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A Kalman Filter for Robust Outlier Detection

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Abstract— **In this paper, we introduce a modified Kalman filter that can perform robust, real-time outlier detection in the observations, without the need for manual parameter tuning by the user. Robotic systems that rely on high quality sensory data can be sensitive to data containing outliers. Since the standard Kalman filter is not robust to outliers, other variations of the Kalman filter have been proposed to overcome this issue, but these methods may require manual parameter tuning, use of heuristics or complicated parameter estimation. Our Kalman filter uses a weighted least squares-like approach by introducing weights for each data sample. A data sample with a smaller weight has a weaker contribution when estimating the current time step's state. We learn the weights and system dynamics using a variational Expectation-Maximization framework. We evaluate our Kalman filter algorithm on data from a robotic dog.**

I. INTRODUCTION

In order to maintain robust control in robotic systems, a high quality of sensory data is needed. While data from sensors such as potentiometers and optical encoders are easily interpretable in their noise characteristics, other sensors such as visual systems, GPS devices and sonar sensors often provide measurements populated with outliers. As a result, robust, reliable detection and removal of outliers is essential in order to process these kinds of data. For example, the application domain of legged locomotion is particularly vulnerable to perceptual data of poor quality, since one undetected outlier can disturb the balance controller to the point that the robot loses stability.

An outlier is generally defined as an observation that "lies outside some overall pattern of distribution" [1]. Outliers may originate from sensor noise (producing values that fall outside a valid range), from temporary sensor failures, or from unanticipated disturbances in the environment (e.g., a brief change of lighting conditions for a visual sensor).

For real-time applications, storing data samples may not be a viable option due to the high frequency of sensory data and insufficient memory resources. In this scenario, sensor data are made available one at a time and must be discarded once they have been observed. Hence, techniques that require access to the entire set of data samples, such as the Kalman smoother (e.g., [2], [3]), are not applicable. Instead, the Kalman filter (e.g., [4], [5]) is a more suitable method, since it assumes that only data samples up to the current time step have been observed. The Kalman filter propagation and update equations are recursive and do not require direct access to previously observed data.

The Kalman filter is a widely used tool for estimating the state of a dynamic system, given noisy measurement data. It is the optimal *linear* estimator for linear Gaussian systems, giving the minimum mean squared error [6]. Using state estimates, the filter can also estimate what the corresponding (output) data are. However, the performance of the Kalman filter degrades when the observed data contains outliers. To address this, previous work has tried to make the Kalman filter more robust to outliers by addressing the sensitivity of the squared error criterion to outliers [7], [8]. One class of approaches considers non-Gaussian distributions for random variables (e.g., [9], [10], [11], [12]), since multivariate Gaussian distributions are known to be susceptible to outliers. For example, [13] use multivariate Student-t distributions. However, the resulting estimation of parameters may be quite complicated for systems with transient disturbances.

Alternatively, it is possible to model the observation and state noise as non-Gaussian, heavy-tailed distributions to account for non-Gaussian noise and outliers (e.g., [14], [15], [16]). Unfortunately, these filters are typically more difficult to implement and may no longer provide the conditional mean of the state vector. Other approaches use resampling techniques (e.g., [17], [18]) or numerical integration (e.g., [19], [20]) but these may require heavy computation not suitable for real-time applications.

Yet another class of methods uses a weighted least squares approach, as done in robust least squares [21], [22], where the measurement residual error is assigned some statistical property. Some of these algorithms fall under the first category of approaches as well, assuming non-Gaussian distributions for variables. Each data sample is assigned a weight that indicates its contribution to the hidden state estimate at each time step. This technique has been used to produce a Kalman filter that is more robust to outliers (e.g., [23], [24]). However, these methods usually model the weights as some heuristic function of the data (e.g., the Huber function [22]) and often require tuning of threshold parameters for optimal performance. Using incorrect or inaccurate estimates for the weights may lead to deteriorated performance, so special attention and care is necessary when using these techniques

In this paper, we are interested in the problem of identifying outliers while tracking the observed data using the Kalman filter. Identifying outliers in the state is different problem entirely, and this is left for another paper. We introduce a modified Kalman filter that can detect outliers in the observed data without the need for parameter tuning or use of heuristic methods on the user's part. This filter learns the weights of each data sample, as well as the system dynamics, using an Expectation-Maximization (EM) framework [25]. For ease of analytical computation, we assume Gaussian distributions for variables and states. We illustrate the performance of this robust Kalman filter on robotic data, comparing it with other robust approaches and demonstrating its effectiveness at detecting outliers in the observations.

II. Outlier Detection in the Kalman Filter

Let us assume we have data $\{\mathbf{z}_k\}_{k=1}^N$, observed over N time steps, and the corresponding hidden states as $\{\boldsymbol{\theta}_k\}_{k=1}^N$ (where $\boldsymbol{\theta}_k \in \Re^{d_2 \times 1}$ and $\mathbf{z}_k \in \Re^{d_1 \times 1}$). Assuming that the system is time-invariant, the Kalman filter system equations are:

$$\begin{aligned} \mathbf{z}_k &= \mathbf{C}\boldsymbol{\theta}_k + \mathbf{v}_k \\ \boldsymbol{\theta}_k &= \mathbf{A}\boldsymbol{\theta}_{k-1} + \mathbf{s}_k \end{aligned} \tag{1}$$

where $\mathbf{C} \in \Re^{d_1 \times d_2}$ is the observation matrix, $\mathbf{A} \in \Re^{d_2 \times d_2}$ is the state transition matrix, $\mathbf{v}_k \in \Re^{d_1 \times 1}$ is the observation noise at time step k, and $\mathbf{s}_k \in \Re^{d_2 \times 1}$ is the state noise at time step k. We assume $\mathbf{v}_k$ and $\mathbf{s}_k$ to be uncorrelated additive mean-zero Gaussian noise, $\mathbf{v}_k \sim \text{Normal}(0, \mathbf{R})$, $\mathbf{s}_k \sim \text{Normal}(0, \mathbf{Q})$, where $\mathbf{R} \in \Re^{d_1 \times d_1}$ is a diagonal matrix with $\mathbf{r} \in \Re^{d_1 \times 1}$ on its diagonal, and $\mathbf{Q} \in \Re^{d_2 \times d_2}$ is a diagonal matrix with $\mathbf{q} \in \Re^{d_2 \times 1}$ on its diagonal. $\mathbf{R}$ and $\mathbf{Q}$ are covariance matrices for the observation and state noise, respectively. The corresponding Kalman filter propagation and update equations are, for $k = 1, .., N$:

Propagation:

$$\boldsymbol{\theta}'_k = \mathbf{A}\left\langle \boldsymbol{\theta}_{k-1} \right\rangle \tag{2}$$

$$\boldsymbol{\Sigma}'_k = \mathbf{A}\boldsymbol{\Sigma}_{k-1}\mathbf{A}^T + \mathbf{Q} \tag{3}$$

Update:

$$\mathbf{S}'_k = \left(\mathbf{C}\boldsymbol{\Sigma}'_k\mathbf{C}^T + \mathbf{R}\right)^{-1} \tag{4}$$

$$K'_k = \boldsymbol{\Sigma}'_k\mathbf{C}^T\mathbf{S}'_k \tag{5}$$

$$\left\langle \boldsymbol{\theta}_k \right\rangle = \boldsymbol{\theta}'_k + K'_k\left(\mathbf{z}_k - \mathbf{C}\boldsymbol{\theta}'_k\right) \tag{6}$$

$$\boldsymbol{\Sigma}_k = \left(\mathbf{I} - K'_k\mathbf{C}\right)\boldsymbol{\Sigma}'_k \tag{7}$$

where $\langle \boldsymbol{\theta}_k \rangle$[1] is the posterior mean vector of the state $\boldsymbol{\theta}_k$, $\boldsymbol{\Sigma}_k$ is the posterior covariance matrix of $\boldsymbol{\theta}_k$, and $\mathbf{S}'_k$ is the covariance matrix of the residual prediction error—all at time step k. In this problem, the system dynamics ($\mathbf{C}$, $\mathbf{A}$, $\mathbf{R}$ and $\mathbf{Q}$) are unknown, and it is possible to use a maximum likelihood framework to estimate these parameter values [26]. Unfortunately, this standard Kalman filter model considers all data samples to be part of the data cloud and is not robust to outliers.

A. Robust Kalman Filtering with Bayesian Weights

To overcome this limitation, we introduce a novel Bayesian algorithm that treats the weights associated with each data sample probabilistically. In particular, we introduce a scalar weight w_k for each observed data sample $\mathbf{z}_k$ such that the variance of $\mathbf{z}_k$ is weighted with w_k, as done in [27]. Gelman et al. [27] consider a weighted least squares regression model and assume that the weights are known and given. In contrast, we model the weights to be Gamma distributed random variables, as done previously in [28] for weighted linear regression. Additionally, we learn estimates for the system dynamics at each time step. We choose a Gamma prior distribution for the weights in order to ensure they remain positive. The prior distributions are:

$$\begin{aligned} \mathbf{z}_k|\boldsymbol{\theta}_k, w_k &\sim \text{Normal}\left(\mathbf{C}\boldsymbol{\theta}_k, \mathbf{R}/w_k\right) \\ \boldsymbol{\theta}_k|\boldsymbol{\theta}_{k-1} &\sim \text{Normal}\left(\mathbf{A}\boldsymbol{\theta}_{k-1}, \mathbf{Q}\right) \\ w_k &\sim \text{Gamma}\left(a_{w_k}, b_{w_k}\right) \end{aligned} \tag{8}$$

We can treat this entire problem as an Expectation-Minimization-like (EM) learning problem [25], [29] and maximize the log likelihood $\log p(\boldsymbol{\theta}_{1:N})$ (otherwise known as the "incomplete" log likelihood with the hidden probabilistic variables marginalized out). Due to analytical issues, we only have access to a lower bound of this measure. This lower bound is based on an expected value of the "complete" data likelihood $\langle \log p\left(\boldsymbol{\theta}_{1:N}, \mathbf{z}_{1:N}, \mathbf{w}\right)\rangle$, formulated over all variables of the learning problem:

$$\begin{aligned} &\log p\left(\boldsymbol{\theta}_{1:N}, \mathbf{z}_{1:N}, \mathbf{w}\right) \\ &= \textstyle\sum_{i=1}^N \log p\left(\mathbf{z}_i|\boldsymbol{\theta}_i, w_i\right) + \sum_{i=1}^N \log p\left(\boldsymbol{\theta}_i|\boldsymbol{\theta}_{i-1}\right) \\ &\quad + \log p\left(\boldsymbol{\theta}_0\right) + \textstyle\sum_{i=1}^N \log p\left(w_i\right) \\ &= \tfrac{1}{2}\textstyle\sum_{i=1}^N \log w_i - \tfrac{N}{2}\log|\mathbf{R}| - \tfrac{N}{2}\log|\mathbf{Q}| \\ &\quad -\tfrac{1}{2}\textstyle\sum_{i=1}^N w_i\left(\mathbf{z}_i - \mathbf{C}\boldsymbol{\theta}_i\right)^T\mathbf{R}^{-1}\left(\mathbf{z}_i - \mathbf{C}\boldsymbol{\theta}_i\right) \\ &\quad -\tfrac{1}{2}\textstyle\sum_{i=1}^N \left(\boldsymbol{\theta}_i - \mathbf{A}\boldsymbol{\theta}_{i-1}\right)^T\mathbf{Q}^{-1}\left(\boldsymbol{\theta}_i - \mathbf{A}\boldsymbol{\theta}_{i-1}\right) \\ &\quad -\tfrac{1}{2}\log|\mathbf{Q}_0| - \tfrac{1}{2}\left(\boldsymbol{\theta}_0 - \hat{\boldsymbol{\theta}}_0\right)^T\mathbf{Q}_0^{-1}\left(\boldsymbol{\theta}_0 - \hat{\boldsymbol{\theta}}_0\right) \\ &\quad + \textstyle\sum_{i=1}^N \left(a_{w_i,0}\right)\log w_i - \sum_{i=1}^N b_{w_i,0}w_i + \text{const} \end{aligned} \tag{9}$$

where $\boldsymbol{\theta}_0$ is the initial state, $\hat{\boldsymbol{\theta}}_0$ is the mean of $\boldsymbol{\theta}_0$, $\mathbf{Q}_0$ is the noise variance of $\boldsymbol{\theta}_0$, $\mathbf{w} \in \Re^{N \times 1}$ has coefficients w_i ($i = 1, .., N$), and $\mathbf{z}_{1:N}$ denotes samples $\{\mathbf{z}_1, \mathbf{z}_2, .., \mathbf{z}_N\}$. Since we are considering this problem as a real-time one (i.e. data samples arrive sequentially, one at a time), we will have observed only data samples $\mathbf{z}_{1:k}$ at time step k. Consequently, in order to estimate the posterior distributions of the random variables and parameter values at time step k, we should consider the log evidence of only the data samples observed to date, i.e., $\log p\left(\boldsymbol{\theta}_{1:k}, \mathbf{z}_{1:k}, \mathbf{w}_{1:k}\right)$.

The expectation of the complete data likelihood should be taken with respect to the true posterior distribution of all hidden variables $Q\left(\mathbf{w}, \boldsymbol{\theta}\right)$. However, since this is an analytically intractable expression, we use a technique from variational calculus to construct a lower bound and make a factorial approximation of the true posterior [29] as follows: $Q\left(\mathbf{w}, \boldsymbol{\theta}\right) = \prod_{i=1}^N Q\left(w_i\right)\prod_{i=1}^N Q\left(\boldsymbol{\theta}_i|\boldsymbol{\theta}_{i-1}\right)Q(\boldsymbol{\theta}_0)$. This factorization of $\boldsymbol{\theta}$ considers the influence of each $\boldsymbol{\theta}_i$ from within its Markov blanket, conserving the Markov property that Kalman filters, by definition, have. While losing a small amount of accuracy, all resulting posterior distributions

[1]Note that $\langle\rangle$ denotes the expectation operator

over hidden variables become analytically tractable. This factorial approximation was chosen purposely so that $Q(w_k)$ is independent from $Q(\boldsymbol{\theta}_k)$; performing joint inference of w_k and $\boldsymbol{\theta}_k$ does not make sense in the context of our generative model. We can derive the final EM update equations from standard manipulations of Normal and Gamma distributions and arrive at the following for time step k:

E-step:

$$\boldsymbol{\Sigma}_k = \left(\langle w_k \rangle \mathbf{C}_k^T \mathbf{R}_k^{-1} \mathbf{C}_k + \mathbf{Q}_k^{-1}\right)^{-1} \tag{10}$$

$$\langle \boldsymbol{\theta}_k \rangle = \boldsymbol{\Sigma}_k \left(\mathbf{Q}_k^{-1} \mathbf{A}_k \langle \boldsymbol{\theta}_{k-1} \rangle + \langle w_k \rangle \mathbf{C}_k^T \mathbf{R}_k^{-1} \mathbf{z}_k\right) \tag{11}$$

$$\langle w_k \rangle = \frac{a_{w_k,0} + \frac{1}{2}}{b_{w_k,0} + \left\langle (\mathbf{z}_k - \mathbf{C}_k \boldsymbol{\theta}_k)^T \mathbf{R}_k^{-1} (\mathbf{z}_k - \mathbf{C}_k \boldsymbol{\theta}_k) \right\rangle} \tag{12}$$

M-step:

$$\mathbf{C}_k = \left(\sum_{i=1}^k \langle w_i \rangle \mathbf{z}_i \langle \boldsymbol{\theta}_i \rangle^T\right) \left(\sum_{i=1}^k \langle w_i \rangle \left\langle \boldsymbol{\theta}_i \boldsymbol{\theta}_i^T \right\rangle\right)^{-1} \tag{13}$$

$$\mathbf{A}_k = \left(\sum_{i=1}^k \langle \boldsymbol{\theta}_i \rangle \langle \boldsymbol{\theta}_{i-1} \rangle^T\right) \left(\sum_{i=1}^k \left\langle \boldsymbol{\theta}_{i-1} \boldsymbol{\theta}_{i-1}^T \right\rangle\right)^{-1} \tag{14}$$

$$r_{km} = \frac{1}{k} \sum_{i=1}^k \langle w_i \rangle \left\langle (\mathbf{z}_{im} - \mathbf{C}_k(m,:)\boldsymbol{\theta}_i)^2 \right\rangle \tag{15}$$

$$q_{kn} = \frac{1}{k} \sum_{i=1}^k \left\langle (\boldsymbol{\theta}_{in} - \mathbf{A}_k(n,:)\boldsymbol{\theta}_{i-1})^2 \right\rangle \tag{16}$$

where $m = 1, ..., d_1$, $n = 1, ...d_2$; r_{km} is the mth coefficient of the vector $\mathbf{r}_k$; q_{kn} is the nth coefficient of the vector $\mathbf{q}_k$; $\mathbf{C}_k(m,:)$ is the mth row of the matrix $\mathbf{C}_k$; $\mathbf{A}_k(n,:)$ is the nth row of the matrix $\mathbf{A}_k$; and $a_{w_k,0}$ and $b_{w_k,0}$ are prior scale parameters for the weight w_k. Equations (10) to (16) need to be computed once for each time step k (e.g., [30] [31]), when the data sample $\mathbf{z}_k$ becomes available.

Since all sensor data cannot be stored in real-time applications, but must be discarded soon after it is received, (13) to (16) need to be re-written in incremental form (i.e., using only values observed, calculated or used in the current time step k). We can do this by collecting sufficient statistics in (13) to (16) and re-writing them as:

$$\mathbf{C}_k = \sum\nolimits_k^{\mathbf{wz}\boldsymbol{\theta}^T} \left(\sum\nolimits_k^{\mathbf{w}\boldsymbol{\theta}\boldsymbol{\theta}^T}\right)^{-1} \tag{17}$$

$$\mathbf{A}_k = \sum\nolimits_k^{\boldsymbol{\theta}\boldsymbol{\theta}'} \left(\sum\nolimits_k^{\boldsymbol{\theta}'\boldsymbol{\theta}'}\right)^{-1} \tag{18}$$

$$\begin{aligned} r_{km} = \frac{1}{k} \Big[& \sum\nolimits_{km}^{\mathbf{w}zz} - 2\mathbf{C}_k(m,:) \left(\sum\nolimits_{km}^{\mathbf{w}z\boldsymbol{\theta}}\right) \\ & + \text{diag}\left\{ \mathbf{C}_k(m,:) \left(\sum\nolimits_k^{\mathbf{w}\boldsymbol{\theta}\boldsymbol{\theta}^T}\right) \mathbf{C}_k(m,:)^T \right\} \Big] \end{aligned} \tag{19}$$

$$\begin{aligned} q_{kn} = \frac{1}{k} \Big[& \sum\nolimits_{kn}^{\theta^2} - 2\mathbf{A}_k(n,:) \left(\sum\nolimits_{kn}^{\theta\boldsymbol{\theta}'}\right) \\ & + \text{diag}\left\{ \mathbf{A}_k(n,:) \left(\sum\nolimits_k^{\boldsymbol{\theta}'\boldsymbol{\theta}'}\right) \mathbf{A}_k(n,:)^T \right\} \Big] \end{aligned} \tag{20}$$

where $m = 1, .., d_1$, $n = 1, .., d_2$, and the sufficient statistics are:

$$\begin{aligned}
\sum\nolimits_k^{\mathbf{wz}\boldsymbol{\theta}^T} &= \langle w_k \rangle \mathbf{z}_k \langle \boldsymbol{\theta}_k \rangle^T + \sum\nolimits_{k-1}^{\mathbf{wz}\boldsymbol{\theta}^T} \\
\sum\nolimits_k^{\mathbf{w}\boldsymbol{\theta}\boldsymbol{\theta}^T} &= \langle w_k \rangle \left\langle \boldsymbol{\theta}_k \boldsymbol{\theta}_k^T \right\rangle + \sum\nolimits_{k-1}^{\mathbf{w}\boldsymbol{\theta}\boldsymbol{\theta}^T} \\
\sum\nolimits_k^{\boldsymbol{\theta}\boldsymbol{\theta}'} &= \langle \boldsymbol{\theta}_k \rangle \langle \boldsymbol{\theta}_{k-1} \rangle^T + \sum\nolimits_{k-1}^{\boldsymbol{\theta}\boldsymbol{\theta}'} \\
\sum\nolimits_k^{\boldsymbol{\theta}'\boldsymbol{\theta}'} &= \left\langle \boldsymbol{\theta}_{k-1} \boldsymbol{\theta}_{k-1}^T \right\rangle + \sum\nolimits_{k-1}^{\boldsymbol{\theta}'\boldsymbol{\theta}'} \\
\sum\nolimits_{km}^{\mathbf{w}zz} &= \langle w_k \rangle z_{km}^2 + \sum\nolimits_{k-1}^{\mathbf{w}zz} \\
\sum\nolimits_{km}^{\mathbf{w}z\boldsymbol{\theta}} &= \langle w_k \rangle z_{km} \boldsymbol{\theta}_k + \sum\nolimits_{k-1,m}^{\mathbf{w}z\boldsymbol{\theta}} \\
\sum\nolimits_{kn}^{\theta^2} &= \left\langle \theta_{kn}^2 \right\rangle + \sum\nolimits_{k-1,n}^{\theta^2} \\
\sum\nolimits_{kn}^{\theta\boldsymbol{\theta}'} &= \langle \theta_{kn} \rangle \langle \boldsymbol{\theta}_{k-1} \rangle + \sum\nolimits_{kn}^{\theta\boldsymbol{\theta}'}
\end{aligned}$$

A few remarks should be made regarding the initialization of priors used in (10) to (12), (17) to (20). In particular, the prior scale parameters $a_{w_k,0}$ and $b_{w_k,0}$ should be selected so that the weights $\langle w_k \rangle$ are 1 with some confidence. That is to say, the algorithm starts by assuming most data samples are inliers. For example, we can set $a_{w_k,0} = 1$ and $b_{w_k,0} = 1$ so that $\langle w_k \rangle$ has a prior mean of $a_{w_k,0}/b_{w_k,0} = 1$ with a variance of $a_{w_k,0}/b_{w_k,0}^2 = 1$. By using these values, the maximum value of $\langle w_k \rangle$ is capped at 1.5. This set of values is generally valid for any data set and/or application and does not need to be modified if no prior information regarding the presence of outliers in the data is available. Otherwise, if the user has prior knowledge regarding the strong or weak presence of outliers in the data set (and hence, a good reason to insert strong biases towards particular parameter values), the prior scale parameters of the weights can be modified accordingly to reflect this. Since some prior knowledge about the observed data's properties must be known in order to distinguish whether a data sample is an outlier or part of the data's structure, this Bayesian approach provides a natural framework to incorporate this information.

Secondly, the algorithm is relatively insensitive to the the initialization of $\mathbf{A}$ and $\mathbf{C}$ and will always converge to the same final solution, regardless of these values. For our experiments, we use $\mathbf{C} = \mathbf{A} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. Finally, the initial values of $\mathbf{R}$ and $\mathbf{Q}$ should be set based on the user's initial estimate of how noisy the observed data is (e.g., $\mathbf{R} = \mathbf{Q} = 0.01\mathbf{I}$ for noisy data, $\mathbf{R} = \mathbf{Q} = 10^{-4}\mathbf{I}$ for less noisy data [32]).

B. Relationship to the Kalman Filter

The equations (10) and (11) for the posterior mean and posterior covariance of $\boldsymbol{\theta}_k$ may not look like the standard Kalman filter equations in (2) to (7), but with a little algebraic manipulation, we can show that the model derived in Section II-A is indeed a variant of the Kalman filter. If we substitute the propagation equations, (2) and (3), into the update equations, (4) to (7), we reach recursive expressions for $\langle \boldsymbol{\theta}_k \rangle$ and $\boldsymbol{\Sigma}_k$. By applying this sequence of algebraic manipulations in reverse order to (10) and (11), we arrive at the following:

Propagation:

$$\boldsymbol{\theta}'_k = \mathbf{A}_k \langle \boldsymbol{\theta}_{k-1} \rangle \tag{21}$$

$$\boldsymbol{\Sigma}'_k = \mathbf{Q}_k \tag{22}$$

Update:

$$\mathbf{S}'_k = \left(\mathbf{C}_k \boldsymbol{\Sigma}'_k \mathbf{C}_k^T + \frac{1}{\langle w_k \rangle} \mathbf{R}_k \right)^{-1} \tag{23}$$

$$K'_k = \boldsymbol{\Sigma}'_k \mathbf{C}_k^T \mathbf{S}'_k \tag{24}$$

$$\langle \boldsymbol{\theta}_k \rangle = \boldsymbol{\theta}'_k + K'_k \left(\mathbf{z}_k - \mathbf{C}_k \boldsymbol{\theta}'_k \right) \tag{25}$$

$$\boldsymbol{\Sigma}_k = \left(\mathbf{I} - K'_k \mathbf{C}_k \right) \boldsymbol{\Sigma}'_k \tag{26}$$

Close examination of the above equations show that (10) and (11) in the Bayesian model correspond to standard Kalman filter equations, with modified expressions for $\boldsymbol{\Sigma}'_k$ and $\mathbf{S}'_k$ and time-varying system dynamics. $\boldsymbol{\Sigma}'_k$ is no longer *explicitly* dependent on $\boldsymbol{\Sigma}_{k-1}$, since $\boldsymbol{\Sigma}_{k-1}$ does not appear in (22). However, the current state's covariance $\boldsymbol{\Sigma}_k$ is still dependent on the previous state's covariance $\boldsymbol{\Sigma}_{k-1}$ (i.e., it is dependent through the other parameters K'_k and $\mathbf{C}_k$).

Additionally, the term $\mathbf{R}_k$ in $\mathbf{S}'_k$ is now weighted. Equation (12) reveals that if the prediction error in $\mathbf{z}_k$ is so large that it dominates the denominator, then the weight $\langle w_k \rangle$ of that data sample will be very small. As this prediction error term in the denominator goes to ∞, $\langle w_k \rangle$ approaches 0. If $\mathbf{z}_k$ has a very small weight $\langle w_k \rangle$, then $\mathbf{S}'_k$, the posterior covariance of the residual prediction error, will be very small, leading to a very small Kalman gain K'_k. In short, the influence of the data sample $\mathbf{z}_k$ will be downweighted when predicting $\boldsymbol{\theta}_k$, the hidden state at time step k.

The resulting Bayesian algorithm has a computational complexity on the same order as that of a standard Kalman filter, since matrix inversions are still needed (for the calculation of covariance matrices), as in the standard Kalman filter. In comparison to other Kalman filters that use heuristics or require more involved computation/implementation, this outlier-robust Kalman filter is principled and easy to implement.

C. An Alternative Kalman Filter

We explored a variation of the previously introduced robust Kalman filter. Instead of performing a full Bayesian treatment of the weighted Kalman filter, we use the standard Kalman filter equations, (2) to (7), and modify (4) so that the output variance for $\mathbf{z}_k$, $\mathbf{R}_k$, is now weighted—as in our original model in (8):

$$\mathbf{S}'_k = \left(\mathbf{C}_k \boldsymbol{\Sigma}'_k \mathbf{C}_k^T + \frac{1}{\langle w_k \rangle} \mathbf{R}_k \right)^{-1} \tag{27}$$

We learn the weights $\langle w_k \rangle$ using (12) from the robust Kalman filter and estimate the system dynamics at each time step using a maximum likelihood framework (i.e., using (17) to (20) from the robust Kalman filter). $\boldsymbol{\Sigma}_k$ is now *explicitly* dependent on $\boldsymbol{\Sigma}_{k-1}$ (i.e., $\boldsymbol{\Sigma}_{k-1}$ appears in the propagation equation for $\boldsymbol{\Sigma}_k$). We introduce this somewhat unprincipled and arbitrarily derived filter for comparison with out weighted Kalman filter.

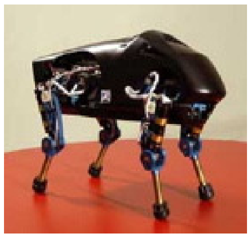


Fig. 1. LittleDog

III. Experimental Results

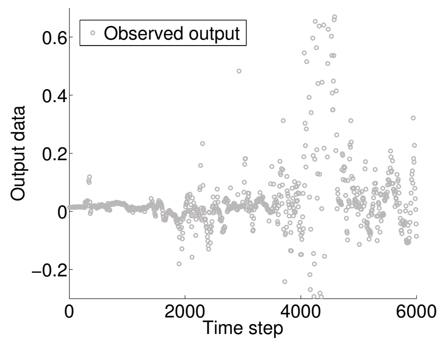
We evaluated our weighted robust Kalman filter on data collected from a robotic dog, LittleDog, manufactured by Boston Dynamics, Inc. (Cambridge, MA), and compared it with three other filters. We omitted the filters of [23] and [24], since we had difficulty implementing them and getting them to work. instead, we used a hand-tuned thresholded Kalman filter to serve as a baseline comparison. The three filters consist of i) the standard Kalman filter, ii) the alternative weighted Kalman filter introduced in Section II-C, and iii) a Kalman filter where outliers are determined by thresholding on the Mahalanobis distance. If the Mahalanobis distance is less than a certain threshold value, then it is considered an inlier and processed. Otherwise, it is an outlier and ignored. This threshold value is *hand-tuned manually* in order to find the optimal value for a particular data set. If we have a priori access to the entire data set and are able to tune this threshold value accordingly, the thresholded Kalman filter gives *near-optimal* performance.

For this paper and these experiments, we are interested in the Kalman filter's prediction of the observed (output) data and the detection of outliers in the observations. We are not interested in the estimation of the system dynamics or in the estimation (or outlier detection) of the states. Estimation of the system matrices for the purpose of parameter identification is a different problem, and details on this difference are highlighted in [33]. Similarly, detecting outliers in the states is a different problem and left to another paper.

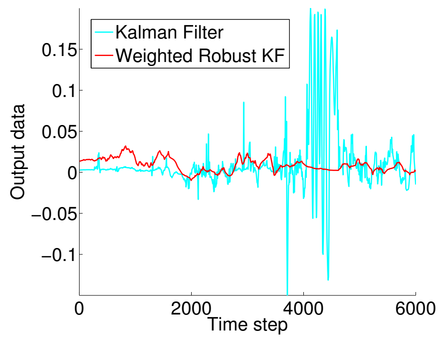
A. LittleDog Robot

We evaluated all filters on a 12 degree-of-freedom (DOF) robotic dog, LittleDog, shown in Figure 1. The robot dog has two sources that measure its orientation: a motion capture (MOCAP) system and an on-board inertia measurement unit (IMU). Both provide a quaternion q of the robot's orientation: q_{MOCAP} from the MOCAP and q_{IMU} from the IMU.

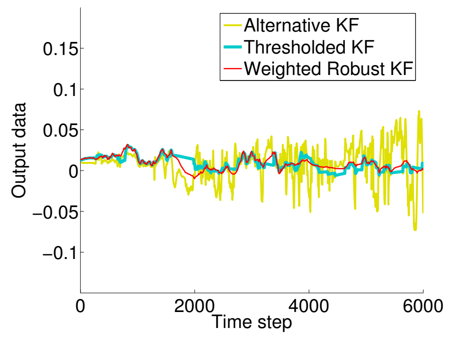
q_{IMU} drifts over time, since the IMU cannot provide stable orientation estimation but its signal is clean. The drift that occurs in the IMU is quite common in systems where sensors collect data that need to be integrated. In contrast, q_{MOCAP} has outliers and noise, but no drift. We would like to estimate the offset between q_{MOCAP} and q_{IMU}, and this offset is a *noisy slowly drifting signal containing outliers*.



(a) Observed data, $(q_{\mathrm{IMU}} - q_{\mathrm{MOCAP}})$, from LittleDog robot: a slowly drifting noisy signal with outliers



(b) Predicted data for the Kalman filter and weighted robust Kalman filter. Note the change of scale in axis from Figure 2(a).



(c) Predicted data for the thresholded Kalman filter, alternative Kalman filter and weighted robust Kalman filter

Fig. 2. Observed vs. predicted data from LittleDog robot shown for all Kalman filters, over 6000 samples

There are various approaches to estimating this slowly drifting signal, depending on the quality of estimate desired. We can estimate it with a straight line, as done in [28]. Alternatively, if we want to estimate the signal more accurately, we can use the proposed outlier-robust Kalman filter to track it. For optimal performance, we manually tuned $\mathbf{C}$, $\mathbf{A}$, $\mathbf{R}$ and $\mathbf{Q}$ for the standard Kalman filter—a tricky and time-consuming process. The system dynamics of the thresholded Kalman filter were learnt using a maximum likelihood framework (i.e., using (17) to (20) without any weights). Its threshold parameter was manually tuned for best performance on this data set.

Figure 2(a) shows the offset data between q_{MOCAP} and q_{IMU} for one of the four quaternion coefficients, collected over 6000 data samples, at 1 sample/time step. As expected, the standard Kalman filter fails to detect and ignore the outliers occurring between the 4000th and 5000th sample, as seen in Figure 2(b). When comparing our weighted robust Kalman filter with the other remaining two filters, Figure 2(c) shows that the thresholded Kalman filter does not react as violently as the standard Kalman filter to outliers and, in fact, appears to perform similarly to the weighted robust Kalman filter. This is to be expected, given that we hand-tuned the threshold parameter for optimal performance (i.e., the thresholded Kalman filter is *near-optimal* in this experiment). Notice that the weighted robust filter does not track noise in the data as closely as the alternative filter. This is a direct result of higher Kalman gains in the alternative filter and a consequence of the dependency on the previous state state's covariance.

In this experiment, the advantages offered by our weighted outlier-robust Kalman filter are clear. It outperforms the traditional Kalman filter and alternative Kalman filter, while achieving a level of performance on par with a thresholded Kalman filter (where the threshold value is manually tuned for optimal performance).

IV. CONCLUSIONS AND FUTURE WORKS

We derived a novel Kalman filter that is robust to outliers in the observations by introducing weights for each data sample. This Kalman filter learns the weights, as well as the system dynamics, without any need for manual parameter tuning by the user, heuristics or sampling. It performs as well as a hand-tuned Kalman filter (that required prior knowledge of the data) on real robotic data. It provides an easy-to-use competitive alternative for robust tracking of sensor data and offers a simple outlier detection mechanism that can be easily applied to more complex, nonlinear filters.

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