Discrete mathematics, Homework 04 **March 14th 2024**

We have shown the following theorem as a part of the BICONNECTIVITY AUGMENTATION PROBLEM algorithm.

Threorem 1 *Let G be a* connected*, and let $B(G)$ be its block tree. Let ℓ be the number of leaves in $B(G)$ and let d be the maximal degree of a c-vertex in $B(G)$. Then*

(a) there exists a set of edges F of cardinality $|F| = \max\{d-1, \lceil \frac{\ell}{2} \rceil\}$ so that $G+F$ is 2-connected, and

(b) for every edge set F' of cardinality $F' < \max\{d-1, \lceil \frac{\ell}{2} \rceil\}$ the graph $G+F'$ is not 2-connected.

In other words, the optimal solution to the BAP (if the input graph is connected but not 2-connected) contains exactly

$$\max\{d-1, \left\lceil \frac{\ell}{2} \right\rceil\} \tag{1}$$

edges.

What if G is not a connected graph in the first place and we would still like to add as few edges as possible in order to make it 2-*connected.* If G has h connected components we may obtain a connected graph by adding exactly $h-1$ edges to G. Yet different choices for the starting $h-1$ edges (which make G connected) may require different numbers of additional edges in order to obtain a 2-connected graph. So, one should choose these edges carefully.

1. Prove the following result:

Threorem 2 *Let G be an undirected graph (on at least 3 vertices) which is not 2-connected, and let $B(G)$ be its* block forest *(the union of block trees of components of G). Assume that ℓ is the number of leaves in $B(G)$, d is the maximal degree of a c-vertex, h is the number of connected components of G, and q is the number of isolated vertices in $B(G)$. Then*

(a) there exists a set of edges F of cardinality $|F| = \max\{d+h-2, \lceil \frac{\ell}{2} \rceil + q\}$ so that $G+F$ is 2-connected, and

(b) for every edge set F' of cardinality $F' < \max\{d+h-2, \lceil \frac{\ell}{2} \rceil + q\}$ the graph $G+F'$ is not 2-connected.

In other words. The optimal solution to the BAP contains exactly

$$\max\{d+h-2, \left\lceil \frac{\ell}{2} \right\rceil + q\} \tag{2}$$

edges.

For (a) you DO NOT need to describe the algorithm. It is enough to find a single edge that will reduce the expression $\max\{d+h-2, \lceil \frac{\ell}{2} \rceil + q\}$ — preferably so that also h and q drop by one. This should inductively bring you to the connected case of Theorem 1.

For (b) note that the terminal block tree is a single (isolated) vertex tree. But if your current block forest is not a single vertex graph, then there is a certain number of edges you need to add so that you reduce the size of the block forest to the trivial level.

The complete solution should be doable on a single handwritten page.