Homework 3

Matic Stare

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Contents

1 Homology

a) Simplicial complex X:

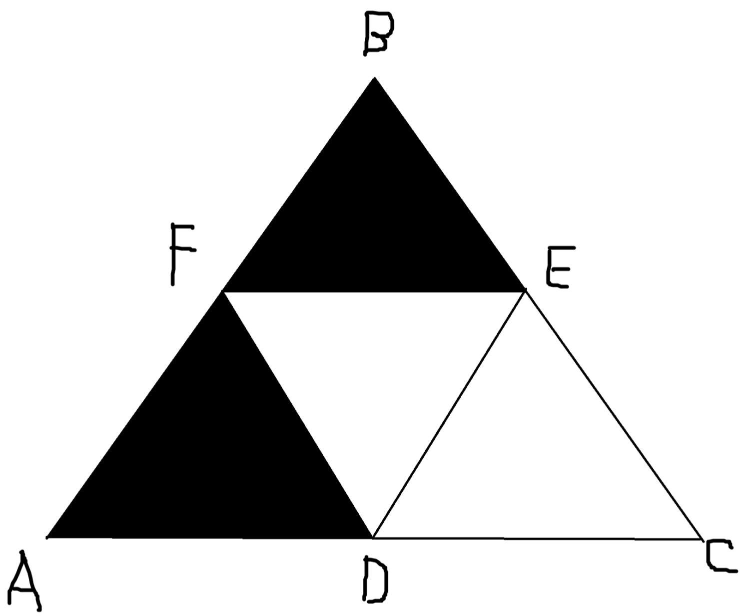


Figure 1: X drawn as a planar 2-dimensional simplicial complex.

b) Chain groups C_n:

- C_0: A, B, C, D, E, F
- C_1: AF, AD, DF, BF, EF, BE, DE, CD, CE
- C_2: ADF, BEF

c) The boundary homomorphisms connect the chain groups into a sequence:

$$\langle 0 \rangle \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} \langle 0 \rangle$$

$\partial_2(ADF) = AD + DF - AF$
$\partial_2(BEF) = BE + EF - BF$

$\partial_1(AF) = F - A$
$\partial_1(AD) = D - A$
$\partial_1(DF) = F - D$
$\partial_1(BF) = F - B$
$\partial_1(EF) = F - E$
$\partial_1(BE) = E - B$
$\partial_1(DE) = E - D$
$\partial_1(CD) = D - C$
$\partial_1(CE) = E - C$

$\partial_0(A) = 0$
$\partial_0(B) = 0$
$\partial_0(C) = 0$
$\partial_0(D) = 0$
$\partial_0(E) = 0$
$\partial_0(F) = 0$

d)
- Cycles Z_n:
  $Z_2 = \langle 0 \rangle$
  $Z_1 = \langle AD + DF - AF, BE + EF - BF, CD + DE - CE, DE + EF - DF \rangle$
  $Z_0 = \langle A, B, C, D, E, F \rangle$
- Boundaries B_n:
  $B_2 = \langle 0 \rangle$
  $B_1 = \langle AD + DF - AF, BE + EF - BF \rangle$
  $B_0 = \langle F - A, D - A, F - D, F - B, F - E, E - B, E - D, D - C, E - C \rangle$

e)
- $H_2(X;\mathbb{Z}) = \frac{Z_2}{B_2} = \frac{\langle 0 \rangle}{\langle 0 \rangle} = \langle 0 \rangle$
- $H_1(X;\mathbb{Z}) = \frac{Z_1}{B_1} = \frac{\langle AD+DF-AF,BE+EF-BF,CD+DE-CE,DE+EF-DF \rangle}{\langle AD+DF-AF,BE+EF-BF \rangle} = \langle CD + DE - CE, DE + EF - DF \rangle$
- $H_0(X;\mathbb{Z}) = \frac{Z_0}{B_0} = \frac{\langle A,B,C,D,E,F \rangle}{\langle F-A,D-A,F-D,F-B,F-E,E-B,E-D,D-C,E-C \rangle} = \langle A \rangle = \mathbb{Z}$

f)
- $H_2(X;\mathbb{Z}_2) = \frac{Z_2}{B_2} = \frac{\langle 0 \rangle}{\langle 0 \rangle} = \langle 0 \rangle$
- $H_1(X;\mathbb{Z}_2) = \frac{Z_1}{B_1} = \frac{\langle AD+DF+AF,BE+EF+BF,CD+DE+CE,DE+EF+DF \rangle}{\langle AD+DF+AF,BE+EF+BF \rangle} = \langle CD + DE + CE, DE + EF + DF \rangle$
- $H_0(X;\mathbb{Z}_2) = \frac{Z_0}{B_0} = \frac{\langle A,B,C,D,E,F \rangle}{\langle F+A,D+A,F+D,F+B,F+E,E+B,E+D,D+C,E+C \rangle} = \langle A \rangle = \mathbb{Z}_2$

g)
- The Betti numbers of X are $b_1 = 2$ and $b_0 = 1$
- The Euler characteristic of X is $\chi(X) = 1 - 2 = -1$

2 Discrete Morse Theory

a) Gradient vector field:

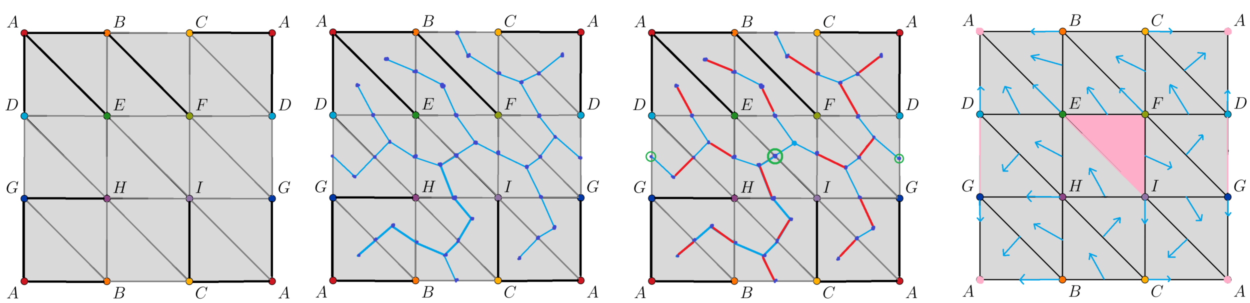


Figure 2: Construction of a vector field on T

b) Our vector field has critical simplices A, EI, DG, EFI, so $c_0 = 1$, $c_1 = 2$, $c_2 = 1$.
The Euler characteristic of T is $\chi(T) = 1 - 2 + 1 = 0$.

c) To compute $\partial_2 : M_2 \to M_1$, we count the 2-paths from $\partial(EFI) = EF + FI + EI$ to EI and DG. There are two 2-paths from FI to DG. There is one 2-path from EI to EI and one 2-path from FI to EI.

$\partial_2(EFI) = 2 \cdot DG + (1 + 1) \cdot EI = 0 \cdot DG + 0 \cdot GH = 0.$

To compute $\partial_1 : M_1 \to M_0$, we count the 1-paths from $\partial(DG) = D + G$ and $\partial(EI) = E + I$ to A. For each of D, G, E and I there is one 1-path to A.

$\partial_1(DG) = (1 + 1) \cdot A = 0$
$\partial_1(EI) = (1 + 1) \cdot A = 0$

Now we can compute the following:

$H_2(M;\mathbb{Z}) = \frac{Z_2(M)}{B_2(M)} = \frac{\ker\partial_2}{\mathrm{im}\partial_3} = \ker\partial_2 = \langle EFI\rangle \cong \mathbb{Z}$

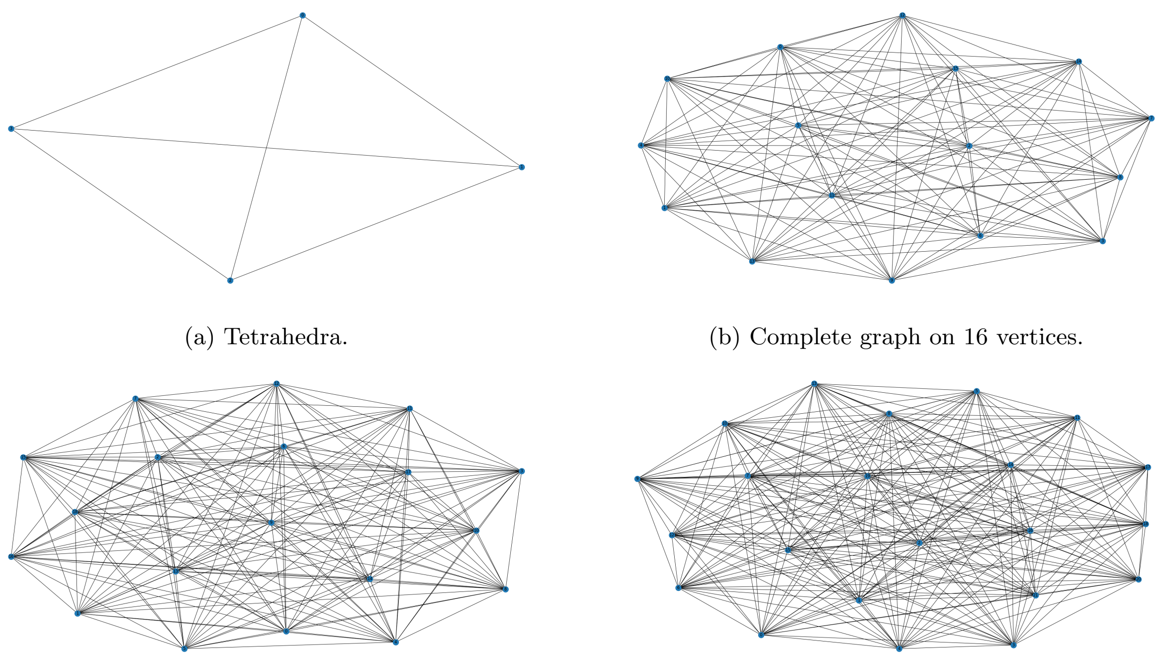
$H_1(M;\mathbb{Z}) = \frac{Z_1(M)}{B_1(M)} = \frac{\ker\partial_1}{\mathrm{im}\partial_2} = \ker\partial_1 = \langle DG, EI\rangle \cong \mathbb{Z}\oplus\mathbb{Z}$

$H_0(M;\mathbb{Z}) = \frac{Z_0(M)}{B_0(M)} = \frac{\ker\partial_0}{\mathrm{im}\partial_1} = \ker\partial_0 = \langle A\rangle \cong \mathbb{Z}$

Betti numbers of T with $\mathbb{Z}$ coefficients are $b_0 = 1$, $b_1 = 2$, $b_2 = 1$.

3 Vietoris-Rips complex

We started this exercise by implementing the cliques function. To optimize it as much as possible, we only checked for every dimension the vertices whose degree is greater or equal to the dimension − 1. Then for every added edge we checked if the construction still forms a clique. We first tested our algorithm on a graph representation of a tetrahedra 3a. The evalvation was instant, because there are only 4 vertices and 6 edges. We figured out, that our algorithm runs the worst, when the given input graph is a complete graph. Because of that, we first tested it on a complete graph of 16 vertices 3b. The algorithm ran around a second. Then we inputed the complete graph of 19 vertices 3c. The algorithm was done in 10 seconds. Lastly, we tried the complete graph of 22 vertices 3d. The algorithm finished in around 100 seconds.



(a) Tetrahedra.

(b) Complete graph on 16 vertices.

(c) Complete graph on 19 vertices.

(d) Complete graph on 22 vertices.

Figure 3: Representation of input data used on cliques algorithm.

We continued by implementing Vietoris-Rips complex. We calculated all edges and then used the cliques function to calculate all simplices in every dimension. We tested our algorithm on the given sample 4a. The algorithm returned these simplices:
Dimension 0 simplices: [[0], [1], [2], [3], [4], [5]]
Dimension 1 simplices: [[0, 1], [0, 3], [1, 2], [1, 3], [1, 4], [2, 5]]
Dimension 2 simplices: [[0, 1, 3]]

Then we randomly generated a graph with 10 vertices 4b. The algorithm returned these simplices:
Dimension 0 simplices: [[0], [1], [2], [3], [4], [5], [6], [7], [8], [9]]
Dimension 1 simplices: [[0, 4], [1, 5], [1, 7], [2, 6], [2, 8], [2, 9], [3, 9], [5, 7], [6, 8], [6, 9], [8, 9]]

Dimension 2 simplices: [[1, 5, 7], [2, 6, 8], [2, 6, 9], [2, 8, 9], [6, 8, 9]]
Dimension 3 simplices: [[2, 6, 8, 9]]

Lastly, we randomly generated a graph with 20 vertices 4c. The algorithm returned these simplices:
Dimension 0 simplices: [[0], [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19]]
Dimension 1 simplices: [[0, 5], [0, 13], [0, 14], [0, 15], [1, 7], [1, 19], [2, 4], [2, 14], [3, 9], [3, 11], [3, 16], [4, 14], [5, 8], [5, 13], [5, 15], [5, 17], [5, 18], [6, 10], [6, 12], [7, 16], [8, 17], [8, 18], [9, 11], [9, 16], [10, 12], [10, 19], [11, 16], [13, 15], [13, 17], [14, 15], [17, 18]]
Dimension 2 simplices: [[0, 5, 13], [0, 5, 15], [0, 13, 15], [0, 14, 15], [2, 4, 14], [3, 9, 11], [3, 9, 16], [3, 11, 16], [5, 8, 17], [5, 8, 18], [5, 13, 15], [5, 13, 17], [5, 17, 18], [6, 10, 12], [8, 17, 18], [9, 11, 16]]
Dimension 3 simplices: [[0, 5, 13, 15], [3, 9, 11, 16], [5, 8, 17, 18]]

(a) Given sample graph.

(b) Randomly generated graph on 10 vertices.

(c) Randomly generated graph on 20 vertices.

Figure 4: Representation of input data used on Vietoris-Rips algorithm.

4 Čech complex

5 Collapsibility