Indefinite Integral

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Type1

$$\boxed{\int \ln x dx = x\ln x - x + c}$$

$$\begin{cases} u = \ln x \Rightarrow du = \dfrac{1}{x}dx \\ dv = dx \Rightarrow v = x \end{cases}$$

$$\Rightarrow \int \ln x dx = uv - \int v du = x\ln x - \int dx = x\ln x - x + c$$

Type 2

$$\boxed{\int \ln^2 x dx = x\ln^2 x - 2x\ln x + 2x + c}$$

$$\begin{cases} u = \ln^2 x \Rightarrow du = \dfrac{2\ln x}{x}dx \\ dv = dx \Rightarrow v = x \end{cases}$$

$$\Rightarrow \int \ln^2 x dx = uv - \int v du = x\ln^2 x - \int 2\ln dx = x\ln^2 x - 2\int \ln x dx =$$

$$= x\ln^2 x - 2(x\ln x - x + c) = x\ln^2 x - 2x\ln x + 2x + \underbrace{2c}_{c'}$$

Type 3

$$\boxed{\int \ln^3 x dx = x\ln^3 x - 3x\ln^2 x + 6x\ln x - 6x + c}$$

$$\begin{cases} u = \ln^3 x \Rightarrow du = \dfrac{3\ln^2 x dx}{x} \\ dv = dx \Rightarrow v = x \end{cases}$$

$$\Rightarrow \int \ln^3 x dx = uv - \int v du = x\ln^3 x - \int 3\ln^2 x dx =$$

$$= x\ln^3 x - 3\int \ln^2 x dx = x\ln^3 x - 3\left(x\ln^2 x - 2x\ln x + 2x + c\right) =$$

$$= x\ln^3 x - 3x\ln^2 x + 6x\ln x - 6x - \underbrace{3c}_{c'}$$

Type 4

$$\boxed{\int \ln^4 x dx = x\ln^4 x - 4x\ln^3 x + 12x\ln^2 x - 24x\ln x + 24x + c}$$

$$\begin{cases} u = \ln^4 x \Rightarrow du = \dfrac{4\ln^3 x dx}{x} \\ dv = dx \Rightarrow v = x \end{cases}$$

$$\Rightarrow \int \ln^4 x dx = uv - \int v du = x\ln^4 x - \int 4\ln^3 x dx =$$

$$= x\ln^4 x - 4\int \ln^3 x dx = x\ln^4 x - 4\left(x\ln^3 x - 3x\ln^2 x + 6x\ln x - 6x + c\right) =$$

$$= x\ln^4 x - 4x\ln^3 x + 12x\ln^2 x - 24x\ln x + 24x - \underbrace{4c}_{c'}$$

Type 5

$$\boxed{\int \ln^n x dx = x\ln^n x - n\int \ln^{n-1} x dx}$$

$$\begin{cases} u = \ln^n x \Rightarrow du = \dfrac{n\ln^{n-1} x}{x} dx \\ dv = dx \Rightarrow v = x \end{cases}$$

$$\int \ln^n x dx = uv - \int v du = x\ln^n x - \int n\ln^{n-1} x dx = x\ln^n x - n\int \ln^{n-1} x dx$$

Examples

$$\int \ln x dx = x\ln x - \int dx = x\ln x - x + c$$

$$\int \ln^2 x dx = x\ln^2 x - 2\int \ln dx = x\ln^2 x - 2(x\ln x - x + c) =$$

$$= x\ln^2 x - 2x\ln x + 2x - \underbrace{2c}_{c'}$$

$$\int \ln^3 x dx = x\ln^3 x - 3\int \ln^2 dx = x\ln^3 x - 3\left(x\ln^2 x - 2x\ln x + x + c\right) =$$
$$= x\ln^3 x - 3\ln^2 x + 6x\ln x - 6x - \underbrace{3c}_{c'}$$

$$\int \ln^4 x dx = x\ln^4 x - 4\int \ln^3 dx = x\ln^4 x - 4\left(x\ln^3 x - 3\ln^2 x + 6x\ln x - 6x + c\right) =$$
$$= x\ln^4 x - 4x\ln^3 x + 12x\ln^2 x - 24x\ln x + 24x - \underbrace{4c}_{c'}$$

Type 6

$$\boxed{\int x\ln x dx = \frac{1}{2}x^2\left(\ln x - \frac{1}{2}\right) + c}$$

$$\begin{cases} u = x\ln x \Rightarrow du = (\ln x + 1)dx \\ dv = dx \Rightarrow v = x \end{cases}$$
$$\Rightarrow \int x\ln x dx = uv - \int v du = x^2\ln x - \int x(\ln x + 1)dx = x^2\ln x - \int x\ln x dx - \int x dx$$
$$\Rightarrow 2\int x\ln x dx = x^2\ln x - \frac{1}{2}x^2 + c$$
$$\Rightarrow \int x\ln x dx = \frac{1}{2}x^2\left(\ln x - \frac{1}{2}\right) + \underbrace{\frac{1}{2}c}_{c'}$$

Type 7

$$\boxed{\int x^2\ln x dx = \frac{1}{3}x^3\left(\ln x - \frac{1}{3}\right) + c}$$

$$\begin{cases} u = x^2\ln x \Rightarrow du = (2x\ln x + x)dx \\ dv = dx \Rightarrow v = x \end{cases}$$
$$\Rightarrow \int x^2\ln x dx = uv - \int v du = x^3\ln x - \int \left(2x^2\ln x + x^2\right)dx =$$
$$= x^3\ln x - 2\int x^2\ln x dx - \int x^2 dx = x^3\ln x - 2\int x^2\ln x dx - \frac{1}{3}x^3 + c$$
$$\Rightarrow 3\int x^2\ln x dx = x^3\ln x - \frac{1}{3}x^3 + c$$
$$\Rightarrow \int x^2\ln x dx = \frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + \underbrace{\frac{1}{3}c}_{c'} = \frac{1}{3}x^3\left(\ln x - \frac{1}{3}\right) + c'$$

Type 8

$$\boxed{\int x^n \ln x dx = \frac{1}{n+1} x^{n+1}\left(\ln x - \frac{1}{n+1}\right) + c}$$

$$\begin{cases} u = x^n \ln x \Rightarrow du = \left(nx^{n-1}\ln x + x^{n-1}\right)dx \\ dv = dx \Rightarrow v = x \end{cases}$$

$$\Rightarrow \int x^n \ln x dx = uv - \int v du = x^{n+1}\ln x - \int \left(nx^n \ln x + x^n\right)dx =$$

$$= x^{n+1}\ln x - n\int x^n \ln x dx - \int x^n dx = x^{n+1}\ln x - n\int x^n \ln x dx - \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow (n+1)\int x^n \ln x dx = x^{n+1}\ln x - \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow \int x^n \ln x dx = \frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2} + \underbrace{\frac{c}{n+1}}_{c'} \Rightarrow \int x^n \ln x dx = \frac{1}{n+1} x^{n+1}\left(\ln x - \frac{1}{n+1}\right) + c'$$

Examples

$$\int x \ln x dx = \frac{1}{1+1} x^{1+1}\left(\ln x - \frac{1}{1+1}\right) + c = \frac{1}{2} x^2\left(\ln x - \frac{1}{2}\right) + c$$

$$\int x^2 \ln x dx = \frac{1}{2+1} x^{2+1}\left(\ln x - \frac{1}{2+1}\right) + c = \frac{1}{3} x^3\left(\ln x - \frac{1}{3}\right) + c$$

$$\int x^3 \ln x dx = \frac{1}{3+1} x^{3+1}\left(\ln x - \frac{1}{3+1}\right) + c = \frac{1}{4} x^4\left(\ln x - \frac{1}{4}\right) + c$$

$$\int x^4 \ln x dx = \frac{1}{4+1} x^{4+1}\left(\ln x - \frac{1}{4+1}\right) + c = \frac{1}{5} x^5\left(\ln x - \frac{1}{5}\right) + c$$

Type 9

$$\boxed{\int \frac{\ln x}{x} dx = \frac{1}{2}\ln^2 x + c}$$

$$u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x} \Rightarrow dx = xdu$$

$$\Rightarrow \int \frac{\ln x}{x} dx = \int \frac{u}{x} x du = \int u du = \frac{1}{2}u^2 + c = \frac{1}{2}\ln^2 x + c$$

Type 10

$$\boxed{\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} - \frac{1}{x} + c}$$

$$\begin{cases} u = \ln x \Rightarrow du = \dfrac{1}{x}dx \\ dv = \dfrac{1}{x^2}dx \Rightarrow v = -\dfrac{1}{x} \end{cases}$$

$$\Rightarrow \int \frac{\ln x}{x^2}\,dx = uv - \int v\,du = -\frac{\ln x}{x} - \int -\frac{1}{x^2}\,dx = -\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx = -\frac{\ln x}{x} - \frac{1}{x} + c$$

Type 11

$$\boxed{\int \frac{\ln^2 x}{x}\,dx = \frac{1}{3}\ln^3 x + c}$$

$$u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x} \Rightarrow dx = x\,du$$

$$\Rightarrow \int \frac{\ln^2 x}{x}\,dx = \int \frac{u^2}{x}\,x\,du = \int u^2\,du = \frac{1}{3}u^3 + c = \frac{1}{3}\ln^3 x + c$$

Type 12

$$\boxed{\int \frac{\ln^n x}{x}\,dx = \frac{1}{n+1}\ln^{n+1} x + c}$$

$$u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x} \Rightarrow dx = x\,du$$

$$\Rightarrow \int \frac{\ln^n x}{x}\,dx = \int \frac{u^n}{x}\,x\,du = \int u^n\,du = \frac{1}{n+1}u^{n+1} + c = \frac{1}{n+1}\ln^{n+1} x + c$$

Examples

$$\int \frac{\ln x}{x}\,dx = \frac{1}{1+1}\ln^{1+1} x + c = \frac{1}{2}\ln^2 x + c$$

$$\int \frac{\ln^2 x}{x}\,dx = \frac{1}{2+1}\ln^{2+1} x + c = \frac{1}{3}\ln^3 x + c$$

$$\int \frac{\ln^3 x}{x} dx = \frac{1}{3+1}\ln^{3+1} x + c = \frac{1}{4}\ln^4 x + c$$

Type 13

$$\boxed{\int \frac{dx}{x \ln x} = \ln|\ln x| + c}$$

$$u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x} \Rightarrow dx = xdu$$

$$\Rightarrow \int \frac{dx}{x \ln x} = \int \frac{xdu}{xu} = \int \frac{du}{u} = \ln|u| + c = \ln|\ln x| + c$$

Type 14

$$\boxed{\int \ln\left(a^2 + x^2\right)dx = x\ln\left(a^2 + x^2\right) - 2x + 2a \arctan\frac{x}{a} + c}$$

$$\begin{cases} u = \ln\left(a^2 + x^2\right) \Rightarrow du = \dfrac{2x}{a^2 + x^2}dx \\ dv = dx \Rightarrow v = x \end{cases}$$

$$\Rightarrow \int \ln\left(a^2 + x^2\right)dx = uv - \int vdu = x\ln\left(a^2 + x^2\right) - \int \frac{2x^2 dx}{a^2 + x^2} =$$

$$= x\ln\left(a^2 + x^2\right) - 2\int \frac{x^2 dx}{a^2 + x^2} = x\ln\left(a^2 + x^2\right) - 2\left(x - a\arctan\frac{x}{a} + c\right) =$$

$$= x\ln\left(a^2 + x^2\right) - 2x + 2a\arctan\frac{x}{a} + \underbrace{2c}_{c'}$$

Type 15

$$\boxed{\int \ln\left(x^2 - a^2\right)dx = x\ln\left(x^2 - a^2\right) - 2x + a\ln\left(\frac{x+a}{x-a}\right) + c}$$

$$\begin{cases} u = \ln\left(x^2 - a^2\right) \Rightarrow du = \dfrac{2x}{x^2 - a^2}\,dx \\ dv = dx \Rightarrow v = x \end{cases}$$

$$\Rightarrow \int \ln\left(x^2 - a^2\right)dx = uv - \int v\,du = x\ln\left(x^2 - a^2\right) - \int \frac{2x^2}{x^2 - a^2}dx =$$

$$= x\ln\left(x^2 - a^2\right) - 2\int \frac{x^2}{x^2 - a^2}\,dx = x\ln\left(x^2 - a^2\right) - 2\left[x + \frac{a}{2}\ln\left(\frac{x-a}{x+a}\right) + c\right] =$$

$$= x\ln\left(x^2 - a^2\right) - 2x - a\ln\left(\frac{x-a}{x+a}\right) + \underbrace{2c}_{c'} = x\ln\left(x^2 - a^2\right) - 2x + a\ln\left(\frac{x+a}{x-a}\right) + c'$$