

UNIVERSIDADE FEDERAL DE MINAS GERAIS
INSTITUTO DE CIÊNCIAS EXATAS
Departamento de Matemática

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**DERIVATIONS OF LIE ALGEBRA EXTENSIONS AND
NON-SINGULAR DERIVATIONS OF LIE ALGEBRAS IN
PRIME CHARACTERISTIC**

Marcos Goulart Lima

Orientador: Csaba Schneider

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MARCOS GOULART LIMA

1. INTRODUCTION

Let L be a Lie algebra and d be a derivation of L . The derivation d is non-singular if it is injective as linear transformation. We are interested in studying what information we can obtain about a Lie algebra if it has a nonsingular derivation. Jacobson's famous theorem [5] states that a finite-dimensional Lie algebra over a field of characteristic zero that admits a non-singular derivation must be nilpotent. It is well-known that this theorem is not valid when the characteristic is non-zero. Non-nilpotent and solvable examples were constructed by Shalev [10] and Mattarei [8], whereas the simple Lie algebras with non-singular derivations were classified by Benkart and her collaborators in [3]. A significant application of Lie algebras with non-singular derivation in characteristic p was presented by Shalev [9]. In his proof of the coclass conjectures of Leddham-Green and Newman for pro- p groups, Shalev uses the fact that finite-dimensional Lie algebras over a field of characteristic $p > 0$ with non-singular derivation d such that $d^{p-1} = 1$, must be nilpotent.

Despite the existing examples, little is known about non-nilpotent Lie algebras with non-singular derivations. In these project we propose to explore the structure of solvable, non-nilpotent Lie algebras with non-singular derivations. In order to study these algebras we develop a theory of derivations of Lie algebra extensions. We adopt the concept of a compatible pair of automorphisms introduced in [2] for derivations of Lie algebras.

Let K and I be Lie algebras such that K acts on I , then we can define the subalgebra $\text{Comp}(K, I)$ of $\text{Der}(K) \oplus \text{Der}(I)$ as the set of derivations of $\text{Der}(K) \oplus \text{Der}(I)$ that are derivations of semi-direct sum $K \ltimes I$. Formally,

$$\text{Comp}(K, I) = \{\alpha + \beta \in \mathfrak{gl}(K) \oplus \mathfrak{gl}(I) \mid \alpha + \beta \in \text{Der}(K \ltimes I)\}.$$

The algebra $\text{Der}(K)$ carries information about the multiplicative structure of K . Analogously, the algebra $\text{Comp}(K, I)$ carries information about the action of K on I . In section 3.4 we present an example of this by exploring the proof of Jacobson's Theorem and we prove a version for Lie algebras representations over a field of characteristic $p > 0$.

Theorem 3.13 *Let K and I be finite dimensional Lie algebras over a field of characteristic p where $p \geq 0$ such that K is nilpotent. Suppose that K act on I by representation $\psi : K \rightarrow \text{Der}(I)$. Let $(\alpha, \beta) \in \text{Comp}(K, I)$ such that α has no eigenvalue 0. If either $p = 0$*

or $p > 0$ and $\dim I < p$ then $\text{Tr}(\psi^n(k)) = 0$, for all $k \in K$ and $n > 0$. In these two cases, $\psi(k), k \in K$ is nilpotent.

We also adapt an algorithm presented by Bettina Eick [2] for calculating the automorphism group of solvable Lie algebras. A key step in the algorithm is the following. Let L be a Lie algebra and I an abelian ideal of L such that I is invariant by $\text{Aut}(L)$. Then there exists a homomorphism $\phi : \text{Aut}(L) \rightarrow \text{Aut}(L)/I \times \text{Aut}(I)$ induced by the actions of $\text{Aut}(L)$ on L/I and I . The image of ϕ can be calculated using $\text{Aut}(L/I)$, while $\text{Ker}(\phi)$ is equal to $Z^1(K, I)$. Then the group $\text{Aut}(L)$ can be obtained applying the first isomorphism theorem to ϕ . It is possible to use this process to derivations.

We can define a Lie algebra homomorphism similar to ψ in the previous paragraph. Let L be a Lie algebra and $I \trianglelefteq L$ an ideal such that I is invariant under $\text{Der}(L)$. Then if $d \in \text{Der}(L)$, d induces derivations α and β of L/I and I , respectively. Hence we obtain a Lie algebra homomorphism

$$\psi : \text{Der}(L) \rightarrow \text{Der}(L/I) \oplus \text{Der}(I).$$

Let K be a Lie algebra and I be a K -module. Let $Z^2(K, I)$ be the vector space of cocycles and $\text{Comp}(K, I)$ the Lie algebra of compatible pairs. Let $(\alpha, \beta) \in \text{Comp}(K, I)$ and $\vartheta \in Z^2(K, I)$. Define an action of $\text{Comp}(K, I)$ over $Z^2(K, I)$ by

$$(\alpha, \beta) \cdot \vartheta(h, k) = \beta(\vartheta(h, k)) - \vartheta(\alpha(k), h) - \vartheta(k, \alpha(h)), \quad \text{for all } h, k \in K.$$

The elements of the annihilator of this action will be called induced pairs and we denote the set of induced pairs by $\text{Indu}(K, I, \vartheta)$. Let $\vartheta \in Z^2(K, I)$ a cocycle and K_ϑ be the Lie algebra extension obtained from K by ϑ . Then we can lift the derivation of $\text{Indu}(K, I, \vartheta)$ to $\text{Der}(K_\vartheta)$. Thus we obtained the following theorem.

Theorem 3.7 *Let K be a Lie algebra and I a K -module. Let $\vartheta \in H^2(K, I)$ and suppose that I , as ideal of K_ϑ , is invariant under derivations of K_ϑ . Let $\phi : \text{Der}(K_\vartheta) \rightarrow \text{Der}(K) \oplus \text{Der}(I)$ given by $\phi(d) = (\alpha, \beta)$. Then:*

- (1) $\text{Im}(\phi) = \text{Indu}(K, I, \vartheta)$
- (2) $\text{Ker}(\phi) \cong Z^1(K, I)$

The details of this construction can be seen in Section 3. There is a significant difference between the application of this approach to automorphisms and to derivations: calculating the automorphism groups of Lie algebras is usually a difficult task that may involve a large orbit-stabilizer calculation, while calculating the algebra $\text{Der}(K_\vartheta)$ can be done by solving a system of linear equations. Thus, to understand the importance of Theorem 3.7 we must discover what additional information of $\text{Der}(K_\vartheta)$ we are able to obtain through information concerning the algebras $\text{Der}(K)$ and $\text{Der}(I)$.

In order facilitate the reading of the text and the references, we added a section with results on the primary decomposition of vector spaces in relation to subalgebras of linear operators and a brief description of the main articles used.

This text is organized as follows: Section 2 is dedicated to literature review. In Section 3, we present compatible pairs and the lifting process of derivations of a Lie algebra K to the Lie algebras K_ϑ such that ϑ is a cocycle. We end this section by applying the compatible pairs to Jacobson's Theorem. Section 4 is composed of some examples and conjectures about modular solvable non-nilpotent Lie algebras with non-singular derivations.

2. NON-SINGULAR DERIVATIONS: KNOWN RESULTS

This section is composed by description of a decomposition of a Lie algebra L relative to a subalgebra K of $\mathfrak{gl}(L)$ and its application in Jacobson's Theorem. Next, we have the calculations presented in Shalev's article [10] about conditions on the order of derivation which guarantee nilpotency of a Lie algebra. The section ends with Mattarei's Theorem that relates the order of non-singular derivations of solvable modular Lie algebras to roots of certain types of polynomials.

2.1. Basic concepts. The symbol ' \oplus ' will be used to denote the direct sum of algebras, while the direct sum of vector spaces will be denoted by ' $\dot{+}$ '.

Let V be a finite-dimensional vector space over a field \mathbb{F} and $a \in \text{End}(V)$. Let $p \in \mathbb{F}[X]$ be a univariate polynomial and define

$$V_0(p(a)) = \{v \in V \mid \text{there is an } m > 0 \text{ such that } p(a)^m v = 0\}.$$

$V_0(p(a))$ is a vector subspace of V invariant under a . Now let A be the associative subalgebra of $\text{End}(V)$ with 1 generated by a . Let p_a be the minimum polynomial of a and suppose that

$$p_a = p_1^{k_1} \cdots p_r^{k_r}$$

is the factorization of p_a into irreducible factors, such that p_i has leading coefficient 1 and $p_i \neq p_j$ for $1 \leq i, j \leq r$. Then V decomposes as a direct sum of subspaces

$$V = V_0(p_1(a)) \dot{+} \cdots \dot{+} V_0(p_r(a)),$$

each space $V_0(p_i(a))$ being invariant under A . Furthermore, the minimum polynomial of the restriction of a to $V_0(p_i(a))$ is $p_i^{k_i}$. A proof of this result can be found in [1] Lemma A.2.2.

We can generalize this decomposition to subalgebras of $\mathfrak{gl}(V)$ generated by more than one element. Let K be a subalgebra of $\mathfrak{gl}(V)$. A decomposition $V = V_1 \oplus \cdots \oplus V_s$ of V into K -modules V_i is said to be *primary* if the minimum polynomial of the restriction of a to V_i is a power of an irreducible polynomial for all $a \in K$ and $1 \leq i \leq s$. The subspaces V_i are called primary components. If for any two components V_i and V_j ($i \neq j$), there is an $x \in K$ such that the minimum polynomials of the restrictions of x to V_i and V_j are powers of different irreducible polynomial, then the decomposition is called *collected*. In general V will not have a primary (or primary collected) decomposition into K -modules but such a decomposition is guaranteed to exist if the base field of V is algebraically closed and $K \leq \mathfrak{gl}(V)$ is nilpotent.

Proposition 2.1 ([1], Theorem 3.1.10). *Let V be finite-dimensional vector space. Let $K \leq \mathfrak{gl}(V)$ be a nilpotent subalgebra. Then V has a unique collected primary decomposition relative to K*

If the vector space V has a collected primary decomposition $V = V_1 \dot{+} \cdots \dot{+} V_s$ then we can characterize the components V_i . For $x \in K$ and $1 \leq i \leq s$ define $p_{x,i}$ to be the irreducible polynomial such that the minimum polynomial of x restricted to V_i is a power of $p_{x,i}$. Then we obtain the equality

$$V_i = \{v \in V \mid \text{for all } x \in K \text{ there is an } m > 0 \text{ such that } p_{x,i}(x)^m v = 0\}.$$

It is worth noting that if the base field of V is algebraically closed, then all irreducible polynomials are of the form $p(X) = (X - \lambda)$, for some $\lambda \in \mathbb{F}$, and hence $p_{x,i} = (X - \lambda_i(x))$, $\lambda_i \in \mathbb{F}^*$. Further, in this case, primary components are of the form

$$V_i = \{v \in V \mid \text{for all } x \in K \text{ there is an } m > 0 \text{ such that } (x - \lambda_i(x)I)^m v = 0\},$$

with $\lambda_i \in K^*$. Its natural to give a name for this case. Let V be a finite-dimensional vector space over a field \mathbb{F} and $K \leq \mathfrak{gl}(V)$ a subalgebra. Let $\lambda \in K^*$. Then

$$V_\lambda = \{v \in V \mid \text{for all } x \in K \text{ there is an } m > 0 \text{ such that } (x - \lambda(x).I)^m v = 0\}.$$

If $V_\lambda \neq 0$ then V_λ is called a *generalized eigenspace* of V associated to the *generalized eigenvalue* $\lambda \in K^*$.

Now we consider a Lie algebra L and a nilpotent subalgebra $K \leq \text{Der}(L)$. Then the decomposition to generalized eigenspaces of D can provide us some information of the multiplicative structure of L .

Proposition 2.2 ([6], Proposition 5 of Chapter III). *Let L be a Lie algebra over an algebraically closed field. Let K be a subalgebra of $\text{Der}(L)$. If $\lambda, \mu : K \rightarrow \mathbb{F}^*$ are generalized eigenvalues of K then $[L_\lambda, L_\mu] \subseteq L_{\lambda+\mu}$ if $\lambda + \mu$ is a generalized eigenvalue of K . Otherwise $[L_\mu, L_\lambda] = 0$.*

2.2. Jacobson's Theorem. In the article *A note on automorphism and derivations of Lie algebras* [5], Jacobson used a variation of Engel's Theorem for weakly closed sets to get sufficient conditions for a Lie algebra to be nilpotent. We recommend the reading of Sections 1 and 2 of Chapter 2 of Jacobson's book [6] as reference for examples and proofs.

Suppose that K and I are Lie algebras and $\psi : K \rightarrow \text{Der}(I)$ is a given Lie algebra homomorphism. Then we say that K *acts* on I or that I is a K -*module*. In this case, the image $\psi(k)(a)$ of $a \in I$ under $k \in K$ will be written simply as $[k, a]$. If I is an ideal of a Lie algebra K , then K acts on I . If $k \in K$, then the image of k under this action will be denoted by ad_k^I or simply by ad_k when the domain of the representation is clear from the context. Thus, for $a \in I$ and for $k \in K$, $\text{ad}_k^I(a) = \text{ad}_k(a) = [k, a]$. The homomorphism $K \rightarrow \text{Der}(I)$ that takes $k \mapsto \text{ad}_k^I$, will be denoted by ad^I .

Example 2.3. Let L be a Lie algebra with an abelian ideal I and set $K = L/I$. Define the Lie algebra representation $\text{ad}^I : K \rightarrow \text{Der}(I)$ by $\text{ad}_{x+I}^I(a) = [x, a]$ for all $x \in L$ and

$a \in I$. This is well defined, since I is abelian. Then I is a K -module. In this case, we say that the action is *induced by the adjoint representation*.

Let A be an associative algebra with 1 over a field \mathbb{F} . A subset S of A is called *weakly closed* if for every ordered pair $(a, b) \in S \times S$, there is an element $\gamma(a, b) \in \mathbb{F}$ such that $ab + \gamma(a, b)ba \in S$. If S is a subset of an Lie or associative algebra X , then $\langle S \rangle$ denotes the Lie or associative, respectively, subalgebra of X generated by S . In the case of associative algebras we assume that $1 \in \langle S \rangle$. This notation may cause confusion when X is an associative and Lie algebra in the same time, in such cases we will indicate clearly if $\langle S \rangle$ denotes associative or Lie subalgebra.

Proposition 2.4 ([6], Theorem 1 of Chapter II). *Let V be a finite-dimensional vector space over a field \mathbb{F} . Let $S \subseteq \text{End}(V)$ be a weakly closed subset such that every $s \in S$ is associative nilpotent, that is, $s^k = 0$, for some positive integer k . Then the associative subalgebra $\langle S \rangle \leq \text{End}(V)$ is nilpotent.*

With this result we can prove Jacobson's Theorem.

Theorem 2.5 ([5], Theorem 3). *Let L be a finite-dimensional Lie algebra over a field of characteristic 0 and suppose that there exists a subalgebra D of the algebra of derivations of L such that*

- (1) D is nilpotent;
- (2) if there is $c \in L$ such that $d(c) = 0$ for all $d \in D$ then $c = 0$.

Then L is nilpotent.

Proof. Let $\overline{\mathbb{F}}$ be the algebraic closure of the base field. We can extend all derivations of L to $\overline{L} = L \otimes \overline{\mathbb{F}}$. If we prove that \overline{L} is nilpotent then L is nilpotent. So we will assume that \mathbb{F} is algebraically closed. In this case the extension of D is nilpotent and without 0 as common eigenvalue, i.e. if there is $c \in L$ such that $d(c) = 0$ for all $d \in D$ then $c = 0$. Let $L = L_{\gamma_1} \dot{+} \cdots \dot{+} L_{\gamma_t}$ be the decomposition of L into generalized eigenspaces of D . By Proposition 2.2 we have $[L_{\gamma_i}, L_{\gamma_j}] \subseteq L_{\gamma_i + \gamma_j}$ if $\gamma_i + \gamma_j$ is a eigenvalue of D and $[L_{\gamma_i}, L_{\gamma_j}] = 0$ otherwise. For a subset $Y \subseteq L$, we let ad_Y denote the set of adjoint mappings induced by elements of Y . Then the inclusion just noted shows that the set $S = \bigcup \text{ad}_{L_{\gamma_j}}$ is a weakly closed set of linear transformations. Let $a \in L_{\gamma_j}$ and $b \in L_{\gamma_i}$. Then $(\text{ad}_a)^s(b) \in L_{\gamma_i + s\gamma_j}$, for all $s \geq 0$.(*)

The generalized eigenvalue $\gamma_j \neq 0$ and \mathbb{F} has characteristic 0 then $\gamma_i + s\gamma_j$, for $s > 0$, are pairwise distinct. Then for some r large enough $(\gamma_i + r\gamma_j)$ is not an eigenvalue and $\text{ad}_a(b) = 0$. Follow that ad_a is nilpotent linear transformation. Thus every element of S is nilpotent. By Proposition 2.4 the associative subalgebra $\langle S \rangle \leq \text{End}(V)$ is nilpotent and hence ad_L is nilpotent. Therefore L is a nilpotent Lie algebra. \square

A review of the proof of Theorem 2.5 shows that the hypothesis of zero characteristic is essential to prove that every element in a homogeneous component is nilpotent. As the following examples shows, Theorem 2.5 fails to hold in characteristic $p > 0$.

Example 2.6. Let \mathbb{F} be the field of 2^m elements and L be the vector space over \mathbb{F} such that

$$L = \langle x_\alpha \mid \alpha \in \mathbb{F}, \alpha \neq 0 \rangle$$

with a basis labeled by nonzero elements of the field \mathbb{F} and with multiplication $[x_\alpha, x_\beta] = (\beta - \alpha)x_{\alpha+\beta}$. Then L is a simple Lie algebra and the map $d \in \text{End}(L)$ given by $d(e_\alpha) = \alpha e_\alpha$ is a non-singular derivation. The calculations of this example and a systematic investigation of simple Lie algebras with nonsingular derivations can be found in [3].

Example 2.7. Let V be a vector space over a field \mathbb{F} of characteristic $p > 0$. Let $B = \{a_1, a_2, \dots, a_p\}$ be a basis of V . Define the linear map $x \in \mathfrak{gl}(V)$ by

$$x(a_i) = a_{i+1 \bmod p}, 1 \leq i \leq p.$$

Let K be the abelian Lie algebra generated by $\{x, x^2, \dots, x^{p-1}\}$. Then V can be considered as K -module with the standard action of $\mathfrak{gl}(V)$ on V . Let L be the semi-direct sum $L = K \oplus V$ then L is an Solvable non-nilpotent Lie algebra of derived length 2. Let $\lambda, \delta \in \mathbb{F}$ both non-zero and $\lambda \neq s\delta$, for all $s \in \mathbb{F}_p$. The linear map $d : L \rightarrow L$ defined by

$$d : \begin{cases} x^j \mapsto j\lambda x^j, & 1 \leq j \leq p-1; \\ a_i \mapsto (\delta + (i-1)\lambda)a_i, & 1 \leq i \leq p, \end{cases}$$

is a non-singular derivation of L .

For Lie algebras over fields of characteristic $p > 3$ we could not find an example of derived length greater than 3 but in characteristic 2 we have the following example.

Example 2.8. Let L be a vector space of dimension 6 over \mathbb{F}_4 . Let $\lambda \in \mathbb{F}_4$ such that $\lambda^2 = \lambda + 1$ and $\{a_1, a_2, a_3, a_4, a_5, a_6\}$ a basis of L over \mathbb{F}_4 . Define the products

$$\begin{aligned} [a_1, a_3] &= \lambda a_5 + a_6, & [a_1, a_4] &= \lambda a_6, & [a_1, a_5] &= \lambda^2 a_3 + a_4, & [a_3, a_5] &= \lambda a_2, \\ [a_1, a_6] &= \lambda^2 a_4, & [a_2, a_3] &= \lambda a_6, & \text{and } [a_2, a_5] &= \lambda^2 a_4. \end{aligned}$$

L is a solvable non-nilpotent Lie algebra of derived length 3. The linear map $d : L \rightarrow L$ defined by

$$d : \begin{cases} a_1 \mapsto a_1 & a_3 \mapsto \lambda a_3 & a_5 \mapsto \lambda^2 a_5 \\ a_2 \mapsto a_2 & a_4 \mapsto \lambda a_4 & a_6 \mapsto \lambda^2 a_6 \end{cases}$$

is a non-singular derivation of L .

Another question is whether the converse of Jacobson's Theorem is true, that is, is it true that all finite-dimensional nilpotent Lie algebras admit non-singular derivation. By Dixmier and Lister [4], there are nilpotent Lie algebras admitting only nilpotent derivations. Bellow we present Dixmier and Lister example of such an algebra.

Example 2.9. Let \mathbb{F} be a field of characteristic 0 and $L = \langle x_1, x_2, \dots, x_8 \rangle$ be a Lie algebra over \mathbb{F} with dimension 8 and multiplication table

$$\begin{aligned} [e_1, e_2] &= e_5 & [e_1, e_3] &= e_6 & [e_1, e_4] &= e_7 & [e_1, e_5] &= -e_8 & [e_2, e_3] &= e_8 & [e_2, e_4] &= e_6 \\ [e_2, e_6] &= -e_7 & [e_3, e_4] &= -e_5 & [e_3, e_5] &= -e_7 & [e_4, e_6] &= -e_8 & [e_i, e_j] &= -[e_j, e_i]. \end{aligned}$$

Moreover, $[e_i, e_j] = 0$ if it is not in table above. Then L is nilpotent with $L^3 \neq 0$, $L^4 = 0$ and every derivation of L is nilpotent.

2.3. Jacobson's Theorem in characteristic $p > 0$. As the examples above shows, Jacobson's Theorem is in general not true in characteristic $p > 0$. However, we have the follow weaker result.

Theorem 2.10. *Let L be a Lie algebra over a field of characteristic $p > 0$ and suppose that there exists a subalgebra $D \leq \text{Der}(L)$ such that*

- (1) D is nilpotent;
- (2) if there is $c \in L$ such that $d(c) = 0$ for all $d \in D$ then $c = 0$.

If D has at most $p - 1$ generalized eigenvalues then L is nilpotent.

Proof. The proof of this theorem is identical to proof of Theorem 2.5 up to point marked by (*). The generalized eigenvalue $\gamma_j \neq 0$ then the set $\{\gamma_i, \gamma_i + \gamma_j, \dots, \gamma_i + (p - 1)\gamma_j\}$ has p distinct elements. As D has at most $p - 1$ generalized eigenvalues then for some r , $0 < r \leq p - 1$, $(\gamma_i + r\gamma_j)$ is not an eigenvalue. Follow that ad_a is nilpotent linear transformation, for every $a \in L_{\gamma_i}$. Thus every element of S is nilpotent. By Proposition 2.4 the associative subalgebra $\langle S \rangle \leq \text{End}(V)$ is nilpotent and hence ad_L is nilpotent. Therefore L is a nilpotent Lie algebra. \square

2.4. The orders of non-singular derivations. An interesting approach by Shalev in article [10] is to study the order of nonsingular derivations, establishing conditions for a Lie algebra over a field of characteristic p with non-singular derivations to be nilpotent. More precisely, Shalev studied the set of orders of nonsingular derivations of non-nilpotent Lie algebras of characteristic p . Later, Mattarei in [8] showed that this set of numbers corresponds to the set of solutions of some polynomial equation over a field of characteristic p . Below we present some results of these articles.

Let L be a Lie algebra over an algebraically closed field of characteristic p . We can characterize the matrix of a non-singular derivation of L . We need a result for derivations in Lie algebras over a field of characteristic p .

Lemma 2.11. *Let L be a Lie algebra over a field \mathbb{F} of characteristic $p > 0$. If $d \in \text{Der}(L)$ then $d^{p^m} \in \text{Der}(L)$, for all $m \geq 1$.*

Proof. If we prove this result for $m = 1$ then the general case when $m \geq 1$ will follow by induction. Let us hence prove the statement only for $m = 1$. Let $d \in \text{Der}(L)$ and $x, y \in L$.

First we prove the Leibniz's formula by induction:

$$d^n([x, y]) = \sum_{k=0}^n \binom{n}{k} [d^k(x), d^{n-k}(y)], \text{ for all } n > 0.$$

The case $n = 1$ follow from derivation's definition. Suppose that Leibniz's formula is valid for n . Then

$$(1) \quad d^n([x, y]) = \sum_{k=0}^n \binom{n}{k} [d^k(x), d^{n-k}(y)].$$

Calculating d in both sides of equation (1) we have

$$(2) \quad d^{n+1}([x, y]) = \sum_{k=0}^n \binom{n}{k} [d^{k+1}(x), d^{n-k}(y)] + \sum_{k=0}^n \binom{n}{k} [d^k(x), d^{n-k+1}(y)].$$

Rearranging the index, the right side of equation (2) can be write as

$$[d^{n+1}(x), y] + \sum_{k=1}^n \left(\binom{n}{k-1} + \binom{n}{k} \right) [d^k(x), d^{n+1-k}(y)] + [x, d^{n+1}(y)].$$

As $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$ then

$$d^{n+1}([x, y]) = \sum_{k=0}^{n+1} \binom{n+1}{k} [d^k(x), d^{n+1-k}(y)].$$

Then by induction Leibniz's formula is proved. As the field \mathbb{F} has characteristic p then setting $n = p^m$ the Leibniz's formula is reduced to

$$d^{p^m}([x, y]) = [d^{p^m}(x), y] + [x, d^{p^m}(y)].$$

□

Proposition 2.12. *Let V be a finite-dimensional vector space over an algebraically closed field of characteristic $p > 0$ and $f \in \text{End}(V)$ non-singular with order r coprime to p . Then f is diagonalizable.*

Proof. Let A be the matrix of the endomorphism f in Jordan normal form and write $A = S + N$ such that S is diagonal, N is nilpotent upper triangular and S, N commute. Denote by M_{ij} the element of a matrix M of the i^{th} line and the j^{th} column. It follows that

- If $S_{ii} = \lambda_i$ then $(S^k)_{ii} = \lambda_i^k$, for all $k > 0$;
- $N_{i(i+j)}^k = 0$, for all $0 \leq j < k$.

As the order of A is r we have $A^r = Id$. Then

$$I = A^r = (S + N)^r = S^r + \binom{r}{1} S^{r-1} N + \binom{r}{2} S^{r-2} N^2 + \cdots + \binom{r}{r-1} S N^{r-1} + N^r.$$

The identity matrix on the left-hand side of the last equation is diagonal, while the summands, with the exception of the first summand, on the right-hand side are nilpotent. Further, if $N \neq 0$, then the second summand $rS^{r-1}N$ is non-zero, and it is the only summand that contains a non-zero entry in a positions $(i, i+1)$ with $i > 0$. However, this implies that if $N \neq 0$, then A^r must contain a non-zero entry in a position $(i, i+1)$, which is a contradiction, as $A^r = I$. Hence $N = 0$ as claimed. Then f is diagonalizable. \square

Let L be a Lie algebra over the field \mathbb{F} of characteristic $p > 0$ such that L has a non-singular derivation d . Let r be the order of d such that $r = sp^t$, with $\gcd(s, p) = 1$. Then by Lemma 2.11 d^{p^t} is a derivation whose order is prime to p and, by Proposition 2.12, d^{p^t} is diagonalizable. So if L is a Lie algebra over an algebraically closed field \mathbb{F} of characteristic $p > 0$ with non-singular derivation then L has a diagonalizable derivation d without eigenvalue 0.

Proposition 2.13 ([10], Lemma 2.2). *Let L be a finite-dimensional Lie algebra in characteristic $p > 0$ which admits a non-singular derivation d whose order n is coprime to p . Suppose that L is not nilpotent. Then there exist $\lambda \in \overline{\mathbb{F}}_p$ such that $(\lambda + \delta)^n = 1$ for all $\delta \in \mathbb{F}_p$.*

Proof. Let $\overline{\mathbb{F}}$ be an algebraic closure of \mathbb{F} and $R = \{\alpha \in \overline{\mathbb{F}}_p \mid \alpha^n = 1\}$. If R is not contained in base field of L then we consider d for the extension $L \otimes \overline{\mathbb{F}}$. By Proposition 2.12, d is diagonalizable. Let $L = L_{\lambda_1} \dot{+} \cdots \dot{+} L_{\lambda_r}$ the decomposition of L to eigenspaces of d . The set $S = \bigcup \text{ad}_{L_{\lambda_j}}$ is weakly closed with $\gamma(\text{ad}_a, \text{ad}_b) = -1$ for all $a \in L_{\lambda_i}, b \in L_{\lambda_j}$. If each ad_a is nilpotent then the associative subalgebra $\langle S \rangle \leq \mathfrak{gl}(L)$ is nilpotent by Proposition 2.4. Hence ad_L is a nilpotent Lie algebra and L is nilpotent. As L is non-nilpotent by hypothesis then there is $a \in L_{\lambda_j}$ and $b \in L_{\lambda_i}$ such that $(\text{ad}_a)^n(b) \neq 0$, $1 \leq n \leq p$. However this implies $(\lambda_i + \delta\lambda_j)$ are eigenvalues of d for $1 \leq \delta \leq p$. Since $|d| = n$ each eigenvalue of d has order n . Thus $(\lambda_i + \delta\lambda_j)^n = 1$, for all $\delta \in \mathbb{F}_p$. As λ_j is an eigenvalue of d , $\lambda_j^n = \lambda_j^{-n} = 1$. Thus $1 = (\lambda_i + \delta\lambda_j)^n \lambda_j^{-n} = (\lambda_i \lambda_j^{-1} + \delta)^n$. Therefore setting $\lambda = \lambda_i \lambda_j^{-1}$, $(\lambda + \delta)^n = 1$ for all $\delta \in \mathbb{F}_p$. \square

Using the same notation as in the proof of Proposition 2.13 and observing that the set R contains precisely the n -th roots of unity in $\overline{\mathbb{F}}$, we write $x^n - 1 = \prod_{\alpha \in R} (x - \alpha)$. As for all $\delta \in \mathbb{F}_p$, $\lambda + \delta \in R$, $\prod_{\delta \in \mathbb{F}_p} (x - \lambda - \delta)$ divides $x^n - 1$. But

$$\prod_{\delta \in \mathbb{F}_p} (x - \lambda - \delta) = (x - \lambda)^p - (x - \lambda) = x^p - x - c,$$

where $c = \lambda^p - \lambda$. The first equation of last display can be seen by observing that the elements $\lambda + \delta$ with $\delta \in \mathbb{F}_p$ are exactly the p roots of the polynomial $(x - \lambda)^p - (x - \lambda)$. Let $g(x) = x^p - x - c$. Then $g(x)$ divides $x^n - 1$, which implies that x^n is congruent to 1 modulo $g(x)$. In this case, Lemma 2.4 of [10] shows that $n \geq p^2 - 1$. Now we can prove the theorem.

Theorem 2.14 ([10], Theorem 1.1). *Let L be a finite dimensional Lie algebra in characteristic $p > 0$ which admits non-singular derivation of order n . Write $n = p^s m$ where m is coprime to p . Suppose $m < p^2 - 1$. Then L is nilpotent.*

Proof. The derivation d^{p^s} has order m . Suppose that L is not nilpotent. Then by the comment above we have $m \geq p^2 - 1$. \square

Mattarei in [8] presented an example of non-nilpotent solvable modular Lie algebra.

Example 2.15. Let $\alpha, \beta \in \overline{\mathbb{F}}_p$ with $\alpha\beta^{-1} \notin \mathbb{F}_p$. Let M be a p -dimensional vector space over $\overline{\mathbb{F}}_p$ with basis e_1, \dots, e_p , and let E, F be the linear transformations of M defined by $E(e_i) = e_{i+1}$ (indices modulo p), and $F(e_i) = (\alpha + i\beta)e_i$. The transformations E and F span a two-dimensional solvable Lie algebra, which admits M as a left module. Let L be the semidirect sum of $\{E\}$ and M with respect to this action. Then F acts on L as a non-singular derivation, with eigenvalues β on $\{E\}$, and $\alpha + \lambda\beta$ for $\lambda \in \mathbb{F}_p$ on M .

The next result links the orders non-singular derivations of Lie algebras of characteristic p to some polynomial equations.

Proposition 2.16. *Let p be a prime number and let n be a positive integer, prime to p . The following statements are equivalent:*

- (1) *there exists a non-nilpotent Lie algebra of characteristic p with a non-singular derivations of order n ;*
- (2) *there exists an element $\alpha \in \overline{\mathbb{F}}_p$ such that $(\alpha + \lambda)^n = 1$ for all $\lambda \in \mathbb{F}_p$*
- (3) *there exist an element $c \in \overline{\mathbb{F}}_p^*$ such that $x^p - x - c$ divides $x^n - 1$ as elements of the polynomial ring $\overline{\mathbb{F}}_p[x]$.*

Mattarei in [8] defines the set N_p of the possible orders of non-singular derivations of non-nilpotent Lie algebras of characteristic p and determine all elements of N_p which are smaller than p^3 , for $p > 3$.

2.5. Objectives of the project. In this section we will present some questions about solvable non-nilpotent modular Lie algebras L with a non-singular derivation d . This questions are based in the examples and results showed in the previous sections. These issues will serve as a reference for further work.

Problem 1. Is there a solvable, non-nilpotent Lie algebra over a field of characteristic $p \geq 3$ with non-singular derivation and derived length greater than 2?

Suppose that the answer to Problem 1 is yes and let L be such Lie algebra. Let $I = L^{(2)}$ and $K = L/I$. As $L^{(3)} = 0$ then I is abelian and so K acts on I by adjoint representation. In this case, K is a solvable Lie algebra of derived length 2 with non-singular derivation. By Proposition 3.1, there is a cocycle $\vartheta \in Z^2(K, I)$ such that $L \cong K_{\vartheta}$. This calculation show us that every Lie algebra that answer Problem 1 can be obtained by an extension of a solvable Lie algebra of derived length 2 with non-singular derivation. So we need to

understand this Lie algebras of derived length 2 to search for an answer of Problem 1. We will study a variation of this question.

Problem 2. Let K be one of the known solvable, non-nilpotent Lie algebra over a field of characteristic $p \geq 3$ with non-singular derivation and derived length 2. Is there a non-trivial K -module I and a cocycle $\vartheta \in Z^2(K, I)$ such that K_ϑ has a non-singular derivation?

As first step to study Problem 2 we will try to describe some cases of abelian Lie algebras K acting over vector spaces. This study defines our next objectives in this project.

Objectives

- To characterize solvable non-nilpotent modular Lie algebras of the form $L = \langle x \rangle \oplus I$ where I is a finite dimensional abelian Lie algebra such that L admits a non-singular derivation; study the extensions of such algebras and obtain ones that admits non-singular derivations; By Corollary 3.14, there is a quotient $Q = L^{(i)}/L^{i+1}$ with $\dim Q \geq p$. Study the number of such quotients.
- How the existence of non-singular derivations affect the structure of $\text{Der}(L)$? Can we define some algebra structure over non-singular derivations of L ?
- Study the general structure of solvable non-nilpotent Lie algebras with non-singular derivations

3. DERIVATIONS AND LIE ALGEBRA EXTENSIONS

3.1. Lie algebra extensions. An *extension* of a Lie algebra K by a Lie algebra I is an exact sequence

$$(3) \quad 0 \rightarrow I \xrightarrow{i} L \xrightarrow{s} K \rightarrow 0$$

of Lie algebras. The Lie algebra L in the middle of the exact sequence contains an ideal $\text{Ker}(s) = \text{Im } i \cong I$ such that $L/I \cong K$. We will write informally that ‘ L is an extension of K by I ’. The extension (3) *splits* if L has a subalgebra S such that $L = S \dot{+} \text{Ker}(s)$. The extension (3) is *trivial* if there exists an ideal S of L such that $L = S \oplus \text{Ker}(s)$. The extension (3) is central if $\text{Ker}(s)$ lies in the center $Z(L)$ of L .

Let K be a Lie algebra over a field \mathbb{F} and let I be a vector space over \mathbb{F} . Denote by $C^2(K, I)$ the vector space of alternating bilinear maps $\vartheta : K \times K \rightarrow I$. If I is a K -module and $\vartheta \in C^2(K, I)$ has the property that

$$(4) \quad \vartheta(x, [y, z]) + \vartheta(y, [z, x]) + \vartheta(z, [x, y]) + [x, \vartheta(y, z)] + [y, \vartheta(z, x)] + [z, \vartheta(x, y)] = 0,$$

for all $x, y, z \in K$, then ϑ is said to be a *cocycle* and the vector space of cocycles is denoted by $Z^2(K, I)$. Let $T : K \rightarrow I$ be a linear transformation and define, $\vartheta_T : K \times K \rightarrow I$ by

$$(5) \quad \vartheta_T(h, k) = T([h, k]) + [k, T(h)] - [h, T(k)] \quad \text{for all } h, k \in K.$$

Then $\vartheta_T \in Z^2(K, I)$ and such a cocycle ϑ_T is said to be a *coboundary*. The set of coboundaries is denoted by $B^2(K, I)$. The set $B^2(K, I)$ is a subspace of $Z^2(K, I)$, and we set

$H^2(K, I) = Z^2(K, I)/B^2(K, I)$ to be the quotient space. The first cohomology group of K and I is defined as

$$Z^1(K, I) = \{\nu \in \text{Hom}(K, I) \mid \nu([h, k]) = [h, \nu(k)] - [k, \nu(h)] \text{ for all } h, k \in K\}.$$

The next result, whose proof can be found, for instance, in [7, Section 4.2], links Lie algebra extensions to cohomology. Let K be a Lie algebra and let I be a K -module. Let $\vartheta \in Z^2(K, I)$ and define the Lie algebra $K_\vartheta = K \dot{+} I$ with the product

$$(6) \quad [x + a, y + b] = [x, y] + \vartheta(x, y) + [a, y] - [b, x] \text{ for all } x, y \in K \text{ and } a, b \in I.$$

Proposition 3.1. *The following hold for the Lie algebra K_ϑ :*

- (1) K_ϑ is a Lie algebra extension of K by I ;
- (2) if $\nu \in B^2(K, I)$, then K_ϑ is isomorphic to $K_{\vartheta+\nu}$;
- (3) if $\vartheta \in B^2(K, I)$, then K_ϑ is a split extension of K by I .

Conversely, let L be a Lie algebra and J be an abelian ideal of L . Then there exists $\vartheta \in Z^2(L/J, J)$ such that $L \cong (L/J)_\vartheta$.

The cocycle ϑ in last the statement of Proposition 3.1 can be constructed as follows. Let $\pi : L \rightarrow L/I$ denote the natural projection, and let $\sigma : L/I \rightarrow L$ be a right inverse of π ; that is, $\pi\sigma = \text{id}_{L/I}$. Then, for $k + I, h + I \in L/I$, set

$$\vartheta(k + I, h + I) = \sigma([k + I, h + I]) - [\sigma(k + I), \sigma(h + I)].$$

Routine calculation shows that $\vartheta \in Z^2(L/I, I)$ and that $L \cong L_\vartheta$.

3.2. Compatible pairs and derivations of semidirect sums. Compatible pairs were introduced in [2] to compute automorphisms of solvable groups and solvable Lie algebras. We adopt the concept for derivations of Lie algebras. Let K and I be Lie algebras such that K acts on I via the homomorphism $\psi : K \rightarrow \text{Der}(I)$. We define the *semidirect sum* $K \oplus_\psi I$ as the vector space $K \dot{+} I$ with the product operation given as

$$[(k_1, a_1), (k_2, a_2)] = ([k_1, k_2], [k_1, a_2] - [k_2, a_1] + [a_1, a_2]).$$

When the K -action on I is clear from the context, then we usually suppress the homomorphism ‘ ψ ’ from the notation and write simply $K \oplus I$. If L is a Lie algebra such that L has an ideal I and a subalgebra K in such a way that $L = K \dot{+} I$, then $L \cong K \oplus_\psi I$ where ψ is the restriction of ad_I to K . In a semidirect sum $K \oplus I$, an element $(k, a) \in K \dot{+} I$ will usually be written as $k + a$.

Suppose that K and I are as in the previous paragraph. The direct sum $\text{Der}(K) \oplus \text{Der}(I)$ of the derivation Lie algebras is a Lie algebra. An element $(\alpha, \beta) \in \text{Der}(K) \oplus \text{Der}(I)$ is said to be a *compatible pair* if

$$(7) \quad \beta([k, a]) = [\alpha(k), a] + [k, \beta(a)] \quad \text{for all } k \in K, a \in I.$$

We let $\text{Comp}(K, I)$ denote the set of compatible pairs in $\text{Der}(K) \oplus \text{Der}(I)$. Using the homomorphism $\psi : K \rightarrow \text{Der}(I)$ associated to the K -action on I , we can write equation

(7) in another form as follows. Writing $[k, a]$ as $\psi(k)(a)$, we have that $(\alpha, \beta) \in \mathbf{Comp}(K, I)$ if and only if the equation

$$\beta\psi(k) = \psi(\alpha(k)) + \psi(k)\beta.$$

holds in $\mathbf{Der}(I)$ for all $k \in K$. Using commutator, this is equivalent to

$$(8) \quad [\beta, \psi(k)] = \psi(\alpha(k)) \quad \text{for all } k \in K.$$

Letting $\mathbf{ad} : \mathbf{Der}(I) \rightarrow \mathbf{Der}(I)$ denote the adjoint representation, equation (8) can be rewritten as

$$(9) \quad \mathbf{ad}_\beta \psi(k) = \psi(\alpha(k)) \quad \text{for all } k \in K.$$

Therefore, $(\alpha, \beta) \in \mathbf{Comp}(K, I)$ if and only if the following diagram commutes:

$$\begin{array}{ccc} K & \xrightarrow{\psi} & \mathbf{Der}(I) \\ \downarrow \alpha & \circlearrowleft & \downarrow \mathbf{ad}_\beta \\ K & \xrightarrow{\psi} & \mathbf{Der}(I). \end{array}$$

A compatible pair $(\alpha, \beta) \in \mathbf{Der}(K) \oplus \mathbf{Der}(I)$ will usually be written as $\alpha + \beta$. If $\alpha + \beta \in \mathbf{Der}(K) \oplus \mathbf{Der}(I)$ as above, then $\alpha + \beta$ can be considered a element of $\mathfrak{gl}(I \oplus K)$ by letting $(\alpha + \beta)(a + k) = \alpha(a) + \beta(k)$ for all $a \in I$ and $k \in K$.

Proposition 3.2. *Using the notation above, we have that*

$$\mathbf{Comp}(K, I) = \{\alpha + \beta \in \mathfrak{gl}(K) \oplus \mathfrak{gl}(I) \mid \alpha + \beta \in \mathbf{Der}(K \oplus I)\}.$$

In particular $\mathbf{Comp}(K, I)$ is a Lie subalgebra of $\mathbf{Der}(K \oplus I)$.

Proof. Suppose that $\alpha + \beta \in \mathbf{Comp}(K, I)$ is a compatible pair and let $k + a, k' + a' \in K \oplus I$. Then

$$\begin{aligned} (\alpha + \beta)[k + a, k' + a'] &= (\alpha + \beta)([k, k'] + ([k, a'] - [k', a] + [a, a'])) \\ &= \alpha([k, k']) + \beta([k, a'] - [k', a] + [a, a']) \\ &= [\alpha(k), k'] + [k, \alpha(k')] + [\alpha(k), a'] - [\alpha(k'), a] \\ &\quad + [\beta(a), a'] + [k, \beta(a')] - [k', \beta(a)] + [a, \beta(a')]. \end{aligned}$$

On the other hand

$$\begin{aligned} [(\alpha + \beta)(k + a), k' + a'] + [k + a, (\alpha + \beta)(k' + a')] &= \\ [\alpha(k), k'] + [\alpha(k), a'] + [\beta(a), k'] + [\beta(a), a'] &+ \\ + [k, \alpha(k')] + [k, \beta(a')] + [a, \alpha(k')] + [a, \beta(a')]. \end{aligned}$$

Thus $\alpha + \beta \in \mathbf{Der}(K \oplus I)$.

Conversely, let $\alpha + \beta \in \mathfrak{gl}(K) \oplus \mathfrak{gl}(I)$ such that $\alpha + \beta$ is a derivation of $K \oplus I$. Then $(\alpha + \beta)|_K = \alpha$ and $(\alpha + \beta)|_I = \beta$, and so $\alpha \in \text{Der}(K)$ and $\beta \in \text{Der}(I)$. Further, if $k \in K$ and $a \in I$, then $[k, a] \in I$, and so

$$\beta([k, a]) = (\alpha + \beta)[k, a] = [(\alpha + \beta)(k), a] + [k, (\alpha + \beta)(a)] = [\alpha(k), a] + [k, \beta(a)].$$

Thus $\alpha + \beta \in \text{Comp}(K, I)$, as required.

The fact that $\text{Comp}(K, I)$ is a Lie subalgebra of $\text{Der}(K \oplus I)$ follows from the fact that $\text{Comp}(K, I)$ is the intersection of two Lie algebras; namely, $\text{Comp}(K, I) = (\mathfrak{gl}(K) \oplus \mathfrak{gl}(I)) \cap \text{Der}(K \oplus I)$. \square

Let K and I be vector spaces. Consider the Lie algebra $\mathfrak{gl}(K) \oplus \mathfrak{gl}(I)$ and define an action of $\mathfrak{gl}(K) \oplus \mathfrak{gl}(I)$ on the vector space $\text{Hom}(K, \mathfrak{gl}(I))$ as follows. Let ad denote the adjoint representation of $\mathfrak{gl}(I)$. Thus, for $\beta, \beta' \in \mathfrak{gl}(I)$ and $\text{ad}_\beta(\beta') = [\beta, \beta']$. For $(\alpha, \beta) \in \mathfrak{gl}(K) \oplus \mathfrak{gl}(I)$ and for $T \in \text{Hom}(K, \mathfrak{gl}(I))$, set

$$(10) \quad (\alpha, \beta) \cdot T = \text{ad}_\beta T - T\alpha.$$

Let us show that this in fact defines a Lie algebra action. First notice that $(\alpha, \beta) \cdot T \in \text{Hom}(K, \mathfrak{gl}(I))$ because it is linear combination of compositions of linear maps. Let us check that the action is compatible with Lie brackets. Let $(\alpha, \beta), (\alpha', \beta') \in \mathfrak{gl}(K) \oplus \mathfrak{gl}(I)$. By definition

$$(\alpha', \beta') \cdot T = \text{ad}_{\beta'} T - T\alpha'.$$

Thus

$$(\alpha, \beta) \cdot ((\alpha', \beta') \cdot T) = \text{ad}_\beta \text{ad}_{\beta'} T - \text{ad}_{\beta'} T\alpha - \text{ad}_\beta T\alpha' + T\alpha'\alpha.$$

In the same way,

$$(\alpha', \beta') \cdot ((\alpha, \beta) \cdot T) = \text{ad}_{\beta'} \text{ad}_\beta T - \text{ad}_\beta T\alpha' - \text{ad}_{\beta'} T\alpha + T\alpha\alpha'.$$

Hence,

$$\begin{aligned} (\alpha, \beta) \cdot ((\alpha', \beta') \cdot T) - (\alpha', \beta') \cdot ((\alpha, \beta) \cdot T) &= \text{ad}_\beta \text{ad}_{\beta'} T - \text{ad}_{\beta'} \text{ad}_\beta T - T\alpha\alpha' + T\alpha'\alpha \\ &= [\text{ad}_\beta, \text{ad}_{\beta'}] T - T[\alpha, \alpha']. \end{aligned}$$

Therefore,

$$[(\alpha, \beta), (\alpha', \beta')] \cdot T = ([\alpha, \alpha'], [\beta, \beta']) \cdot T.$$

Now, if K and I are Lie algebras and I is a K -module, then there is a corresponding homomorphism $\psi \in \text{Hom}(K, \text{Der}(I))$. Now suppose that $\alpha + \beta \in \mathfrak{gl}(K) \oplus \mathfrak{gl}(I)$ such that $\alpha + \beta \in \text{Der}(K) \oplus \text{Der}(I)$. Then, for $k \in K$, we have $\text{ad}_\beta T(k) + T\alpha(k)$ is a derivation of I since $\text{ad}_\beta T(k), T\alpha(k) \in \text{Der}(I)$.

If X is a subalgebra of $\text{Der}(K) \oplus \text{Der}(I)$, then the annihilator $\text{Ann}_X(\psi)$ of ψ in X is defined as

$$\text{Ann}_X(\psi) = \{(\alpha, \beta) \in X \mid (\alpha, \beta) \cdot \psi = 0\}.$$

Computing the annihilator of ψ in $\text{Der}(K) \oplus \text{Der}(I)$ explicitly, we obtain

$$\begin{aligned} \text{Ann}_{\text{Der}(K) \oplus \text{Der}(I)}(\psi) &= \{(\alpha, \beta) \in \text{Der}(K) \oplus \text{Der}(I) \mid (\alpha, \beta) \cdot \psi = 0\} \\ &= \{(\alpha, \beta) \in \text{Der}(K) \oplus \text{Der}(I) \mid \text{ad}_\beta \psi - \psi \alpha = 0\} = \text{Comp}(K, I). \end{aligned}$$

The last equality follows from (9). Hence we have proved the following proposition.

Proposition 3.3. *Let K and I be Lie algebras such that I is also a K -module via the representation $\psi \in \text{Hom}(K, \text{Der}(I))$. Then $\text{Comp}(K, I) = \text{Ann}_{\text{Der}(K) \oplus \text{Der}(I)}(\psi)$, where the action of $\text{Der}(K) \oplus \text{Der}(I)$ on $\text{Hom}(K, \text{Der}(I))$ is given by (10).*

3.3. Derivations of K_ϑ . In this section we present a method to describe the derivations of an extension K_ϑ presented in Proposition 3.1 from the derivations of the Lie algebra K . By an adaptation of the process used by Eick in [2], we set conditions which guarantee that a derivation of K can be lifted to a derivation of K_ϑ . It is first necessary to define an action of $\mathfrak{gl}(K) \oplus \mathfrak{gl}(I)$ on the vector space of alternating bilinear maps.

Let K and I be vector spaces. Let (α, β) be an element of the Lie algebra $\mathfrak{gl}(K) \oplus \mathfrak{gl}(I)$ and $\vartheta \in \mathcal{C}^2(K, I)$. Define an action of $\mathfrak{gl}(K) \oplus \mathfrak{gl}(I)$ on $\mathcal{C}^2(K, I)$ by setting for $\vartheta \in \mathcal{C}^2(K, I)$

$$(11) \quad (\alpha, \beta) \cdot \vartheta(h, k) = \beta(\vartheta(h, k)) - \vartheta(\alpha(h), k) - \vartheta(h, \alpha(k)), \quad \text{for all } h, k \in K.$$

Let $(\alpha', \beta') \in \mathfrak{gl}(K) \oplus \mathfrak{gl}(I)$ then

$$(12) \quad (\alpha, \beta) \cdot ((\alpha', \beta') \cdot \vartheta(h, k)) = (\alpha, \beta) \cdot (\beta'(\vartheta(h, k)) - \vartheta(\alpha'(h), k) - \vartheta(h, \alpha'(k))).$$

Applying the action in each summand of the right-hand of equation (12) we have

$$(\alpha, \beta) \cdot \beta'(\vartheta(h, k)) = \beta\beta'\vartheta(h, k) - \beta'\vartheta(\alpha(h), k) - \beta'\vartheta(h, \alpha(k)),$$

$$(\alpha, \beta) \cdot \vartheta(\alpha'(h), k) = \beta\vartheta(\alpha'(h), k) - \vartheta(\alpha'\alpha(h), k) - \vartheta(\alpha'(h), \alpha(k)),$$

$$(\alpha, \beta) \cdot \vartheta(h, \alpha'(k)) = \beta\vartheta(h, \alpha'(k)) - \vartheta(\alpha(h), \alpha'(k)) - \vartheta(h, \alpha'\alpha(k)).$$

Then

$$\begin{aligned} (\alpha, \beta) \cdot ((\alpha', \beta') \cdot \vartheta(h, k)) &= \beta\beta'\vartheta(h, k) - \beta'\vartheta(\alpha(h), k) - \beta'\vartheta(h, \alpha(k)) \\ &\quad - \beta\vartheta(\alpha'(h), k) + \vartheta(\alpha'\alpha(h), k) + \vartheta(\alpha'(h), \alpha(k)) \\ &\quad - \beta\vartheta(h, \alpha'(k)) + \vartheta(\alpha(h), \alpha'(k)) + \vartheta(h, \alpha'\alpha(k)). \end{aligned}$$

It follows

$$\begin{aligned} [(\alpha, \beta), (\alpha', \beta')] \cdot \vartheta(h, k) &= [\beta, \beta']\vartheta(h, k) - \vartheta([\alpha, \alpha'](h), k) - \vartheta(h, [\alpha, \alpha'](k)) \\ &= ([\alpha, \alpha'], [\beta, \beta']) \cdot \vartheta(h, k). \end{aligned}$$

Therefore, the action presented in (11) is well defined.

Our goal now is to study the action of compatible pairs $\mathbf{Comp}(K, I)$ on subspaces $Z^2(K, I)$ and $B^2(K, I)$ of $C^2(K, I)$. For this, assume that K is a Lie algebra and I is a K -module. Then for all $h, k, l \in K$, $(\alpha, \beta) \in \mathbf{Comp}(K, I)$ and $\vartheta \in Z^2(K, I)$ we have

$$\begin{aligned} (\alpha, \beta) \cdot \vartheta(k, [h, l]) &= \beta(\vartheta(k, [h, l])) - \vartheta(\alpha(k), [h, l]) - \vartheta(k, \alpha([h, l])) \\ &= \beta(\vartheta(k, [h, l])) - \vartheta(\alpha(k), [h, l]) - \vartheta(k, [\alpha(h), l]) - \vartheta(k, [h, \alpha(l)]). \end{aligned}$$

If

$$X = (\alpha, \beta) \cdot \vartheta(k, [h, l]) + (\alpha, \beta) \cdot \vartheta(h, [l, k]) + (\alpha, \beta) \cdot \vartheta(l, [k, h]),$$

then

$$\begin{aligned} X &= \beta(\vartheta(k, [h, l])) + \beta(\vartheta(h, [l, k])) + \beta(\vartheta(l, [k, h])) \\ &\quad - \vartheta(\alpha(k), [h, l]) - \vartheta(\alpha(h), [l, k]) - \vartheta(\alpha(l), [k, h]) \\ &\quad - \vartheta(k, [\alpha(h), l]) - \vartheta(h, [\alpha(l), k]) - \vartheta(l, [\alpha(k), h]) \\ &\quad - \vartheta(k, [h, \alpha(l)]) - \vartheta(h, [l, \alpha(k)]) - \vartheta(l, [k, \alpha(h)]). \end{aligned}$$

Using that β is linear and the definition of cocycles (4)

$$\begin{aligned} X &= -\beta([k, \vartheta(h, l)]) - \beta([h, \vartheta(l, k)]) - \beta([l, \vartheta(k, h)]) \\ &\quad + [\alpha(k), \vartheta(h, l)] + [\alpha(h), \vartheta(l, k)] + [\alpha(l), \vartheta(k, h)] \\ &\quad + [k, \vartheta(\alpha(h), l)] + [h, \vartheta(\alpha(l), k)] + [l, \vartheta(\alpha(k), h)] \\ &\quad + [k, \vartheta(h, \alpha(l))] + [h, \vartheta(l, \alpha(k))] + [l, \vartheta(k, \alpha(h))]. \end{aligned}$$

Since (α, β) is a compatible pair we have by (7)

$$\begin{aligned} \beta([k, \vartheta(h, l)]) &= [\alpha(k), \vartheta(h, l)] + [k, \beta(\vartheta(h, l))]; \\ \beta([h, \vartheta(l, k)]) &= [\alpha(h), \vartheta(l, k)] + [h, \beta(\vartheta(l, k))]; \\ \beta([l, \vartheta(k, h)]) &= [\alpha(l), \vartheta(k, h)] + [l, \beta(\vartheta(k, h))]. \end{aligned}$$

Hence we obtain combining the last two displayed systems of equations

$$\begin{aligned} X &= -[k, \beta(\vartheta(h, l))] - [h, \beta(\vartheta(l, k))] - [l, \beta(\vartheta(k, h))] \\ &\quad + [k, \vartheta(\alpha(h), l)] + [h, \vartheta(\alpha(l), k)] + [l, \vartheta(\alpha(k), h)] \\ &\quad + [k, \vartheta(h, \alpha(l))] + [h, \vartheta(l, \alpha(k))] + [l, \vartheta(k, \alpha(h))]. \end{aligned}$$

Again, by the definition of the action in (11)

$$X = -[k, (\alpha, \beta) \cdot \vartheta(h, l)] - [h, (\alpha, \beta) \cdot \vartheta(l, k)] - [l, (\alpha, \beta) \cdot \vartheta(k, h)].$$

So $(\alpha, \beta) \cdot \vartheta \in Z^2(K, I)$.

Now suppose that $\vartheta \in \mathbf{B}^2(K, I)$. By definition (5) there is a linear map $T : K \rightarrow I$ such that $\vartheta = \vartheta_T$. Hence

$$(13) \quad \vartheta_T(h, k) = T([h, k]) + [k, T(h)] - [h, T(k)].$$

Let $Y = (\alpha, \beta) \cdot \vartheta_T(h, k)$. By (13) we have

$$(14) \quad Y = \beta(\vartheta_T(h, k)) - \vartheta_T(\alpha(h), k) - \vartheta_T(h, \alpha(k)).$$

Using the definition of ϑ_T we have

$$(15) \quad \begin{aligned} \beta(\vartheta_T(h, k)) &= \beta T([h, k]) + \beta[k, T(h)] - \beta[h, T(k)], \\ \vartheta_T(\alpha(h), k) &= T([\alpha(h), k]) + [k, T\alpha(h)] - [\alpha(h), T(k)], \\ \vartheta_T(h, \alpha(k)) &= T([h, \alpha(k)]) + [\alpha(k), T(h)] - [h, T\alpha(k)]. \end{aligned}$$

We can use that (α, β) is a compatible pair in equation (15) to write

$$\beta(\vartheta_T(h, k)) = \beta T([h, k]) + [\alpha(k), T(h)] + [k, \beta T(h)] - [\alpha(h), T(k)] - [h, \beta T(k)].$$

Then

$$\begin{aligned} Y &= \beta T([h, k]) + [\alpha(k), T(h)] + [k, \beta T(h)] - [\alpha(h), T(k)] - [h, \beta T(k)] \\ &\quad - T([\alpha(h), k]) - [k, T\alpha(h)] + [\alpha(h), T(k)] \\ &\quad - T([h, \alpha(k)]) - [\alpha(k), T(h)] + [h, T\alpha(k)]. \end{aligned}$$

Making the cancellations, Y can be written as

$$\begin{aligned} Y &= \beta T([h, k]) - T([\alpha(h), k]) - T([h, \alpha(k)]) \\ &\quad + [k, \beta T(h)] - [k, T\alpha(h)] + [h, T\alpha(k)] - [h, \beta T(k)]. \end{aligned}$$

Now we use that T and the action are linear to obtain

$$Y = \beta T([h, k]) - T([\alpha(h), k] + [h, \alpha(k)]) + [k, \beta T(h) - T\alpha(h)] - [h, \beta T(k) - T\alpha(k)].$$

Hence,

$$Y = (\beta T - T\alpha)([h, k]) + [k, (\beta T - T\alpha)(h)] - [h, (\beta T - T\alpha)(k)].$$

If $U = \beta T - T\alpha : K \rightarrow I$ then

$$(\alpha, \beta) \cdot \vartheta(h, k) = U([h, k]) - [k, U(h)] - [h, U(k)].$$

Therefore, $(\alpha, \beta) \cdot \vartheta \in \mathbf{B}^2(K, I)$. We just proof

Proposition 3.4. *Let K be a Lie algebra and let I be a K -module. Consider the action of $\text{Comp}(K, I)$ on $\mathbf{C}^2(K, I)$ defined in (11). Then the vector spaces $\mathbf{Z}^2(K, I)$ and $\mathbf{B}^2(K, I)$ are invariants by this action.*

This result allows us to define an action of $\mathbf{Comp}(K, I)$ on $H^2(K, I)$: let $\vartheta \in Z^2(K, I)$ and $(\alpha, \beta) \in \mathbf{Comp}(K, I)$. Define the action

$$(16) \quad (\alpha, \beta) \cdot (\vartheta + B^2(K, I)) = ((\alpha, \beta) \cdot \vartheta) + B^2(K, I).$$

This is well defined by Proposition 3.4.

Definition 3.5. Let K be a Lie algebra and I a K -module. Let $\vartheta \in Z^2(K, I)$ and consider the action of $\mathbf{Comp}(K, I)$ on $H^2(K, I)$ defined in (16). Define the set of induced pairs of $\mathbf{Comp}(K, I)$ by

$$\text{Indu}(K, I, \vartheta) = \text{Ann}_{\mathbf{Comp}(K, I)}(\vartheta + B^2(K, I)).$$

Now we have the tools needed to describe the Lie algebra $\text{Der}(K_\vartheta)$ from the Lie algebra $\text{Der}(K)$. We will define a homomorphism $\phi : \text{Der}(K_\vartheta) \rightarrow \text{Der}(K)$, whose kernel is known and the image coincides with the induced pairs defined above. So, using the First Isomorphism Theorem for Lie algebras we have $\text{Der}(K_\vartheta)$ is isomorphic to $\text{Ker}(\phi) \oplus \text{Im}(\phi)$ but these subspaces correspond to structures: $\text{Ker}(\phi) \cong Z^1(K, I)$ and $\text{Im}(\phi) \cong \text{Indu}(K, I, \vartheta)$. One application of this type of construction is using known information about the algebra $\text{Der}(K)$ to obtain information about the algebra $\text{Der}(K_\vartheta)$ as the existence of non-singular derivations. Therefore, this method will allow us to study some properties of Lie algebra extensions by cocycles. First we define ϕ .

Let K be a Lie algebra and I a K -module. Let $\vartheta \in H^2(K, I)$ and $d \in \text{Der}(K_\vartheta)$. Suppose that I , as ideal of K_ϑ , is invariant under d . Recall that $K_\vartheta = K \oplus I$ and let $\pi_K : K_\vartheta \rightarrow K$ and $\pi_I : K_\vartheta \rightarrow I$ to be the natural vector space projections of K_ϑ onto K and K_ϑ onto I . Then define the maps

- $\alpha : K \rightarrow K$ by $\alpha(h) = \pi_K d(h)$, for all $h \in K$;
- $\beta : I \rightarrow I$ by $\beta(a) = d(a)$, for all $a \in I$;
- $\eta : K \rightarrow I$ by $\eta(h) = \pi_I d(h)$, for all $h \in K$.

For each $h + a \in K_\vartheta$ we have

$$(17) \quad d(h + a) = \alpha(h) + \eta(h) + \beta(a) \text{ for all } h \in K \text{ and } a \in I.$$

We can see that β is a derivation of I because it is restriction of d to I . To see that $\alpha \in \text{Der}(K)$ let $x, y \in K$. To make our calculation more clear, we will denote $[\cdot, \cdot]_K$ the product in K , and by $[\cdot, \cdot]_\vartheta$ the product in K_ϑ . Then by product definition on K_ϑ

$$d([h, k]_\vartheta) = d([h, k]_K + \vartheta(h, k)).$$

By the decomposition showed in (17)

$$(18) \quad d([h, k]_\vartheta) = \alpha([h, k]_K) + \eta([h, k]_K) + \beta(\vartheta(h, k)).$$

We can calculate

$$(19) \quad [d(h), k]_\vartheta + [h, d(k)]_\vartheta = [\alpha(h) + \eta(h), k]_\vartheta + [h, \alpha(k) + \eta(k)]_\vartheta,$$

and use the definition of the product in equation (19) to get

$$(20) \quad [d(h), k]_{\vartheta} + [h, d(k)]_{\vartheta} = [\alpha(h), k]_K + \vartheta(\alpha(h), k) - [k, \eta(h)]_{\vartheta} \\ + [h, \alpha(k)]_K + \vartheta(h, \alpha(k)) + [h, \eta(k)]_{\vartheta}.$$

Comparing the components of K in (18) and (20) we have

$$\alpha([h, k]_K) = [\alpha(h), k]_K + [h, \alpha(k)]_K,$$

and $\alpha \in \text{Der}(K)$.

Now it is possible to define our homomorphism ϕ . Let K be a Lie algebra and I a K -module. Let $\vartheta \in \mathbf{H}^2(K, I)$ and suppose that I , as an ideal of K_{ϑ} , is invariant under derivations. For all $x + a \in K_{\vartheta}$ and $d \in \text{Der}(K)_{\vartheta}$ write $d(x + a) = \alpha(x) + \eta(x) + \beta(a)$ with $\alpha \in \text{Der}(K)$ and $\beta \in \text{Der}(I)$. Then define $\phi : \text{Der}(K_{\vartheta}) \rightarrow \text{Der}(K) \oplus \text{Der}(I)$ by

$$(21) \quad \phi(d) = (\alpha, \beta).$$

The following calculation will check that ϕ is a Lie algebra morphism. Let $d, d' \in \text{Der}(K_{\vartheta})$ such that

$$\begin{aligned} d(h + a) &= \alpha(h) + \eta(h) + \beta(a) \\ d'(h + a) &= \alpha'(h) + \eta'(h) + \beta'(a), \end{aligned}$$

Then

$$\begin{aligned} dd'(h) &= d(\alpha'(h) + \eta'(h) + \beta'(a)) \\ &= \alpha\alpha'(h) + \eta(\alpha'(h)) + \beta(\eta'(h) + \beta'(a)). \end{aligned}$$

Hence, $\pi_K dd'(h) = \alpha\alpha'(h)$. Analogously, $\pi_K d'd(h) = \alpha'\alpha(h)$. So $\pi_K[d, d'] = [\alpha, \alpha']$. As β and β' are defined by restriction of d and d' to I , respectively, then $\pi_I[d, d'] = [\beta, \beta']$. Therefore,

$$\phi([d, d']) = ([\alpha, \alpha'], [\beta, \beta']) = [(\alpha, \beta), (\alpha', \beta')] = [\phi(d), \phi(d')].$$

The next result presents the first connection between compatible pairs and the homomorphism ϕ .

Lemma 3.6. *Let K be a Lie algebra and I a K -module. Let $\vartheta \in \mathbf{H}^2(K, I)$ and suppose that I , as an ideal of K_{ϑ} , is invariant under derivations. Let $\phi : \text{Der}(K_{\vartheta}) \rightarrow \text{Der}(K) \oplus \text{Der}(I)$ given by $\phi(d) = (\alpha, \beta)$, defined in (21). Then $\text{Im}(\phi) \leq \text{Comp}(K, I)$.*

Proof. Let $(\alpha, \beta) \in \text{Im}(\phi)$. Then there is $d \in \text{Der}(K_{\vartheta})$ such that $\phi(d) = (\alpha, \beta)$. If $h \in K$ and $a \in I$ then

$$\begin{aligned} \beta([h, a]_{\vartheta}) &= d([h, a]_{\vartheta}) && (\text{since } [h, a] \in I) \\ &= [d(h), a]_{\vartheta} + [h, d(a)]_{\vartheta} && (d \in \text{Der}(K_{\vartheta})) \\ &= [\alpha(h) + \eta(h), a]_{\vartheta} + [h, \beta(a)]_{\vartheta} \\ &= [\alpha(h), a]_{\vartheta} + [h, \beta(a)]_{\vartheta} && (\text{since } I \text{ is abelian}). \end{aligned}$$

□

Now we present the main theorem of this section. Recall that for a Lie algebra K , for a K -module I , and for $\vartheta \in Z^2(K, I)$, $\text{Indu}(K, I, \vartheta)$ was defined in Definition 3.5.

Theorem 3.7. *Let K be a Lie algebra and let I be a K -module. Let $\vartheta \in H^2(K, I)$ and suppose that I , as ideal of K_ϑ , is invariant by derivations. Let $\phi : \text{Der}(K_\vartheta) \rightarrow \text{Der}(K) \oplus \text{Der}(I)$ be defined as above. Then:*

- (1) $\text{Im}(\phi) = \text{Indu}(K, I, \vartheta)$
- (2) $\text{Ker}(\phi) \cong Z^1(K, I)$

Proof. In this proof we will denote the product in K_ϑ of the $h \in K$ and $a \in I$ just by the action $[h, a]$ of K on I , since $[h, a]_\vartheta = [h, a]$.

1) Let $(\alpha, \beta) \in \text{Indu}(K, I, \vartheta)$. By definition

$$(\alpha, \beta) \cdot \vartheta = 0 \text{ mod } B^2(K, I).$$

Then there is a linear map $T : K \rightarrow I$ such that, for all $h, k \in K$,

$$(22) \quad \beta(\vartheta(h, k)) - \vartheta(\alpha(h), k) - \vartheta(h, \alpha(k)) = T([h, k]) + [k, T(h)] - [h, T(k)].$$

Let $h \in K$, $a \in I$ and define the linear map $(\alpha, \beta)^* : K_\vartheta \rightarrow K_\vartheta$ by

$$(23) \quad (\alpha, \beta)^*(h + a) = \alpha(h) - T(h) + \beta(a).$$

Let's check that $(\alpha, \beta)^*$ is a derivation of K_ϑ . Let $k + b \in K_\vartheta$. If

$$X = (\alpha, \beta)^*([h + a, k + b]_\vartheta)$$

then

$$\begin{aligned} X &= (\alpha, \beta)^*([h, k]_K + \vartheta(h, k) + [h, b] - [k, a]) \\ &= \alpha([h, k]_K) - T([h, k]_K) + \beta(\vartheta(h, k)) + \beta([h, b]) - \beta([k, a]). \end{aligned}$$

Now, let

$$Y = [(\alpha + \beta)^*(h + a), k + b]_\vartheta + [h + a, (\alpha + \beta)^*(k + b)]_\vartheta.$$

By definition (23)

$$[(\alpha + \beta)^*(h + a), k + b]_\vartheta = [\alpha(h) - T(h) + \beta(a), k + b]_\vartheta.$$

Hence, by product definition in (6)

$$[\alpha(h) - T(h) + \beta(a), k + b]_\vartheta = [\alpha(h), k]_K + \vartheta(\alpha(h), k) + [\alpha(h), b] - [k, -T(h) + \beta(a)]$$

and

$$[(\alpha + \beta)^*(h + a), k + b]_\vartheta = [\alpha(h), k]_K + \vartheta(\alpha(h), k) + [\alpha(h), b] - [k, -T(h) + \beta(a)].$$

Analogously,

$$[h + a, (\alpha + \beta)^*(k + b)]_\vartheta = [h, \alpha(k)]_K + \vartheta(h, \alpha(k)) + [h, -T(k) + \beta(b)] - [\alpha(k), a].$$

It follows

$$\begin{aligned} Y &= [\alpha(h), k]_K + [h, \alpha(k)]_K + \vartheta(\alpha(h), k) + \vartheta(h, \alpha(k)) \\ &\quad + [\alpha(h), b] + [h, \beta(b)] - [k, \beta(a)] - [\alpha(k), a] - [h, T(k)] + [k, T(h)]. \end{aligned}$$

We can use that $(\alpha, \beta) \in \mathbf{Comp}(K, I)$ to write Y as

$$Y = \alpha([h, k]_K) + \vartheta(\alpha(h), k) + \vartheta(h, \alpha(h)) \\ + \beta([h, b]) - \beta([k, a]) - [h, T(k)] + [k, T(h)].$$

By equation (22)

$$\vartheta(\alpha(h), k) + \vartheta(h, \alpha(k)) = \beta(\vartheta(h, k)) - T([h, k]) - [k, T(h)] + [h, T(k)].$$

Then

$$Y = [\alpha(h), k]_K + [h, \alpha(k)]_K + \beta(\vartheta(h, k)) - T([h, k]) - [k, T(h)] + [h, T(k)] \\ + \beta([h, b]) - \beta([k, a]) - [h, T(k)] + [k, T(h)].$$

As $X = Y$ then $(\alpha, \beta)^*$ is a derivation.

Besides, observe that $\pi_K(\alpha, \beta)^* = \alpha$ and $\pi_I(\alpha, \beta)^* = \beta$. Hence $\phi((\alpha + \beta)^*) = (\alpha, \beta)$, that is, $\mathbf{Indu}(K, I, \vartheta) \subseteq \mathbf{Im}(\phi)$.

Now, suppose that $(\alpha, \beta) \in \mathbf{Im}(\phi)$. Then there is $d \in \mathbf{Der}(K_\vartheta)$ such that

$$\phi(d) = (\alpha, \beta).$$

By Theorem 3.6 we have $\mathbf{Im}(\phi) \subseteq \mathbf{Comp}(K, I)$. Then it is enough to show that there is a linear map $T : K \rightarrow I$ such that the equation (22) is satisfied.

For each $h + a \in K_\vartheta$ we can use the decomposition defined in (17) to write

$$d(h + a) = \alpha(h) + \eta(h) + \beta(a).$$

Then

$$[d(h + a), k + b]_\vartheta = [\alpha(h) + \eta(h) + \beta(a), k + b]_\vartheta.$$

By product definition in (6) we get

$$[\alpha(h) + \eta(h) + \beta(a), k + b]_\vartheta = [\alpha(h), k]_K + \vartheta(\alpha(h), k) + [\alpha(h), b] - [k, \eta(h) + \beta(a)].$$

Hence

$$[d(h + a), k + b]_\vartheta = [\alpha(h), k]_K + \vartheta(\alpha(h), k) + [\alpha(h), b] - [k, \eta(h) + \beta(a)].$$

Analogously,

$$[h + a, d(k + b)]_\vartheta = [h, \alpha(k)]_K + \vartheta(h, \alpha(k)) + [h, \eta(k) + \beta(b)] - [\alpha(k), a].$$

Therefore

$$(24) \quad [d(h + a), k + b]_\vartheta + [h + a, d(k + b)]_\vartheta = [\alpha(h), k]_K + [h, \alpha(k)]_K + \vartheta(\alpha(h), k) + \vartheta(h, \alpha(k)) \\ + [\alpha(h), b] + [h, \beta(b)] - [\alpha(k), a] - [k, \beta(a)] - [k, \eta(h)] + [h, \eta(k)].$$

We can use that $(\alpha, \beta) \in \mathbf{Comp}(K, I)$ in the last displayed equation to write

$$[d(h + a), k + b]_\vartheta + [h + a, d(k + b)]_\vartheta = \alpha([h, k]_K) + \vartheta(\alpha(h), k) + \vartheta(h, \alpha(k)) \\ + \beta([h, b]) - \beta([k, a]) - [k, \eta(h)] + [h, \eta(k)].$$

Now we will calculate $d([k + a, h + b]_{\vartheta})$. By product definition

$$d([h + a, k + b]_{\vartheta}) = d([h, k]_K + \vartheta(h, k) + [h, b] - [k, a]).$$

Hence

$$d([h, k]_K + \vartheta(h, k) + [h, b] - [k, a]) = \alpha([h, k]_K) + \eta([h, k]_K) + \beta(\vartheta(h, k)) + \beta([h, b]) - \beta([k, a]).$$

As d is a derivation then we have equality

$$d([h + a, k + b]_{\vartheta}) = [d(h + a), k + b]_{\vartheta} + [h + a, d(k + b)]_{\vartheta}.$$

It follows

$$\vartheta(\alpha(h), k) + \vartheta(h, \alpha(k)) - [k, \eta(h)] + [h, \eta(k)] = \eta([h, k]_K) + \beta(\vartheta(h, k)).$$

We can rearrange the last displayed equation to get

$$-(\eta([h, k]_K) + [k, \eta(h)] - [h, \eta(k)]) = \beta(\vartheta(h, k)) - \vartheta(\alpha(h), k) - \vartheta(h, \alpha(k)).$$

Therefore $T = -\eta$ satisfies the equation (22) e $\text{Im}(\phi) \subseteq \text{Indu}(\mathbf{K}, \mathbf{l}, \vartheta)$.

2) Let $d \in \text{Ker}(\phi)$. The decomposition showed in (17) provide us

$$d(h) = \eta(h), h \in K.$$

Let $h, k \in K$. By definition of derivation

$$(25) \quad d([h, k]_{\vartheta}) = [d(h), k]_{\vartheta} + [h, d(k)]_{\vartheta}.$$

We can use product definition in K_{ϑ} to write

$$d([h, k]_{\vartheta}) = d([h, k]_K + \vartheta(h, k)).$$

Since $d \in \text{Ker}(\phi)$ then

$$d([h, k]_{\vartheta}) = \eta([h, k]_K).$$

By other hand,

$$[d(h), k]_{\vartheta} + [h, d(k)]_{\vartheta} = [\eta(h), k]_{\vartheta} + [h, \eta(k)]_{\vartheta}.$$

Then (25) it is equal to

$$\eta([k, h]_K) = [k, \eta(h)] - [h, \eta(k)],$$

and $\eta \in Z^1(\mathbf{K}, \mathbf{l})$. Observe that η is the restriction of d to K . Denote the restriction of d to K by $d|_K$. Therefore, if $d \in \text{Ker}(\phi)$ then $d|_K \in Z^1(K, I)$.

Let $d \in \text{Ker}(\phi)$ and define $\sigma : \text{Ker}(\phi) \rightarrow (Z^1(K, I), +)$ by $\sigma(d) = d|_K$. Then $\sigma(\text{Ker}(\phi)) \subseteq Z^1(\mathbf{K}, \mathbf{l})$. Let $d' \in \text{Ker}(\phi)$. Then

$$\sigma(d + d') = (d + d')|_K = d|_K + d'|_K = \sigma(d) + \sigma(d').$$

So σ it is group homomorphism.

First we will show that σ is injective. Let $d, d' \in \text{Ker}(\phi)$ such that $\sigma(d) = \sigma(d')$. Let $h + a \in K_{\vartheta}$. Then

$$d(h + a) = d(h) = d|_K(h) = d'|_K(h) = d'(h) = d(h + a).$$

Hence $d = d'$. Now, to prove that σ is onto, let $\eta \in Z^1(K, I)$ and define a linear map $d : K_{\vartheta} \rightarrow K_{\vartheta}$ by

$$d(h + a) = T(x), h \in K, a \in I.$$

We will show that d is a derivation. Observe that, for all $h + a, k + b \in K_{\vartheta}$ we have

$$d([h + a, k + b]_{\vartheta}) = d([h, k]_K + \vartheta(h, k) + [h, b] - [k, a]) = \eta([h, k]_K).$$

By other hand,

$$\begin{aligned} [d(h + a), k + b]_{\vartheta} + [h + a, d(k + b)]_{\vartheta} &= [\eta(h), k + b]_{\vartheta} + [h + a, \eta(k)]_{\vartheta} \\ &= -[k, \eta(h)] + [h, \eta(k)]. \end{aligned}$$

Since $\eta \in Z^1(K, I)$ then $d([h + a, k + b]_{\vartheta}) = [d(h + a), k + b]_{\vartheta} + [h + a, d(k + b)]_{\vartheta}$, hence $d \in \text{Der}(K_{\vartheta})$. It is immediate that $\phi(d) = 0$. So $d \in \text{Ker}(\phi)$. As by definition, $\sigma(d) = \eta$ then σ is onto and, therefore, is an isomorphism. \square

3.4. Compatible pairs and Jacobson Theorem. In this section we show some examples of the use of compatible pairs.

Example 3.8. Let K and I be finite dimensional Lie algebras over an algebraically closed field \mathbb{F} . Suppose that K act on I by representation $\psi : K \rightarrow \text{Der}(I)$. Let $D \subseteq \text{Comp}(K, I)$ be a subalgebra. By Proposition 3.2, $D \subseteq \text{Der}(L)$. If D is nilpotent then L has a decomposition in generalized eigenspaces of D . This decomposition induces decompositions in K and I , because as subspaces of L they are invariants by D . Hence,

$$L = K_{\lambda_1} \oplus \cdots \oplus K_{\lambda_r} \oplus I_{\mu_1} \cdots \oplus I_{\mu_s}.$$

In particular, we have $[K_{\lambda_i}, I_{\mu_j}] \subseteq I_{\lambda_i + \mu_j}$ if $\lambda_i + \mu_j$ is eigenvalue of D in I . Otherwise $[K_{\lambda_i}, I_{\mu_j}] = 0$.

From this example we can state a result:

Proposition 3.9. *Let K and I be finite dimensional Lie algebras over an algebraically closed field \mathbb{F} . Suppose that K act on I by representation $\psi : K \rightarrow \text{Der}(I)$. Let $D \subseteq \text{Comp}(K, I)$ be a subalgebra. Suppose that 0 is not generalized eigenvalue of D . Then if either characteristic of \mathbb{F} is zero or either characteristic of \mathbb{F} is p and D has at most $p - 1$ generalized eigenvalues the $\psi(K)$ is nilpotent.*

Proof. Let $L = K_{\lambda_1} \oplus \cdots \oplus K_{\lambda_r} \oplus I_{\mu_1} \cdots \oplus I_{\mu_s}$ the eigenspace decomposition present in Example 3.8. Suppose that 0 is not generalized eigenvalue of D . Let $E_K = \{\lambda_1, \dots, \lambda_r\}$ and $E_I = \{\mu_1, \dots, \mu_s\}$ be generalized eigenvalue of D in K and I , respectively. Let $k \in K_{\alpha_j}, a \in I_{\mu_i}$ then

$$\begin{cases} \psi^n(k)(a) \in I_{\mu_i + n\lambda_j} & \text{if } \mu_i + n\lambda_j \in E_I \\ \psi^n(k)(a) = 0 & \text{if } \mu_i + n\lambda_j \notin E_I \end{cases}$$

- If characteristic of \mathbb{F} is zero then the linear functions $\mu_i + \lambda_j, \mu_i + 2\lambda_j, \dots, \mu_i + n\lambda_j \dots$ are all distinct because $\lambda_j \neq 0$, so $\mu_i + n\lambda_j \notin E_I$ for some n and $\psi(k)^n = 0$.
- If $\text{char}(\mathbb{F}) = p$ and $s < p$ the set $\{\mu_i + \lambda_j, \mu_i + 2\lambda_j, \dots, \mu_i + (p-1)\lambda_j, \mu_i\}$ has p distinct elements and E_I has at most $p-1$, then $\psi^n(k) = 0$ for some n with $1 \leq n \leq p$.

In both cases $\psi(k)$ is nilpotent for all $k \in K_{\lambda_j}$, $1 \leq j \leq r$. Let $S = \bigcup \psi(K_{\lambda_j})$. S is a weakly closed set such that each element is associative nilpotent then $\psi(K)$ is nilpotent. \square

For our next example we need some result about traces of matrices.

Proposition 3.10. *Let \mathbb{F} be a field of characteristic p . Suppose that $A \in M(n, \mathbb{F})$ with $n < p$ or $p=0$. Then A is nilpotent if, and only if, the trace of matrices A^r is zero, for $1 \leq r \leq n$.*

Proof. Let $\overline{\mathbb{F}}$ the algebraic closure of \mathbb{F} e consider A in its Jordan normal form. This can be done because Jordan normal form is obtained from A by conjugation of matrices over \mathbb{F} . But since trace and nilpotency of matrices are invariants by conjugation our results still valid for A . We will use that a matrix is nilpotent if, and only if, zero is its only eigenvalue.

A can be seen as a diagonal block matrix where each block is formed by grouping the blocks associated to same eigenvalue. Denote by A_j the block associated to eigenvalue $\lambda_t \in \overline{\mathbb{F}}$ and by n_j its order. Let $\lambda_1, \dots, \lambda_k$ be the non-zero eigenvalues of A . Then

$$(26) \quad \text{tr}(A^r) = n_1 \lambda_1^n + \dots + n_k \lambda_k^n$$

Suppose that A is nilpotent. Then zero is the only eigenvalue of A and by equation (26) we have $\text{tr}(A^r) = 0$ for $1 \leq r \leq n$.

Conversely, suppose that $\text{tr}(A^r) = 0$ for $1 \leq r \leq n$. From equation (26) we can extract the system

$$(27) \quad n_1 \lambda_1^r + \dots + n_k \lambda_k^r = 0, \quad 1 \leq r \leq k,$$

in the variables n_1, \dots, n_k , whose matrix of coefficients is

$$C = \begin{bmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \dots & \lambda_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^k & \lambda_2^k & \dots & \lambda_k^k \end{bmatrix}.$$

Denote by $m_i(\lambda)$ the operation that multiplies the line i of a matrix by λ and A^t the transposed matrix of A . So we can write

$$C = m_1(\lambda_1).m_2(\lambda_2) \dots m_k(\lambda_k).V,$$

where

$$V = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{k-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_k & \lambda_k^2 & \cdots & \lambda_k^{k-1} \end{bmatrix}$$

is the Vandermonde matrix in the variables $\lambda_1, \lambda_2, \dots, \lambda_k$ whose determinant is $\det V = \prod_{1 \leq i < j \leq k} (\lambda_j - \lambda_i)$. As λ_i are distinct we have that $\det V$ is non-zero. Then the determinant of C is $\lambda_1 \cdot \lambda_2 \cdots \lambda_k \cdot \det V$ and C is non-singular. Follow that the system (27) has only trivial solution. Therefore each n_j is zero. If $p = 0$ then zero is the only eigenvector of A , but if $p \neq 0$ then $n_j = 0$ modulo p doesn't imply $n_j = 0$ and its necessary to use that each $n_j < p$ to conclude that zero is the only eigenvalue of A . \square

Proposition 3.11. *Let \mathbb{F} be a field of characteristic p . Let $A, B, C \in M(n, \mathbb{F})$ with $p = 0$ or $n < p$. If $[A, B] = C + \lambda B$, $\lambda \in \mathbb{F}$ and $[B, C] = 0$ then $[A, B^r] = rB^{r-1}C + \lambda rB^r$ for all $r \geq 1$. In particular, if $\lambda \neq 0$ and C is nilpotent then B is nilpotent.*

Proof. We proof this result by induction on r . The case $r = 1$ follow from hypotheses. Suppose that result is valid for $(r - 1)$. Then, $[A, B^{r-1}] = (r - 1)B^{r-2}C + \lambda(r - 1)B^{r-1}$. We can rewrite this equation as

$$\lambda(r - 1)B^{r-1} = AB^{r-1} - B^{r-1}A - (r - 1)B^{r-2}C.$$

Multiplying last equation to right by B we have

$$\lambda(r - 1)B^r = AB^r - B^{r-1}(AB) - (r - 1)B^{r-2}(CB),$$

From hypotheses we can write $AB = BA + C + \lambda B$ and $CB = BC$. Replacing them above we obtain

$$\lambda(r - 1)B^r = AB^r - B^rA - B^{r-1}C - \lambda B^r - (r - 1)B^{r-1}C.$$

Therefore,

$$AB^r - B^rA = \lambda rB^r + rB^{r-1}C.$$

For the second result suppose $\lambda \neq 0$ and C nilpotent with nilpotency index m . Using first part we have

$$B^r = (1/\lambda r)[A, B^r] - (1/\lambda)B^{r-1}C, \text{ for all } r \geq 1.$$

Observe that $(B^{r-1}C)^m = (B^{r-1})^m(C)^m = 0$, Hence, for all $r \geq 1$ $B^{r-1}C$ is nilpotent and has trace zero by Proposition 3.10. As trace of commutators are always zero then $\text{tr}([A, B^r]) = 0$ for all $r \geq 1$. Follows that $\text{tr}(B^r) = 0$ for all $r \geq 1$ and again by Proposition 3.10 we conclude that B is nilpotent. \square

Proposition 3.12. *Let L be a Lie algebra, I an ideal of L such that L/I is nilpotent and such that $\text{ad}_x^I : I \rightarrow I$ is nilpotent for all $x \in L$. Then L is nilpotent.*

Proof. As L/I is nilpotent then for each $x \in L$, $(\text{ad}_{x+I}^I)^n$ is a nilpotent endomorphism in $\text{End}(L/I)$, i.e., there is $n > 0$ such that $(\text{ad}_x)^n(a) \in I$, for all $x \in L, a \in I$. In the other hand, ad_x^I is nilpotent, so we have a m such that $(\text{ad}_x^I)^m(\text{ad}_x)^n = 0$, i.e., $(\text{ad}_x^I)^{m+n} = 0$. So ad_x is a nilpotent endomorphism in $\mathfrak{gl}(L)$. By Engel's theorem, L is nilpotent. \square

Now we can present a similar result the proposition 3.9 but with a new proof using compatible pairs.

Theorem 3.13. *Let K and I be finite dimensional Lie algebras over a field of characteristic p such that K is nilpotent. Suppose that K act on I by representation $\psi : K \rightarrow \text{Der}(I)$. Let $(\alpha, \beta) \in \text{Comp}(K, I)$ such that α has no eigenvalue 0. If either $p = 0$ or $p > 0$ and dimension of I is less than p then $\text{Tr}(\psi^n(k)) = 0$, for all $k \in K$. In these two cases, $\psi(k)$ is nilpotent.*

Proof. As α has no eigenvalue 0 then it is non-singular and by Proposition 2.12 α is diagonalizable. Let x_1, \dots, x_s be a basis of K such that $\alpha(x_i) = \lambda_i x_i$. For all $a \in \mathfrak{gl}(I)$ denote by $[a]$ the matrix of a in this base. Then

$$[[\beta], [\psi(x_i)]] = \lambda_i [\psi(x_i)].$$

We can apply Proposition 3.11 in this last equation for $A = \beta$, $B = \psi(x_i)$, $C = 0$ and $\lambda = \lambda_i \neq 0$ to conclude that $\psi(x_i)$ is nilpotent for $1 \leq i \leq s$. Now we observe that if K is a nilpotent Lie algebra in either characteristic is 0 or characteristic p with dimension of L less than p then Lie theorem is valid. Lie theorem grants that there is a basis of I such that all matrices of representation ψ is upper triangular. Therefore, the matrices $[\psi(x_i)]$ are strictly upper triangular. Then all $\psi(k)$, for all $k \in K$, has only 0 in diagonal, because they are linear combination of $\psi(x_i)$. Hence every $\psi(k)$ is nilpotent. \square

Corollary 3.14. *Let L be a solvable Lie algebra over a field \mathbb{F} of characteristic p . Suppose that L has a nonsingular derivation. If either $p = 0$ or $p > 0$ and dimension of $L^{(i)}/L^{(i+1)} < p$ then L is nilpotent.*

Proof. Suppose that $L \geq L^{(1)} \geq \dots \geq L^{(k)} \geq L^{(k+1)} = 0$ is the derived series of L . Define $L_0 = L$ and $L_i = L_{i-1}/L_{i-1}^{(k+1-i)}$, $1 \leq i \leq k-1$. As each term of derived series are invariant by derivations then each L_i has a non-singular derivation. In particular, L_{k-1} is an solvable Lie algebra of derived length 2 with non-singular derivation. Then by theorem 3.13 ad_k is nilpotent for all $k \in L_{k-1}$ and by Proposition 3.12 L_{k-1} is nilpotent. By induction we have that L_i is nilpotent for every $0 \leq i \leq k-1$. Hence L is nilpotent \square

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DEPARTAMENTO DE CIÊNCIAS EXATAS E APLICADAS, INSTITUTO DE CIÊNCIAS EXATAS E APLICADAS,
UNIVERSIDADE FEDERAL DE OURO PRETO, RUA 37, 115, JOÃO MONLEVADE, MG, BRAZIL, MARCOS-
GOULART@DECEA.UFOP.BR