

# A NOTE ON AUTOMORPHISMS AND DERIVATIONS OF LIE ALGEBRAS

N. JACOBSON

In a recent paper Borel and Serre proved the theorem: If  $\mathfrak{L}$  is a Lie algebra of characteristic 0 and  $\mathfrak{L}$  has an automorphism of prime period without fixed points  $\neq 0$ , then  $\mathfrak{L}$  is nilpotent.<sup>1</sup> In this note we give a proof valid also for characteristic  $p \neq 0$ . By the same method we can prove several other similar results on automorphisms and derivations. Our method is based on decompositions of the Lie algebra which determine weakly closed sets of linear transformations. Such a set  $\mathfrak{W}$  has, by definition, the closure property that if  $A, B \in \mathfrak{W}$  then there exists a  $\gamma(A, B)$  in the base field such that  $AB + \gamma BA \in \mathfrak{W}$ . The main result we shall need is the generalized Engel theorem that if  $\mathfrak{W}$  is a weakly closed set of nilpotent linear transformations in a finite-dimensional vector space, then the enveloping associative algebra  $\mathfrak{W}^*$  of  $\mathfrak{W}$  is nilpotent [3].

**THEOREM 1.** *If  $\mathfrak{L}$  is a Lie algebra with an automorphism  $\sigma$  of prime period  $l$  and  $\sigma$  has no fixed points  $\neq 0$ , then  $\mathfrak{L}$  is nilpotent.*

**PROOF.** If  $\Omega$  is the algebraic closure of the base field, then we can extend  $\sigma$  to  $\mathfrak{L}_\Omega$ . This extension has order  $l$  and no fixed points  $\neq 0$ . Hence we may as well suppose that the base field is algebraically closed. Then  $\mathfrak{L} = \sum_{\zeta} \mathfrak{L}_{\zeta}$  where  $\mathfrak{L}_{\zeta}$  is the subspace of  $\mathfrak{L}$  corresponding to the characteristic root  $\zeta$  of  $\sigma$ . Now  $[\mathfrak{L}_{\zeta}, \mathfrak{L}_{\zeta'}] \subseteq \mathfrak{L}_{\zeta\zeta'}$  if  $\zeta\zeta'$  is a characteristic root and  $[\mathfrak{L}_{\zeta}, \mathfrak{L}_{\zeta'}] = 0$  otherwise. Let  $\text{Ad } (\mathfrak{L}_{\zeta})$  denote the set of adjoint mappings determined by the elements of  $\mathfrak{L}_{\zeta}$ . Then the relation just noted shows that the set  $\mathfrak{W} = \bigcup \text{Ad } (\mathfrak{L}_{\zeta})$  is a weakly closed set of linear transformations (with  $\gamma(A, B) = -1$ ). The roots  $\zeta$  are the  $p$ th roots of 1 and, by assumption, no  $\zeta = 1$ . If  $l = p$  the characteristic of the base field, then 1 is the only  $p$ th root of 1 and  $\mathfrak{L} = 0$ . Hence assume  $l \neq p$ . If  $x \in \mathfrak{L}_{\zeta}$  and  $a \in \mathfrak{L}_{\zeta'}$ , then  $x(\text{Ad } a) = [xa]$  is either 0 or is in  $\mathfrak{L}_{\zeta\zeta'}$ . Hence  $x(\text{Ad } a)^k$  is either 0 or is in  $\mathfrak{L}_{\zeta(\zeta')^k}$ . Since  $l \neq p$ ,  $\zeta'$  is primitive and  $k$  can be chosen so that  $\zeta(\zeta')^k = 1$ . This gives  $x(\text{Ad } a)^k = 0$ . Since  $\zeta'$  is arbitrary this shows that  $\text{Ad } a$  is nilpotent for every  $a \in \mathfrak{L}_{\zeta'}$ . Hence every element of  $\mathfrak{W}$  is nilpotent. We can therefore conclude that the enveloping associative algebra of  $\mathfrak{W}$  and hence of  $\text{Ad } (\mathfrak{L})$  is nilpotent. Hence  $\mathfrak{L}$  is a nilpotent Lie algebra.

Received by the editors May 27, 1954.

<sup>1</sup> [1, Prop. 4]. Borel and Serre state this only for base field the reals or complexes. However, their proof is valid for any base field of characteristic 0.

**THEOREM 2.** *Let  $\mathfrak{L}$  be a Lie algebra which possesses an automorphism  $\sigma$  none of whose characteristic roots are roots of unity. Then  $\mathfrak{L}$  is nilpotent.*

**PROOF.** As in the preceding proof we may assume the base field is algebraically closed and we have  $\mathfrak{L} = \sum \mathfrak{L}_\zeta$ ,  $\mathfrak{L}_\zeta$  the space of the characteristic root  $\zeta$  of  $\sigma$ . Since the elements  $\zeta(\zeta')^k$ ,  $k=1, 2, \dots$ , are unequal, not all of these are roots. This implies that  $\text{Ad } a$  is nilpotent for any  $a \in \mathfrak{L}_\zeta$ . The remainder of the argument is identical with that of the preceding proof.

**THEOREM 3.** *Let  $\mathfrak{L}$  be a Lie algebra of characteristic 0 and suppose that there exists a subalgebra  $\mathfrak{D}$  of the algebra of derivations of  $\mathfrak{L}$  such that (i)  $\mathfrak{D}$  is nilpotent and (ii) there are no  $\mathfrak{D}$ -constants  $\neq 0$  ( $cD=0$  for all  $D \in \mathfrak{D} \Rightarrow c=0$ ). Then  $\mathfrak{L}$  is nilpotent. Next let  $\mathfrak{L}$  be a restricted Lie algebra of characteristic  $p \neq 0$  and  $\mathfrak{D}$  an algebra of restricted derivations in  $\mathfrak{L}$ . Suppose (i)  $\mathfrak{D}$  is nilpotent as an ordinary Lie algebra and (ii) there are no  $\mathfrak{D}$ -constants  $\neq 0$ .<sup>2</sup> Then  $\mathfrak{L}$  is a nilpotent restricted Lie algebra.*

**PROOF.** As before we may assume the base field is algebraically closed. Since  $\mathfrak{D}$  is nilpotent, a result of Zassenhaus [4, p. 28] states that  $\mathfrak{L} = \sum \mathfrak{L}_\alpha$  where the  $\mathfrak{L}_\alpha$  are the weight space of  $\mathfrak{D}$ . By assumption 0 is not a weight. Also we have the relation  $[\mathfrak{L}_\alpha \mathfrak{L}_\beta] = 0$  if  $\alpha + \beta$  is not a weight and  $[\mathfrak{L}_\alpha \mathfrak{L}_\beta] \subseteq \mathfrak{L}_{\alpha+\beta}$  if  $\alpha + \beta$  is a weight. This implies that the set  $\bigcup_\alpha \text{Ad } (\mathfrak{L}_\alpha)$  is weakly closed. Also, in the characteristic 0 case, the relation implies that  $\text{Ad } a$  is nilpotent for every  $a \in \mathfrak{L}_\beta$ . Thus every element of  $\bigcup_\alpha \text{Ad } (\mathfrak{L}_\alpha)$  is nilpotent and the result follows as before. Suppose next that the characteristic is  $p \neq 0$  and that  $\mathfrak{L}$  is restricted and the given derivations are restricted. In this case we assert that  $\mathfrak{L}_\alpha^p = 0$ . Let  $x \in \mathfrak{L}_\alpha$ ,  $D \in \mathfrak{D}$ . Then  $x(D - \alpha)^N = 0$  for  $N$  sufficiently large. Since  $\mathfrak{D}$  is restricted, so is  $D^{p^k}$ ; hence

$$\begin{aligned} x^p D^{p^k} &= [\dots [x D^{p^k}, x], \dots, x] && (p-1 \text{ } x\text{'s}) \\ &= [\dots [x(D^{p^k} - \alpha^{p^k}), x], \dots, x] && (p-1 \text{ } x\text{'s}) \\ &= [\dots [x(D - \alpha)^{p^k}, x] \dots x] = 0 \end{aligned}$$

if  $p^k \geq N$ . Thus  $x^p$  is in the space of the characteristic root 0 for  $D$ . Since this holds for all  $D \in \mathfrak{D}$ ,  $x^p = 0$ . This relation implies  $(\text{Ad } x)^p = 0$  so that again every element of the weakly closed set  $\bigcup_\alpha \text{Ad } (\mathfrak{L}_\alpha)$  is nilpotent. It follows that  $\mathfrak{L}$  is nilpotent as an ordinary Lie algebra. We can now see from the identity for  $(a+b)^{p^k}$  that every element of

<sup>2</sup> Cf. [2] for definitions and elementary properties of restricted Lie algebras.

$\mathfrak{L}$  is nilpotent in the sense that  $a^{p^k}=0$  for  $k$  large. Hence  $\mathfrak{L}$  is a nilpotent restricted Lie algebra [2, p. 24].

REMARKS. It would be interesting to know if the last theorem is characteristic of nilpotent (restricted) Lie algebras. More precisely: If  $\mathfrak{L}$  is a nilpotent (restricted) Lie algebra, then does  $\mathfrak{L}$  have a derivation without constants  $\neq 0$ ? Since a derivation without constants  $\neq 0$  is not inner the following theorem, due to Schenkman, may be considered a partial answer to our question.

**THEOREM 4.** *Every nilpotent Lie algebra has a derivation  $D$  which is not inner.*

PROOF. Write  $\mathfrak{L} = \mathfrak{M} \oplus \Phi e$  where  $\mathfrak{M}$  is an ideal in  $\mathfrak{L}$ . Let  $\mathfrak{Z}$  be the subalgebra of  $\mathfrak{L}$  of elements commuting with every  $m \in \mathfrak{M}$ . Let  $n$  be chosen so that  $\mathfrak{Z} \subseteq \mathfrak{L}^n$ ,  $\nsubseteq \mathfrak{L}^{n+1}$  and choose  $z \in \mathfrak{Z}$ ,  $\nsubseteq \mathfrak{L}^{n+1}$ . Then the linear transformation  $D$  sending  $l = m + \lambda e$ ,  $m \in \mathfrak{M}$ ,  $\lambda \in \Phi$ , onto  $\lambda z$  is a derivation, as is readily verified. This derivation is outer. For, otherwise we would have  $D = \text{Ad } q$ ,  $q \in \mathfrak{L}$ . Now  $q \in \mathfrak{Z}$  and  $[e, q] = z$ . Thus  $q \in \mathfrak{L}^n$  and hence  $z = [eq] \in \mathfrak{L}^{n+1}$  contrary to its choice.

#### REFERENCES

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YALE UNIVERSITY