MAXIMAL AUTOMORPHISM GROUPS OF SURFACES

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Abstract. We give an overview of the full classification of pairs $(S, \mathrm{Aut}^\circ(S))$, where S is a relatively minimal surface and $\mathrm{Aut}^\circ(S)$ is a maximal connected algebraic subgroup of $\mathrm{Bir}(S)$. When S is not rational, this classification is obtained through two articles [Fon24, FM24] and holds over any algebraically closed field of arbitrary characteristic.

1. Introduction

This is a survey note on the classification of maximal connected algebraic subgroups of groups of birational transformations of surfaces over $\mathbf{k}$, an algebraically closed field of arbitrary characteristic. We denote the automorphism group of a projective surface S as $\mathrm{Aut}(S)$, which is equipped with a natural structure of smooth algebraic group. With respect to this structure, we write $\mathrm{Aut}^\circ(S)$ for the neutral component. The notation $\mathrm{Bir}(S)$ indicates the group of birational maps from S into itself.

In [Enr93], Enriques classified the connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^2)$ over $\mathbb{C}$. We may recover his classification over $\mathbf{k}$ and obtain the following: every connected algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ is contained in a maximal one, and the latter is conjugate to one among

- $\mathrm{Aut}^\circ(\mathbb{P}^2) = \mathrm{PGL}_3$,
- $\mathrm{Aut}^\circ(\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)))$ with $n \neq 1$.

Notice that $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}) = \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathrm{Aut}^\circ(\mathbb{P}^1 \times \mathbb{P}^1) = {\mathrm{PGL}_2}^2$. If $n > 0$, then the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ has a unique minimal section of negative self-intersection and the group

$$\mathrm{Aut}^\circ(\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))) \simeq (\mathrm{GL}_2/\mu_n) \ltimes \mathbf{k}^{n+1}$$

acts on $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ with two orbits: the minimal section and its complement.

A modern strategy to study connected algebraic subgroups works as follows. We start by employing a regularization result together with the minimal model program. Let S' be a projective surface and G be a connected algebraic subgroup of $\mathrm{Bir}(S')$. By a refined version of Weil regularization theorem (see [Wei55, Bri17]), there exists a smooth projective surface S'', G-birationally equivalent to S', on which G acts by automorphisms. Running an MMP, a fortiori G-equivariant, yields that G is conjugate to an algebraic subgroup of $\mathrm{Aut}^\circ(S)$, where S is a relatively minimal surface (namely, without (-1)-curves) birational to S'. Thus, we are reduced to the study of automorphism groups of relatively minimal surfaces and equivariant birational maps between them.

In this note, our focus is on the classification of pairs $(S, \mathrm{Aut}^\circ(S))$, where S is a non-rational *relatively minimal surface* and $\mathrm{Aut}^\circ(S)$ is a *maximal connected algebraic subgroup* of $\mathrm{Bir}(S)$.

2. Automorphism groups of ruled surfaces

We first consider surfaces birational to $C \times \mathbb{P}^1$, where C is a smooth projective curve of genus $g \geq 1$; those surfaces are of negative Kodaira dimension and not rational. The associated relatively minimal surfaces are ruled surfaces over C, i.e., smooth projective surfaces S equipped with a structure of $\mathbb{P}^1$-bundle $\pi\colon S \to C$. The classification result is the following:

Theorem A. [Fon24, Theorem A] *Let C be a curve of genus g and $\pi\colon S \to C$ be a $\mathbb{P}^1$-bundle. If $g = 1$, then $\mathrm{Aut}^\circ(S)$ is maximal if and only if S is one of the following:*

(1) $C \times \mathbb{P}^1$,
(2) $\mathbb{P}(\mathcal{O}_C \oplus \mathcal{L})$ *with* $\mathcal{L}$ *a non-trivial line bundle over* C *of degree zero,*
(3) *the Atiyah's ruled surface* $\mathcal{A}_0$,
(4) *the Atiyah's ruled surface* $\mathcal{A}_1$.

On the other hand, if $g \geq 2$ *then* $\mathrm{Aut}^\circ(S)$ *is maximal if and only if* $S = C \times \mathbb{P}^1$.

In the statement above, we distinguish two special ruled surfaces over an elliptic curve C, denoted $\mathcal{A}_0$ and $\mathcal{A}_1$, which correspond to the two indecomposable $\mathbb{P}^1$-bundles over C classified by Atiyah in [Ati57]. If $g = 1$ and $\mathrm{Aut}^\circ(S)$ is maximal, then $\mathrm{Aut}^\circ(S)$ acts transitively on C via the structural morphism π. A description of the maximal automorphism groups in Theorem A as extensions of $\mathrm{Aut}^\circ(C)$ by a linear group is given in [Mar71]: it follows that the maximal connected algebraic subgroups of $\mathrm{Bir}(S)$ are of dimension at most 4.

Consider now the ruled surface $S = \mathbb{P}(\mathcal{O}_C \oplus \mathcal{L})$ with $\mathcal{L}$ a line bundle of positive degree. Then the automorphism group $\mathrm{Aut}^\circ(S)$ is not maximal and it acts trivially on C through the structural morphism π. Hence the automorphisms of S commute with the transition matrices of the $\mathbb{P}^1$-bundle π, which may be chosen diagonal. A local computation gives that $\mathrm{Aut}^\circ(S) \simeq \mathbb{G}_m \rtimes \Gamma(C, \mathcal{L})$ (for details, see [FZ23]). In particular, we may choose $\mathcal{L}$ such that $\mathrm{Aut}^\circ(S)$ is of arbitrary large dimension and not contained in any maximal connected algebraic subgroup of $\mathrm{Bir}(S)$.

On the other hand, if S is a relatively minimal surface of non-negative Kodaira dimension, then S is the unique minimal model in its birational class, hence $\mathrm{Aut}^\circ(S)$ is maximal. We obtain:

Corollary B. [Fon24, Theorem C] *Let* S *be a surface. Every connected algebraic subgroup of* $\mathrm{Bir}(S)$ *is contained in a maximal one if and only if* S *is not birational to* $C \times \mathbb{P}^1$ *for some smooth projective curve* C *of genus* $g \geq 1$.

Thus, the structure of $\mathrm{Bir}(C \times \mathbb{P}^1)$ is very special in dimension two and in this case, we cannot reduce the study of connected algebraic subgroups to the maximal ones. For analog results in higher dimensions, see [FZ23, FFZ23, Kol24].

3. Automorphism groups of surfaces with $\kappa \geq 0$

In this section, we consider relatively minimal surfaces S with Kodaira dimension $\kappa \geq 0$. The automorphism group $\mathrm{Aut}^\circ(S)$ is an Abelian variety of dimension ≤ 2, with equality if and only if $S = \mathrm{Aut}^\circ(S)$ is an Abelian surface acting on itself by translations.

Let us assume that $\mathrm{Aut}^\circ(S) = E$ is an elliptic curve. Then S is a contracted product

$$E \times^G X,$$

where $G \subset E$ is a finite subgroup scheme and X is a G-*normal curve*, namely every finite birational G-equivariant morphism with target X is an isomorphism: see [Bri24].

If $\mathrm{char}(\mathbf{k}) = 0$, then $G \subset E$ is a finite constant group and X is a smooth projective curve. In this case, the maximal automorphism groups are classified in [Fon24].

If $\mathrm{char}(\mathbf{k}) = p > 0$, then the notion of equivariantly normal curves really comes into play, because G *can be non-reduced* and X can admit cusps as singularities. However, the quotient $Y = X/G$ is a smooth projective curve. In [FM24], the Betti numbers of S are computed as follows. Using the comparison theorem, a result of lifting to characteristic zero for curves, together with the proper base change theorem for étale cohomology, we reduce to singular cohomology. These numerical invariants are enough to locate the birational class of the surface S.

Theorem C. [FM24, Theorem A] *Assume that* $\mathrm{char}(\mathbf{k}) = p > 0$. *The elliptic surface* S *satisfies* $\kappa \leq 1$ *and has the following Betti numbers:*

$$b_1(S) = 2 + 2g(Y), \quad b_2(S) = 2 + 4g(Y).$$

- *If $\kappa = 0$, then either the surface S is quasi-hyperelliptic and X is a rational curve with a cusp, or S is hyperelliptic or an Abelian surface: in both latter cases, X is an elliptic curve.*
- *If $\kappa = 1$, then S is a proper elliptic surface, and X can be any other G-normal curve with arithmetic genus $p_a(X) \geq 2$.*

In the cases of Abelian and hyperelliptic surfaces, X is an elliptic curve, on which the finite subgroup scheme $G \subset E$ acts by translations in the first case, but not in the second.

Using Theorem C, we get the classification of maximal connected algebraic subgroups of $\mathrm{Bir}(S)$ in positive characteristic (see [FM24, Corollary B]). Together with [Fon24, Theorem D], we may now state the classification in any characteristic:

Corollary D. *Let S be a relatively minimal surface with $\kappa \geq 0$. Then $\mathrm{Aut}^\circ(S)$ is a maximal connected algebraic subgroup of $\mathrm{Bir}(S)$, isomorphic to an Abelian variety; and every connected algebraic subgroup of $\mathrm{Bir}(S)$ is conjugate to an algebraic subgroup of $\mathrm{Aut}^\circ(S)$.*
The pairs $(S, \mathrm{Aut}^\circ(S))$ are classified as follows:

$\kappa(S)$	S	$\mathrm{Aut}^\circ(S)$
0	*Enriques surface*	*Trivial*
0	*K3 surface*	*Trivial*
0	*Hyperelliptic surface*	*Elliptic curve*
0	*Quasi-hyperelliptic surface*	*Elliptic curve*
0	*Abelian surface*	*Abelian surface*
1	*Proper elliptic surface*	*Elliptic curve or trivial*
2	*General type surface*	*Trivial*

The case where S is a quasi-hyperelliptic surface occurs only if $\mathrm{char}(\mathbf{k}) \in \{2, 3\}$. Moreover, if S is a proper elliptic surface, then $\mathrm{Aut}^\circ(S)$ is an elliptic curve if and only if S is a contracted product $E \times^G X$, with $G \subset E$ a finite subgroup scheme and X a G-normal curve with arithmetic genus $p_a(X) \geq 2$.

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