Indefinite Integral

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Type1

$$\boxed{\int \frac{dx}{a^2+x^2}=\frac{1}{a}\arctan\frac{x}{a}+c}$$

$$\int \frac{dx}{a^2+x^2}=\int \frac{dx}{a^2\left(1+\frac{x^2}{a^2}\right)}=\frac{1}{a^2}\int \frac{dx}{1+\frac{x^2}{a^2}}=I$$

$$u^2=\frac{x^2}{a^2}\Rightarrow u=\frac{x}{a}\Rightarrow \frac{du}{dx}=\frac{1}{a}\Rightarrow dx=adu$$

$$\Rightarrow I=\frac{1}{a^2}\int \frac{adu}{1+u^2}=\frac{1}{a}\int \frac{du}{1+u^2}=\frac{1}{a}\arctan u+c=\frac{1}{a}\arctan\frac{x}{a}+c$$

Example

$$\int \frac{dx}{4+x^2}=\frac{1}{2}\arctan\frac{x}{2}+c$$

$$\int \frac{dx}{9+x^2}=\frac{1}{3}\arctan\frac{x}{3}+c$$

Type2

$$\boxed{\int \frac{dx}{a^2+b^2x^2}=\frac{1}{ab}\arctan\frac{b}{a}x+c}$$

$$\int \frac{dx}{a^2+b^2x^2}=\int \frac{dx}{a^2\left(1+\frac{b^2}{a^2}x^2\right)}=\frac{1}{a^2}\int \frac{dx}{1+\frac{b^2}{a^2}x^2}=I$$

$$u^2=\frac{b^2}{a^2}x^2\Rightarrow u=\frac{b}{a}x\Rightarrow \frac{du}{dx}=\frac{b}{a}\Rightarrow dx=\frac{a}{b}du$$

$$\Rightarrow I=\frac{1}{a^2}\int\frac{\frac{a}{b}du}{1+u^2}=\frac{1}{ab}\int\frac{du}{1+u^2}=\frac{1}{ab}\arctan u+c=\frac{1}{ab}\arctan\frac{b}{a}x+c$$

Example

$$\int\frac{dx}{4+9x^2}=\frac{1}{6}\arctan\frac{3}{2}x+c$$

$$\int\frac{dx}{9+4x^2}=\frac{1}{6}\arctan\frac{2}{3}x+c$$

Type3

$$\boxed{\int\frac{dx}{\sqrt{a^2-x^2}}=\arcsin\frac{x}{a}+c\quad;\quad a>0}$$

$$\int\frac{dx}{\sqrt{4-x^2}}=\int\frac{dx}{\sqrt{4\left(1-\frac{x^2}{4}\right)}}=\int\frac{dx}{2\sqrt{1-\frac{x^2}{4}}}=\frac{1}{2}\int\frac{dx}{\sqrt{1-\frac{x^2}{4}}}=I$$

$$u^2=\frac{x^2}{4}\Rightarrow u=\frac{x}{2}\Rightarrow\frac{du}{dx}=\frac{1}{2}\Rightarrow dx=2du$$

$$\Rightarrow I=\frac{1}{2}\int\frac{2du}{\sqrt{1-u^2}}=\int\frac{du}{\sqrt{1-u^2}}=\arcsin u+c=\arcsin\frac{x}{2}+c$$

Example

$$\int\frac{dx}{\sqrt{4-x^2}}=\arcsin\frac{x}{2}+c$$

$$\int\frac{dx}{\sqrt{9-x^2}}=\arcsin\frac{x}{3}+c$$

Type4

$$\boxed{\int\frac{dx}{\sqrt{a^2+b^2x^2}}=\frac{1}{b}\arcsin\frac{b}{a}x+c\quad;\quad a>0}$$

$$\int\frac{dx}{\sqrt{a^2-b^2x^2}}=\int\frac{dx}{\sqrt{a^2\left(1-\frac{b^2}{a^2}x^2\right)}}=\int\frac{dx}{a\sqrt{1-\frac{b^2}{a^2}x^2}}=\frac{1}{a}\int\frac{dx}{\sqrt{1-\frac{b^2}{a^2}x^2}}=I$$

$$u^2=\frac{b^2}{a^2}x^2\Rightarrow u=\frac{b}{a}x\Rightarrow\frac{du}{dx}=\frac{b}{a}\Rightarrow dx=\frac{a}{b}du$$

$$\Rightarrow I=\frac{1}{a}\int\frac{\frac{a}{b}du}{\sqrt{1-u^2}}=\frac{1}{b}\int\frac{du}{\sqrt{1-u^2}}=\frac{1}{b}\arcsin u+c=\frac{1}{b}\arcsin\frac{b}{a}x+c$$

Example

$$\int\frac{dx}{\sqrt{4+9x^2}}=\frac{1}{3}\arcsin\frac{3}{2}x+c$$

$$\int\frac{dx}{\sqrt{9+4x^2}}=\frac{1}{2}\arcsin\frac{2}{3}x+c$$

Type5

$$\boxed{\int\frac{dx}{\sqrt{x^2-a^2}}=\ln\left|\sqrt{x^2-a^2}+x\right|+c\quad;\quad a>0}$$

$$x=a\sec t\,;0\le t<\frac{\pi}{2}$$

$$\Rightarrow\begin{cases}\dfrac{dx}{dt}=a\sec t\tan t\Rightarrow dx=a\sec t\tan t\,dt\\ x^2-a^2=a^2\sec^2 t-a^2=a^2\left(\sec^2 t-1\right)=a^2\tan^2 t\end{cases}$$

$$\int\frac{dx}{\sqrt{x^2-a^2}}=\int\frac{a\sec t\tan t\,dt}{\sqrt{a^2\tan^2 t}}=\int\frac{a\sec t\tan t\,dt}{a\tan t}=\int\sec t\,dt=\ln\left|\sec t+\tan t\right|+c=$$

$$=\ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|+c=\ln\left|\frac{\sqrt{x^2-a^2}+x}{a}\right|+c=\ln\left|\sqrt{x^2-a^2}+x\right|-\underbrace{\ln a+c}_{c'}=\ln\left|\sqrt{x^2-a^2}+x\right|+c$$

Example

$$\int\frac{dx}{\sqrt{x^2-4}}=\ln\left|\sqrt{x^2-4}+x\right|+c$$

Type6

$$\boxed{\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + c \quad ; \quad a>0}$$

$$x = a\sin t \; ; -\frac{\pi}{2} < t < \frac{\pi}{2}$$

$$\Rightarrow \begin{cases} \dfrac{dx}{dt} = a\cos t \Rightarrow dx = a\cos t dt \\ a^2 - x^2 = a^2 - a^2\sin^2 t = a^2\left(1-\sin^2 t\right) = a^2\cos^2 t \end{cases}$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\cos t dt}{\sqrt{a^2\cos^2 t}} = \int \frac{a\cos t dt}{a\cos t} = \int dt = t + c = \arcsin\frac{x}{a} + c$$

$$= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + c = \ln\left|\frac{\sqrt{x^2-a^2}+x}{a}\right| + c = \ln\left|\sqrt{x^2-a^2}+x\right| - \underbrace{\ln a + c}_{c'} = \ln\left|\sqrt{x^2-a^2}+x\right| + c$$

Example

$$\int \frac{dx}{\sqrt{4-x^2}} = \arcsin\frac{x}{2} + c$$

$$\int \frac{dx}{\sqrt{9-x^2}} = \arcsin\frac{x}{3} + c$$

Type7

$$\boxed{\int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left|\sqrt{a^2+x^2}+x\right| + c \quad ; \quad a>0}$$

$$x = a\tan t \; ; -\frac{\pi}{2} < t < \frac{\pi}{2}$$

$$\Rightarrow \begin{cases} \dfrac{dx}{dt} = a\sec^2 t \Rightarrow dx = a\sec^2 t dt \\ a^2 + x^2 = a^2 + a^2\tan^2 t = a^2\left(1+\tan^2 t\right) = a^2\sec^2 t \end{cases}$$

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \int \frac{a\sec^2 t dt}{\sqrt{a^2\sec^2 t}} = \int \frac{a\sec^2 dt}{a\sec t} = \int \sec t\, dt = \ln\left|\sec t + \tan t\right| + c =$$

$$\ln\left|\frac{\sqrt{a^2+x^2}}{a}+\frac{x}{a}\right|+c=\ln\left|\frac{\sqrt{a^2+x^2}+x}{a}\right|+c=\ln\left|\sqrt{a^2+x^2}+x\right|\underbrace{-\ln a+c}_{c'}=\ln\left|\sqrt{a^2+x^2}+x\right|+c$$

Example

$$\int\frac{dx}{\sqrt{4+x^2}}=\ln\left|\sqrt{4+x^2}+x\right|+c$$

$$\int\frac{dx}{\sqrt{9+x^2}}=\ln\left|\sqrt{9+x^2}+x\right|+c$$

Type8

$$\boxed{\int\sqrt{x^2-a^2}\,dx=\frac{1}{2}x\sqrt{x^2-a^2}-\frac{1}{2}a^2\ln\left|\sqrt{x^2-a^2}+x\right|+c\quad;\quad a>0}$$

$$x=a\sec t\,;0\le t<\frac{\pi}{2}$$

$$\Rightarrow\begin{cases}\dfrac{dx}{dt}=a\sec t\tan t\Rightarrow dx=a\sec t\tan t\,dt\\ x^2-a^2=a^2\sec^2 t-a^2=a^2\left(\sec^2 t-1\right)=a^2\tan^2 t\end{cases}$$

$$\int\sqrt{x^2-a^2}\,dx=\int\sqrt{a^2\tan^2 t}\times a\sec t\tan t\,dt=\int a\tan t\times a\sec t\tan t\,dt=$$

$$=a^2\int\sec t\tan^2 t\,dt=a^2\int\sec t\left(\sec^2 t-1\right)dt=a^2\int\left(\sec^3 t-\sec t\right)dt=$$

$$=a^2\left(\frac{1}{2}\sec t\tan t+\frac{1}{2}\ln\left|\sec t+\tan t\right|-\ln\left|\sec t+\tan t\right|+c\right)=$$

$$=\frac{1}{2}a^2\sec t\tan t-\frac{1}{2}a^2\ln\left|\sec t+\tan t\right|+a^2c=$$

$$=\frac{1}{2}a^2\frac{x}{a}\times\frac{\sqrt{x^2-a^2}}{a}-\frac{1}{2}a^2\ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|+a^2c=$$

$$=\frac{1}{2}x\sqrt{x^2-a^2}-\frac{1}{2}a^2\ln\left|\sqrt{x^2-a^2}+x\right|+\underbrace{\frac{1}{2}a^2\ln a+a^2c}_{c'}$$

$$= \frac{1}{2}x\sqrt{x^2-a^2} - \frac{1}{2}a^2 \ln\left|\sqrt{x^2-a^2}+x\right| + c$$

Example

$$\int \sqrt{x^2-4}\,dx = \frac{1}{2}x\sqrt{x^2-4} - 2\ln\left|\sqrt{x^2-4}+x\right| + c$$

Type9

$$\boxed{\int \sqrt{a^2-x^2}dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2 \arcsin\frac{x}{a} + c \quad ; \quad a>0}$$

$$x = a\sin t \; ; -\frac{\pi}{2} \le t \le \frac{\pi}{2}$$

$$\Rightarrow \begin{cases} \dfrac{dx}{dt} = a\cos t \Rightarrow dx = a\cos t dt \\ a^2 - x^2 = a^2 - a^2\sin^2 t = a^2\left(1-\sin^2 t\right) = a^2\cos^2 t \end{cases}$$

$$\int \sqrt{a^2-x^2}\,dx = \int \sqrt{a^2\cos^2 t} \times a\cos t\,dt = \int a\cos t \times a\cos td =$$

$$= \int a^2\cos^2 t\,dt = a^2\int \cos^2 t\,dt = a^2\left(\frac{1}{2}\sin t\cos t + \frac{1}{2}t + c\right) =$$

$$= \frac{1}{2}a^2\sin t\cos t + \frac{1}{2}a^2 t + \underbrace{a^2 c}_{c'} = \frac{1}{2}a^2 \times \frac{x}{a} \times \frac{\sqrt{a^2-x^2}}{a} + \frac{1}{2}a^2\arcsin\frac{x}{a} + c' =$$

$$= \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\arcsin\frac{x}{a} + c$$

Example

$$\int \sqrt{4-x^2}\,dx = \frac{1}{2}x\sqrt{4-x^2} + 2\arcsin\frac{x}{2} + c$$

Type10

$$\boxed{\int \sqrt{a^2+x^2}dx = \frac{1}{2}x\sqrt{a^2+x^2} + \frac{1}{2}a^2 \ln\left(\sqrt{a^2+x^2}+x\right) + c \quad ; \quad a>0}$$

$$x = a\tan t \; ; -\frac{\pi}{2} < t < \frac{\pi}{2}$$

$$\Rightarrow \begin{cases} \dfrac{dx}{dt} = a\sec^2 t \Rightarrow dx = a\sec^2 t dt \\ a^2 + x^2 = a^2 + a^2\tan^2 t = a^2\left(1 + \tan^2 t\right) = a^2\sec^2 t \end{cases}$$

$$\int \sqrt{a^2 + x^2}\, dx = \int \sqrt{a^2\sec^2 t} \times a\sec^2 t\, dt = \int a\sec t \times a\sec^2 t\, dt =$$

$$= \int a^2\sec^3 t\, dt = a^2 \int \sec^3 t\, dt = a^2\left(\frac{1}{2}\sec t\tan t + \frac{1}{2}\ln\left|\sec t + \tan t\right|\right) + c =$$

$$= \frac{1}{2}a^2\sec t\tan t + \frac{1}{2}a^2\ln\left|\sec t + \tan t\right| + c = \frac{1}{2}a^2 \times \frac{\sqrt{a^2 + x^2}}{a} \times \frac{x}{a} + \frac{1}{2}a^2\ln\left|\frac{\sqrt{a^2 + x^2}}{a} + \frac{x}{a}\right| + c =$$

$$= \frac{1}{2}x\sqrt{a^2 + x^2} + \frac{1}{2}a^2\ln\left|\sqrt{a^2 + x^2} + x\right| \underbrace{- \frac{1}{2}a^2\ln a + c}_{c'}$$

$$= \frac{1}{2}x\sqrt{a^2 + x^2} + \frac{1}{2}a^2\ln\left|\sqrt{a^2 + x^2} + x\right| - c$$

Example

$$\int \sqrt{4 + x^2}\, dx = \frac{1}{2}x\sqrt{4 + x^2} + 2\ln\left|\sqrt{4 + x^2} + x\right| + c$$