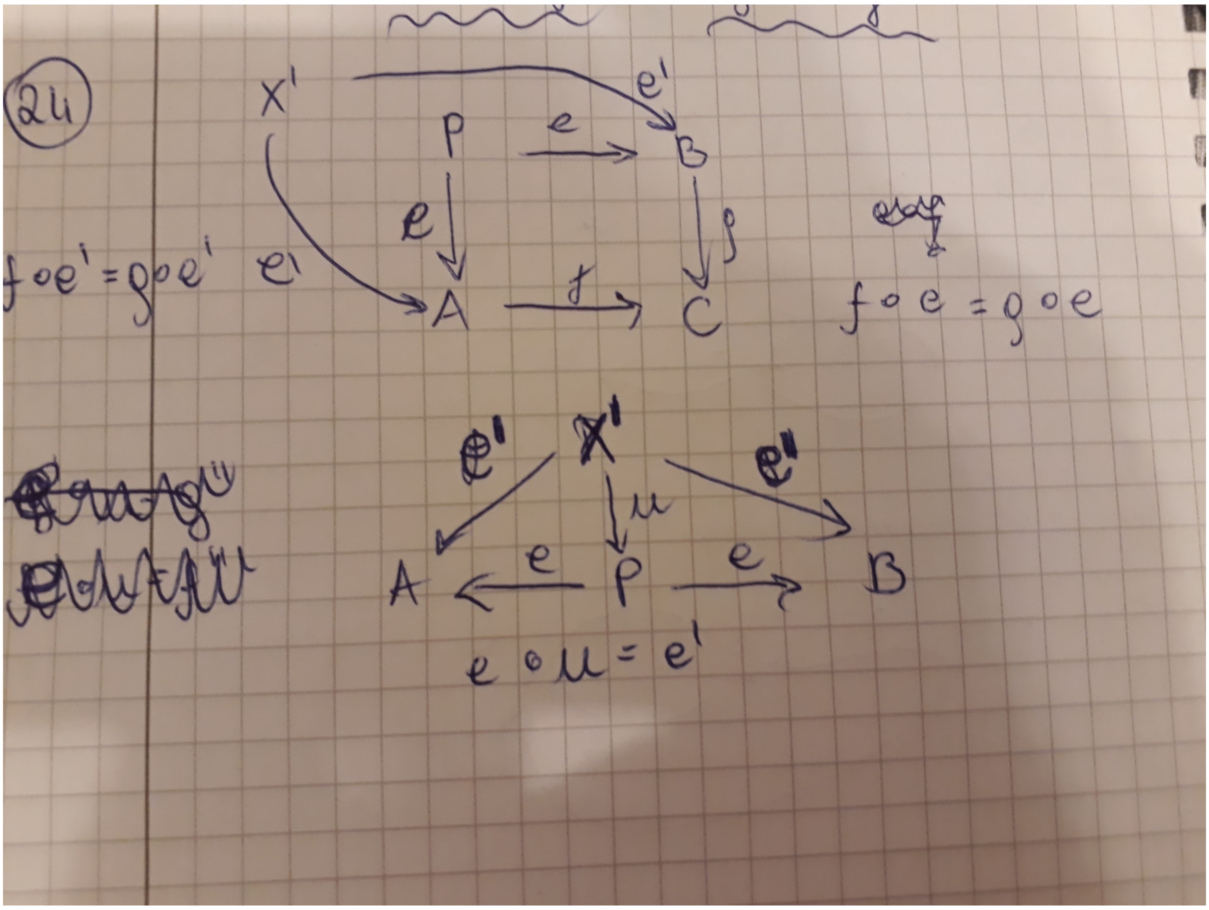
Każdy inny zbiór D': $D' \subseteq A \wedge D' \subseteq B \Rightarrow D' \subseteq D = A \cap B$

Exercise 24. Show that if the following diagram

$$\begin{tikzcd}
P \arrow[r, "e"] \arrow[d, "e"] & B \arrow[d, "g"] \\
A \arrow[r, "f"] & C
\end{tikzcd}$$

depicts a pullback then (P, e) is the equalizer of arrows f and g.

24.



if
$$\begin{tikzcd} P \arrow[r, "e"] \arrow[d, "e"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$
Depicts a pullback of arrows f and g then (P, e) is an equalizer

$$\begin{tikzcd} P' \arrow[dr, dashed, "h\text{-uniq}"] \arrow[drr, bend left, "e'"] \arrow[ddr, bend right, "e'"'] & & \\ & P \arrow[r, "e"] \arrow[d, "e"'] & B \arrow[d, "g"] \\ & A \arrow[r, "f"] & C \end{tikzcd}$$

$\underline{f \circ e = g \circ e} \wedge f \circ e' = g \circ e' \wedge e' = e \circ h \rightarrow$ from pullback

$A = B$ because p after $e = B = A$

Conditions to be an equaliser

1) $g \circ e = f \circ e \Rightarrow$ true

2) for any $e' : P' \rightarrow A$, $\underline{f \circ e' = g \circ e'}$ ✓ there is a unique $\underline{h : P' \rightarrow P}$ that $\underline{e' = e \circ h}$ ✓

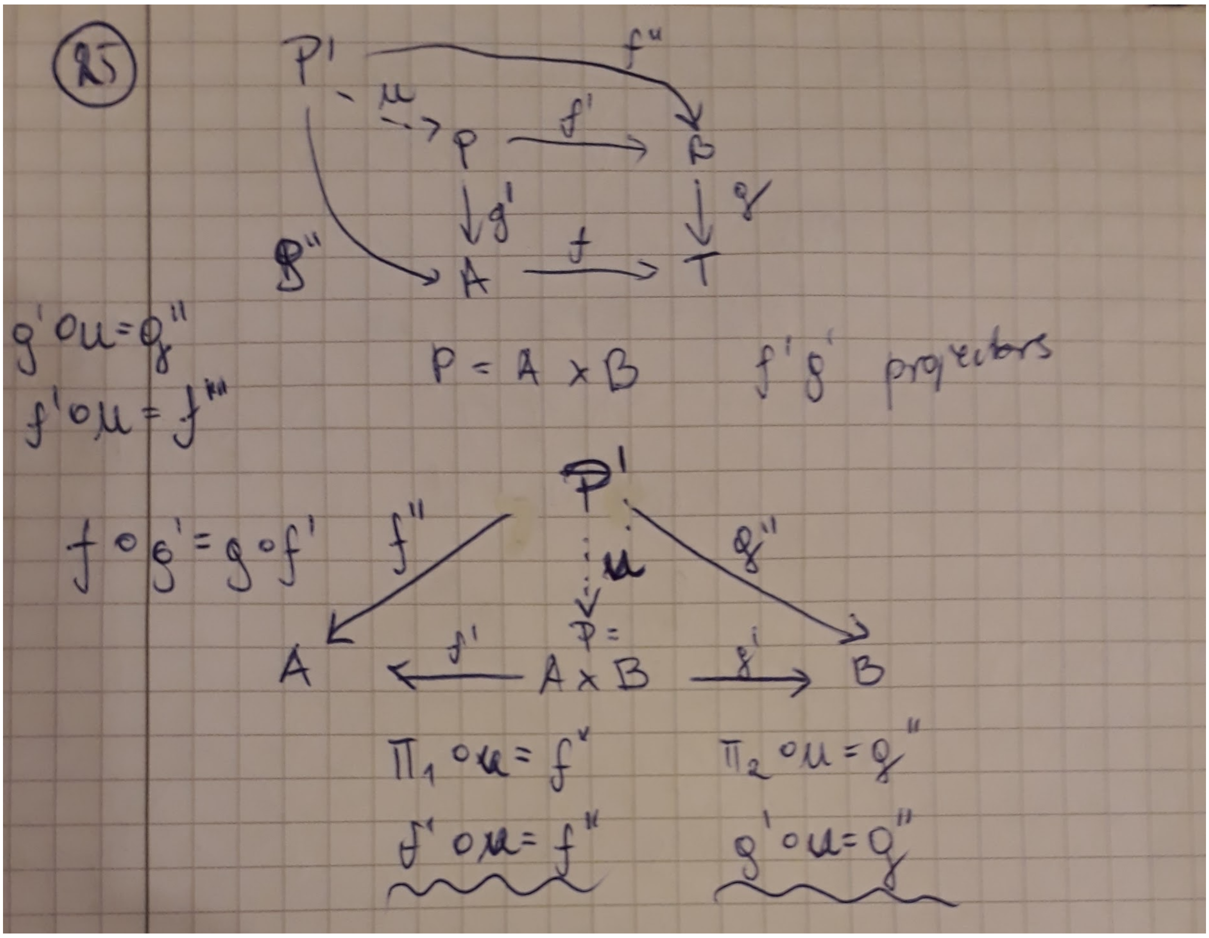
So (P, e) is an equalizer

Exercise 25. Let T be a terminal object. Show that if the following diagram

$$\begin{tikzcd} P \arrow[r, "f'"] \arrow[d, "g'"] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & T \end{tikzcd}$$

depicts a pullback then $P = A \times B$ and f' and g' are projections.

25.



https://math.stackexchange.com/questions/186737/terminal-objects-and-pullbacks
wszystkie rozwiązania używają definicji limitu….