Indefinite Integral

Hossein Rahimzadeh
www.cafeplanck.com
info@cafeplanck.com

Type 1

$$\boxed{\int x \arcsin x dx = \left(\frac{1}{2}x^2 - \frac{1}{4}\right)\arcsin x + \frac{1}{4}x\sqrt{1-x^2} + c}$$

$$\int x \arcsin x dx = I$$

$$\begin{cases} u = x \Rightarrow du = dx \\ dv = \arcsin x dx \Rightarrow v = x \arcsin x + \sqrt{1-x^2} \end{cases}$$

$$\Rightarrow I = uv - \int vdu = x^2 \arcsin x + x\sqrt{1-x^2} - \int \left(x \arcsin x + \sqrt{1-x^2}\right)dx =$$

$$= x^2 \arcsin x + x\sqrt{1-x^2} - \int x \arcsin x dx - \int \sqrt{1-x^2}\, dx =$$

$$= x^2 \arcsin x + x\sqrt{1-x^2} - \int x \arcsin x dx - \left(\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\arcsin x + c\right)$$

$$\Rightarrow 2\int x \arcsin x dx = x^2 \arcsin x + \frac{1}{2}x\sqrt{1-x^2} - \frac{1}{2}\arcsin x + c$$

$$\Rightarrow \int x \arcsin x dx = \frac{1}{2}x^2 \arcsin x + \frac{1}{4}\sqrt{1-x^2} - \frac{1}{4}\arcsin x + \underbrace{\frac{c}{2}}_{c'} = \left(\frac{1}{2}x^2 - \frac{1}{4}\right)\arcsin x + \frac{1}{4}x\sqrt{1-x^2} + c'$$

Type 2

$$\boxed{\int x^2 \arcsin x dx = \frac{1}{3}x^3 \arcsin x + \frac{1}{9}\left(x^2+2\right)\sqrt{1-x^2} + c}$$

$$\int x^2 \arcsin x dx = I$$

$$\begin{cases} u = x^2 \Rightarrow du = 2xdx \\ dv = \arcsin x dx \Rightarrow v = x \arcsin x + \sqrt{1-x^2} \end{cases}$$

$$\Rightarrow I = uv - \int v du =$$

$$= x^3 \arcsin x + x^2\sqrt{1-x^2} - \int\left(2x^2 \arcsin x dx + 2x\sqrt{1-x^2}\right)dx =$$

$$= x^3 \arcsin x + x^2\sqrt{1-x^2} - 2\int x^2 \arcsin x dx - \int 2x\sqrt{1-x^2 dx} =$$

$$\Rightarrow 3\int x^2 \arcsin x dx = x^3 \arcsin x + x^2\sqrt{1-x^2} - \left[-\frac{2}{3}\left(1-x^2\right)\sqrt{1-x^2} + c\right] =$$

$$= x^3 \arcsin x + x^2\sqrt{1-x^2} + \frac{2}{3}\left(1-x^2\right)\sqrt{1-x^2} - c =$$

$$= x^3 \arcsin x + \frac{1}{3}\left(x^2+2\right)\sqrt{1-x^2} + c'$$

$$\Rightarrow \int x^2 \arcsin x dx = \frac{1}{3}x^3 \arcsin x + \frac{1}{9}\left(x^2+2\right)\sqrt{1-x^2} + \underbrace{\frac{1}{3}c'}_{c''}$$

Type 3

$$\boxed{\int x\arccos x\,dx = \left(\frac{1}{2}x^2 - \frac{1}{4}\right)\arccos x - \frac{1}{4}x\sqrt{1-x^2} + c}$$

$$\int x\arccos x\,dx = I$$

$$\begin{cases} u = x \Rightarrow du = dx \\ dv = \arccos x\,dx \Rightarrow v = x\arccos x - \sqrt{1-x^2} \end{cases}$$

$$\Rightarrow I = uv - \int v\,du =$$

$$= x^2\arccos x - x\sqrt{1-x^2} - \int\left(x\arccos x - \sqrt{1-x^2}\right)dx =$$

$$= x^2\arccos x - x\sqrt{1-x^2} - \int x\arccos x\,dx + \int\sqrt{1-x^2}\,dx =$$

$$= x^2\arccos x - x\sqrt{1-x^2} - \int x\arccos x\,dx + \left(\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\arcsin x + c\right)$$

$$\Rightarrow 2\int x\arccos x\,dx = x^2\arccos x - \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\left(\frac{\pi}{2} - \arccos x\right) + c$$

$$\Rightarrow \int x\arccos x\,dx = \frac{1}{2}x^2\arccos x - \frac{1}{4}x\sqrt{1-x^2} - \frac{1}{4}\arccos x + \underbrace{c + \frac{\pi}{8}}_{c'}$$

$$\Rightarrow \int x\arccos x\,dx = \left(\frac{1}{2}x^2 - \frac{1}{4}\right)\arccos x - \frac{1}{4}x\sqrt{1-x^2} + c'$$

Type 4

$$\boxed{\int x^2 \arccos x dx = \frac{1}{3}x^3 \arccos x - \frac{1}{9}\left(x^2+2\right)\sqrt{1-x^2}+c}$$

$$\int x^2 \arccos x dx = I$$

$$\begin{cases} u = x^2 \Rightarrow du = 2x dx \\ dv = \arccos x dx \Rightarrow v = x\arccos x - \sqrt{1-x^2} \end{cases}$$

$$\Rightarrow I = uv - \int v du =$$

$$= x^3 \arccos x - x^2\sqrt{1-x^2} - \int\left(2x^2 \arccos x - 2x\sqrt{1-x^2}\right)dx =$$

$$= x^3 \arccos x - x^2\sqrt{1-x^2} - 2\int x^2 \arccos x dx + \int 2x\sqrt{1-x^2}\, dx =$$

$$\Rightarrow 3\int x^2 \arccos x dx = x^3 \arccos x - x^2\sqrt{1-x^2} + \left[-\frac{2}{3}\left(1-x^2\right)\sqrt{1-x^2}+c\right] =$$

$$= x^3 \arccos x - x^2\sqrt{1-x^2} - \frac{2}{3}\left(1-x^2\right)\sqrt{1-x^2}+c =$$

$$= x^3 \arccos x - \frac{1}{3}\left(x^2+2\right)\sqrt{1-x^2}+c$$

$$\Rightarrow \int x^2 \arccos x dx = \frac{1}{3}x^3 \arccos x - \frac{1}{9}\left(x^2+2\right)\sqrt{1-x^2}+\underbrace{\frac{1}{3}c}_{c'}$$

Type 5

$$\boxed{\int x\arctan x dx=\frac{1}{2}\left(1+x^2\right)\arctan x-\frac{1}{2}x+c}$$

$$\int x\arctan x dx = I$$

$$\begin{cases} u = x \Rightarrow dx = du \\ dv = \arctan x dx \Rightarrow v = x\arctan x - \frac{1}{2}\ln\left(1+x^2\right) \end{cases}$$

$$\Rightarrow I = uv - \int v du =$$

$$= x^2\arctan x - \frac{1}{2}x\ln\left(1+x^2\right) - \int\left[x\arctan x - \frac{1}{2}\ln\left(1+x^2\right)\right]dx =$$

$$= x^2\arctan x - \frac{1}{2}x\ln\left(1+x^2\right) - \int x\arctan x dx + \frac{1}{2}\int \ln\left(1+x^2\right)dx =$$

$$= x^2\arctan x - \frac{1}{2}x\ln\left(1+x^2\right) - \int x\arctan x dx + \frac{1}{2}\left[x\ln\left(1+x^2\right) - 2x + 2\arctan x + c\right]$$

$$= x^2\arctan x - \frac{1}{2}x\ln\left(1+x^2\right) - \int x\arctan x dx + \frac{1}{2}x\ln\left(1+x^2\right) - x + \arctan x + \underbrace{\frac{1}{2}c}_{c'}$$

$$\Rightarrow 2\int x\arctan x dx = x^2\arctan x + \arctan x - x + c' = \left(1+x^2\right)\arctan x - x + c'$$

$$\Rightarrow \int x\arctan x dx = \frac{1}{2}\left(1+x^2\right)\arctan x - \frac{1}{2}x + \underbrace{\frac{1}{2}c'}_{c''}$$

Type 6

$$\boxed{\int x^2 \arctan x dx = \frac{1}{3}x^3 \arctan x + \frac{1}{6}\ln\left(1+x^2\right) - \frac{1}{6}x^2 + c}$$

$$\int x^2 \arctan x dx = I$$

$$\begin{cases} u = x^2 \Rightarrow du = 2xdx \\ dv = \arctan x\, xdx \Rightarrow v = x\arctan x - \dfrac{1}{2}\ln\left(1+x^2\right) \end{cases}$$

$$\Rightarrow I = uv - \int vdu =$$

$$= x^3 \arctan x - \frac{1}{2}x^2 \ln\left(1+x^2\right) - \int \left[2x^2 \arctan x - x\ln\left(1+x^2\right)\right]dx =$$

$$= x^3 \arctan x - \frac{1}{2}x^2 \ln\left(1+x^2\right) - 2\int x^2 \arctan x dx + \int x\ln\left(1+x^2\right)dx =$$

$$= x^3 \arctan x + \frac{1}{2}x^2 \ln\left(1+x^2\right) - 2\int x^2 \arctan x dx + \left[\frac{1}{2}\left(1+x^2\right)\ln\left(1+x^2\right) - \frac{1}{2}\left(1+x^2\right) + c\right]$$

$$\Rightarrow 3\int x^2 \arctan x dx = x^3 \arctan x + \frac{1}{2}\ln\left(1+x^2\right) - \frac{1}{2}x^2 \underbrace{- \frac{1}{2} + c}_{c'}$$

$$\Rightarrow \int x^2 \arctan x dx = \frac{1}{3}x^3 \arctan x + \frac{1}{6}\ln\left(1+x^2\right) - \frac{1}{6}x^2 + \underbrace{\frac{1}{3}c'}_{c''}$$

Type 7

$$\boxed{\int x \operatorname{arccot} x dx = \frac{1}{2}\left(1+x^2\right)\operatorname{arccot} x + \frac{1}{2}x + c}$$

$$\int x \operatorname{arccot} x dx = I$$

$$\begin{cases} u = x \Rightarrow du = dx \\ dv = \operatorname{arccot} x dx \Rightarrow v = x \operatorname{arccot} x + \frac{1}{2}\ln\left(1+x^2\right) \end{cases}$$

$$I = uv - \int v du =$$

$$= x^2 \operatorname{arccot} x + \frac{1}{2}x\ln\left(1+x^2\right) - \int \left[x \operatorname{arccot} x + \frac{1}{2}\ln\left(1+x^2\right)\right]dx =$$

$$= x^2 \operatorname{arccot} x + \frac{1}{2}x\ln\left(1+x^2\right) - \int x \operatorname{arccot} x dx - \frac{1}{2}\int \ln\left(1+x^2\right)dx =$$

$$= x^2 \operatorname{arccot} x + \frac{1}{2}x\ln\left(1+x^2\right) - \int x \operatorname{arccot} x dx - \frac{1}{2}\left[x\ln\left(1+x^2\right) - 2x + 2\operatorname{arccot} x + c\right]$$

$$\Rightarrow 2\int x \operatorname{arccot} x dx = x^2 \operatorname{arccot} x + x - \operatorname{arccot} x + c =$$

$$= x^2 \operatorname{arccot} x + x - \left(\frac{\pi}{2} - \operatorname{arccot} x\right) + c = \left(1+x^2\right)\operatorname{arccot} x + x \underbrace{- \frac{\pi}{2} + c}_{c'}$$

$$\Rightarrow \int x \operatorname{arccot} x = \frac{1}{2}\left(1+x^2\right)\operatorname{arccot} x + \frac{1}{2}x + c'$$

Type 8

$$\boxed{\int x^2 \arctan x dx = \frac{1}{3}x^3 \operatorname{arccot} x - \frac{1}{6}\ln\left(1+x^2\right) + \frac{1}{6}x^2 + c}$$

$$\int x^2 \operatorname{arccot} x dx = I$$

$$\begin{cases} u = x^2 \Rightarrow du = 2xdx \\ dv = \operatorname{arccot} x dx \Rightarrow v = x \operatorname{arccot} x + \frac{1}{2}\ln\left(1+x^2\right) \end{cases}$$

$$\Rightarrow I = uv - \int vdu =$$

$$= x^3 \operatorname{arccot} x + \frac{1}{2}x^2 \ln\left(1+x^2\right) - \int \left[2x^2 \operatorname{arccot} x + x\ln\left(1+x^2\right)\right] dx =$$

$$= x^3 \operatorname{arccot} x + \frac{1}{2}x^2 \ln\left(1+x^2\right) - 2\int x^2 \operatorname{arccot} x dx - \int x\ln\left(1+x^2\right) dx =$$

$$= x^3 \operatorname{arccot} x + \frac{1}{2}x^2 \ln\left(1+x^2\right) - 2\int x^2 \operatorname{arccot} x dx - \left[\frac{1}{2}\left(1+x^2\right)\ln\left(1+x^2\right) - \frac{1}{2}\left(1+x^2\right) + c\right]$$

$$\Rightarrow 3\int x^2 \operatorname{arccot} x dx = x^3 \operatorname{arccot} x - \frac{1}{2}\ln\left(1+x^2\right) + \frac{1}{2}x^2 + \underbrace{\frac{1}{2} + c}_{c'}$$

$$\Rightarrow \int x^2 \operatorname{arccot} x dx = \frac{1}{3}x^3 \operatorname{arccot} x - \frac{1}{6}\ln\left(1+x^2\right) + \frac{1}{6}x^2 + \underbrace{\frac{1}{3}c'}_{c''}$$