MSc thesis

Sobol' sensitivity analysis for a parametrized diffusion process

Long LI

Under supervision of Clémentine PRIEUR and Pierre ÉTORÉ

Master of Science in Industrial and Applied Mathematics (MSIAM)

Grenoble, June 21 2017

Review of SDEs

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A diffusion process $X : [0, T] \times \Theta \to \mathbb{R}$ $(T \in \mathbb{R}_+)$ satisfies an autonomous SDE,

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t \text{ with } X_0 = Z,$$

where $b : \mathbb{R} \to \mathbb{R}$, $\sigma : \mathbb{R} \to \mathbb{R}_+$, W_t defined on $(\Theta, \mathcal{F}_\Theta, \mathbb{P}_\theta)$.

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Existence and uniqueness

$$\exists K \in \mathbb{R}_+ \text{ s.t. } \forall (x,y),\ |b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq K|y - x|.$$

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$$L^2(\Theta, \mathbb{P}_\theta) = \{v : \Theta \to \mathbb{R} \text{ s.t. } \mathbb{E}(v^2) := \int_\Theta v^2(x) d\mathbb{P}_\theta(x) < +\infty\}.$$

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Condition for $X_t \in L^2(\Theta, \mathbb{P}_\theta)$

Z is a r.v. independent of W_t s.t. $\mathbb{E}\left(|Z|^2\right) < +\infty$.

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where X_t defined on $(\Omega, \mathcal{F}, \mathbb{P})$, W_t defined on $(\Theta, \mathcal{F}_\Theta, \mathbb{P}_\theta)$, $\xi = (\xi_1, \ldots, \xi_d)^T \in \mathbb{R}^d$ defined on $(\Xi, \mathcal{B}_\Xi, \mathbb{P}_\xi)$, ξ_i defined on $(\Xi_i, \mathcal{B}_{\Xi_i}, \mathbb{P}_{\xi_i})$.

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Fondamental assumptions

- ξ and W_t are independent.

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Consequences :

$$\Omega = \Theta \times \Xi, \ \mathbb{P} = \mathbb{P}_\theta \otimes \mathbb{P}_\xi, L^2(\Omega, \mathbb{P}) = L^2(\Theta, \mathbb{P}_\theta) \otimes L^2(\Xi, \mathbb{P}_\xi).$$

$$\Xi = \prod_{i=1}^{d} \Xi_i, \ \mathbb{P}_\xi = \otimes_{i=1}^{d} \mathbb{P}_{\xi_i}, \ L^2(\Xi, \mathbb{P}_\xi) = \otimes_{i=1}^{d} L^2(\Xi_i, \mathbb{P}_{\xi_i}).$$

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Models for SA

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Inputs	Models	Outputs
$\omega = (\xi, \theta)$	$\mathcal{M}_0$	$X(t, \omega)$

$$\omega = (\xi, \theta) \longrightarrow \mathcal{M}_0 \longrightarrow X(t, \omega)$$

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$$\boxed{\xi} \longrightarrow \boxed{\mathcal{M}_1} \longrightarrow \boxed{\mathcal{M}_1(t, x, \xi) = \mathbb{E}^x(X_t \mid \xi)}$$

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where $\mathbb{E}^x(\bullet) = \mathbb{E}(\bullet \mid X_0 = x)$.

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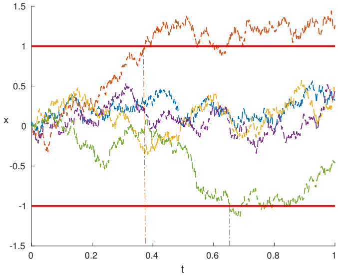

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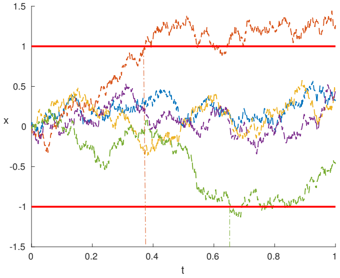

Inputs	Model	Outputs
ξ	$\mathcal{M}_2$	$\mathcal{M}_2(x,\xi) = \mathbb{E}^x(\tau_D \mid \xi)$

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- $y = f(\xi) \in \mathbb{R}$, $\xi = (\xi_1, \dots, \xi_d)^T$ defined on $(\Xi, \mathcal{B}_\Xi, \mathbb{P}_\xi)$, $\xi_i, i = 1, \dots, d$ independent, $f \in L^2(\Xi, \mathbb{P}_\xi)$.

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$$f(\xi) = f_0 + \sum_{i=1}^{d} f_i(\xi_i) + \sum_{1 \le i \le j \le d} f_{i,j}(\xi_i, \xi_j) + \cdots + f_{1,\ldots,d}(\xi_1, \ldots, \xi_d) = \sum_{I \subseteq \{1,\ldots,d\}} f_I(\xi_I),$$

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First order Sobol' indices

$$S_i = \frac{Var\left(\mathbb{E}(y|\xi_i)\right)}{Var(y)}, \ i = 1, \ldots, d$$

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$$\begin{cases} \mathcal{M}_1(t,x,\xi) = \mathbb{E}^x(X_t \mid \xi) \\ S_I\left(\mathcal{M}_1(t,x,\xi)\right) = \dfrac{Var(\mathbb{E}(\mathcal{M}_1 \mid \xi_I))}{Var(\mathcal{M}_1)}, \end{cases}$$

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$\forall I \subseteq \{1,\ldots,d\},\ \xi_I = (\xi_i, i \in I)$.

$$\begin{cases} \mathcal{M}_2(x,\xi) = \mathbb{E}^x(\tau_D \mid \xi) \\ S_I\left(\mathcal{M}_2(x,\xi)\right) = \dfrac{Var(\mathbb{E}(\mathcal{M}_2 \mid \xi_I))}{Var(\mathcal{M}_2)} \end{cases}$$

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Monte Carlo estimator

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Idea : $\xi'_{(-i)}$ independent copy of $\xi_{(-i)}$, $y = f(\xi_i, \xi_{(-i)})$, $y^i = f(\xi_i, \xi'_{(-i)})$, $S_i = \frac{Cov(y, y^i)}{Var(y)}$.

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1. Draw 2 independent samples :

$$A = \begin{pmatrix} \xi^A_{1,1} & \cdots & \xi^A_{d,1} \\ \vdots & \vdots & \vdots \\ \xi^A_{1,n} & \cdots & \xi^A_{d,n} \end{pmatrix} \quad B = \begin{pmatrix} \xi^B_{1,1} & \cdots & \xi^B_{d,1} \\ \vdots & \vdots & \vdots \\ \xi^B_{1,n} & \cdots & \xi^B_{d,n} \end{pmatrix},$$

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3. Compute the $(d+1) \times n$ scenarios via f :

$$y^B = \begin{pmatrix} y^B_1 \\ \vdots \\ y^B_n \end{pmatrix}, \; y^{C_i} = \begin{pmatrix} y^{C_i}_1 \\ \vdots \\ y^{C_i}_n \end{pmatrix}, \; \forall 1 \le i \le d.$$

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4. Compute the estimator (A. Janon et al., 2014) :

$$\hat{S}_i = \frac{\frac{1}{n}\sum_{j=1}^n y^B_j y^{C_i}_j - \left(\frac{1}{n}\sum_{j=1}^n \frac{y^B_j + y^{C_i}_j}{2}\right)^2}{\frac{1}{n}\sum_{j=1}^n \frac{(y^B_j)^2 + (y^{C_i}_j)^2}{2} - \left(\frac{1}{n}\sum_{j=1}^n \frac{y^B_j + y^{C_i}_j}{2}\right)^2}, \; i = 1, \ldots, d$$

Orthogonal polynomials

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$$\langle f, g \rangle := \langle f, g \rangle_{L^2(\Xi, \mathbb{P}_\xi)} = \mathbb{E}(fg) = \int_\Xi f(x) g(x) d\mathbb{P}_\xi(x).$$

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- Orthogonality :

$$\langle \Psi_n, \Psi_m \rangle = h_n \delta_{nm}, \; n, m \in \mathbb{N}$$

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- Recurrence :

$$-x\Psi_n(x) = A_n \Psi_{n+1}(x) - (A_n + C_n)\Psi_n(x) + C_n \Psi_{n-1}(x), \; n \geq 1, \; A_n, C_n \neq 0, \frac{C_n}{A_{n-1}} > 0.$$

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$$\langle \Psi_n, \Psi_m\rangle = h_n\delta_{nm},\ n,m\in\mathbb{N}$$

- Recurrence :

$$-x\Psi_n(x) = A_n\Psi_{n+1}(x) - (A_n + C_n)\Psi_n(x) + C_n\Psi_{n-1}(x),\ n\geq 1,\ A_n, C_n \neq 0,\ \frac{C_n}{A_{n-1}} > 0.$$

Law	support	density function	Polynomials
Uniform	$[-1,1]$	$\frac{1}{2}$	Legendre
Gaussian	$\mathbb{R}$	$\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$	Hermite
Gamma	$\mathbb{R}_+$	$\frac{x^{\gamma-1}e^{-x}}{\Gamma(\gamma)}$	Laguerre
Beta	$[-1,1]$	$\frac{(1+y)^{\alpha-1}(1-y)^{\beta-1}}{2^{\alpha+\beta-1}B(\alpha,\beta)}$	Jacobi

Some standard distributions and classical orthogonal polynomials.

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Orthogonal polynomials

Orthogonal polynomials

Multidimensional orthogonal polynomials

$$\Psi_\alpha(\xi) = \prod_{i=1}^{d} \Psi^{(i)}_{\alpha_i}(\xi_i) \text{ with } \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d,$$

due to the independence of $\xi_i, \ i = 1, \ldots, d$.

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Approximate space of $L^2(\Xi, \mathbb{P}_\xi)$

$$\mathcal{Q}_{d,p} = span\{\xi^\alpha = \prod_{j=1}^{d} \xi_j^{\alpha_j} \mid \alpha \in \mathcal{J}_{d,p}\},\ \mathcal{J}_{d,p} = \{\alpha \in \mathbb{N}^d \mid |\alpha| := \sum_{j=1}^{d} \alpha_j \le p\},\ \dim(\mathcal{Q}_{d,p}) = \binom{d+p}{p}.$$

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$$\Rightarrow \text{ a truncated stochastic basis } \mathcal{S}_P := \{\Psi_k(\xi), k \in K\},\ K = \{0, 1, \dots, P\}$$

via an indexing function

$$\kappa : \begin{cases} \mathcal{J}_{d,p} \to K \\ \quad \alpha \mapsto k = \kappa(\alpha). \end{cases}$$

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Additional hypothesis

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Bounded conditions

$\forall x, \xi,\ \boldsymbol{b}(x, \xi)$ and $\boldsymbol{\sigma}(x, \xi)$ are bounded.

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Additional hypothesis

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$\forall x, \xi,\ \boldsymbol{b}(x,\xi)$ and $\boldsymbol{\sigma}(x,\xi)$ are bounded.

Affine decomposition hypothesis

$$\forall x, \xi,\ \boldsymbol{b}(x,\xi) = \sum_{q=1}^{Q} \tilde{b}_q(x)\tilde{U}_q(\xi),\ \boldsymbol{\sigma}(x,\xi) = \sum_{q'=1}^{Q'} \tilde{\sigma}_{q'}(x)\bar{U}_{q'}(\xi),$$

where $Q, Q' \in \mathbb{N}^*$, $\tilde{b}_q, \tilde{U}_q, \tilde{\sigma}_{q'}, \bar{U}_{q'}, q = 1, \dots, Q, q' = 1, \dots, Q'$ are given functions.

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Stochastic Galerkin projection

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$$\langle \mathcal{A}\left(t,x,\xi;\sum_{l=0}^{P} u_l \Psi_l\right), \Psi_k \rangle = \langle f(t,x,\xi), \Psi_k \rangle, \;\; k = 0,\ldots,P.$$

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A system of $(P+1)$ coupled deterministic differential equations for $u_k, \ k = 0, \ldots, P$.

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Classical PDEs schemes : finite differences, finite elements, spectral approach, etc.

Parabolic case

Feymann-Kac formula

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t,$$

where b and σ are bounded and satisfy the lipschitz condition. Assume moreover $0 < m \le \sigma^2(x),\ \forall x \in \mathbb{R}$.

$$\begin{cases} \partial_t u(t,x) + k(x)u(t,x) = \mathcal{A}u(t,x) + g(x),\ \forall (t,x) \in (0,T] \times \mathbb{R} \\ u(0,x) = f(x),\ \forall x \in \mathbb{R} \\ \lim\limits_{|x|\to\infty} |u(t,x)| = 0,\ \forall t \in [0,T], \end{cases}$$

where $f, g, k : \mathbb{R} \to \mathbb{R}$ are continuous bounded, $f \in L^2(\mathbb{R})$ s.t. $\lim_{|x|\to\infty} |f(x)| = 0$, the operator $\mathcal{A}$ acting on functions in $\varphi \in \mathcal{C}^2(\mathbb{R})$ by

$$(\mathcal{A}\varphi)(t,x) = \frac{1}{2}\sigma^2(x)\partial_{xx}\varphi(t,x) + b(x)\partial_x\varphi(t,x),\ \forall (t,x) \in (0,T] \times \mathbb{R}.$$

Then $u(t,x) \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}; \mathbb{R})$ and satisfies

$$u(t,x) = \mathbb{E}^x \left[f(X_t) \exp\left(-\int_0^t k(X_s)ds \right) + \int_0^t g(X_s) \exp\left(-\int_0^s k(X_r)dr \right) ds \right].$$

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Parametrized parabolic PDE

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Setting $k = g \equiv 0$ and $f(x) = x, \ \forall x \in \mathbb{R}$,

$$u(t,x,\xi) = \mathbb{E}^x(X_t \mid \xi) = \mathcal{M}_1(t,x,\xi)$$

is the solution of the parametrized PDE,

$$\begin{cases} \partial_t u(t,x,\xi) = \dfrac{1}{2}\boldsymbol{\sigma}^2(x,\xi)\partial^2_{xx}u(t,x,\xi) + \boldsymbol{b}(x,\xi)\partial_x u(t,x,\xi), \ \forall (t,x) \in (0,T] \times \mathbb{R} \\ u(0,x,\xi) = x, \ \forall x \in \mathbb{R}. \end{cases}$$

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In practice, we work in a bounded open domain $D = (l,r) \subset \mathbb{R}$,

$$\begin{cases} \partial_t u(t,x,\xi) = \dfrac{1}{2}\boldsymbol{\sigma}^2(x,\xi)\partial^2_{xx}u(t,x,\xi) + \boldsymbol{b}(x,\xi)\partial_x u(t,x,\xi), \ \forall (t,x) \in (0,T] \times D \\ u(0,x,\xi) = x, \ \forall x \in \mathbb{R}. \\ u(t,l,\xi) = \mathbb{E}^{x=l}(X_t \mid \xi), \ \forall t \in (0,T] \\ u(t,r,\xi) = \mathbb{E}^{x=r}(X_t \mid \xi), \ \forall t \in (0,T] \end{cases}$$

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PC expansion and Sobol' SA

PC expansion and Sobol' SA

PC expansion of $u(t, x, \xi)$ in the appoximate space $\mathcal{S}_P$:

$$u(t, x, \xi) \approx \sum_{k=0}^{P} u_k(t, x) \Psi_k(\xi).$$

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$$\mathbb{E}(u(t,x,\xi)) \approx \mathbb{E}\left(\sum_{k=0}^{P} u_k(t,x)\Psi_k(\xi)\right) = \sum_{k=0}^{P} u_k(t,x)\mathbb{E}(\Psi_k(\xi)) = u_0(t,x),$$

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where $I \subseteq \{1,\dots,d\}$ and $K_I := \{k \in \{1,\dots,P\} \mid \Psi_k(z) = \Psi_k(z = \xi_I)\}$,

$$Var\left[\mathbb{E}\left(u(t,x,\xi) \mid \xi = \xi_I\right)\right] \approx \sum_{k' \in K_I} u_{k'}^2(t,x).$$

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$$S_I\left(\mathcal{M}_1(t,x,\xi)\right) = \frac{Var[\mathbb{E}(\mathcal{M}_1 \mid \xi_I)]}{Var(\mathcal{M}_1)} \approx \frac{\sum_{k' \in K_I} u_{k'}^2(t,x)}{\sum_{k=1}^{P} u_k^2(t,x)}.$$

Elliptic case

Feymann-Kac formula

$$\begin{cases} \mathcal{A}(x)u(x) - k(x)u(x) = -g(x), \ \forall x \in D \\ u(x) = f(x), \ \forall x \in \partial D, \end{cases}$$

where $\mathcal{A}$ is an elliptic operator of type

$$\mathcal{A}(x)\varphi(x) = \frac{1}{2}\sigma^2(x)\varphi''(x) + b(x)\varphi'(x), \ \forall \varphi \in \mathcal{C}^2(D;\mathbb{R}), \ \forall x \in D,$$

and b, σ are bounded s.t.

$$0 < m \leq \sigma^2(x) \leq M < \infty, \ |(\sigma^2)'(x)| \leq M, \ |b'(x)| \leq M, \ \forall x \in D.$$

Assuming also $k \geq 0, g \in \mathcal{C}^2(\bar{D};\mathbb{R})$. Then $u \in \mathcal{C}^2(\bar{D};\mathbb{R})$ and satisfies

$$u(x) = \mathbb{E}^x\left[f(X_{\tau_D})\exp\left(-\int_0^{\tau_D} k(X_r)dr\right) + \int_0^{\tau_D} g(X_s)\exp\left(-\int_0^s k(X_r)dr\right)ds\right].$$

Parametrized parabolic PDE, PC expansion and Sobol' SA

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Setting $f = k \equiv 0$ and $g \equiv 1$,

$$u(x,\xi) = \mathbb{E}^x[\tau_D \mid \xi] = \mathcal{M}_2(x,\xi)$$

becomes the solution of the parametrized PDE,

$$\begin{cases} -\dfrac{1}{2}\boldsymbol{\sigma}^2(x,\xi)u''(x,\xi) - \boldsymbol{b}(x,\xi)u'(x,\xi) = 1, \ \forall x \in D \\ \hfill u(x,\xi) = 0, \ \forall x \in \partial D. \end{cases}$$

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Numerical exemple : parametrized OU process

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Propriities of the OU process

$$dX_t = -\alpha X_t dt + \beta dW_t,\ t > 0 \text{ with } X_0 = x,$$

where $\alpha, x \in \mathbb{R}$, and $\beta \in \mathbb{R}_+$.

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- explicit solution :

$$X_t = xe^{\alpha t} + \beta \int_0^t e^{-\alpha(t-s)} dW_s.$$

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- exact scheme :

$$X_{i+1} = X_i e^{-\alpha \Delta t} + \beta \sqrt{\frac{1 - e^{-2\alpha \Delta t}}{2b}} \zeta_i,\ \zeta \sim \mathcal{N}(0,1) \text{ i.i.d.},\ i = 0, \dots, N-1.$$

Numerical exemple : parametrized OU process

$$dX_t = -\alpha(\xi_1)X_t dt + \beta(\xi_2)dW_t,\ t > 0 \text{ with } X_0 = x \in \mathbb{R},$$

where $\xi_i \sim \mathcal{U}([0,1])$ i.i.d., $i = 1, 2$, α, β characterized by their mean $\mu_i \in \mathbb{R}$ and standard deviation $\sigma_i \in \mathbb{R}$.

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Numerical exemple : parametrized OU process

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where $\xi_i \sim \mathcal{U}([0,1])$ i.i.d., $i = 1, 2$, α, β characterized by their mean $\mu_i \in \mathbb{R}$ and standard deviation $\sigma_i \in \mathbb{R}$.

$$\begin{cases} \alpha(\xi_1) = \mu_1 + \sqrt{3}\sigma_1(2\xi_1 - 1) \sim \mathcal{U}\left([\mu_1 - \sqrt{3}\sigma_1, \mu_1 + \sqrt{3}\sigma_1]\right) \\ \beta(\xi_2) = \mu_2 + \sqrt{3}\sigma_2(2\xi_2 - 1) \sim \mathcal{U}\left([\mu_2 - \sqrt{3}\sigma_2, \mu_2 + \sqrt{3}\sigma_2]\right) \end{cases}$$

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Numerical exemple : parametrized OU process

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Parabolic case

Parabolic case

Model and Sobol' indices

$$\begin{cases} \mathcal{M}_1(t,x,\xi) = \mathbb{E}^x\left(X_t \mid \xi\right) \\ S_i\left(\mathcal{M}_1(t,x,\xi)\right) = \dfrac{Var(\mathbb{E}(\mathcal{M}_1 \mid \xi_i))}{Var(\mathcal{M}_1)} \\ S_{12}\left(\mathcal{M}_1(t,x,\xi)\right) = 1 - \displaystyle\sum_{i=1}^{2} S_i\left(\mathcal{M}_1(t,x,\xi)\right) \end{cases}$$

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Explicit solutions

$$E_2 := \mathbb{E}\left[\left(\mathbb{E}^x(X_t \mid \xi)\right)^2\right] = x^2 \mathbb{E}\left(e^{-2\alpha(\xi_1)t}\right) = x^2 e^{-2\mu_1 t} \frac{\sinh(2\sqrt{3}\sigma_1 t)}{2\sqrt{3}\sigma_1 t},$$

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Parabolic case

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Parabolic case

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Explicit solutions

$$E_2 := \mathbb{E}\left[\left(\mathbb{E}^x(X_t \mid \xi)\right)^2\right] = x^2\mathbb{E}\left(e^{-2\alpha(\xi_1)t}\right) = x^2 e^{-2\mu_1 t}\frac{\sinh(2\sqrt{3}\sigma_1 t)}{2\sqrt{3}\sigma_1 t},$$

$$Var(\mathcal{M}_1) = Var\left(\mathbb{E}^x(X_t \mid \xi)\right) = E_2 - E_1^2,$$

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Parabolic case

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$S_1(\mathcal{M}_1(t,x,\xi)) = 1,\ S_2(\mathcal{M}_1(t,x,\xi)) = 0$ and $S_{12}(\mathcal{M}_1(t,x,\xi)) = 0.$

Parametrized parabolic PDE

- Feymann-Kac formula : $u(t,x,\xi) = \mathbb{E}^x[X_t(\omega,\xi) \mid \xi]$ is the solution of

$$(\mathcal{P}_u)\begin{cases} \partial_t u(t,x,\xi) = -\alpha(\xi_1)x\partial_x u(t,x,\xi) + \frac{1}{2}\beta(\xi_2)^2\partial^2_{x,x}u(t,x,\xi), \ \forall (t,x) \in (0,T] \times (0,1) \\ u(0,x,\xi) = x, \ \forall x \in (0,1) \\ u(t,0,\xi) = \mathbb{E}^{x=0}_\omega[X_t \mid \xi] = 0, \ \forall t \in (0,T] \\ u(t,1,\xi) = \mathbb{E}^{x=1}_\omega[X_t \mid \xi] = e^{-\alpha(\xi_1)t}, \ \forall t \in (0,T]. \end{cases}$$

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- Stochastic Garlerkin projection in $\mathcal{S}_P$:

$$(\mathcal{P}_U)\begin{cases} \partial_t U(t,x) = -\boldsymbol{A}x\partial_x U(t,x) + \boldsymbol{B}\partial^2_{x,x}U(t,x), \ \forall (t,x) \in (0,T]\times(0,1) \\ U(0,x) = (\langle x,\Psi_0\rangle,\dots,\langle x,\Psi_P\rangle)^T = (x,0,\dots,0)^T \in \mathbb{R}^{P+1}, \ \forall x \in (0,1) \\ U(t,0) = \mathbf{0}^{P+1}, \ \forall t \in (0,T] \\ U(t,1) = \left(\langle e^{-\alpha t},\Psi_0\rangle,\dots,\langle e^{-\alpha t},\Psi_P\rangle\right)^T \in \mathbb{R}^{P+1}, \ \forall t \in (0,T], \end{cases}$$

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where $U = (u_0,\dots,u_P)^T$, $\boldsymbol{A} = (a_{ij})$, $\boldsymbol{B} = (b_{ij}) \in \mathcal{M}_{P+1}(\mathbb{R})$ s.t.

$$\begin{cases} a_{ij} = \mu_1\delta_{ij} + \sigma_1\langle\Psi_1\Psi_i,\Psi_j\rangle \\ b_{ij} = \frac{1}{2}\mu_2^2\delta_{ij} + \mu_2\sigma_2\langle\Psi_2\Psi_i,\Psi_j\rangle + \frac{1}{2}\sigma_2^2\langle\Psi_2^2\Psi_i,\Psi_j\rangle \end{cases}$$

Crank-Nicholson scheme

Consider $0 = t_0 < t_1 < \cdots < t_N = T, \Delta t = \frac{T}{N}, 0 = x_0 < x_1 < \cdots < x_M = 1, \Delta x = \frac{1}{M}$ and $U(t_i, x_j) \approx U^{i,j}, \forall (t_i, x_j)$.

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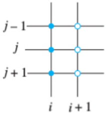

$$\begin{aligned}\frac{U^{i+1,j} - U^{i,j}}{\Delta t} &= \frac{1}{2}\left(-\boldsymbol{A}x_j \frac{U^{i,j+1} - U^{i,j-1}}{2\Delta x} + \boldsymbol{B}\frac{U^{i,j-1} - 2U^{i,j} + U^{i,j+1}}{(\Delta x)^2}\right) \cdots \\ &+ \frac{1}{2}\left(-\boldsymbol{A}x_j \frac{U^{i+1,j+1} - U^{i+1,j-1}}{2\Delta x} + \boldsymbol{B}\frac{U^{i+1,j-1} - 2U^{i+1,j} + U^{i+1,j+1}}{(\Delta x)^2}\right),\end{aligned}$$

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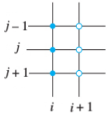

$$\frac{U^{i+1,j} - U^{i,j}}{\Delta t} = \frac{1}{2}\left(-\boldsymbol{A}x_j \frac{U^{i,j+1} - U^{i,j-1}}{2\Delta x} + \boldsymbol{B}\frac{U^{i,j-1} - 2U^{i,j} + U^{i,j+1}}{(\Delta x)^2}\right) \cdots$$
$$+ \frac{1}{2}\left(-\boldsymbol{A}x_j \frac{U^{i+1,j+1} - U^{i+1,j-1}}{2\Delta x} + \boldsymbol{B}\frac{U^{i+1,j-1} - 2U^{i+1,j} + U^{i+1,j+1}}{(\Delta x)^2}\right),$$

$$\boldsymbol{L}U^{i+1} = \boldsymbol{R}U^i + (F^{i+1} + F^i), \; i = 0, \dots, N-1$$

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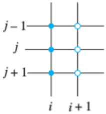
Consider $0 = t_0 < t_1 < \cdots < t_N = T, \Delta t = \frac{T}{N}, 0 = x_0 < x_1 < \cdots < x_M = 1, \Delta x = \frac{1}{M}$ and $U(t_i, x_j) \approx U^{i,j}, \forall (t_i, x_j)$.



$$\begin{aligned}\frac{U^{i+1,j} - U^{i,j}}{\Delta t} = \frac{1}{2}\left(-\boldsymbol{A}x_j \frac{U^{i,j+1} - U^{i,j-1}}{2\Delta x} + \boldsymbol{B}\frac{U^{i,j-1} - 2U^{i,j} + U^{i,j+1}}{(\Delta x)^2}\right) \ldots \\ + \frac{1}{2}\left(-\boldsymbol{A}x_j \frac{U^{i+1,j+1} - U^{i+1,j-1}}{2\Delta x} + \boldsymbol{B}\frac{U^{i+1,j-1} - 2U^{i+1,j} + U^{i+1,j+1}}{(\Delta x)^2}\right),\end{aligned}$$

$$\boldsymbol{L}U^{i+1} = \boldsymbol{R}U^{i} + (F^{i+1} + F^{i}),\ i = 0, \ldots, N-1$$

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Avantage : unconditionally stable in L^2 norm and second order both in time and in space.

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Numerical results

$\mu_1 = 1, \mu_2 = 0.1, \sigma_1 = \sigma_2 = 0.05, t \in [0, 10], x \in [0, 1], \Delta t = \Delta x = 0.01.$

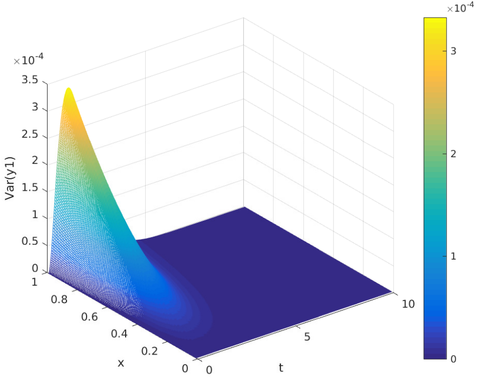

Figure: $Var\left(\mathcal{M}_1(t, x, \xi)\right)$ estimated by PC PDE for $P = 9$

Numerical results

$$X_0 = x = 0.99, n = 1000, m = 10000, P = 9.$$

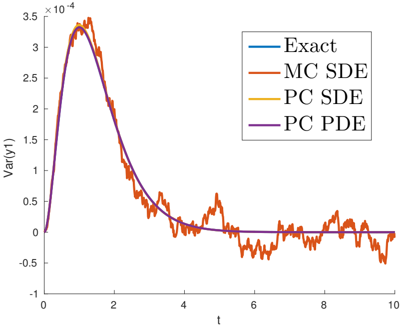

Figure: Evolution of $Var(\mathcal{M}_1)$ obtained by various methods

Numerical results

$$\sigma_{y_1}^{T,x} := Var(\mathcal{M}_1(T,x,\xi)), \; x = 0.99$$

$$\varepsilon_{\text{SDE}}^{T,x}(P) = |\hat{\sigma}_{\text{SDE}}^{T,x}(P) - \sigma_{y_1}^{T,x}|, \text{ and } \varepsilon_{\text{PDE}}^{T,x}(P) = |\hat{\sigma}_{\text{PDE}}^{T,x}(P) - \sigma_{y_1}^{T,x}|.$$

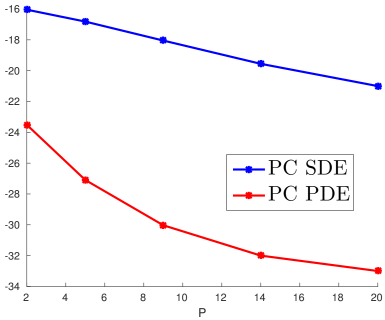

Figure: $\log\left(\varepsilon_{\text{SDE}}^{T,x}(P)\right)$ and $\log\left(\varepsilon_{\text{PDE}}^{T,x}(P)\right)$

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Numerical results

$$\varepsilon_1^{T,x} = |1 - \hat{s}_1^{T,x}| \text{ and } \varepsilon_2^{T,x} = |\hat{s}_2^{T,x}|.$$

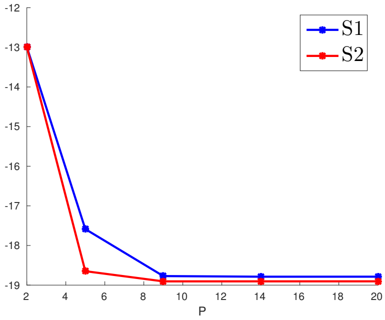

Figure: $\log\left(\varepsilon_1^{T,x}\right)$ and $\log\left(\varepsilon_2^{T,x}\right)$

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Elliptic case

Model and Sobol' indices

$$\begin{cases} \mathcal{M}_2(x,\xi) = \mathbb{E}^x\left(\tau_D \mid \xi\right) \\ S_i\left(\mathcal{M}_2(x,\xi)\right) = \dfrac{Var(\mathbb{E}(\mathcal{M}_2 \mid \xi_i))}{Var(\mathcal{M}_2)} \\ S_{12}\left(\mathcal{M}_2(x,\xi)\right) = 1 - \sum\limits_{i=1}^{2} S_i\left(\mathcal{M}_2(x,\xi)\right) \end{cases}$$

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S_i\left(\mathcal{M}_2(x,\xi)\right) = \dfrac{Var(\mathbb{E}(\mathcal{M}_2 \mid \xi_i))}{Var(\mathcal{M}_2)} \\
S_{12}\left(\mathcal{M}_2(x,\xi)\right) = 1 - \sum_{i=1}^{2} S_i\left(\mathcal{M}_2(x,\xi)\right)
\end{cases}
$$

Feymann-Kac formula

$$
u(x,\xi) = \mathbb{E}^x[\tau_D \mid \xi] = \mathcal{M}_2(x,\xi)
$$

is the solution of the parametrized PDE,

$$
\begin{cases}
\alpha(\xi_1) x u'(x,\xi) - \dfrac{\beta(\xi_2)^2}{2} u''(x,\xi) = 1, \ \forall x \in D \\
u(x,\xi) = 0, \ \forall x \in \partial D.
\end{cases}
$$

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$$n = 10000, P = 9.$$

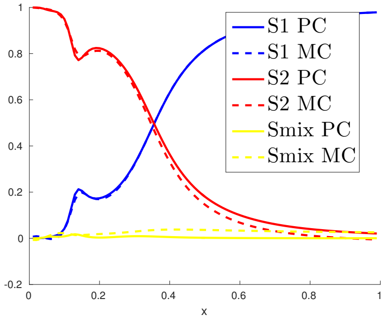

Figure: Sobol' indices of the mean exit time from the domain $[0, 1]$ of the OU process, respectively explained by ξ_1, ξ_2 and their interaction

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Thanks for your attentions !