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Hub location problems in transportation networks

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A B S T R A C T

In this paper we propose a 4-index formulation for the uncapacitated multiple allocation hub location problem tailored for urban transport and liner shipping network design. This formulation is very tight and most of the tractable instances for MIP solvers are optimally solvable at the root node. While the existing state-of-the-art MIP solvers fail to solve even small size instances of problem, our accelerated and efficient primal (Benders) decomposition solves larger ones. In addition, a very efficient greedy heuristic, proven to be capable of obtaining high quality solutions, is proposed. We also introduce fixed cost values for Australian Post (AP) dataset.

1. Introduction

In the last two decades, due to an enormous increase in the body of telecommunications, transportation and logistics, several different cooperative strategies such as alliances and coalitions are either formed or investigated and many industrial studies are devoted to these areas. In such studies, hub-and-spoke structures have received a lot of attention as they offer possibilities of efficient capacity sharing and fleet management on different legs of transport routes. This leads to a better utilization of transporters such as vehicles and vessels. While public transport networks and liner shipping industries evidently operate on hub-and-spoke network structures, Hub Location Problems (HLP) have not received significant attention in these areas in the literature.

Aiming at minimizing the total costs, maximizing utilization of transporters, maximizing the service level, etc., in a hub-and-spoke network, the flow between O–D pairs is routed through some selected intermediate nodes (called *hub* nodes) and edges (called *hub edges*) connecting the hubs. Once the hubs are chosen, the non-hub nodes (called *spoke* nodes) are allocated to them in order to transship flow via the sub-graph composed of all the hub nodes and the hub links connecting them (called *hub-level (sub-)network*). The allocation scheme is either single or multiple based on the particular nature of application. In a single allocation scheme, a spoke node is allocated to a single hub, while such restriction is relaxed in a multiple allocation scheme. Such a hub-and-spoke structure avoids direct shipping, which results in underutilized (or at least very infrequently utilized) use of vehicles/vessels operating on some (or perhaps many) of the direct links and drops the

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underutilized links in favor of concentrating flow on hub edges and better utilization of facilities operating there. As a result of this flow concentration, economies of scale can be exploited by using more efficient transporters on the hub links.

In classical HLP models, four main assumptions are usually considered (Nickel et al., 2001):

(i) The hub-level network is a complete graph.
(ii) Using inter-hub connections has a lower price per unit than using spoke connections. That is, it benefits from a discount factor α,($0 < \alpha < 1$).
(iii) The direct connections between the spoke nodes are not allowed.
(iv) The triangle inequality holds in the cost structure and costs are proportional to the distance.

In applications like public transport and in particular liner shipping which are the main areas our model addresses, the structure of hub-level sub-network plays a major role in the level of service offered to the users and also vital for the sustainability of business in such highly competitive environments.

1.1. Urban transport

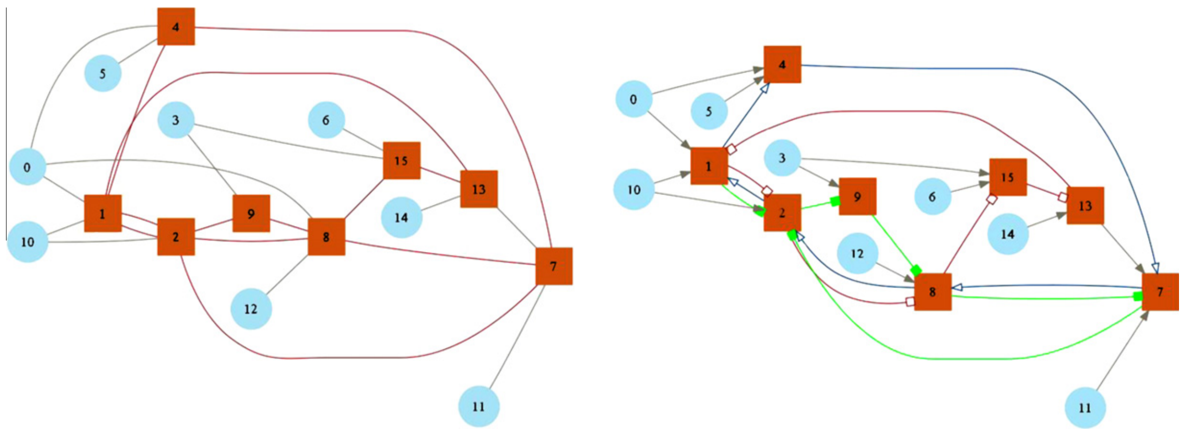
Most of hub-and-spoke network structures proposed in the literature are based on a point-to-point connection between hub nodes (complete hub-level sub-graph), and are very rarely applicable to real-life cases. In a city with at least two choices of service (e.g. bus, metro, subway and train, etc.), the hub nodes are usually chosen from among the nodes where bus lines intersect or pass by a reasonable proximity of other fast-lane services. The fast-lane tracks are considered to be the hub edges while the bus lines are assumed to be spoke edges. Given that such nodes are potential hub nodes, direct point-to-point connections between all the fast-lane stops is rarely available (the fast-lane sub-network is not usually a complete graph). Moreover, due to a priori existence of infrastructures (or even historical architecture, environmental barriers, parks, etc.) it may not be possible to have complete hub-level sub-graphs (even if economically advantageous).

A typical hub-and-spoke structure in public transport is depicted in Fig. 1. In this figure the rectangles are hub nodes and the circles are spoke nodes. The allocation follows the multiple assignment scheme and the hub-level is not complete. Once a passenger/commodity departs from origin and arrives at the first hub node, the number of hub level links which must be traversed before arriving at the destination is not restricted to at most two (as the case in the classical HLP models).

Three different operating fast-lane tracks of the plain hub-and-spoke structure in Fig. 1a distinguishable by different rectangular arrow-heads are depicted in Fig. 1b. We must also note that, there might exist spoke connections between two hub nodes which coincide with the existence of bus lines between two fast-lane stops, if advantageous. This is depicted in the figures by the spoke line between nodes 7 and 13.

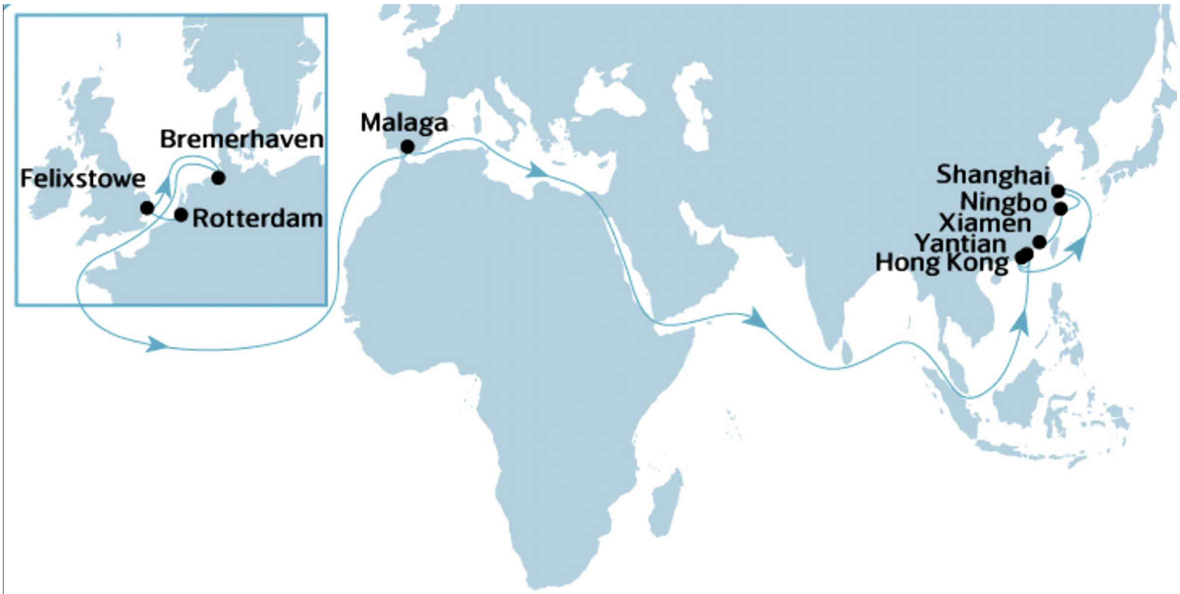
1.2. Liner shipping

Up to 90 percent of global trade volume is transported as containerized cargo on vessels where again up to 90 percent of such vessels are fully cellular container vessels (UNCTAD, 2008). The network structure is based on hub-and-spoke models. As shown in Fig. 2a in an east-bound voyage (in this case, the *Asia–Europe (AE7)-East Bound*) a vessel starts from European Container Terminal (ECT) in Rotterdam while feeder vessels (spoke-link-operating vessels) have supplied it with containers from different smaller ports destined to east Asia. The vessel later calls Felixstowe in UK where it delivers all containers that



(a) Plain hub-and-spoke structure. (b) Transportation hub-and-spoke structure.

Fig. 1. A typical hub-and-spoke structure.



(a) East Bound line AE7 of Maersk Line.

(b) West Bound line AE7 of Maersk Line.

Fig. 2. Line AE7 of Maersk Line. Source: www.maerskline.com.

have been destined to this port as well as all the feeder (spoke) ports allocated to it. Furthermore, it also loads the containers from Felixstowe and also those arrived from smaller ports using feeder vessels. This process is repeated at every port along the route towards Bremerhaven, Malaga, Yantian, Hong Kong, Shanghai, Ningbo and finally Xiamen. But, the west-bound ports may not exactly coincide with the east-bound path (see Fig. 2b). So far, except (Nickel et al., 2001), almost none of the hub location models in the literature address such flexible structures.

Here, one again observes that in such applications there is no necessity for the hub-level network to be a complete network.

In principle, *no* particular structure is available for such network *except* that the *hub-level network is connected*.

2. Literature review

Hakimi (1964) showed that in order to find the optimal location of a single switching center that minimizes the total wire length in a communication network, one can limit oneself to finding the vertex median of the corresponding graph. Hakimi (1965) proved that optimal locations of switching center(s) in a graph of communication network are at p-medians of the corresponding weighted graph. In general he emphasized the *node optimality* of one-median and p-median problems in a weighted graph.

Goldman (1969) proposed models for the problem of finding p centers among a set of candidates and assigning flows of the rest to the nominated center(s), minimizing transportation cost. His problem was in fact the *hub median* problem while he refer to it by *center*, instead.

HLPs were first addressed by O'Kelly (1986a,b). In discrete hub location problem, the first work is again due to O'Kelly (1987), where he proposed a quadratic model for the *Single Allocation p-Hub Median Problem* (SApHMP) (also known as *Uncapacitated Single Allocation p-Hub Median Problem* (USApHMP)). There are some reviews dedicated to HLPs on discrete networks. Among them, we refer the readers to the latest two reviews (Campbell et al., 2002; Alumur and Kara, 2008) and references therein.

In the formulation context, the first linear integer programming for pHMP was proposed by Campbell (1994b). Again, Campbell (1996) presented another integer programming (IP) formulations for the SApHMP. Skorin-Kapov et al. (1996), proposed a tight MIP formulation for the USApHMP. Ernst and Krishnamoorthy (1996) presented an LP formulation for the SApHMP which required fewer variables and constraints. O'Kelly et al. (1996) employed other existing formulations and improved the linearization scheme for both single and multiple allocations. Ebery (2001) presented another MIP for USApHMP.

Other work by Sohn and Park (1998) deals with the USApHMP. They studied the case of symmetric unit flow cost which is proportional to the distance. They improved the formulation in O'Kelly et al. (1996). The first model for the multiple allocation problem was due to Campbell (1992). Campbell (1994b) showed that in the absence of capacity constraints, the total flow from each origin to each destination will be routed via the least cost hub pair. Therefore, all the n^4 binary variables can be relaxed in the MApHMP. He proposed formulations for the UpHMP and UHLP. Campbell (1996), Skorin-Kapov et al. (1996), Ernst and Krishnamoorthy (1998a) and Sohn and Park (1998) proposed other models for the UMApHMP.

The efforts aiming at finding the optimal number of hubs for a given set of interactions between a number of fixed nodes leads to incorporating additional aspects into the problem. Trying to make the number of hubs an endogenous part of the problem, one can either make the operating cost of hub facilities explicit or consider limited budget for establishment. O'Kelly (1992) first introduced incorporation of fixed costs as hub node setup costs. Campbell (1994b) also suggested using a threshold approach and incorporated fixed costs for spoke edges in the pHMP. Sohn and Park (1998) proposed improved MIP formulations for UMApHMP and USApHMP where fixed costs for hub edges were considered.

In cases where the number of hubs is not fixed *a priori*, in addition to the multiple and single allocation schemes, some capacity policies might also exist. These cases are studied in Campbell (1994b), Aykin (1994), Ernst and Krishnamoorthy (1999), Ebery et al., 2000, Labbé et al., 2005, Yaman, 2005, Yaman and Carello, 2005, Marín, 2005a and Costa et al., 2008.

2.1. Applications

When comparing different application areas where HLPs are employed, HLPs have received less attention in the transportation sector. Nevertheless, we cite some of the major ones in the following.

2.1.1. Freight transport

Powell and Sheffi (1983) deal with the load planning problem of less-than-truckload motor carriers through determination of freight routes on a network to minimize costs while maintaining the service. Hall (1989) examines the impact of overnight restrictions and time zones on the configuration of an air freight network and location of hub terminal. Iyer and Ratliff (1990) find the location of accumulation centers in a guaranteed time distribution system. O'Kelly and Miller (1991) considered a single facility minimax hub location problem for sitting a hub facility in order to minimize the most costly interaction between a set of fixed nodes. Campbell (2005) takes into account the strategic network design for motor carriers in a hub-and-spoke framework. Racunica and Wynter (2005) formulated a nonlinear concave-cost hub location problem to identify the optimal location of intermodal freight hubs. Jeong et al. (2007) applies the hub-and-spoke network problem for railroad freight to find transport routes, frequency of service, length of trains to be used, and transportation volume, etc. Cunha and Silva (2007) models a hub-and-spoke network design for LTL trucking company in Brazil. Recent papers include CalIk et al. (2009) for a hub covering problem over incomplete hub networks, Yaman (in press) for a hierarchical hub median problem with single assignment, Campbell (2009) for the time definite transportation services and Alumur et al. (2009) for a single allocation incomplete hub networks for LTL trucking in Turkey.

2.1.2. Public transport

Mathematical models and solution approaches are presented in Gelareh and Nickel (2007) and Nickel et al. (2001). Gelareh (2008) proposed several variants of the hub location problems with a variety of hub-level configurations in particular addressing public transport planning (see Gelareh and Nickel, 2008a). They also proposed the first multi-period hub location problem for the same application (see Gelareh and Nickel, 2008b). Exact and heuristic solution methods for both single and multi-period models were developed, and computational results indicated that both are very efficient.

2.1.3. Air transport

Jaillet et al. (1996) introduced flow-based models for designing capacitated networks and routing policies suggesting the presence of hubs whenever they are cost efficient. The network design problem is concerned with the operation of a single airline with a fixed share of the market. Kuby and Gray (1993) considered a hub network design problem with stopovers and feeders for Federal Express. A recent work is due to Eiselt and Marianov (2009) dealing with a competitive hub location problem where customers have gravity-like utility functions.

2.1.4. Maritime transport

Aversa et al. (2005) proposed a mixed integer programming (MIP) model for locating a hub port in the East coast of South America. In 2008, Takano and Arai (2009) applied the quadratic model of O'Kelly and developed a genetic algorithm to solve instances of the problem. As opposed to many existing tight linear formulations, this work applies a nonlinear formulation; it neither allows more than one hub edge being used along any O–D path nor does it allow any spoke–spoke connection. Imai et al. (2009) presented a model for simultaneous hub-and-spoke network design and fleet deployment problems in liner shipping. Their emphasis is on the empty container repositioning and their model hardly resembles a standard hub-and-spoke model. Imai et al. (2006) studies the viability of deploying mega-vessels by employing a non-zero sum two-person game with hub-and-spoke networks strategy for mega-vessels and multi-port calling for conventional ship size. Other results can be found in Hsu and Hsieh (2007) for a two-objective model to determine the optimal liner routing, ship size, and sailing frequency for container carriers by minimizing shipping costs and inventory costs. Konings (2006) tries to investigate whether hub-and-spoke services are fruitful tools in improving the performance of container-on-barge transport and so in gaining market share. Recently, Gelareh et al. (2010) proposed a multi-criteria competitive hub location problem for liner shipping industries. Other works addressing liner shipping network design, not necessarily the hub-and-spoke design, include Choong et al. (2002) for empty container management of intermodal transportation networks.

The rest of the paper is organized as follows: In the next section, we are going to present a new mathematical model for application of HLPs in transportation. We will compare our model with the most similar ones available in the literature, in Section 3. In Section 4, we will propose a Benders decomposition schemes. In addition to that, we will propose a greedy neighborhood search heuristic equipped with intensification and diversification strategies. The computational experiments are reported in Section 5. In Section 6, we conclude our work and propose some research directions.

3. A hub location model arising in transportation applications

As mentioned earlier, to the best of our knowledge, the models in Nickel et al. (2001) were the first MIP models for HLPs in urban traffic networks. They proposed two models which are known as the Public Transport (PT) and Generalized Public Transport (GPT). In their models, some classical assumptions of HLPs are relaxed and models are customized for the public transport planning.

Here, we propose another MIP model with similar application. We refer to this model by Hub Location Model for Public Transport (HLPPT).

The variables in this model are defined as follows: $x_{ijkl} = 1, i \neq j, k \neq l$ if the optimal path from i to j that traverses the hub edge (k, l), 0 otherwise. Also, $a_{ijk} = 1\, j \neq i,\, k \neq i, j$ if the optimal path from i to j traverses (i, k), where i is not a hub and k is a hub, 0 otherwise and $b_{ijk} = 1, j \neq i,\, k \neq i, j$ if the optimal path from i to j traverses (k, j), where k is a hub node and j is not a hub, 0 otherwise. In addition, $e_{ij} = 1, i \neq j$ if the optimal path from i to j traverses (i, j) where at most one of i and j is a spoke node, 0 otherwise. For the hub-level variables, $y_{kl} = 1, k < l$ if the hub edge $\{k, l\}$ is established, 0 otherwise and $h_k = 1$ if node k is chosen to be a hub, 0 otherwise.

The transportation cost per unit of flow with origin i and destination j amounts to the sum of: (i) the cost of sending flow from i to the first hub node (if i is a spoke node assigned to a hub node) in the path to j, (ii) the cost incurred by traversing one or more hub edges in the hub-level network discounted by the factor α, $0 < \alpha < 1$, and (iii) the cost of transition on the last spoke edge (if destination is a spoke node). We introduce W_{ij} as the flow from i to j, C_{ij} the cost per unit of flow on link i to j, F_k fixed cost for establishing hub at node k and I_{kl} the fixed cost for establishing hub link between nodes k and l. The proposed mathematical formulation follows:

(HLPPT)

$$\min \sum_i \sum_{j \neq i} \sum_k \sum_{l \neq k} \alpha W_{ij} C_{kl} x_{ijkl} + \sum_i \sum_{j \neq i} \sum_{k \neq i,j} W_{ij} C_{ik} a_{ijk} + \sum_i \sum_{j \neq i} \sum_{k \neq i,j} W_{ij} C_{kj} b_{ijk}$$

$$+ \sum_i \sum_{j \neq i} W_{ij} C_{ij} e_{ij} + \sum_k F_k h_k + \sum_k \sum_{l > k} I_{kl} y_{kl}, \tag{1}$$

$$\text{s.t.}\, y_{kl} \leqslant h_k, y_{kl} \leqslant h_l, \quad k, l > k, \tag{2}$$

$$\sum_{l\neq i} x_{ijil} + \sum_{l\neq i,j} a_{ijl} + e_{ij} = 1, \quad i,j\neq i, \tag{3}$$

$$\sum_{l\neq j} x_{ijlj} + \sum_{l\neq i,j} b_{ijl} + e_{ij} = 1, \quad i,j\neq i, \tag{4}$$

$$\sum_{l\neq k,i} x_{ijkl} + b_{ijk} = \sum_{l\neq k,j} x_{ijlk} + a_{ijk}, \quad i,j\neq i,k\neq i,j, \tag{5}$$

$$x_{ijkl} + x_{ijlk} \leqslant y_{kl}, \quad i,j\neq i,k,l>k, \tag{6}$$

$$\sum_{l\neq k} x_{kjkl} \leqslant h_k, \quad j,k\neq j, \tag{7}$$

$$\sum_{k\neq l} x_{ilkl} \leqslant h_l, \quad i,l\neq i, \tag{8}$$

$$a_{ijk} \leq 1 - h_i, \quad i,j\neq i,k\neq i,j, \tag{9}$$

$$b_{ijl} \leq 1 - h_j, \quad i,j\neq i,l\neq i,j, \tag{10}$$

$$a_{ijk} + \sum_{l\neq j,k} x_{ijlk} \leqslant h_k, \quad i,j\neq i,k\neq i,j, \tag{11}$$

$$b_{ijk} + \sum_{l\neq k,i} x_{ijkl} \leqslant h_k, \quad i,j\neq i,k\neq i,j, \tag{12}$$

$$e_{ij} + 2x_{ijij} + \sum_{l\neq j,i} x_{ijil} + \sum_{l\neq i,j} x_{ijlj} \leqslant h_i + h_j, \quad i,j\neq i, \tag{13}$$

$$x_{ijkl}, y_{kl}, h_k, a_{ijk}, b_{ijk}, e_{ij} \in \{0,1\}. \tag{14}$$

The objective (1) calculates the total cost of transportation plus hub node and edge setup costs. Constraints (2) ensure that both end-points of a hub edge are hub nodes. Constraints (3)–(5) are the flow conservation constraints. The constraints (6) ensure that a hub edge should exist before being used in any flow path. In (7) (resp. (8)) it is ensured that only a flow with origin (destination) of hub type is allowed to select a hub edge to leave the origin (arrive at the destination). Constraints (9) and (10) check the end-points of spoke edges. Any flow from i to j that enters to (depart from) a node other than i and j, the node should be a hub node. This is ensured by (11) and (12). The choice of edges on the path between origin and destination (i and j) depends on the status of i and j: whether both, none or just one of them is a hub. This is checked by (13). In an uncapacitated environment, as also mentioned in Campbell (1994b), only hub node and hub edge variables may need to be considered as binary variables. Therefore, the (14) can be replaced by,

$$x_{ijkl}, a_{ijk}, b_{ijk}, e_{ij} \in (0,1), \qquad h_k, y_{kl} \in \{0,1\}. \tag{15}$$

Henceforward, whenever we refer to the HLPPT as (1)-(13) together with the constraint (15).

3.1. HLPPT vs. PT

In order to justify the necessity of proposing new formulation we need to compare HLPPT and PT formulation. The definition of variables in PT is the same as in HLPPT except for s_{ijkl} which is the fraction of flow from i to j traverses spoke edge (k,l).

(PT)

$$\min \quad \sum_i \sum_{j\neq i} \sum_k \sum_{l\neq k} W_{ij} d_{kl} (\alpha x_{ijkl} + x_{ijkl}) + \sum_k \sum_{l>k} I_{kl} y_{k,l} + \sum_k F_k h_k \tag{16}$$

$$\text{s.t.} \quad \sum_{l\in N} (x_{ijkl} + s_{ijkl} - x_{ijlk} - s_{ijlk}) = \begin{cases} +1, & i,j,k \in V : k = i, \quad i\neq j, \\ -1, & i,j,k \in V : k = j, \quad i\neq j, \\ 0, & i,j,k \in V : k\neq i, \quad k\neq j, \end{cases} \tag{17}$$

$$x_{ijkl} \leqslant y_{kl} \quad i,j,k,l\neq k, \tag{18}$$

Table 1
Comparison between HLPPT and PT.

	Number of constraints	Number of variables	
		Binary	Continuous
PT	$2n^4 + 3n^3 + 5n^2$	$\frac{n(n-1)}{2} + n$	$2n^4$
HLPPT	$n^4 + 5n^3 + 7n^2$	$\frac{n(n-1)}{2} + n$	$n^4 + 2n^3 + n^2$

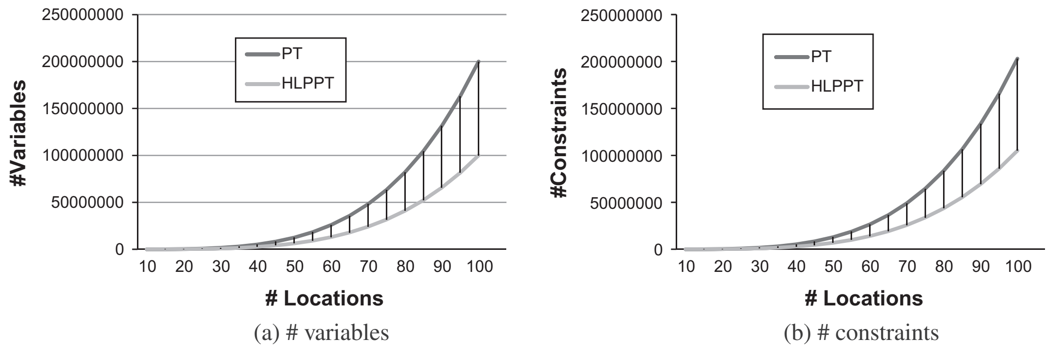


Fig. 3. PT vs. HLPPT.

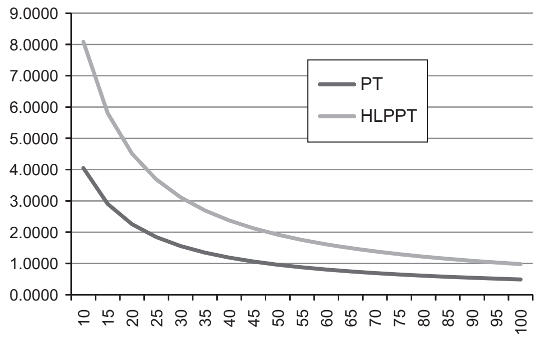


Fig. 4. Ratio of $\frac{\text{no. basic variables}}{\text{no. total number of variables}} \times 100$.

$$x_{ijlk} \leqslant y_{kl} \quad i,j,k,l \neq k, \tag{19}$$

$$s_{ijik} \leqslant h_k \quad i,j,k : k \neq j, \tag{20}$$

$$s_{ijkj} \leqslant h_k \quad i,j,k : k \neq i, \tag{21}$$

$$s_{ijij} \leqslant h_i + h_j \quad i,j, \tag{22}$$

$$y_{kl} \leqslant h_k \quad k,l \neq k, \tag{23}$$

$$y_{kl} \leqslant h_l \quad k,l \neq k, \tag{24}$$

$$s_{ijkl}, x_{ijkl} \geqslant 0 \quad i,j,k,l, \tag{25}$$

$$y_{kl}, h_k \in \{0,1\} \quad k,l. \tag{26}$$

(i) PT does not guarantee the connectivity in the hub-level network,
(ii) as shown in Table 1, although both PT and HLPPT both are 4-index formulations, the number of variables and constraints in HLPPT is considerably less as n grows. This is depicted in Fig. 3a for the variables and Fig. 3b for the constraints.

Table 2
HLPPT vs MAHLP and SAHLP.

	HLPPT	MAHLP	SAHLP
Locating hubs	√	√	√
Selecting hub edges	√[a]	×	×
Allocation	Polynomial	Polynomial	NP-hard [b]

[a] In a special case reduces to QAP.
[b] QAP.

Table 1 sheds more light on this comparison.

It must be noted that in any feasible solution to the problem less than 0.4% of the flow variables will be in the basis. By some extreme assumption, if a flow between a given O–D pair traverses $n-1$ links from origin to destination then the total number of flow variables in the basis will be strictly less than $n(n-1)^2$. Fig. 4 depicts the ratio of maximum number of basic variables to the total number of variables for both formulations. One observes that this ratio is higher for HLPPT meaning that HLPPT is more compact than PT and utilizes a larger fraction of variables.

Therefore, our new formulation is superior to PT both with respect to the number of variables and constraints and the utilization of variables used to describe the polytope.

3.1.1. complexity of model versus classical models

In traditional HLPs with the assumptions of (i)–(iv), once the hub nodes are nominated, the hub-level configuration is known to be a complete sub-graph (clique) and the remaining problem in the multiple allocation will be to find the cheapest routes which are composed of one, two or at most three links. In the single assignment scheme it leads to the Quadratic Assignment Problem (QAP) which is again an NP-hard problem.

In our model, HLPPT, three levels of decisions are made simultaneously: (i) locating the hub nodes, (ii) choosing the connecting hub edges such that it results in a connected hub-level graph, and (iii) routing the flows through the cheapest paths which is not restricted to one, two or three links. In (ii), neither the number of hub edges nor the way in which they must be connected (to make an optimal connected graph) is known. In fact, (ii) can be considered as the problem of assigning an unknown and finite number of edges to pairs of hub nodes in such a way that the resulted graph becomes connected. Therefore, in a special case, it reduces to the QAP. Table 2 sheds some light on this fact by taking into account both single allocation (*SAHLP*) and multiple allocation (*MAHLP*) hub location problems.

Thus, in terms of difficulty of problem, one can say that HLPPT is more challenging than MAHLP. Compared to SAHLP, if not more complex, it is definitely not easier since allocation of $n-q$ spoke nodes to q hub nodes must not be more challenging than allocation of an unknown and finite number of hub edges to pairs of hub nodes to make a connected graph. We expect it to be $NP-hard$, too.

4. Solution methods

It is well-known that the HLPs are NP-hard problems. Although there have been a lot of efforts to push the limit of solvability of instance of variants of HLPs, even, small size instances cannot be solved to optimality in a reasonable amount of time using general–purpose MIP solvers. Yet, almost all such efforts only consider classical models that are less complex compared to HLPPT. The structure of HLPPT formulation fits handsomely in the context of primal decomposition methods such as Benders for exact resolution of larger size instances. Nevertheless, we also propose a greedy heuristic algorithm to solve larger size instances.

4.1. Benders decomposition

Benders algorithm was proposed by Benders (1962) as a method for partitioning variables in an MIP model. This approach which is an iterative algorithm has been applied to numerous optimization problems in real-world applications. In classical view, it exploits the decomposable structure of problems and iterates between solving a master problem to optimality and consequently solving a sub-problem as the original one having the master-level variables fixed as parameters. The method makes use of exchange of information in terms of cuts between these two smaller problems to reach the optimality of the original model (Benders, 1962).

Benders decomposition approaches for UMAHLP for the formulation of Hamacher et al. (2004) is considered by Camargo et al. (2008). They decomposed the problem following the Benders scheme and solved the sub-problem for each origin and destination by inspection. Rodríguez-Martín and Salazar-González (2008) also presented an MIP model and proposed a double Benders decomposition.

Here we let the master problem be responsible for generating the hub-level network. Subsequently, the sub-problem (SP) reduces to a minimum cost multi-commodity flow problem on a graph amended by the solution to the master problem.

4.1.1. Master problem

In contrast with many classical hub location models, here it is not sufficient for the master problem to identify which hub nodes to be nominated and sent to the sub-problem. Rather, the connectivity of the master problem graph must be guaranteed. A trivial part of original model as a potential master problem for the Benders reformulation which contains hub-level decisions is the following:

(**MP1**)

$$\min \sum_k F_k h_k + \sum_k \sum_{l>k} I_{kl} y_{kl} + \eta, \tag{27}$$

$$\text{s.t. } y_{kl} \leqslant h_k, \quad k, l > k, \tag{28}$$

$$y_{kl} \leqslant h_l, \quad k, l > k, \tag{29}$$

$$\sum_k \sum_{l>k} y_{kl} \geqslant 1, \tag{30}$$

$$\text{optimality cuts} \qquad \text{for all extreme points}, \tag{31}$$

$$\text{feasibility cuts} \qquad \text{for all extreme rays}, \tag{32}$$

$$y_{kl}, \qquad h_k \in \{0,1\}, \quad \eta \geqslant 0. \tag{33}$$

A solution to MP1 can easily be a graph with more than one component. Even if one considers adding feasibility cuts in addition to optimality cuts the number of such cuts is prohibitive. In addition to this, it is not expected that the LP solution to the above formulation results in a feasible SP (i.e. bounded SP dual) and this avoids adding cuts generated from LP solutions.

Maculan et al. (2003) proposed a polynomial formulation for obtaining a connected sub-graph of a given graph. Here, we make use of a modified version of the model in Maculan et al. (2003) for generating a connected hub-level network and obtaining feasible master problem solution, even for the LP relaxation. The resulting master problem is depicted in the following.

Let $G(V,E)$ be a connected graph, where $V = \{1,2,3,\ldots,n\}$ is the set of nodes or vertices and E the set of edges. Let $G_d = (V, A)$ be a directed graph derived from G, where $A = \{(i,j),(j,i) | \{i,j\} \in E\}$, that is, each edge u is associated with two arcs (i,j) and $(j,i) \in A$. Two new graphs $G^0 = (V_0, E_0)$ and $G_d^0 = (V_0, A_0)$ where $V_0 = V \cup \{0\}$, $E_0 = E \cup \{\{0,j\} | j \in V\}$, $A_0 = A \cup \{(0,j) | j \in V\}$, are introduced.

Let $h = (h_i)_{i\in V} \in \{0,1\}^{|V|}$, $y = (y_u)_{u\in E_0} \in \{0,1\}^{|E_0|}$ two $0-1$ vectors, and $z_{ij}^k \geqslant 0, (i,j) \in A_0$, $k \in V'$, where V' is a subset of V, and z_{ij}^k is a real flow in the arc $(i,j) \in A_0$, having *0* as source and k as destination. $E(i)$ is considered as the set of edges $u \in E$ such that an endpoint is i, $\Gamma^+(i) = \{j | (i,j) \in A_0\}$ and $\Gamma^-(i) = \{j | (j,i) \in A_0\}$, $m = |E|$ and $n = |V|$ (Maculan et al., 2003).

Henceforward, we will refer to the following model as (MP).

(**MP**)

$$\min \sum_k F_k h_k + \sum_k \sum_{l>k} I_{kl} y_{kl} + \eta, \tag{34}$$

$$\text{s.t. } \sum_{j\in\Gamma^+(0)} z_{0j}^k - h_k = 0, \quad k \in V, \tag{35}$$

$$\sum_{j\in\Gamma^+(i)} z_{ij}^k - \sum_{j\in\Gamma^-(i)} z_{ji}^k = 0, \quad i \in V - \{k\}, k \in V, \tag{36}$$

$$\sum_{j\in\Gamma^+(k)} z_{kj}^k - \sum_{j\in\Gamma^-(k)} z_{jk}^k + h_k = 0, \quad k \in V, \tag{37}$$

$$z_{ij}^k \leqslant y_{ij}, \quad \{i,j\} \in E_0, \quad k \in V, \tag{38}$$

$$z_{ji}^k \leqslant y_{ij}, \quad \{i,j\} \in E_0, \quad k \in V, \tag{39}$$

$$y_{ij} \leqslant h_i, \quad \{i,j\} \in E, \tag{40}$$

$$y_{ij} \leqslant h_j, \quad \{i,j\} \in E, \tag{41}$$

$$\sum_{j\in V} y_{0j} = 1, \tag{42}$$

$$\sum_{j\neq i,k} z_{ji}^k \leqslant h_i, \quad i,k \in V, \tag{43}$$

$$y_{j0} = 0, \quad j \in V, \tag{44}$$

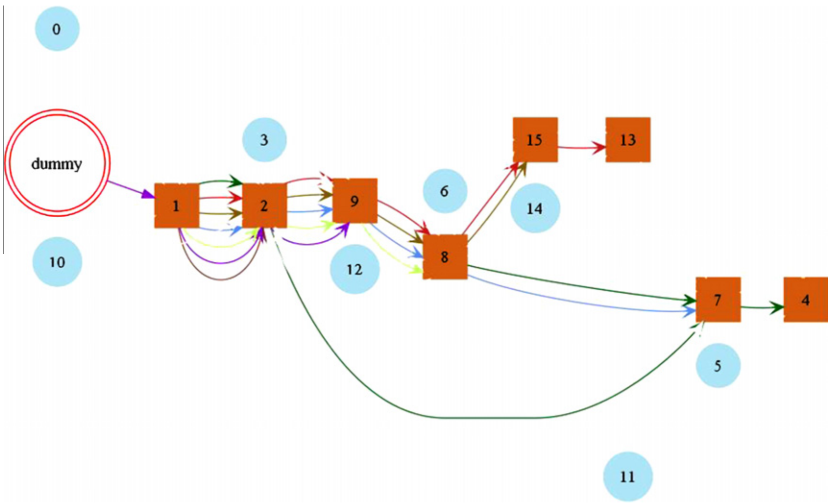


Fig. 5. The dummy node sends one unit flow to all nominated hubs.

$$\sum_{l\neq k}(y_{kl}+y_{lk})\geqslant h_k,\quad k\in V, \tag{45}$$

$$z^k_{ij}\geqslant 0,\quad (i,j)\in A_0,\quad k\in V, \tag{46}$$

$$y_{ij}\in\{0,1\},\quad \{i,j\}\in E_0,\quad h_k\in\{0,1\},\quad k\in V,\quad \eta\geqslant 0. \tag{47}$$

The objective function (34) minimizes the fixed cost of establishments plus the linking variable. (35) guarantees that one unit of flow is destined to every hub node from the dummy node. (36) ensures that the flow conservation holds at any non-destination node along the path destined to any hub node. (37) indicates that the destination hub node receives its demand from the dummy node and does not let any fraction of the flow to depart. (38)–(40) ensure that the flow only travels on hub links and every hub link has two hub end-points, respectively. From the dummy node, only one dummy hub edge is established (42). (43) also guarantee that the flow from dummy node to a hub k may pass through i only if it is a hub node.

Without (45), if integrality condition is maintained the solution gives a feasible SP but this does not hold for fractional LP solution. It must be guaranteed that if a fraction of a node is opened as hub then the total links encompassed from and arrived to it must be at least equal to it (45) (in fractional sense of LP relaxation).

Fig. 5 depicts a solution to the MP where the dummy node sends the flow to all nominated hub nodes.

The dual variables u^1_{ij}, u^2_{ij} and $u^3_{ijk}\in\mathbb{R},\ u^4_{ij},\ u^5_{ij}, u^6_{ij},\ u^7_{ij},\ u^8_{ij},\ u^9_{ij},\ u^{10}_{ij}$ and $u^{11}_{ij}\in\mathbb{R}^+$ are introduced in correspondence with the indices of the constraints (3)–(5) (in fact $u^3_{ijk},\ \forall i,j,k$ can be omitted since the RHS is zero. They might be needed, e.g. in the case that the subgradient algorithm is going to be used to solve the sub-problem), (6)–(13), respectively. Also $u^e_{ij},\ u^a_{ijk},\ u^b_{ijk},\ u^x_{ijkl}\in\mathbb{R}$ are considered for bound constraints on variables e_{ij}, a_{ijk}, b_{ijk}, x_{ijkl}.

Moreover, since any solution (fractional or integer) results in a feasible flow problem, we are only dealing with optimality cuts of the following form added to the MP.

For the fixed parameters $\bar{h}_k,\ \forall k$ and $\bar{y}_{kl},\ \forall k,\ l>k$, a cut generated for the MP follows:

$$-\sum_{i,j\neq i}\Bigg(u^1_{ij}+u^2_{ij}+u^{11}_{ij}(h_i+h_j)+u^5_{ij}h_i+u^6_{ij}h_j+ue_{ij}+\sum_{k\neq i,j}\left(u^7_{ijk}(1-h_i)+u^8_{ijk}(1-h_j)+u^9_{jik}h_k+u^{10}_{ijk}h_k++ua_{ijk}+ub_{ijk}\right)$$
$$+\sum_k\sum_{l>k}u^4_{ijkl}(y_{kl})+\sum_{k\neq l,l\neq i,k\neq j}ux_{ijkl}\Bigg)\leqslant\eta,$$

where the LHS is composed of $\frac{n(n-1)}{2}$ strictly independent terms. If the cost matrix is symmetric then we can generate $\frac{n(n-1)}{2}$ cuts from this, otherwise the $n(n-1)$ cuts are separated corresponding to every $i,j\neq i$. For a given $i,j\neq i$ we can generate the following part of the cut:

$$-\left(u^1_{ij}+u^2_{ij}+u^{11}_{ij}(h_i+h_j)+u^5_{ij}h_i+u^6_{ij}h_j+ue_{ij}\right)+\sum_{k\neq i,j}\left(u^7_{ijk}(1-h_i)+u^8_{ijk}(1-h_j)\right)h_k+u^9_{jik}h_k+u^{10}_{ijk}h_k++ua_{ijk}+ub_{ijk}$$
$$+\sum_k\sum_{l>k}u^4_{ijkl}(y_{kl})+\sum_{k\neq l,l\neq i,k\neq j}ux_{ijkl}\leqslant\eta_{ij}, \tag{48}$$

where $\eta_{ij}\geqslant 0,\ \sum_{ij\neq i}\eta_{ij}=\eta$.

The dual of the flow sub-problem can be disaggregated for every O–D and resulting in $n(n-1)$ smaller problems. However, as n grows this becomes prohibitive. Hence, we adopt an aggregation strategy by separating single cut for every origin (O) and all destinations instead of separating cuts for every O–D. This aggregates $n-1$ cuts into a single one. This makes a trade-off between the sharpness of cuts and the computational time required to solve a quadratic number of sub-problems.

The SP is well-known due to its degeneracy and the SPD for its multiple optimality. Consequently, these cuts may be the cuts dominated by the other ones corresponding to the other solutions. Therefore, we apply the technique of identifying pareto-optimal cuts proposed by Magnanti and Wong (1981).

4.1.2. Implementation

We implemented the benders decomposition in branch-and-cut style as done by Bai and Rubin (2009), Fortz and Poss (2009) and Gelareh and Pisinger (2010). We benefit from the advanced features of modern MIP solvers like CPLEX callbacks. As soon as a solution to the MP is detected we separate the cuts and add it to the master problem.

4.1.3. Lagrangian relaxation

If an integer feasible solution is captured inside the callback then one knows that the resulting minimum cost network flow problem with integrality property can be efficiently solved using shortest path algorithms. After invoking the shortest path algorithm one can extract the solution to e_{ij}, a_{ijk}, b_{ijk} and x_{ijkl}. The optimal solution to the primal problem are the optimal lagrangian multiplier for the dual problem. Therefore, one can dualize all the constraints in the sub-problem dual for which the primal variables are non-zero and solve an unconstrained optimization problem which is computationally inexpensive. However, due to the existence of unsigned dual variables u^1_{ij} and u^2_{ij} with positive RHS in the primal problem a trivial solution to this lagrangian problem will be that only the unsigned variables will take non-zero value. Nevertheless, this can be easily rectified after invoking the Magnanti–Wong procedure to find alternate solutions.

4.2. Greedy neighborhood search

A simple greedy neighborhood search is presented in this section so that high quality solutions can be obtained for larger instances in a reasonable amount of time.

As mentioned earlier, our problem can be interpreted as the problem of finding a connected hub-level network followed by a minimum cost network flow problem. Obviously, the latter obeys the former. This means that the way O–D flows are transferred is induced by the hub-level network configuration. Therefore, without loss of generality we concentrate on the search for the best (or as good as possible) hub-level configuration.

Definition 1 (*Hamming distance*). The Hamming distance between two strings of equal length is the number of positions for which the corresponding symbols are different. In other words, it measures the number of substitutions required to change one into the other.

Definition 2 (*Edge vector*). An edge vector $\mathbf{a}$, is a binary $\frac{n(n-1)}{2}$ entry vector, where $\mathbf{a}_i = 1$ if the edge corresponding to ith element of vector receives a hub edge and 0 otherwise.

Now, we translate our problem into the necessary components of a greedy algorithm.

- *set of candidates*: set of all the potential edges to appear as hub edge,
- *selection function*: $\Delta = f^{new} - f^{cur}$,
- *feasibility function*: a functionality for checking the connectivity, and,
- *objective function*: calculates the objective function of HLPPT (hub-level network setup cost plus the flow cost).

In fact this greedy algorithm is a Hill Climbing algorithm exploring a neighborhood induced by the Hamming metric defined on the set of *edge vectors*. In our algorithm, the internal iterations checks for a new neighbor with distance of 1 from the current solution. The neighborhood of a solution s is introduced as $N(s) = \{s' \in S: Hamm(s, s') = 1\}$.

Although, the size of this neighborhood is equal to $\frac{n(n-1)}{2}$, a significant fraction of neighboring solutions do not make connected hub-level network and as a result the computational burden of calculating objective function for them is eliminated.

In this algorithm we unify the *feasibility function* and the *objective function* and introduce the *Eval* function that returns the objective value for the feasible trials and ∞ for the infeasible ones. In this way, our concern would not be to move from a feasible solution to another feasible neighbor.

The greedy algorithm is depicted in Algorithm (1).

Algorithm 1. A simple greedy algorithm for HLPPT

Input: *init_sol*

Output: x^*

$\bar{x}$:=Create_initial_solution();
min := Eval($\bar{x}$) ;
$last\_min$:=∞ ;
$repeated\_min$:=0 ;

while ($repeated\_min$ = 0) **do**
  $\bar{f} := Eval(\bar{x})$;
  if $\bar{f} \leqslant min$ **then**
    $min := \bar{f}$;
    $x^* := \bar{x}$;
  for each i = 1 *to* $nrLocations*(nrLocations - 1)/2$ **do**
    Δf:=0 ;
    $x' := \bar{x}$;
    $x'_i := 1 - x'_i$;
    if is_not_feasible (x') **then**
      Δf:=∞ ;
    else
      Δf:=$Eval(x') - min$;
    if $\Delta f < 0$ **then**
      x^*:=x';
      min:=$Eval(x')$;
  if $min = last\_min$ **then**
    $repeated\_min$:= $repeated\_min$ + 1;
  $last\_min$:=min;
  $\bar{x} := x^*$;
stop.

4.2.1. Initial solution

As we can see in Algorithm 1, an initial solution is needed for our algorithm to proceed. Our experiments have revealed that starting from a random initial solution may not be the best idea. Extensive experiments with instances of HLPPT, revealed that:

- The number of hubs in the optimal solution is an unknown factor of the discount factor (see Section 5). That is, the number of hubs is in a direct relationship with the discount considered for using hub edges; The higher the discount is, the higher tendency towards having more hub edges and consequently hub nodes.

- It is more likely for the most geographically center oriented and busiest (in terms of total flow arriving to and departing from) locations to become hub. In our experience with our data, there was at least one hub node in the set composed of $n \times 0.2$ most central nodes in union with $n \times 0.2$ busiest nodes.

I.e. for every discount factor α, we select $max(n \times 0.2, 2)$ geographically most central nodes and $max(n \times 0.2, 2)$ of the busiest (highest incoming–outgoing) locations as the initial hubs. Preferably, the hub level network should be a complete graph of these selected locations which we call them hubs. This initial solution is supplied to Algorithm 1.

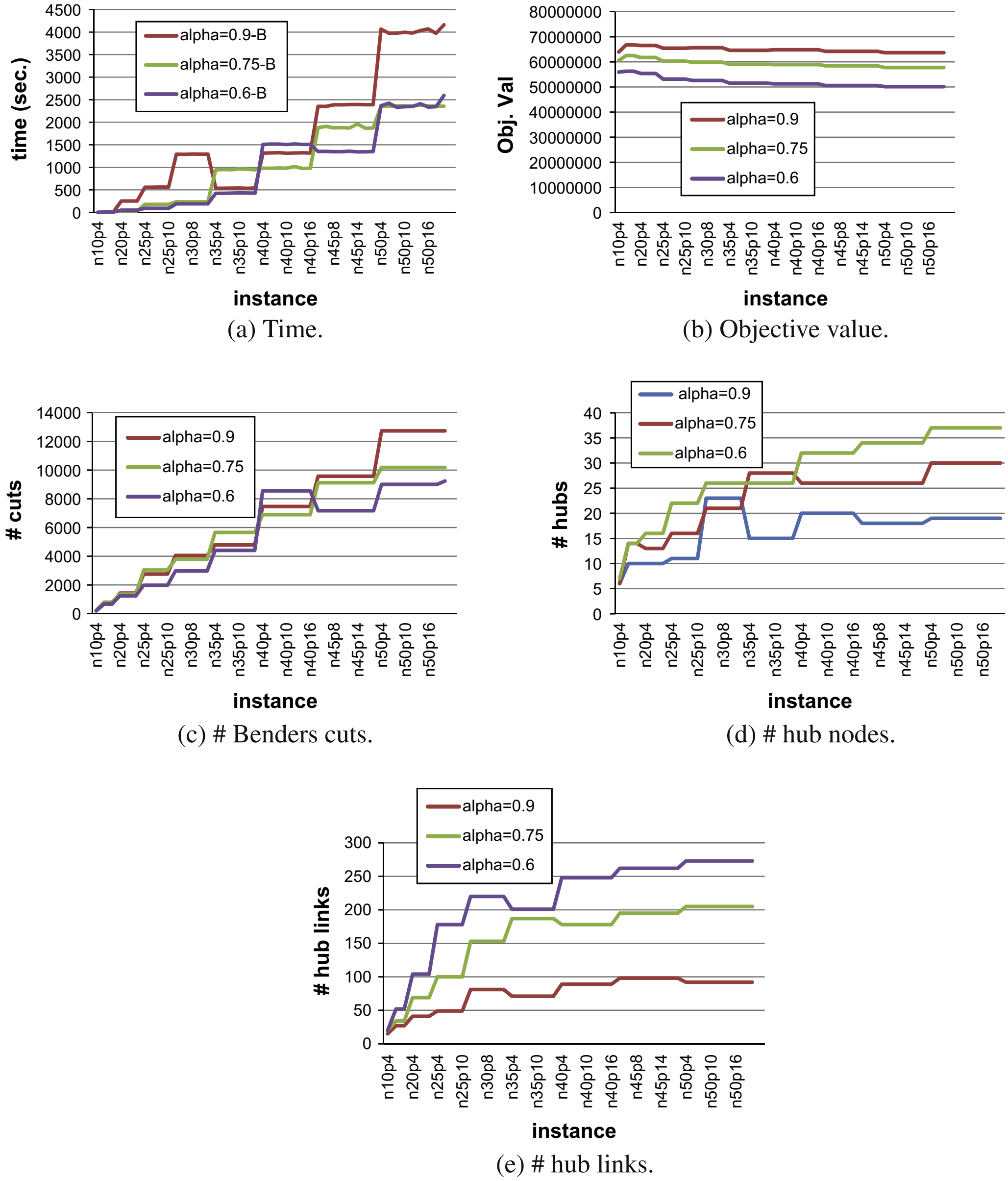
4.2.2. Computational complexity

Since the hub-level network is an undirected graph, we have $\frac{n(n-1)}{2}$ possible hub edges. We assume two configurations to be neighbors if their Hamming distance is equal to 1. As a result, cardinality of the set of neighbors of a given configuration is in general $\frac{n(n-1)}{2}$. Therefore, the size of the neighborhood in the worst case is $\mathcal{O}(n^2)$.

At each iteration of the outmost loop, the internal loop finds the best feasible move from among a total of $\frac{n(n-1)}{2}$ moves. Therefore, at each iteration at most one move will take place. Theoretically, the outer loop can iterate for all hub-level

Table 3
Numerical experiments on AP instances.

Instance	Time			Obj. val.			# B & B nodes			# Benders cuts			# Hub nodes			# Hub links		
	$\alpha = 0.9$	$\alpha = 0.75$	$\alpha = 0.6$	$\alpha = 0.9$	$\alpha = 0.75$	$\alpha = 0.6$	$\alpha = 0.9$	$\alpha = 0.75$	$\alpha = 0.6$	$\alpha = 0.9$	$\alpha = 0.75$	$\alpha = 0.6$	$\alpha = 0.9$	$\alpha = 0.75$	$\alpha = 0.6$	$\alpha = 0.9$	$\alpha = 0.75$	$\alpha = 0.6$
n10p4	1.61	2.12	1.47	63927260.64	60553300.77	55928404.76	0	0	0	204	223	196	6	7	7	15	20	20
n15p4	17.92	15.31	8.8	66737758.49	62514734.34	56284860.4	0	0	0	664	800	683	8	9	11	27	34	52
n15p6	17.77	15.54	9.07	66737758.49	62514734.34	56284860.4	0	0	0	664	800	683	8	9	11	27	34	52
n20p4	252.86	34	52.37	66538208.48	61747225.62	55382450.77	0	0	0	1428	1356	1235	10	13	16	41	69	104
n20p6	253.49	34.22	51.86	66538208.48	61747225.62	55382450.77	0	0	0	1428	1356	1235	10	13	16	41	69	104
n20p8	253.51	33.8	51.43	66538208.48	61747225.62	55382450.77	0	0	0	128	1356	1235	10	13	16	41	69	104
n25p4	559.29	178.42	91.05	65452923.71	60278237.6	53152920.57	0	0	0	2757	3036	1976	11	16	22	49	100	178
n25p6	558.75	178.54	91.84	65452923.71	60278237.6	53152920.57	0	0	0	2757	3036	1976	11	16	22	49	100	178
n25p8	562.53	178.83	91.01	65452923.71	60278237.6	53152920.57	0	0	0	2757	3036	1976	11	16	22	49	100	178
n25p10	560.06	178.23	91.52	65452923.71	60278237.6	53152920.57	0	0	0	2757	3036	1976	11	16	22	49	100	178
n30p4	1290.46	236.58	191.58	65635822.17	59887496.04	52586007.85	0	0	0	4051	3792	2971	14	21	26	81	153	220
n30p6	1287.78	235.1	190.12	65635822.17	59887496.04	52586007.85	0	0	0	4051	3792	2971	14	21	26	81	153	220
n30p8	1294.44	235.3	190.88	65635822.17	59887496.04	52586007.85	0	0	0	4051	3792	2971	14	21	26	81	153	220
n30p10	1292.75	235.28	191.18	65635822.17	59887496.04	52586007.85	0	0	0	4051	3792	2971	14	21	26	81	153	220
n30p12	1293.06	235.21	190.75	65635822.17	59887496.04	52586007.85	0	0	0	4051	3792	2971	14	21	26	81	153	220
n35p4	534.91	944.43	423.67	64604938.12	59047573.29	51545922.31	0	0	0	4790	5658	4408	15	24	26	71	187	201
n35p6	536.5	951.72	424.78	64604938.12	59047573.29	51545922.31	0	0	0	4790	5658	4408	15	24	26	71	187	201
n35p8	538.68	947.77	429.02	64604938.12	59047573.29	51545922.31	0	0	0	4790	5658	4408	15	24	26	71	187	201
n35p10	540.68	964.64	432.84	64604938.12	59047573.29	51545922.31	0	0	0	4790	5658	4408	15	24	26	71	187	201
n35p12	534.85	959.51	429.9	64604938.12	59047573.29	51545922.31	0	0	0	4790	5658	4408	15	24	26	71	187	201
n35p14	541.24	945.94	428.33	64604938.12	59047573.29	51545922.31	0	0	0	4790	5658	4408	15	24	26	71	187	201
n40p4	1316.23	980.19	1508.09	64818901.86	58925413.63	51262319.28	0	0	0	7464	6897	8556	19	26	33	89	178	248
n40p6	1319.96	979.5	1514.35	64818901.86	58925413.63	51262319.28	0	0	0	7464	6897	8556	19	26	33	89	178	248
n40p8	1325.18	984.6	1514.73	64818901.86	58925413.63	51262319.28	0	0	0	7464	6897	8556	19	26	33	89	178	248
n40p10	1313.15	981.45	1507.46	64818901.86	58925413.63	51262319.28	0	0	0	7464	6897	8556	19	26	33	89	178	248
n40p12	1318.21	1016.97	1516.95	64818901.86	58925413.63	51262319.28	0	0	0	7464	6897	8556	19	26	33	89	178	248
n40p14	1322.56	976.17	1508.39	64818901.86	58925413.63	51262319.28	0	0	0	7464	6897	8556	19	26	33	89	178	248
n40p16	1318.41	977.97	1507.83	64818901.86	58925413.63	51262319.28	0	0	0	7464	6897	8556	19	26	33	89	178	248
n45p4	2352.55	1879.44	1351.09	64193518.38	58423423.71	50565481.23	0	0	0	9573	9123	7172	19	29	34	98	195	262
n45p6	2351.23	1907.78	1352.66	64193518.38	58423423.71	50565481.23	0	0	0	9573	9123	7172	19	29	34	98	195	262
n45p8	2387.48	1876.02	1347.97	64193518.38	58423423.71	50565481.23	0	0	0	9573	9123	7172	19	29	34	98	195	262
n45p10	2387.33	1879.69	1346.3	64193518.38	58423423.71	50565481.23	0	0	0	9573	9123	7172	19	29	34	98	195	262
n45p12	2390.8	1872.39	1358.74	64193518.38	58423423.71	50565481.23	0	0	0	9573	9123	7172	19	29	34	98	195	262
n45p14	2393.13	1960.41	1341.93	64193518.38	58423423.71	50565481.23	0	0	0	9573	9123	7172	19	29	34	98	195	262
n45p16	2388.76	1867.89	1345.21	64193518.38	58423423.71	50565481.23	0	0	0	9573	9123	7172	19	29	34	98	195	262
n45p18	2389.3	1871.9	1349.72	64193518.38	58423423.71	50565481.23	0	0	0	9573	9123	7172	19	29	34	98	195	262
n50p4	4064.6	2353.04	2366.06	63646347.05	57748587.76	50121773.87	0	0	0	12,733	10,183	9006	19	30	37	92	205	273
n50p6	3973.57	2356.68	2421.6	63646347.05	57748587.76	50121773.87	0	0	0	12,733	10,183	9006	19	30	37	92	205	273
n50p8	3976.31	2362.53	2332.76	63646347.05	57748587.76	50121773.87	0	0	0	12,733	10,183	9006	19	30	37	92	205	273
n50p10	3994.64	2368.39	2343.31	63646347.05	57748587.76	50121773.87	0	0	0	12,733	10,183	9006	19	30	37	92	205	273
n50p12	3978.78	2357.4	2350.59	63646347.05	57748587.76	50121773.87	0	0	0	12,733	10,183	9006	19	30	37	92	205	273
n50p14	4032.27	2379.8	2411.27	63646347.05	57748587.76	50121773.87	0	0	0	12,733	10,183	9006	19	30	37	92	205	273
n50p16	4068.83	2361.41	2335.59	63646347.05	57748587.76	50121773.87	0	0	0	12,733	10,183	9006	19	30	37	92	205	273
n50p18	3972.09	2359.95	2346.02	63646347.05	57748587.76	50121773.87	0	0	0	12,733	10,183	9006	19	30	37	92	205	273
n50p20	4161.35	2355.95	2595.37	63646347.05	57748587.76	50121773.88	0	0	0	12,733	10,183	9238	19	30	37	92	205	273



(a) Time. (b) Objective value. (c) # Benders cuts. (d) # hub nodes. (e) # hub links.

Fig. 6. Numerical results of Benders decomposition.

configurations namely $\sum_{i=1}^{\frac{n(n-1)}{2}} \binom{\frac{n(n-1)}{2}}{i}$. But not all of them are feasible configurations and also a stopping criteria will be met much before even a fraction of is investigated. On the other hand, for each *feasible* neighbor (a feasible neighbor is a neighbor with connected hub-level graph) and for each pair of origin–destination (i, j) a shortest path Dijkstra's algorithm is applied. The complexity of Dijkstra's algorithm is $\mathcal{O}(|E| + |V|log|V|)$, where $|E| \leqslant q + p(n-p) \leqslant \frac{n(n-1)}{2}$ and $|V| = n$. Here, q is the number of hub edges and p is the number of hub nodes in the feasible solution. Dijkstra must be applied for $n - 1$ nodes.

4.3. Diversification and intensification

Due to the heuristic nature of the method and myopic characteristic of greedy algorithms, this possibility always exists that the search process gets stuck in a local optimum. It is always worth trying to invoke a prudent diversification of the search and increase the likelihood of obtaining better solution(s) by a more thorough investigation in the search space.

Definition 3 (*Neighborhood I*). For a given hub-level structure and a hub node, i, the new structure resulting from replacing the given hub with a non-hub node having pth ($p = 1\ldots3$) highest value of $\frac{\sum_{j\neq k} W_{kj}}{\sum_{j\neq k} C_{kj}}$, $k \neq i$ and switching the assignments of all the incoming and outgoing edges of i to this pth closest non-hub is called the pth level neighbor of the given hub node with respect to Neighborhood I for the given hub-level structure.

The process follows: there exists a spoke node as best of pth level neighbors for each hub node. The best of all these '*best*' ones is the one that can become hub and replace the current hub node. The resulting configuration may have a degraded objective value but it is locally the best choice (imitating the diversification process); if the new trial point is worse, possibly it is standing on a previously non-explored peak which can drop to a deeper valley if is subjected to an exploration with respect to the original neighborhood structure. By moving to this neighbor with respect to the neighborhood I and intersecting this new structure to the original neighborhood structure (by delivering this new trial point to the greedy search) we may have a better hub-level structure. That is, greedy search may remove some components in favor of other ones (here we used $p = 1$).

Alternatively, there is another neighborhood structure that can be applied in the case that Neighborhood I gets stuck in a local optimum or even not being able to improve the best-known solution of greedy algorithm.

Definition 4. Neighborhood IIFor a given hub-level structure and for a given hub node i, the new structure resulted by replacing the given hub with the pth ($p = 1 \ldots 3$) closest non-hub node to i and switching the assignments of all the incoming and outgoing edges of the given hub to the pth closest non-hub is called the pth level neighbor of i with respect to the Neighborhood II for the existing hub-level structure.

5. Numerical experiments

We ran our Benders approach on instances of the AP dataset. We use the public code of AP generator for the UMApHMP from or-library.[1] This code has been made for generating instances for p-HMP. Nevertheless, in order to make the comparison easier for future studies, we use this code while noting that the choice of p for the generator does not have any particular meaning in our experiments with the HLPPT. The fixed costs are generated as in the following:

1. Hub nodes: $F_k = \left(\sum_j W_{ij}/\max_k d_k\right) \times 10e8$ where d_k is the distance of most remote location to k,
2. Hub edges: $I_{kl} = \left(\frac{d_{kl}/w_{kl}}{\max_{i,j \neq i} d_{ij}/w_{ij}}\right) \times 10e7$.

We also examine the performance of our heuristic against the Benders approach.

All experiments have been run on an Intel Xeon CPU 2.66 GHz processor with 8 GB RAM using single thread.

5.1. Benders decomposition

We chose $\alpha \in \{0.6, 0.75, 0.9\}$, $n \in \{10, 15, 20, \ldots, 50\}$ and for using UMApHMP instance generator p is chosen from the set $\{2m, m \in \{2, \lceil 0.4n \rceil /2 + 1\}\}$.

Every instance is referred to as *nxxpyy* where *xx* is the number of nodes and *yy* is the p chosen for generator. We ran Benders for all three values of α and solve all instances to optimality. CPLEX parameters have been finely tuned to avoid numerical instabilities.

The numerical results are depicted in Table 3. The first column indicates the instance name, the next three reports the computational time in seconds for all three values of α. In the next six columns, we report the objective values and number of nodes required for the branch-and-bound in order to prove optimality after adding cuts at the root node, respectively. The total number of disaggregated Benders cuts added in the course of our algorithm is shown in the following three columns. The next three columns report the number of hub nodes while the number of hub links are reported in the last three columns.

The highest computational time is 4161.35 s which is bit more than an hour. Hence, all instances can be solved in approximately one hour.

One can see in Table 3 and Fig. 6a that the complexity of problem is related to the value of α: the higher α is, the higher the complexity and consequently the higher computational time needed to prove the optimality. In addition to this some kind of stepwise pattern can be observed within the instances with the same number of nodes. That means that the AP instances with the proposed fixed cost have similar structure which is independent of the value of p, chosen in the instance generator of Australian Post dataset. Fig. 6b indicates that objective function is in general decreasing. In general, the higher the size of instance is the lower the objective function becomes.

Fig. 6c shows that, in general, no matter how α behaves, the number of *effective* cuts are almost linearly increasing as the size of the instance grows. By *effective* cuts we mean a cut that separates part of the MP polytope.

All instances are solved at the root node. The tightness of the formulation helps in solving all instances at the root node by separating effective cuts.

[1] http://people.brunel.ac.uk/mastjjb/jeb/info.html.

Table 4
Numerical experiments with local search on AP instances

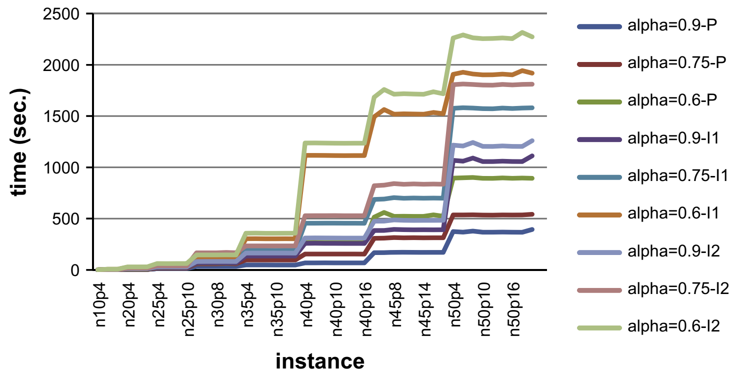
Instance	LS-P						LS-I1						LS-I2					
	Obj. val			Gap (%)			Obj. val			Gap (%)			Obj. val			Gap (%)		
	$\alpha = 0.9$	$\alpha = 0.75$	$\alpha = 0.6$	$\alpha = 0.9$	$\alpha = 0.75$	$\alpha = 0.6$	$\alpha = 0.9$	$\alpha = 0.75$	$\alpha = 0.6$	$\alpha = 0.9$	$\alpha = 0.75$	$\alpha = 0.6$	$\alpha = 0.9$	$\alpha = 0.75$	$\alpha = 0.6$			
n10p4	67497298.21	62556459.51	56705320.07	5.29	3.31	1.39	63976442.08	60617311.44	55984418.68	0.08	0.11	0.10	63976442.08	60617311.44	55984418.68	0.08	0.11	0.10
n15p4	67637102.77	63080978.82	57379690.36	1.33	0.91	1.95	66958463.33	63080978.82	56577038.33	0.33	0.91	0.52	66958463.33	63080978.82	56577038.33	0.33	0.91	0.52
n15p6	67637102.77	63080978.82	57379690.36	1.33	0.91	1.95	66958463.33	63080978.82	56577038.33	0.33	0.91	0.52	66958463.33	63080978.82	56577038.33	0.33	0.91	0.52
n20p4	69672916.22	64908770.22	58023559.4	4.50	5.12	4.77	67443958.63	63220029.19	56031299.72	1.34	2.39	1.17	67443958.63	63220029.19	56031299.72	1.34	2.39	1.17
n20p6	69672916.22	64908770.22	58023559.4	4.50	5.12	4.77	67443958.63	63220029.19	56031299.72	1.34	2.39	1.17	67443958.63	63220029.19	56031299.72	1.34	2.39	1.17
n20p8	69672916.22	64908770.22	58023559.4	4.50	5.12	4.77	67443958.63	63220029.19	56031299.72	1.34	2.39	1.17	67443958.63	63220029.19	56031299.72	1.34	2.39	1.17
n25p4	67526556.43	62318211.34	55459996.39	3.07	3.38	4.34	65866873.12	61302972.53	54375861.11	0.63	1.70	2.30	65866873.12	61302972.53	54375861.11	0.63	1.70	2.30
n25p6	67526556.43	62318211.34	55459996.39	3.07	3.38	4.34	65866873.12	61302972.53	54375861.11	0.63	1.70	2.30	65866873.12	61302972.53	54375861.11	0.63	1.70	2.30
n25p8	67526556.43	62318211.34	55459996.39	3.07	3.38	4.34	65866873.12	61302972.53	54375861.11	0.63	1.70	2.30	65866873.12	61302972.53	54375861.11	0.63	1.70	2.30
n25p10	67526556.43	62318211.34	55459996.39	3.07	3.38	4.34	65866873.12	61302972.53	54375861.11	0.63	1.70	2.30	65866873.12	61302972.53	54375861.11	0.63	1.70	2.30
n30p4	68599150.14	62512047.89	55304771.58	4.32	4.38	5.17	66996787.44	61281204.22	54691313.15	2.03	2.33	4.00	66996787.44	61281204.22	54691313.15	2.03	2.33	4.00
n30p6	68599150.14	62512047.89	55304771.58	4.32	4.38	5.17	66996787.44	61281204.22	54691313.15	2.03	2.33	4.00	66996787.44	61281204.22	54691313.15	2.03	2.33	4.00
n30p8	68599150.14	62512047.89	55304771.58	4.32	4.38	5.17	66996787.44	61281204.22	54691313.15	2.03	2.33	4.00	66996787.44	61281204.22	54691313.15	2.03	2.33	4.00
n30p10	68599150.14	62512047.89	55304771.58	4.32	4.38	5.17	66996787.44	61281204.22	54691313.15	2.03	2.33	4.00	66996787.44	61281204.22	54691313.15	2.03	2.33	4.00
n30p12	68599150.14	62512047.89	55304771.58	4.32	4.38	5.17	66996787.44	61281204.22	54691313.15	2.03	2.33	4.00	66996787.44	61281204.22	54691313.15	2.03	2.33	4.00
n35p4	67429793.49	61714555.16	53939968.34	4.19	4.52	4.64	65621560.97	60396873.02	53009116.41	1.55	2.29	2.84	65621560.97	60396873.02	53009116.41	1.55	2.29	2.84
n35p6	67429793.49	61714555.16	53939968.34	4.19	4.52	4.64	65621560.97	60396873.02	53009116.41	1.55	2.29	2.84	65621560.97	60396873.02	53009116.41	1.55	2.29	2.84
n35p8	67429793.49	61714555.16	53939968.34	4.19	4.52	4.64	65621560.97	60396873.02	53009116.41	1.55	2.29	2.84	65621560.97	60396873.02	53009116.41	1.55	2.29	2.84
n35p10	67429793.49	61714555.16	53939968.34	4.19	4.52	4.64	65621560.97	60396873.02	53009116.41	1.55	2.29	2.84	65621560.97	60396873.02	53009116.41	1.55	2.29	2.84
n35p12	67429793.49	61714555.16	53939968.34	4.19	4.52	4.64	65621560.97	60396873.02	53009116.41	1.55	2.29	2.84	65621560.97	60396873.02	53009116.41	1.55	2.29	2.84
n35p14	67429793.49	61714555.16	53939968.34	4.19	4.52	4.64	65621560.97	60396873.02	53009116.41	1.55	2.29	2.84	65621560.97	60396873.02	53009116.41	1.55	2.29	2.84
n40p4	68098920.76	62406969.51	54909586.08	4.82	5.91	7.11	66657564.1	61416955.42	53484120.38	2.76	4.23	4.33	66657564.1	61416955.42	53484120.38	2.76	4.23	4.33
n40p6	68098920.76	62406969.51	54909586.08	4.82	5.91	7.11	66657564.1	61416955.42	53484120.38	2.76	4.23	4.33	66657564.1	61416955.42	53484120.38	2.76	4.23	4.33
n40p8	68098920.76	62406969.51	54909586.08	4.82	5.91	7.11	66657564.1	61416955.42	53484120.38	2.76	4.23	4.33	66657564.1	61416955.42	53484120.38	2.76	4.23	4.33
n40p10	68098920.76	62406969.51	54909586.08	4.82	5.91	7.11	66657564.1	61416955.42	53484120.38	2.76	4.23	4.33	66657564.1	61416955.42	53484120.38	2.76	4.23	4.33
n40p12	68098920.76	62406969.51	54909586.08	4.82	5.91	7.11	66657564.1	61416955.42	53484120.38	2.76	4.23	4.33	66657564.1	61416955.42	53484120.38	2.76	4.23	4.33
n40p14	68098920.76	62406969.51	54909586.08	4.82	5.91	7.11	66657564.1	61416955.42	53484120.38	2.76	4.23	4.33	66657564.1	61416955.42	53484120.38	2.76	4.23	4.33
n40p16	68098920.76	62406969.51	54909586.08	4.82	5.91	7.11	66657564.1	61416955.42	53484120.38	2.76	4.23	4.33	66657564.1	61416955.42	53484120.38	2.76	4.23	4.33
n45p4	67443711.56	61928629.03	54282187.57	4.82	6.00	7.35	66176171.79	60816856.81	52918913.2	3.00	4.10	4.65	66176171.79	60816856.81	52918913.2	3.00	4.10	4.65
n45p6	67443711.56	61928629.03	54282187.57	4.82	6.00	7.35	66176171.79	60816856.81	52918913.2	3.00	4.10	4.65	66176171.79	60816856.81	52918913.2	3.00	4.10	4.65
n45p8	67443711.56	61928629.03	54282187.57	4.82	6.00	7.35	66176171.79	60816856.81	52918913.2	3.00	4.10	4.65	66176171.79	60816856.81	52918913.2	3.00	4.10	4.65
n45p10	67443711.56	61928629.03	54282187.57	4.82	6.00	7.35	66176171.79	60816856.81	52918913.2	3.00	4.10	4.65	66176171.79	60816856.81	52918913.2	3.00	4.10	4.65
n45p12	67443711.56	61928629.03	54282187.57	4.82	6.00	7.35	66176171.79	60816856.81	52918913.2	3.00	4.10	4.65	66176171.79	60816856.81	52918913.2	3.00	4.10	4.65
n45p14	67443711.56	61928629.03	54282187.57	4.82	6.00	7.35	66176171.79	60816856.81	52918913.2	3.00	4.10	4.65	66176171.79	60816856.81	52918913.2	3.00	4.10	4.65
n45p16	67443711.56	61928629.03	54282187.57	4.82	6.00	7.35	66176171.79	60816856.81	52918913.2	3.00	4.10	4.65	66176171.79	60816856.81	52918913.2	3.00	4.10	4.65
n45p18	67443711.56	61928629.03	54282187.57	4.82	6.00	7.35	66176171.79	60816856.81	52918913.2	3.00	4.10	4.65	66176171.79	60816856.81	52918913.2	3.00	4.10	4.65
n50p4	67564565.54	62191236.01	54498038.06	5.80	7.69	8.73	65967456.94	61160004.41	53252964.13	3.52	5.91	6.25	65967456.94	61160004.41	53252964.13	3.52	5.91	6.25
n50p6	67564565.54	62191236.01	54498038.06	5.80	7.69	8.73	65967456.94	61160004.41	53252964.13	3.52	5.91	6.25	65967456.94	61160004.41	53252964.13	3.52	5.91	6.25
n50p8	67564565.54	62191236.01	54498038.06	5.80	7.69	8.73	65967456.94	61160004.41	53252964.13	3.52	5.91	6.25	65967456.94	61160004.41	53252964.13	3.52	5.91	6.25
n50p10	67564565.54	62191236.01	54498038.06	5.80	7.69	8.73	65967456.94	61160004.41	53252964.13	3.52	5.91	6.25	65967456.94	61160004.41	53252964.13	3.52	5.91	6.25
n50p12	67564565.54	62191236.01	54498038.06	5.80	7.69	8.73	65967456.94	61160004.41	53252964.13	3.52	5.91	6.25	65967456.94	61160004.41	53252964.13	3.52	5.91	6.25
n50p14	67564565.54	62191236.01	54498038.06	5.80	7.69	8.73	65967456.94	61160004.41	53252964.13	3.52	5.91	6.25	65967456.94	61160004.41	53252964.13	3.52	5.91	6.25
n50p16	67564565.54	62191236.01	54498038.06	5.80	7.69	8.73	65967456.94	61160004.41	53252964.13	3.52	5.91	6.25	65967456.94	61160004.41	53252964.13	3.52	5.91	6.25
n50p18	67564565.54	62191236.01	54498038.06	5.80	7.69	8.73	65967456.94	61160004.41	53252964.13	3.52	5.91	6.25	65967456.94	61160004.41	53252964.13	3.52	5.91	6.25
n50p20	67564565.54	62191236.01	54498038.06	5.80	7.69	8.73	65967456.94	61160004.41	53252964.13	3.52	5.91	6.25	65967456.94	61160004.41	53252964.13	3.52	5.91	6.25

Fig. 6d shows that for smaller (larger) α the increase in the number of hub nodes in the optimal solution is higher (lower) and as the instance size increases this becomes more significant. However, the general trend is an increase in the number of hubs. Similar and even more aggressive trend can be observed for hub links in Fig. 6e.

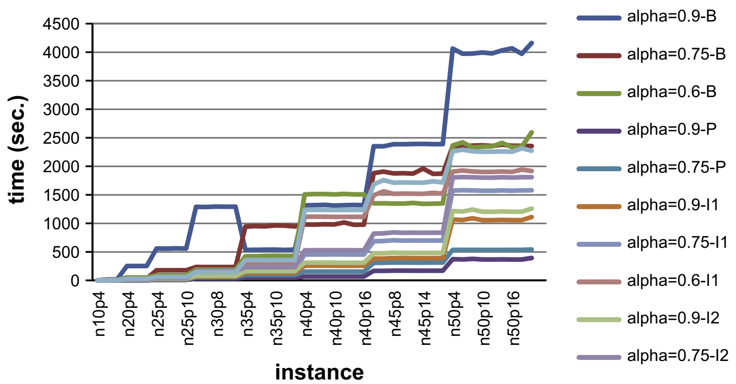
One can conclude that the Benders approach is a powerful technique for solving instances of this problem.

5.2. Local search

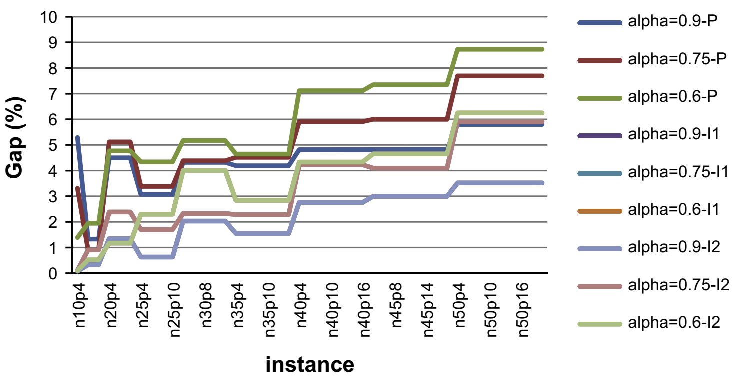
We also compare the greedy heuristic against our Benders approach. The motivation is to have an alternate solution approach capable of obtaining a good approximation of the optimal solution for instances of the problem while being less



(a) Time in local search.



(b) Time comparison.



(c) Gap (%).

Fig. 7. Numerical results of greedy local search.

resource demanding. Therefore, the main focus here is quality rather than computational time. The results are reported in Table 4.

Three variants of the algorithm are considered: (1) pure greedy local search (P) (Algorithm 1), (2) greedy local search equipped with *NeighborhoodI* structure (I1), and (3) greedy algorithm equipped with both *NeighborhoodI* and *NeighborhoodII* (I2).

The first three columns after the name of instances show the objective values of the best known solution for three different values of α in the *P*. This is followed by the optimality gaps, respectively. Similarly, other results are also reported in the remaining columns for (I1) and (I2).

The gaps are calculated using the optimal solution of Benders and best-known solution of the heuristics.

As Fig. 7a shows as instance size grows the computational time increases and of course the more neighborhood structure is used the higher computational time is. However, the higher α is the less computational time is required.

There is not a fair comparison between the Benders approach and the greedy algorithm, nevertheless, Fig. 7b tries to shed some light on a general comparison between the Benders and all three variants of greedy algorithms in terms of time. Here *B* stands for Benders results.

The most important aspect of this comparison is the solution quality. As shown in Fig. 7c, the gaps never exceed 9 percent. The general trend is an increase in the gap as instance size grows. Yet, the lowest gap in each algorithm is obtained for larger values of α. I.e. the more a problem looks like a pure network problem the more efficient the algorithm is.

Hence, we conclude that the greedy heuristic is a reliable method for this problem and is significantly less resource demanding approach.

6. Summary and conclusion

We proposed a new HLP model with suitable flexibilities for transportation applications. Here, some of the classical assumptions of the HLPs have been relaxed in order to obtain a more realistic and practical model. The model has real world application especially in the network design of liner shipping companies and also within urban transport network.

A Benders decomposition approach which is efficiently implemented in form of cutting plane methods and makes use of modern features of general–purpose solvers to solve the master problem is proposed. The Benders algorithm exploits the decomposable structure of the flow problem and separates multiple Benders cuts so that almost all instances are solved at the root node without resorting to branching tree.

We also proposed a local search equipped with diversification and intensification mechanisms which proves to be efficiently capable of obtaining high quality solution of larger size.

The contribution of the paper to the literature are the following:

- While to the best of our knowledge no similar model with these properties is known in literature, our model is superior to the most similar one (Nickel et al., 2001) (as shown in Figs. 3 and 4) both in terms of dimension and also compactness of formulation.
- While general–purpose solvers fail to solve instances of even small size of this 4-index formulation, our Benders decomposition approach efficiently tackle significantly large scale ones in about 1 h.
- An efficient local search is proposed which is a variant of variable neighborhood search and is capable of obtaining high quality solutions in reasonable CPU time.

We conclude that the model is a good approximation of real-world application as shown in the introduction section. It offers enough flexibility in design such that it can be used in public transport as well as liner shipping applications. We will extend the model to include more details of the applications, especially in liner shipping. Some long-term memory heuristics deserve attention in this context.

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