Linear Control Systems
25411

Instructor: Behzad Ahi — **Assignment 1**

Fall 1403 — **Due Date: 1403/8/7**

Matin M.babaei   400102114

1 Laplace Transform

1.1 Find the Laplace transforms of the following functions

1. $f_1(t) = te^{-10t} + e^{-20t}$
2. $f_2(t) = 1 + \cosh 5t$
3. $f_3(t) = \sum_{n=0}^{\infty}(-1)^n u(t-n)$
4. $f_4(t) = \sqrt{t} + 3t$

---

1. $e^{-10t} \longleftrightarrow \dfrac{1}{s+10}$ $\qquad$ $e^{-20t} \longleftrightarrow \dfrac{1}{s+20}$

$te^{-10t} \longleftrightarrow -\dfrac{d}{ds}\left(\dfrac{1}{s+10}\right) = +\dfrac{1}{(s+10)^2}$

$F_1(s) = \dfrac{1}{(s+10)^2} + \dfrac{1}{s+20}$

با فرض $t \geq 0$ بود، در غیر این صورت ← $\tilde{f}_i(t) = f_i(t)u(t) + f_i(t)u(-t)$ ← بقیه ماجرا

2.

$1 \longleftrightarrow \dfrac{1}{s}$ $\qquad$ $\cosh(5t) = \dfrac{e^{5t}+e^{-5t}}{2} \longleftrightarrow \dfrac{1}{2}\left(\dfrac{1}{s-5} + \dfrac{1}{s+5}\right)$

$F_2(s) = \dfrac{1}{s} + \dfrac{1}{2}\left(\dfrac{1}{s-5} + \dfrac{1}{s+5}\right) = \dfrac{1}{s} + \dfrac{s}{s^2-25}$

با فرض $t \geq 0$ بود، در غیر این صورت ← $\tilde{f}_i(t) = f_i(t)u(t) + f_i(t)u(-t)$ ← بقیه ماجرا

3.

$$f_p(t) = \sum_{n=0}^{\infty} (-1)^n u(t-n) = u(t) - u(t-1) + u(t-2) - \dots + \dots$$

$$u(t-a) \longrightarrow \frac{e^{-as}}{s}$$

$$F_p(s) = \frac{1}{s} - \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} \dots = \frac{1}{s}\left(1 - e^{-s} + (e^{-s})^2 - \dots\right)$$

$$a_0 = 1 \quad q = -e^{-s} \xrightarrow{|q|<1} \frac{1}{1+e^{-s}} \rightarrow F_p(s) = \frac{1}{s}\left(\frac{1}{1+e^{-s}}\right)$$

4.

$$f_2(t) = \sqrt{t} + 3t \qquad (\text{به ازای } t \geq 0)$$

First Note: (we know $t \leftrightarrow \frac{1}{s^2}$)

$$\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} = \int_{0^-}^{\infty} \frac{1}{\sqrt{t}} e^{-st}\, dt = \frac{1}{s}\int_{0^-}^{\infty} \frac{\sqrt{s}}{\sqrt{u}} e^{-u}\, du$$

$u = st \rightarrow dt = \frac{1}{s} du$ (انتگرال گوسی)

$$w = \sqrt{u} \rightarrow u = w^2, \; dw = \frac{1}{2\sqrt{u}} du \rightarrow \int_0^{\infty} e^{-w^2} dw = \frac{\sqrt{\pi}}{2}$$

$$\rightarrow \mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} = \frac{2}{\sqrt{s}}\left(\frac{\sqrt{\pi}}{2}\right) = \frac{\sqrt{\pi}}{\sqrt{s}} \qquad (s>0)$$

Now See:

$$\mathcal{L}\{\sqrt{t}\} = \mathcal{L}\{t\frac{1}{\sqrt{t}}\} = -\frac{d}{ds}\mathcal{L}\{\frac{1}{\sqrt{t}}\} = -\frac{d}{ds}\left(\frac{\sqrt{\pi}}{\sqrt{s}}\right) = \frac{\sqrt{\pi}}{2 s^{3/2}} \quad (s>0)$$

$$\rightarrow F_4(s) = \frac{3}{s^2} + \frac{\sqrt{\pi}}{2 s^{3/2}}$$

1.2 Find the inverse Laplace transform of the following functions.

1. $T_1(s) = \dfrac{3s+1}{s^2+4}$
2. $T_2(s) = \dfrac{1}{s(1-e^{-s})}$
3. $T_3(s) = \dfrac{120(s^2+3)}{(s^2+4)(s^2+9)}$

1.

$$T_1(s) = \frac{3s+1}{s^2+4} = 3\frac{s}{s^2+2^2} + \frac{1}{s^2+2^2}$$

Recall,

$$\cos(at) \longleftrightarrow \frac{a}{s^2+a^2}, \quad \sin(at) \longleftrightarrow \frac{s}{s^2+a^2}$$

$$T_1(s) = \left(3\frac{s}{s^2+2^2} + \frac{1}{2}\,\frac{2}{s^2+2^2}\right) u(t)$$

$$\mathcal{L}^{-1}(T_1(s)) = 3\cos(2t) + \frac{1}{2}\sin(2t)$$

2. $f_2(t) \rightarrow T_2(s)$

$$T_2(s) = \frac{1}{s(1-e^{-s})} = \frac{1}{s}\,\frac{1}{1-e^{-s}}$$

$$\underset{\langle T\rangle}{f_2(t)} = \tilde{f}_2(t) \longrightarrow F_2(s) = \frac{\tilde{F}_2(s)}{1-e^{sT}}$$

$$\longrightarrow \tilde{T}_2(s)\ \frac{1}{1-e^{-Ts}} = \frac{1}{s}\cdot\frac{1}{1-e^{-s}} \qquad \begin{cases} \tilde{T}_2(s) = \frac{1}{s} \\ T = 1 \end{cases}$$ دوره تناوب

$$\tilde{T}_2(s) = \frac{1}{s} \longrightarrow \tilde{f}_2(t) = 1 \longrightarrow$$ Fail! Period $= \infty$

So, we consider another way:

$$T_2(s) = \frac{1}{s}\cdot\frac{1}{1-e^{-s}} \overset{\substack{s>0 \\ |e^{-s}|<1}}{=\!=} \frac{1}{s}\left(1+e^{-s}+e^{-2s}+\dots\right)$$

$$\longrightarrow f_2(t) = u(t)+u(t-1)+\dots = \sum_{n\in W} u(t-n)$$

3. $$T_3(s) = \frac{120\,(s^2+3)}{(s^2+4)(s^2+9)}$$

تفکیک کسر :

$$\frac{s^2+3}{(s^2+4)(s^2+9)} = \frac{A}{s^2+4} + \frac{B}{s^2+9}$$

$$\begin{cases} A+B = 1 \\ 9A+4B = 3 \end{cases} \longrightarrow 5A = -1 \longrightarrow A = -\frac{1}{5},\ B = \frac{6}{5}$$

$$\rightarrow T_3(s) = 120 \left(\frac{-\frac{1}{5}}{s^2+4} + \frac{4/5}{s^2+9} \right)$$

$$T_3(s) = 24 \left(-\frac{1}{2} \cdot \frac{2}{s^2+4} + 2 \cdot \frac{3}{s^2+9} \right)$$

$$\rightarrow \mathcal{L}^{-1}(T_3(s)) = 24 \left(-\frac{1}{2} \sin(2t) + 2 \sin(3t) \right)$$

$$\mathcal{L}^{-1}\{T_3(s)\} = \left(-12 \sin(2t) + 48 \sin(3t) \right) u(t)$$

2 Laplace in Differential Equations

Use the Laplace transform in order to solve the following differential equations where x and y are functions of t.

1. $x'' - x' - 2x = 0: \ x(0) = 0, x'(0) = 2$
2. $x'' + x = \sin 2t: \ x(0) = x'(0) = 0$
3. $x' = x + 2y', y' = 6x + 3y: \ x(0) = 1, y(0) = -2$
4. $x'' + 2x' + 4y = 0, y'' + 6x + 2y = 0: \ x(0) = y(0) = 0, x'(0) = y'(0) = -1$

1.

$$x'' - x' - 2x = 0 \qquad x(0) = 0, \quad x'(0) = 2$$

$\mathcal{L}$:

$$s^2X(s) - s\cancel{x(0)} - x'(0) - (sX(s) - \cancel{x(0)}) - 2X(s) = 0$$

$$\rightarrow s^2X(s) - sX(s) - 2X(s) = 2$$

$$\rightarrow X(s)(s^2 - s - 2) = 2 \rightarrow X(s)(s-2)(s+1) = 2$$

$$\rightarrow X(s) = \frac{2}{(s-2)(s+1)} = 2\left(\frac{A}{s-2} + \frac{B}{s+1} \right)$$

$$\begin{cases} A + B = 0 \\ A - 2B = 1 \end{cases} \longrightarrow A - 2(-A) = 1 \rightarrow A = \frac{1}{3},\ B = \frac{-1}{3}$$

$$\longrightarrow X(s) = \frac{2}{3}\left(\frac{1}{s-2} - \frac{1}{s+1}\right)$$

$$\longrightarrow x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{2}{3}\left(e^{2t} - e^{-t}\right) u(t) \quad (t \geq 0)$$

2. $x'' + x = \sin 2t \qquad x(0) = x'(0) = 0$

$$\longrightarrow s^2 X(s) + X(s) = \frac{2}{s^2+4} \longrightarrow X(s) = 2\,\frac{1}{(s^2+1)(s^2+4)}$$

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{A}{s^2+1} + \frac{B}{s^2+4} = \frac{1}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right)$$

$$\begin{cases} A + B = 0 \\ 4A + B = 1 \end{cases} \longrightarrow 3A = 1 \rightarrow A = \frac{1}{3},\ B = \frac{-1}{3}$$

$$X(s) = \frac{2}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right) = \frac{2}{3}\left(\frac{1}{s^2+1} - \frac{1}{2}\,\frac{2}{s^2+4}\right)$$

$$\longrightarrow x(t) = \frac{2}{3}\left(\sin(t) - \frac{1}{2}\sin(2t)\right) u(t) \quad (t \geq 0)$$

3. $$\begin{cases} ۲y' = x' - x & x(۰) = ۱ \ , \ y(۰) = -۲ \\ y' - ۳y = ۶x \end{cases}$$

$\mathcal{L}$:

$$\begin{cases} ۲sY(s) - ۲y(۰) = sX(s) - x(۰) - X(s) \\ sY(s) - y(۰) - ۳Y(s) = ۶X(s) \end{cases}$$

$$\begin{cases} ۲sY(s) + ۴ = sX(s) - X(s) - ۱ \\ sY(s) + ۲ - ۳Y(s) = ۶X(s) \end{cases}$$

$$\begin{cases} ۲sY(s) + (۱-s)X(s) = -۵ \\ (s-۳)Y(s) + (-۶)X(s) = -۲ \end{cases}$$

حل ۲ معادله ۲ مجهول $\longrightarrow$ $X(s) = \dfrac{s-۱۵}{s^۲-۱۶s+۳} \ , \ Y(s) = \dfrac{-۲s+۳۲}{s^۲-۱۶s+۳}$

یافتن ریشه‌ها به کمک تجزیه و تفکیک کسر $\longrightarrow$ $\boxed{x(t) = \mathcal{L}^{-1}\{X(s)\} = \dfrac{(۷+\sqrt{۶۱})e^{\alpha_۱ t} + (\sqrt{۶۱}-۷)e^{\alpha_۲ t}}{۲\sqrt{۶۱}}}$

$$\boxed{\begin{aligned} \alpha_۱ &= ۸-\sqrt{۶۱} \\ \alpha_۲ &= ۸+\sqrt{۶۱} \end{aligned}}$$

$$\boxed{y(t) = -\frac{(۸+\sqrt{۶۱})e^{\alpha_۱ t} + (\sqrt{۶۱}-۸)e^{\alpha_۲ t}}{\sqrt{۶۱}}} \quad t>۰$$

4. 4. $x'' + 2x' + 4y = 0, y'' + 6x + 2y = 0 :\ x(0) = y(0) = 0, x'(0) = y'(0) = -1$

$$\begin{cases} s^2 X(s) + 1 + 2sX(s) + 4Y(s) = 0 \\ s^2 Y(s) + 1 + 6X(s) + 2Y(s) = 0 \end{cases}$$

$$\begin{cases} (s^2 + 2s)X(s) + 4Y(s) = -1 \\ 6X(s) + (s^2+2)Y(s) = -1 \end{cases}$$

$$\begin{pmatrix} s^2+2s & 4 \\ 6 & s^2+2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \frac{1}{(s^2+2s)(s^2+2) - 24} \begin{pmatrix} s^2+2 & -4 \\ -6 & s^2+2s \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$= \frac{1}{s(s+2)(s^2+2) - 24} \begin{pmatrix} s^2+2 & -4 \\ -6 & s^2+2s \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$\xrightarrow{\text{با حل}} \begin{cases} X(s) = \dfrac{-s^2+2}{s^4+2s^3+2s^2+4s-24} \\ Y(s) = \dfrac{s^2+2s-6}{s^4+2s^3+2s^2+4s-24} \end{cases}$$

ابتدا باید $X(s)$ و $Y(s)$ را باز کرد:

$$X = \frac{N_1}{s + D_1} + \frac{N_2}{s + D_2} + \frac{N_3 s + N_4}{s^2 + D_3 s + D_4}$$

با ضرب کردن دو طرف در ریشه های صفرشونده، ضرایب مجهول پیدا میشوند،

E.g. $N_1 = \lim_{s \to -D_1} \frac{2 - s^2}{(s+D_2)(s^2+D_3 s+D_4)}$ مثلاً در اینجا مقادیر جالبی ندارند!

$$X(s) = \frac{N_1}{s+D_1} + \frac{N_2}{s+D_2} + \frac{N_3\left(s+\frac{D_3}{2}\right) + N_4 - \frac{N_3 D_3}{2}}{\left(s+\frac{D_3}{2}\right)^2 + D_4 - \frac{D_3^2}{4}}$$

$$\rightarrow x(t) = \Bigg(N_1 e^{-D_1 t} + N_2 e^{-D_2 t} + N_3 e^{-\frac{D_3}{2}t} \cos\left(\sqrt{D_4 - \frac{D_3^2}{4}}\, t\right)$$

$$+ \frac{N_4 - \frac{N_3 D_3}{2}}{\sqrt{D_4 - \frac{D_3^2}{4}}} e^{-\frac{D_3 t}{2}} \sin\left(\sqrt{D_4 - \frac{D_3^2}{4}}\, t\right) \Bigg) u(t)$$

اگر $Y(s)$ را هم به شکل قبلی Decompose کنیم، فرم جواب مانند $x(t)$ صرفاً با ضرایب دیگری است.

$$Y(s) = \frac{N_1}{s+D_1} + \frac{N_2}{s+D_2} + \frac{N_3\left(s+\frac{D_3}{2}\right) + N_4 - \frac{N_3 D_3}{2}}{\left(s+\frac{D_3}{2}\right)^2 + D_4 - \frac{D_3^2}{4}}$$

Then:

$$y(t) = \Bigg(N_1 e^{-D_1 t} + N_2 e^{-D_2 t} + N_3 e^{-\frac{D_3}{2}t} \cos\left(\sqrt{D_4 - \frac{D_3^2}{4}}\, t\right)$$

$$+ \frac{N_4 - \frac{N_3 D_3}{2}}{\sqrt{D_4 - \frac{D_3^2}{4}}} e^{-\frac{D_3 t}{2}} \sin\left(\sqrt{D_4 - \frac{D_3^2}{4}}\, t\right) \Bigg) u(t)$$

What happens if the boundary conditions are split? For example could you solve the problem in the first case if we had boundary conditions $x(0) = 0, x(10) = 2$ instead of $x(0) = 0, x'(0) = 2$? In *optimal control* course, you will learn how to numerically solve a general nonlinear differential equation with the split boundary conditions.

Yes, they are const terms.

If we change the boundary conditions, e.g. by setting $x(0) = 0$ and $x(10) = 2$, the problem becomes more complex because we lose initial conditions suited for the Laplace transform, which is generally applied when we have all conditions at $t = 0$.

In optimal control theory, such problems with split boundary conditions (where conditions are specified at different points in time) can be handled numerically. This typically involves transforming the problem into a boundary value problem ([illegible]) (BVP) and using methods like the shooting method or finite difference method to solve it. These techniques iteratively adjust the conditions and solving the system until both boundary conditions are satisfied.

3 Derivative of Laplace Transform

Find $f(0^+)$, $f'(0^+)$ and $f''(0^+)$ for the function whose Laplace transform is given below. Assume that $f(t) = 0$ within the interval $t < 0$.

$$F(s) = \frac{s+2}{s^2(s+1)(s+2)}$$

$$F(s) = \frac{s+2}{s^2(s+1)(s+2)}$$

$$f(0^+) = \lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{s(s+2)}{s^2(s+1)(s+2)}$$

$$= 0$$

$$f'(0^+) = \lim_{t\to 0^+} f'(t) = \lim_{s\to\infty} s^2F(s) - s\cancelto{0}{f(0)}$$

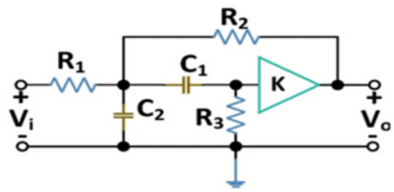
$$= \lim_{s\to\infty} \frac{s^2(s+2)}{s^2(s+1)(s+2)} = 0$$

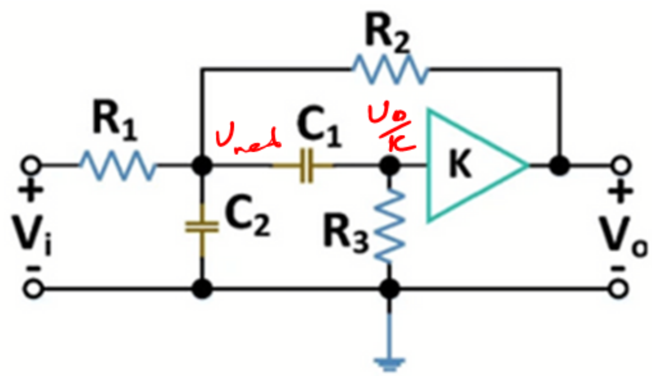
$$f''(0^+) = \lim_{s\to\infty} s^3F(s) - s^2\cancelto{0}{f(0^+)} - s\cancelto{0}{f'(0^+)}$$

$$= \lim_{s\to\infty} \frac{s^3(s+2)}{s^2(s+1)(s+2)} = 1$$

4 Transfer Function

Find the transfer function $G(s) = \dfrac{V_o(s)}{V_i(s)}$ in the following circuit assuming the amplifier to be ideal. Sketch the block diagram of the close-loop system if we want that the output voltage V_o follows a reference voltage V_{ref}.





$u = V_i \qquad x = V_{ref} \qquad y = V_o$

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

$$\begin{cases} \dfrac{V_i - V_{ref}}{R_1} = \dfrac{V_{ref}}{\frac{1}{C_2 s}} + \dfrac{V_{ref} - \frac{V_o}{K}}{\frac{1}{C_1 s}} + \dfrac{V_{ref} - V_o}{R_2} \\[2ex] \dfrac{V_{ref} - \frac{V_o}{K}}{\frac{1}{C_1 s}} = \dfrac{V_o/K}{R_3} \end{cases}$$

$$\longrightarrow V_{ref} - \frac{V_o}{K} = \frac{V_o}{K C_1 R_3 s} \longrightarrow V_{ref} = \frac{V_o}{K}\left(1 + \frac{1}{C_1 R_3 s}\right)$$

$$\frac{U_i}{R_1} = U_{ref}\left(\frac{1}{R_1} + C_2 S + C_1 S + \frac{1}{R_2}\right) - U_o\left(\frac{1}{K C_1 S} + \frac{1}{R_2}\right)$$

$$\frac{U_i}{R_1} = \frac{U_o}{K}\left(1 + \frac{1}{C_1 R_3 S}\right)\left((C_1 + C_2)S + \frac{1}{R_1} + \frac{1}{R_2}\right) - U_o\left(\frac{1}{K C_1 S} + \frac{1}{R_2}\right)$$

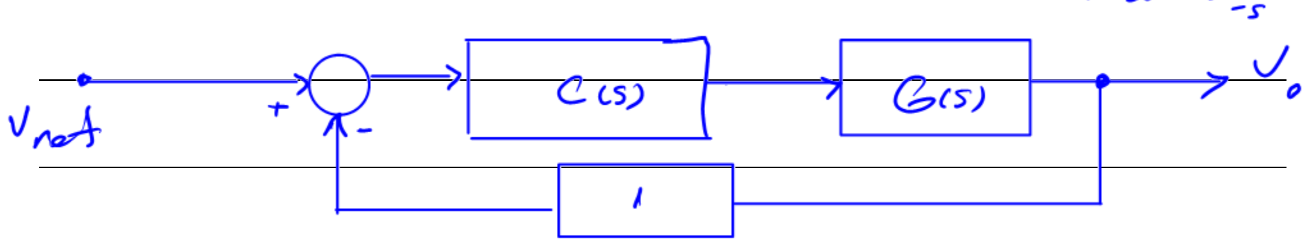
$$\rightarrow \frac{U_i}{R_1} = U_o\left(\frac{1}{K}\left(1 + \frac{1}{C_1 R_3 S}\right)\left((C_1 + C_2)S + \frac{1}{R_1} + \frac{1}{R_2}\right) - \left(\frac{1}{K C_1 S} + \frac{1}{R_2}\right)\right)$$

$$\rightarrow \boxed{\frac{U_o}{U_i} = \frac{1}{R_1\left(\frac{1}{K}\left(1 + \frac{1}{C_1 R_3 S}\right)\left((C_1 + C_2)S + \frac{1}{R_1} + \frac{1}{R_2}\right) - \left(\frac{1}{K C_1 S} + \frac{1}{R_2}\right)\right)}}$$

ساده سازی :

$$G(s) = \frac{U_o(s)}{U_i(s)} = \frac{K R_2 R_3 C_1 S}{R_1 + R_2 + S R_1 R_2 (C_1 + C_2) + C_1 R_3 S\left(R_1(1-K) + R_2\right) + C_1 C_2 S^2 R_1 R_2 R_3}$$

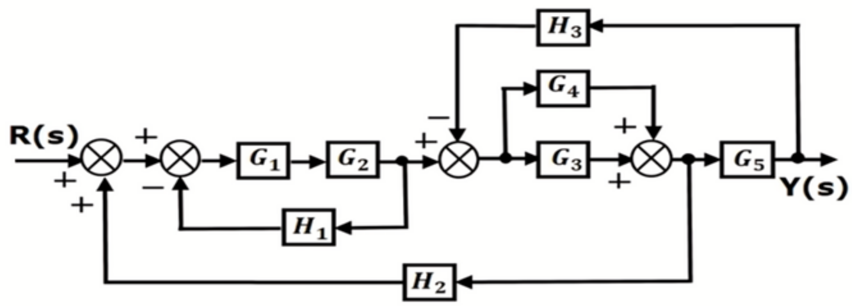
حال می‌خواهیم به نحوی یک سیستم فیدبک دار U_o از U_{ref}
سروو کنیم :



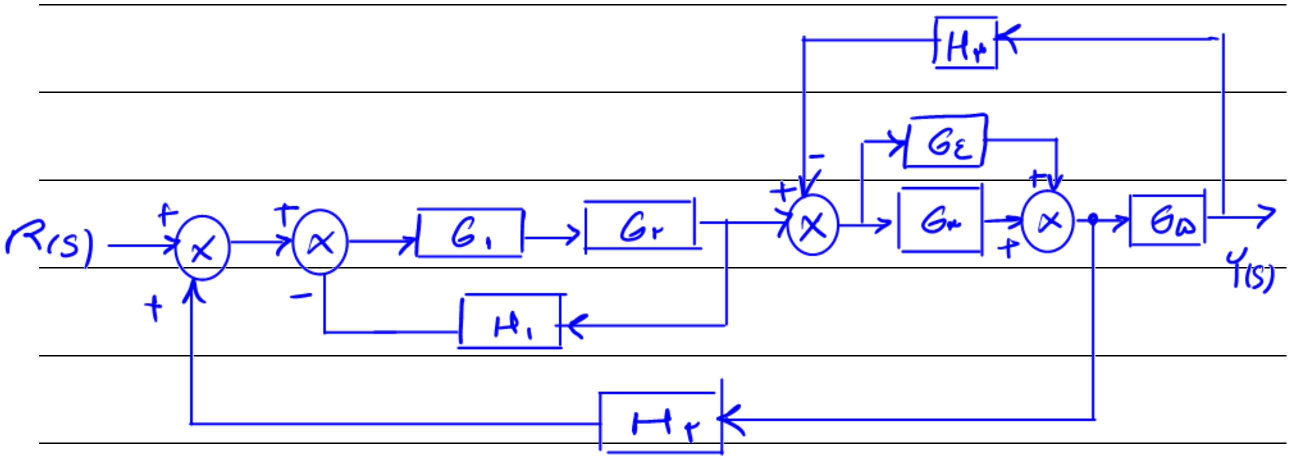
Suiteble $C(s)$ has to be implemented.

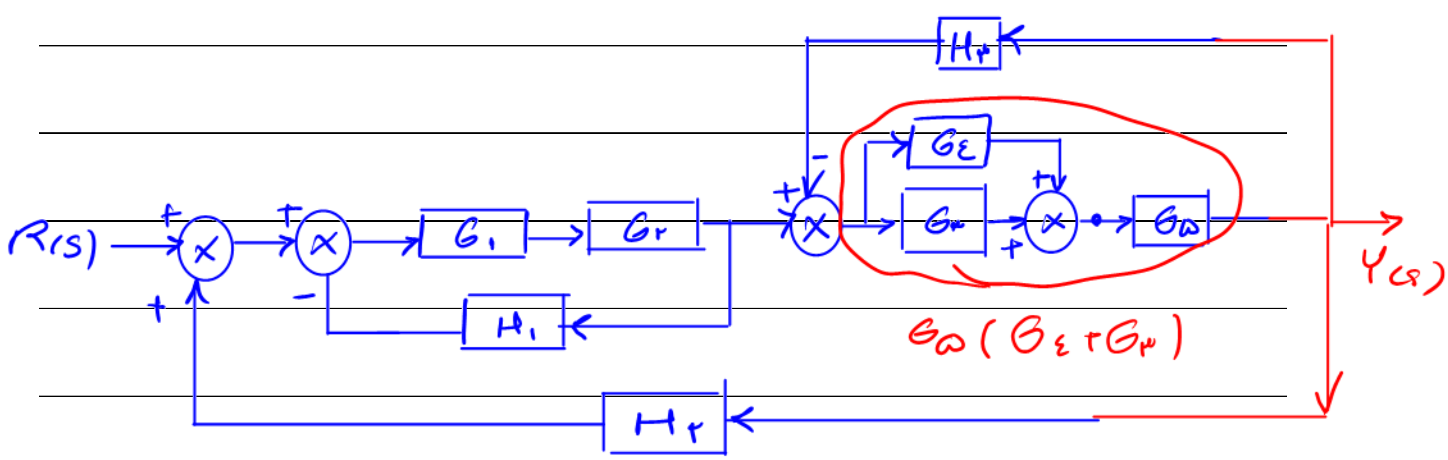
5 Block Diagram Reduction

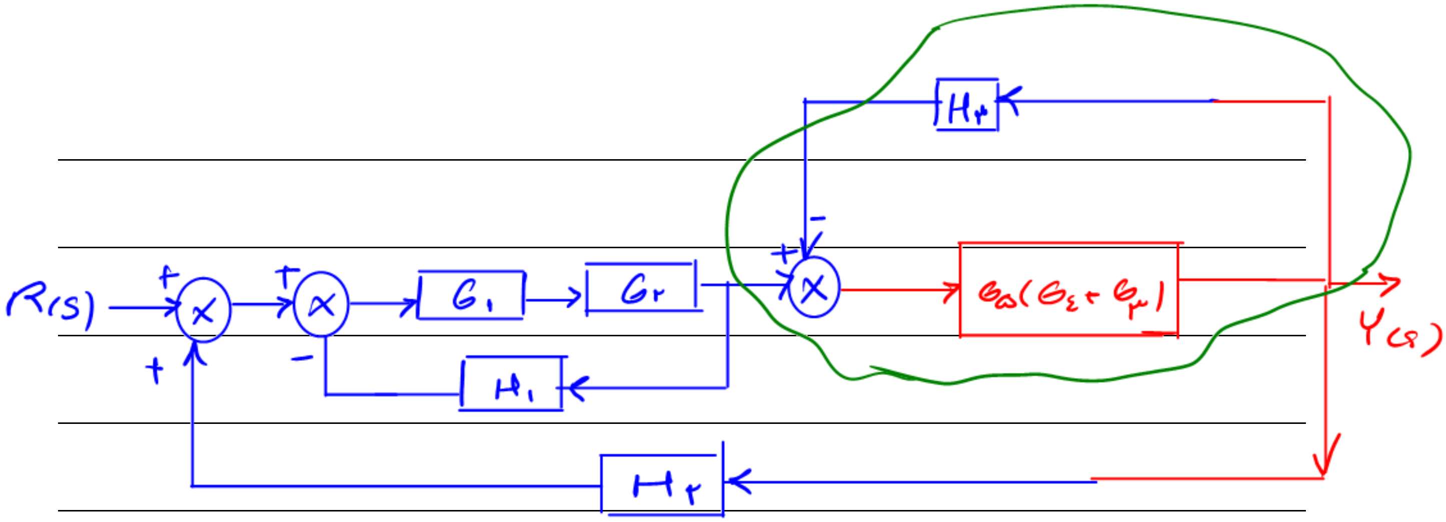
Determine the $Y(s)$ in the following block diagram. Use two methods and show the similarity of results.

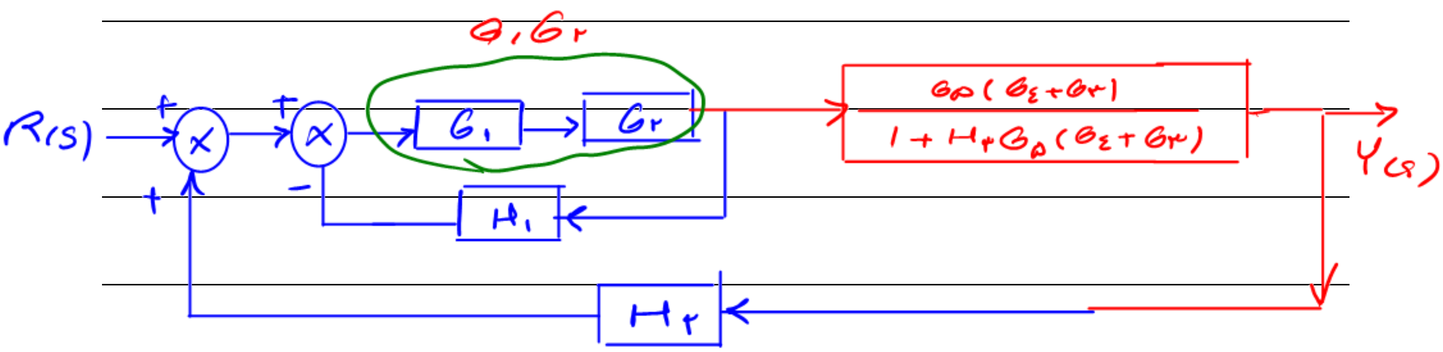


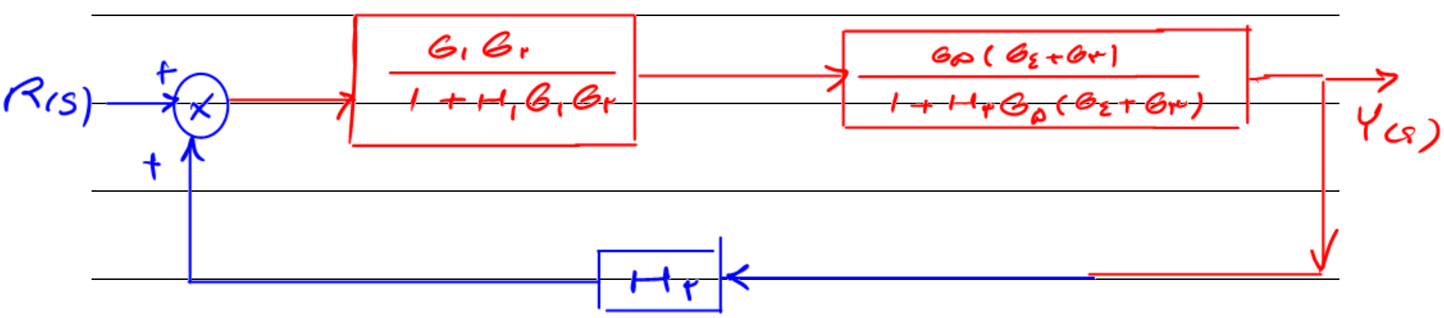
روش اول) ساده‌سازی بلوک دیاگرام :

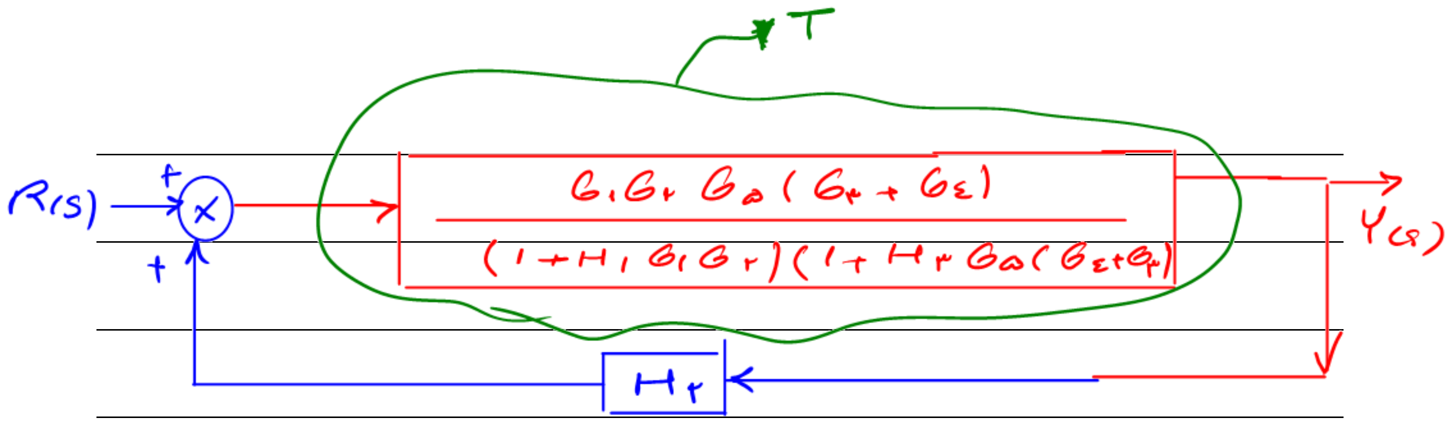





R(s)
G1
Gr
H1
Hr
Y(s)


R(s)
G1
Gr
H1
Hr
Y(s)


R(s)
Hr
Y(s)

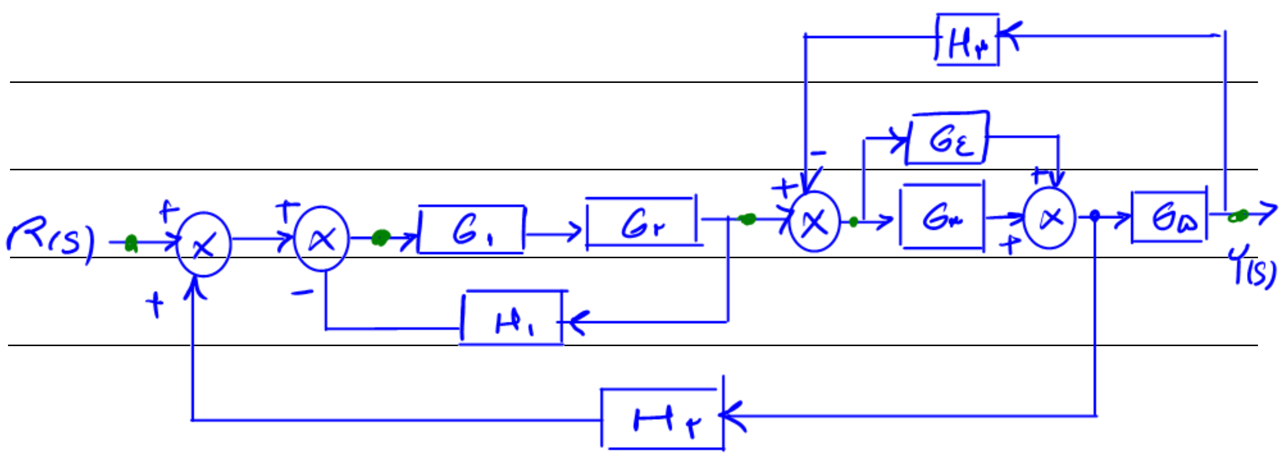


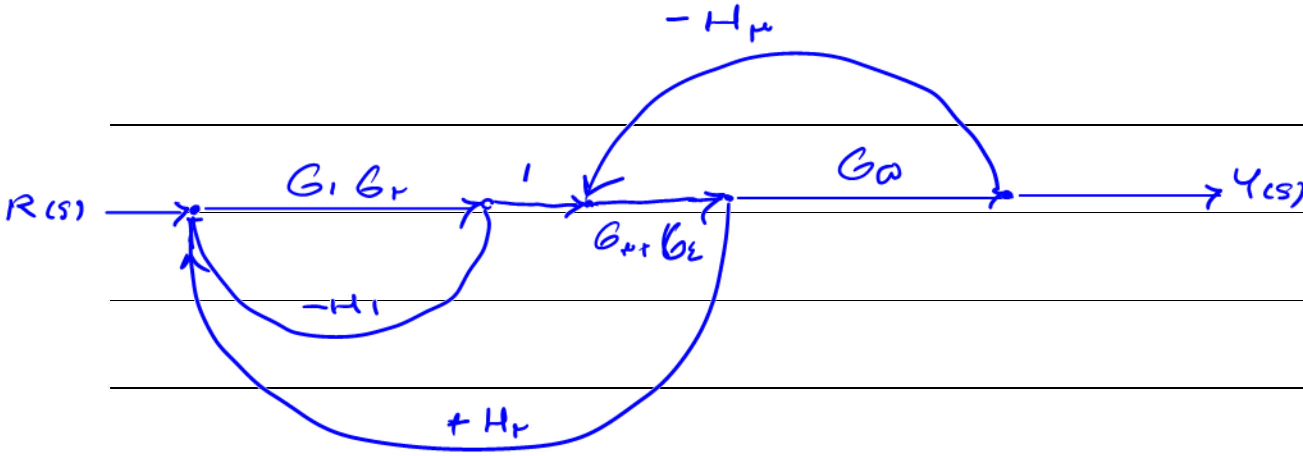
$$\frac{Y(s)}{R(s)} = \frac{T}{1-TH_2} = \frac{1}{\frac{1}{T} - H_2}$$

$$= \frac{1}{\frac{(1+H_1G_1G_2)(1+H_3G_5(G_3+G_4))}{G_1G_2G_5(G_3+G_4)} - H_2}$$

$$= \frac{G_1G_2G_5(G_3+G_4)}{1+H_3G_5(G_3+G_4)+H_1G_1G_2+H_1H_3G_1G_2G_5(G_3+G_4)-H_2G_1G_2G_5(G_3+G_4)}$$

روش دوم) قانون میسون





$$P_1 = G_1G_2(G_3+G_4)G_5$$

$$L_1 = -H_1G_1G_2$$

$$L_2 = +G_1G_2(G_3+G_4)H_2$$

$$L_3 = -H_3(G_3+G_4)G_5$$

Non-touching $\longrightarrow L_1L_3$

$$\Delta(s) = 1-(L_1+L_2+L_3)+L_1L_3$$

$$= 1+H_1G_1G_2 - G_1G_2(G_3+G_4)H_2 + H_3(G_3+G_4)G_5$$

$$+H_1H_3G_1G_2(G_3+G_4)G_5$$

$$\Delta_1 = 1$$

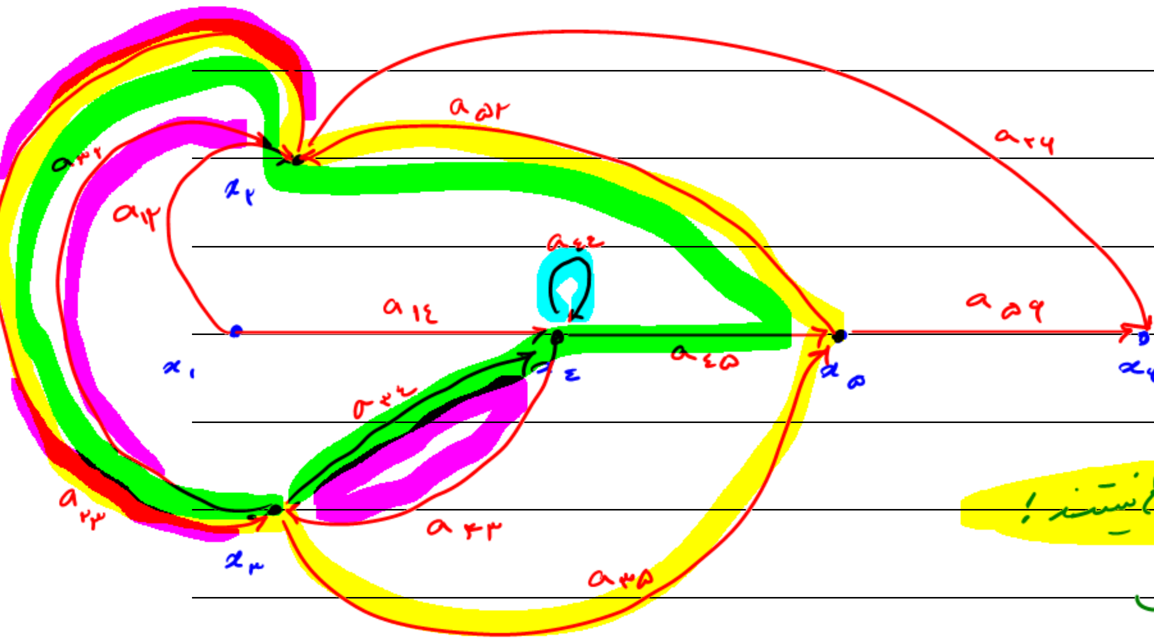
$$\rightarrow \frac{Y(s)}{R(s)} = \frac{P_1}{\Delta(s)} = \frac{G_1G_2(G_3+G_4)G_5}{\Delta(s)}$$

We can see that both methods have the same result.

6 Signal Flow Graph

Represent the following set of equations by a signal flow graph and determine the overall gain $\frac{x_6}{x_1}$ using Mason's gain formula

$$
\begin{aligned}
x_2 &= a_{12}x_1 + a_{32}x_3 + a_{52}x_5 \\
x_3 &= a_{23}x_2 + a_{43}x_4 \\
x_4 &= a_{14}x_1 + a_{34}x_3 + a_{44}x_4 \\
x_5 &= a_{35}x_3 + a_{45}x_4 \\
x_6 &= a_{26}x_2 + a_{56}x_5
\end{aligned}
$$



x_6 و x_1 در حلقه‌ها نیستند!

ورودی

خروجی

همه مسیرها :

P_k :

$P_1 = a_{12}\, a_{26}$ $\quad$ $P_2 = a_{12}\, a_{23}\, a_{35}\, a_{56}$

$P_3 = a_{12}\, a_{23}\, a_{34}\, a_{45}\, a_{56}$ $\quad$ $P_5 = a_{14}\, a_{43}\, a_{35}\, a_{56}$

$P_4 = a_{14}\, a_{45}\, a_{56}$ $\quad$ $P_7 = a_{14}\, a_{43}\, a_{35}\, a_{52}\, a_{26}$

$P_6 = a_{14}\, a_{43}\, a_{32}\, a_{26}$ $\quad$ $P_8 = a_{14}\, a_{45}\, a_{52}\, a_{26}$

همه حلقه‌ها :

$L_1 = a_{44}$ $\quad$ $L_2 = a_{34}\, a_{43}$ $\quad$ $L_3 = a_{35}\, a_{52}\, a_{23}$

$L_4 = a_{34}\, a_{45}\, a_{52}\, a_{23}$ $\quad$ $L_5 = a_{23}\, a_{32}$

تعیین ۲ به ۲ non touching ⟵ $L_1 L_3$ و $L_1 L_5$

Δ :

$$\boxed{\Delta = 1 - L_1 - L_2 - L_3 - L_4 - L_5 + L_1 L_5 + L_1 L_3}$$

Δ_i :

$$\boxed{\begin{array}{l} \Delta_1 = 1 - L_2 - L_3 \qquad \Delta_2 = 1 - L_3 \qquad \Delta_4 = 1 - L_5 \\ \Delta_3 = 1 - 0 = 1 \qquad \Delta_5 = \Delta_6 = \Delta_7 = \Delta_8 = 1 \end{array}}$$

$$T(s) = \frac{\sum_{k=1}^{8} P_k \Delta_k}{\Delta}$$ ⟵ جایگذاری مقادیر بالا

$$\Rightarrow \frac{x_4}{x_1} = \frac{\sum_{i=1}^{8} P_i \Delta_i}{\Delta} \longrightarrow y_i \,;\, P_i \,,\, \Delta_i$$ ⟶ بالا موجود است

همچنین خود Δ

7 Response of a Linear System

Given the transfer function $T(s)$ and the input signal $u(t)$, determine the output signal $y(t)$ for the time interval $0 \le t \le 16$.

$$T(s) = \frac{6}{(0.2s+1)(s+1)}$$

$$u(t) = \begin{cases} 2 & 2 \le t \le 6 \\ -1 & 8 \le t \le 12 \\ 0 & O.W. \end{cases}$$

$$y(t) = ?$$

$h(t) \rightarrow$ heaviside $\rightarrow$ step function

$$u(t) = ۲\big(h(t-۲) - h(t-۶)\big) - \big(h(t-۸) - h(t-۱۲)\big)$$

$$U(s) = \frac{۲}{s}e^{۲s} - \frac{۲}{s}e^{۶s} - \frac{1}{s}e^{۸s} - \frac{1}{s}e^{۱۲s}$$

$$= \frac{1}{s}\left(۲e^{۲s} - ۲e^{۶s} - e^{۸s} - e^{۱۲s}\right)$$

$$Y(s) = U(s)\,H(s) = \frac{۶}{(۰/۲s+1)(s+1)} \cdot \frac{1}{s}\left(۲e^{۲s} - ۲e^{۶s} - e^{۸s} + e^{۱۲s}\right)$$

$$\frac{1}{(۰/۲s+1)(s+1)\ s} = \frac{A}{۰/۲s+1} + \frac{B}{s+1} + \frac{C}{s}$$

$$\times s\Big|_{s=0} \longrightarrow 1 = C$$

$$\times s+1\Big|_{s=-1} \longrightarrow \frac{1}{۰/۸\,(-1)} = B \longrightarrow B = -\frac{۱۰}{۸}$$

$$\times ۰/۲s+1\Big|_{s=-۵} \longrightarrow \frac{1}{(-۴)(-۵)} = A \rightarrow A = \frac{1}{۲۰}$$

$$Y(s) = 4\left(\frac{1}{20}\,\frac{1}{0.2s+1} - \frac{10}{8}\,\frac{1}{s+1} + \frac{1}{s}\right)\left(2e^{2s} - 2e^{4s} - e^{8s} + e^{12s}\right)$$

$Z(s)$ خصوص اصلی

طبق Heaviside $\leftarrow$

$$Z(s) = 4\left(\frac{1}{20}\,\frac{5}{s+5} - \frac{10}{8}\,\frac{1}{s+1} + \frac{1}{s}\right)$$

$$z(t) = 4\left(\frac{1}{4}\,e^{-5t} - \frac{10}{8}\,e^{-t} + 1\right)$$

Therefore:

$$\boxed{y(t) = 4\left(\frac{1}{4}e^{-5t} - \frac{10}{8}e^{-t} + 1\right)\left(2h(t-2) - 2h(t-4) - h(t-8) + h(t-12)\right)}$$

h: heaviside function ✓

Linear Control System

Matin Mb - 400102114

CHW1

Assignment N.O. 1

Objectives:

- *To understand numerical ways of solving* **Ordinary Differential Equations**
- *Defining and calling functions in matlab,*
- *To apply a signal to a system in practice using* ***lsim*** *and ...,*
- *To apply a signal to a system in practice using Simulink*

Section 8 **(Continues-Time Signals)*:***

To plot these continuous-time signals in MATLAB, we need to sample each signal with an appropriate sampling rate T_s, which may vary depending on the nature and frequency of the signal. Here's how you can approach each signal and generate the plots.

General Steps:

1. **Define the time interval:** t goes from 0 to 12.
2. **Choose sampling time** T_s**:** Choose T_s based on the smoothness of each signal's variations. A smaller T_s results in a higher sampling frequency, capturing more detail.
3. **Define each function and plot:** Compute the values of each function at the time points and plot them.

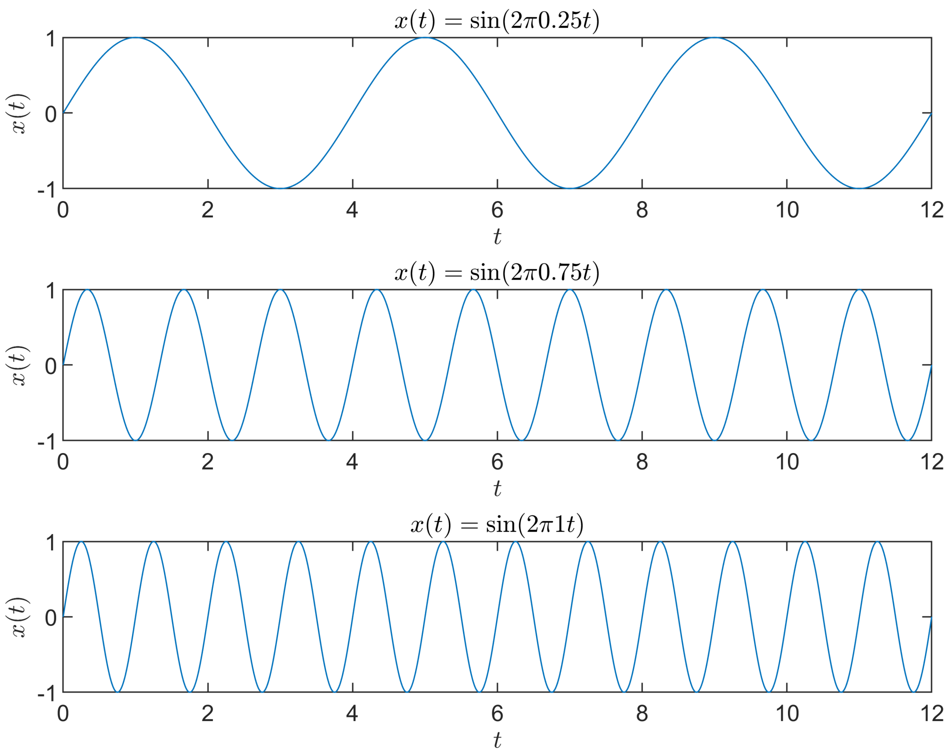
Initial Code:

```
clc; clear; close all;
```

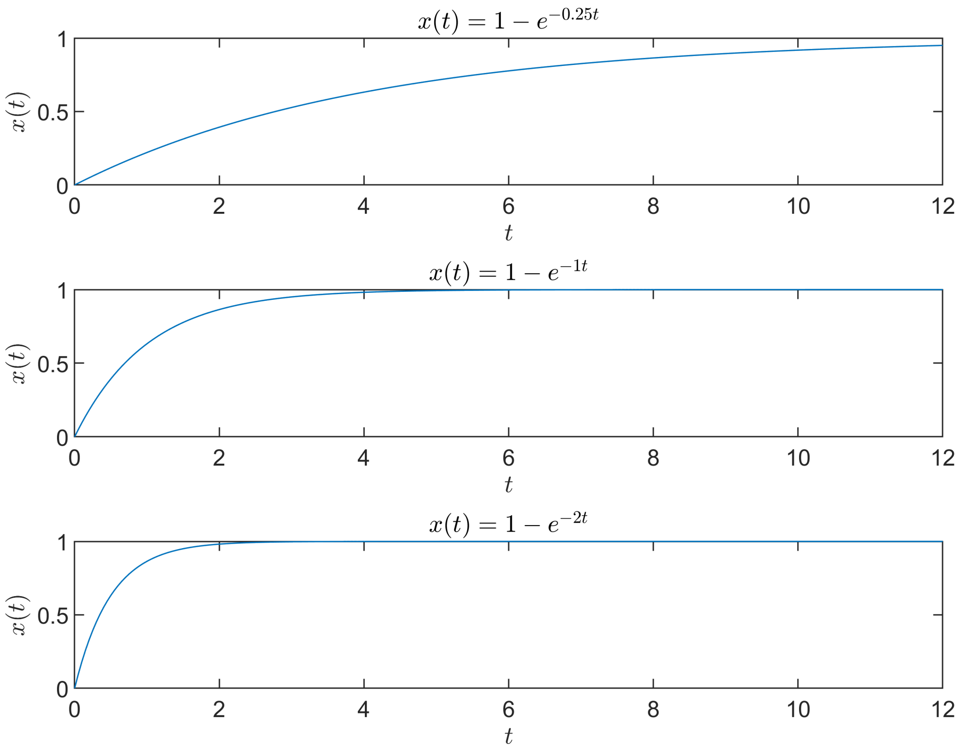
Code:

```
% Time settings
t_start = 0;
t_end = 12;

% Signal 1: x(t) = sin(2 * pi * alpha * t), alpha = 0.25, 0.75, 1
alpha_values = [0.25, 0.75, 1];
Ts1 = 0.01; % Adequate sampling rate for a smooth sinusoidal function
t1 = t_start:Ts1:t_end;

figure;
for i = 1:length(alpha_values)
    alpha = alpha_values(i);
```

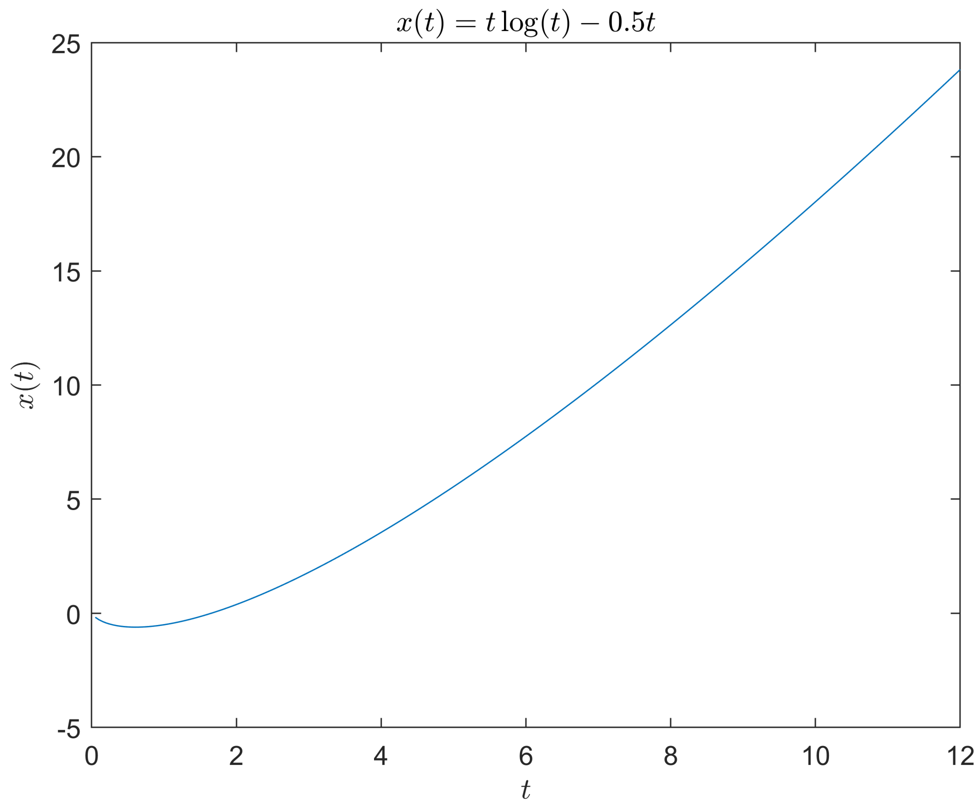
```
    x1 = sin(2 * pi * alpha * t1);
    subplot(3, 1, i);
    plot(t1, x1);
    title(['$x(t) = \sin(2\pi ' num2str(alpha) ' t)$'], 'Interpreter', 'latex');
    xlabel('$t$', 'Interpreter', 'latex');
    ylabel('$x(t)$', 'Interpreter', 'latex');
end
```



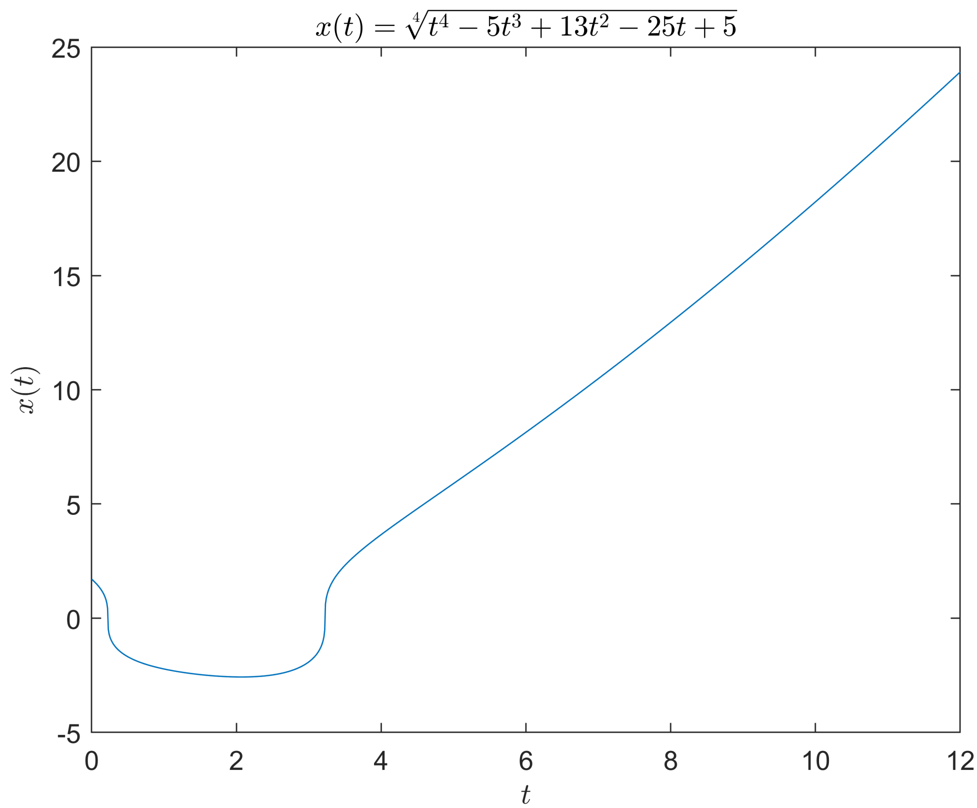

```
% Signal 2: x(t) = 1 - exp(-beta * t), beta = 0.25, 1, 2
beta_values = [0.25, 1, 2];
Ts2 = 0.05; % Choose a sampling rate appropriate for exponential decay
t2 = t_start:Ts2:t_end;

figure;
for i = 1:length(beta_values)
    beta = beta_values(i);
    x2 = 1 - exp(-beta * t2);
    subplot(3, 1, i);
    plot(t2, x2);
    title(['$x(t) = 1 - e^{-' num2str(beta) ' t}$'], 'Interpreter', 'latex');
    xlabel('$t$', 'Interpreter', 'latex');
    ylabel('$x(t)$', 'Interpreter', 'latex');
end
```



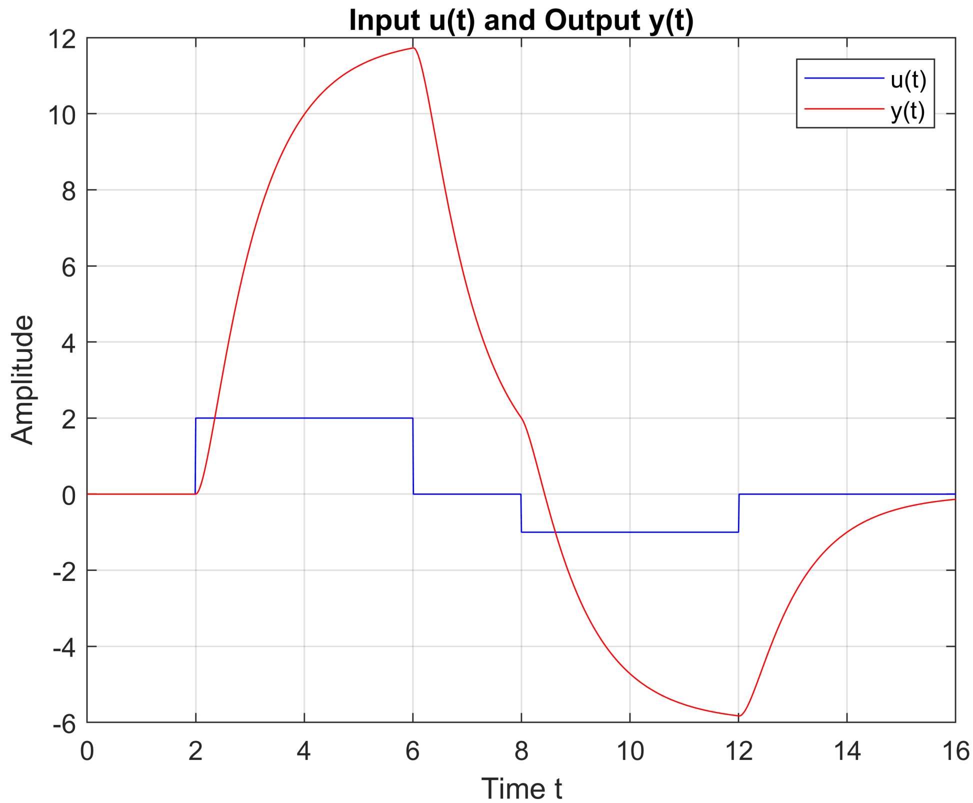

```
% Signal 3: x(t) = t * log(t) - 0.5 * t
Ts3 = 0.05; % Adjusted to capture the non-linear nature of the function
t3 = t_start + Ts3:Ts3:t_end; % Avoid t=0 to prevent log(0) issue
x3 = t3 .* log(t3) - 0.5 * t3;

figure;
plot(t3, x3);
title('$x(t) = t \log(t) - 0.5 t$', 'Interpreter', 'latex');
xlabel('$t$', 'Interpreter', 'latex');
ylabel('$x(t)$', 'Interpreter', 'latex');
```




```
% Signal 4: x(t) = sqrt(4 * t^4 - 5 * t^3 + 13 * t^2 - 25 * t + 5)
Ts4 = 0.01; % High sampling rate to capture the polynomial behavior
t4 = t_start:Ts4:t_end;
x4 = nthroot(t4.^4 - 5 * t4.^3 + 13 * t4.^2 - 25 * t4 + 5, 3);


figure;
plot(t4, x4);
title('$x(t) = \sqrt[4]{t^4 - 5 t^3 + 13 t^2 - 25 t + 5}$', 'Interpreter', 'latex');
xlabel('$t$', 'Interpreter', 'latex');
ylabel('$x(t)$', 'Interpreter', 'latex');
```




```
% Signal 5: (Question 7)

% Define time vector for plotting
t = 0:0.01:16;

% Define input signal u(t)
u_t = (2 * (t >= 2 & t <= 6)) - (1 * (t >= 8 & t <= 12));

% Define transfer function T(s) using symbolic variables
s = tf('s');
T_s = 6 / ((0.2 * s + 1) * (s + 1));

% Find output response y(t) by applying input u(t) through T(s)
% Using the lsim function to compute the time-domain response
y_t = lsim(T_s, u_t, t);

% Plotting both u(t) and y(t) on the same axes
figure;
plot(t, u_t, 'b', 'DisplayName', 'u(t)'); hold on;
plot(t, y_t, 'r', 'DisplayName', 'y(t)');
xlabel('Time t');
ylabel('Amplitude');
title('Input u(t) and Output y(t)');
legend;
grid on;
```



Note that the output of lsim() is exactly the output we calculated in Theoretical part.

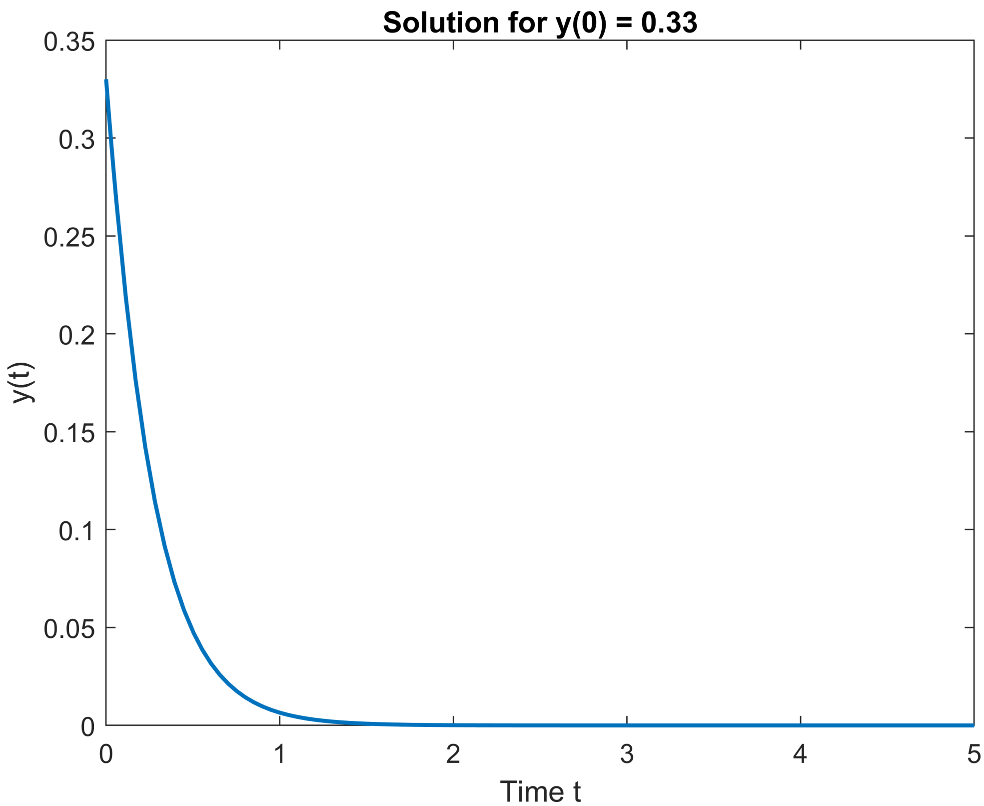
Section 9 (**Numerical Solution of Ordinary Differential Equations*****):***

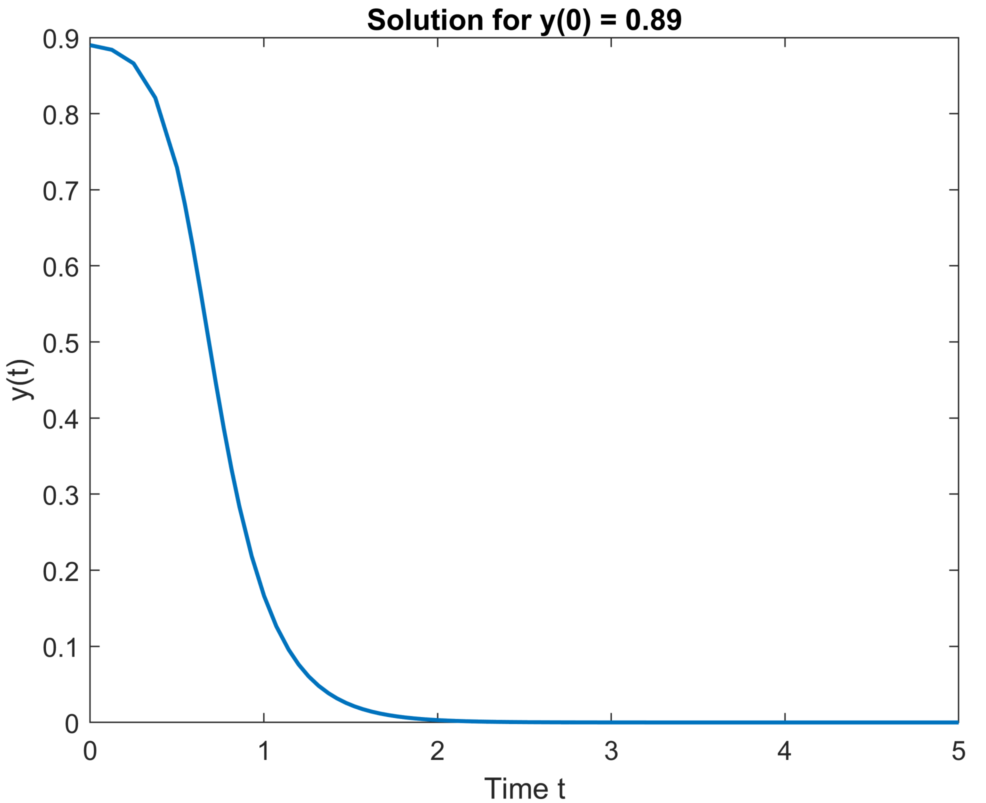
```
clc;
clear all;
close all;
```

Code:

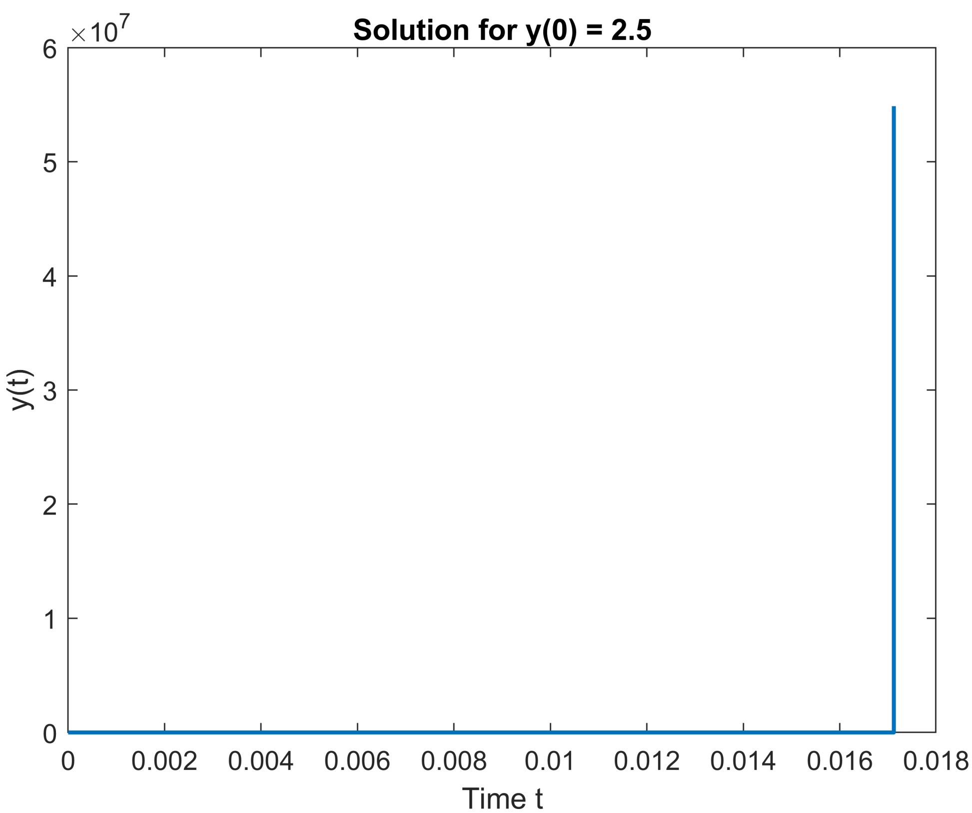
```
% Define the time span
tspan = [0 5];

% Initial conditions
initial_conditions = [0.33, 0.89, 2.5];


% Loop over each initial condition
for i = 1:length(initial_conditions)
    y0 = initial_conditions(i);

    % Solve the ODE
    [t, y] = ode45(@myODE, tspan, y0);

    % Plot the solution
```

```
    figure;
    plot(t, y, 'LineWidth', 1.5);
    title(['Solution for y(0) = ', num2str(y0)]);
    xlabel('Time t');
    ylabel('y(t)');
end
```






```
Warning: Failure at t=1.712082e-02.  Unable to meet integration tolerances without reducing the step size
below the smallest value allowed (5.551115e-17) at time t.
```



As we can see in the third initial value, we diverge because system is not stable for this input.

Section 10 (*Numerical Methods to Find the Response of a Linear System*):

```
clc;
clear all;
close all;
```

$H(s) = \frac{Y(s)}{X(s)} = \frac{30}{(s+5)(s+1)} = \frac{-7.5}{s+5} + \frac{7.5}{s+1}$ which suggests $30X(s) = (s^2 + 6s + 5)Y(s)$ which implies

$30x(t) = y''(t) + 6y'(t) + 5y(t)$ (Initial Conditions are neglected) . Now if we let $y_d[n] = y(t = nh)$ then we get:

$y'(t) \equiv \frac{y_d[n] - y_d[n-1]}{h}$ and $y''(t) \equiv \frac{y_d[n] - 2y_d[n-1] + y_d[n-2]}{h^2}$.

Which will result in:

$$30x(t) = 30x_d[n] = \frac{1}{h^2}(y_d[n] - 2y_d[n-1] + y_d[n-2]) + \frac{6}{h}(y_d[n] - y_d[n-1]) + 5y_d[n]$$

which follows:

$$y_d[n](\frac{1}{h^2} + \frac{6}{h} + 5) = (\frac{2}{h^2} + \frac{6}{h})y_d[n-1] - \frac{1}{h^2}y_d[n-2] + 30x[n]$$

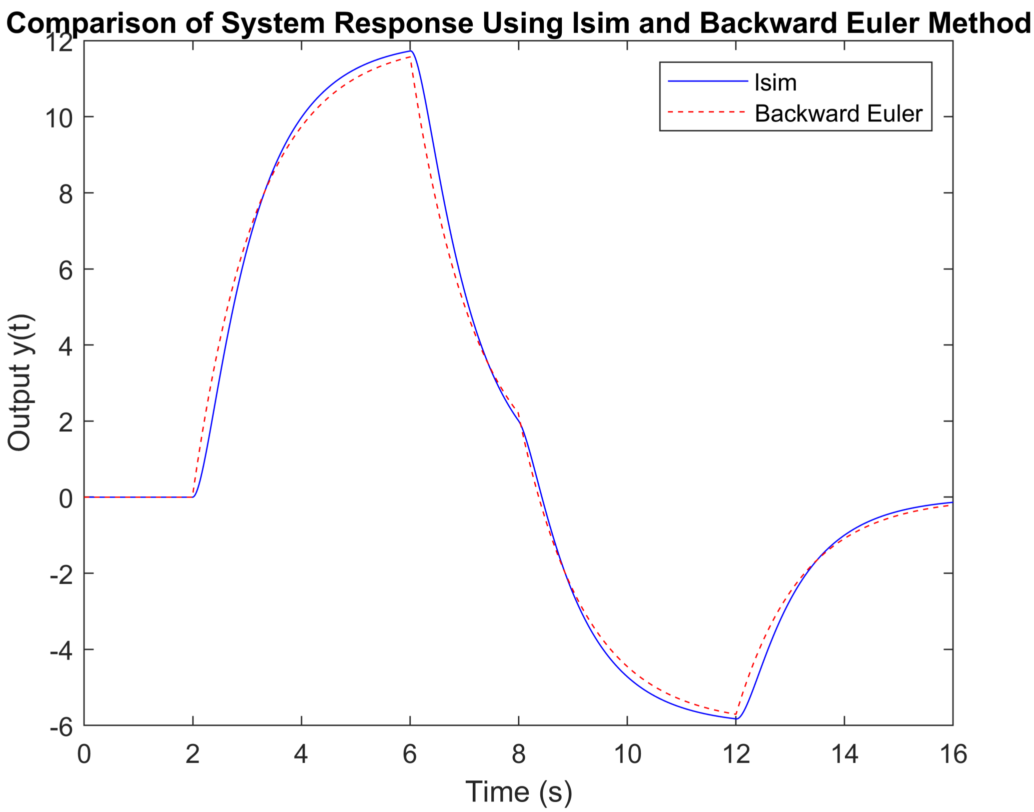
and by approximation we get:

$y_d[n](1 + 6h + 5h^2) = 30h^2x[n] + (2 - 6h)y_d[n-1] - y_d[n-2]$.

Finally, this is the implementation of the Euler's method for the differential equation:

```
numerator = 6;
denominator = conv([0.2, 1], [1, 1]);
sys = tf(numerator, denominator);

Ts = 0.01;  % Sampling time

t = 0:Ts:16;
u = zeros(size(t));
u(t >= 2 & t <= 6) = 2;
u(t >= 8 & t <= 12) = -1;

y_lsim = lsim(sys, u, t);


% Given parameters
Y = zeros(size(t));  % Output signal initialization
Y(1) = 0;            % Initial condition
Y(2) = 0;            % Second initial condition (assumed zero)
h = Ts;              % Time step (sampling time)

X = 2* ((t >= 2) & (t <= 6)) - ((t >= 8) & (t <= 12));
```

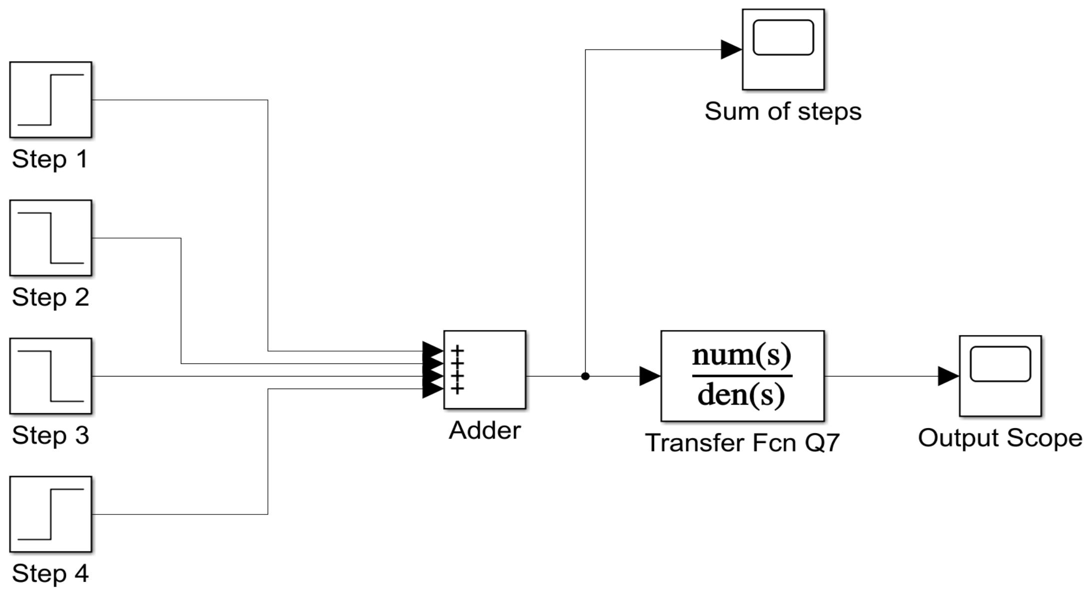
```
for i = 3:length(t)
    % Backward Euler update formula
    Y(i) = g(i, Y(i-1), Y(i-2), X(i), h);
end


figure;
plot(t, y_lsim, 'b', 'DisplayName', 'lsim');
hold on;
plot(t, Y, 'r--', 'DisplayName', 'Backward Euler');
xlabel('Time (s)');
ylabel('Output y(t)');
legend;
title('Comparison of System Response Using lsim and Backward Euler Method');
hold off;
```



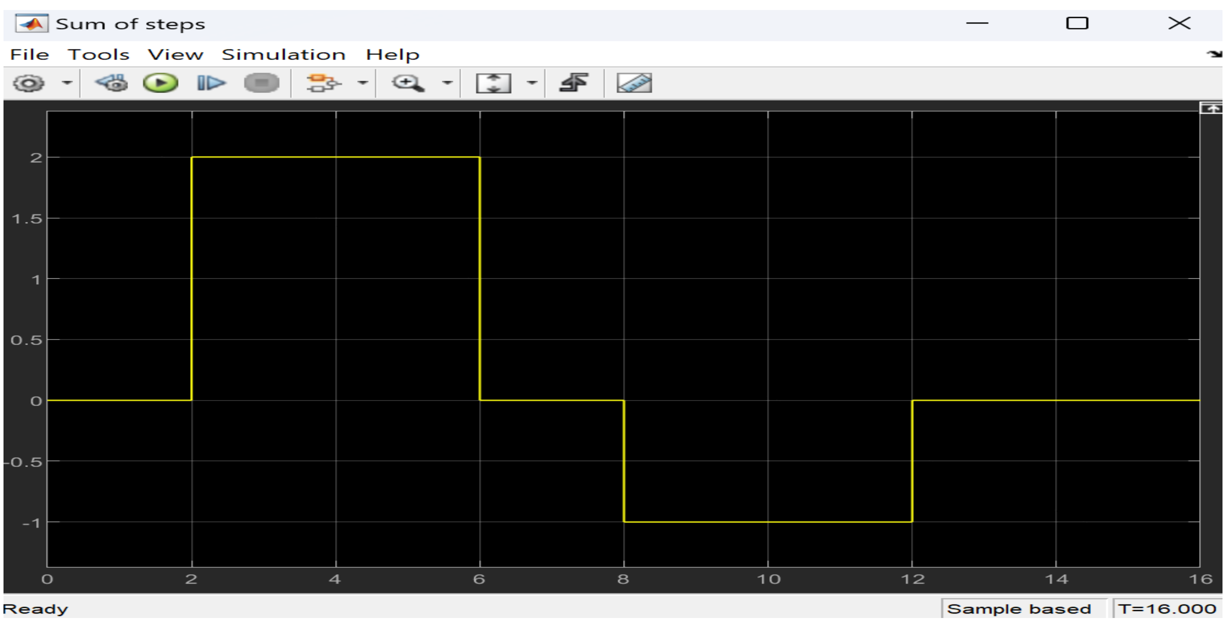
As we can see in the plot, both look like the same. In fact, Backward Euler tries to fit the lsim but as we used some approximation described above,

Section 11 (Simulink):

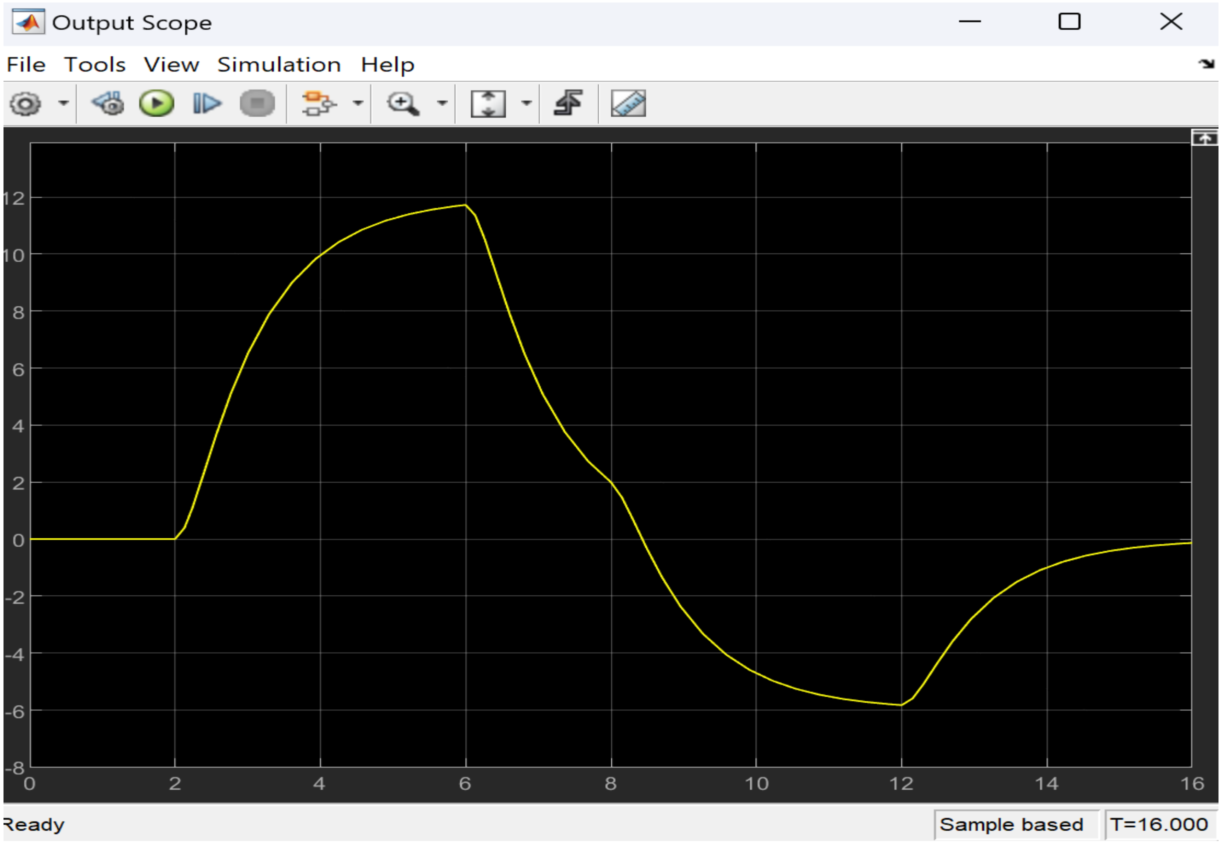
Our Simulink model is shown below in details(Blocks and Modules) and is also available on the folder.



My input to the System(Transfer Fcn):



The output of the System:



As we can see, all plots look the same but different solvers and approximations. Numerical ways look the poorest and lsim and Simulink are way better (With same Sample size).

Used Functions:

```
function dydt = myODE(t, y)
    dydt = 5*y^3 - 4*y;
end
```

```
function [result] = g(i, y_i1, y_i2, u , h)
result =  (30*h*u- y_i2 + (+8)*y_i1)/( 7 + 5*h);
end
```