Formularium

1 Probability theory

Formula	**Description**
$F_X(x) = Pr[x \leq x]$	Cumulative distribution function of a random variable, this is non-decreasing and right-continues
$\lim_{x->\infty} F_X(x) = 1$	The normalization condition of this CDF
$\mathrm{f}_X(x) = \frac{dF_X(x)}{dx}; \quad \mathrm{f}_X(x)dx = Pr[x < X \leq x + dx]$	The density function, the derivative of the CDF
$\int_0^\infty \mathrm{f}_X(x)dx = 1$	The normalization condition of the density function
$p_X(n) = Pr[X = n], n \in \mathbb{N}$	The probability mass function (discrete RV)
$\sum_{n=0}^\infty p_X(n) = 1$	The normalization function (discrete RV)
$F_{XY}(x, y) = Pr[X \leq x, Y \leq y] \quad x, y \in \mathbb{R}_{\geq 0}$	joint cumulative distribution function of two random variables
$Pr[X \leq x, Y \leq y] \neq Pr[X \leq x]Pr[Y \leq y]$, if they are independent RV it is equal, thus $F_{XY}(x, y) = F_X(x)F_Y(y); \quad F_X(x) = F_{XY}(x, \infty); \quad F_Y(y) = F_{XY}(\infty, y)$	properties of the joint CDF
$f_{XY}(x, y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}$ and have that $f_{XY}(x, y)dxdy = Pr[x < X \leq x + dx, y < Y \leq y + dy]$	Joint density function
$p_{XY}(n, m) = Pr[X = n, Y = m]$	joint mass function (discrete RV)
$f_{X\|Y}(x\|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$ and have that $f_{X\|Y}(x\|y)dx = Pr[x < X \leq x + dx\|Y = y]$	Conditional density function
$p_{X\|Y}(n\|m) = Pr[x = n\|Y = m] = \frac{p_{XY}(n,m)}{p_Y(m)}$	Conditional mass function (discrete RV)
$E[X] = \int_0^\infty x dF_X(x) = \int_0^\infty x f_X(x)dx$	Mean or expected value of a RV, it is the summary of a complete probability distribution
$E[X] = \sum_{n=0}^\infty n p_X(n)$	Mean or expected value of a discrete RV, it is the summary of a complete probability distribution
$E[aX + bY] = aE[X] + bE[Y]$	$E[.]$ is a linear operator
$E[XY] = E[X]E[Y]$	If RV X and Y are independent

Formula	Description
$Var[X] = E[(X - E[X])^2] = E[X^2] - E[X]^2$	The variance of a RV
$E[X] = E[E[X \vert Y]]$	The conditional expectation of X given Y

2 Terminology

Name	Description
Arrival instant	The time at which a customer arrives at the queue.
Service instant	The time at which a customer leaves the system after being served completely.
Queue content	The number of customers in the queue waiting for service.
System content	The number of customers in the total system.
Queue capacity	The maximum number of customers in the queue.
System capacity	The maximum number of customers in the system.
Service time	The amount of time that the customer occupies a server.
Waiting time	The amount of time a customer waits in the queue before starting service.
Delay or sojourn time	The amount of time a customer resides in the system.

3 Distributions

Name	Density Function	Mean	Variance
Binomial	$Pr[X = k] = \binom{n}{k} p^k (1-p)^{n-k}$ with $\binom{n}{k} = \frac{n!}{k!(n-k)!}$	np	$np(1-p)$
Geometric	$Pr[X = k] = (1-p)^{k-1} p$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Normal	$f(x \vert \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	μ	σ^2
Uniform	$f(x \vert a, b) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b, \\ 0 & \text{for } x < a \text{ or } x > b. \end{cases}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential	$f(x \vert \lambda) = \lambda e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Poisson	$f(k \vert \lambda) = \frac{e^{-\lambda} \lambda^k}{k!}$	λ	λ

3.1 Memoryless property

Formula	**Description**
$r(t) = \frac{Pr[X \leq t+dt \mid X>t]}{dt} = \frac{x(t)}{1-X(t)}$	The hazard rate function of a RV
$r(t) = \lambda$	The hazard rate function of an exponential function

4 Markov property

Formula	**Description**
$Pr[X(t) = n \mid X(t_1) = n_1, \ldots, X(t_k) = n_k] = Pr[X(t) = n \mid X(t_k) = n_k]$	The Markov property, stating that the futur state only depends on the present state and not on the past states.

5 M/M/1 queue

Formula	**Description**
$\lambda s(n) = \mu s(n+1), \quad n = 0, 1, 2, \ldots$	The local balance equations for the M/M/1 queue
$s(n) = \rho^n s(0)$ with $\rho = \frac{\lambda}{\mu}$	Recursive solution of the balance equations, with ρ the load on the system.
$\sum_{n=0}^{\infty} s(n) = 1 = \sum_{n=0}^{\infty} \rho^n s(0) = \frac{s(0)}{1-\rho}$	Using the normalization function, giving us a solution for $s(0)$
$s(0) = 1 - \rho$	The solution for $s(0)$.
$s(n) = (1-\rho)\rho^n$	The solution for $s(n)$.
$E[S] = \frac{\rho}{1-\rho}$	The mean system content of an M/M/1 queue.
$Var[S] = \frac{\rho}{(1-\rho)^2}$	The variance system content of an M/M/1 queue.
$q(0) = s(0) + s(1)$	The queue content, when no customer in the queue, we have either situation where there is no customer in the system or one customer that is being served.
$q(n) = s(n+1), \quad n = 1, 2, \ldots$	When there are customers in the system, the queue has one less than the system.
$q(0) = 1 - \rho^2, \quad q(n) = (1-\rho)\rho^{n+1} \quad n = 1, 2, \ldots$	Filled in using previous formulas.
$E[Q] = \frac{\rho^2}{1-\rho}$	The mean queue content.
$Var[Q] = \frac{\rho^2}{(1-\rho)^2}(1 + \rho - \rho^2)$	The variance of the queue content.

6 The generic BD-Queueing system

Formula	Description
$\lambda_n s(n) = \mu_{n+1} s(n+1)$	The local balance equations for a generic Birth-death queue.
$s(n) = s(0)\prod_{m=0}^{n-1} \frac{\lambda_m}{\mu_{m+1}}$	The local balance equations recursively solved.
$\sum_{n=0}^{\infty} s(n) = 1 = s(0)(1 + \sum_{n=1}^{\infty}\prod_{m=0}^{n-1}\frac{\lambda_m}{\mu_{m+1}})$	Using the normalization property, we get an extra equation.
$s(0) = (1 + \sum_{l=1}^{\infty}\prod_{m=0}^{l-1}\frac{\lambda_m}{\mu_{m+1}})^{-1}$	Solution for s(0).
$s(n) = \frac{\prod_{m=0}^{n-1}\frac{\lambda_m}{\mu_{m+1}}}{1+\sum_{l=1}^{\infty}\prod_{m=0}^{l-1}\frac{\lambda_m}{\mu_{m+1}}}$	Solution for s(n).
$E[S] = \sum_{n=1}^{\infty} n\frac{\prod_{m=0}^{n-1}\frac{\lambda_m}{\mu_{m+1}}}{1+\sum_{l=1}^{\infty}\prod_{m=0}^{l-1}\frac{\lambda_m}{\mu_{m+1}}}$	Mean system content.
$Var[S] = \sum_{n=1}^{\infty} n^2\frac{\prod_{m=0}^{n-1}\frac{\lambda_m}{\mu_{m+1}}}{1+\sum_{l=1}^{\infty}\prod_{m=0}^{l-1}\frac{\lambda_m}{\mu_{m+1}}} - (\sum_{n=1}^{\infty} n\frac{\prod_{m=0}^{n-1}\frac{\lambda_m}{\mu_{m+1}}}{1+\sum_{l=1}^{\infty}\prod_{m=0}^{l-1}\frac{\lambda_m}{\mu_{m+1}}})^2$	Mean system content.
$\sum_{n=1}^{\infty}\prod_{m=0}^{n-1}\frac{\lambda_m}{\mu_{m+1}}$	Stability condition, this converges when the system is stable.

7 The M/M/c queue

Formula	Description
$s(n) = \begin{cases} s(0)\frac{(c\rho)^n}{n!} & \text{for } n \leq c, \\ s(0)\frac{(c\rho)^c}{c!}\rho^{n-c} & \text{for } n > c. \end{cases}$	System content probabilities.
$s(0) = (\sum_{l=0}^{c-1}\frac{(c\rho)^l}{l!} + \frac{(c\rho)^c}{c!}\frac{1}{1-\rho})$	Normalization condition.
$\rho = \frac{\lambda}{c*\mu} < 1$	Stability condition.
$p^Q = (\sum_{l=0}^{c-1}\frac{(c\rho)^l}{l!} + \frac{(c\rho)^c}{c!}\frac{1}{1-\rho})^{-1}\frac{(c\rho)^c}{c!}\frac{1}{1-\rho}$	Erlang C formula. It is the probability that an arriving customer has to wait.

Formula	**Description**
$E[Q] = p^Q \frac{\rho}{1-\rho}$	Mean queue content.
$E[S] = p^Q \frac{\rho}{1-\rho} + c\rho$	Mean system content.
$c\rho$	Average number of customer being served in the system.

8 The M/M/1/K queue

Formula	**Description**
$n = 1, \ldots, K$	For all formulas.
$s(n) = \frac{1-\rho}{1-\rho^{K+1}} \rho^n$	System content probabilities.
$E[S] = \frac{\rho}{1-\rho} \frac{1-(K(1-\rho)+1)\rho^K}{1-\rho^{K+1}}$	System content probabilities.
$l_p = \frac{1-\rho}{1-\rho^{K+1}} \rho^K$	Loss rate or loss probability.