

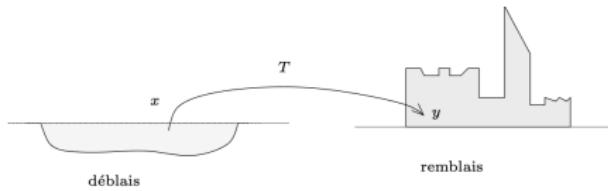
Relaxed intro to Optimal Transport

BAMS

December 9, 2025

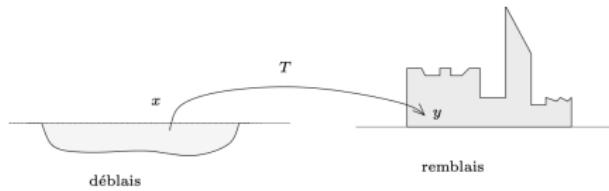
Introduction

- ▶ Problem of transporting mass in an efficient way
- ▶ Monge 1781: most efficient way of transporting soil from the ground to a given place
- ▶ Kantorovich 1975: Nobel prize for Economics “for their contributions to the theory of optimum allocation of resources”
- ▶ Many formulations: we will focus on two, discrete and continuous versions
- ▶ Many applications!!!



Introduction

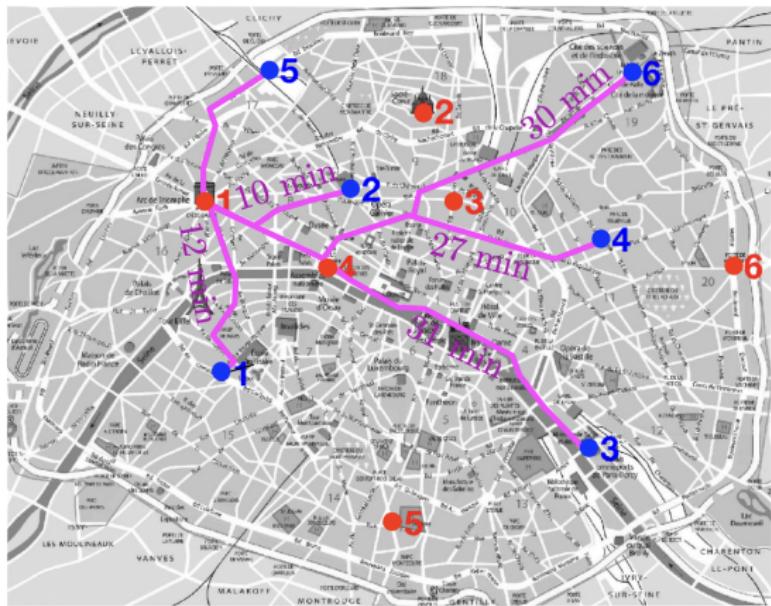
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Disclaimer: attempted to be a soft and not 100% formal introduction to OT! :)

Why are we doing this !

Fom bakeries to cafés

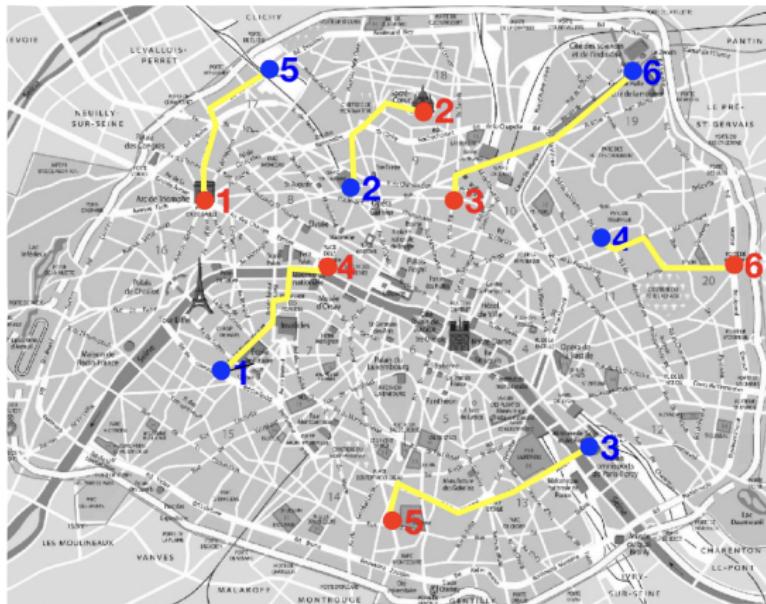


c_{ij}	y_1	y_2	y_3	y_4	y_5	y_6
x_1	12	10	31	27	10	30
x_2	22	7	25	15	11	14
x_3	19	7	19	10	15	15
x_4	10	6	21	19	14	24
x_5	15	23	14	24	31	34
x_6	35	26	16	9	34	15

Figure: Supplying all cafés from bakery 1

Why are we doing this II

Fom bakeries to cafés



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Figure: Organizing supply using optimal transport!

Why are we doing this III

In 3-D: Color Image Palette Equalization

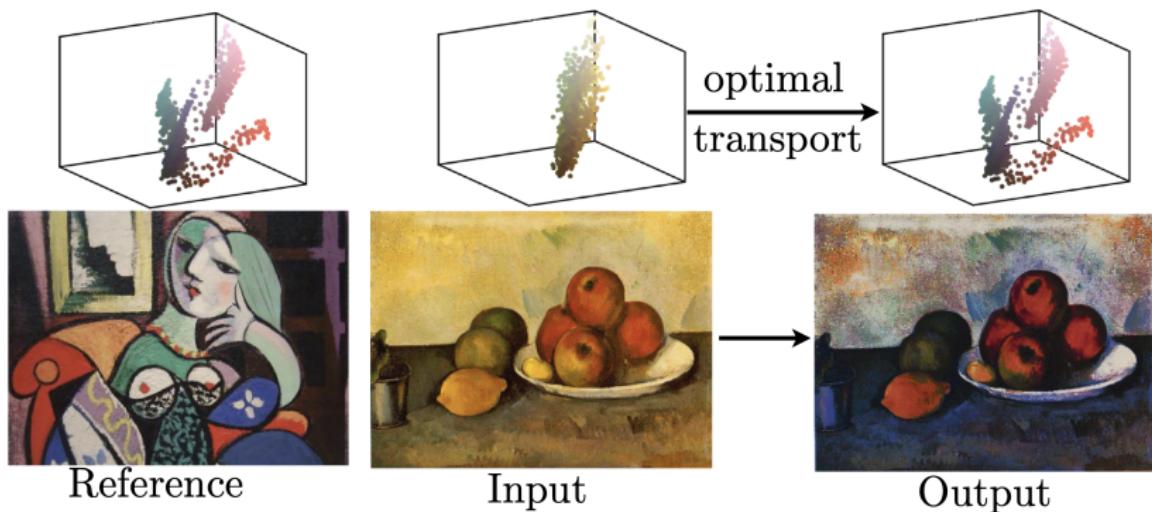


Figure: Optimally allocating color

Convexity I

Convex Set

A set $C \subseteq \mathbb{R}^n$ is convex if for any $x, y \in C$ and any $\lambda \in [0, 1]$, we have

$$\lambda x + (1 - \lambda)y \in C.$$

Intuitively, any line segment between two points in C stays inside C .

Extremal Points

A point $x \in C$ is an *extremal point* of a convex set C if

$$x = \lambda y + (1 - \lambda)z \Rightarrow y = z$$

for $y, z \in C$ and $\lambda \in (0, 1)$.

E.g. the extremal points of a polygon are its vertices.

Convexity II

Convex Function

A function $f : C \rightarrow \mathbb{R}$ defined on a convex set C is convex if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all $x, y \in C$, $\lambda \in [0, 1]$. Geometrically, its graph lies below the chord joining any two points.

Polyhedra

A polyhedron is the intersection of finitely many hyperplanes, i.e. for $b \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$

$$P = \{x \in \mathbb{R}^n : Ax \leq b\}.$$

If P is bounded, it is a polytope.

Convexity III

Polytope

A polytope is a bounded convex set obtained as the convex hull of finitely many points. Example: a triangle, cube, or any convex polygon/polyhedron.

Convex Optimization

A convex optimization problem is one of the form

$$\min_{x \in C} f(x)$$

where C is convex and f is convex. Nice property: any local minimum is also a global minimum.

Discrete Monge formulation

- ▶ Two clouds of points $\{x_i\}_{i=1,\dots,n}$, $\{y_i\}_{i=1,\dots,n}$
- ▶ Two probability measures (distribution of masses) on them
- ▶ The cost of transporting mass x_i to y_j is C_{ij}

Permutation

Given an interval $I \subset \mathbb{N}$, $I = \{1, \dots, n\}$, a permutation is a bijective function $\sigma : I \rightarrow I$.

We indicate $\text{Sym}(I)$ or just $\text{Sym}(n)$ the set of all permutations on $I = \{1, \dots, n\}$.

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Monge's formulation

$$\min_{\sigma \in \text{Sym}(n)} \sum_{i,j} C_{i,\sigma(j)} \quad (\text{D1})$$

Discrete Kantorovich formulation

Kantorovich's extension: we are allowed to "divide" masses! yay!

Bistochastic matrices

A matrix $P \subset \mathbb{R}_+^{n \times n}$ is a bistochastic matrix if $\sum_j P_{ij} = 1$, $\sum_i P_{ij} = 1$, i.e. its rows and its columns sum to 1.
We indicate B_n the set of bistochastic matrices in $\mathbb{R}_+^{n \times n}$.

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Kantorovich's formulation

$$\min_{P \in B_n} \sum_{ij} P_{ij} C_{ij} \tag{D2}$$

Connecting Monge and Kantorovich

We can rewrite permutations as in Monge's formulation via permutations matrices

$$P_n := B_n \cap \{0, 1\}^{n \times n}$$

Then Monge's problem becomes:

$$\min_{P \in P_n} \sum_{ij} P_{ij} C_{ij} \tag{D1'}$$

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Important remarks

1. $B_n \subset [0, 1]^{n \times n}$ is a polyhedra
2. $P_n = \text{Ext}(B_n)$ (Von Neumann)
3. Kantorovich formulation is a **convex relaxation** of Monge's one
4. OT problem as in ?? can be solved efficiently via linear programming

Probability measures

Probability measure

A probability measure on (a σ -algebra \mathcal{A} of) a set X is a measure $\mu : \mathcal{A} \rightarrow [0, 1]$ such that $\mu(X) = 1$. We indicate as $\mathcal{P}(X)$ the set of measures on X .

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Push-forward measure

Given a function $T : X \rightarrow Y$, we define the push forward operator $T_{\#} : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ by

$$T_{\#}\mu(A) = \mu(T^{-1}(A)) \quad \forall A \in \mathcal{B}(Y)$$

and call $T_{\#}\mu$ push forward measure.

The push forward measure is the measure assigning to a set the measure of its **pre-image** according to a map T .

Transport via functions

Given

- ▶ two spaces X and Y
- ▶ probability measures $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$
- ▶ cost function $c : X \rightarrow Y$

Monge's formulation

$$\min_{T|T\#\mu=\nu} \int_X c(x, T(x)) \, d\mu(x) \quad (\text{C1})$$

Transport via transport plans I

Transport plan

A transport plan is a probability measure $\pi \in \mathcal{P}(X \times Y)$ such that $\pi(A \times Y) = \mu(A)$, $\pi(X \times B) = \nu(B)$.

We indicate $\Gamma(\mu, \nu)$ the set of all transport plans for μ and ν .

Transport plans are another way of carrying mass from X to Y , and in particular $\pi(A \times B)$ is the mass that was in $A \subset X$ and has been sent to $B \subset Y$.

For stats buddies: transport plans are like joint probability measures!

Kantorovich's formulation

$$\min_{\pi \in \Gamma(\mu, \nu)} \int_{X \times Y} c(x, y) \, d\pi(x, y) \quad (\text{C2})$$

Connecting Monge and Kantorovich, again

For any transport *map* T , there exists a transport *plan* π_T :

$$T \longmapsto (\text{id} \times T)_\# \mu =: \pi_T$$

Moreover, it can be checked $\mathcal{C}(T) = \mathcal{C}(\pi_T)$ where \mathcal{C} is the total cost attained by a transport map/plan.

Thus,

$$\inf_M \geq \inf_K$$

Useful resources

- ▶ Brué Ambrosio Semola Lecture on Optimal Transport
- ▶ Villani Optimal Transport Old and New
- ▶ Peyré and Cuturi Computational Optimal Transport
- ▶ Peyré slides on discrete part
- ▶ Mati's notes chapter 3

Beware: there's a sea out there, only check out topics we went over or ask for more!