Formularium

1 Probability theory

Formula	**Description**
$F_X(x) = Pr[x \leq x]$	Cumulative distribution function of a random variable, this is non-decreasing and right-continues
$\lim_{x->\infty} F_X(x) = 1$	The normalization condition of this CDF
$\mathrm{f}_X(x) = \frac{dF_X(x)}{dx}$; $\mathrm{f}_X(x)dx = Pr[x < X \leq x + dx]$	The density function, the derivative of the CDF
$\int_0^\infty \mathrm{f}_X(x)dx = 1$	The normalization condition of the density function
$p_X(n) = Pr[X = n], n \in \mathbb{N}$	The probability mass function (discrete RV)
$\sum_{n=0}^\infty p_X(n) = 1$	The normalization function (discrete RV)
$F_{XY}(x, y) = Pr[X \leq x, Y \leq y] \quad x, y \in \mathbb{R}_{\geq 0}$	joint cumulative distribution function of two random variables
$Pr[X \leq x, Y \leq y] \neq Pr[X \leq x]Pr[Y \leq y]$, if they are independent RV it is equal, thus $F_{XY}(x, y) = F_X(x)F_Y(y)$; $F_X(x) = F_{XY}(x, \infty)$; $F_Y(y) = F_{XY}(\infty, y)$	properties of the joint CDF
$f_{XY}(x, y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}$ and have that $f_{XY}(x, y)dxdy = Pr[x < X \leq x + dx, y < Y \leq y + dy]$	Joint density function
$p_{XY}(n, m) = Pr[X = n, Y = m]$	joint mass function (discrete RV)
$f_{X\|Y}(x\|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$ and have that $f_{X\|Y}(x\|y)dx = Pr[x < X \leq x + dx\|Y = y]$	Conditional density function
$p_{X\|Y}(n\|m) = Pr[x = n\|Y = m] = \frac{p_{XY}(n,m)}{p_Y(m)}$	Conditional mass function (discrete RV)
$E[X] = \int_0^\infty x dF_X(x) = \int_0^\infty x f_X(x)dx$	Mean or expected value of a RV, it is the summary of a complete probability distribution
$E[X] = \sum_{n=0}^\infty n p_X(n)$	Mean or expected value of a discrete RV, it is the summary of a complete probability distribution
$E[aX + bY] = aE[X] + bE[Y]$	$E[.]$ is a linear operator
$E[XY] = E[X]E[Y]$	If RV X and Y are independent

Formula	Description
$Var[X] = E[(X - E[X])^2] = E[X^2] - E[X]^2$	The variance of a RV
$E[X] = E[E[X\|Y]]$	The conditional expectation of X given Y

2 Terminology

Name	Description
Arrival instant	The time at which a customer arrives at the queue.
Service instant	The time at which a customer leaves the system after being served completely.
Queue content	The number of customers in the queue waiting for service.
System content	The number of customers in the total system.
Queue capacity	The maximum number of customers in the queue.
System capacity	The maximum number of customers in the system.
Service time	The amount of time that the customer occupies a server.
Waiting time	The amount of time a customer waits in the queue before starting service.
Delay or sojourn time	The amount of time a customer resides in the system.

3 Distributions

Name	Density Function	Mean	Variance
Binomial	$Pr[X = k] = \binom{n}{k} p^k (1-p)^{n-k}$ with $\binom{n}{k} = \frac{n!}{k!(n-k)!}$	np	$np(1-p)$
Geometric	$Pr[X = k] = (1-p)^{k-1} p$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Normal	$f(x\|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	μ	σ^2
Uniform	$f(x\|a,b) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b, \\ 0 & \text{for } x < a \text{ or } x > b. \end{cases}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential	$f(x\|\lambda) = \lambda e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Poisson	$f(k\|\lambda) = \frac{e^{-\lambda}\lambda^k}{k!}$	λ	λ

Name	Density Function	Mean	Variance
Erlang	$f_N(x) = \frac{\mu^N x^{N-1} e^{-\mu x}}{(N-1)!}$	$N\frac{1}{\mu}$	$N\frac{1}{\mu^2}$

3.1 Memoryless property

Formula	Description
$r(t) = \frac{Pr[X \leq t+dt \mid X>t]}{dt} = \frac{x(t)}{1-X(t)}$	The hazard rate function of a RV
$r(t) = \lambda$	The hazard rate function of an exponential function

4 Markov property

Formula	Description
$Pr[X(t) = n \mid X(t_1) = n_1, \ldots, X(t_k) = n_k] = Pr[X(t) = n \mid X(t_k) = n_k]$	The Markov property, stating that the futur state only depends on the present state and not on the past states.

5 M/M/1 queue

Formula	Description
$\lambda s(n) = \mu s(n+1), \quad n = 0, 1, 2, \ldots$	The local balance equations for the M/M/1 queue
$s(n) = \rho^n s(0)$ with $\rho = \frac{\lambda}{\mu}$	Recursive solution of the balance equations, with ρ the load on the system.
$\sum_{n=0}^{\infty} s(n) = 1 = \sum_{n=0}^{\infty} \rho^n s(0) = \frac{s(0)}{1-\rho}$	Using the normalization function, giving us a solution for $s(0)$
$s(0) = 1 - \rho$	The solution for $s(0)$.
$s(n) = (1-\rho)\rho^n$	The solution for $s(n)$.
$E[S] = \frac{\rho}{1-\rho}$	The mean system content of an M/M/1 queue.
$Var[S] = \frac{\rho}{(1-\rho)^2}$	The variance system content of an M/M/1 queue.
$q(0) = s(0) + s(1)$	The queue content, when no customer in the queue, we have either situation where there is no customer in the system or one customer that is being served.
$q(n) = s(n+1), \quad n = 1, 2, \ldots$	When there are customers in the system, the queue has one less than the system.

Formula	Description
$q(0) = 1-\rho^2, \quad q(n) = (1-\rho)\rho^{n+1} \quad n = 1,2,\dots$	Filled in using previous formulas.
$E[Q] = \frac{\rho^2}{1-\rho}$	The mean queue content.
$Var[Q] = \frac{\rho^2}{(1-\rho)^2}(1+\rho-\rho^2)$	The variance of the queue content.
$E[in] = E[out]$	Operational law

6 The generic BD-Queueing system

Formula	Description
$\lambda_n s(n) = \mu_{n+1} s(n+1)$	The local balance equations for a generic Birth-death queue.
$s(n) = s(0)\prod_{m=0}^{n-1}\frac{\lambda_m}{\mu_{m+1}}$	The local balance equations recursively solved.
$\sum_{n=0}^{\infty} s(n) = 1 = s(0)(1+\sum_{n=1}^{\infty}\prod_{m=0}^{n-1}\frac{\lambda_m}{\mu_{m+1}})$	Using the normalization property, we get an extra equation.
$s(0) = (1+\sum_{l=1}^{\infty}\prod_{m=0}^{l-1}\frac{\lambda_m}{\mu_{m+1}})^{-1}$	Solution for s(0).
$s(n) = \frac{\prod_{m=0}^{n-1}\frac{\lambda_m}{\mu_{m+1}}}{1+\sum_{l=1}^{\infty}\prod_{m=0}^{l-1}\frac{\lambda_m}{\mu_{m+1}}}$	Solution for s(n).
$E[S] = \sum_{n=1}^{\infty} n\frac{\prod_{m=0}^{n-1}\frac{\lambda_m}{\mu_{m+1}}}{1+\sum_{l=1}^{\infty}\prod_{m=0}^{l-1}\frac{\lambda_m}{\mu_{m+1}}}$	Mean system content.
$Var[S] = \sum_{n=1}^{\infty} n^2\frac{\prod_{m=0}^{n-1}\frac{\lambda_m}{\mu_{m+1}}}{1+\sum_{l=1}^{\infty}\prod_{m=0}^{l-1}\frac{\lambda_m}{\mu_{m+1}}} - (\sum_{n=1}^{\infty} n\frac{\prod_{m=0}^{n-1}\frac{\lambda_m}{\mu_{m+1}}}{1+\sum_{l=1}^{\infty}\prod_{m=0}^{l-1}\frac{\lambda_m}{\mu_{m+1}}})^2$	Mean system content.
$\sum_{n=1}^{\infty}\prod_{m=0}^{n-1}\frac{\lambda_m}{\mu_{m+1}}$	Stability condition, this converges when the system is stable.

7 The M/M/c queue

Formula	**Description**
$s(n) = \begin{cases} s(0)\frac{(c\rho)^n}{n!} & \text{for } n \leq c, \\ s(0)\frac{(c\rho)^c}{c!}\rho^{n-c} & \text{for } n > c. \end{cases}$	System content probabilities.
$s(0) = (\sum_{l=0}^{c-1} \frac{(c\rho)^l}{l!} + \frac{(c\rho)^c}{c!}\frac{1}{1-\rho})$	Normalization condition.
$\rho = \frac{\lambda}{c * \mu} < 1$	Stability condition.
$p^Q = (\sum_{l=0}^{c-1} \frac{(c\rho)^l}{l!} + \frac{(c\rho)^c}{c!}\frac{1}{1-\rho})^{-1}\frac{(c\rho)^c}{c!}\frac{1}{1-\rho}$	Erlang C formula. It is the probability that an arriving customer has to wait.
$E[Q] = p^Q\frac{\rho}{1-\rho}$	Mean queue content.
$E[S] = p^Q\frac{\rho}{1-\rho} + c\rho$	Mean system content.
$c\rho$	Average number of customer being served in the system.

8 The M/M/1/K queue

Formula	**Description**
$n = 1, \ldots, K$	For all formulas.
$s(n) = \frac{1-\rho}{1-\rho^{K+1}}\rho^n$	System content probabilities.
$E[S] = \frac{\rho}{1-\rho}\frac{1-(K(1-\rho)+1)\rho^K}{1-\rho^{K+1}}$	System content probabilities.
$l_p = \frac{1-\rho}{1-\rho^{K+1}}\rho^K$	Loss rate or loss probability.
$\lambda(1 - l_p) = \mu(1 - s(0))$	The operational law.
$l_p = 1 - \frac{1-s(0)}{\rho}$ with $\rho \geq 1 \rightarrow l_p \geq 1 - \frac{1}{\rho}$	Alternative solution for the loss.

Formula	Description
$\lambda_{eff} = \lambda(1 - l_p)$ and $\rho_{eff} = \rho(1 - l_p)$	The effective arrival rate and effective load.

9 Waiting time and delay

Formula	Description
$D = \sum_{n=1}^{S^A+1} B_n$	The delay is equal to all the service times of the customers in the system when the tagged customer arrives plus the service time of that tagged customer.
$Pr[D \leq t] = E[Pr[D \leq t \vert S^A]] = \sum_{m=0}^{\infty} Pr[D \leq t \vert S^A = m] Pr[S^A = m]$	The law of total probability.
$Pr[D \leq t \vert S^A] = 1 - \sum_{n=0}^{S^A} \frac{1}{n!} e^{-\mu t} (\mu t)^n$	Delay is Erlang distributed with $S^A + 1$ stages and rate μ
$Pr[S^A = m] = s(m) = (1 - \rho)\rho^m$	Using PASTA for the stationary distribution of the system.
$Pr[D \leq t] = 1 - e^{-(\mu - \lambda)t}$	Going further on the law of total probability we derive that the delay is exponentially distributed with rate $\mu - \lambda$. So Delay is memoryless.
$E[D] = \frac{1}{\mu - \lambda}$	The mean delay.
$Var[D] = \frac{1}{(\mu - \lambda)^2}$	The variance of the delay.
$E[D] = \frac{1}{\lambda} E[S]$	Little's law.
$E[W] = \frac{1}{\lambda} E[Q]$	The mean waiting time in relation to the mean queue content.
$Var[W] = \frac{\rho(2 - \rho)}{(\mu - \lambda)^2}$	The variance of the waiting time.

10 Little's law

Little's law states that the average number of customers in a stationary system is equal to the average arrival rate times the average time a customer spends in the system.

This is valid for all queueing systems! For $\lambda < \infty$, $\overline{D} < \infty$ and $\overline{S} < \infty$.

$$\overline{S} = \lambda \overline{D}$$

Make sure that for example in finite systems, the effective arrival rate λ_{eff} is used.