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Lab Course
Scientific Computing

Worksheet 2

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due: Sun., 04.12.2022, 23:59 (submission on the Moodle page)
oral examination: Tue., 06.12.2022 (exact time slots announced on the Moodle page)

In this worksheet we work with two ODEs.

First, we examine **Dahlquist's test equation**:

$$\dot{x} = \lambda x \tag{1}$$

with initial condition

$$x(0) = 1. \tag{2}$$

The analytical solution is given by

$$x(t) = e^{\lambda t}.$$

For this worksheet we set $\lambda = -1$.

We use this rather simple equation with a known exact solution to examine the properties of different numerical methods.

a) Use MATLAB to plot the function $x(t)$ for $t \in [0, 5]$ in a graph.

b) Consider a general initial value problem

$$\dot{y} = f(y), \quad y(0) = y_0.$$

Implement the following explicit numerical methods

1) explicit Euler method,
2) method of Heun,
3) Runge-Kutta method (fourth order)

as a MATLAB function depending on the initial value y_0, the time step size δt, the end time t_{end}, and the right hand side $f(y)$[1], i.e.:
`function y = expl_euler(y_0, dt, t_end, f).`
The output of the function shall be a vector containing all approximate values for y, including y_0.

c) For each of the three methods implemented, compute approximate solutions x_k for equation (1) with initial conditions (2), end time $t_{end} = 5$. We want to investigate the behavior of the approximate solutions for different time step sizes $\delta t = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$.

i) For each of the three methods, create a separate figure. Each figure contains the four computed solutions for the four different time steps δt, all four in one plot. Add also the analytical solution for reference in each figure.

ii) For each case, compute the approximation error defined as

$$E := \sqrt{\frac{\delta t}{t_{end}} \sum_k \left(x_k - x_{k,exact}\right)^2},$$

where x_k denotes the approximation, $x_{exact,k}$ the exact solution at $t = \delta t \cdot k$.

iii) For each of the three methods, determine the factor by which the error E is reduced if the step size δt is halved. Write down all the results (errors and error reductions) in the tables below. Also write all information needed in the tables to the MATLAB console in a readable way.

Hint: the factors should be greater than one if the errors are reduced.

[1] Use a *MATLAB Function Handle* as argument.

explicit Euler method ($q = 1$)				
δt	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
error				
error red.	—			

method of Heun ($q = 2$)				
δt	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
error				
error red.	—			

Runge-Kutta method ($q = 4$)				
δt	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
error				
error red.	—			

Now we examine the **Van-der-Pol-Oscillator**:

$$\ddot{x} - \mu \left(1 - x^2\right) \dot{x} + x = 0. \tag{3}$$

This nonlinear second-order ODE can be transformed to a first-order form by introducing a second variable $y = \dot{x}$, resulting in the system:

$$\begin{cases} \dot{x} = y \\ \dot{y} = \mu(1 - x^2)y - x \end{cases} \tag{4}$$

with initial conditions

$$\begin{aligned} x(0) &= 1, \\ y(0) &= 1. \end{aligned} \tag{5}$$

For this worksheet we set $\mu = 1$.

d) Extend the Heun solver from part **b)** to also account for vector valued functions.

Hint: keep the function signature from the original version

e) Compute approximate solutions for equation (4) with initial conditions (5), end time $t_{end} = 20$ and time step size $\delta t = 0.1$, using the now extended method of Heun from **d)**. In the same figure, graph in separate subplots: x vs. t, y vs. t, and also the trajectory in phase space (y vs. x).

Optional: In a separate figure, create a quiver plot to visualize the phase space as a vector field. (Hint: use a meshgrid)

Questions:

Q1 By which factor is the error reduced for each halving of δt if you apply a

1) first order ($O(\delta t)$),

2) second order ($O\left(\delta t^2\right)$),

3) third order ($O\left(\delta t^3\right)$),

4) fourth order ($O\left(\delta t^4\right)$)

method.

Q2 For which integer q can you conclude that the error of the

1) explicit Euler method,

2) method of Heun,

3) Runge-Kutta method (fourth order)

behaves like $O\left(\delta t^q\right)$?

Q3 Is a higher order method always more accurate than a lower order method (for the same step size δt)?

Q4 How can we approximate the error if we don't have access to an analytical solution?

Q5 Assume you have to compute the solution up to a certain prescribed accuracy limit and that you see that you can do with less time steps if you use the Runge-Kutta-method than if you use Euler or the method of Heun. Can you conclude in this case that the Runge-Kutta method is the most efficient one of the three alternatives?

Q6 What do you observe when playing around with the initial values in **e)**?