Source Terms

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Seminar Course - Fundamentals of Wave Simulation - Solving Hyperbolic Systems of PDEs

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Table of contents

From Conservation Laws to Balance Laws

Our reference equation is

$$q_t + f(q)_x = \psi(q) \tag{1}$$

where

- the homogeneous equation $q_t + f(q)_x = 0$ is **hyperbolic**
- $\psi(q)$ (the **source terms**) don't depend on derivatives of q
  - $\Rightarrow q_t = \psi(q)$ is an independent system of ODEs

The Advection-Reaction Equation

A standard example that will be used to illustrate the following numerical methods is the **advection-reaction equation**

$$q_t + \bar{u} q_x = -\beta q. \tag{2}$$

It can be seen as the model for the transport along a flow of a radioactive substance, where

- β is the **decay rate**
- $\bar{u}$ is the (constant) **transport speed**
- $q(x, 0) = \mathring{q}(x)$ is the **initial condition**.

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Exact solution

Along the characteristic $\frac{dx}{dt} = \bar{u}$ we have $\frac{dq}{dt} = -\beta q$ and it follows that

$$q(x, t) = e^{-\beta t} \mathring{q}(x - \bar{u} t). \tag{3}$$

The Advection-Reaction Equation: Plot

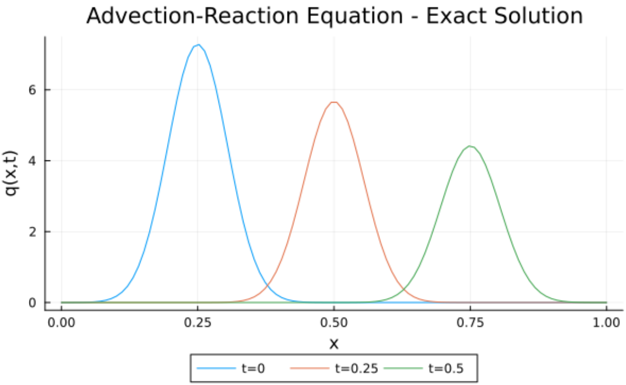

Figure: Evolution of the exact solution of the advection-reaction equation with $\bar{u} = 1$, $\beta = 1$, and $\mathring{q} = \text{Gaussian}(0.25, 0.003)$.

The Unsplit Method

Dor this specific example we can easily compute an **unsplit method**

$$
\begin{aligned}
q_t &= -\bar{u}q_x - \beta q \\
\frac{Q_i^{n+1} - Q_i^n}{\Delta t} &= -\bar{u}\frac{Q_i^n - Q_{i-1}^n}{\Delta x} - \beta Q_i^n \\
Q_i^{n+1} &= Q_i^n - \bar{u}\frac{\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n) - \Delta t \beta Q_i^n
\end{aligned}
$$

which is first-order accurate and stable for $0 < \bar{u}\frac{\Delta t}{\Delta x} \leq 1$.

Taylor Expansion of the Exact Solution

Note

The full Taylor expansion of (2) can be written formally as

$$e^{-\Delta t(\bar{u}\partial_x+\beta)}q(x,t) := q(x,t+\Delta t) = \\ = \sum_{j=0}^{\infty}\frac{(\Delta t)^j}{j!}\partial_t^j q(x,t) = \sum_{j=0}^{\infty}\frac{(\Delta t)^j}{j!}(-\bar{u}\partial_x-\beta)^j q(x,t). \tag{4}$$

The operator $e^{-\Delta t(\bar{u}\partial_x+\beta)}$ is called **solution operator** for the equation (2) over a time step of length Δt.

Godunov Splitting

In the case of the advection equation, we can split it into two subproblems:

$$\text{Problem A: } q_t + \bar{u} q_x = 0, \tag{5}$$

$$\text{Problem B: } q_t = -\beta q. \tag{6}$$

The idea is to apply the two methods in an alternating manner, using standard solving stategies, e.g.:

$$\text{A-step: } Q_i^* = Q_i^n - \frac{\bar{u}\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n), \tag{7}$$

$$\text{B-step: } Q_i^{n+1} = Q_i^* - \beta \Delta t Q_i^*. \tag{8}$$

Unsplit Method vs Godunov Splitting

One may think that given that both Q_i^* and Q_i^{n+1} are calculated using Δt, the solution is valid for time $2\Delta t$, but it is not really the case: in fact if we combine the two stages and eliminate Q_i^*, we obtain

$$Q_i^{n+1} = Q_i^n - \frac{\bar{u}\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n) - \beta \Delta t Q_i^n + \frac{\bar{u}\beta\Delta t^2}{\Delta x}(Q_i^n - Q_{i-1}^n),$$

which differs from the unsplit method for the last term:

$$Q_i^{n+1} = Q_i^n - \bar{u}\frac{\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n) - \Delta t \beta Q_i^n$$

Commuting vs Non-commuting operators

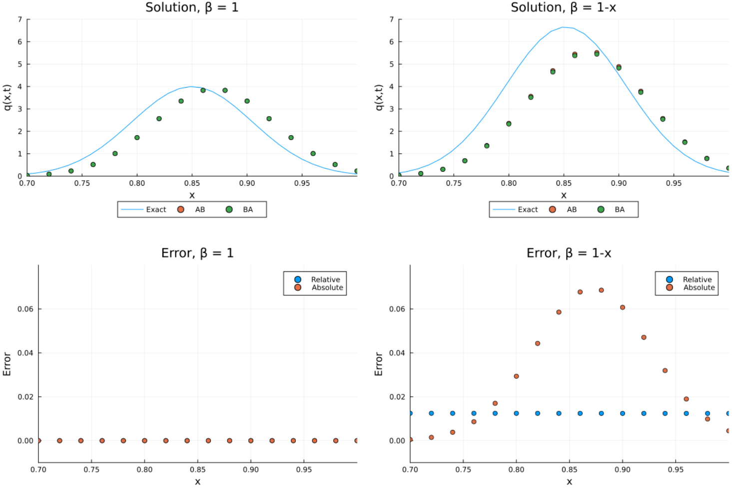

Figure: Comparison between the exact solution of the advection-reaction equation and the split method with the two different orders of steps. The problem has $\bar{u} = 1$, $\mathring{q} = \text{Gaussian}(0.25, 0.003)$, $\Delta x = \Delta t = 0.02$, $t = 0.6$.

General Formulation

Consider the more general formulation

$$q_t = (\mathcal{A} + \mathcal{B})q \tag{9}$$

where $\mathcal{A}$ and $\mathcal{B}$ can be differential operators. We assume that they don't explicitly depend on t, so that we can write

$$q_{tt} = (\mathcal{A} + \mathcal{B})q_t = (\mathcal{A} + \mathcal{B})^2 q \tag{10}$$

If we Taylor expand the solution at time t and use the notation defined in (4), we easily get to

$$q(x, \Delta t) = \sum_{j=0}^{\infty} \frac{\Delta t^j}{j!}(\mathcal{A} + \mathcal{B})^j q(x, 0) = e^{\Delta t(\mathcal{A}+\mathcal{B})} q(x, 0). \tag{11}$$

General Formulation II

With Godunov Splitting, we obtain

$$q^*(x, \Delta t) = e^{\Delta t \mathcal{A}} q(x, 0) \tag{12}$$

and

$$q^{**}(x, \Delta t) = e^{\Delta t \mathcal{B}} q^*(x, \Delta t) = e^{\Delta t \mathcal{B}} e^{\Delta t \mathcal{A}} q(x, 0). \tag{13}$$

The splitting error is then

$$q(x, \Delta t) - q^{**}(x, \Delta t) = (e^{\Delta t(\mathcal{A}+\mathcal{B})} - e^{\Delta t \mathcal{B}} e^{\Delta t \mathcal{A}}) q(x, 0). \tag{14}$$

If we Taylor expand qx^{**}, we obtain

$$q^{**}(x, \Delta t) = (I + \Delta t(\mathcal{A} + \mathcal{B}) + \frac{1}{2}\Delta t^2(\mathcal{A}^2 + 2\mathcal{B}\mathcal{A} + \mathcal{B}^2) + \dots) q(x, 0). \tag{15}$$

Strang Splitting

With the *Strang splitting* we are approximating $e^{\Delta t(\mathcal{A}+\mathcal{B})}$ by $e^{\frac{1}{2}\Delta t\mathcal{A}}e^{\Delta t\mathcal{B}}e^{\frac{1}{2}\Delta t\mathcal{A}}$. The Taylor expansion shows in fact that

$$e^{\frac{1}{2}\Delta t\mathcal{A}}e^{\Delta t\mathcal{B}}e^{\frac{1}{2}\Delta t\mathcal{A}} = I+\Delta t(\mathcal{A}+\mathcal{B})+\frac{1}{2}\Delta t^2(\mathcal{A}^2+\mathcal{A}\mathcal{B}+\mathcal{B}\mathcal{A}+\mathcal{B}^2)+\mathcal{O}(\Delta t^3). \tag{16}$$

After n time steps we obtain

$$Q^n = \underbrace{(e^{\frac{1}{2}\Delta t\mathcal{A}}e^{\Delta t\mathcal{B}}e^{\frac{1}{2}\Delta t\mathcal{A}})(e^{\frac{1}{2}\Delta t\mathcal{A}}e^{\Delta t\mathcal{B}}e^{\frac{1}{2}\Delta t\mathcal{A}})\dots(e^{\frac{1}{2}\Delta t\mathcal{A}}e^{\Delta t\mathcal{B}}e^{\frac{1}{2}\Delta t\mathcal{A}})}_{n \text{ times}} Q^0. \tag{17}$$

Another way of obtaining the same result is by alternating the oder of application of $\mathcal{A}$ and $\mathcal{B}$, in each time step, i.e.

$$\begin{aligned} Q^1 &= e^{\Delta t\mathcal{A}}e^{\Delta t\mathcal{B}}Q^0 \\ Q^2 &= e^{\Delta t\mathcal{B}}e^{\Delta t\mathcal{A}}Q^1 \\ &\;\;\vdots \end{aligned} \tag{18}$$

Bibliography

[1] R. J. LeVeque, **Finite Volume Methods for Hyperbolic Problems**. Cambridge: Cambridge University Press, 2002.

[2] `https://github.com/matilde-t/SeminarCourse-FundamentalsOfWaveSimulation`

[3] `https://github.com/clawpack/apps/tree/master/fvmbook/chap17`

[4] `https://github.com/clawpack/riemann_book`