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**DERIVATIONS OF LIE ALGEBRA EXTENSIONS AND  
NON-SINGULAR DERIVATIONS OF LIE ALGEBRAS IN  
PRIME CHARACTERISTIC**

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# DERIVATIONS OF LIE ALGEBRA EXTENSIONS AND NON-SINGULAR DERIVATIONS OF LIE ALGEBRAS IN PRIME CHARACTERISTIC

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## 1. INTRODUCTION

Let  $L$  be a Lie algebra and  $d$  be a derivation of  $L$ . The derivation  $d$  is non-singular if it is injective as linear transformation. We are interested in studying what information we can obtain about a Lie algebra if it has a nonsingular derivation. Jacobson's famous theorem [7] states that a finite-dimensional Lie algebra over a field of characteristic zero that admits a non-singular derivation must be nilpotent. It is well-known that this theorem is not valid when the characteristic is non-zero. Non-nilpotent and solvable examples were constructed by Shalev [12] and Mattarei [10], whereas the simple Lie algebras with non-singular derivations were classified by Benkart and her collaborators in [4]. A significant application of Lie algebras with non-singular derivation in characteristic  $p$  was presented by Shalev [11]. In his proof of the coclass conjectures of Leddham-Green and Newman for pro- $p$  groups, Shalev uses the fact that finite-dimensional Lie algebras over a field of characteristic  $p > 0$  with non-singular derivation  $d$  such that  $d^{p-1} = 1$ , must be nilpotent.

Despite the existing examples, little is known about non-nilpotent Lie algebras with non-singular derivations. In this project we propose to explore the structure of solvable, non-nilpotent Lie algebras with non-singular derivations. In order to study these algebras we develop a theory of derivations of Lie algebra extensions. We adopt the concept of a compatible pair of automorphisms introduced in [3] for derivations of Lie algebras.

Let  $K$  and  $I$  be Lie algebras such that  $K$  acts on  $I$ , then we can define the subalgebra  $\text{Comp}(K, I)$  of *compatible pairs* of  $\text{Der}(K) \oplus \text{Der}(I)$  as the set of derivations of  $\text{Der}(K) \oplus \text{Der}(I)$  that are derivations of the semi-direct sum  $K \oplus I$ . Formally,

$$\text{Comp}(K, I) = \{\alpha + \beta \in \mathfrak{gl}(K) \oplus \mathfrak{gl}(I) \mid \alpha + \beta \in \text{Der}(K \oplus I)\}.$$

The algebra  $\text{Der}(K)$  carries information about the multiplicative structure of  $K$ . Analogously, the algebra  $\text{Comp}(K, I)$  carries information about the action of  $K$  on  $I$ . In Section 3.4 we present an example of this by exploring the proof of Jacobson's Theorem and we prove a version for Lie algebra representations over a field of characteristic  $p > 0$ .

**Theorem 3.14** *Let  $K$  and  $I$  be finite dimensional Lie algebras over a field of characteristic  $p$  where  $p \geq 0$  such that  $K$  is nilpotent. Suppose that  $K$  act on  $I$  by representation  $\psi : K \rightarrow \text{Der}(I)$ . Let  $(\alpha, \beta) \in \text{Comp}(K, I)$  such that  $\alpha$  has no eigenvalue 0. If either  $p = 0$*

or  $p > 0$  and  $\dim I < p$  then  $\text{Tr}(\psi^n(k)) = 0$ , for all  $k \in K$  and  $n > 0$ . In these two cases,  $\psi(k)$  is nilpotent for all  $k \in K$ .

We also adapt an algorithm presented by Bettina Eick [3] for calculating the automorphism group of solvable Lie algebras. A key step in the algorithm is the following. Let  $L$  be a Lie algebra and  $I$  an abelian ideal of  $L$  such that  $I$  is invariant under  $\text{Aut}(L)$ . Then there exists a homomorphism  $\phi : \text{Aut}(L) \rightarrow \text{Aut}(L/I) \times \text{Aut}(I)$  induced by the actions of  $\text{Aut}(L)$  on  $L/I$  and  $I$ . The image of  $\phi$  can be calculated using  $\text{Aut}(L/I)$ , while  $\text{Ker}(\phi)$  is equal to  $Z^1(K, I)$ . Then the group  $\text{Aut}(L)$  can be obtained applying the first isomorphism theorem to  $\phi$ . It is possible to use this process to derivations.

We can define a Lie algebra homomorphism similar to  $\psi$  in the previous paragraph. Let  $L$  be a Lie algebra and  $I \trianglelefteq L$  an ideal such that  $I$  is invariant under  $\text{Der}(L)$ . Then if  $d \in \text{Der}(L)$ ,  $d$  induces derivations  $\alpha$  and  $\beta$  of  $L/I$  and  $I$ , respectively. Hence we obtain a Lie algebra homomorphism

$$\psi : \text{Der}(L) \rightarrow \text{Der}(L/I) \oplus \text{Der}(I).$$

Let  $K$  be a Lie algebra and  $I$  be a  $K$ -module. Let  $Z^2(K, I)$  be the vector space of cocycles and  $\text{Comp}(K, I)$  the Lie algebra of compatible pairs. Let  $(\alpha, \beta) \in \text{Comp}(K, I)$  and  $\vartheta \in Z^2(K, I)$ . Define an action of  $\text{Comp}(K, I)$  on  $Z^2(K, I)$  by

$$(\alpha, \beta) \cdot \vartheta(h, k) = \beta(\vartheta(h, k)) - \vartheta(\alpha(h), k) - \vartheta(h, \alpha(k)), \quad \text{for all } h, k \in K.$$

The elements of the annihilator of  $\vartheta$  under this action will be called *induced pairs* and we denote the set of induced pairs by  $\text{Indu}(K, I, \vartheta)$ . Let  $\vartheta \in Z^2(K, I)$  a cocycle and  $K_\vartheta$  be the Lie algebra extension obtained from  $K$  by  $\vartheta$ . Then we can lift the derivation of  $\text{Indu}(K, I, \vartheta)$  to  $\text{Der}(K_\vartheta)$ . Thus we obtained the following theorem.

**Theorem 3.8** *Let  $K$  be a Lie algebra and  $I$  a  $K$ -module. Let  $\vartheta \in H^2(K, I)$  and suppose that  $I$ , as an ideal of  $K_\vartheta$ , is invariant under derivations of  $K_\vartheta$ . Let  $\phi : \text{Der}(K_\vartheta) \rightarrow \text{Der}(K) \oplus \text{Der}(I)$  the natural homomorphism. Then:*

- (1)  $\text{Im}(\phi) = \text{Indu}(K, I, \vartheta)$
- (2)  $\text{Ker}(\phi) \cong Z^1(K, I)$ .

The details of this construction can be seen in Section 3. There is a significant difference between the application of this approach to automorphisms and to derivations: calculating the automorphism groups of Lie algebras is usually a difficult task that may involve a large orbit-stabilizer calculation, while calculating the algebra  $\text{Der}(K_\vartheta)$  can be done by solving a system of linear equations. Thus, to understand the importance of Theorem 3.8 we must discover what additional information of  $\text{Der}(K_\vartheta)$  we are able to obtain through information concerning the algebras  $\text{Der}(K)$  and  $\text{Der}(I)$ .

In order facilitate the reading of the text and the references, we added a section with results on the primary decomposition of vector spaces in relation to subalgebras of linear operators and a brief description of the main articles used.

This text is organized as follows: Section 2 is dedicated to literature review. In Section 3, we present compatible pairs and the lifting process of derivations of a Lie algebra  $K$  to the Lie algebras  $K_\vartheta$  such that  $\vartheta$  is a cocycle. We end this section by applying the compatible pairs to Jacobson's Theorem. Section 4 is composed of some examples and conjectures about modular solvable non-nilpotent Lie algebras with non-singular derivations.

## 2. NON-SINGULAR DERIVATIONS: KNOWN RESULTS

This section is composed by description of a decomposition of a Lie algebra  $L$  relative to a subalgebra  $K$  of  $\mathfrak{gl}(L)$  and its application in Jacobson's Theorem. Next, we have the calculations presented in Shalev's article [12] about conditions on the order of derivation which guarantee nilpotency of a Lie algebra. The section ends with Mattarei's Theorem that relates the order of non-singular derivations of solvable modular Lie algebras to roots of certain types of polynomials.

**2.1. Basic concepts.** The symbol ' $\oplus$ ' will be used to denote the direct sum of algebras, while the direct sum of vector spaces will be denoted by ' $\dot{+}$ '.

Let  $V$  be a finite-dimensional vector space over a field  $\mathbb{F}$  and  $a \in \text{End}(V)$ . Let  $p \in \mathbb{F}[X]$  be a univariate polynomial and define

$$V_0(p(a)) = \{v \in V \mid \text{there is an } m > 0 \text{ such that } p(a)^m v = 0\}.$$

$V_0(p(a))$  is a vector subspace of  $V$  invariant under  $a$ . Now let  $A$  be the associative subalgebra of  $\text{End}(V)$  with 1 generated by  $a$ . Let  $p_a$  be the minimum polynomial of  $a$  and suppose that

$$p_a = p_1^{k_1} \cdots p_r^{k_r}$$

is the factorization of  $p_a$  into irreducible factors, such that  $p_i$  has leading coefficient 1 and  $p_i \neq p_j$  for  $1 \leq i, j \leq r$ . Then  $V$  decomposes as a direct sum of subspaces

$$V = V_0(p_1(a)) \dot{+} \cdots \dot{+} V_0(p_r(a)),$$

each space  $V_0(p_i(a))$  being invariant under  $A$ . Furthermore, the minimum polynomial of the restriction of  $a$  to  $V_0(p_i(a))$  is  $p_i^{k_i}$ . A proof of this result can be found in [2] Lemma A.2.2.

We can generalize this decomposition to subalgebras of  $\mathfrak{gl}(V)$  generated by more than one element. Let  $K$  be a subalgebra of  $\mathfrak{gl}(V)$ . A decomposition  $V = V_1 \dot{+} \cdots \dot{+} V_s$  of  $V$  into  $K$ -modules  $V_i$  is said to be *primary* if the minimum polynomial of the restriction of  $a$  to  $V_i$  is a power of an irreducible polynomial for all  $a \in K$  and  $1 \leq i \leq s$ . The subspaces  $V_i$  are called *primary components*. If for any two components  $V_i$  and  $V_j$  ( $i \neq j$ ), there is an  $x \in K$  such that the minimum polynomials of the restrictions of  $x$  to  $V_i$  and  $V_j$  are powers of different irreducible polynomial, then the decomposition is called *collected*. In general  $V$  will not have a primary (or primary collected) decomposition into  $K$ -modules but such a decomposition is guaranteed to exist if the base field of  $V$  is algebraically closed and  $K \leq \mathfrak{gl}(V)$  is nilpotent.

**Proposition 2.1** ([2], Theorem 3.1.10). *Let  $V$  be finite-dimensional vector space. Let  $K \leq \mathfrak{gl}(V)$  be a nilpotent subalgebra. Then  $V$  has a unique collected primary decomposition relative to  $K$*

If the vector space  $V$  has a collected primary decomposition  $V = V_1 \dot{+} \cdots \dot{+} V_s$  then we can characterize the components  $V_i$ . For  $x \in K$  and  $1 \leq i \leq s$  define  $p_{x,i}$  to be the irreducible polynomial such that the minimum polynomial of  $x$  restricted to  $V_i$  is a power of  $p_{x,i}$ . Then we obtain the equality

$$V_i = \{v \in V \mid \text{for all } x \in K \text{ there is an } m > 0 \text{ such that } p_{x,i}(x)^m v = 0\}.$$

It is worth noting that if the base field of  $V$  is algebraically closed, then all irreducible polynomials are of the form  $p(X) = (X - \lambda)$ , for some  $\lambda \in \mathbb{F}$ , and hence  $p_{x,i} = (X - \lambda_i(x))$ ,  $\lambda_i \in \mathbb{F}^*$ . Further, in this case, primary components are of the form

$$V_i = \{v \in V \mid \text{for all } x \in K \text{ there is an } m > 0 \text{ such that } (x - \lambda_i(x)I)^m v = 0\},$$

with  $\lambda_i \in K^*$ . Its natural to give a name for this case. Let  $V$  be a finite-dimensional vector space over a field  $\mathbb{F}$  and  $K \leq \mathfrak{gl}(V)$  a subalgebra. Let  $\lambda \in K^*$ . Then

$$V_\lambda = \{v \in V \mid \text{for all } x \in K \text{ there is an } m > 0 \text{ such that } (x - \lambda(x)I)^m v = 0\}.$$

If  $V_\lambda \neq 0$  then  $V_\lambda$  is called a *generalized eigenspace* of  $V$  associated to the *generalized eigenvalue*  $\lambda \in K^*$ .

Now we consider a Lie algebra  $L$  and a nilpotent subalgebra  $K \leq \text{Der}(L)$ . Then the decomposition to generalized eigenspaces of  $D$  can provide us some information of the multiplicative structure of  $L$ .

**Proposition 2.2** ([8], Proposition 5 of Chapter III). *Let  $L$  be a Lie algebra over an algebraically closed field. Let  $K$  be a subalgebra of  $\text{Der}(L)$ . If  $\lambda, \mu : K \rightarrow \mathbb{F}^*$  are generalized eigenvalues of  $K$  then  $[L_\lambda, L_\mu] \subseteq L_{\lambda+\mu}$  if  $\lambda + \mu$  is a generalized eigenvalue of  $K$ . Otherwise  $[L_\mu, L_\lambda] = 0$ .*

Following we present some general results about Lie algebras that will be used in the this text.

**Proposition 2.3.** *Let  $L$  be a Lie algebra, let  $I$  be an ideal of  $L$  such that  $L/I$  is nilpotent and such that  $\text{ad}_x^I : I \rightarrow I$  is nilpotent for all  $x \in L$ . Then  $L$  is nilpotent.*

*Proof.* As  $L/I$  is nilpotent then for each  $x \in L$ ,  $(\text{ad}_{x+I})^n$  is a nilpotent endomorphism in  $\text{End}(L/I)$ , i.e., there is  $n > 0$  such that  $(\text{ad}_x)^n(a) \in I$ , for all  $x \in L, a \in I$ . On the other hand,  $\text{ad}_x^I$  is nilpotent, so we have a  $m$  such that  $(\text{ad}_x^I)^m(\text{ad}_x)^n = 0$ , i.e.,  $(\text{ad}_x)^{m+n} = 0$ . So  $\text{ad}_x$  is a nilpotent endomorphism in  $\mathfrak{gl}(L)$ . By Engel's Theorem,  $L$  is nilpotent.  $\square$

**Theorem 2.4** ([5], Theorem 4.1). *Let  $L$  be a solvable subalgebra of  $\mathfrak{gl}(V)$ ,  $V$  finite-dimensional. If  $V \neq 0$ , then  $V$  contains a common eigenvector for all the endomorphism in  $L$ .*

**Theorem 2.5** ([5], Corollary A of Theorem 4.1). *(Lie) Let  $L$  be a finite-dimensional solvable Lie algebra over an algebraically closed field  $\mathbb{F}$  of characteristic 0. Let  $\psi : L \rightarrow \mathfrak{gl}(V)$  be a finite-dimensional representation of  $L$ . Then there is a basis of  $V$  relative to which the matrix of all  $\psi(x)$  for all  $x \in L$  are all upper triangular.*

The Theorems 2.4 and 2.5 still valid in characteristic  $p > 0$  if  $\dim V < p$ . The proof of Theorem 2.4 in prime characteristic goes through as in the Humphreys' book except for the last sentence. In the book's version, we have  $n\lambda([x, y]) = 0$  and conclude  $\lambda([x, y]) = 0$  because the characteristic of  $\mathbb{F}$  is 0. In this case, since  $p > \dim V = n$ , we can still make the same conclusion since  $n$  will not be a zero divisor. The proof of Theorem 2.5 goes through exactly as in the book. For future reference in the text we will report a version of the Lie's Theorem in positive characteristic.

**Theorem 2.6.** *Let  $L$  be a finite-dimensional solvable Lie algebra over an algebraically closed field  $\mathbb{F}$  of characteristic  $p > 0$ . Let  $V$  be a finite-dimensional vector space of dimension  $n < p$ . Let  $\psi : L \rightarrow \mathfrak{gl}(V)$  be a finite-dimensional representation of  $L$ . Then there is a basis of  $V$  relative to which the matrix of all  $\psi(x)$  for all  $x \in L$  are all upper triangular.*

**2.2. Jacobson's Theorem.** In the article *A note on automorphism and derivations of Lie algebras* [7], Jacobson used a variation of Engel's Theorem for weakly closed sets to get sufficient conditions for a Lie algebra to be nilpotent. We recommend the reading of Sections 1 and 2 of Chapter 2 of Jacobson's book [8] as reference for examples and proofs.

Suppose that  $K$  and  $I$  are Lie algebras and  $\psi : K \rightarrow \text{Der}(I)$  is a given Lie algebra homomorphism. Then we say that  $K$  acts on  $I$  or that  $I$  is a  $K$ -module. In this case, the image  $\psi(k)(a)$  of  $a \in I$  under  $k \in K$  will be written simply as  $[k, a]$ . If  $I$  is an ideal of a Lie algebra  $K$ , then  $K$  acts on  $I$ . If  $k \in K$ , then the image of  $k$  under this action will be denoted by  $\text{ad}_k^I$  or simply by  $\text{ad}_k$  when the domain of the representation is clear from the context. Thus, for  $a \in I$  and for  $k \in K$ ,  $\text{ad}_k^I(a) = \text{ad}_k(a) = [k, a]$ . The homomorphism  $K \rightarrow \text{Der}(I)$  that takes  $k \mapsto \text{ad}_k^I$ , will be denoted by  $\text{ad}^I$ .

**Example 2.7.** Let  $L$  be a Lie algebra with an abelian ideal  $I$  and set  $K = L/I$ . Define the Lie algebra representation  $\text{ad}^I : K \rightarrow \text{Der}(I)$  by  $\text{ad}_{x+I}^I(a) = [x, a]$  for all  $x \in L$  and  $a \in I$ . This is well defined, since  $I$  is abelian. Then  $I$  is a  $K$ -module. In this case, we say that the action is *induced by the adjoint representation*.

Let  $A$  be an associative algebra with 1 over a field  $\mathbb{F}$ . A subset  $S$  of  $A$  is called *weakly closed* if for every ordered pair  $(a, b) \in S \times S$ , there is an element  $\gamma(a, b) \in \mathbb{F}$  such that  $ab + \gamma(a, b)ba \in S$ . If  $S$  is a subset of an Lie or associative algebra  $X$ , then  $\langle S \rangle$  denotes the Lie or associative, respectively, subalgebra of  $X$  generated by  $S$ . In the case of associative algebras we assume that  $1 \in \langle S \rangle$ . This notation may cause confusion when  $X$  is an associative and Lie algebra in the same time, in such cases we will indicate clearly if  $\langle S \rangle$  denotes associative or Lie subalgebra.

**Proposition 2.8** ([8], Theorem 1 of Chapter II). *Let  $V$  be a finite-dimensional vector space over a field  $\mathbb{F}$ . Let  $S \subseteq \text{End}(V)$  be a weakly closed subset such that every  $s \in S$  is*

associative nilpotent, that is,  $s^k = 0$ , for some positive integer  $k$ . Then the associative subalgebra  $\langle S \rangle \leq \text{End}(V)$  is nilpotent.

With this result we can prove Jacobson's Theorem.

**Theorem 2.9** ([7], Theorem 3). *Let  $L$  be a finite-dimensional Lie algebra over a field of characteristic 0 and suppose that there exists a subalgebra  $D$  of the algebra of derivations of  $L$  such that*

- (1)  $D$  is nilpotent;
- (2) if there is  $c \in L$  such that  $d(c) = 0$  for all  $d \in D$  then  $c = 0$ .

*Then  $L$  is nilpotent.*

*Proof.* Let  $\overline{\mathbb{F}}$  be the algebraic closure of the base field. We can extend all derivations of  $L$  to  $\overline{L} = L \otimes \overline{\mathbb{F}}$ . If we prove that  $\overline{L}$  is nilpotent then  $L$  is nilpotent. So we will assume that  $\mathbb{F}$  is algebraically closed. In this case the extension of  $D$  is nilpotent and without 0 as common eigenvalue, i.e. if there is  $c \in L$  such that  $d(c) = 0$  for all  $d \in D$  then  $c = 0$ . Let  $L = L_{\gamma_1} \dot{+} \cdots \dot{+} L_{\gamma_t}$  be the decomposition of  $L$  into generalized eigenspaces of  $D$ . By Proposition 2.2 we have  $[L_{\gamma_i}, L_{\gamma_j}] \subseteq L_{\gamma_i + \gamma_j}$  if  $\gamma_i + \gamma_j$  is a eigenvalue of  $D$  and  $[L_{\gamma_i}, L_{\gamma_j}] = 0$  otherwise. For a subset  $Y \subseteq L$ , we let  $\text{ad}_Y$  denote the set of adjoint mappings induced by elements of  $Y$ . Then the inclusion just noted shows that the set  $S = \bigcup \text{ad}_{L_{\gamma_j}}$  is a weakly closed set of linear transformations. Let  $a \in L_{\gamma_j}$  and  $b \in L_{\gamma_i}$ . Then  $(\text{ad}_a)^s(b) \in L_{\gamma_i + s\gamma_j}$ , for all  $s \geq 0$ .(\*)

The generalized eigenvalue  $\gamma_j \neq 0$  and  $\mathbb{F}$  has characteristic 0 then  $\gamma_i + s\gamma_j$ , for  $s > 0$ , are pairwise distinct. Then for some  $r$  large enough  $(\gamma_i + r\gamma_j)$  is not an eigenvalue and  $\text{ad}_a(b) = 0$ . Follow that  $\text{ad}_a$  is nilpotent linear transformation. Thus every element of  $S$  is nilpotent. By Proposition 2.8 the associative subalgebra  $\langle S \rangle \leq \text{End}(V)$  is nilpotent. Observe that the Lie subalgebra  $\langle S \rangle$  is subset of the associative subalgebra  $\langle S \rangle$ , then  $\langle S \rangle$  is nilpotent as Lie subalgebra. But  $\langle S \rangle = \text{ad}_L$  implies that  $L$  is a nilpotent Lie algebra.  $\square$

A review of the proof of Theorem 2.9 shows that the hypothesis of zero characteristic is essential to prove that every element in a homogeneous component is nilpotent. As the following examples shows, Theorem 2.9 fails to hold in characteristic  $p > 0$ .

**Example 2.10.** Let  $\mathbb{F}$  be the field of  $2^m$  elements and  $L$  be the vector space over  $\mathbb{F}$  such that

$$L = \langle x_\alpha \mid \alpha \in \mathbb{F}, \alpha \neq 0 \rangle$$

with a basis labeled by nonzero elements of the field  $\mathbb{F}$  and with multiplication  $[x_\alpha, x_\beta] = (\beta - \alpha)x_{\alpha + \beta}$ . Then  $L$  is a simple Lie algebra and the map  $d \in \text{End}(L)$  given by  $d(e_\alpha) = \alpha e_\alpha$  is a non-singular derivation. The calculations of this example and a systematic investigation of simple Lie algebras with nonsingular derivations can be found in [4].

**Example 2.11.** Let  $V$  be a vector space over a field  $\mathbb{F}$  of characteristic  $p > 0$ . Let  $B = \{a_1, a_2, \dots, a_p\}$  be a basis of  $V$ . Define the linear map  $x \in \mathfrak{gl}(V)$  by

$$x(a_i) = a_{i+1 \pmod p}, 1 \leq i \leq p.$$

Let  $K$  be the abelian Lie algebra generated by  $\{x, x^2, \dots, x^{p-1}\}$ . Then  $V$  can be considered as  $K$ -module with the standard action of  $\mathfrak{gl}(V)$  on  $V$ . Let  $L$  be the semi-direct sum  $L = K \oplus V$  then  $L$  is an Solvable non-nilpotent Lie algebra of derived length 2. Let  $\lambda, \delta \in \mathbb{F}$  both non-zero and  $\lambda \neq s\delta$ , for all  $s \in \mathbb{F}_p$ . The linear map  $d : L \rightarrow L$  defined by

$$d : \begin{cases} x^j \mapsto j\lambda x^j, & 1 \leq j \leq p-1; \\ a_i \mapsto (\delta + (i-1)\lambda)a_i, & 1 \leq i \leq p, \end{cases}$$

is a non-singular derivation of  $L$ .

For Lie algebras over fields of characteristic  $p > 3$  we could not find an example of derived length greater than 3 but in characteristic 2 we have the following example.

**Example 2.12.** Let  $L$  be a vector space of dimension 6 over  $\mathbb{F}_4$ . Let  $\lambda \in \mathbb{F}_4$  such that  $\lambda^2 = \lambda + 1$  and  $\{a_1, a_2, a_3, a_4, a_5, a_6\}$  a basis of  $L$  over  $\mathbb{F}_4$ . Define the products

$$\begin{aligned} [a_1, a_3] &= \lambda a_5 + a_6, & [a_1, a_4] &= \lambda a_6, & [a_1, a_5] &= \lambda^2 a_3 + a_4, & [a_3, a_5] &= \lambda a_2, \\ [a_1, a_6] &= \lambda^2 a_4, & [a_2, a_3] &= \lambda a_6, & \text{and } [a_2, a_5] &= \lambda^2 a_4. \end{aligned}$$

$L$  is a solvable non-nilpotent Lie algebra of derived length 3. The linear map  $d : L \rightarrow L$  defined by

$$d : \begin{cases} a_1 \mapsto a_1 & a_3 \mapsto \lambda a_3 & a_5 \mapsto \lambda^2 a_5 \\ a_2 \mapsto a_2 & a_4 \mapsto \lambda a_4 & a_6 \mapsto \lambda^2 a_6 \end{cases}$$

is a non-singular derivation of  $L$ .

Another question is whether the converse of Jacobson's Theorem is true, that is, is it true that all finite-dimensional nilpotent Lie algebras admit non-singular derivation. By Dixmier and Lister [6], there are nilpotent Lie algebras admitting only nilpotent derivations. Bellow we present Dixmier and Lister example of such an algebra.

**Example 2.13.** Let  $\mathbb{F}$  be a field of characteristic 0 and  $L = \langle x_1, x_2, \dots, x_8 \rangle$  be a Lie algebra over  $\mathbb{F}$  with dimension 8 and multiplication table

$$\begin{aligned} [e_1, e_2] &= e_5 & [e_1, e_3] &= e_6 & [e_1, e_4] &= e_7 & [e_1, e_5] &= -e_8 & [e_2, e_3] &= e_8 & [e_2, e_4] &= e_6 \\ [e_2, e_6] &= -e_7 & [e_3, e_4] &= -e_5 & [e_3, e_5] &= -e_7 & [e_4, e_6] &= -e_8 & [e_i, e_j] &= -[e_j, e_i]. \end{aligned}$$

Moreover,  $[e_i, e_j] = 0$  if it is not in table above. Then  $L$  is nilpotent with  $L^3 \neq 0$ ,  $L^4 = 0$  and every derivation of  $L$  is nilpotent.

**2.3. Jacobson's Theorem in characteristic  $p > 0$ .** As the examples above shows, Jacobson's Theorem is in general not true in characteristic  $p > 0$ . However, we have the follow weaker result.



**Theorem 2.14.** *Let  $L$  be a Lie algebra over a field of characteristic  $p > 0$  and suppose that there exists a subalgebra  $D \leq \text{Der}(L)$  such that*

- (1)  $D$  is nilpotent;
- (2) if there is  $c \in L$  such that  $d(c) = 0$  for all  $d \in D$  then  $c = 0$ .

*If  $D$  has at most  $p - 1$  generalized eigenvalues then  $L$  is nilpotent.*

*Proof.* The proof of this theorem is identical to proof of Theorem 2.9 up to point marked by (\*). The generalized eigenvalue  $\gamma_j \neq 0$  then the set  $\{\gamma_i, \gamma_i + \gamma_j, \dots, \gamma_i + (p - 1)\gamma_j\}$  has  $p$  distinct elements. As  $D$  has at most  $p - 1$  generalized eigenvalues then for some  $r$ ,  $0 < r \leq p - 1$ ,  $(\gamma_i + r\gamma_j)$  is not an eigenvalue. Follow that  $\text{ad}_a$  is nilpotent linear transformation, for every  $a \in L_{\gamma_i}$ . Thus every element of  $S$  is nilpotent. By Proposition 2.8 the associative subalgebra  $\langle S \rangle \leq \text{End}(V)$  is nilpotent and hence  $\text{ad}_L$  is nilpotent. Therefore  $L$  is a nilpotent Lie algebra.  $\square$

**2.4. The orders of non-singular derivations.** An interesting approach by Shalev in article [12] is to study the order of nonsingular derivations, establishing conditions for a Lie algebra over a field of characteristic  $p$  with non-singular derivations to be nilpotent. More precisely, Shalev studied the set of orders of nonsingular derivations of non-nilpotent Lie algebras of characteristic  $p > 0$ . Later, Mattarei in [10] showed that this set of numbers corresponds to the set of solutions of some polynomial equation over a field of characteristic  $p$ . Below we present some results of these articles.

Let  $L$  be a Lie algebra over an algebraically closed field of characteristic  $p$ . We can characterize the matrix of a non-singular derivation of  $L$ . We need a result for derivations in Lie algebras over a field of characteristic  $p > 0$ .

**Lemma 2.15.** *Let  $L$  be a Lie algebra over a field  $\mathbb{F}$  of characteristic  $p > 0$ . If  $d \in \text{Der}(L)$  then  $d^{p^m} \in \text{Der}(L)$ , for all  $m \geq 1$ .*

*Proof.* If we prove this result for  $m = 1$  then the general case when  $m \geq 1$  will follow by induction. Let us hence prove the statement only for  $m = 1$ . Let  $d \in \text{Der}(L)$  and  $x, y \in L$ . First we prove the Leibniz's formula by induction:

$$d^n([x, y]) = \sum_{k=0}^n \binom{n}{k} [d^k(x), d^{n-k}(y)], \text{ for all } n > 0.$$

The case  $n = 1$  follow from derivation's definition. Suppose that Leibniz's formula is valid for  $n$ . Then

$$(1) \quad d^n([x, y]) = \sum_{k=0}^n \binom{n}{k} [d^k(x), d^{n-k}(y)].$$

Calculating  $d$  in both sides of equation (1) we have

$$(2) \quad d^{n+1}([x, y]) = \sum_{k=0}^n \binom{n}{k} [d^{k+1}(x), d^{n-k}(y)] + \sum_{k=0}^n \binom{n}{k} [d^k(x), d^{n-k+1}(y)].$$

Rearranging the index, the right side of equation (2) can be write as

$$[d^{n+1}(x), y] + \sum_{k=1}^n \left( \binom{n}{k-1} + \binom{n}{k} \right) [d^k(x), d^{n+1-k}(y)] + [x, d^{n+1}(y)].$$

As  $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$  then

$$d^{n+1}([x, y]) = \sum_{k=0}^{n+1} \binom{n+1}{k} [d^k(x), d^{n+1-k}(y)].$$

Then by induction Leibniz's formula is proved. As the field  $\mathbb{F}$  has characteristic  $p > 0$  then setting  $n = p^m$  the Leibniz's formula is reduced to

$$d^{p^m}([x, y]) = [d^{p^m}(x), y] + [x, d^{p^m}(y)].$$

□

**Proposition 2.16.** *Let  $V$  be a finite-dimensional vector space over an algebraically closed field of characteristic  $p > 0$  and  $f \in \text{End}(V)$  non-singular with order  $r$  coprime to  $p$ . Then  $f$  is diagonalizable.*

*Proof.* Let  $A$  be the matrix of the endomorphism  $f$  in Jordan normal form and write  $A = S + N$  such that  $S$  is diagonal,  $N$  is nilpotent upper triangular and  $S, N$  commute. Denote by  $M_{ij}$  the element of a matrix  $M$  of the  $i^{\text{th}}$  line and the  $j^{\text{th}}$  column. It follows that

- If  $S_{ii} = \lambda_i$  then  $(S^k)_{ii} = \lambda_i^k$ , for all  $k > 0$ ;
- $N_{i(i+j)}^k = 0$ , for all  $0 \leq j < k$ .

As the order of  $A$  is  $r$  we have  $A^r = Id$ . Then

$$I = A^r = (S + N)^r = S^r + \binom{r}{1} S^{r-1} N + \binom{r}{2} S^{r-2} N^2 + \cdots + \binom{r}{r-1} S N^{r-1} + N^r.$$

The identity matrix on the left-hand side of the last equation is diagonal, while the summands, with the exception of the first summand, on the right-hand side are nilpotent. Further, if  $N \neq 0$ , then the second summand  $rS^{r-1}N$  is non-zero, and it is the only summand that contains a non-zero entry in a positions  $(i, i+1)$  with  $i > 0$ . However, this implies that if  $N \neq 0$ , then  $A^r$  must contain a non-zero entry in a position  $(i, i+1)$ , which is a contradiction, as  $A^r = I$ . Hence  $N = 0$  as claimed. Then  $f$  is diagonalizable. □

Let  $L$  be a Lie algebra over the field  $\mathbb{F}$  of characteristic  $p > 0$  such that  $L$  has a non-singular derivation  $d$ . Let  $r$  be the order of  $d$  such that  $r = sp^t$ , with  $\gcd(s, p) = 1$ . Then by Lemma 2.15  $d^{p^t}$  is a derivation whose order is prime to  $p$  and, by Proposition 2.16,  $d^{p^t}$  is diagonalizable. So if  $L$  is a Lie algebra over an algebraically closed field  $\mathbb{F}$  of characteristic  $p > 0$  with non-singular derivation then  $L$  has a diagonalizable derivation  $d$  without eigenvalue 0.

**Proposition 2.17** ([12], Lemma 2.2 ). *Let  $L$  be a finite-dimensional Lie algebra in characteristic  $p > 0$  which admits a non-singular derivation  $d$  whose order  $n$  is coprime to  $p$ . Suppose that  $L$  is not nilpotent. Then there exist  $\lambda \in \overline{\mathbb{F}}_p$  such that  $(\lambda + \delta)^n = 1$  for all  $\delta \in \mathbb{F}_p$ .*

*Proof.* Let  $\overline{\mathbb{F}}$  be an algebraic closure of  $\mathbb{F}$  and  $R = \{\alpha \in \overline{\mathbb{F}}_p \mid \alpha^n = 1\}$ . If  $R$  is not contained in the base field of  $L$  then we consider  $d$  for the extension  $L \otimes \overline{\mathbb{F}}$ . By Proposition 2.16,  $d$  is diagonalizable. Let  $L = L_{\lambda_1} \dot{+} \cdots \dot{+} L_{\lambda_r}$  the decomposition of  $L$  to eigenspaces of  $d$ . The set  $S = \bigcup \text{ad}_{L_{\lambda_j}}$  is weakly closed with  $\gamma(\text{ad}_a, \text{ad}_b) = -1$  for all  $a \in L_{\lambda_i}, b \in L_{\lambda_j}$ . If each  $\text{ad}_a$  is nilpotent then the associative subalgebra  $\langle S \rangle \leq \mathfrak{gl}(L)$  is nilpotent by Proposition 2.8. Hence  $\text{ad}_L$  is a nilpotent Lie algebra and  $L$  is nilpotent. As  $L$  is non-nilpotent by hypothesis then there is  $a \in L_{\lambda_j}$  and  $b \in L_{\lambda_i}$  such that  $(\text{ad}_a)^n(b) \neq 0$ ,  $1 \leq n \leq p$ . However this implies  $(\lambda_i + \delta\lambda_j)$  are eigenvalues of  $d$  for  $1 \leq \delta \leq p$ . Since  $|d| = n$  each eigenvalue of  $d$  has order  $n$ . Thus  $(\lambda_i + \delta\lambda_j)^n = 1$ , for all  $\delta \in \mathbb{F}_p$ . As  $\lambda_j$  is an eigenvalue of  $d$ ,  $\lambda_j^n = \lambda_j^{-n} = 1$ . Thus  $1 = (\lambda_i + \delta\lambda_j)^n \lambda_j^{-n} = (\lambda_i \lambda_j^{-1} + \delta)^n$ . Therefore setting  $\lambda = \lambda_i \lambda_j^{-1}$ ,  $(\lambda + \delta)^n = 1$  for all  $\delta \in \mathbb{F}_p$ .  $\square$

Using the same notation as in the proof of Proposition 2.17 and observing that the set  $R$  contains precisely the  $n$ -th roots of unity in  $\overline{\mathbb{F}}$ , we write  $x^n - 1 = \prod_{\alpha \in R} (x - \alpha)$ . As for all  $\delta \in \mathbb{F}_p$ ,  $\lambda + \delta \in R$ ,  $\prod_{\delta \in \mathbb{F}_p} (x - \lambda - \delta)$  divides  $x^n - 1$ . But

$$\prod_{\delta \in \mathbb{F}_p} (x - \lambda - \delta) = (x - \lambda)^p - (x - \lambda) = x^p - x - c,$$

where  $c = \lambda^p - \lambda$ . The first equation of last display can be seen by observing that the elements  $\lambda + \delta$  with  $\delta \in \mathbb{F}_p$  are exactly the  $p$  roots of the polynomial  $(x - \lambda)^p - (x - \lambda)$ . Let  $g(x) = x^p - x - c$ . Then  $g(x)$  divides  $x^n - 1$ , which implies that  $x^n$  is congruent to 1 modulo  $g(x)$ . In this case, Lemma 2.4 of [12] shows that  $n \geq p^2 - 1$ . Now we can prove the theorem.

**Theorem 2.18** ([12], Theorem 1.1). *Let  $L$  be a finite dimensional Lie algebra in characteristic  $p > 0$  which admits non-singular derivation of order  $n$ . Write  $n = p^s m$  where  $m$  is coprime to  $p$ . Suppose  $m < p^2 - 1$ . Then  $L$  is nilpotent.*

*Proof.* The derivation  $d^{p^s}$  has order  $m$ . Suppose that  $L$  is not nilpotent. Then by the comment above we have  $m \geq p^2 - 1$ .  $\square$

Mattarei in [10] presented an example of non-nilpotent solvable modular Lie algebra.

**Example 2.19.** Let  $\alpha, \beta \in \overline{\mathbb{F}}_p$  with  $\alpha\beta^{-1} \notin \mathbb{F}_p$ . Let  $M$  be a  $p$ -dimensional vector space over  $\overline{\mathbb{F}}_p$  with basis  $e_1, \dots, e_p$ , and let  $E, F$  be the linear transformations of  $M$  defined by  $E(e_i) = e_{i+1}$  (indices modulo  $p$ ), and  $F(e_i) = (\alpha + i\beta)e_i$ . The transformations  $E$  and  $F$  span a two-dimensional solvable Lie algebra, which admits  $M$  as a left module. Let  $L$  be the semidirect sum of  $\{E\}$  and  $M$  with respect to this action. Then  $F$  acts on  $L$  as a non-singular derivation, with eigenvalues  $\beta$  on  $\{E\}$ , and  $\alpha + \lambda\beta$  for  $\lambda \in \mathbb{F}_p$  on  $M$ .

The next result links the orders non-singular derivations of Lie algebras of characteristic  $p$  to some polynomial equations.

**Proposition 2.20.** *Let  $p$  be a prime number and let  $n$  be a positive integer, prime to  $p$ . The following statements are equivalent:*

- (1) *there exists a non-nilpotent Lie algebra of characteristic  $p$  with a non-singular derivations of order  $n$ ;*
- (2) *there exists an element  $\alpha \in \overline{\mathbb{F}}_p$  such that  $(\alpha + \lambda)^n = 1$  for all  $\lambda \in \mathbb{F}_p$*
- (3) *there exist an element  $c \in \overline{\mathbb{F}}_p^*$  such that  $x^p - x - c$  divides  $x^n - 1$  as elements of the polynomial ring  $\overline{\mathbb{F}}_p[x]$ .*

Mattarei in [10] defines the set  $N_p$  of the possible orders of non-singular derivations of non-nilpotent Lie algebras of characteristic  $p$  and determine all elements of  $N_p$  which are smaller than  $p^3$ , for  $p > 3$ .

**2.5. Objectives of the project.** In this section we will present some questions about solvable non-nilpotent modular Lie algebras  $L$  with a non-singular derivation  $d$ . This questions are based in the examples and results showed in the previous sections. These issues will serve as a reference for further work.

**Problem 1.** Is there a solvable, non-nilpotent Lie algebra over a field of characteristic  $p \geq 3$  with non-singular derivation and derived length greater than 2?

Suppose that the answer to Problem 1 is yes and let  $L$  be such Lie algebra. Let  $I = L^{(2)}$  and  $K = L/I$ . As  $L^{(3)} = 0$  then  $I$  is abelian and so  $K$  acts on  $I$  by adjoint representation. In this case,  $K$  is a solvable Lie algebra of derived length 2 with non-singular derivation. By Proposition 3.1, there is a cocycle  $\vartheta \in Z^2(K, I)$  such that  $L \cong K_{\vartheta}$ . This calculation show us that every Lie algebra that answer Problem 1 can be obtained by an extension of a solvable Lie algebra of derived length 2 with non-singular derivation. So we need to understand this Lie algebras of derived length 2 to search for an answer of Problem 1. We will study a variation of this question.

**Problem 2.** Let  $K$  be one of the known solvable, non-nilpotent Lie algebra over a field of characteristic  $p \geq 3$  with non-singular derivation and derived length 2. Is there a non-trivial  $K$ -module  $I$  and a cocycle  $\vartheta \in Z^2(K, I)$  such that  $K_{\vartheta}$  has a non-singular derivation?

As first step to study Problem 2 we will try to describe some cases of abelian Lie algebras  $K$  acting over vector spaces. This study defines our next objectives in this project.

### Objectives

- To characterize solvable non-nilpotent modular Lie algebras of the form  $L = \langle x \rangle \oplus I$  where  $I$  is a finite dimensional abelian Lie algebra such that  $L$  admits a non-singular derivation; study the extensions of such algebras and obtain ones that admits non-singular derivations; By Corollary 3.15, there is a quotient  $Q = L^{(i)}/L^{i+1}$  with  $\dim Q \geq p$ . Study the number of such quotients.

- How the existence of non-singular derivations affect the structure of  $\text{Der}(L)$ ? Can we define some algebra structure over non-singular derivations of  $L$ ?
- Study the general structure of solvable non-nilpotent Lie algebras with non-singular derivations

### 3. DERIVATIONS AND LIE ALGEBRA EXTENSIONS

**3.1. Lie algebra extensions.** An *extension* of a Lie algebra  $K$  by a Lie algebra  $I$  is an exact sequence

$$(3) \quad 0 \rightarrow I \xrightarrow{i} L \xrightarrow{s} K \rightarrow 0$$

of Lie algebras. The Lie algebra  $L$  in the middle of the exact sequence contains an ideal  $\text{Ker}(s) = \text{Im } i \cong I$  such that  $L/I \cong K$ . We will write informally that ‘ $L$  is an extension of  $K$  by  $I$ ’. The extension (3) *splits* if  $L$  has a subalgebra  $S$  such that  $L = S \dot{+} \text{Ker}(s)$ . The extension (3) is *trivial* if there exists an ideal  $S$  of  $L$  such that  $L = S \oplus \text{Ker}(s)$ . The extension (3) is central if  $\text{Ker}(s)$  lies in the center  $Z(L)$  of  $L$ .

Let  $K$  be a Lie algebra over a field  $\mathbb{F}$  and let  $I$  be a vector space over  $\mathbb{F}$ . Denote by  $\mathcal{C}^2(K, I)$  the vector space of alternating bilinear maps  $\vartheta : K \times K \rightarrow I$ . If  $I$  is a  $K$ -module and  $\vartheta \in \mathcal{C}^2(K, I)$  has the property that

$$(4) \quad \vartheta(x, [y, z]) + \vartheta(y, [z, x]) + \vartheta(z, [x, y]) + [x, \vartheta(y, z)] + [y, \vartheta(z, x)] + [z, \vartheta(x, y)] = 0,$$

for all  $x, y, z \in K$ , then  $\vartheta$  is said to be a *cocycle* and the vector space of cocycles is denoted by  $Z^2(K, I)$ . Let  $T : K \rightarrow I$  be a linear transformation and define,  $\vartheta_T : K \times K \rightarrow I$  by

$$(5) \quad \vartheta_T(h, k) = T([h, k]) + [k, T(h)] - [h, T(k)] \quad \text{for all } h, k \in K.$$

Then  $\vartheta_T \in Z^2(K, I)$  and such a cocycle  $\vartheta_T$  is said to be a *coboundary*. The set of coboundaries is denoted by  $B^2(K, I)$ . The set  $B^2(K, I)$  is a subspace of  $Z^2(K, I)$ , and we set  $H^2(K, I) = Z^2(K, I)/B^2(K, I)$  to be the quotient space. The first cohomology group of  $K$  and  $I$  is defined as

$$Z^1(K, I) = \{\nu \in \text{Hom}(K, I) \mid \nu([h, k]) = [h, \nu(k)] - [k, \nu(h)] \text{ for all } h, k \in K\}.$$

The next result, whose proof can be found, for instance, in [9, Section 4.2], links Lie algebra extensions to cohomology. Let  $K$  be a Lie algebra and let  $I$  be a  $K$ -module. Let  $\vartheta \in Z^2(K, I)$  and define the Lie algebra  $K_\vartheta = K \dot{+} I$  with the product

$$(6) \quad [x + a, y + b] = [x, y] + \vartheta(x, y) + [a, y] - [b, x] \text{ for all } x, y \in K \text{ and } a, b \in I.$$

**Proposition 3.1.** *The following hold for the Lie algebra  $K_\vartheta$ :*

- (1)  $K_\vartheta$  is a Lie algebra extension of  $K$  by  $I$ ;
- (2) if  $\nu \in B^2(K, I)$ , then  $K_\vartheta$  is isomorphic to  $K_{\vartheta+\nu}$ ;
- (3) if  $\vartheta \in B^2(K, I)$ , then  $K_\vartheta$  is a split extension of  $K$  by  $I$ .

*Conversely, let  $L$  be a Lie algebra and  $J$  be an abelian ideal of  $L$ . Then there exists  $\vartheta \in Z^2(L/J, J)$  such that  $L \cong (L/J)_\vartheta$ .*

The cocycle  $\vartheta$  in last the statement of Proposition 3.1 can be constructed as follows. Let  $\pi : L \rightarrow L/I$  denote the natural projection, and let  $\sigma : L/I \rightarrow L$  be a right inverse of  $\pi$ ; that is,  $\pi\sigma = \text{id}_{L/I}$ . Then, for  $k + I, h + I \in L/I$ , set

$$\vartheta(h + I, k + I) = \sigma([h + I, k + I]) - [\sigma(h + I), \sigma(k + I)].$$

Routine calculation shows that  $\vartheta \in Z^2(L/I, I)$  and that  $L \cong L_{\vartheta}$ .

**3.2. Compatible pairs and derivations of semidirect sums.** Compatible pairs were introduced in [3] to compute automorphisms of solvable groups and solvable Lie algebras. We adopt the concept for derivations of Lie algebras. Let  $K$  and  $I$  be Lie algebras such that  $K$  acts on  $I$  via the homomorphism  $\psi : K \rightarrow \text{Der}(I)$ . We define the *semidirect sum*  $K \oplus_{\psi} I$  as the vector space  $K + I$  with the product operation given as

$$[(k_1, a_1), (k_2, a_2)] = ([k_1, k_2], [k_1, a_2] - [k_2, a_1] + [a_1, a_2]).$$

When the  $K$ -action on  $I$  is clear from the context, then we usually suppress the homomorphism ‘ $\psi$ ’ from the notation and write simply  $K \oplus I$ . If  $L$  is a Lie algebra such that  $L$  has an ideal  $I$  and a subalgebra  $K$  in such a way that  $L = K + I$ , then  $L \cong K \oplus_{\psi} I$  where  $\psi$  is the restriction of  $\text{ad}_I$  to  $K$ . In a semidirect sum  $K \oplus I$ , an element  $(k, a) \in K + I$  will usually be written as  $k + a$ .

Suppose that  $K$  and  $I$  are as in the previous paragraph. The direct sum  $\text{Der}(K) \oplus \text{Der}(I)$  of the derivation Lie algebras is a Lie algebra. An element  $(\alpha, \beta) \in \text{Der}(K) \oplus \text{Der}(I)$  is said to be a *compatible pair* if

$$(7) \quad \beta([k, a]) = [\alpha(k), a] + [k, \beta(a)] \quad \text{for all } k \in K, a \in I.$$

We let  $\text{Comp}(K, I)$  denote the set of compatible pairs in  $\text{Der}(K) \oplus \text{Der}(I)$ . Using the homomorphism  $\psi : K \rightarrow \text{Der}(I)$  associated to the  $K$ -action on  $I$ , we can write equation (7) in another form as follows. Writing  $[k, a]$  as  $\psi(k)(a)$ , we have that  $(\alpha, \beta) \in \text{Comp}(K, I)$  if and only if the equation

$$\beta\psi(k) = \psi(\alpha(k)) + \psi(k)\beta.$$

holds in  $\text{Der}(I)$  for all  $k \in K$ . Using commutator, this is equivalent to

$$(8) \quad [\beta, \psi(k)] = \psi(\alpha(k)) \quad \text{for all } k \in K.$$

Letting  $\text{ad} : \text{Der}(I) \rightarrow \text{Der}(I)$  denote the adjoint representation, equation (8) can be rewritten as

$$(9) \quad \text{ad}_{\beta}\psi(k) = \psi(\alpha(k)) \quad \text{for all } k \in K.$$

Therefore,  $(\alpha, \beta) \in \text{Comp}(K, I)$  if and only if the following diagram commutes:

$$\begin{array}{ccc} K & \xrightarrow{\psi} & \text{Der}(I) \\ \downarrow \alpha & \circlearrowleft & \downarrow \text{ad}_{\beta} \\ K & \xrightarrow{\psi} & \text{Der}(I). \end{array}$$

A compatible pair  $(\alpha, \beta) \in \text{Der}(K) \oplus \text{Der}(I)$  will usually be written as  $\alpha + \beta$ . If  $\alpha + \beta \in \text{Der}(K) \oplus \text{Der}(I)$  as above, then  $\alpha + \beta$  can be considered a element of  $\mathfrak{gl}(I \oplus K)$  by letting  $(\alpha + \beta)(k + a) = \alpha(k) + \beta(a)$  for all  $a \in I$  and  $k \in K$ .

**Proposition 3.2.** *Using the notation above, we have that*

$$\text{Comp}(K, I) = \{\alpha + \beta \in \mathfrak{gl}(K) \oplus \mathfrak{gl}(I) \mid \alpha + \beta \in \text{Der}(K \oplus I)\}.$$

*In particular  $\text{Comp}(K, I)$  is a Lie subalgebra of  $\text{Der}(K \oplus I)$ .*

*Proof.* Suppose that  $\alpha + \beta \in \text{Comp}(K, I)$  is a compatible pair and let  $k + a, k' + a' \in K \oplus I$ . Then

$$\begin{aligned} (\alpha + \beta)[k + a, k' + a'] &= (\alpha + \beta)([k, k'] + ([k, a'] - [k', a] + [a, a'])) \\ &= \alpha([k, k']) + \beta([k, a'] - [k', a] + [a, a']) \\ &= [\alpha(k), k'] + [k, \alpha(k')] + [\alpha(k), a'] - [\alpha(k'), a] \\ &\quad + [\beta(a), a'] + [k, \beta(a')] - [k', \beta(a)] + [a, \beta(a')]. \end{aligned}$$

On the other hand

$$\begin{aligned} [(\alpha + \beta)(k + a), k' + a'] + [k + a, (\alpha + \beta)(k' + a')] &= \\ [\alpha(k), k'] + [\alpha(k), a'] + [\beta(a), k'] + [\beta(a), a'] &+ \\ + [k, \alpha(k')] + [k, \beta(a')] + [a, \alpha(k')] + [a, \beta(a')]. \end{aligned}$$

Thus  $\alpha + \beta \in \text{Der}(K \oplus I)$ .

Conversely, let  $\alpha + \beta \in \mathfrak{gl}(K) \oplus \mathfrak{gl}(I)$  such that  $\alpha + \beta$  is a derivation of  $K \oplus I$ . Then  $(\alpha + \beta)|_K = \alpha$  and  $(\alpha + \beta)|_I = \beta$ , and so  $\alpha \in \text{Der}(K)$  and  $\beta \in \text{Der}(I)$ . Further, if  $k \in K$  and  $a \in I$ , then  $[k, a] \in I$ , and so

$$\beta([k, a]) = (\alpha + \beta)[k, a] = [(\alpha + \beta)(k), a] + [k, (\alpha + \beta)(a)] = [\alpha(k), a] + [k, \beta(a)].$$

Thus  $\alpha + \beta \in \text{Comp}(K, I)$ , as required.

The fact that  $\text{Comp}(K, I)$  is a Lie subalgebra of  $\text{Der}(K \oplus I)$  follows from the fact that  $\text{Comp}(K, I)$  is the intersection of two Lie algebras; namely,  $\text{Comp}(K, I) = (\mathfrak{gl}(K) \oplus \mathfrak{gl}(I)) \cap \text{Der}(K \oplus I)$ .  $\square$

**Lemma 3.3.** *Let  $K$  and  $I$  be Lie algebras over a field  $\mathbb{F}$  of characteristic  $p > 0$ . If  $(\alpha, \beta) \in \text{Comp}(K, I)$  then  $(\alpha, \beta)^{p^t} \in \text{Comp}(K, I)$  for all  $t \geq 0$ .*

*Proof.* Let  $L = K \oplus I$  be the semi-direct sum of  $K$  and  $I$ . By Proposition 3.2,  $(\alpha, \beta) \in \text{Der}(L)$ . Then by Lemma 2.15,  $(\alpha, \beta)^{p^t} \in \text{Der}(L)$ , for all  $t \geq 0$ . Hence, by Proposition 3.2,  $(\alpha, \beta)^{p^t} \in \text{Comp}(K, I)$ .  $\square$

Let  $K$  and  $I$  be vector spaces. Consider the Lie algebra  $\mathfrak{gl}(K) \oplus \mathfrak{gl}(I)$  and define an action of  $\mathfrak{gl}(K) \oplus \mathfrak{gl}(I)$  on the vector space  $\text{Hom}(K, \mathfrak{gl}(I))$  as follows. Let  $\text{ad}$  denote the

adjoint representation of  $\mathfrak{gl}(I)$ . Thus, for  $\beta, \beta' \in \mathfrak{gl}(I)$  and  $\text{ad}_\beta(\beta') = [\beta, \beta']$ . For  $(\alpha, \beta) \in \mathfrak{gl}(K) \oplus \mathfrak{gl}(I)$  and for  $T \in \text{Hom}(K, \mathfrak{gl}(I))$ , set

$$(10) \quad (\alpha, \beta) \cdot T = \text{ad}_\beta T - T\alpha.$$

Let us show that this in fact defines a Lie algebra action. First notice that  $(\alpha, \beta) \cdot T \in \text{Hom}(K, \mathfrak{gl}(I))$  because it is linear combination of compositions of linear maps. Let us check that the action is compatible with Lie brackets. Let  $(\alpha, \beta), (\alpha', \beta') \in \mathfrak{gl}(K) \oplus \mathfrak{gl}(I)$ . By definition

$$(\alpha', \beta') \cdot T = \text{ad}_{\beta'} T - T\alpha'.$$

Thus

$$(\alpha, \beta) \cdot ((\alpha', \beta') \cdot T) = \text{ad}_\beta \text{ad}_{\beta'} T - \text{ad}_{\beta'} T\alpha - \text{ad}_\beta T\alpha' + T\alpha'\alpha.$$

In the same way,

$$(\alpha', \beta') \cdot ((\alpha, \beta) \cdot T) = \text{ad}_{\beta'} \text{ad}_\beta T - \text{ad}_\beta T\alpha' - \text{ad}_{\beta'} T\alpha + T\alpha\alpha'.$$

Hence,

$$\begin{aligned} (\alpha, \beta) \cdot ((\alpha', \beta') \cdot T) - (\alpha', \beta') \cdot ((\alpha, \beta) \cdot T) &= \text{ad}_\beta \text{ad}_{\beta'} T - \text{ad}_{\beta'} \text{ad}_\beta T - T\alpha\alpha' + T\alpha'\alpha \\ &= [\text{ad}_\beta, \text{ad}_{\beta'}]T - T[\alpha, \alpha']. \end{aligned}$$

Therefore,

$$[(\alpha, \beta), (\alpha', \beta')] \cdot T = ([\alpha, \alpha'], [\beta, \beta']) \cdot T.$$

Now, if  $K$  and  $I$  are Lie algebras and  $I$  is a  $K$ -module, then there is a corresponding homomorphism  $\psi \in \text{Hom}(K, \text{Der}(I))$ . Now suppose that  $\alpha + \beta \in \mathfrak{gl}(K) \oplus \mathfrak{gl}(I)$  such that  $\alpha + \beta \in \text{Der}(K) \oplus \text{Der}(I)$ . Then, for  $k \in K$ , we have  $\text{ad}_\beta T(k) + T\alpha(k)$  is a derivation of  $I$  since  $\text{ad}_\beta T(k), T\alpha(k) \in \text{Der}(I)$ .

If  $X$  is a subalgebra of  $\text{Der}(K) \oplus \text{Der}(I)$ , then the annihilator  $\text{Ann}_X(\psi)$  of  $\psi$  in  $X$  is defined as

$$\text{Ann}_X(\psi) = \{(\alpha, \beta) \in X \mid (\alpha, \beta) \cdot \psi = 0\}.$$

Computing the annihilator of  $\psi$  in  $\text{Der}(K) \oplus \text{Der}(I)$  explicitly, we obtain

$$\begin{aligned} \text{Ann}_{\text{Der}(K) \oplus \text{Der}(I)}(\psi) &= \{(\alpha, \beta) \in \text{Der}(K) \oplus \text{Der}(I) \mid (\alpha, \beta) \cdot \psi = 0\} \\ &= \{(\alpha, \beta) \in \text{Der}(K) \oplus \text{Der}(I) \mid \text{ad}_\beta \psi - \psi\alpha = 0\} = \text{Comp}(K, I). \end{aligned}$$

The last equality follows from (9). Hence we have proved the following proposition.

**Proposition 3.4.** *Let  $K$  and  $I$  be Lie algebras such that  $I$  is also a  $K$ -module via the representation  $\psi \in \text{Hom}(K, \text{Der}(I))$ . Then  $\text{Comp}(K, I) = \text{Ann}_{\text{Der}(K) \oplus \text{Der}(I)}(\psi)$ , where the action of  $\text{Der}(K) \oplus \text{Der}(I)$  on  $\text{Hom}(K, \text{Der}(I))$  is given by (10).*



**3.3. Derivations of  $K_\vartheta$ .** In this section we present a method to describe the derivations of an extension  $K_\vartheta$  presented in Proposition 3.1 from the derivations of the Lie algebra  $K$ . By an adaptation of the process used by Eick in [3], we set conditions which guarantee that a derivation of  $K$  can be lifted to a derivation of  $K_\vartheta$ . It is first necessary to define an action of  $\mathfrak{gl}(K) \oplus \mathfrak{gl}(I)$  on the vector space of alternating bilinear maps.

Let  $K$  and  $I$  be vector spaces. Let  $(\alpha, \beta)$  be an element of the Lie algebra  $\mathfrak{gl}(K) \oplus \mathfrak{gl}(I)$  and  $\vartheta \in \mathcal{C}^2(K, I)$ . Define an action of  $\mathfrak{gl}(K) \oplus \mathfrak{gl}(I)$  on  $\mathcal{C}^2(K, I)$  by setting for  $\vartheta \in \mathcal{C}^2(K, I)$

$$(11) \quad (\alpha, \beta) \cdot \vartheta(h, k) = \beta(\vartheta(h, k)) - \vartheta(\alpha(h), k) - \vartheta(h, \alpha(k)), \quad \text{for all } h, k \in K.$$

Let  $(\alpha', \beta') \in \mathfrak{gl}(K) \oplus \mathfrak{gl}(I)$ , then

$$(12) \quad (\alpha, \beta) \cdot ((\alpha', \beta') \cdot \vartheta(h, k)) = (\alpha, \beta) \cdot (\beta'(\vartheta(h, k)) - \vartheta(\alpha'(h), k) - \vartheta(h, \alpha'(k))).$$

Applying the action in each summand of the right-hand of equation (12) we have

$$(\alpha, \beta) \cdot \beta'(\vartheta(h, k)) = \beta\beta'(\vartheta(h, k)) - \beta'\vartheta(\alpha(h), k) - \beta'\vartheta(h, \alpha(k)),$$

$$(\alpha, \beta) \cdot \vartheta(\alpha'(h), k) = \beta\vartheta(\alpha'(h), k) - \vartheta(\alpha'\alpha(h), k) - \vartheta(\alpha'(h), \alpha(k)),$$

$$(\alpha, \beta) \cdot \vartheta(h, \alpha'(k)) = \beta\vartheta(h, \alpha'(k)) - \vartheta(\alpha(h), \alpha'(k)) - \vartheta(h, \alpha'\alpha(k)).$$

Then

$$\begin{aligned} (\alpha, \beta) \cdot ((\alpha', \beta') \cdot \vartheta(h, k)) &= \beta\beta'(\vartheta(h, k)) - \beta'\vartheta(\alpha(h), k) - \beta'\vartheta(h, \alpha(k)) \\ &\quad - \beta\vartheta(\alpha'(h), k) + \vartheta(\alpha'\alpha(h), k) + \vartheta(\alpha'(h), \alpha(k)) \\ &\quad - \beta\vartheta(h, \alpha'(k)) + \vartheta(\alpha(h), \alpha'(k)) + \vartheta(h, \alpha'\alpha(k)). \end{aligned}$$

It follows

$$\begin{aligned} [(\alpha, \beta), (\alpha', \beta')] \cdot \vartheta(h, k) &= [\beta, \beta']\vartheta(h, k) - \vartheta([\alpha, \alpha'](h), k) - \vartheta(h, [\alpha, \alpha'](k)) \\ &= ([\alpha, \alpha'], [\beta, \beta']) \cdot \vartheta(h, k). \end{aligned}$$

Therefore, the action presented in (11) is well defined.

Our goal now is to study the action of compatible pairs  $\mathbf{Comp}(K, I)$  on subspaces  $\mathbf{Z}^2(K, I)$  and  $\mathbf{B}^2(K, I)$  of  $\mathcal{C}^2(K, I)$ . For this, assume that  $K$  is a Lie algebra and  $I$  is a  $K$ -module. Then for all  $h, k, l \in K$ ,  $(\alpha, \beta) \in \mathbf{Comp}(K, I)$  and  $\vartheta \in \mathbf{Z}^2(K, I)$  we have

$$\begin{aligned} (\alpha, \beta) \cdot \vartheta(k, [h, l]) &= \beta(\vartheta(k, [h, l])) - \vartheta(\alpha(k), [h, l]) - \vartheta(k, \alpha([h, l])) \\ &= \beta(\vartheta(k, [h, l])) - \vartheta(\alpha(k), [h, l]) - \vartheta(k, [\alpha(h), l]) - \vartheta(k, [h, \alpha(l)]). \end{aligned}$$

If

$$X = (\alpha, \beta) \cdot \vartheta(k, [h, l]) + (\alpha, \beta) \cdot \vartheta(h, [l, k]) + (\alpha, \beta) \cdot \vartheta(l, [k, h]),$$

then

$$\begin{aligned} X = & \beta(\vartheta(k, [h, l])) + \beta(\vartheta(h, [l, k])) + \beta(\vartheta(l, [k, h])) \\ & - \vartheta(\alpha(k), [h, l]) - \vartheta(\alpha(h), [l, k]) - \vartheta(\alpha(l), [k, h]) \\ & - \vartheta(k, [\alpha(h), l]) - \vartheta(h, [\alpha(l), k]) - \vartheta(l, [\alpha(k), h]) \\ & - \vartheta(k, [h, \alpha(l)]) - \vartheta(h, [l, \alpha(k)]) - \vartheta(l, [k, \alpha(h)]). \end{aligned}$$

Using that  $\beta$  is linear and the definition of cocycles (4)

$$\begin{aligned} X = & -\beta([k, \vartheta(h, l)]) - \beta([h, \vartheta(l, k)]) - \beta([l, \vartheta(k, h)]) \\ & + [\alpha(k), \vartheta(h, l)] + [\alpha(h), \vartheta(l, k)] + [\alpha(l), \vartheta(k, h)] \\ & + [k, \vartheta(\alpha(h), l)] + [h, \vartheta(\alpha(l), k)] + [l, \vartheta(\alpha(k), h)] \\ & + [k, \vartheta(h, \alpha(l))] + [h, \vartheta(l, \alpha(k))] + [l, \vartheta(k, \alpha(h))]. \end{aligned}$$

Since  $(\alpha, \beta)$  is a compatible pair we have by (7)

$$\begin{aligned} \beta([k, \vartheta(h, l)]) &= [\alpha(k), \vartheta(h, l)] + [k, \beta(\vartheta(h, l))]; \\ \beta([h, \vartheta(l, k)]) &= [\alpha(h), \vartheta(l, k)] + [h, \beta(\vartheta(l, k))]; \\ \beta([l, \vartheta(k, h)]) &= [\alpha(l), \vartheta(k, h)] + [l, \beta(\vartheta(k, h))]. \end{aligned}$$

Hence we obtain combining the last two displayed systems of equations

$$\begin{aligned} X = & -[k, \beta(\vartheta(h, l))] - [h, \beta(\vartheta(l, k))] - [l, \beta(\vartheta(k, h))] \\ & + [k, \vartheta(\alpha(h), l)] + [h, \vartheta(\alpha(l), k)] + [l, \vartheta(\alpha(k), h)] \\ & + [k, \vartheta(h, \alpha(l))] + [h, \vartheta(l, \alpha(k))] + [l, \vartheta(k, \alpha(h))]. \end{aligned}$$

Again, by the definition of the action in (11)

$$X = -[k, (\alpha, \beta) \cdot \vartheta(h, l)] - [h, (\alpha, \beta) \cdot \vartheta(l, k)] - [l, (\alpha, \beta) \cdot \vartheta(k, h)].$$

So  $(\alpha, \beta) \cdot \vartheta \in \mathbf{Z}^2(K, I)$ .

Now suppose that  $\vartheta \in \mathbf{B}^2(K, I)$ . By definition (5) there is a linear map  $T : K \rightarrow I$  such that  $\vartheta = \vartheta_T$ . Hence

$$(13) \quad \vartheta_T(h, k) = T([h, k]) + [k, T(h)] - [h, T(k)].$$

Let  $Y = (\alpha, \beta) \cdot \vartheta_T(h, k)$ . By (13) we have

$$(14) \quad Y = \beta(\vartheta_T(h, k)) - \vartheta_T(\alpha(h), k) - \vartheta_T(h, \alpha(k)).$$

Using the definition of  $\vartheta_T$  we have

$$(15) \quad \begin{aligned} \beta(\vartheta_T(h, k)) &= \beta T([h, k]) + \beta[k, T(h)] - \beta[h, T(k)], \\ \vartheta_T(\alpha(h), k) &= T([\alpha(h), k]) + [k, T\alpha(h)] - [\alpha(h), T(k)], \\ \vartheta_T(h, \alpha(k)) &= T([h, \alpha(k)]) + [\alpha(k), T(h)] - [h, T\alpha(k)]. \end{aligned}$$

We can use that  $(\alpha, \beta)$  is a compatible pair in equation (15) to write

$$\beta(\vartheta_T(h, k)) = \beta T([h, k]) + [\alpha(k), T(h)] + [k, \beta T(h)] - [\alpha(h), T(k)] - [h, \beta T(k)].$$

Then

$$\begin{aligned} Y &= \beta T([h, k]) + [\alpha(k), T(h)] + [k, \beta T(h)] - [\alpha(h), T(k)] - [h, \beta T(k)] \\ &\quad - T([\alpha(h), k]) - [k, T\alpha(h)] + [\alpha(h), T(k)] \\ &\quad - T([h, \alpha(k)]) - [\alpha(k), T(h)] + [h, T\alpha(k)]. \end{aligned}$$

Making the cancellations,  $Y$  can be written as

$$\begin{aligned} Y &= \beta T([h, k]) - T([\alpha(h), k]) - T([h, \alpha(k)]) \\ &\quad + [k, \beta T(h)] - [k, T\alpha(h)] + [h, T\alpha(k)] - [h, \beta T(k)]. \end{aligned}$$

Now we use that  $T$  and the action are linear to obtain

$$Y = \beta T([h, k]) - T([\alpha(h), k] + [h, \alpha(k)]) + [k, \beta T(h) - T\alpha(h)] - [h, \beta T(k) - T\alpha(k)].$$

Hence,

$$Y = (\beta T - T\alpha)([h, k]) + [k, (\beta T - T\alpha)(h)] - [h, (\beta T - T\alpha)(k)].$$

If  $U = \beta T - T\alpha : K \rightarrow I$  then

$$(\alpha, \beta) \cdot \vartheta(h, k) = U([h, k]) - [k, U(h)] - [h, U(k)].$$

Therefore,  $(\alpha, \beta) \cdot \vartheta \in \mathbf{B}^2(K, I)$ . We just proof

**Proposition 3.5.** *Let  $K$  be a Lie algebra and let  $I$  be a  $K$ -module. Consider the action of  $\mathbf{Comp}(K, I)$  on  $\mathbf{C}^2(K, I)$  defined in (11). Then the vector spaces  $\mathbf{Z}^2(K, I)$  and  $\mathbf{B}^2(K, I)$  are invariants by this action.*

This result allows us to define an action of  $\mathbf{Comp}(K, I)$  on  $\mathbf{H}^2(K, I)$ : let  $\vartheta \in \mathbf{Z}^2(K, I)$  and  $(\alpha, \beta) \in \mathbf{Comp}(K, I)$ . Define the action

$$(16) \quad (\alpha, \beta) \cdot (\vartheta + \mathbf{B}^2(K, I)) = ((\alpha, \beta) \cdot \vartheta) + \mathbf{B}^2(K, I).$$

This is well defined by Proposition 3.5.

**Definition 3.6.** Let  $K$  be a Lie algebra and  $I$  a  $K$ -module. Let  $\vartheta \in \mathbf{Z}^2(K, I)$  and consider the action of  $\mathbf{Comp}(K, I)$  on  $\mathbf{H}^2(K, I)$  defined in (16). Define the set of induced pairs of  $\mathbf{Comp}(K, I)$  by

$$\mathbf{Indu}(K, I, \vartheta) = \mathbf{Ann}_{\mathbf{Comp}(K, I)}(\vartheta + \mathbf{B}^2(K, I)).$$

Now we have the tools needed to describe the Lie algebra  $\text{Der}(K_\vartheta)$  from the Lie algebra  $\text{Der}(K)$ . We will define a homomorphism  $\phi : \text{Der}(K_\vartheta) \rightarrow \text{Der}(K)$ , whose kernel is known and the image coincides with the induced pairs defined above. So, using the First Isomorphism Theorem for Lie algebras we have  $\text{Der}(K_\vartheta)$  is isomorphic to  $\text{Ker}(\phi) \oplus \text{Im}(\phi)$  but these subspaces correspond to structures:  $\text{Ker}(\phi) \cong Z^1(K, I)$  and  $\text{Im}(\phi) \cong \text{Indu}(K, I, \vartheta)$ . One application of this type of construction is using known information about the algebra  $\text{Der}(K)$  to obtain information about the algebra  $\text{Der}(K_\vartheta)$  as the existence of non-singular derivations. Therefore, this method will allow us to study some properties of Lie algebra extensions by cocycles. First we define  $\phi$ .

Let  $K$  be a Lie algebra and  $I$  a  $K$ -module. Let  $\vartheta \in H^2(K, I)$  and  $d \in \text{Der}(K_\vartheta)$ . Suppose that  $I$ , as ideal of  $K_\vartheta$ , is invariant under  $d$ . Recall that  $K_\vartheta = K \oplus I$  and let  $\pi_K : K_\vartheta \rightarrow K$  and  $\pi_I : K_\vartheta \rightarrow I$  to be the natural vector space projections of  $K_\vartheta$  onto  $K$  and  $K_\vartheta$  onto  $I$ . Then define the maps

- $\alpha : K \rightarrow K$  by  $\alpha(h) = \pi_K d(h)$ , for all  $h \in K$ ;
- $\beta : I \rightarrow I$  by  $\beta(a) = d(a)$ , for all  $a \in I$ ;
- $\eta : K \rightarrow I$  by  $\eta(h) = \pi_I d(h)$ , for all  $h \in K$ .

For each  $h + a \in K_\vartheta$  we have

$$(17) \quad d(h + a) = \alpha(h) + \eta(h) + \beta(a) \text{ for all } h \in K \text{ and } a \in I.$$

We can see that  $\beta$  is a derivation of  $I$  because it is restriction of  $d$  to  $I$ . To see that  $\alpha \in \text{Der}(K)$  let  $x, y \in K$ . To make our calculation more clear, we will denote  $[\cdot, \cdot]_K$  the product in  $K$ , and by  $[\cdot, \cdot]_\vartheta$  the product in  $K_\vartheta$ . Then by product definition on  $K_\vartheta$

$$d([h, k]_\vartheta) = d([h, k]_K + \vartheta(h, k)).$$

By the decomposition showed in (17)

$$(18) \quad d([h, k]_\vartheta) = \alpha([h, k]_K) + \eta([h, k]_K) + \beta(\vartheta(h, k)).$$

We can calculate

$$(19) \quad [d(h), k]_\vartheta + [h, d(k)]_\vartheta = [\alpha(h) + \eta(h), k]_\vartheta + [h, \alpha(k) + \eta(k)]_\vartheta,$$

and use the definition of the product in equation (19) to get

$$(20) \quad [d(h), k]_\vartheta + [h, d(k)]_\vartheta = [\alpha(h), k]_K + \vartheta(\alpha(h), k) - [k, \eta(h)]_\vartheta \\ + [h, \alpha(k)]_K + \vartheta(h, \alpha(k)) + [h, \eta(k)]_\vartheta.$$

Comparing the components of  $K$  in (18) and (20) we have

$$\alpha([h, k]_K) = [\alpha(h), k]_K + [h, \alpha(k)]_K,$$

and  $\alpha \in \text{Der}(K)$ .

Now it is possible define our homomorphism  $\phi$ . Let  $K$  be a Lie algebra and  $I$  a  $K$ -module. Let  $\vartheta \in H^2(K, I)$  and suppose that  $I$ , as an ideal of  $K_\vartheta$ , is invariant under derivations. For

all  $x + a \in K_{\vartheta}$  and  $d \in \text{Der}(K)_{\vartheta}$  write  $d(h + a) = \alpha(h) + \eta(h) + \beta(a)$  with  $\alpha \in \text{Der}(K)$  and  $\beta \in \text{Der}(I)$ . Then define  $\phi : \text{Der}(K_{\vartheta}) \rightarrow \text{Der}(K) \oplus \text{Der}(I)$  by

$$(21) \quad \phi(d) = (\alpha, \beta).$$

The following calculation will check that  $\phi$  is a Lie algebra morphism. Let  $d, d' \in \text{Der}(K_{\vartheta})$  such that

$$\begin{aligned} d(h + a) &= \alpha(h) + \eta(h) + \beta(a) \\ d'(h + a) &= \alpha'(h) + \eta'(h) + \beta'(a), \end{aligned}$$

Then

$$\begin{aligned} dd'(h) &= d(\alpha'(h) + \eta'(h) + \beta'(a)) \\ &= \alpha\alpha'(h) + \eta(\alpha'(h)) + \beta(\eta'(h) + \beta'(a)). \end{aligned}$$

Hence,  $\pi_K dd'(h) = \alpha\alpha'(h)$ . Analogously,  $\pi_K d'd(h) = \alpha'\alpha(h)$ . So  $\pi_K[d, d'] = [\alpha, \alpha']$ . As  $\beta$  and  $\beta'$  are defined by restriction of  $d$  and  $d'$  to  $I$ , respectively, then  $\pi_I[d, d'] = [\beta, \beta']$ . Therefore,

$$\phi([d, d']) = ([\alpha, \alpha'], [\beta, \beta']) = [(\alpha, \beta), (\alpha', \beta')] = [\phi(d), \phi(d')],$$

and  $\phi$  is indeed a Lie algebra homomorphism.

The next result presents the first connection between compatible pairs and the homomorphism  $\phi$ .

**Lemma 3.7.** *Let  $K$  be a Lie algebra and  $I$  a  $K$ -module. Let  $\vartheta \in \text{H}^2(K, I)$  and suppose that  $I$ , as an ideal of  $K_{\vartheta}$ , is invariant under derivations. Let  $\phi : \text{Der}(K_{\vartheta}) \rightarrow \text{Der}(K) \oplus \text{Der}(I)$  given by  $\phi(d) = (\alpha, \beta)$ , defined in (21). Then  $\text{Im}(\phi) \leq \text{Comp}(K, I)$ .*

*Proof.* Let  $(\alpha, \beta) \in \text{Im}(\phi)$ . Then there is  $d \in \text{Der}(K_{\vartheta})$  such that  $\phi(d) = (\alpha, \beta)$ . If  $h \in K$  and  $a \in I$  then

$$\begin{aligned} \beta([h, a]_{\vartheta}) &= d([h, a]_{\vartheta}) && (\text{since } [h, a] \in I) \\ &= [d(h), a]_{\vartheta} + [h, d(a)]_{\vartheta} && (d \in \text{Der}(K_{\vartheta})) \\ &= [\alpha(h) + \eta(h), a]_{\vartheta} + [h, \beta(a)]_{\vartheta} \\ &= [\alpha(h), a]_{\vartheta} + [h, \beta(a)]_{\vartheta} && (\text{since } I \text{ is abelian}). \end{aligned}$$

□

Now we present the main theorem of this section. Recall that for a Lie algebra  $K$ , for a  $K$ -module  $I$ , and for  $\vartheta \in \text{Z}^2(K, I)$ ,  $\text{Indu}(K, I, \vartheta)$  was defined in Definition 3.6.

**Theorem 3.8.** *Let  $K$  be a Lie algebra and let  $I$  be a  $K$ -module. Let  $\vartheta \in \text{H}^2(K, I)$  and suppose that  $I$ , as ideal of  $K_{\vartheta}$ , is invariant by derivations. Let  $\phi : \text{Der}(K_{\vartheta}) \rightarrow \text{Der}(K) \oplus \text{Der}(I)$  be defined as above. Then:*

- (1)  $\text{Im}(\phi) = \text{Indu}(K, I, \vartheta)$
- (2)  $\text{Ker}(\phi) \cong \text{Z}^1(K, I)$

*Proof.* In this proof we will denote the product in  $K_\vartheta$  of  $h \in K$  and  $a \in I$  just by the action  $[h, a]$  of  $K$  on  $I$ , since  $[h, a]_\vartheta = [h, a]$ .

1) Let  $(\alpha, \beta) \in \text{Indu}(K, I, \vartheta)$ . By definition

$$(\alpha, \beta) \cdot \vartheta = 0 \pmod{B^2(K, I)}.$$

Then there is a linear map  $T : K \rightarrow I$  such that, for all  $h, k \in K$ ,

$$(22) \quad \beta(\vartheta(h, k)) - \vartheta(\alpha(h), k) - \vartheta(h, \alpha(k)) = T([h, k]) + [k, T(h)] - [h, T(k)].$$

Let  $h \in K$ ,  $a \in I$  and define the linear map  $(\alpha, \beta)^* : K_\vartheta \rightarrow K_\vartheta$  by

$$(23) \quad (\alpha, \beta)^*(h + a) = \alpha(h) - T(h) + \beta(a).$$

Let's check that  $(\alpha, \beta)^*$  is a derivation of  $K_\vartheta$ . Let  $k + b \in K_\vartheta$ . If

$$X = (\alpha, \beta)^*([h + a, k + b]_\vartheta)$$

then

$$\begin{aligned} X &= (\alpha, \beta)^*([h, k]_K + \vartheta(h, k) + [h, b] - [k, a]) \\ &= \alpha([h, k]_K) - T([h, k]_K) + \beta(\vartheta(h, k)) + \beta([h, b]) - \beta([k, a]). \end{aligned}$$

Now, let

$$Y = [(\alpha + \beta)^*(h + a), k + b]_\vartheta + [h + a, (\alpha + \beta)^*(k + b)]_\vartheta.$$

By definition (23)

$$[(\alpha + \beta)^*(h + a), k + b]_\vartheta = [\alpha(h) - T(h) + \beta(a), k + b]_\vartheta.$$

Hence, by product definition in (6)

$$[\alpha(h) - T(h) + \beta(a), k + b]_\vartheta = [\alpha(h), k]_K + \vartheta(\alpha(h), k) + [\alpha(h), b] - [k, -T(h) + \beta(a)]$$

and

$$[(\alpha + \beta)^*(h + a), k + b]_\vartheta = [\alpha(h), k]_K + \vartheta(\alpha(h), k) + [\alpha(h), b] - [k, -T(h) + \beta(a)].$$

Analogously,

$$[h + a, (\alpha + \beta)^*(k + b)]_\vartheta = [h, \alpha(k)]_K + \vartheta(h, \alpha(k)) + [h, -T(k) + \beta(b)] - [\alpha(k), a].$$

It follows

$$\begin{aligned} Y &= [\alpha(h), k]_K + [h, \alpha(k)]_K + \vartheta(\alpha(h), k) + \vartheta(h, \alpha(k)) \\ &\quad + [\alpha(h), b] + [h, \beta(b)] - [k, \beta(a)] - [\alpha(k), a] - [h, T(k)] + [k, T(h)]. \end{aligned}$$

We can use that  $(\alpha, \beta) \in \text{Comp}(K, I)$  to write  $Y$  as

$$\begin{aligned} Y &= \alpha([h, k]_K) + \vartheta(\alpha(h), k) + \vartheta(h, \alpha(k)) \\ &\quad + \beta([h, b]) - \beta([k, a]) - [h, T(k)] + [k, T(h)]. \end{aligned}$$

By equation (22)

$$\vartheta(\alpha(h), k) + \vartheta(h, \alpha(k)) = \beta(\vartheta(h, k)) - T([h, k]) - [k, T(h)] + [h, T(k)].$$

Then

$$Y = [\alpha(h), k]_K + [h, \alpha(k)]_K + \beta(\vartheta(h, k)) - T([h, k]) - [k, T(h)] + [h, T(k)] \\ + \beta([h, b]) - \beta([k, a]) - [h, T(k)] + [k, T(h)].$$

As  $X = Y$  then  $(\alpha, \beta)^*$  is a derivation.

Besides, observe that  $\pi_K(\alpha, \beta)^* = \alpha$  and  $\pi_I(\alpha, \beta)^* = \beta$ . Hence  $\phi((\alpha, \beta)^*) = (\alpha, \beta)$ , that is,  $\text{Indu}(K, I, \vartheta) \subseteq \text{Im}(\phi)$ .

Now, suppose that  $(\alpha, \beta) \in \text{Im}(\phi)$ . Then there is  $d \in \text{Der}(K_\vartheta)$  such that

$$\phi(d) = (\alpha, \beta).$$

By Theorem 3.7 we have  $\text{Im}(\phi) \subseteq \text{Comp}(K, I)$ . Then it is enough to show that there is a linear map  $T : K \rightarrow I$  such that the equation (22) is satisfied.

For each  $h + a \in K_\vartheta$  we can use the decomposition defined in (17) to write

$$d(h + a) = \alpha(h) + \eta(h) + \beta(a).$$

Then

$$[d(h + a), k + b]_\vartheta = [\alpha(h) + \eta(h) + \beta(a), k + b]_\vartheta.$$

By product definition in (6) we get

$$[\alpha(h) + \eta(h) + \beta(a), k + b]_\vartheta = [\alpha(h), k]_K + \vartheta(\alpha(h), k) + [\alpha(h), b] - [k, \eta(h) + \beta(a)].$$

Hence

$$[d(h + a), k + b]_\vartheta = [\alpha(h), k]_K + \vartheta(\alpha(h), k) + [\alpha(h), b] - [k, \eta(h) + \beta(a)].$$

Analogously,

$$[h + a, d(k + b)]_\vartheta = [h, \alpha(k)]_K + \vartheta(h, \alpha(k)) + [h, \eta(k) + \beta(b)] - [\alpha(k), a].$$

Therefore

(24)

$$[d(h + a), k + b]_\vartheta + [h + a, d(k + b)]_\vartheta = [\alpha(h), k]_K + [h, \alpha(k)]_K + \vartheta(\alpha(h), k) + \vartheta(h, \alpha(k)) \\ + [\alpha(h), b] + [h, \beta(b)] - [\alpha(k), a] - [k, \beta(a)] - [k, \eta(h)] + [h, \eta(k)].$$

We can use that  $(\alpha, \beta) \in \text{Comp}(K, I)$  in the last displayed equation to write

$$[d(h + a), k + b]_\vartheta + [h + a, d(k + b)]_\vartheta = \alpha([h, k]_K) + \vartheta(\alpha(h), k) + \vartheta(h, \alpha(k)) \\ + \beta([h, b]) - \beta([k, a]) - [k, \eta(h)] + [h, \eta(k)].$$

Now we will calculate  $d([k + a, h + b]_\vartheta)$ . By product definition

$$d([h + a, k + b]_\vartheta) = d([h, k]_K + \vartheta(h, k) + [h, b] - [k, a]).$$

Hence

$$d([h, k]_K + \vartheta(h, k) + [h, b] - [k, a]) = \alpha([h, k]_K) + \eta([h, k]_K) + \beta(\vartheta(h, k)) + \beta([h, b]) - \beta([k, a]).$$

As  $d$  is a derivation then we have equality

$$d([h + a, k + b]_{\vartheta}) = [d(h + a), k + b]_{\vartheta} + [h + a, d(k + b)]_{\vartheta}.$$

It follows

$$\vartheta(\alpha(h), k) + \vartheta(h, \alpha(k)) - [k, \eta(h)] + [h, \eta(k)] = \eta([h, k]_K) + \beta(\vartheta(h, k)).$$

We can rearrange the last displayed equation to get

$$-(\eta([h, k]_K) + [k, \eta(h)] - [h, \eta(k)]) = \beta(\vartheta(h, k)) - \vartheta(\alpha(h), k) - \vartheta(h, \alpha(k)).$$

Therefore  $T = -\eta$  satisfies the equation (22) e  $\text{Im}(\phi) \subseteq \text{Indu}(K, I, \vartheta)$ .

2) Let  $d \in \text{Ker}(\phi)$ . The decomposition showed in (17) provide us

$$d(h) = \eta(h), h \in K.$$

Let  $h, k \in K$ . By definition of derivation

$$(25) \quad d([h, k]_{\vartheta}) = [d(h), k]_{\vartheta} + [h, d(k)]_{\vartheta}.$$

We can use product definition in  $K_{\vartheta}$  to write

$$d([h, k]_{\vartheta}) = d([h, k]_K + \vartheta(h, k)).$$

Since  $d \in \text{Ker}(\phi)$  then

$$d([h, k]_{\vartheta}) = \eta([h, k]_K).$$

By other hand,

$$[d(h), k]_{\vartheta} + [h, d(k)]_{\vartheta} = [\eta(h), k]_{\vartheta} + [h, \eta(k)]_{\vartheta}.$$

Then (25) it is equal to

$$\eta([k, h]_K) = [k, \eta(h)] - [h, \eta(k)],$$

and  $\eta \in Z^1(K, I)$ . Observe that  $\eta$  is the restriction of  $d$  to  $K$ . Denote the restriction of  $d$  to  $K$  by  $d|_K$ . Therefore, if  $d \in \text{Ker}(\phi)$  then  $d|_K \in Z^1(K, I)$ .

Let  $d \in \text{Ker}(\phi)$  and define  $\sigma : \text{Ker}(\phi) \rightarrow (Z^1(K, I), +)$  by  $\sigma(d) = d|_K$ . Then  $\sigma(\text{Ker}(\phi)) \subseteq Z^1(K, I)$ . Let  $d' \in \text{Ker}(\phi)$ . Then

$$\sigma(d + d') = (d + d')|_K = d|_K + d'|_K = \sigma(d) + \sigma(d').$$

So  $\sigma$  it is group homomorphism.

First we will show that  $\sigma$  is injective. Let  $d, d' \in \text{Ker}(\phi)$  such that  $\sigma(d) = \sigma(d')$ . Let  $h + a \in K_{\vartheta}$ . Then

$$d(h + a) = d(h) = d|_K(h) = d'|_K(h) = d'(h) = d(h + a).$$

Hence  $d = d'$ . Now, to prove that  $\sigma$  is onto, let  $\eta \in Z^1(K, I)$  and define a linear map  $d : K_{\vartheta} \rightarrow K_{\vartheta}$  by

$$d(h + a) = T(x), h \in K, a \in I.$$

We will show that  $d$  is a derivation. Observe that, for all  $h + a, k + b \in K_{\vartheta}$  we have

$$d([h + a, k + b]_{\vartheta}) = d([h, k]_K + \vartheta(h, k) + [h, b] - [k, a]) = \eta([h, k]_K).$$



By other hand,

$$\begin{aligned} [d(h+a), k+b]_{\vartheta} + [h+a, d(k+b)]_{\vartheta} &= [\eta(h), k+b]_{\vartheta} + [h+a, \eta(k)]_{\vartheta} \\ &= -[k, \eta(h)] + [h, \eta(k)]. \end{aligned}$$

Since  $\eta \in Z^1(K, I)$  then  $d([h+a, k+b]_{\vartheta}) = [d(h+a), k+b]_{\vartheta} + [h+a, d(k+b)]_{\vartheta}$ , hence  $d \in \text{Der}(K_{\vartheta})$ . It is immediate that  $\phi(d) = 0$ . So  $d \in \text{Ker}(\phi)$ . As by definition,  $\sigma(d) = \eta$  then  $\sigma$  is onto and, therefore, is an isomorphism.  $\square$

**Example 3.9.** Let  $L$  be a Lie algebra with an abelian ideal  $I$  invariant by derivations. Set  $K = L/I$ . By Proposition 3.1, there is a  $\vartheta \in Z^2(K, I)$  such that  $L \cong K_{\vartheta}$ . Then we can apply the map  $\phi : \text{Der}(L) \rightarrow \text{Der}(L/I) \oplus \text{Der}(I)$  defined in Theorem 3.8. Further, if  $d \in \text{Der}(L)$  then  $\phi(d) = (\alpha, \beta) \in \text{Comp}(L/I, I)$ . Hence, each derivation of  $L$  gives rise to a pair of derivations  $\alpha \in \text{Der}(L/I)$  and  $\beta \in I$ . In particular, if  $d$  is non-singular then  $\alpha$  and  $\beta$  are non-singular.

**3.4. Compatible pairs and Jacobson Theorem.** In this section we show some examples of the use of compatible pairs in the study of non-singular derivations.

**Example 3.10.** Let  $K$  and  $I$  be finite-dimensional Lie algebras over an algebraically closed field  $\mathbb{F}$ . Suppose that  $K$  acts on  $I$  by representation  $\psi : K \rightarrow \text{Der}(I)$ . Let  $D \leq \text{Comp}(K, I)$  be a subalgebra. Define  $L = K \oplus I$ . By Proposition 3.2,  $D \leq \text{Der}(L)$ . If  $D$  is nilpotent then  $L$  has a decomposition into generalized eigenspaces of  $D$ . This decomposition induces decompositions in  $K$  and  $I$ , since as  $K$  and  $I$  are invariants under  $D$ . Hence,

$$L = K_{\lambda_1} \oplus \cdots \oplus K_{\lambda_r} \oplus I_{\mu_1} \oplus \cdots \oplus I_{\mu_s}.$$

In particular, we have  $[K_{\lambda_i}, I_{\mu_j}] \subseteq I_{\lambda_i + \mu_j}$  if  $\lambda_i + \mu_j$  is eigenvalue of  $D$  in  $I$ . Otherwise  $[K_{\lambda_i}, I_{\mu_j}] = 0$ .

From this example we can state a result:

**Proposition 3.11.** *Let  $K$  and  $I$  be finite-dimensional Lie algebras over an algebraically closed field  $\mathbb{F}$ . Suppose that  $K$  acts on  $I$  by representation  $\psi : K \rightarrow \text{Der}(I)$ . Let  $D \leq \text{Comp}(K, I)$  be a subalgebra. Suppose that 0 is not generalized eigenvalue of  $D$ . Then if either the characteristic of  $\mathbb{F}$  is zero or the characteristic of  $\mathbb{F}$  is  $p$  and  $D$  has at most  $p-1$  generalized eigenvalues, then the Lie subalgebra  $\psi(K) \leq \mathfrak{gl}(I)$  is nilpotent.*

*Proof.* Let  $L = K_{\lambda_1} \dot{+} \cdots \dot{+} K_{\lambda_r} \dot{+} I_{\mu_1} \oplus \cdots \oplus I_{\mu_s}$  the generalized eigenspace decomposition presented in Example 3.10. Suppose that 0 is not generalized eigenvalues of  $D$ . Let  $E_K = \{\lambda_1, \dots, \lambda_r\}$  and  $E_I = \{\mu_1, \dots, \mu_s\}$  be the generalized eigenvalues of  $D$  in  $K$  and  $I$ , respectively. Let  $k \in K_{\alpha_j}, a \in I_{\mu_i}$  then

$$\begin{cases} \psi^n(k)(a) \in I_{\mu_i + n\lambda_j} & \text{if } \mu_i + n\lambda_j \in E_I \\ \psi^n(k)(a) = 0 & \text{if } \mu_i + n\lambda_j \notin E_I. \end{cases}$$

- If the characteristic of  $\mathbb{F}$  is 0 then the linear functions  $\mu_i + \lambda_j, \mu_i + 2\lambda_j, \dots, \mu_i + n\lambda_j, \dots$  are all distinct since  $\lambda_j \neq 0$ . Since  $\dim I$  is finite, so  $\mu_i + n\lambda_j \notin E_I$  for some  $n > 0$ . Hence  $\psi(k)^n = 0$ .
- If the characteristic of  $\mathbb{F}$  is  $p > 0$  and  $s < p$  then the linear forms  $\{\mu_i + \lambda_j, \mu_i + 2\lambda_j, \dots, \mu_i + (p-1)\lambda_j, \mu_i\}$  cannot be all non-trivial, and so  $\mu_i + n\lambda_j = 0$  for some  $1 \leq n \leq p$ , and so  $\psi^n(k) = 0$  for some  $n, 1 \leq n \leq p$ .

In both cases  $\psi(k)$  is nilpotent for all  $k \in K_{\lambda_j}, 1 \leq j \leq r$ . Let  $S = \bigcup \psi(K_{\lambda_j})$ . Since  $S$  is a weakly closed set such that each element is nilpotent. Then the associative subalgebra  $\langle S \rangle \leq \text{End}(I)$  is nilpotent. We conclude that the Lie algebra  $\langle S \rangle = \psi(K) \leq \mathfrak{gl}(I)$  is nilpotent.  $\square$

For our next example we need some result about traces of matrices.

**Proposition 3.12.** *Let  $\mathbb{F}$  be a field of characteristic  $p \geq 0$ . Suppose that  $A \in \mathbf{M}(n, \mathbb{F})$  with  $n < p$  or  $p = 0$ . Then  $A$  is nilpotent if, and only if, the trace of matrices  $A^r$  is zero, for  $1 \leq r \leq n$ .*

*Proof.* Let  $\overline{\mathbb{F}}$  be the algebraic closure of  $\mathbb{F}$  and assume without loss of generality that  $A$  is in Jordan normal form. We will use that a matrix is nilpotent if, and only if, zero is the only eigenvalue of  $A$ .

Hence  $A$  is as a diagonal block matrix where each block is formed by grouping the Jordan blocks associated to same eigenvalue. Let  $\lambda_1, \dots, \lambda_k$  be the non-zero eigenvalues of  $A$ . Denote by  $A_t$  the diagonal block in  $A$  associated with eigenvalue  $\lambda_t$  and let assume that  $A_t$  is an  $n_j \times n_j$ -matrix. Then

$$(26) \quad \text{tr}(A^r) = n_1 \lambda_1^n + \dots + n_k \lambda_k^n.$$

Suppose that  $A$  is nilpotent. Then zero is the only eigenvalue of  $A$ , and also of  $A^r$  for all  $r \geq 1$ , and by equation (26) we have  $\text{tr}(A^r) = 0$  for  $1 \leq r \leq n$ .

Conversely, suppose that  $\text{tr}(A^r) = 0$  for  $1 \leq r \leq n$ . From equation (26) we can extract the system

$$(27) \quad n_1 \lambda_1^r + \dots + n_k \lambda_k^r = 0, \quad 1 \leq r \leq k,$$

of linear equations in the variables  $n_1, \dots, n_k$  over  $\mathbb{F}$  considering each  $n_j$  as the element  $n_j \cdot 1$  in  $\mathbb{F}$ , whose matrix of coefficients is

$$C = \begin{bmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \dots & \lambda_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^k & \lambda_2^k & \dots & \lambda_k^k \end{bmatrix}.$$

Denote by  $m_i(\lambda)$  the operation that multiplies line  $i$  of a matrix by  $\lambda$  and  $A^t$  the transposed matrix of  $A$ . So we can write

$$C = m_1(\lambda_1).m_2(\lambda_2) \dots m_k(\lambda_k).V^t,$$

where

$$V = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{k-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_k & \lambda_k^2 & \cdots & \lambda_k^{k-1} \end{bmatrix}$$

is the Vandermonde matrix in the variables  $\lambda_1, \lambda_2, \dots, \lambda_k$  whose determinant is  $\det V = \prod_{1 \leq i < j \leq k} (\lambda_j - \lambda_i)$ . As the  $\lambda_i$  are pairwise distinct we have that  $\det V$  is non-zero. Then the determinant of  $C$  is  $\lambda_1 \lambda_2 \cdots \lambda_k \cdot \det V$ . As we assume that  $\lambda_i \neq 0$  for  $1, \dots, k$ ,  $C$  is non-singular. It follows that the system (27) has only trivial solution. Therefore, considered as an element of  $\mathbb{F}$ , each  $n_j$  is zero. If  $p = 0$  then zero is the only eigenvector of  $A$ . If  $p > 0$ , then, since we assume that  $n < p$ , we also have that  $n_j < p$  for all  $j$ . Hence the fact that  $n_j = 0$  in  $\mathbb{F}$ , implies that  $n_j = 0$  as a natural number. Conclude that zero is the only eigenvalue of  $A$ .  $\square$

**Proposition 3.13** ([1], Fact 3.17.13). *Let  $\mathbb{F}$  be a field of characteristic  $p > 0$ . Let  $A, B, C \in \mathbf{M}(n, \mathbb{F})$  with  $p = 0$  or  $n < p$ . If  $[A, B] = C + \lambda B$ , for some  $\lambda \in \mathbb{F}$  and  $[B, C] = 0$  then  $[A, B^r] = rB^{r-1}C + \lambda rB^r$  for all  $r \geq 1$ . In particular, if  $\lambda \neq 0$  and  $C$  is nilpotent then  $B$  is nilpotent.*

*Proof.* We prove this result by induction on  $r$ . The case  $r = 1$  follows from the conditions. Suppose that result is valid for  $(r - 1)$ . Then,

$$[A, B^{r-1}] = (r - 1)B^{r-2}C + \lambda(r - 1)B^{r-1}.$$

We can rewrite this equation as

$$\lambda(r - 1)B^{r-1} = AB^{r-1} - B^{r-1}A - (r - 1)B^{r-2}C.$$

Multiplying the the last equation on the right by  $B$  we have

$$\lambda(r - 1)B^r = AB^r - B^{r-1}(AB) - (r - 1)B^{r-2}(CB).$$

By the conditions we can write  $AB = BA + C + \lambda B$  and  $CB = BC$ . Replacing these terms above we obtain

$$\lambda(r - 1)B^r = AB^r - B^rA - B^{r-1}C - \lambda B^r - (r - 1)B^{r-1}C.$$

Therefore,

$$AB^r - B^rA = \lambda rB^r + rB^{r-1}C.$$

For the second statement suppose  $\lambda \neq 0$  and  $C$  is nilpotent with nilpotency index  $m$ . Using the first assertion we have

$$B^r = (1/\lambda r)[A, B^r] - (1/\lambda)B^{r-1}C, \text{ for all } r \geq 1.$$

Since,  $B$  and  $C$  commute,  $(B^{r-1}C)^m = (B^{r-1})^m(C)^m = 0$ , Hence, for all  $r \geq 1$   $B^{r-1}C$  is nilpotent and has trace zero by Proposition 3.12. As the trace of commutators is always

zero then  $\text{tr}([A, B^r]) = 0$  for all  $r \geq 1$ . It follows that  $\text{tr}(B^r) = 0$  for all  $r \geq 1$  and again by Proposition 3.12 we conclude that  $B$  is nilpotent.  $\square$

Now we can present a result similar to the Proposition 3.11 but with a new proof using compatible pairs.

**Theorem 3.14.** *Let  $K$  and  $I$  be finite dimensional Lie algebras over a field of characteristic  $p \geq 0$  such that  $K$  is solvable. Suppose that  $K$  acts on  $I$  by representation  $\psi : K \rightarrow \text{Der}(I)$ . Let  $(\alpha, \beta) \in \text{Comp}(K, I)$  such that  $\alpha$  has no eigenvalue 0. If either  $p = 0$  or  $p > 0$  and dimension of  $I$  is less than  $p$  then  $\text{Tr}(\psi^n(k)) = 0$ , for all  $k \in K$ . In these two cases,  $\psi(k)$  is nilpotent for all  $k \in K$ .*

*Proof.* As  $\alpha$  has no eigenvalue 0, it is non-singular. Suppose that the order of  $\alpha$ , considered as an endomorphism of  $I$ , is  $p^t m$ . Then by Lemma 3.3,  $(\alpha, \beta)^{p^t} = (\alpha^{p^t}, \beta^{p^t})$  is a compatible pair and by Proposition 2.16,  $\alpha^{p^t}$  is diagonalizable. Hence by possibly replacing  $(\alpha, \beta)$  by  $(\alpha, \beta)^{p^t}$ , we may assume without loss of generality that  $\alpha$  is diagonalizable. Let  $x_1, \dots, x_s$  be a basis of  $K$  such that  $\alpha(x_i) = \lambda_i x_i$ . For all  $a \in \mathfrak{gl}(I)$  denote by  $[a]$  the matrix of  $a$  in this basis. Then, by equation (8),

$$[[\beta], [\psi(x_i)]] = \lambda_i [\psi(x_i)].$$

We can apply Proposition 3.13 to this last equation for  $A = [\beta]$ ,  $B = [\psi(x_i)]$ ,  $C = 0$  and  $\lambda = \lambda_i \neq 0$  to conclude that  $\psi(x_i)$  is nilpotent for  $1 \leq i \leq s$ . Now we observe that if  $K$  is a nilpotent Lie algebra in either characteristic is 0 or characteristic  $p$  with dimension of  $L$  less than  $p$  then Lie's Theorem (Theorem 2.6) is valid. Lie's Theorem grants that there is a basis of  $I$  such that the image of  $\psi$  lies in the subalgebra of  $\mathfrak{gl}(I)$  formed by upper triangular matrices. Since  $[\psi(x_i)]$  is nilpotent and upper triangular, it must be strictly upper triangular (that is, it contains zeros in the diagonal). Then all  $\psi(k)$ , for all  $k \in K$ , are also strictly upper triangular matrices, since they are linear combinations of the  $\psi(x_i)$ . Hence every  $\psi(k)$  is nilpotent.  $\square$

**Corollary 3.15.** *Let  $L$  be a solvable Lie algebra over a field  $\mathbb{F}$  of characteristic  $p \geq 0$ . Suppose that  $L$  has a nonsingular derivation. If either  $p = 0$  or  $p > 0$  and dimension of  $L^{(i)}/L^{(i+1)} < p$ , for all  $i$ , then  $L$  is nilpotent.*

*Proof.* Suppose that  $L > L^{(1)} > \dots > L^{(k)} > L^{(k+1)} = 0$  is the derived series of  $L$ . We prove this result by induction on  $k$ . When  $k = 0$ , then  $L$  is clearly nilpotent, as it is actually abelian. Suppose that the result holds for Lie algebras of derived length  $k - 1$  and assume that  $L$  has derived length  $k$ . Then  $I = L^{(k)}$  is an abelian ideal of  $L$ . Setting  $K = L/I$ , we have that  $K$  acts on  $I$  (see Example 2.7) and let us call the corresponding representation  $\psi$ . Further, since the terms of the derived series are invariant under derivations, a non-singular derivation  $\delta \in \text{Der}(L)$  gives rise to a compatible pair  $(\alpha, \beta) \in \text{Comp}(K, I)$  as explained in Example 3.9. Since  $\delta$  is non-singular, so are  $\alpha \in \text{Der}(K)$  and  $\beta \in \text{Der}(I)$ . Note that  $K$  is solvable of solvable length  $k - 1$  and  $K^{(i)}/K^{(i+1)} \cong L^{(i)}/L^{(i+1)}$  for all  $i \leq k - 1$ . Hence the induction hypothesis is valid for  $K$  and we obtain that  $K$  is nilpotent. Further, since

$\dim I < p$ , we have that  $\psi(k)$  is nilpotent for all  $k$ . Now Proposition 2.3 implies that  $L$  is nilpotent.  $\square$

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