Source Terms

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Seminar Course - Fundamentals of Wave Simulation - Solving Hyperbolic Systems of PDEs

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From Conservation Laws to Balance Laws

Our reference equation is

$$q_t + f(q)_x = \psi(q) \tag{1}$$

where

- the homogeneous equation $q_t + f(q)_x = 0$ is **hyperbolic**
- $\psi(q)$ (the **source terms**) don't depend on derivatives of q
  - $\Rightarrow q_t = \psi(q)$ is an independent system of ODEs

The Advection-Reaction Equation

A standard example that will be used to illustrate the following numerical methods is the **advection-reaction equation**

$$q_t + \bar{u} q_x = -\beta q. \tag{2}$$

It can be seen as the model for the transport along a flow of a radioactive substance, where

- β is the **decay** rate
- $\bar{u}$ is the (constant) **transport speed**
- $q(x, 0) = \mathring{q}(x)$ is the initial condition.

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Exact solution

Along the characteristic $\frac{dx}{dt} = \bar{u}$ we have $\frac{dq}{dt} = -\beta q$ and it follows that

$$q(x, t) = e^{-\beta t} \mathring{q}(x - \bar{u} t). \tag{3}$$

The Advection-Reaction Equation: Plot

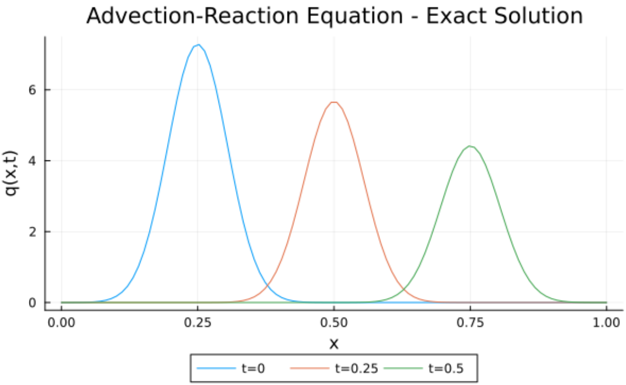

Figure: Evolution of the exact solution of the advection-reaction equation with $\bar{u} = 1$, $\beta = 1$, and $\mathring{q} = \text{Gaussian}(0.25, 0.003)$.

The Unsplit Method

Dor this specific example we can easily compute an **unsplit method**

$$
\begin{aligned}
q_t &= -\bar{u} q_x - \beta q \\
\frac{Q_i^{n+1} - Q_i^n}{\Delta t} &= -\bar{u} \frac{Q_i^n - Q_{i-1}^n}{\Delta x} - \beta Q_i^n \\
Q_i^{n+1} &= Q_i^n - \bar{u} \frac{\Delta t}{\Delta x} (Q_i^n - Q_{i-1}^n) - \Delta t \beta Q_i^n
\end{aligned}
$$

which is first-order accurate and stable for $0 < \bar{u} \frac{\Delta t}{\Delta x} \leq 1$.

Taylor Expansion of the Exact Solution

Note

The full Taylor expansion of (2) can be written formally as

$$e^{-\Delta t(\bar{u}\partial_x+\beta)}q(x,t) := q(x,t+\Delta t) = \\ = \sum_{j=0}^{\infty}\frac{(\Delta t)^j}{j!}\partial_t^j q(x,t) = \sum_{j=0}^{\infty}\frac{(\Delta t)^j}{j!}(-\bar{u}\partial_x-\beta)^j q(x,t). \tag{4}$$

The operator $e^{-\Delta t(\bar{u}\partial_x+\beta)}$ is called **solution operator** for the equation (2) over a time step of length Δt.

Bibliography

[1] R. J. LeVeque, **Finite Volume Methods for Hyperbolic Problems**. Cambridge: Cambridge University Press, 2002.

[2] `https://github.com/matilde-t/SeminarCourse-FundamentalsOfWaveSimulation`

[3] `https://github.com/clawpack/apps/tree/master/fvmbook/chap17`

[4] `https://github.com/clawpack/riemann_book`