# Poincaré, modified logarithmic Sobolev and isoperimetric inequalities for Markov chains with non-negative Ricci curvature

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# Overview of the talk

Main result: convergence to equilibrium of the zero-range process

- The zero-range process
- Bound on mixing time and interpretation

From geometry to inequalities

- Setup
- Ricci for discrete processes
- Modified Logarithmic Sobolev inequality for discrete processes with non-neg Ricci

Further inequalities

- Poincaré inequality and the spectral gap
- Isoperimetric inequality and Cheeger constant

Other applications

# The zero-range process I

Consider $K$ interacting particles on the complete graph with $L$ sites.

The state space is $\mathcal{X}_{K,L} = \{\eta \in \mathbb{N}^L : \sum_i \eta_i = K\}$.

The zero-range process with **constant rates** can be described as:

1. Choose a site $i$ uniformly at random
   1.1 If $\eta_i = 0$, do nothing
   1.2 Else, choose another site $j$ uniformly at random and move 1 particle from $i$ to $j$

$\eta^{i,j}$ denotes the new configuration after such a move.

$$Q_{K,L}(\eta, \theta) = \begin{cases} \frac{1}{L} & \theta = \eta^{i,j} \text{ for some } i, j \\ 0 & \text{else} \end{cases}$$

Invariant measure is the uniform measure $\pi_{K,L}$.

# The zero-range process II

Good model for:

- traffic flow
- population dynamics
- particle physics
- ... any system in which deciding to move depends on the surroundings

# Main result and interpretation

## Convergence to equilibrium

The constant-rate zero range process with $K$ particles and $L$ sites has mixing time bounded by

$$\tau_{mix}(\varepsilon) \leq KL \log L \left( \frac{1}{8} - \frac{\log \varepsilon}{c} \right) \tag{1}$$

for some universal constant $c$.

# Main result and interpretation

## Convergence to equilibrium

The constant-rate zero range process with $K$ particles and $L$ sites has mixing time bounded by

$$\tau_{mix}(\varepsilon) \leq KL\log L\left(\frac{1}{8} - \frac{\log\varepsilon}{c}\right) \quad (1)$$

for some universal constant $c$.

$\star$ This is convergence to equilibrium! $\star$

# Setup

Finite space of states $\mathcal{X}$, irreducible, reversible Markov kernel $Q$ and stationary measure $\pi$

$$Q(x,y)\pi(x) = Q(y,x)\pi(y)$$

Operator $L$ acting on functions $\psi : \mathcal{X} \to \mathbb{R}$ is

$$L\psi(x) = \sum_{y\in\mathcal{X}} (\psi(y) - \psi(x))Q(x,y)$$

is the generator of a continuous time Markov chain.
Space of probability densities

$$\mathscr{P}(\mathcal{X}) := \left\{\rho : \mathcal{X} \to \mathbb{R}_+ : \sum_{x\in\mathcal{X}} \rho(x)\,\pi(x) = 1\right\}$$

## Distance $\mathcal{W}$

We define a discrete transport distance, by analogy with the Wasserstein distance. For $\rho_0, \rho_1 \in \mathscr{P}(\mathcal{X})$,

$$\mathcal{W}(\rho_0, \rho_1)^2 := \inf_{\rho,\psi} \frac{1}{2} \int_0^1 \sum_{x,y \in X} \big(\psi_t(x) - \psi_t(y)\big)^2 \hat{\rho}_t(x,y) \, Q(x,y) \, \pi(x) \, dt.$$

where the infimum runs over all sufficiently regular curves satisfying a continuity equation

$$\begin{cases} \dfrac{d}{dt}\rho_t(x) + \displaystyle\sum_{y \in \mathcal{X}} \big(\psi_t(y) - \psi_t(x)\big)\hat{\rho}_t(x,y) Q(x,y) = 0, \quad \forall x \in \mathcal{X}, \\ \rho|_{t=0} = \rho_0, \quad \rho|_{t=1} = \rho_1. \end{cases}$$

and $\hat{\rho}(x,y) = \theta(\rho(x), \rho(y))$ which is the logarithmic mean.

Endowing $\mathcal{X}$ with this distance, we obtain that the Markov semigroup $P_t = e^{tL}$ is the gradient flow of Shannon's entropy $\mathcal{H}(\rho) = \sum_{x \in \mathcal{X}} \rho(x) \log \rho(x) \pi(x)$

# Ricci for discrete processes

## Entropic Ricci curvature for discrete space

$(\mathcal{X}, Q, \pi)$ has entropic Ricci curvature bounded from below by $\kappa \in \mathbb{R}$ if for any $\rho_t \in (\mathscr{P}(\mathcal{X}), \mathcal{W})$ we have

$$\mathcal{H}(\rho_t) \leq (1-t)\mathcal{H}(\rho_0) + t\mathcal{H}(\rho_1) - \frac{\kappa}{2}t(1-t)\mathcal{W}(\rho_0, \rho_1)^2$$

In this case, we write $\mathrm{Ric}(\mathcal{X}, Q, \pi) \geq \kappa$.

# Ricci for the zero-range process

In another contribution, M. Fathi and J. Maas provide explicit ways of computing bounds on Ricci for discrete processes

They show that for the ZRP with *increasing* rates, Ricci is positive

The ZRP with *constant* rates has $\mathrm{Ric} \geq 0$

# The Modified log-Sobolev inequality I

## Modified logarithmic Sobolev inequality

Bound on the entropy in terms of a norm of the gradient of our observable:

$$\mathcal{H}(\rho) \leq \frac{1}{2\lambda}\left\|\nabla \log \rho\right\|^2 = \frac{1}{2\lambda}\mathcal{I}(\rho) \qquad \text{(MLSI)}$$

where $\mathcal{I}$ is the Fisher information.

Implications:

- encodes the distribution of the eigenvalues of the Laplacian (generator)
- entropy decay along the heat flow $\mathcal{H}(P_t\rho) \leq e^{-2\lambda t}\mathcal{H}(\rho)$
- with Poincaré ineq,, it implies **convergence to equilibrium**
- convergence in Wassertsein distance $\mathcal{W}(\rho_t, \sigma_t) \leq e^{-2\lambda t}\mathcal{W}(\rho_0, \sigma_0)$
- hypercontractivity $\|P_t\rho\| \leq \|\rho\|$

# The Modified log-Sobolev inequality II

We look at convergence to equilibrium in terms of the **mixing time** of the Markov chain.

## Total variation mixing time of a Markov chain

For $\varepsilon > 0$,

$$\tau_{mix}(\varepsilon) := \inf\left\{t > 0 : \|P_t^*\delta_x - \pi\|_{TV} < \varepsilon \quad \forall x \in \mathcal{X}\right\}$$

Since

$$\|\nu - \pi\|_{TV}^2 \leq \frac{1}{2}\mathcal{H}(\nu) \qquad \text{(Pinsker inequality)}$$

we can use MLSI to bound $\tau_{mix}$.

# The Modified log-Sobolev inequality III

## Convergence to equilibrium

Assume $\mathrm{Ric}(\mathcal{X}, Q, \pi) \geq 0$ and the diameter of $(\mathcal{X}, d_{\mathcal{W}})$ is $\leq D$. If a MLSI with constant $\lambda = \frac{c}{D^2}$ holds, then

$$\tau_{mix}(\varepsilon) \leq D^2(1/8 - c \log \varepsilon)$$

for some universal constant $c$.

# Modified Log-Sobolev inequality for the zero-range process

### Diameter bound

There is a constant $c > 0$ such that the diameter of $(\mathcal{X}_{K,L}, d_{\mathcal{W}})$ is bounded by $cK\sqrt{L \log L}$

So, for the zero range process with constant uniform rates, a MLSI with constant $\frac{c}{K^2 L \log L}$ holds.

### Convergence of the ZRP

The total variation mixing time of the zero range process on $\mathcal{X}_{K,L}$ with constant uniform rates is

$$\tau_{mix}(\varepsilon) \leq KL \log L \left( \frac{1}{8} - \frac{\log \varepsilon}{c} \right)$$

for some universal constant $c$.

# Poincaré inequality

# Isoperimetric inequality and Cheeger constant

# example 1

# example 2