Minkowski Diagrams

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1 Introduction

Your textbook presents the Lorentz transformation using an algebraic approach. Equation 1.37, for instance, gives the coordinates of an object in a coordinate system moving in the x direction with speed v.

$$x' = \gamma(x - vt) \tag{1}$$
$$y' = y \tag{2}$$
$$z' = z \tag{3}$$
$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \tag{4}$$

This is called a Lorentz transformation. Since, y and z aren't modified by the transformation, let's just consider what happens to x and t as a result of Equations 1 and 4. These two equations are linear in x and t (the variables t and x appear to the first power and are only multiplied by constant functions of γ, v, and c.

One can get helpful insights by looking at this transformation graphically. Then the ideas of time dilation, length contraction, simultaneity constraints, and and other relativistic phenomena become evident geometrical relationships. Events and moving objects have constant locations on these graphs, independent of the coordinate system (i.e. independent of the inertial reference frame). Their times and lengths measured in a given inertial reference frame are determined by their coordinates as measured on the coordinate axes appropriate for that reference frame. This kind of graphical transformation is called a Minkowski diagram. A graphical representation of an object's position and time is called a space-time diagram.

In this handout, I will start by discussing what an event, a moving point, and a stationary or moving extended object look like in a single reference frame which is stationary with respect to the observer. Section 3 then discusses how this graph changes when the x and t coordinates are transformed according to Equations 1 and 4. Section 4 then applies this transformed coordinate graph to show graphically how length contraction, time dilation, speed limitations, and simultaneity issues appear in this graphical format.

The graphics shown in the handout were generate using the java program `special.jar`, which you can launch from our course web page using the link "Minkowski." You can also run it directly from the URL http://volta.byu.edu/special.

You will find definitions of special nomenclature used in this article on page 14.

You can find an alternative explanation of Minkowski diagrams in Wikipedia at http://en.wikipedia.org/wiki/Minkowsk$_$diagram.

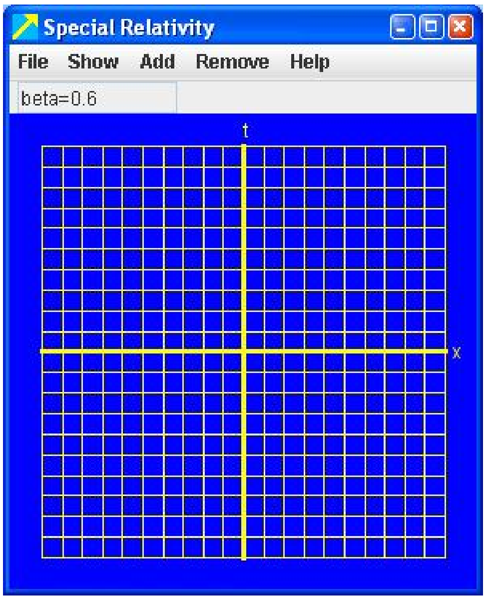


Figure 1: Minkowski diagram for a stationary coordinate system

2 Stationary Coordinate System

You are probably already familiar with graphical representations of objects and their motion in a coordinate system which is stationary relative to the observer. We would like to plot the time and the distance on axes that have the same units. To do so, instead of plotting the time directly, what we'll really be plotting is the speed of light times the time ct. Another way of thinking of this is choosing units for time where $c = 1$. In this case, the velocity is measured in units of the speed of light so that $v = \beta$. Equations 1 and 4 become

$$x' = \gamma(x - \beta t) \tag{5}$$
$$t' = \gamma\left(t - \beta x\right) \ . \tag{6}$$

We'll let the vertical axis be the time and the horizontal axis be the position of the particle. Such a coordinate system is shown in Figure 1 . For simplicity, I haven't labeled the axes with particular coordinates. With our choice of units, one unit in the horizontal direction is equal to the distance light travels during the same unit of time in the horizontal direction. In other words, if I choose the vertical units as years, the horizontal units will be light-years. If I choose the vertical units as nanoseconds, the horizontal units will be feet (approximately how far light travels in 10^{-9} seconds).

2.1 Event

Figure 2 shows an event, something that happens at a definite location and moment in time. Since both are fixed, an event is just a point in space-time. The particular event in the figure occurs at a time $t = 4$ and a position of $x = 6$.

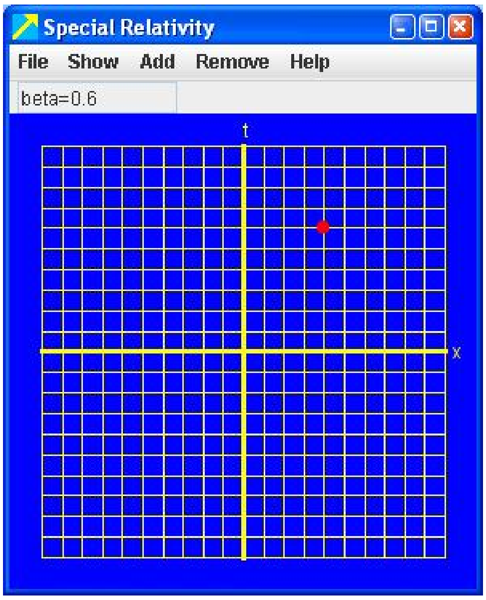


Figure 2: A single event in a stationary reference frame

2.2 Moving Object

Using our units, an object moving with a constant speed β have x and t related by

$$x = x_0 + \beta t \tag{7}$$

$$t = -\frac{x_0}{\beta} + \frac{1}{\beta}x \tag{8}$$

In other words, the object will appear as a line on the space-time diagram with a slope $1/\beta$ and an intercept of $t_0 = -x_0/\beta$. As an example, Figure 3 shows an object moving with a speed of $\beta = 1/4$. It has a position of $x = 2$ at $t = 0$ giving $x_0 = 2$ and $t_0 = -8$.

We call the line representing the history of the position of an object for all time its "world line." A valuable feature of Minkowski diagrams is that the world line of an object does not depend on the reference frame selected. Since changing reference frames is just changing coordinate axes, the world line itself remains constant.

A special case of a moving object would be a stationary object, or an object with $v = 0$. In this case, the slope of the line would be $1/\beta = 1/0 = \infty$. In other words, it would be a vertical line. Another way of thinking about the object is as something with a fixed value of x for all time. An example of an object which is stationary at the position $x = 2$ is shown in Figure 4 .

If the object was accelerating, the world line would be a curved line rather than a straight line. I didn't include the ability to draw lines for accelerating particles in my program, so we won't worry about what these look like graphically here.

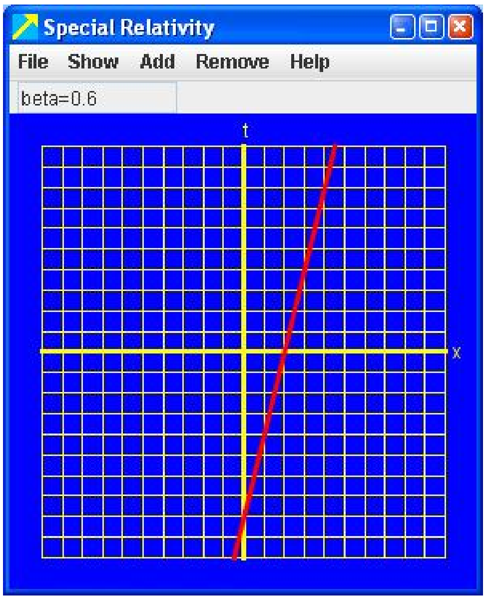


Figure 3: An object moving with a constant speed of $c/4$ in a stationary coordinate system

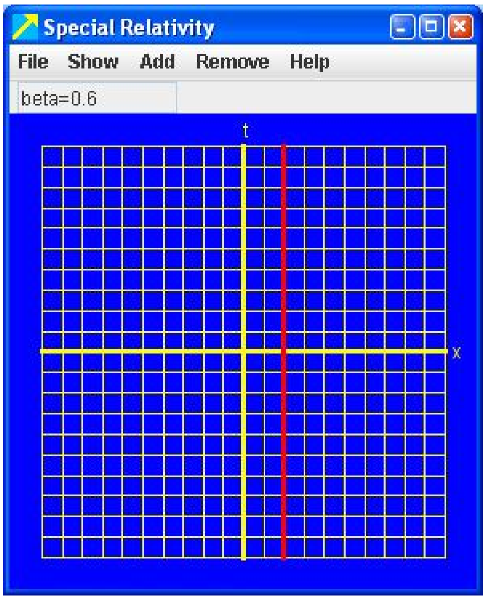


Figure 4: Stationary object at the position $x = 2$.

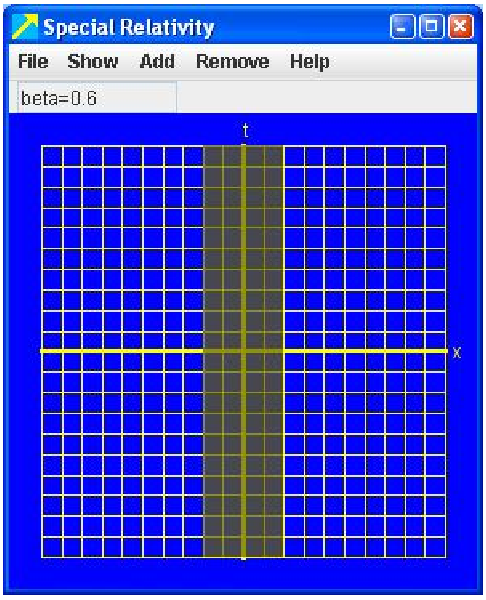


Figure 5: Stationary object of length 4 starting at $x = -2$

2.3 Extended Object

An extended object is one with a range of positions and times. With our single spatial dimension, it would look like a stick with x having a range between the two ends of stick at positions x_1 and x_2. A stationary stick would have its end points at constant time locations. The diagram therefore looks like a rectangle of with a range $x_1 \leq x \leq x_2$ and $-\infty < t < \infty$. This is shown in Figure 5 for a stationary stick with $x_1 = -2$ and $x_2 = 2$.

If the extended object is moving with a constant speed, then the two endpoints of the object will move like the constant speed objects discussed in Section 2.2. In other words, they will be slanted lines with a slope of $1/\beta$. Figure 6 shows the diagram of the same object as in Figure 5, but moving the a speed of $c/4$.

3 Coordinate Transformation

In this section, I'll discuss how to construct the x' and t' axes for a coordinate system which is moving relative to the observer. Equation 5 says that the units on the x' axis will be stretched by a factor of γ and will be tilted with an angle of slope of $\alpha = \arctan \beta$ relative to the x axis. Equation 6 says thet the units on the t' axis will also be stretched by a factor of γ and will be tilted by the same angle α relative to the t axis (or by the angle $\arctan(1/\beta)$ with respect to the x axis).

The direction of the t' axis shouldn't be too surprising if you consider what it represents. Imagine an object which is moving with a speed β relative to the observer and is at the origin at the time $t = t' = 0$. In the coordinate system also moving with a speed β relative to the observer, this object should be moving along the t' axis. These are the

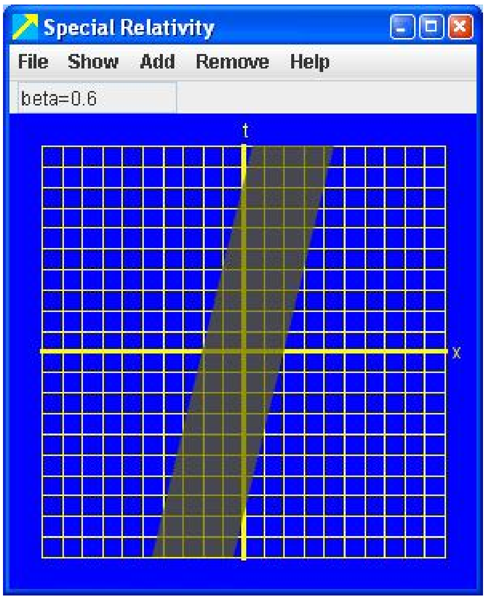


Figure 6: Object of length 4 centered on the origin at time $t = 0$ and moving with a speed of $v = c/4$

points with $x' = 0$ for all time t'. In other words, the world line for a moving object which is at the origin at $t = t' = 0$ should be along the t' axis.

Figure 7 shows what the x' and t' axis would look like for a coordinate system moving at half the speed of light ($\beta = 0.5$). I have also included the grid lines representing space-time coordinates with constant x' and constant t'. Note how the grid lines are farther apart on the x' and t' axes than they are measured along the t and x axes. This is because of the factor of γ in Equations 5 and 6. If the coordinate system is moving with a speed $-\beta$ instead of β, the x' and t' axes will be rotated to the other side of the x and t axes.

3.1 Light Cone

In interesting set of reference points in space-time diagrams are the paths light beams would follow. These paths lie along what is called the "light cone." These are the points moving at the speed of light with $\beta = 1$. In three spatial dimensions (four total dimensions) these paths have the equation

$$x^2 + y^2 + z^2 = t^2 \tag{9}$$

which describes a cone at a 45° angle between an two axes. In two dimensions, this just reduces to

$$\begin{aligned} x^2 &= t^2 && (10) \\ x &= \pm t\,. && (11) \end{aligned}$$

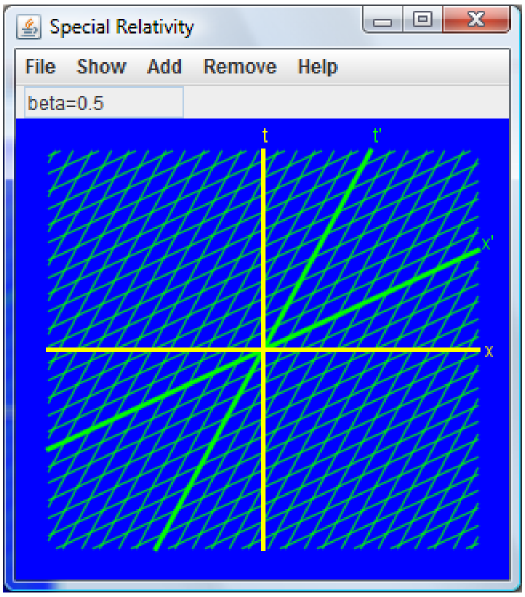


Figure 7: Rotated coordinate axes x' and t' for a reference frame moving at half the speed of light

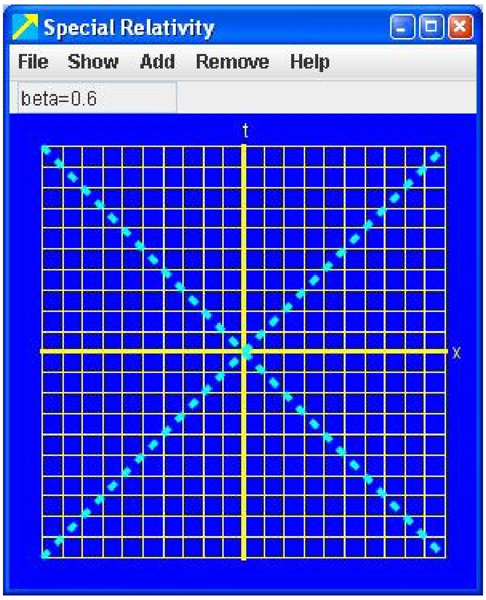


Figure 8: light cone

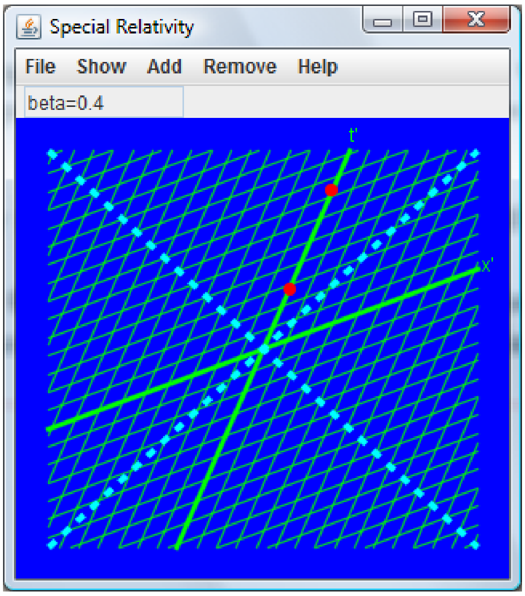


Figure 9: two events in Region 1 which happen at the same location in a reference frame with $\beta = 0.4$

The two lines representing the light cone are plotted as cyan dashed lines in Figure 8 .
The light cone divides space into three regions:

1. The region containing the t and t' axes. If two events are both in this region, one could find a reference frame where both events happen at the same position. The region with positive t and t' represents events in the future. The region with negative t and t' represent the past.

2. The light cone itself.

3. The region containing the x and x' axes. If two events are both in this region, one could find a reference frame where both events happened at the same time (they were simultaneous). The region with positive x and x' represent things in front of us. The region with negative x and x' represent things behind us.

Since the speed of light is the same for all obeservers, the light cones are the same for any reference frame. They are always half-way between the coordinate axes for x' and t'.
Figure 9 shows two events in Region 1 with the coordinate system in which they happen at the same position. Figure shows two events in Region 3 along with the coordinate system in which they are simultaneous. Figure shows two events which cannot be at the same place or at the same time in any reference frame since they are in different regions.

3.2 Accelerating Reference Frames

As an aside, recall from Section 2.2 that the world line for an accelerating object is curved line rather than a straight line, as is the case with objects moving at constant speed. An

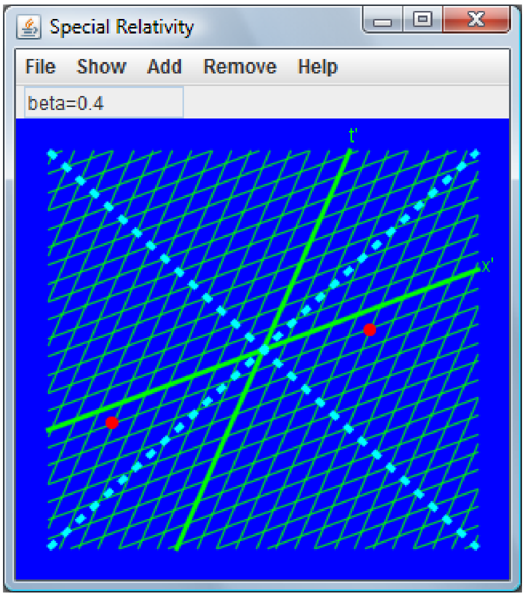


Figure 10: two events in Region 3 which happen at the same time in a reference frame with $\beta = 0.4$

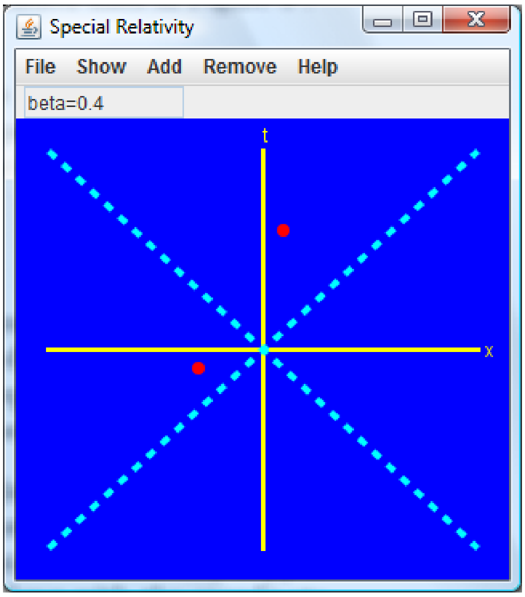


Figure 11: two events which occur in different regions

accelerating coordinate system moving with the object would be one in which such an accelerating object would be stationary. In this case, the coordiate axis would not only have to be rotated, but also warped so that they would no longer be straight lines, but rather would follow the trajectory of an accelerating object.

General Relativity is Einstein's extension of Special Relativity to include the effects of accelerating coordinate systems. It turns out that accelerating coordinate systems are what is needed to correctly handle gravitational forces. (I'd love to explain why, but that will take me way beyond the focus of this handout...) What I want to point out, however, is that the geometry of accelerating reference frames involves coordinates that have to curve to match the acceleration of this frame. This is essentially the idea behind the phrase "warping of space and time" people sometimes use to describe General Relativity. It's really nothing mysterious or magic, just a reflection of the fact that the trajectories of accelerating objects are curved paths in space-time diagrams.

It may be giving you a headache to think about coordinate systems that are warped as well as being rotated. If so, don't worry about it for now. You won't have the mathematical tools of differential geometry needed to describe these kinds of coordinate systems for a while yet. You can just file this idea away as an interesting tidbit you'll learn more about later.

4 Special Relativity Effects

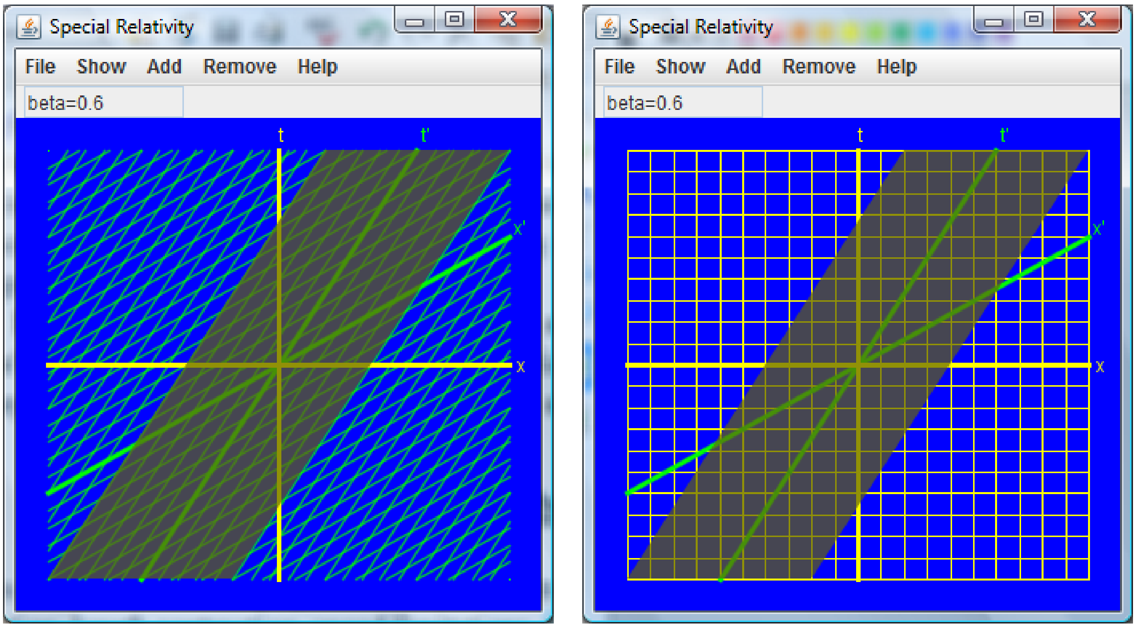
This section uses the graphical coordinate transformations developed in Section 3 to demonstrate the effects of time dilation, length contraction, adding speeds, and simultaneity issues. Each of these effects is easily seen geometrically in Minkowski diagrams.

4.1 Length Contraction

To see length contraction, let's create an object of length 8 in a coordinate system moving at $\beta = 0.6$. The object has an extent from -5 to 5 as measured along the x' axis in the moving frame as shown in Figure 13a . The same object is shown in Figure 13b with the grid drawn for the stationary coordinate system. An observer on the ground measures the same stick to have a length of 8. It goes from -4 to 4 along the x axis.

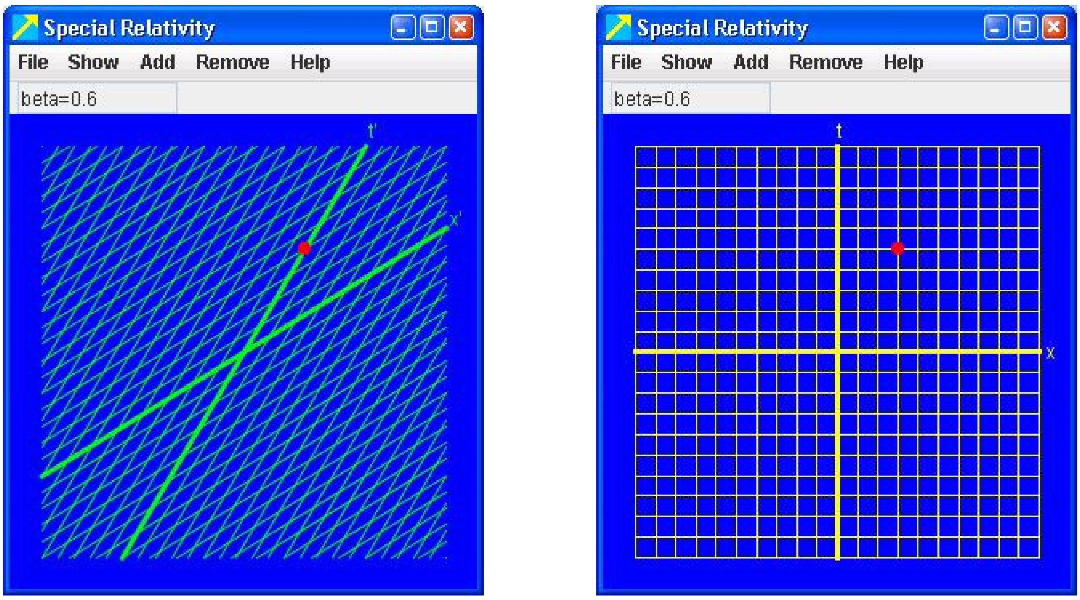
4.2 Time Dilation

I will give an example of time dilation using a specific example. Let's take an object at rest and at the origin in a reference frame moving at $\beta = 0.6$. Four seconds later, as measured in that reference frame, the object will be at the coordinate $(x' = 0, t' = 4)$. This is illustrated in Figure 13a . Looking at this same event in the observer's reference frame (see Figure 13b), we see that five seconds have elapsed rather than the four seconds measured in the moving frame. (Since the frame is moving, the object is also no longer at the origin.)



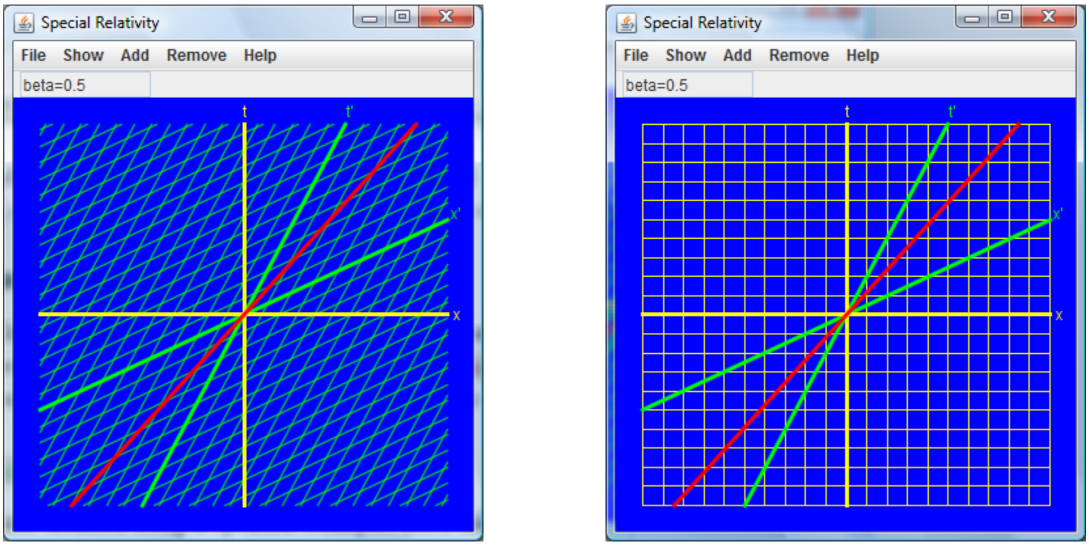
(a) moving coordinate system (b) stationary coordinate system

Figure 12: object of length 10 moving at $\beta = 0.6$



(a) moving reference frame (b) stationary reference frame

Figure 13: object four seconds after starting at the origin and moving at a speed $\beta = 0.6$



(a) moving coordinate system (b) stationary coordinate system

Figure 14: world line of an object moving at $\beta = 0.6$ in a reference frame moving at speed $\beta = 0.5$

4.3 Adding Speeds

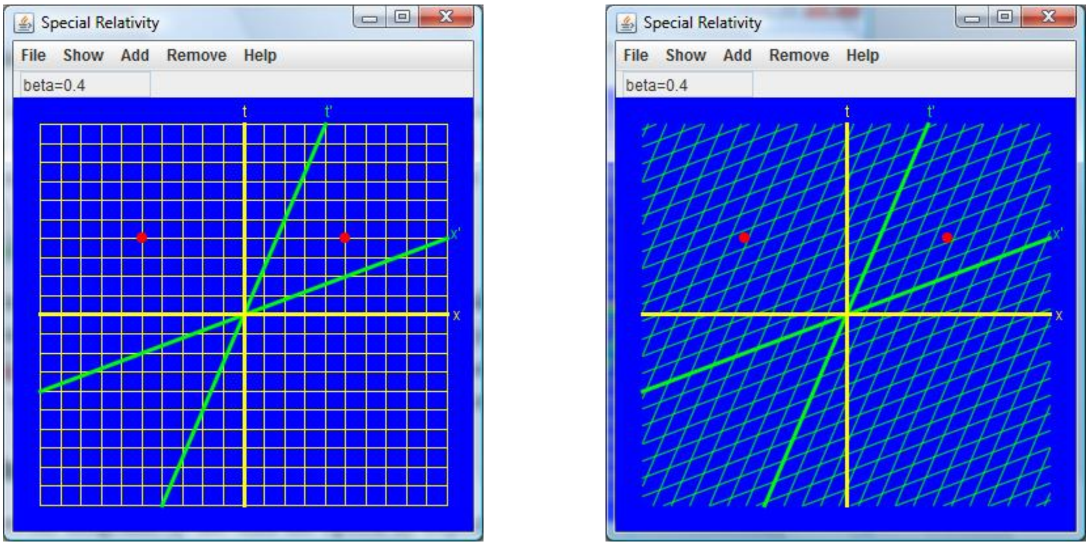
It is possible to use Minkowski diagrams to the find the speed of objects in different reference frames. For example, let's start with an object moving at $\beta = 0.6$ in a reference frame moving at a speed $\beta = 0.5$. We want to find out how fast it's going relative to a stationary observer. As explained in Section 2.2, the world-line of a moving obvect has a slope of $1/\beta$, where β is the object's speed in units of the speed of light. Therefore an object with a speed of $\beta = 0.6$ has a slope of $1/(3/5) = 5/3$. Such a world line is shown in Figure 14a along with the coordinate system of the moving reference frame. Note that it has a slope of $5/3$ since it passes through the origin an the point $(x' = 3, t' = 5)$. The same world line is shown in Figure 14b, but this time in the stationary coordinate system. The slope of the line as measured in this coordinate system is 1.2 corresponding to a speed $\beta = 0.83$. This is pretty close to the analytical answer using Equation 1.43 in your textbook which gives

$$u_x = \frac{u'_x + \beta}{1 + u_x \beta} = 0.85 \tag{12}$$

for our units.

4.4 Simultaneity

It is also easy to compare whether two events are simulaneous in different coordinate systems. Consider the two events shown in Figure 15a which happen at the same time, $t = 4$ in the stationary coordinate system, but in different positions. Figure 15b shows



(a) stationary reference frame

(b) moving reference frame

Figure 15: two simultaneous events

that one event happens at time $t' = 2.2$ and the other at $t' = 6.7$ in a rerence frame moving at $\beta = 0.4$.

Nomenclature

c: the speed of light

event: something that occurs at a specific location and time

General Relativity: a generalization of special relativity in which reference frames need not be in uniform motion (i.e. they can be accelerating). This generalization is needed to adequately explain gravitational forces.

light cone: the set of paths light beams follow in space-time diagrams. In four dimensions, these are cones where $x^2 + y^2 + z^2 = t^2$. In two space-time dimensions, these are just the lines with $x^2 = t^2$ $(x = \pm t)$.

linear equation: an equation in which all of the variables appear to first order and are only multiplied by constants (not by each other)

Minkowski diagram: a graphical way to visualize Special Relativity Effects by looking at events in two different coordinate systems.

space-time diagram: a graphical representation of an object's position and time

world line: the line representing the position of an object as a function of time. In a Minkowski diagram, the world line of an object is independent of the coordinate system (reference frame) chosen.

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