Chapter 9

Multi-asset options

1 Introduction

This chapter covers options which depend on more than one risky asset. We focus on two special and important types of multi-asset options: exchange options and cross-currency options.

2 Two-assets options

We assume the following real-world dynamics of the two risky assets (without dividends):

$$\begin{aligned} dX_t &= \mu_X X_t\,dt + \sigma_X X_t\,dW_t^X, && (1)\\ dY_t &= \mu_Y Y_t\,dt + \sigma_Y Y_t\,dW_t^Y, && (2) \end{aligned}$$

where r is the constant risk-free rate, σ_X and σ_Y are the constant volatility of X_t and Y_t respectively. Here, the Brownian motions W_t^X and W_t^Y are correlated:

$$dW_t^X\,dW_t^Y = \rho\,dt, \qquad (3)$$

where ρ is the correlation coefficient.

Consider a European option whose payoff function $f(X, Y)$ depends on the terminal asset price X_T and Y_T. Let $V(X, Y, t)$ denote the option value at time t. As before, we can perform the delta-hedging argument to derive a PDE for V. Since there are two sources of randomness, the hedging portfolio must contain positions in two of the risky assets.

The delta-hedging argument:

Consider the self-financing portfolio Π_t of holding one unit of the option, short ϕ_t units of asset X and short ψ_t units of asset Y. The goal is to choose ϕ_t and ψ_t so that Π_t is riskless.

The increment of the portfolio is

$$d\Pi_t = dV_t - \phi_t X_t - \psi_t dY_t. \tag{4}$$

By the multivariate Ito's Lemma,

$$\begin{aligned} dV_t &= \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial X} dX_t + \frac{\partial V}{\partial Y} dY_t \\ &\quad \frac{1}{2}\frac{\partial^2 V}{\partial X^2}(dX_t)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial Y^2}(dY_t)^2 + \frac{\partial^2 V}{\partial X \partial Y} dX_t\, dY_t, \\ &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_X^2 X^2 \frac{\partial^2 V}{\partial X^2} + \frac{1}{2}\sigma_Y^2 Y^2 \frac{\partial^2 V}{\partial Y^2} + \rho\sigma_X\sigma_Y XY \frac{\partial^2 V}{\partial X \partial Y}\right) dt \\ &\quad + \frac{\partial V}{\partial X} dX_t + \frac{\partial V}{\partial Y} dY_t. \end{aligned} \tag{5}$$

Substituting (5) into (4) and setting

$$\phi_t = \frac{\partial V}{\partial X}, \quad \psi_t = \frac{\partial V}{\partial Y},$$

we then obtain

$$d\Pi_t = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_X^2 X^2 \frac{\partial^2 V}{\partial X^2} + \frac{1}{2}\sigma_Y^2 Y^2 \frac{\partial^2 V}{\partial Y^2} + \rho\sigma_X\sigma_Y XY \frac{\partial^2 V}{\partial X \partial Y}\right) dt. \tag{6}$$

Since Π_t is riskless, it must earn the risk-free rate:

$$d\Pi_t = r\Pi_t\, dt = r\left(V_t - \frac{\partial V}{\partial X} X_t - \frac{\partial V}{\partial Y} Y_t\right) dt. \tag{7}$$

Equating (6) and (7), we arrive at the following PDE:

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma_X^2 X^2 \frac{\partial^2 V}{\partial X^2} + \frac{1}{2}\sigma_Y^2 Y^2 \frac{\partial^2 V}{\partial Y^2} + \rho\sigma_X\sigma_Y XY \frac{\partial^2 V}{\partial X \partial Y} & \\ + rX\frac{\partial V}{\partial X} + rY\frac{\partial V}{\partial Y} - rV &= 0. \end{aligned} \tag{8}$$

The solution domain is

$$\{X > 0, \quad Y > 0, \quad t \in [0, T)\}.$$

The terminal condition is

$$V(X, Y, T) = f(X, Y).$$

3 Exchange Option

A **European exchange option** is an option that gives the holder the right but not the obligation to exchange one risky asset for another risky asset at maturity. Let X_t and Y_t be the price processes of the two risky assets. The terminal payoff of a European exchange option at maturity date T of exchanging Y_T for X_T is given by

$$V(X_T, Y_T, T) = \max\{X_T - Y_T, 0\}.$$

We can simplify the PDE (8) governing the price $V(X, Y, t)$ by a similarity reduction. Suppose we can write the price function in the following form:

$$V(X, Y, t) = YH(Z, t),$$

where the new variable is

$$Z = \frac{X}{Y}.$$

It is enough to find a solution to H. Let us compute the partial derivatives of V in terms of those of H.

$$
\begin{aligned}
\frac{\partial V}{\partial t} &= Y\frac{\partial H}{\partial t}, && (9)\\
\frac{\partial V}{\partial X} &= Y\frac{\partial H}{\partial Z}\frac{\partial Z}{\partial X} \\
&= \frac{\partial H}{\partial Z}, && (10)\\
\frac{\partial^2 V}{\partial X^2} &= \frac{\partial^2 H}{\partial Z^2}\frac{\partial Z}{\partial X} \\
&= \frac{1}{Y}\frac{\partial^2 H}{\partial Z^2}, && (11)\\
\frac{\partial V}{\partial Y} &= H + Y\frac{\partial H}{\partial Z}\frac{\partial Z}{\partial Y} \\
&= H - Z\frac{\partial H}{\partial Z}, && (12)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 V}{\partial Y^2} &= \frac{\partial H}{\partial Z}\frac{\partial Z}{\partial Y} - Z\frac{\partial^2 H}{\partial Z^2}\frac{\partial Z}{\partial Y} - \frac{\partial Z}{\partial Y}\frac{\partial H}{\partial Z} \\
&= \frac{1}{Y}Z^2\frac{\partial^2 H}{\partial Z^2}, && (13) \\
\frac{\partial^2 V}{\partial X \partial Y} &= \frac{\partial^2 H}{\partial Z^2}\frac{\partial Z}{\partial Y} \\
&= -\frac{X}{Y^2}\frac{\partial^2 H}{\partial Z^2}. && (14)
\end{aligned}
$$

Substituting the above partial derivatives into (8), we arrive at the following PDE governing H:

$$
\frac{\partial H}{\partial t} + \frac{1}{2}\overline{\sigma} Z^2 \frac{\partial^2 H}{\partial Z^2} = 0, \tag{15}
$$

where

$$
\overline{\sigma} = \sqrt{\sigma_X^2 + \sigma_Y^2 - 2\rho\sigma_X\sigma_Y}.
$$

This is just the usual Black-Scholes PDE where the underlying asset Z has volatility $\overline{\sigma}$ and the interes rate is 0!. Moreover, the terminal condition for H is

$$
H(Z, T) = (Z - 1)^+.
$$

Hence, we can regard H as a European call option on Z where the strike price is 1, interest rate is 0 and the volatility of Z is $\overline{\sigma}$. This yields the Black-Scholes like price for H:

$$
H(Z, t) = ZN(d_1) - KN(d_2), \tag{16}
$$

where

$$
\begin{aligned}
d_1 &= \frac{\ln Z + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}, && (17) \\
d_2 &= d_1 - \sigma\sqrt{T-t}, && (18) \\
\overline{\sigma} &= \sqrt{\sigma_X^2 + \sigma_Y^2 - 2\rho\sigma_X\sigma_Y}. && (19)
\end{aligned}
$$

Consequently, the price of the exchange option is

$$
V(X, Y, t) = XN(d_1) - YN(d_2). \tag{20}
$$

4 Cross-currency options

This section deals with options whose underlying asset S_t is denominated in a foreign currency f, but pays off in the domestic currency d. These options are called **quanto** options in general.

For convenience, we use US dollars as our domestic currency. The general payoff can be expressed as a function of S_T and F_T, where S_T is the terminal asset price in foreign currency and F_T is the terminal exchange rate in dollars per one unit of foreign currency:

$$V(S_T, F_T, T).$$

Thus, the value of a quanto contract is exposed to the exchange rate risk as well as the underlying asset price risk. Let $V(S, F, t)$ denote the time-t value of this contract in US dollars. We assume that S_t and F_t follows Geometric brownian motion (in the real world):

$$\begin{aligned} dS_t &= \mu_S S_t \, dt + \sigma_S S_t \, dW_t^S \\ dF_t &= \mu_F F_t \, dt + \sigma_F F_t \, dW_t^F. \end{aligned}$$

with

$$dW_t^S \, dW_t^F = \rho \, dt$$

where ρ is the correlation coefficient between them.

Since there are two sources of randomness, we hedge a quanto contract using positions in the asset and positions in the exchange rates. We cannot buy exchange rates directly, so we will invest in foreign currency. To be precise, consider the portfolio consisting of long one unit of the quanto $V(S, F, t)$, short $\triangle_S$ units of the asset, and short $\triangle_F$ units of foreign currency. The value of the portfolio in US dollars is

$$\Pi = V(S, F, t) - \triangle_F F - \triangle_S S \cdot F.$$

The change in the portfolio is due to the change in the value of the components and interest received on the foreign currency. Suppose r_f is the foreign risk-free rate. Then (dropping all subscripts t for stochastic variables):

$$\begin{aligned} d\Pi &= dV - \triangle_F \, dF - \triangle_F r_f F \; dt - \triangle_S d(SF) \\ &= dV - \triangle_F \, dF - \triangle_F r_f F \, dt - \triangle_S \left(S \, dF + F \, dS + \, dSdF\right) \\ &= dV - \triangle_F \, dF - \triangle_F r_f F \, dt - \triangle_S \left(S \, dF + F \, dS + \rho\sigma_S\sigma_F SF \, dt\right) \end{aligned}$$

Since

$$dV = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\sigma_F^2 F^2 \frac{\partial^2 V}{\partial F^2} + \rho\sigma_S\sigma_F SF \frac{\partial^2 V}{\partial F \partial S} \right) dt + \frac{\partial V}{\partial S}\, dS + \frac{\partial V}{\partial F}\, dF$$

we deduce that

$$\begin{aligned} d\Pi \;=\; & \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\sigma_F^2 F^2 \frac{\partial^2 V}{\partial F^2} + \rho\sigma_S\sigma_F SF \frac{\partial^2 V}{\partial F \partial S} - \rho\sigma_S\sigma_F \triangle_S SF - \triangle_F r_f F \right) dt \\ & + \left(\frac{\partial V}{\partial S} - \triangle_S F \right) dS \\ & + \left(\frac{\partial V}{\partial F} - \triangle_F - \triangle_S S \right) dF. \end{aligned} \tag{21}$$

To make the portfolio riskless, we choose

$$\triangle_S = \frac{1}{F}\frac{\partial V}{\partial S}$$

and

$$\triangle_F = \frac{\partial V}{\partial F} - \triangle_S S = \frac{\partial V}{\partial F} - \frac{S}{F}\frac{\partial V}{\partial S}.$$

Since Π is in US dollars, it must earn the domestic risk-free rate r_d. Hence

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\sigma_F^2 F^2 \frac{\partial^2 V}{\partial F^2} + \rho\sigma_S\sigma_F SF \frac{\partial^2 V}{\partial F \partial S} - \rho\sigma_S\sigma_F \triangle_S SF - \triangle_F r_f F$$
$$= r_d \left(V - \triangle_F F - \triangle_S SF \right).$$

After simplifications, we have

$$\begin{aligned} & \frac{\partial V}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\sigma_F^2 F^2 \frac{\partial^2 V}{\partial F^2} + \rho\sigma_S\sigma_F SF \frac{\partial^2 V}{\partial F \partial S} \\ & \quad + (r_d - r_f) F \frac{\partial V}{\partial F} + \left(r_f - \rho\sigma_S\sigma_F S \right) \frac{\partial V}{\partial S} - r_d V \;=\; 0. \end{aligned} \tag{22}$$

5 Correlated Brownian motions

Correlated Brownian motion are necessary when describing several simultaneous processes governed by SDE's, where upward (downward) movements in one generally result in upward (downward) movements of the other. For example, two foreign currencies may be modelled whose countries have closely linked economies. Correlations could be negative, in which case upward movement of one generally implies downward movement of the other.

Generating correlated Brownian motions is simply a matter of generating correlated normal random variables.

5.1 Generating correlated normal random variables

Suppose $\boldsymbol{x} = (x_1, \ldots, x_n)^T$ is a random vector. The **variance-covariance matrix** $\boldsymbol{C}[\boldsymbol{x}]$ of $\boldsymbol{x}$ is the $n \times n$ matrix that has (i, j)-th entry given by

$$\boldsymbol{C} := \boldsymbol{C}[\boldsymbol{x}]_{ij} := \mathrm{Cov}(x_i, x_j).$$

The variance-covariance matrix is the expected value, element by element, of the $n \times n$ matrix computed as

$$(\boldsymbol{x} - \mathbb{E}[\boldsymbol{x}])(\boldsymbol{x} - \mathbb{E}[\boldsymbol{x}])^T.$$

Some properties of the covariance matrix $\boldsymbol{C}$:

(i) It is symmetric: $\boldsymbol{C}^T = \boldsymbol{C}$.

(ii) It is *positive semi-definite*: $\boldsymbol{y}^T \boldsymbol{C} \boldsymbol{y} \geq 0$ for all $\boldsymbol{y} \in \mathbb{R}^n$. In fact, it is *positive definite* ($\boldsymbol{y}^T \boldsymbol{C} \boldsymbol{y} > 0$ for all non-zero column vectors $\boldsymbol{y} \in \mathbb{R}^n$) unless one variable is an exact linear combination of the others.

Suppose we want to generate the random vector $\boldsymbol{x} = (x_1, \ldots, x_n)^T$, where $\boldsymbol{x} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{C})$? Here, $\mathcal{N}(\boldsymbol{0}, \boldsymbol{C})$ stands for the distribution of multivariate normal random variables with mean vector $\boldsymbol{0}$ and variance-covariance matrix $\boldsymbol{C}$. Note that it is then easy to handle the case where $\mathbb{E}[\boldsymbol{x}] \neq \boldsymbol{0}$.

By way of motivation, suppose we have a vector $\boldsymbol{z} = (z_1, \ldots, z_n)^T$ of n i.i.d. standard normal variables $z_i \sim \mathcal{N}(0, 1)$ for $i = 1, \ldots, n$. Then, for any given scalars $l_1, \ldots, l_n$, we have

$$(l_1, \ldots, l_n)\boldsymbol{z} = l_1 z_1 + \cdots + l_n z_n \sim \mathcal{N}(0, l_1^2 + \cdots + l_n^2)$$

That is, a linear combination of independent standard normal random variables is again normal.

More generally, let $\boldsymbol{L}$ be an $n \times n$ matrix. Consider the new vector $\boldsymbol{y} = \boldsymbol{L}\boldsymbol{z}$. Clearly, each element of $\boldsymbol{y}$ is normally distributed, and $\mathbb{E}[\boldsymbol{y}] = \mathbb{E}[\boldsymbol{L}\boldsymbol{z}] = \boldsymbol{L}\mathbb{E}[\boldsymbol{z}] = \boldsymbol{0}$.

Furthermore, the variance-covariance matrix of $\boldsymbol{y}$ is given by

$$\begin{aligned}
\mathbb{E}[(\boldsymbol{y}-\mathbb{E}[\boldsymbol{y}])(\boldsymbol{y}-\mathbb{E}[\boldsymbol{y}])^T] &= \mathbb{E}[\boldsymbol{y}\boldsymbol{y}^T] \\
&= \mathbb{E}[\boldsymbol{L}\boldsymbol{z}\boldsymbol{z}^T\boldsymbol{L}^T] \\
&= \boldsymbol{L}\mathbb{E}[\boldsymbol{z}\boldsymbol{z}^T]\boldsymbol{L}^T \\
&= \boldsymbol{L}\boldsymbol{L}^T
\end{aligned}$$

since $\mathbb{E}[\boldsymbol{z}\boldsymbol{z}^T] = 1$ because

- elements of $\boldsymbol{z}$ are independent: $\mathbb{E}[z_i z_j] = 0$ for all $i \neq j$.
- elements of $\boldsymbol{z}$ have unit variance: $\mathbb{E}[z_i^2] = 1$ for all i.

To summarize:

> If $\boldsymbol{z} = (z_1, \ldots, z_n)^T$ is a vector of n i.i.d. standard normal variables, then
>
> $$\boldsymbol{L}\boldsymbol{z} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{L}\boldsymbol{L}^T).$$

Therefore, if we want to generate a random vector $\boldsymbol{x}$ with $\mathbb{E}[\boldsymbol{x}] = \boldsymbol{0}$ and variance-covariance matrix $\boldsymbol{C}$, then we can do the following:

> (1) Generate a vector $\mathbf{z} = (z_1, \ldots, z_n)^T$ of n i.i.d. standard normal variables.
>
> (2) Write $\boldsymbol{C}$ as:
> $$\boldsymbol{C} = \boldsymbol{L}\boldsymbol{L}^T.$$
>
> (3) It then follows that
> $$\boldsymbol{L}\mathbf{z} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{C}).$$

Writing $\boldsymbol{C} = \boldsymbol{L}\boldsymbol{L}^T$ requires us to find the Cholesky decomposition of $\boldsymbol{C}$.

5.2 The Cholesky Decomposition of real symmetric positive definite matrix

For our purposes, the variance-covariance matrix $\boldsymbol{C}$ will be real symmetric and positive definite.

A well-known fact from linear algebra is that the matrix $\boldsymbol{C} = (c_{ij})$ is positive definite if and only if there exists a unique lower triangular matrix $\boldsymbol{L} = (l_{ij})$, with real and strictly positive diagonal elements, such that $\boldsymbol{C} = \boldsymbol{L}\boldsymbol{L}^T$. This factorization is called the **Cholesky decomposition** of $\boldsymbol{C}$.

The algorithm: First, we calculate the values of $\boldsymbol{L} = (l_{ij})$ on the main diagonals. Subsequently, we calculate the off-diagonals for the elements below the diagonal:

$$\begin{aligned} l_{kk} &= \sqrt{c_{kk} - \sum_{j=1}^{k-1} l_{kj}^2} \\ l_{ik} &= \frac{1}{l_{kk}} \left(c_{ik} - \sum_{j=1}^{k-1} l_{ij} l_{kj} \right), \quad i > k. \end{aligned} \tag{23}$$

5.3 Generating correlated Brownian motion

Suppose $\boldsymbol{Z}_t$ is a (column) vector of n independent Brownian motions. Then its variance-covariance matrix is

$$\mathbb{E}[\boldsymbol{Z}_t \boldsymbol{Z}_t^T] = \begin{pmatrix} t & 0 & 0 & \cdots & 0 \\ 0 & t & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 0 & t \end{pmatrix} = t\boldsymbol{I}_n$$

where $\boldsymbol{I}_n$ is the $n \times n$ identity matrix.

Generally, if $\boldsymbol{X}_t$ is a (column) vector of n correlated Brownian motions, then the variance-covariance matrix is

$$\mathbb{E}[\boldsymbol{X}_t \boldsymbol{X}_t^T] = t \begin{pmatrix} 1 & \rho_{12} & \rho_{13} & \cdots & \rho_{1n} \\ \rho_{21} & 1 & \rho_{23} & \cdots & \rho_{2n} \\ \vdots & & & & \vdots \\ \rho_{n1} & \rho_{n2} & \rho_{n3} & \cdots & 1 \end{pmatrix}$$

where $\mathbb{E}[z_i z_j] = \rho_{ij} t = \rho_{ji} t = \mathbb{E}[z_j z_i]$. The matrix

$$\boldsymbol{C}^* \;=\; \begin{pmatrix} 1 & \rho_{12} & \rho_{13} & \cdots & \rho_{1n} \\ \rho_{21} & 1 & \rho_{23} & \cdots & \rho_{2n} \\ \vdots & & & & \vdots \\ \rho_{n1} & \rho_{n2} & \rho_{n3} & \cdots & 1 \end{pmatrix} \tag{24}$$

is called the **correlation matrix** of $\boldsymbol{X}_t$.

Theorem 5.1. Suppose $\boldsymbol{Z}$ is a vector of n independent Brownian motion, and $\boldsymbol{C}^*$ is the correlation matrix given by (24). Suppose the Cholesky decomposition of $\boldsymbol{C}^*$ is given by

$$\boldsymbol{C}^* = \boldsymbol{L}\boldsymbol{L}^T.$$

Then $\boldsymbol{L}\boldsymbol{Z}_t$ is a vector of n Brownian motion whose correlation matrix is $\boldsymbol{C}^*$.

Corollary 5.2. Suppose dW_t^X and dW_t^Y are two standard Brownian motion with $dW_t^X dW_t^Y = \rho\, dt$. There exists a standard Brownian motion dW_t, independent of dW_t^X, such that

$$dW_t^Y = \rho\, dW_t^X + \sqrt{1-\rho^2}\, dW_t.$$

Proof. The correlation matrix of $(W_t^X, W_t^Y)^T$ is

$$\boldsymbol{C}^* = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

The Cholesky decomposition of $\boldsymbol{C}^*$ is

$$\boldsymbol{C}^* = \boldsymbol{L}\boldsymbol{L}^T$$

where

$$\boldsymbol{L} = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}.$$

By Theorem 5.1, there exist independent Brownian motion $Z_{1,t}$ and $Z_{2,t}$ such that

$$\begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \cdot \begin{pmatrix} Z_{1,t} \\ Z_{2,t} \end{pmatrix} = \begin{pmatrix} W_t^X \\ W_t^Y \end{pmatrix}$$

It follows that $Z_{1,t} = W_t^X$ and

$$\rho Z_{1,t} + \sqrt{1-\rho^2}\; Z_{2,t} = W_t^Y.$$

$\square$

6 Generalized Black-Scholes Market

From the previous section, we see that correlated Brownian motions can be decomposed into independent ones. Thus, it is sufficient to just consider a market model driven by independent Brownian motions. As a generalisation of the Black-Scholes market with one risky asset (driven by one Brownian motion) and the money market account, we consider a market consisting of $N+1$ assets with the following time-t prices:

- the risk-less asset: M_t (money market account)
- N risky assets: $S_{1,t}$, $S_{2,t}$, …, $S_{N,t}$.

Furthermore, we assume that prices are driven by K random factors which are modelled by K **independent** Brownian motions $Z_{1,t}$, $Z_{2,t}$, …, $Z_{K,t}$. Notice that we have used the letter Z instead of the usual W for these Brownian motions; For the sake of clarity, we reserve the letter W for correlated Brownian motions.

A **generalized Black-Scholes market** assumes that prices of the underlying assets in the market, under the real-world measure $\mathbb{P}$, are given by the following system of SDE:

$$dM_t = rM_t\,dt \tag{25}$$

$$dS_{i,t} = \mu_i S_{i,t}\,dt + S_i\left(\sum_{j=1}^{K}\sigma_{ij}\,dZ_{j,t}\right), \quad 1 \le i \le N. \tag{26}$$

Few important remarks:

- The filtration generated by the independent Brownian motions is the information available in the market as time passes. The filtration at any time t is denoted by $\mathcal{F}_t$, and it depends only on the Brownian motions up to time t: $Z_{1,s}$, …, $Z_{K,s}$ for $s \le t$.
- For convenience, we assume that the interest rate r, the drifts μ_i and volatilities σ_{ij} are constants. This is just to help fix ideas more easily, since allowing them to be time-dependent and stochastic (adapted to filtration) would force us to consider some technicalities which are rather distracting.
- In general, we do not assume that $N = K$. As we will see later, we need to impose some restriction on N in terms of K so that the market is *complete* (i.e. every contingent claim can be replicated by a self-financing portfolio – which is roughly saying that all risks can be traded).

When dealing with quantities with many dimensions, it is often convenient to simplify the notation using vectors. All vectors will be represented by bold letters (big or small), which by default, are column vectors. A row vector will be constructed by transposing a column vector. For example, we have

$$\begin{aligned}
\mathbf{Z}_t &= \begin{pmatrix} Z_{1,t} \\ \vdots \\ Z_{K,t} \end{pmatrix}, \quad \text{(a } K \times 1 \text{ vector)}, \\
\mathbf{S}_t &= \begin{pmatrix} S_{1,t} \\ \vdots \\ S_{N,t} \end{pmatrix}, \quad \text{(a } N \times 1 \text{ vector)}.
\end{aligned}$$

We now introduce the $N \times K$-matrix of the σ_{ij}'s:

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1K} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2K} \\ \vdots & \vdots & & \vdots \\ \sigma_{N1} & \sigma_{N2} & \cdots & \sigma_{NK} \end{pmatrix}. \tag{27}$$

We can then write (26) as follows:

$$\begin{pmatrix} \frac{dS_{1,t}}{S_{1,t}} \\ \vdots \\ \frac{dS_{N,t}}{S_{N,t}} \end{pmatrix} = \begin{pmatrix} \mu_1\, dt \\ \mu_2\, dt \\ \vdots \\ \mu_N\, dt \end{pmatrix} + \mathbf{\Sigma} \begin{pmatrix} dZ_{1,t} \\ dZ_{2,t} \\ \vdots \\ dZ_{K,t} \end{pmatrix} \tag{28}$$

Definition 6.1 (A general European contingent claim). A **European contingent claim** V in the generalised Black-Scholes market is a financial instrument whose payoff V_T at time T is a $\mathcal{F}_T$-measurable random variable. In other words, V is only allowed to depend on the historical paths of $\mathbf{Z}_t$ for $t \le T$.

An example of such a claim is

$$V_T = \max\left(\sum_{i=1}^{N} S_{i,T} - K, 0\right),$$

which is called a **basket option**. We could also consider path-dependent options with payoff

$$V_T = \max\left(\frac{1}{T}\int_0^T (a_1 S_{1,t} + \cdots + a_N S_{N,t})\, dt - K, 0\right),$$

where a_1, ..., a_N are some prescribed real constants.

The problem is to price a European contingent claim written on the assets $\mathbf{S}_t$. As we have seen in the one-factor Black-Scholes market, the risk-neutral valuation formula is valid because of two important facts:

- The existence of the risk-neutral measure $\mathbb{Q}$ (by making use of the Girsanov Theorem);
- The market is *complete* (by making use of the Martingale Representation Theorem).

The idea is to extend the above to the generalized Black-Scholes market.

6.1 Existence of the risk-neutral measure

We first define what do we mean by a risk-neutral measure in our model with multiple assets.

Definition 6.2 (Risk-neutral/Equivalent Martingale Measure). A probability measure $\mathbb{Q}$ is called a **risk-neutral measure** if $\mathbb{Q}$ is equivalent to $\mathbb{P}$ and that the *discounted* process

$$\widetilde{S}_{i,t} := \frac{S_{i,t}}{M_t}, \quad \text{for all } 1 \leq i \leq N,$$

is a $\mathbb{Q}$-martingale.

To show that a risk-neutral measure exists, we require a multidimensional version of the Girsanov Theorem.

Theorem 6.3 (Multi-dimensional Girsanov Theorem). *Let $(\theta_1, \ldots, \theta_K)$ be a vector of constants. There exists a measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, such that the process $\widetilde{Z}_t$, defined by*

$$d\widetilde{Z}_{i,t} = \theta_i \, dt + dZ_{i,t}, \quad \text{for } 1 \leq i \leq K,$$

are independent Brownian motion with respect to $\mathbb{Q}$.

In general, the constants θ_i's can be replaced by some adapted processes $\theta_{i,t}$, and the conclusion of the theorem will still hold, provided that these $\theta_{i,t}$'s satisfy some technical condition.

We now proceed to find a risk-neutral measure $\mathbb{Q}$. Assuming such a measure exists, we "reverse engineer" to determine the required θ_i's.

By Ito's Lemma,

$$d\widetilde{S}_{i,t} = (\mu_i - r)\widetilde{S}_{i,t}\,dt + \widetilde{S}_{i,t}\sum_{j=1}^{K}\sigma_{ij}\,dZ_{j,t}.$$

Substituting

$$d\widetilde{Z}_{j,t} = \theta_j\,dt + dZ_{j,t}, \quad 1 \le i \le K,$$

we have

$$d\widetilde{S}_{i,t} = \left(\mu_i - r - \sum_{j=1}^{K}\sigma_{ij}\theta_j\right)\widetilde{S}_{i,t}\,dt + \widetilde{S}_{i,t}\sum_{j=1}^{K}\sigma_{ij}\,d\widetilde{Z}_{j,t}.$$

The Girsanov Theorem ensures that there is an equivalent measure $\mathbb{Q}$ under which each $d\widetilde{Z}_{j,t}$ is a Brownian motion. Since we insist that $\widetilde{S}_{i,t}$ is a $\mathbb{Q}$-martingale, the drift must be zero, i.e.

$$\sum_{j=1}^{K}\sigma_{ij}\theta_j \;=\; \mu_i - r. \tag{29}$$

Such a condition must hold for every i, $1 \le i \le N$. We can write this condition in the following matrix form:

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1K} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2K} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \sigma_{N1} & \sigma_{N2} & \cdots & \sigma_{NK} \end{pmatrix} \cdot \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_K \end{pmatrix} = \begin{pmatrix} \mu_1 - r \\ \mu_2 - r \\ \vdots \\ \vdots \\ \mu_N - r \end{pmatrix} \tag{30}$$

Setting

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \vdots \\ \mu_N \end{pmatrix}, \quad \boldsymbol{\theta} = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_K \end{pmatrix},$$

the above condition can be written in a more compact form as:

$$\boldsymbol{\Sigma\theta} = \boldsymbol{\mu} - r\mathbf{1},$$

where $\mathbf{1}$ is the all-one $N \times 1$ vector.

We have proved the following.

Theorem 6.4. *With the above notations, if there exists a $K \times 1$ vector $\boldsymbol{\theta}$ such that $\boldsymbol{\Sigma}\boldsymbol{\theta} = \boldsymbol{\mu} - r\mathbf{1}$, then there exists a risk-neutral measure $\mathbb{Q}$ for the generalised Black-Scholes market.*

Remark.

(1) Unfortunately, the equation $\boldsymbol{\Sigma x} = \boldsymbol{\mu} - r\mathbf{1}$ in the variable $\boldsymbol{x} = (x_1, \ldots, x_K)^T$ does not always admit a solution in general. This was never an issue in the one-factor model since we can always solve for θ:

$$\theta = \frac{\mu - r}{\sigma},$$

because we always assume $\sigma \neq 0$.

(2) In the generalised setting, there may NOT exist any solution to the equation $\boldsymbol{\Sigma x} = \boldsymbol{\mu} - r\mathbf{1}$, even if we assume all σ_{ij}'s are non-zero.

Here is one example: Take $N = 2$, $K = 1$, $\boldsymbol{\mu} = (\mu_1, \mu_2)^T$ where $\mu_1 \neq \mu_2$, and

$$\Sigma = \begin{pmatrix} \sigma \\ \sigma \end{pmatrix}.$$

If there is a solution $\boldsymbol{\theta} = (\theta)$ to the equation $\boldsymbol{\Sigma x} = \boldsymbol{\mu} - r\mathbf{1}$, then

$$\begin{cases} \sigma\theta &= \mu_1 - r, \\ \sigma\theta &= \mu_2 - r. \end{cases}$$

This implies that $\mu_1 = \mu_2$, which is a contradiction.

Therefore, we cannot guarantee that risk-neutral measures exist. Luckily, this won't happen if there is no arbitrage in the market.

A portfolio can be regarded as a vector-valued process $(\phi_{0,t}, \phi_{1,t}, \ldots, \phi_{N,t})$ such that at any time t, the investor holds the following quantities of the underlying assets:

- $\phi_{0,t}$ units of money market accounts;
- $\phi_{i,t}$ units of the asset $S_{i,t}$, for $1 \leq i \leq N$.

Clearly, we require each $\phi_{i,t}$ to be adapted to the filtration $\mathcal{F}_t$. At any time t, the value of a portfolio is given by

$$\Pi_t = \phi_{0,t} M_t + \sum_{i=1}^{N} \phi_{i,t} S_{i,t}$$

The portfolio is **self-financing** if

$$d\Pi_t \;=\; \phi_{0,t}\, dM_t + \sum_{i=1}^{N} \phi_{i,t}\, dS_{i,t}. \tag{31}$$

Theorem 6.5. *Suppose there is no arbitrage opportunities in the generalised Black-Scholes market. Then there exists a risk-neutral measure $\mathbb{Q}$.*

Proof. We shall prove the contrapositive. Suppose there does not exist a risk-neutral measure. By Theorem 6.4,

$$\boldsymbol{\Sigma x} \neq \boldsymbol{\mu} - r\mathbf{1} \text{ for all vector } \boldsymbol{x} \in \mathbb{R}^K.$$

This says that $\boldsymbol{\mu} - r\mathbf{1}$ is not in the image $\mathrm{Im}(\boldsymbol{\Sigma})$ of $\boldsymbol{\Sigma} : \mathbb{R}^K \to \mathbb{R}^N$.

By a general theorem in linear algebra,

$$\mathrm{Im}(\boldsymbol{\Sigma}) = \mathrm{Ker}(\boldsymbol{\Sigma}^T)^{\perp},$$

where $\mathrm{Ker}(\boldsymbol{\Sigma}^T)^{\perp}$ is the set of $N \times 1$ vectors that are orthogonal to the kernel of $\boldsymbol{\Sigma}^T : \mathbb{R}^N \to \mathbb{R}^K$. It follows that $\mathrm{Ker}(\boldsymbol{\Sigma}^T) \neq \{\mathbf{0}\}$, and so there exists a vector $\boldsymbol{\phi} \in \mathbb{R}^N$:

$$\boldsymbol{\phi} = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix} \neq \mathbf{0}$$

such that $\boldsymbol{\phi} \in \mathrm{Ker}(\boldsymbol{\Sigma}^T)$ (i.e. $\boldsymbol{\Sigma}^T\boldsymbol{\phi} = \mathbf{0}$) and $\boldsymbol{\phi} \cdot (\boldsymbol{\mu} - r\mathbf{1}) \neq 0$. Without loss of generality, we may assume that

$$\boldsymbol{\Sigma}^T\boldsymbol{\phi} \;=\; \mathbf{0}, \tag{32}$$

$$\boldsymbol{\phi} \cdot (\boldsymbol{\mu} - r\mathbf{1}) \;>\; 0, \tag{33}$$

since if $\boldsymbol{\phi} \cdot (\boldsymbol{\mu} - r\mathbf{1}) < 0$, we can replace $\boldsymbol{\phi}$ by $-\boldsymbol{\phi}$.

We will now construct a self-financing portfolio that generates an arbitrage opportunity. Consider the portfolio Π_t, $0 \leq t \leq T$, given by:

$$\Pi_t = \psi_{0,t}\, M_t + \sum_{i=1}^{N} \psi_{i,t} S_{i,t},$$

where we have set

$$\psi_{i,t} = \frac{\phi_{i,t} M_t}{S_{i,t}}, \quad \text{for } 1 \leq i \leq N,$$

and have chosen $\psi_{0,t}$ so that the initial value of the portfolio is 0 and that the portfolio is self-financing – this can always be done since we can always borrow from the money market if there is a shortfall from positions in the risky assets, and invest in the money market in if there is a surplus. Thus,

$$d\Pi_t \quad = \quad \psi_{0,t} r M_t \, dt + \sum_{i=1}^{N} \psi_{i,t} dS_{i,t}. \tag{34}$$

By Ito's Lemma,

$$d\left(\frac{\Pi_t}{M_t}\right) \quad = \quad -r\frac{\Pi_t}{M_t}\, dt + \frac{1}{M_t}\, d\Pi_t. \tag{35}$$

Substituting (34) into (35), we obtain

$$\begin{aligned}
d\left(\frac{\Pi_t}{M_t}\right) &= -r\frac{\Pi_t}{M_t}\, dt + \frac{1}{M_t}\left(\psi_{0,t} r M_t\, dt + \sum_{i=1}^{N}\psi_{i,t} dS_{i,t}\right) \\
&= -r\frac{\Pi_t}{M_t}\, dt + r\psi_{0,t}\, dt + \sum_{i=1}^{N}\psi_{i,t}\frac{dS_{i,t}}{M_t} && (36)\\
&= -r\frac{\Pi_t}{M_t}\, dt + r\psi_{0,t}\, dt + \sum_{i=1}^{N}\psi_{i,t}\left(d\left(\frac{S_t}{M_t}\right) + r\frac{S_t}{M_t}\, dt\right) && (37)\\
&= -r\frac{\Pi_t}{M_t}\, dt + r\left(\frac{\psi_{0,t} M_t + \sum_{i=1}^{N}\psi_{i,t} S_{i,t}}{M_t}\right) dt + \sum_{i=1}^{N}\psi_{i,t} d\left(\frac{S_t}{M_t}\right) && (38)\\
&= -r\frac{\Pi_t}{M_t}\, dt + r\frac{\Pi_t}{M_t}\, dt + \sum_{i=1}^{N}\psi_{i,t} d\left(\frac{S_t}{M_t}\right) && (39)\\
&= \sum_{i=1}^{N}\psi_{i,t} d\widetilde{S}_{i,t} && (40)\\
&= \sum_{i=1}^{N}\psi_{i,t}\left((\mu_i - r)\widetilde{S}_{i,t}\, dt + \widetilde{S}_{i,t}\sum_{j=1}^{K}\sigma_{ij}\, dZ_{j,t}\right) && (41)\\
&= \sum_{i=1}^{N}\phi_{i,t}\left((\mu_i - r)\, dt + \sum_{j=1}^{K}\sigma_{ij}\, dZ_{j,t}\right) && (42)\\
&= \sum_{i=1}^{N}\phi_{i,t}(\mu_i - r)\, dt + \sum_{i=1}^{N}\sum_{j=1}^{K}\phi_{i,t}\sigma_{ij}\, dZ_{j,t} && (43)
\end{aligned}$$

$$
\begin{aligned}
d\left(\frac{\Pi_t}{M_t}\right) &= \sum_{i=1}^{N} \phi_{i,t}(\mu_i - r)\,dt + \sum_{j=1}^{K}\sum_{i=1}^{N} \phi_{i,t}\sigma_{ij}\,dZ_{j,t} &\quad (44)\\
&= \sum_{i=1}^{N} \phi_{i,t}(\mu_i - r)\,dt + \sum_{i=1}^{K} \left(\boldsymbol{\Sigma}^T\boldsymbol{\phi}\right)_j\,dZ_{j,t} &\quad (45)
\end{aligned}
$$

where $\left(\boldsymbol{\Sigma}^T\boldsymbol{\phi}\right)_j$ is the j-th entry of the $K \times 1$ vector $\left(\boldsymbol{\Sigma}^T\boldsymbol{\phi}\right)$. By (32), we deduce that

$$
\begin{aligned}
d\left(\frac{\Pi_t}{M_t}\right) &= \sum_{i=1}^{N} \phi_{i,t}(\mu_i - r)\,dt \\
&= \boldsymbol{\phi}\cdot(\boldsymbol{\mu} - r\mathbf{1})\,dt &\quad (46)
\end{aligned}
$$

It follows from (35) that

$$
\begin{aligned}
d\Pi_t &= r\Pi_t\,dt + M_t \cdot d\left(\frac{\Pi_t}{M_t}\right) \\
&= r\Pi_t\,dt + \boldsymbol{\phi}\cdot(\boldsymbol{\mu} - r\mathbf{1})M_t\,dt \\
&> r\Pi_t\,dt. &\quad (47)
\end{aligned}
$$

We have just shown that the portfolio Π_t is risk-less with 0 initial outlay and positive growth rate. This is obviously an arbitrage opportunity! $\square$

In fact, the converse of the above theorem is also true (Exercise). We summarise these results as follows.

Theorem 6.6. *The following are equivalent in the generalised Black-Scholes market:*

(i) There is no arbitrage opportunities.

(ii) There exists a risk-neutral measure.

(iii) There exists a solution to the equation $\boldsymbol{\Sigma}\boldsymbol{x} = \boldsymbol{\mu} - r\mathbf{1}$.

In the special case when $N = K$ *and* $\boldsymbol{\Sigma}$ *is invertible, then all of the above hold. Moreover, there is a unique solution to* $\boldsymbol{x} = \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1})$.

6.2 Market Completeness

Form now on, we are going to assume that our market has a risk-neutral measure $\mathbb{Q}$. This implies that

$$d\widetilde{S}_{i,t} \quad = \quad \widetilde{S}_{i,t} \sum_{j=1}^{K} \sigma_{ij}\, d\widetilde{Z}_{j,t}.$$

Consider a European contingent claim V. We say that V is **replicable** (or **attainable**) if there exists a self-financing portfolio Π_t such that $\Pi_T = V_T$. Let us now discuss what it takes in order to be able to replicate V.

Define the variable U_t by

$$U_t = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{V_T}{M_T}\right].$$

By iterated conditioning, U_t is a $\mathbb{Q}$-martingale. By a suitable extension of the Martingale Representation Theorem, we will be able to write U_t as

$$dU_t = \sum_{j=1}^{K} \eta_{j,t}\, d\widetilde{Z}_{j,t}$$

for some adapted vector processes $(\eta_{1,t}, \ldots, \eta_{K,t})$.

The key step in replicating V is to convert $d\widetilde{Z}_{j,t}$ into $d\widetilde{S}_{j,t}$ (all the other steps are similar to those in the one-factor model and can be readily extended, so we will not discuss them here). In the one-factor Black-Scholes model, this step was straight forward. However in the multi-dimensional setting, we are actually dealing with the solution to a system of linear equations. In particular, we want to go from

$$\frac{d\widetilde{S}_{i,t}}{\widetilde{S}_{i,t}} \quad = \quad \sum_{j=1}^{K} \sigma_{ij}\, d\widetilde{Z}_{j,t}, \quad 1 \leq i \leq N \tag{48}$$

to

$$d\widetilde{Z}_{j,t} \quad = \quad \sum_{l=1}^{n} \gamma_{jl} \frac{d\widetilde{S}_{l,t}}{\widetilde{S}_{l,t}}, \quad 1 \leq j \leq K \tag{49}$$

for suitable coefficients γ_{jl}. The equations (48) and (49) demand that

$$\begin{pmatrix} d\widetilde{Z}_{1,t} \\ \vdots \\ d\widetilde{Z}_{K,t} \end{pmatrix}$$

is the only solution to the following system of linear equations in the variable $\boldsymbol{x} = (x_1, \ldots, x_K)^T$:

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1K} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \sigma_{N1} & \sigma_{N2} & \cdots & \sigma_{NK} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_K \end{pmatrix} = \begin{pmatrix} \frac{d\widetilde{S}_{1,t}}{\widetilde{S}_{1,t}} \\ \vdots \\ \vdots \\ \vdots \\ \frac{d\widetilde{S}_{N,t}}{\widetilde{S}_{N,t}} \end{pmatrix} \tag{50}$$

Therefore, by standard linear algebra, this is only possible if

$$\operatorname{rank}(\boldsymbol{\Sigma}) = \text{number of rows in } \boldsymbol{x} = K.$$

Recall that $\min\{N, K\} \geq \operatorname{rank}(\boldsymbol{\Sigma})$. This amounts to saying that

- $N \geq K$; and
- the K column vectors of $\boldsymbol{\Sigma}$ are all independent, i.e. $\boldsymbol{\Sigma}$ has full column rank (so it has left inverse).

In this case, the solution $\boldsymbol{\Sigma x} = \boldsymbol{\mu} - r\mathbf{1}$ will be uique as well, and thus there will be exactly one unique risk-neutral measure. The condition $N \geq K$ says that in order to have a complete market it is necessary to have at least as many assets as there are risk factors.

Definition 6.7 (**Complete Market**). *The generalised Black-Scholes market is* **complete** *if every European contingent claim can be replicated by some self-financing portfolio (trading strategy).*

We now summarize what we have shown:

Theorem 6.8 (**Risk-neutral pricing**). *Suppose that the generalized Black-Scholes market does not admit arbitrage. Then there exists a risk-neutral measure $\mathbb{Q}$. Moreover, this market is complete if and only if*

- *$N \geq K$, and*
- *the $N \times K$ matrix $\boldsymbol{\Sigma}$ has full column rank K.*

In this case, the measure $\mathbb{Q}$ is unique and we can price any European contingent claim V by the formula

$$V_t = M_t \mathbb{E}_t^{\mathbb{Q}} \left[\frac{V_T}{M_T} \right].$$

Note that if $N = K$, the condition on $\boldsymbol{\Sigma}$ is simply that $\boldsymbol{\Sigma}$ is an invertible square matrix.