

Source Terms

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Seminar Course - Fundamentals of Wave Simulation - Solving Hyperbolic Systems of PDEs

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Our reference equation is

$$q_t + f(q)_x = \psi(q) \quad (1)$$

where

- the homogeneous equation $q_t + f(q)_x = 0$ is **hyperbolic**
- $\psi(q)$ (the **source terms**) don't depend on derivatives of q
 - $\Rightarrow q_t = \psi(q)$ is an independent system of ODEs

A standard example that will be used to illustrate the following numerical methods is the **advection-reaction equation**

$$q_t + \bar{u}q_x = -\beta q. \quad (2)$$

It can be seen as the model for the transport along a flow of a radioactive substance, where

- β is the **decay rate**
- \bar{u} is the (constant) **transport speed**
- $q(x, 0) = \hat{q}(x)$ is the **initial condition**.

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Exact solution

Along the characteristic $\frac{dx}{dt} = \bar{u}$ we have $\frac{dq}{dt} = -\beta q$ and it follows that

$$q(x, t) = e^{-\beta t} \hat{q}(x - \bar{u}t). \quad (3)$$

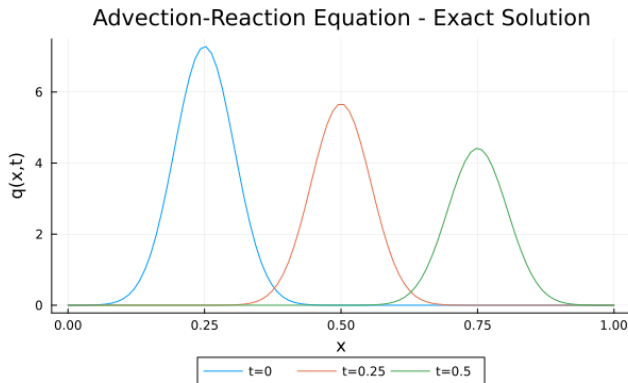


Figure: Evolution of the exact solution of the advection-reaction equation with $\bar{u} = 1$, $\beta = 1$, and $\hat{q} = \text{Gaussian}(0.25, 0.003)$.

For this specific example we can easily compute an **unsplit method**

$$\begin{aligned}q_t &= -\bar{u}q_x - \beta q \\ \frac{Q_i^{n+1} - Q_i^n}{\Delta t} &= -\bar{u} \frac{Q_i^n - Q_{i-1}^n}{\Delta x} - \beta Q_i^n \\ Q_i^{n+1} &= Q_i^n - \bar{u} \frac{\Delta t}{\Delta x} (Q_i^n - Q_{i-1}^n) - \Delta t \beta Q_i^n\end{aligned}$$

which is first-order accurate and stable for $0 < \bar{u} \frac{\Delta t}{\Delta x} \leq 1$.

Note

The full Taylor expansion of (2) can be written formally as

$$\begin{aligned} e^{-\Delta t(\bar{u}\partial_x + \beta)} q(x, t) &:= q(x, t + \Delta t) = \\ &= \sum_{j=0}^{\infty} \frac{(\Delta t)^j}{j!} \partial_t^j q(x, t) = \sum_{j=0}^{\infty} \frac{(\Delta t)^j}{j!} (-\bar{u}\partial_x - \beta)^j q(x, t). \end{aligned} \quad (4)$$

The operator $e^{-\Delta t(\bar{u}\partial_x + \beta)}$ is called **solution operator** for the equation (2) over a time step of length Δt .

In the case of the advection equation, we can split it into two subproblems:

$$\text{Problem A: } q_t + \bar{u}q_x = 0, \quad (5)$$

$$\text{Problem B: } q_t = -\beta q. \quad (6)$$

The idea is to apply the two methods in an alternating manner, using standard solving strategies, e.g.:

$$\text{A-step: } Q_i^* = Q_i^n - \frac{\bar{u}\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n), \quad (7)$$

$$\text{B-step: } Q_i^{n+1} = Q_i^* - \beta\Delta t Q_i^*. \quad (8)$$

One may think that given that both Q_i^* and Q_i^{n+1} are calculated using Δt , the solution is valid for time $2\Delta t$, but it is not really the case: in fact if we combine the two stages and eliminate Q_i^* , we obtain

$$Q_i^{n+1} = Q_i^n - \frac{\bar{u}\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n) - \beta\Delta t Q_i^n + \frac{\bar{u}\beta\Delta t^2}{\Delta x}(Q_i^n - Q_{i-1}^n),$$

which differs from the unsplit method for the last term:

$$Q_i^{n+1} = Q_i^n - \bar{u}\frac{\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n) - \Delta t\beta Q_i^n$$

- [1] R. J. LeVeque, **Finite Volume Methods for Hyperbolic Problems**. Cambridge: Cambridge University Press, 2002.
- [2] <https://github.com/matilde-t/SeminarCourse-FundamentalsOfWaveSimulation>
- [3] <https://github.com/clawpack/apps/tree/master/fvmbook/chap17>
- [4] https://github.com/clawpack/riemann_book