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Numerical Methods for Stochastic ODEs

A Stochastic Ordinary Differential Equation (SODE) is an ordinary differential equation with a random forcing, usually white noise $\xi(t)$, chosen so that the random forces are uncorrelated at distinct times t.

example $\quad \frac{du}{dt} = -\lambda u + \sigma \xi(t) \quad , \quad u(0) = u_0 \quad , \quad \lambda, \sigma > 0, \quad u_0 \in \mathbb{R}$

For a Brownian motion, $\xi(t) = \frac{dW(t)}{dt}$

General form

$$\underline{u}(t) = \underline{u_0} + \int_0^t \underline{f}(\underline{u}(s))\, ds + \int_0^t G(\underline{u}(s))\, d\underline{W}(s) \qquad (*)$$

or differential form

$$d\underline{u} = \underline{f}(\underline{u})\, dt + G(\underline{u})\, d\underline{W}(t) \quad , \quad \underline{u}(0) = \underline{u_0}$$

- $\underline{f} : \mathbb{R}^d \to \mathbb{R}^d$ drift
- $G : \mathbb{R}^d \to \mathbb{R}^{d \times m}$ diffusion
  - $G(\underline{u})$ independent of $\underline{u}$ (e.g. $G(\underline{u}) = \sigma$) = additive noise $\Rightarrow \int_0^t G\, d\underline{W}(s) = G\underline{W}(t)$
  - $G(\underline{u})$ varies with $\underline{u}$ = multiplicative noise $\Rightarrow$ integrand $G(\underline{u}(s))$ is random
- $\underline{u_0} \in \mathbb{R}^d$ initial condition
- $\underline{W}(t) = [W_1(t), \dots, W_m(t)]^T, \quad W_i(t)$ i.i.d. Brownian motions

Example 1: mean-reverting Ornstein-Uhlenbeck process (additive noise)

$$du = \lambda(\mu - u)\, dt + \sigma\, dW(t) \quad , \quad u(0) = u_0 \quad , \quad \lambda, \mu, \sigma \in \mathbb{R}$$

$$d\varphi = \frac{\partial \varphi}{\partial t} dt + \frac{\partial \varphi}{\partial u} du + \frac{1}{2} \frac{\partial^2 \varphi}{\partial u^2} g^2\, dt \qquad \text{Itô formula}$$

$$\varphi(t,u) = e^{\lambda t} u \quad \Rightarrow \quad \frac{\partial \varphi}{\partial t} = \lambda e^{\lambda t} u, \quad \frac{\partial \varphi}{\partial u} = e^{\lambda t}, \quad \frac{\partial^2 \varphi}{\partial u^2} = 0$$

$$d\varphi = \lambda e^{\lambda t} u \, dt + e^{\lambda t}(\lambda(\mu - u)\, dt + \sigma \, dW(t))$$

$$\varphi(t, u(t)) - \varphi(0, u_0) = e^{\lambda t} u - e^{\lambda \cdot 0} u_0 = \int_0^t \lambda e^{\lambda s} u \, ds + \int_0^t e^{\lambda s} \lambda(\mu - u)\, ds + \int_0^t e^{\lambda s} \sigma \, dW(s)$$

$$e^{\lambda t} u - u_0 = \mu (e^{\lambda t} - 1) + \sigma \int_0^t e^{\lambda s} \, dW(s)$$

$$u = e^{-\lambda t} u_0 + \mu(1 - e^{-\lambda t}) + \sigma \int_0^t e^{\lambda(s-t)} \, dW(s)$$

$$\mu(t) = \mathbb{E}[u(t)] = e^{-\lambda t} u_0 + \mu(1 - e^{-\lambda t}) \quad \Rightarrow \quad \mu(t) \to \mu \text{ for } t \to \infty$$

Example 2: geometric Brownian motion (multiplicative noise)

Deterministic case : $du = ru \, dt$ $\qquad u(0) = u_0$

$\downarrow \; r \mapsto r + \sigma \xi(t)$

Stochastic case : $du = ru \, dt + \sigma u \, dW(t)$

if $u_0 = 0 \Rightarrow du = 0 \Rightarrow u \equiv 0$

if $u_0 > 0$, we take $\varphi(t, u) = \log(u)$, $\frac{\partial \varphi}{\partial t} = 0$, $\frac{\partial \varphi}{\partial u} = \frac{1}{u}$, $\frac{\partial^2 \varphi}{\partial u^2} = -\frac{1}{u^2}$

$$d\varphi = \frac{1}{u} du + \frac{1}{2}\left(-\frac{1}{u^2}\right)\sigma^2 u^2 \, dt = \frac{1}{u}(ru \, dt + \sigma u \, dW(t)) - \frac{1}{2}\sigma^2 \, dt$$

$$\varphi(t, u) - \varphi(0, u_0) = \int_0^t r \, ds + \int_0^t \sigma \, dW(s) - \frac{1}{2}\sigma^2 \int_0^t ds$$

$$\log(u) = \log(u_0) + \left(r - \frac{1}{2}\sigma^2\right)t + \sigma W(t)$$

$$u = u_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right) \geq 0 \quad \forall t \geq 0$$

Euler - Maruyama method

From $(*)$, $\underline{u}(t_{n+1}) = \underline{u_0} + \int_0^{t_{n+1}} f(\underline{u}(s))\, ds + \int_0^{t_{n+1}} G(\underline{u}(s))\, d\underline{W}(s)$ $\qquad t_n = n\Delta t$, $n = 0, 1, 2, \dots$

$\Delta t > 0$

and $\underline{u}(t_n) = \underline{u_0} + \int_0^{t_n} \underline{f}(\underline{u}(s))\, ds + \int_0^{t_n} G(\underline{u}(s))\, d\underline{W}(s)$

subtracting, $\underline{u}(t_{n+1}) - \underline{u}(t_n) = \int_{t_n}^{t_{n+1}} \underline{f}(\underline{u}(s))\, ds + \int_{t_n}^{t_{n+1}} G(\underline{u}(s))\, d\underline{W}(s)$

by approximating $\underline{f}(\underline{u}(t))$ with $\underline{f}(\underline{u}_n)$ and $G(\underline{u}(t))$ with $G(\underline{u}_n)$ ($\underline{u}_n := \underline{u}(t_n)$) on $[t_n, t_{n+1})$

we obtain the Euler - Maruyama method :

$$\underline{u}_{n+1} = \underline{u}_n + \underline{f}(\underline{u}_n)\Delta t + G(\underline{u}_n)\Delta \underline{W}_n \quad , \quad \Delta \underline{W}_n := \underline{W}(t_{n+1}) - \underline{W}(t_n) , \quad \Delta \underline{W}_n \overset{iid}{\sim} \mathcal{N}(\underline{0}, \Delta t I_m)$$

DERIVATION

From Taylor's theorem, $\underline{f}(\underline{u}(r)) = \underline{f}(\underline{u}(s)) + D\underline{f}(\underline{u}(s))(\underline{u}(r) - \underline{u}(s)) +$ remainder

$G(\underline{u}(r)) = G(\underline{u}(s)) + DG(\underline{u}(s))(\underline{u}(r) - \underline{u}(s)) +$ remainder

Stability of Euler - Maruyama method

$Y \sim \mathcal{N}(0, \sigma^2) \Rightarrow X = e^Y$

log-normal r.v. with

$\mathbb{E}[X] = e^{\sigma^2/2}$,

$Var(X) = (e^{\sigma^2} - 1)e^{\sigma^2}$

Geometric Brownian motion with $d = m = 1$

$$du = ru\,dt + \sigma u\,dW(t), \quad u(0) = u_0$$

solution (from example 2) $u = u_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right)$

$$\mathbb{E}[u(t)^2] = \mathbb{E}\left[u_0^2 \exp\left((2r - \sigma^2)t\right)\exp\left(2\sigma W(t)\right)\right] = u_0^2 \exp\left((2r - \sigma^2)t\right)\mathbb{E}\left[\exp\left(2\sigma W(t)\right)\right] =$$

$$= u_0^2 \exp\left((2r - \sigma^2)t\right)\exp\left(2\sigma^2 t\right) = u_0^2 \exp\left((2r + \sigma^2)t\right)$$

$2\sigma W(t) \sim \mathcal{N}(0, 4\sigma^2 t)$

$$\Rightarrow \mathbb{E}[u(t)^2] = u_0^2 \exp\left((2r + \sigma^2)t\right) \underset{t \to \infty}{\longrightarrow} 0 \iff 2r + \sigma^2 < 0$$

Question: what is the behaviour of $\mathbb{E}[u_n^2]$, $n \to \infty$?

$$u_{n+1} = u_n + r u_n \Delta t + \sigma u_n \Delta W_n = (1 + r\Delta t + \sigma \Delta W_n)u_n \Rightarrow u_n = \prod_{j=0}^{n-1}(1 + r\Delta t + \sigma \Delta W_j)u_0$$

$$\mathbb{E}[u_n^2] = \mathbb{E}\left[\prod_{j=0}^{n-1}(1 + r\Delta t + \sigma \Delta W_j)^2 u_0^2\right] = u_0^2 \prod_{j=0}^{n-1}\mathbb{E}[(1 + r\Delta t + \sigma \Delta W_j)^2] =$$

$$= u_0^2 \prod_{j=0}^{n-1}\mathbb{E}[(1 + r\Delta t)^2 + \sigma^2 \Delta W_j^2 + 2(1 + r\Delta t)\Delta W_j] = u_0^2 \prod_{j=0}^{n-1}((1 + r\Delta t)^2 + \sigma^2 \Delta t)$$

$\mathbb{E}[\Delta W_j^2] = Var(\Delta W_j) + \mathbb{E}[\Delta W_j]$

$$\Rightarrow \mathbb{E}[u_n^2] = u_0^2 ((1 + r\Delta t)^2 + \sigma^2 \Delta t)^n \underset{n \to \infty}{\longrightarrow} 0 \iff |(1 + r\Delta t)^2 + \sigma^2 \Delta t| < 1$$

$$-1 < 1 + 2r\Delta t + r^2 \Delta t^2 + \sigma^2 \Delta t < 1$$

$$(-2 <)\ \Delta t(2r + r^2 \Delta t + \sigma^2) < 0$$

$$\boxed{(0 <)\ \Delta t < \frac{-\sigma^2 - 2r}{r^2}}$$

θ-Euler-Maruyama

$$\underline{u}_{n+1} = \underline{u}_n + \left((1-\theta)\,\underline{f}(\underline{u}_n) + \theta\,\underline{f}(\underline{u}_{n+1}) \right) \Delta t + G(\underline{u}_n)\,\Delta \underline{W}_n \quad , \quad \Delta t > 0, \quad u_0 \in \mathbb{R}, \quad \theta \in [0,1]$$

Exercise: stability of θ-Euler-Maruyama method

$$du = r u\,dt + \sigma u\,dW(t), \qquad u(0) = u_0$$

$$u_{n+1} = u_n + \left((1-\theta) r u_n + \theta r u_{n+1} \right) \Delta t + \sigma u_n \Delta W_n$$

$$u_{n+1} = u_n + \Delta t (1-\theta) r u_n + \Delta t \theta r u_{n+1} + \sigma u_n \Delta W_n$$

$$(1 - \Delta t \theta r)\, u_{n+1} = (1 + \Delta t (1-\theta) r + \sigma \Delta W_n)\, u_n$$

$$u_{n+1} = \frac{1 + \Delta t (1-\theta) r + \sigma \Delta W_n}{1 - \Delta t \theta r}\, u_n \Rightarrow u_n = \prod_{j=0}^{n-1} \frac{1 + \Delta t (1-\theta) r + \sigma \Delta W_n}{1 - \Delta t \theta r}\, u_0$$

$$\mathbb{E}[u_n^2] = \ldots = \frac{u_0^2}{(1-\Delta t\,\theta r)^2} \prod_{j=0}^{n-1} \mathbb{E}\left[(1 + \Delta t (1-\theta) r)^2 + \sigma^2 \Delta W_n^2 + 2(1 + \Delta t (1-\theta) r)\sigma \Delta W_n \right] =$$

$$= \frac{u_0^2}{(1-\Delta t \theta r)^2} \prod_{j=0}^{n-1} \left((1 + \Delta t (1-\theta) r)^2 + \sigma^2 \Delta t \right) \xrightarrow[n\to\infty]{} 0 \iff \left| \frac{(1 + \Delta t (1-\theta) r)^2 + \sigma^2 \Delta t}{(1 - \Delta t \theta r)^2} \right| < 1$$

$$1 + \Delta t^2 (1-\theta)^2 r^2 + 2\Delta t (1-\theta) r + \sigma^2 \Delta t < 1 + \Delta t^2 \theta^2 r^2 - 2 \Delta t \theta r$$

$$\Delta t (1-\theta)^2 r^2 + 2(1-\theta) r + \sigma^2 < \Delta t \theta^2 r^2 - 2\theta r$$

$$\Delta t r^2 + \Delta t r^2 \theta^2 - 2\Delta t r^2 \theta + 2r - 2r\theta + \sigma^2 < \Delta t \theta^2 r^2 - 2\theta r$$

$$\Delta t (r^2 - 2 r^2 \theta) + 2r + \sigma^2 < 0$$

$$\Delta t r^2 (1 - 2\theta) + 2r + \sigma^2 < 0$$

$$\theta = \frac{1}{2} \Rightarrow 2r + \sigma^2 < 0 \quad \text{!!}$$

Calculating the L^2 error numerically

$$\boxed{\| \underline{u}(T) - \underline{u}_N \|_{L^2(\Omega, \mathbb{R}^d)} \approx \left(\frac{1}{M} \sum_{j=1}^{M} \| \underline{u}^j(T) - \underline{u}^j_N \|_2^2 \right)^{\frac{1}{2}}}$$

- $\underline{u}^j(T) := \underline{u}(T, \omega_j)$ independent samples of the exact solution $\underline{u}(T)$
- $\underline{u}^j_N := \underline{u}_N(\omega_j)$ numerical approximation to $\underline{u}(T, \omega_j)$ for $T = N \Delta t$

Issues:

1. need to simulate $\underline{u}_N^j$ using the same Brownian path as $\underline{u}^j(T)$ $\rightarrow$ choose independent sample paths $\underline{W}(\cdot, \omega_j)$ of the Brownian motion $\underline{W}(t)$, making sure that the same sample path is used for both samples

2. usually no explicit solution $\rightarrow$ $\underline{u}^j(T)$ approximated by accurate reference solutions $\underline{u}^j_{N\mathrm{ref}}$ computed with a small time step $\Delta t_{\mathrm{ref}} = \frac{T}{N_{\mathrm{ref}}}$ $(\Delta t = K \Delta t_{\mathrm{ref}})$

$$\underline{W}((n+1)\Delta t) - \underline{W}(n\Delta t) = \sum_{j=nK}^{(n+1)K-1} \left(\underline{W}((j+1)\Delta t_{\mathrm{ref}}) - \underline{W}(j\Delta t_{\mathrm{ref}}) \right)$$

3. need large number of samples M and different timesteps Δt