Steady State Values of RBC with endogenous labour supply

Matthias Schön*

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Abstract

This is part of the first assignment for the course ECON 714-001 Quantitative Macro Theory by Jesus Fernandez Villaverde. It contains the theoretical foundation of the model and and the calculaton of the steady state values of leisure to calibrate the model. The programming code for the whole model is provided for Matlab and Fortran.

*CMR, University of Cologne; Albertus-Magnus-Platz; 50923 Köln; Germany; E-mail: m.schoen@wiso.uni-koeln.de

Description of the RBC Model

Consider a simple RBC model consisting of a representative household whose utility is given by

$$U = \sum_{t=1}^{T} \beta^{t-1} u\left(c_t, 1 - l_t\right),$$

where c is consumption and l is time spend in production.

The household has a constant returns to scale technology for producing output given by

$$y_t = z_t k_t^{\alpha} l_t^{1-\alpha}$$

where k_t ist capital and $z_t$is a productivity parameter follwing an AR(1) process. It also faces a budget constraint

$$k_{t+1} = \left(1 - \delta\right) k_t + y_t - c_t.$$

The constraints collapses to

$$k_{t+1} = \left(1 - \delta\right) k_t + z_t k_t^{\alpha} l_t^{1-\alpha} - c_t.$$

Let λ_t be the multiplier of this constraint and form

$$V\left(k_t, z_t\right) = u\left(c_t, 1 - l_t\right) + \beta E_t V\left(k_{t+1}, z_{t+1}\right) + \lambda_t \left(\left(1 - \delta\right) k_t + z_t k_t^{\alpha} l_t^{1-\alpha} - c_t - k_{t+1}\right).$$

The first order condition with respect to c is

$$\frac{\partial u}{\partial c_t} - \lambda_t \left(-1\right) = 0 \Leftrightarrow \frac{\partial u}{\partial c} = \lambda_t \tag{1}$$

w.r.t. l is

$$\frac{\partial u}{\partial l_t} + \lambda_t z_t \left(1 - \alpha\right) k_t^{\alpha} l_t^{-\alpha} = 0 \tag{2}$$

w.r.t. $k_{t+1}$is

$$\beta E_t \frac{\partial V}{\partial k}\left(k_{t+1}, z_{t+1}\right) + \lambda_t \left(-1\right) = 0 \Leftrightarrow \lambda_t = \beta E_t \frac{\partial V}{\partial k}\left(k_{t+1}, z_{t+1}\right). \tag{3}$$

The envelope condition is

$$\frac{\partial V}{\partial k}\left(k_t, z_t\right) = \lambda_t \left(\left(1 - \delta\right) + z_t \alpha k_t^{\alpha-1} l_t^{1-\alpha}\right). \tag{4}$$

If we combine (1) and (2) yields

$$-\frac{\partial u}{\partial l_t} = \frac{\partial u}{\partial c} z_t \left(1 - \alpha\right) k_t^{\alpha} l_t^{-\alpha} \tag{5}$$

and (1) and (3) lead to

$$\frac{\partial u}{\partial c} = \beta E_t \frac{\partial V}{\partial k}\left(k_{t+1}, z_{t+1}\right). \tag{6}$$

Shifting (4) forward one period in time and taking expectations gives

$$E_t \frac{\partial V}{\partial k}\left(k_{t+1}, z_{t+1}\right) = E_t \lambda_{t+1}\left((1-\delta) + z_{t+1}\alpha k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha}\right) \tag{7}$$

Now let assume that $u\left(c_t, 1-l_t\right) = \log(c_t)$-$\psi\frac{l^2}{2}$ and $f(k_t, l_t) = k^\alpha l^{1-\alpha}$. Equation (6) becomes

$$\frac{1}{c_t} = \beta E_t \frac{\partial V}{\partial k}\left(k_{t+1}, z_{t+1}\right) \tag{8}$$

equation (5) becomes

$$\psi l_t = \frac{1}{c_t} z_t \left(1-\alpha\right) k_t^\alpha l_t^{-\alpha}$$

and (7)

$$E_t \frac{\partial V}{\partial k}\left(k_{t+1}, z_{t+1}\right) = E_t \frac{1}{c_{t+1}}\left((1-\delta) + z_{t+1}\alpha k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha}\right) \tag{9}$$

Combing (8) and (9)yields

$$\frac{1}{c_t} = \beta E_t \frac{1}{c_{t+1}}\left((1-\delta) + z_{t+1}\alpha k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha}\right)$$

In the nonstochastic steady state holds $c = c_t = c_{t+1}$, $k = k_t = k_{t+1}$, $l = l_t = l_{t+1}$ and $z_t = z_{t+1} = 1$. So we can write the steady state equations as

$$\begin{aligned} \frac{1}{c} &= \frac{1}{c}\beta\left((1-\delta) + \alpha k^{\alpha-1} l^{1-\alpha}\right) \\ k &= \left(\frac{\alpha\beta}{1-\beta+\beta\delta}\right)^{\frac{1}{1-\alpha}} l \Leftrightarrow \frac{k}{l} = \left(\frac{\alpha\beta}{1-\beta+\beta\delta}\right)^{\frac{1}{1-\alpha}} \end{aligned} \tag{10}$$

and

$$c = \frac{(1-\alpha)}{\psi}\left(\frac{k}{l}\right)^\alpha l^{-1} = \frac{(1-\alpha)}{\psi}\left(\frac{\alpha\beta}{1-\beta+\beta\delta}\right)^{\frac{\alpha}{1-\alpha}} l^{-1} \tag{11}$$

By pluging 10 and 11 in the budget constraint

$$\delta\left(\frac{\alpha\beta}{1-\beta+\beta\delta}\right)^{\frac{1}{1-\alpha}} l = \left(\left(\frac{\alpha\beta}{1-\beta+\beta\delta}\right)^{\frac{1}{1-\alpha}} l\right)^\alpha l^{1-\alpha} - \frac{(1-\alpha)}{\psi}\left(\frac{k}{l}\right)^\alpha l^{-1}$$

we can now compute the steady state values of labour, capital, consumption and output.

$$\begin{aligned}
l_{ss} &= \left[\frac{(1-\alpha)}{\psi}\frac{1}{\left(1-\delta\frac{\alpha\beta}{1-\beta+\beta\delta}\right)}\right]^{1/2} \\
k_{ss} &= \left(\frac{\alpha\beta}{1-\beta+\beta\delta}\right)^{\frac{1}{1-\alpha}} l_{ss} \\
c_{ss} &= \frac{(1-\alpha)}{\psi}\left(\frac{\alpha\beta}{1-\beta+\beta\delta}\right)^{\frac{\alpha}{1-\alpha}} l_{ss}^{-1} \\
y_{ss} &= \left(\frac{\alpha\beta}{1-\beta+\beta\delta}\right)^{\frac{\alpha}{1-\alpha}} l_{ss}
\end{aligned}$$

As we want to have a steady state labour supply of 1/3 we have to calibrate the utility parameter ψ accordingly. By rearranging the steady state equation for labour we get

$$\psi = \frac{(1-\alpha)}{l_{ss}^2}\frac{1}{\left(1-\delta\frac{\alpha\beta}{1-\beta+\beta\delta}\right)}.$$

Plugging in the parameter values and and $l_{ss} = 1/3$ yields

$$\psi = 7.5981.$$

The corresponding other steady state values are

$$\begin{aligned}
k_{ss} &= 1.1909 \\
c_{ss} &= 0.4024 \\
y_{ss} &= 0.5096.
\end{aligned}$$