Puits de potentiel infini entre 0 et L

Mécanique Quantique

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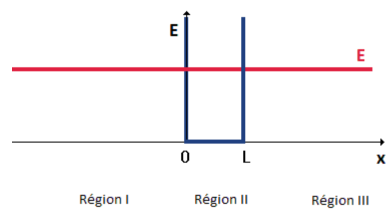
Puits de potentiel infini en une dimension entre 0 et L

$$V(x)=\begin{cases}\infty & , \quad x<0\\ 0 & , \quad 0<x<L\\ \infty & , \quad x>L\end{cases}$$

Le cas $E>0$



Région I	Frontière X= 0	Région II	Frontière X= L	Région III
$\hat{H}u(x)=Eu(x)$		$\hat{H}u(x)=Eu(x)$		$\hat{H}u(x)=Eu(x)$
$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)\right]u(x)=Eu(x)$		$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)\right]u(x)=Eu(x)$		$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)\right]u(x)=Eu(x)$
$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}u(x)+V(x)u(x)=Eu(x)$		$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}u(x)+V(x)u(x)=Eu(x)$		$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}u(x)+V(x)u(x)=Eu(x)$
$V(x)=\infty$		$V(x)=0$		$V(x)=\infty$
		$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}u(x)=Eu(x)$		
		$\frac{d^2}{dx^2}u(x)+\frac{2mE}{\hbar^2}u(x)=0$		
		$\frac{d^2}{dx^2}u(x)+k^2u(x)=0$ où $k^2=\frac{2mE}{\hbar^2}$		
$u_I(x)=0$		$u_{II}(x)=A\sin(kx)+B\cos(kx)$ où $k^2=\frac{2mE}{\hbar^2}$		$u_{III}(x)=0$

Les conditions de continuités :

À $x = 0$:

1. $u_I(0) = u_{II}(0) \Rightarrow 0 = A\sin 0 + B\cos 0 \Rightarrow B = 0$

À $x = L$:

2. $u_{II}(L) = u_{III}(L) \Rightarrow A\sin(kL) + \underbrace{B}_{0}\cos(kL) = 0 \Rightarrow k_n L = n\pi \text{ , } n = 1, 2, 3, \cdots$

Donc,

$$k_n = \frac{n\pi}{L} \text{ , } n = 1, 2, 3, \cdots$$

On trouve E_n :

$$k^2 = \frac{2mE}{\hbar^2} \Rightarrow \left(\frac{n\pi}{L}\right)^2 = \frac{2mE_n}{\hbar^2} \text{ , } n = 1, 2, 3, \cdots$$

$$\boxed{E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} \text{ , } n = 1, 2, 3, \cdots}$$

On trouve les fonctions propres $u_n(x)$:

$$u(x) = A\sin(kx) + B\cos(kx)$$

$$u_n(x) = A_n \sin\left(\frac{n\pi}{L}x\right) \text{, } n = 1, 2, 3, \cdots$$

On trouve A_n :

Les fonctions propres $u_n(x)$ sont orthonormées :

$$\int_0^L dx u_m^*(x) u_n(x) = \delta_{mn}$$

$$\int_0^L dx \left[A_m \sin\left(\frac{m\pi}{L}x\right) \right]^* A_n \sin\left(\frac{n\pi}{L}x\right) = \delta_{mn}$$

$$A_m A_n \int_0^L dx \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) = \delta_{mn}$$

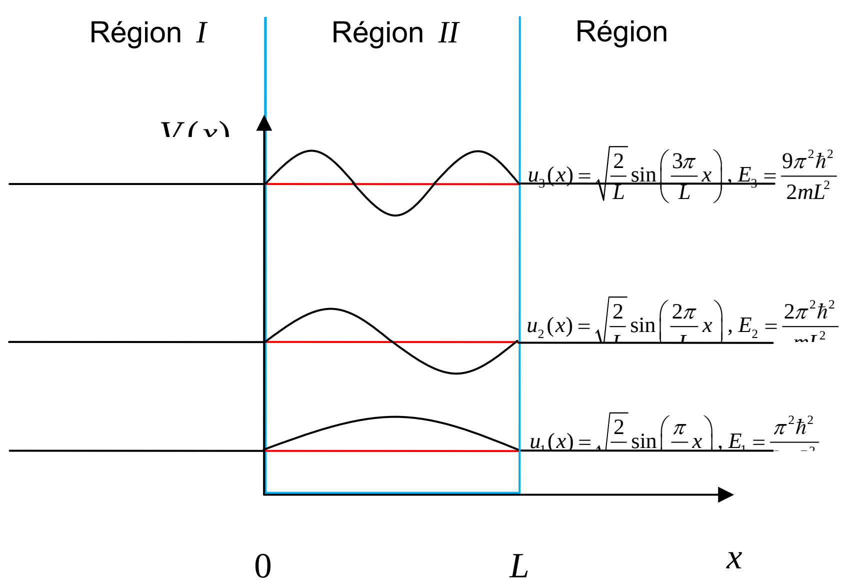
$$A_n^2 \int_0^L dx \sin^2\left(\frac{n\pi}{L}x\right) = 1 \quad , m = n$$

$$A_n^2 \frac{L}{2} = 1$$

$$A_n = \sqrt{\frac{2}{L}} \, , n = 1, 2, 3, \cdots$$

Les fonctions propres :

$$\boxed{u_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right) , n = 1, 2, 3, \cdots}$$

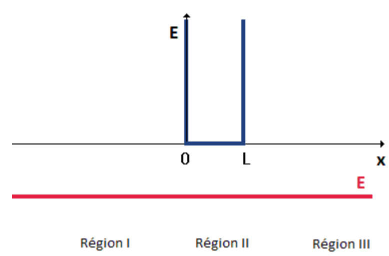


La solution générale :

$$\psi(x)=\sum_{n=1}^{\infty}\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right)$$

Et dans le temps :

$$\boxed{\psi(x,t)=\sum_{n=1}^{\infty}\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right)e^{-\frac{iE_nt}{\hbar}}\ ,\ E_n=\frac{n^2\pi^2\hbar^2}{2mL^2}}$$

<table>
<tr><th colspan="5">Le cas $E<0$</th></tr>
<tr><td colspan="5">
</td></tr>
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<tr><td></td><td>$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}u(x)=Eu(x)$</td><td></td></tr>
<tr><td></td><td>$\frac{d^2}{dx^2}u(x)-\frac{2m\lvert E\rvert}{\hbar^2}u(x)=0$</td><td></td></tr>
<tr><td></td><td>$\frac{d^2}{dx^2}u(x)-\kappa^2u(x)=0$ où
$\kappa^2=\frac{2m\lvert E\rvert}{\hbar^2}$</td><td></td></tr>
<tr><td>$u_I(x)=0$</td><td>$u_{II}(x)=Ae^{\kappa x}+Be^{-\kappa x}$ où $\kappa^2=\frac{2m\lvert E\rvert}{\hbar^2}$</td><td>$u_{III}(x)=0$</td></tr>
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Les conditions de continuités :

À $x = 0$:

1. $u_I(0) = u_{II}(0) \;\Rightarrow\; 0 = A + B$

À $x = L$:

2. $u_{II}(L) = u_{III}(L) \Rightarrow 0 = Ae^{\kappa L} + Be^{-\kappa L}$

$\begin{pmatrix} 1 & 1 \\ e^{\kappa L} & e^{-\kappa L} \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ Alors, Il n’y a pas de solution pour $E < 0$