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Projeto de Tese de Doutorado

DERIVATIONS OF LIE ALGEBRA EXTENSIONS AND NON-SINGULAR DERIVATIONS OF LIE ALGEBRAS IN PRIME CHARACTERISTIC

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Belo Horizonte
Junho de 2016

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1. Introduction

Let L be a Lie algebra and d be a derivation of L. The derivation d is non-singular if is injective as linear transformation. We are interested in studying what information we can obtain about a Lie algebra if it has a nonsingular derivation. Jacobson's famous theorem [5] states that a finite-dimensional Lie algebra over a field of characteristic zero that admits a non-singular derivation must be nilpotent. It is well-known that this theorem is not valid is not valid when the characteristic is non-zero. Non-nilpotent and solvable examples were constructed by Shalev [10] and Mattarei [8], whereas the simple Lie algebras with non-singular derivations were classified by Benkart and her collaborators in [3]. A significant application of non-singular derivation was presented by Shalev [9]. In his proof of coclass conjectures of Leddham-Green and Newman for pro-p groups, Shalev uses that finite-dimensional Lie algebras over a field of characteristic $p > 0$ with non-singular derivation d such that $d^{p-1} = 1$, must be nilpotent.

Despite the existing examples, little is known about non-nilpotent Lie algebras with non-singular derivations. In these notes we explore the structure of solvable, non-nilpotent Lie algebras with non-singular derivations. In order to study this algebras we develop a theory of the derivations of Lie algebra extensions. We adopt the concept of a compatible pair of automorphisms introduced in [2] for derivations of Lie algebras.

Let K and I be Lie algebras such that K acts on I, then we can define a subalgebra $\mathsf{Comp}(L, I)$ of derivations of semi-direct sum $K \oplus I$

$$\mathsf{Comp}(K, I) = \{\alpha + \beta \in \mathfrak{gl}(K) \oplus \mathfrak{gl}(I) \mid \alpha + \beta \in \mathsf{Der}(K \oplus I)\}.$$

The algebra $\mathsf{Der}(K)$ carries information about the multiplicative structure of K. Analogously, the algebra $\mathsf{Comp}(K, I)$ carries information about the action of K on I. In section 3.4 we present an example of this by exploring the proof of Jacobson's Theorem and we prove a version for Lie algebras representations over a field of characteristic $p > 0$.

Theorem 3.14 *Let K and I be finite dimensional Lie algebras over a field of characteristic p such that K is nilpotent. Suppose that K act on I by representation $\psi : K \to \mathsf{Der}(I)$. Let $(\alpha, \beta) \in \mathsf{Comp}(K, I)$ such that α has no eigenvalue 0. If either $p = 0$ or $p > 0$ and dimension of I is less than p then $Tr(\psi^n(k)) = 0$, for all $k \in K$. In these two cases, $\psi(k)$ is nilpotent.*

We also adapt an algorithm presented by Bettina Eick [2] for calculating the automorphism group of solvable Lie algebras. A key step in the algorithm is the following. Let L be a Lie algebra and I an abelian ideal of L such that I is invariant by $Aut(L)$. Then there exists a homomorphism $\phi : Aut(L) \to Aut(L/I) \times Aut(I)$ induced by the actions of $Aut(L)$ on L/I and I. The image of ϕ can be calculated using $Aut(L/I)$, while $\mathsf{Ker}(\phi)$ is equal to $\mathsf{Z}^1(K, I)$. Then the group $Aut(L)$ can be obtained applying the first isomorphism theorem to ϕ. It is possible to use this process to derivations.

Let K be a Lie algebra and I be a K-module. Let $\mathsf{Z}^2(K, I)$ be the vector space of cocycles and $\mathsf{Comp}(K, I)$ the subalgebra of compatible pairs. Let $(\alpha, \beta) \in \mathsf{Comp}(K, I)$ and $\vartheta \in \mathsf{Z}^2(K, I)$. Define an action of $\mathsf{Comp}(K, I)$ over $\mathsf{Z}^2(K, I)$ by

$$(\alpha, \beta) \cdot \vartheta(h, k) = \beta(\vartheta(h, k)) - \vartheta(\alpha(k), h) - \vartheta(k, \alpha(h)), \qquad \text{for all } h, k \in K.$$

The elements of annihilator of this action will be called induced pairs and we denote the set of induced pairs by $\mathsf{Indu}(K, I, \vartheta)$. Let $\vartheta \in \mathsf{Z}^2(K, I)$ a cocycle and K_θ be the Lie algebra extension obtained from K by ϑ. Then we can lift the derivation of $\mathsf{Indu}(K, I, \vartheta)$ to $Der(K_\theta)$. Thus we obtained the following theorem.

Theorem 3.8 *Let K be a Lie algebra and I a K-module. Let $\vartheta \in \mathsf{H}^2(K, I)$ and suppose that I, as ideal of K_ϑ, invariant under derivations of K_ϑ. Let $\phi : \mathsf{Der}(K_\vartheta) \to \mathsf{Der}(K) \oplus \mathsf{Der}(I)$ given by $\phi(d) = (\alpha, \beta)$. Then:*

(1) $\mathsf{Im}(\phi) = \mathsf{Indu}(K, I, \vartheta)$
(2) $\mathsf{Ker}(\phi) \cong \mathsf{Z}^1(K, I)$

The details of this construction can be seen in Section 3. There is a significant difference between the application of this approach to automorphisms and to derivations: calculating the automorphism groups of a Lie algebra is usually a difficult task that may involve a large orbit-stabilizer calculation, while calculating the algebra $\mathsf{Der}(K_\vartheta)$ can be done by solving a system of linear equations. Thus, to understand the importance of Theorem 3.8 we must discover whose additional information of $\mathsf{Der}(K_\vartheta)$ we are able to obtain through information of algebras $\mathsf{Der}(K)$ and $\mathsf{Der}(I)$.

In order facilitate the reading of the text and the references, we added a section with results on the decomposition of vector spaces in relation to subalgebras of linear operators and a brief description of the main articles used.

This text is organized as follows: section 2 is dedicated to literature review. In Section 3, we present compatible pairs and the lifting process of derivations of a Lie algebra K to the Lie algebras K_ϑ such that ϑ is a cocycle. We end this section by applying the compatible pairs to Jacobson's Theorem. The Section 4 is composed of some examples and conjectures about modular solvable non-nilpotent Lie algebras with non-singular derivations.

2. Non-singular derivations: known results

This section is composed by description of a decomposition of a Lie algebra L relative to a subalgebra K of $\mathfrak{gl}(L)$ and it utilization in Jacobson's theorem. Next, we have the calculations of Shalev article [10] about conditions on the order of derivation which guarantees nilpotency of a Lie algebra. It finishes with Mattarei theorem that relates the order of derivations of solvable modular Lie algebra with non-singular derivation with roots of some polynomials.

2.1. **Basic concepts.** Let V be a finite-dimensional vector space over field $\mathbb{F}$ and $a \in \mathsf{End}(V)$. Let $p \in \mathbb{F}[X]$ be a univariate polynomial and define

$$V_0(p(a)) = \{v \in V \mid \text{ there is an } m > 0 \text{ such that } p(a)^m v = 0\}.$$

$V_0(p(a))$ is a vector subspace of V invariant under a. Now let A be the associative algebra with 1 generated by a. Let p_a be the minimum polynomial of a and suppose that

$$p_a = p_1^{k_1} \cdots p_r^{k_r}$$

is the factorization of p_a into irreducible factors, such that p_i has leading coefficient 1 and $p_i \neq p_j$ for $1 \leqslant i, j \leqslant r$. Then V decomposes as a direct sum of subspaces

$$V = V_0(p_1(a)) \oplus \cdots \oplus V_0(p_r(a)),$$

each space $V_0(p_i(a))$ being invariant under A. Furthermore, the minimum polynomial of the restriction of a to $V_0(p_i(a))$ is $p_i^{k_i}$. A proof of this result can be found in [1] Lemma A.2.2.

We can generalize this decomposition, instead of us consider an element $a \in \mathsf{End}(V)$ we can consider a subalgebra $K \leqslant \mathfrak{gl}(V)$. A decomposition $V = V_1 \oplus \cdots \oplus V_s$ of V into K-modules V_i is said to be primary if the minimum polynomial of the restriction of a to V_i is a power of an irreducible polynomial for all $a \in K$ and $1 \leqslant i \leqslant s$. The subspaces V_i are called primary components. If for any two components V_i and V_j $(i \neq j)$, there is an $x \in K$ such that the minimum polynomials of the restrictions of x to V_i and V_j are powers of different irreducible polynomials then the decomposition is called collected. In general V will not have a primary (or primary collected) decomposition into $K-$modules but it is guaranteed if the base field of V is algebraically closed and $K \leqslant \mathfrak{gl}(V)$ is nilpotent.

Proposition 2.1. *Let V be a vector space of finite-dimension. Let $K \leqslant \mathfrak{gl}(V)$ be a nilpotent subalgebra. Then V has a unique collected primary decomposition relative to K*

A proof of this result can be found in Theorem 3.1.10 of [1].

If the vector space V has a primary collected decomposition $V = V_1 \oplus \cdots \oplus V_s$ then we can characterize the components V_i. For $x \in K$ and $1 \leqslant i \leqslant s$ define $p_{x,i}$ to be the irreducible polynomial such that the minimum polynomial of x restricted to V_i is a power of $p_{x,i}$. Then

$$V_i = \{v \in V \mid \text{ for all } x \in K \text{ there is an } m > 0 \text{ such that } p_{x,i}(x)^m v = 0\}.$$

It is worth noting the case that the base field of V is algebraically closed, then all irreducible polynomials are of the form $p_{x,i} = (X - \lambda_i(x))$, $\lambda_i(x) \in \mathbb{F}$ and the primary components are of the form

$$V_i = \{v \in V \mid \text{ for all } x \in K \text{ there is an } m > 0 \text{ such that } (x - \lambda_i(x).Id)^m v = 0\},$$

with $\lambda_i \in K^*$. Its natural define a name for this case. Let V be a finite-dimensional vector space over field $\mathbb{F}$ and $K \in \mathfrak{gl}(V)$ a subalgebra. Let $\lambda \in K^*$. If

$$V_\lambda = \{v \in V \mid \text{ for all } x \in K \text{ there is an } m > 0 \text{ such that } (x - \lambda(x).Id)^m v = 0\} \neq 0,$$

then V_λ is called generalized eigenspace of V associated to eigenvalue $\lambda \in K^*$.

Now we consider a Lie algebra L and a nilpotent subalgebra $K \leqslant \mathsf{Der}(L)$. Then the decomposition to generalized eigenspaces of D can provide us some information of the multiplicative structure of L.

Proposition 2.2. *Let L be a Lie algebra over an algebraically closed field. Let K be a subalgebra of* $\mathsf{Der}(L)$. *If $\lambda, \mu : K \to \mathbb{F}$ are eigenvalues of K then $[L_\lambda, L_\mu] \subseteq L_{\lambda+\mu}$ if $\lambda + \mu$ is a eigenvalue of K. Otherwise $[L_\mu, L_\lambda] = 0$.*

A proof of this result can be found Section 2 of Chapter 3 of [6].

2.2. **Jacobson Theorem.** In the article *A note on automorphism and derivations of Lie algebras* [5], Jacobson used a variation of Engel's theorem for weakly closed sets to get sufficient conditions for a Lie algebra to be nilpotent. We recommend the reading of the Section 1 and 2 of Chapter 2 of Jacobson's book [6] as reference for examples and proofs.

Let A be an associative algebra with 1 over a field $\mathbb{F}$. A subset S of A is called weakly closed if for every ordered pair $(a, b) \in S \times S$, there is an element $\gamma(a, b) \in \mathbb{F}$ such that $ab + \gamma(a, b)ba \in M$. If S is a subset of A we denote by $\langle S \rangle$ the subalgebra of A (subalgebra containing 1) generated by S.

Proposition 2.3. *Let V be a finite-dimensional vector space over a field $\mathbb{F}$. Let $S \subseteq \mathfrak{gl}(V)$ be a weakly closed subset such that every $s \in S$ is associative nilpotent, that is, $s^k = 0$, for some positive integer k. Then the subalgebra $\langle S \rangle$ is nilpotent.*

A proof of this theorem can be found in Section 1 of Chapter 2 of [6]. With this result we can prove the following theorem.

Theorem 2.4. *(Jacobson) Let L be a finite-dimensional Lie algebra over a field of characteristic 0 and suppose that there exists a subalgebra D of the algebra of derivations of L such that*

(1) *D is nilpotent;*
(2) *if there is $c \in L$ such that $d(c) = 0$ for all $d \in D$ then $c = 0$.*

Then L is nilpotent.

Proof. Let $\bar{\mathbb{F}}$ be the algebraic closure of the base field. We can extend all derivations of L to $\bar{L} = L \otimes \bar{\mathbb{F}}$. If we prove that $\bar{L}$ is nilpotent then L is nilpotent. So we will assume that $\mathbb{F}$ is algebraically closed. In this case the extension of D is nilpotent and without 0 as eigenvalue, i. e. if there is $c \in L$ such that $d(c) = 0$ for all $d \in D$ then $c = 0$. Let $L = L_{\gamma_1} \oplus \cdots \oplus L_{\gamma_r}$ be the decomposition of L into generalized eigenspaces of D. By Proposition 2.2 we have $[L_{\gamma_i}, L_{\gamma_j}] \subseteq L_{\gamma_i+\gamma_j}$ if $\gamma_i + \gamma_j$ is a eigenvalue of D and $[L_{\gamma_i}, L_{\gamma_j}] = 0$ otherwise. Let $\mathsf{ad}_{L_{\gamma_j}}$ denote the set of adjoint mappings induced by elements of L_{γ_j}. Then the relation just noted shows that the set $S = \bigcup \mathsf{ad}_{L_{\gamma_j}}$ is a weakly closed set of linear transformations. Let $a \in L_{\gamma_j}$ and $b \in L_{\gamma_i}$. Then $(\mathsf{ad}_a)^s(b) \in L_{\gamma_i+s\gamma_j}$, for all $s \geqslant 0$. By assumption, 0 is not a eigenvalue and so the elements $\{\gamma_i + s\gamma_j\}_{s\geqslant 0}$ are all distinct. Then for some r greater enough $(\gamma_i + r\gamma_j)$ is not an eigenvalue and $\mathsf{ad}_a(b) = 0$. Follow that ad_a is nilpotent. Thus every element of S is nilpotent. By Proposition 2.3 we can conclude that the algebra $\langle S\rangle$ is nilpotent and hence ad_L is nilpotent. Therefore L is a nilpotent Lie algebra. □

A review of the proof of theorem 2.4 shows that the hypothesis of zero characteristic is essential to prove that every element in a homogeneous component is nilpotent. However, we can guarantee this in characteristic p by requiring that D has at most $p-1$ eigenvalues or that dimension of L is less than p. This leads to the following corollary fo Jacobson theorem.

Corollary 2.5. *Let L be a Lie algebra over a field of characteristic $p > 0$ and suppose that there exists a subalgebra D of the algebra of derivations of L such that*

(1) *D is nilpotent;*
(2) *if there is $c \in L$ such that $d(c) = 0$ for all $d \in D$ then $c = 0$.*

If D has at most $p-1$ eigenvalues then L is nilpotent.

One question that arises from the theorem is its validity in characteristic $p \neq 0$. However this does not happen, we have simple modular Lie algebras with non-singular derivations. For example, let $\mathbb{F}$ be the field of 2^m elements and L be the vector space over $\mathbb{F}$ such that

$$L = \langle x_\alpha \mid \alpha \in \mathbb{F}, \alpha \neq 0\rangle$$

with a basis labelled by nonzero elements of the field $\mathbb{F}$ and with multiplication $[x_\alpha, x_\beta] = (\beta-\alpha)x_{\alpha+\beta}$. Then L is a simple Lie algebr and the map $d \in \mathsf{End}(L)$ given by $d(e_\alpha) = \alpha e_\alpha$ is a non-singular derivation. A systematic investigation of simple Lie algebras with nonsingular derivations can be found in [3]. Another question is whether the converse is true. By Dixmer and Lister [4], there are nilpotent Lie algebras admitting only nilpotent derivations. We present one example of such an algebra: suppose that $\mathbb{F}$ is a field of characteristic 0 and L is the Lie algebra over $\mathbb{F}$

$$L = \langle x_1, x_2, \cdots, x_8\rangle$$

with dimension 8 and multiplication table

$$[e_1,e_2]=e_5 \quad [e_1,e_3]=e_6 \quad [e_1,e_4]=e_7 \quad [e_1,e_5]=-e_8 \quad [e_2,e_3]=e_8 \quad [e_2,e_4]=e_6$$
$$[e_2,e_6]=-e_7 \quad [e_3,e_4]=-e_5 \quad [e_3,e_5]=-e_7 \quad [e_4,e_6]=-e_8 \quad [e_i,e_j]=-[e_j,e_i]$$

and $[e_i,e_j]=0$ if it is not in table above. L is nilpotent with $L^3\neq 0$, $L^4=0$ and every derivation of L is nilpotent.

2.3. **The orders of non-singular derivations.** An interesting approach by Shalev in article [10] is to study the order of nonsingular derivations, establishing conditions for a Lie algebra over a field of characteristic p with non-singular derivations to be nilpotent. More precisely, Shalev studied the set of orders of nonsingular derivations of non-nilpotent Lie algebras of characteristic p. Later, Mattarei in [8] showed that this set of numbers corresponds to the solution of some equation modulo p. Below we present some results of these articles.

Let L be a Lie algebra over an algebraically closed field of characteristic p. We can characterize the matrix of a non-singular derivation of L. We need a result for derivations in Lie algebras over a field of characteristic p.

Proposition 2.6. *Let L be a Lie algebra over a field $\mathbb{F}$ of characteristic $p>0$. If $d\in\mathsf{Der}(L)$ then $d^p\in\mathsf{Der}(L)$.*

Proof. Let $d\in\mathsf{Der}(L)$. Then by Leibniz formula
$$d^n([x,y])=\sum_{k-0}^{n}\binom{n}{k}d^k(x)d^{n-k}(y),\ \text{for all n>0}.$$
Set $n=p$. Then $d^p([x,y])=[d^p(x),y]+[x,d^p(y)]$ and $d^p\in\mathsf{Der}(L)$. □

Proposition 2.7. *Let be V a finite dimension vector space over an algebraically closed field of characteristic $p>0$ and $f\in\mathsf{End}(V)$ non-singular with order r coprime to p. Then f is diagonalizable.*

Proof. Let A be the matrix of the endomorphism f in Jordan normal form and write $A=S+N$ such that S is diagonalizable, N is nilpotent and S,N commute. Denote by M_{ij} the element of matrix M of i^{th} line and j^{th} column. Follow that

- If $S_{ii}=\lambda_i$ then $S^k_{ii}=\lambda^k_i$, for all $k>0$;
- $N^k_{i(i+j)}=0$, for all $0\leqslant j<k$ and all $k>0$.

As the order of A is r we have $A^r=Id$. Then
$$Id=A^r=(S+N)^r=S^r+\binom{r}{1}S^{r-1}N+\binom{r}{2}s^{r-2}N^2+\cdots+\binom{r}{r-1}SN^{r-1}+N^r.$$
So $\binom{r}{1}S^{r-1}N=r(s^{r-1}n)=0$. But r and the eigenvalues of S^{r-1} are nonzero, this implies $S=0$. Then f is diagonalizable. □

Let L be a Lie algebra over the field $\mathbb{F}$ such that L has a non-singular derivation d. Let r be the order of d such that $r = sp^t$, with $\gcd(s,p) = 1$. Then d^{p^t} is a derivation whose order is prime to p and, by Proposition 2.7, it is diagonalizable. So if L is a Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic $p > 0$ with non-singular derivation then L has a diagonalizable derivation d without eigenvalue 0.

Proposition 2.8. *Let L be a finite-dimensional Lie algebra in characteristic $p > 0$ which admits a non-singular derivation d whose order n is coprime to p. Suppose L is not nilpotent. Then there exist $\lambda \in \bar{\mathbb{F}}_p$ such that $(\lambda + \delta)^n = 1$ for all $\delta \in \mathbb{F}_p$.*

Proof. Let $R = \{\alpha \in \bar{\mathbb{F}}_p \mid \alpha^n = 1\}$. If R is not contained in base field of L then we consider d for the extension $L \otimes \bar{\mathbb{F}}$, such that $F \subset \bar{\mathbb{F}}$. By Proposition 2.7, d is diagonalizable. Let $L = L_{\lambda 1} \dotplus \cdots \dotplus L_{\lambda_r}$ the decomposition of L to eigenspaces of d. The set $S = \bigcup \mathsf{ad}_{L_{\lambda_j}}$ is weakly closed with $\gamma(\mathsf{ad}_a, \mathsf{ad}_b) = -1$ for all $a \in L_{\lambda_i}, b \in L_{\lambda_j}$. If each ad_a is nilpotent then L is nilpotent by Proposition 2.3. As L is non-nilpotent by hypothesis then there is $a \in L_{\lambda_j}$ and $b \in L_{\lambda_i}$ such that $(\mathsf{ad}_a)^n(b) \neq 0$, $1 \leqslant n \leqslant p$. However this implies , $(\lambda_i + \delta\lambda_j)$ are eigenvalues of d for $1 \leqslant r \leqslant p$. Set $\lambda = \lambda_i\lambda_j^{-1}$. Then $(\lambda + \delta)^n = 1$ for all $\delta \in \mathbb{F}_p$. ☐

Usying the same notation of proof of Proposition 2.8 write $x^n - 1 = \prod_{\alpha \in R}(x - \alpha)$. Hence $\prod_{\delta \in \mathbb{F}_p}(x - \lambda - \delta)$ divides $x^n - 1$. But

$$\prod_{\delta \in \mathbb{F}_p}(x - \lambda - \delta) = (x - \lambda)^p - (x - \lambda) = x^p - x - c,$$

where $c = \lambda^p - \lambda$. Let $g(x) = x^p - x - c$. Then $g(x)$ divides $x^n - 1$, which implies that x^n is congruent to 1 modulo $g(x)$. In this case, Lemma 2.4 of [10] shows that $n \geqslant p^2 - 1$. Now we can prove the theorem.

Theorem 2.9. *Let L be a finite dimensional Lie algebra in characteristic $p > 0$ which admits non-singular derivation of order n. Write $n = p^s m$ where m is coprime to p. Suppose $m < p^2 - 1$. Then L is nilpotent.*

Proof. The derivation d^{p^s} has order m. Suppose that L is not nilpotent. Then by the comment above we have $m \geqslant p^2 - 1$. ☐

Mattarei in [8] presented an example of non-nilpotent solvable modular Lie algebra.

Example 2.10. Let $\alpha, \beta \in \bar{\mathbb{F}}_p$ with $\alpha\beta^{-1} \notin \mathbb{F}_p$. Let M be a p-dimensional vector space over $\bar{\mathbb{F}}_p$ with basis $e_1, \cdots, e_p$, and let E, F be the linear transformations of M defined by $E(e_i) = e_{i+1}$(indices modulo p), and $F(e_i) = (\alpha + i\beta)e_i$. The transformations E and F span a two-dimensional solvable Lie algebra, which has M as a left module. Let L be the semidirect sum of $\{E\}$ and M with respect to this action. Then F acts on L as a non-singular derivation, with eigenvalues β on $\{E\}$, and $\alpha + \lambda\beta$ for $\lambda \in \mathbb{F}_p$ on M.

The next result links the orders non-singular derivations of Lie algebras of characteristic p and two polynomials.

Proposition 2.11. *Let p a prime number and let n be a positive integer, prime to o. The following statements are equivalent:*

(1) *there exists a non-nilpotent Lie algebra of characteristic p with a non-singular derivations of order n;*
(2) *there exists an element $\alpha \in \bar{\mathbb{F}}_p$ such that $(\alpha+\lambda)^n = 1$ for all $\lambda \in \mathbb{F}_p$*
(3) *there exist an element $c \in \bar{\mathbb{F}}_p^*$ such that $x^p - x - c$ divides $x^n - 1$ as elements of the polynomial ring $\bar{\mathbb{F}}_p[x]$.*

Mattarei in [8] defines the set N_p of the possible orders of non-singular derivations of non-nilpotent Lie algebras of characteristic p and determine all elements of N_p which are smaller than p^3, for $p > 3$.

3. Derivations and Lie algebra extensions

3.1. **Lie algebra extensions.** The symbol '$\oplus$' will be used to denote the direct sum of algebras, while the direct sum of vector spaces will be denoted by '$\dotplus$'.

An *extension* of a Lie algebra K by a Lie algebra I is an exact sequence

$$0 \to I \xrightarrow{i} L \xrightarrow{s} K \to 0 \tag{1}$$

of Lie algebras. The Lie algebra L in the middle of the exact sequence contains an ideal $\mathsf{Ker}(s) = \operatorname{Im} i \cong I$ such that $L/I \cong K$. We will write informally that 'L is an extension of K by I'. The extension (1) *splits* if L has a subalgebra S such that $L = S \dotplus \mathsf{Ker}(s)$. The extension (1) is *trivial* if there exists an ideal S of L such that $L = S \oplus \mathsf{Ker}(s)$. The extension (1) is central if $\mathsf{Ker}(s)$ lies in the center $Z(L)$ of L.

Suppose that K and I are Lie algebras and $\psi : K \to \mathsf{Der}(I)$ is a given Lie algebra homomorphism. Then we say that K *acts* on I or that I is a K*-module*. In this case, the image $\psi(k)(a)$ of $a \in I$ under $k \in K$ will be written simply as $[k, a]$. If I is an ideal of a Lie algebra K, then K acts on I. If $k \in K$, then the image of k under this action will be denoted by ad_k^I or simply by ad_k when the domain of the representation is clear from the context. Thus, for $a \in I$ and for $k \in K$, $\mathsf{ad}_k^I(a) = \mathsf{ad}_k(a) = [k, a]$. The homomorphism $K \to \mathsf{Der}(I)$ that takes $k \mapsto \mathsf{ad}_k^I$, will be denoted by ad^I.

Example 3.1. Let L be a Lie algebra with an abelian ideal I and set $K = L/I$. Define the Lie algebra representation $\mathsf{ad}^I : K \to \mathsf{Der}(I)$ by $\mathsf{ad}_{x+I}^I(a) = [x, a]$ for all $x \in L$ and $a \in I$. This is well defined, since I is abelian. Then I is a K-module. In this case, we say that the action is *induced by the adjoint representation.*

Let K be a Lie algebra over a field $\mathbb{F}$ and let I be a vector space over $\mathbb{F}$. Denote by $\mathsf{C}^2(K, I)$ the vector space of alternating bilinear maps $\vartheta : K \times K \to I$. If I is a K-module and $\vartheta \in \mathsf{C}^2(K, I)$ has the property that

$$\vartheta(x,[y,z]) + \vartheta(y,[z,x]) + \vartheta(z,[x,y]) + [x,\vartheta(y,z)] + [y,\vartheta(z,x)] + [z,\vartheta(x,y)] = 0,$$

for all $x,\ y,\ z \in K$, then ϑ is said to be a *cocycle* and the vector space of coclycles is denoted by $\mathsf{Z}^2(K, I)$. Let $T : K \to I$ be a linear transformation and define, $\vartheta_T : K \times K \to I$ by

$$\vartheta_T(k, h) = T([k, h]) + [h, T(k)] - [k, T(h)] \quad \text{for all} \quad k,\ h \in K.$$

Then $\vartheta_T \in \mathsf{Z}^2(K, I)$ and such a cocycle ϑ_T is said to be a *coboundary.* The set of coboundaries is denoted by $\mathsf{B}^2(K, I)$. The set $\mathsf{B}^2(K, I)$ is a subspace of $\mathsf{Z}^2(K, I)$, and we set $\mathsf{H}^2(K, I) = \mathsf{Z}^2(K, I)/\mathsf{B}^2(K, I)$ to be the quotient space. The first cohomology group of K and I is defined as

$$\mathsf{Z}^1(K, I) = \{\nu \in \mathsf{Hom}(K, I) \mid \nu([k, h]) = [k, \nu(h)] - [h, \nu(k)] \text{ for all } k,\ h \in K\}.$$

The next result, whose proofs can be found, for instance, in [7, Section 4.2], links Lie algebra extensions to cohomology. Let K be a Lie algebra and let I be a K-module. Let $\vartheta \in \mathsf{Z}^2(K, I)$ and define the Lie algebra $K_\vartheta = K \dotplus I$ with the product

$$[x + a, y + b] = [x, y] + \vartheta(x, y) + [a, y] - [b, x] \text{ for all } x,\ y \in K \text{ and } a,\ b \in I. \tag{2}$$

Proposition 3.2. *The following hold for the Lie algebra K_ϑ:*

(1) *K_ϑ is a Lie algebra extension of K by I;*
(2) *if $\nu \in \mathsf{B}^2(K, I)$, then K_ϑ is isomorphic to $K_{\vartheta+\nu}$;*
(3) *if $\vartheta \in \mathsf{B}^2(K, I)$, then K_ϑ is a split extension of K by I.*

Conversely, let L be a Lie algebra and J be an abelian ideal of L. Then there exists $\vartheta \in \mathsf{Z}^2(L/J, J)$ such that $L \cong (L/J)_\vartheta$.

The cocycle ϑ in last statement of Proposition 3.2 can be constructed as follows. Let $\pi : L \to L/I$ denote the natural projection, and let $\sigma : L/I \to L$ be a right inverse of π; that is, $\pi\sigma = \mathrm{id}_{L/I}$. Then, for $k + I,\ h + I \in L/I$, set

$$\vartheta(k + I, h + I) = \sigma([k + I, h + I]) - [\sigma(k + I), \sigma(h + I)].$$

Routine calculation shows that $\vartheta \in \mathsf{Z}^2(L/I, I)$ and that $L \cong L_\vartheta$.

3.2. **Compatible pairs and derivations of semidirect sums.** Compatible pairs were introduced in [2] to compute automorphisms of solvable groups and solvable Lie algebras. We adopt the concept for derivations of Lie algebras. Let K and I be Lie algebras such that K acts on I via the homomorphism $\psi : K \to \mathsf{Der}(I)$. We define the *semidirect sum* $K \oplus_\psi I$ as the vector space $K \dotplus I$ with the product operation given as

$$[(k_1, a_1), (k_2, a_2)] = ([k_1, k_2], [k_1, a_2] - [k_2, a_1] + [a_1, a_2]).$$

When the K-action on I is clear from the context, then we usually suppress the homomorphism 'ψ' from the notation and write simply $K \oplus I$. If L is a Lie algebra such that L has an ideal I and a subalgebra K in such a way that $L = K \dotplus I$, then $L \cong K \oplus_\psi I$ where ψ is the restriction of ad_I to K. In a semidirect sum $K \oplus I$, an element $(k, a) \in K \dotplus I$ will usually be written as $k + a$.

Suppose that K and I are as in the previous paragraph. The direct sum $\mathsf{Der}(K)\oplus\mathsf{Der}(I)$ of the derivation Lie algebras is a Lie algebra. An element $(\alpha,\beta)\in\mathsf{Der}(K)\oplus\mathsf{Der}(I)$ is said to be a *compatible pair* if

$$\beta([k,a]) = [\alpha(k),a] + [k,\beta(a)] \quad \text{for all} \quad k\in K,\ a\in I. \tag{3}$$

We let $\mathsf{Comp}(K,I)$ denote the set of compatible pairs in $\mathsf{Der}(K)\oplus\mathsf{Der}(I)$. Using the homomorphism $\psi : K \to \mathsf{Der}(I)$ associated to the K-action on I, we can write equation (3) in another form as follows. Writing $[k,a]$ as $\psi(k)(a)$, we have that $(\alpha,\beta)\in\mathsf{Comp}(K,I)$ if and only if the equation

$$\beta\psi(k) = \psi(\alpha(k)) + \psi(k)\beta.$$

holds in $\mathsf{Der}(I)$ for all $k\in K$. Using commutator, this is equivalent to

$$[\beta,\psi(k)] = \psi(\alpha(k)) \quad \text{for all} \quad k\in K. \tag{4}$$

Letting $\mathsf{ad} : \mathsf{Der}(I) \to \mathsf{Der}(I)$ denote the adjoint representation, equation (4) can be rewritten as

$$\mathsf{ad}_\beta\psi(k) = \psi(\alpha(k)) \quad \text{for all} \quad k\in K.$$

Therefore, $(\alpha,\beta)\in\mathsf{Comp}(K,I)$ if and only if the following diagram commutes:

$$\begin{tikzcd} K \arrow[r, "\psi"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathsf{Der}(I) \arrow[d, "\mathsf{ad}_\beta"] \\ K \arrow[r, "\psi"] & \mathsf{Der}(I). \end{tikzcd}$$

A compatible pair $(\alpha,\beta)\in\mathsf{Der}(K)\oplus\mathsf{Der}(I)$ will usually be written as $\alpha+\beta$. If $\alpha+\beta\in\mathsf{Der}(K)\oplus\mathsf{Der}(I)$ as above, then $\alpha+\beta$ can be considered a element of $\mathfrak{gl}(I\oplus K)$ by letting $(\alpha+\beta)(a+k) = \alpha(a)+\beta(k)$ for all $a\in I$ and $k\in K$.

Proposition 3.3. *Using the notation above, we have that*

$$\mathsf{Comp}(K,I) = \{\alpha+\beta\in\mathfrak{gl}(K)\oplus\mathfrak{gl}(I) \mid \alpha+\beta\in\mathsf{Der}(K\oplus I)\}.$$

In particular $\mathsf{Comp}(K,I)$ *is a Lie subalgebra of* $\mathsf{Der}(K\oplus I)$.

Proof. Suppose that $\alpha+\beta\in\mathsf{Comp}(K,I)$ is a compatible pair and let $k+a,\ k'+a'\in K\oplus I$. Then

$$\begin{aligned}(\alpha+\beta)[k+a,k'+a'] &= (\alpha+\beta)([k,k'] + ([k,a'] - [k',a] + [a,a']))\\ &= \alpha([k,k']) + \beta([k,a'] - [k',a] + [a,a'])\\ &= [\alpha(k),k'] + [k,\alpha(k')] + [\alpha(k),a'] - [\alpha(k'),a]\\ &\quad + [\beta(a),a'] + [k,\beta(a')] - [k',\beta(a)] + [a,\beta(a')].\end{aligned}$$

On the other hand

$$
\begin{aligned}
[(\alpha+\beta)(k+a), k'+a'] + [k+a, (\alpha+\beta)(k'+a')] = \\
[\alpha(k), k'] + [\alpha(k), a'] + [\beta(a), k'] + [\beta(a), a'] \\
+ [k, \alpha(k')] + [k, \beta(a')] + [a, \alpha(k')] + [a, \beta(a')].
\end{aligned}
$$

Thus $\alpha + \beta \in \mathsf{Der}(K \oplus I)$.

Conversely, let $\alpha + \beta \in \mathfrak{gl}(K) \oplus \mathfrak{gl}(I)$ such that $\alpha + \beta$ is a derivation of $K \oplus I$. Then $(\alpha + \beta)|_K = \alpha$ and $(\alpha + \beta)|_I = \beta$, and so $\alpha \in \mathsf{Der}(K)$ and $\beta \in \mathsf{Der}(I)$. Further, if $k \in K$ and $a \in I$, then $[k, a] \in I$, and so

$$\beta([k, a]) = (\alpha + \beta)[k, a] = [(\alpha + \beta)(k), a] + [k, (\alpha + \beta)(a)] = [\alpha(k), a] + [k, \beta(a)].$$

Thus $\alpha + \beta \in \mathsf{Comp}(K, I)$, as required.

The fact that $\mathsf{Comp}(K, I)$ is a Lie subalgebra of $\mathsf{Der}(K \oplus I)$ follows from the fact that $\mathsf{Comp}(K, I)$ is the intersection of two Lie algebras; namely, $\mathsf{Comp}(K, I) = (\mathfrak{gl}(K) \oplus \mathfrak{gl}(I)) \cap \mathsf{Der}(K \oplus I)$. □

Let K and I be vector spaces. Consider the Lie algebra $\mathfrak{gl}(K) \oplus \mathfrak{gl}(I)$ and define an action of $\mathfrak{gl}(K) \oplus \mathfrak{gl}(I)$ on the vector space $\mathsf{Hom}(K, \mathfrak{gl}(I))$ as follows. Let ad denote the adjoint representation of $\mathfrak{gl}(I)$. Thus, for $\beta,\ \beta' \in \mathfrak{gl}(I)$ and $\mathsf{ad}_\beta(\beta') = [\beta, \beta']$. For $(\alpha, \beta) \in \mathfrak{gl}(K) \oplus \mathfrak{gl}(I)$ and for $T \in \mathsf{Hom}(K, \mathfrak{gl}(I))$, set

$$(\alpha, \beta) \cdot T = \mathsf{ad}_\beta T - T\alpha. \tag{5}$$

Let us show that this in fact defines a Lie algebra action. First notice that $(\alpha, \beta) \cdot T$ is a linear map because is linear combination of composition and sums of linear maps. Let us check that it preserves Lie brackets. Let $(\alpha, \beta),\ (\alpha', \beta') \in \mathfrak{gl}(K) \oplus \mathfrak{gl}(I)$ and $k \in K$. By definition

$$(\alpha', \beta') \cdot T = \mathsf{ad}_{\beta'} T - T\alpha'.$$

So

$$(\alpha, \beta) \cdot ((\alpha', \beta') \cdot T) = \mathsf{ad}_\beta \mathsf{ad}_{\beta'} T - \mathsf{ad}_{\beta'} T\alpha - \mathsf{ad}_\beta T\alpha' + T\alpha'\alpha.$$

In the same way,

$$(\alpha', \beta') \cdot ((\alpha, \beta) \cdot T) = \mathsf{ad}_{\beta'} \mathsf{ad}_\beta T - \mathsf{ad}_\beta T\alpha' - \mathsf{ad}_{\beta'} T\alpha + T\alpha\alpha'.$$

Hence,

$$
\begin{aligned}
(\alpha, \beta) \cdot ((\alpha', \beta') \cdot T) - (\alpha', \beta') \cdot ((\alpha, \beta) \cdot T) &= \mathsf{ad}_\beta \mathsf{ad}_{\beta'} T - \mathsf{ad}_{\beta'} \mathsf{ad}_\beta T + T\alpha\alpha' - T\alpha'\alpha \\
&= [\mathsf{ad}_\beta, \mathsf{ad}_{\beta'}] T + T[\alpha, \alpha'].
\end{aligned}
$$

Therefore,

$$[(\alpha, \beta), (\alpha', \beta')] \cdot T = ([\alpha, \alpha'], [\beta, \beta']) \cdot T.$$

Now, if K is a Lie algebra and I is a K-module, then there is a corresponding homomorphism $\psi \in \mathsf{Hom}(K, \mathsf{Der}(I))$. Now suppose that $\alpha + \beta \in \mathfrak{gl}(K) \oplus \mathfrak{gl}(I)$ such that $\alpha + \beta \in \mathsf{Der}(K) \oplus \mathsf{Der}(I)$. Then, for $k \in K$, we have $\mathsf{ad}_\beta T(k) + T\alpha(k)$ is a derivation of I since $\mathsf{ad}_\beta T(k),\ T\alpha(k) \in \mathsf{Der}(I)$.

If X is a subalgebra of $\mathsf{Der}(K) \oplus \mathsf{Der}(I)$, then the annihilator $\mathsf{Ann}_K(\psi)$ of ψ in X is defined as

$$\mathsf{Ann}_X(\psi) = \{(\alpha, \beta) \in X \mid (\alpha, \beta) \cdot \psi = 0\}.$$

Computing the annihilator of ψ in $\mathsf{Der}(K) \oplus \mathsf{Der}(I)$ explicitly, we obtain

$$\begin{aligned}\mathsf{Ann}_{\mathsf{Der}(K)\oplus\mathsf{Der}(I)}(\psi) &= \{(\alpha, \beta) \in \mathsf{Der}(K) \oplus \mathsf{Der}(I) \mid (\alpha, \beta) \cdot \psi = 0\} \\ &= \{(\alpha, \beta) \in \mathsf{Der}(K) \oplus \mathsf{Der}(I) \mid \mathsf{ad}_\beta\psi - \psi\alpha = 0\} = \mathsf{Comp}(K, I).\end{aligned}$$

The last equality follows from (4). Hence we have proved the following proposition.

Proposition 3.4. *Let K and I be Lie algebras such that I is also a K-module via the representation $\psi \in \mathsf{Hom}(K, \mathsf{Der}(I))$. Then $\mathsf{Comp}(K, I) = \mathsf{Ann}_{\mathsf{Der}(K)\oplus\mathsf{Der}(I)}(\psi)$, where the action of $\mathsf{Der}(K) \oplus \mathsf{Der}(I)$ on $\mathsf{Hom}(K, \mathsf{Der}(I))$ is given by (5).*

3.3. **Derivations of K_ϑ.** In this section we present a method to describe the derivations of extension K_ϑ presented in Proposition 3.2 from derivations of Lie algebra K. By an adaptation of the process used by Eick in [2], we set conditions for a derivation in K can be lifted to a derivation of K_ϑ. It is first necessary define an action of $\mathfrak{gl}(K) \oplus \mathfrak{gl}(I)$ on vector space of alternating bilinear maps.

Let K and I be vector spaces. Let (α, β) be an element of Lie algebra $\mathfrak{gl}(K) \oplus \mathfrak{gl}(I)$ and $\vartheta \in \mathsf{C}^2(K, I)$, define an action $\mathfrak{gl}(K) \oplus \mathfrak{gl}(I)$ on $\vartheta \in \mathsf{C}^2(K, I)$ by

$$(\alpha, \beta) \cdot \vartheta(h, k) = \beta(\vartheta(h, k)) - \vartheta(\alpha(k), h) - \vartheta(k, \alpha(h)), \quad \text{for all } h, k \in K. \tag{6}$$

If $(\alpha', \beta') \in \mathfrak{gl}(K) \oplus \mathfrak{gl}(I)$ then by our definition

$$\begin{aligned}(\alpha, \beta)(\alpha', \beta') \cdot \vartheta(h, k) = \beta\beta'\vartheta(h, k)) &- \beta'\vartheta(\alpha(k), h) - \beta'\vartheta(k, \alpha(h)) \\ &- \beta\vartheta(\alpha'(h), k)) + \vartheta(\alpha'\alpha(k), h) - \vartheta(\alpha'(k), \alpha(h)) \\ &\beta\vartheta(h, \alpha'(k)) - \vartheta(\alpha(k), \alpha'(h)) - \vartheta(k, \alpha'\alpha(h)).\end{aligned}$$

Follow that

$$[(\alpha, \beta), (\alpha', \beta')] \cdot \vartheta(h, k) = [\beta, \beta']\vartheta(h, k)) - \vartheta([\alpha', \alpha](k), h) - \vartheta(k, [\alpha', \alpha](h)).$$

Therefore, the action presented in (6) is well defined.

Our goal now is to study the action of compatible pairs $\mathsf{Comp}(K, I)$ on subspaces $\mathsf{Z}^2(K, I)$ and $\mathsf{B}^2(K, I)$ of $\mathsf{C}^2(K, I)$. For this, consider that K is a Lie algebra and I a K-module. Then for all $k, h, l \in K$, $(\alpha, \beta) \in \mathsf{Comp}(K, I)$ and $\vartheta \in Z^2(K, I)$ we have

$$\begin{aligned}(\alpha, \beta) \cdot \vartheta(k, [h, l]) &= \beta(\vartheta(k, [h, l])) - \vartheta(\alpha(k), [h, l]) - \vartheta(k, \alpha([h, l])) \\ &= \beta(\vartheta(k, [h, l])) - \vartheta(\alpha(k), [h, l]) - \vartheta(k, [\alpha(h), l]) - \vartheta(k, [h, \alpha(l)]).\end{aligned}$$

If

$$X = (\alpha, \beta) \cdot \vartheta(k, [h, l]) + (\alpha, \beta) \cdot \vartheta(h, [l, k]) + (\alpha, \beta) \cdot \vartheta(l, [k, h]),$$

then

$$\begin{aligned}X = \beta(\vartheta(k,[h,l])) + \beta(\vartheta(h,[l,k])) + \beta(\vartheta(l,[k,h]))\\ - \vartheta(\alpha(k),[h,l]) - \vartheta(\alpha(h),[l,k]) - \vartheta(\alpha(l),[k,h])\\ - \vartheta(k,[\alpha(h),l]) - \vartheta(h,[\alpha(l),k]) - \vartheta(l,[\alpha(k),h])\\ - \vartheta(k,[h,\alpha(l)]) - \vartheta(h,[l,\alpha(k)]) - \vartheta(l,[k,\alpha(h)]).\end{aligned}$$

Using cocycle definition

$$\begin{aligned}X = -\beta([k,\vartheta(h,l)]) - \beta([h,\vartheta(l,k)]) - \beta([l,\vartheta(k,h)])\\ + [\alpha(k),\vartheta(h,l)] + [\alpha(h),\vartheta(l,k)] + [\alpha(l),\vartheta(k,h)]\\ + [k,\vartheta(\alpha(h),l)] + [h,\vartheta(\alpha(l),k)] + [l,\vartheta(\alpha(k),h)]\\ + [k,\vartheta(h,\alpha(l))] + [h,\vartheta(l,\alpha(k))] + [l,\vartheta(k,\alpha(h))].\end{aligned}$$

(α,β) is a compatible pair then we can replace in X the equalities

$$\beta([k,\vartheta(h,l)]) = [\alpha(k),\vartheta(h,l)] + [k,\beta(\vartheta(h,l))];$$

$$\beta([h,\vartheta(l,k)]) = [\alpha(h),\vartheta(l,k)] + [h,\beta(\vartheta(l,k))];$$

$$\beta([l,\vartheta(k,h)]) = [\alpha(l),\vartheta(k,h)] + [l,\beta(\vartheta(k,h))];$$

Hence

$$\begin{aligned}X = -[k,\beta(\vartheta(h,l))] - [h,\beta(\vartheta(l,k))] - [l,\beta(\vartheta(k,h))]\\ + [k,\vartheta(\alpha(h),l)] + [h,\vartheta(\alpha(l),k)] + [l,\vartheta(\alpha(k),h)]\\ + [k,\vartheta(h,\alpha(l))] + [h,\vartheta(l,\alpha(k))] + [l,\vartheta(k,\alpha(h))].\end{aligned}$$

Again, by action definition we obtain

$$X = -[k,(\alpha,\beta)\cdot\vartheta(h,l)] - [h,(\alpha,\beta)\cdot\vartheta(l,k)] - [l,(\alpha,\beta)\cdot\vartheta(k,h)].$$

So $(\alpha,\beta)\cdot\vartheta \in Z^2(K,I)$.

Now suppose that $\vartheta \in \mathsf{B}^2(K,I)$. Then there is a linear map $T : K \to I$ such that

(7) $$\vartheta(k,h) = T([k,h]) + [h,T(k)] - [k,T(h)].$$

Let $Y = (\alpha,\beta)\cdot\vartheta(k,h)$. By (7) we have

$$Y = (\alpha,\beta)\cdot(T([k,h]) + [h,T(k)] - [k,T(h)]).$$

Using action definition we have

$$\begin{aligned} Y = \beta T([k,h]) + \beta([h,T(k)]) - \beta([k,T(h)]) \\ - T([\alpha(h),k]) - [\alpha(h),T(k)] + [\alpha(k),T(h)] \\ - T([k,\alpha(h)]) - [h,T\alpha(k)] + [k,T\alpha(h)]. \end{aligned}$$

We can use that (α,β) is a compatible pair in last equation

$$\begin{aligned} Y = \beta T([k,h]) + [\alpha(h),T(k)] + [h,\beta T(k)] - [\alpha(k),T(h)] - [k,\beta T(h)] \\ - T([\alpha(k),h]) - [\alpha(h),T(k)] + [\alpha(k),T(h)] \\ - T([k,\alpha(h)]) - [h,T\alpha(k)] + [k,T\alpha(h)] \\ = \beta T([k,h]) + [h,\beta T(k)] - [k,\beta T(h)] \\ - T([\alpha(k),h]) - T([k,\alpha(h)]) - [h,T\alpha(k)] + [k,T\alpha(h)] \end{aligned}$$

Hence,

$$Y = (\beta T - T\alpha)([k,h]) - [h,(\beta T - T\alpha)(k)] + [k,(\beta T - T\alpha)(h)].$$

If $U = \beta T - T\alpha : K \to I$ then

$$(\alpha,\beta)\cdot\vartheta(k,h) = U([k,h]) - [h,U(h)] - [k,U(h)].$$

Therefore, $(\alpha,\beta)\cdot\vartheta \in \mathsf{B}^2(K,I)$. We just proof

Proposition 3.5. *Let K be a Lie algebra and I a K-module. Consider the action of $\mathsf{Comp}(K,I)$ on $C^2(K,I)$ defined in (6). Then the vector spaces $Z^2(K,I)$ and $\mathsf{B}^2(K,I)$ are invariants by this action.*

This result allow us to define an action of $\mathsf{Comp}(K,I)$ on $H^2(K,I)$: let $\vartheta \in Z^2(K,I)$ and $(\alpha,\beta) \in \mathsf{Comp}(K,I)$. Define the action

$$(\alpha,\beta)\cdot(\vartheta + \mathsf{B}^2(K,I)) = ((\alpha,\beta)\cdot\vartheta) + \mathsf{B}^2(K,I). \tag{8}$$

This is well defined by Proposition 3.5.

Definition 3.6. Let K be a Lie algebra and I a K-module. Let $\vartheta \in Z^2(K,I)$ and consider the action of $\mathsf{Comp}(K,I)$ on $H^2(K,I)$ defined in (8). Define the set of induced pairs of $\mathsf{Comp}(K,I)$ by

$$Indu(K,I,\vartheta) = Ann_{\mathsf{Comp}(K,I)}(\vartheta + \mathsf{B}^2(K,I)).$$

Now we have the tools needed to describe the Lie algebra $\mathsf{Der}(K_\vartheta)$ from the Lie algebra $\mathsf{Der}(K)$. We will define a homomorphism $\phi : \mathsf{Der}(K_\vartheta) \to \mathsf{Der}(K)$, which kernel is known and the image coincides with the induced pairs defined above. So, using the first theorem of isomorphisms for Lie algebras we have $\mathsf{Der}(K_\vartheta)$ is isomorphic to $\mathsf{Ker}(\phi) \oplus \mathsf{Im}(\phi)$ but these subspaces correspond to structures: $\mathsf{Ker}(\phi) \cong \mathsf{Z}^1(\mathsf{K},\mathsf{I})$ and $\mathsf{Im}(\phi) \cong \mathsf{Indu}(\mathsf{K},\mathsf{I},\vartheta)$.

One application of this type of construction is use known information of algebra $\mathsf{Der}(K)$ to obtain information about algebra $\mathsf{Der}(K)_\vartheta$ as the existence of non-singular derivations. Therefore, this method will allow us to study some properties of Lie algebras extensions by cocycles. First we define ϕ.

Let K be a Lie algebra and I a K-module. Let $\vartheta \in \mathsf{H}^2(K, I)$ and $d \in \mathsf{Der}(K)_\vartheta$. Suppose that I, as ideal of K_ϑ, it is invariant by derivation d. Set $P_K : K_\vartheta \to K$ and $P_I : K_\vartheta \to I$ to be the natural projections of K_ϑ on K and K_ϑ on I then define the maps

- $\alpha : K \to K$ by $\alpha(k) = P_K d(k)$, for all $k \in K$;
- $\beta : I \to I$ by $\beta(a) = d(a)$, for all $a \in I$;
- $\varphi : K \to I$ by $\varphi(k) = P_I d(k)$, for all $k \in K$.

For each $x + a \in K_\vartheta$ we have

$$d(x + a) = \alpha(x) + \varphi(x) + \beta(a) \text{ for all } a \in I \text{ and } x \in K. \tag{9}$$

We can see that β is a derivation of I because it is restriction of d to I. To see that $\alpha \in \mathsf{Der}(K)$ let $x, y \in K$. Then by product definition on K_ϑ

$$d([x, y]_\vartheta) = d([x, y]_K + \vartheta(x, y)).$$

By decomposition showed in (9)

$$d([x, y]_\vartheta) = \alpha([x, y]_K) + \varphi([x, y]_K) + \beta(\vartheta(x, y)).$$

We can calculate

$$[d(x), y]_\vartheta + [x, d(y)]_\vartheta = [\alpha(x) + \varphi(x), y] + [x, \alpha(y) + \varphi(y)], \tag{10}$$

and use product definition of K_ϑ to get

$$\begin{aligned} [d(x), y]_\vartheta + [x, d(y)]_\vartheta = {} & [\alpha(x), y]_K + [x, \alpha(y)]_K + \vartheta(\alpha(x), y) \\ & + \vartheta(y, \alpha(x)) + [\varphi(x), \alpha(y)] - [\varphi(y), \alpha(x)]. \end{aligned} \tag{11}$$

Comparing the components of K in (10) and (11) we have

$$\alpha([x, y]_K) = [\alpha(x), y]_K + [x, \alpha(y)]_K,$$

and $\alpha \in \mathsf{Der}(K)$.

Now it's possible define our homomorphism ϕ. Let K be a Lie algebra and I a K-module. Let $\vartheta \in H^2(K, I)$ and suppose that I, as ideal of K_ϑ, it is invariant by derivations. For all $x + a \in K_\vartheta$ and $d \in \mathsf{Der}(K)_\vartheta$ write $d(x + a) = \alpha(x) + \beta(a) + \varphi(x)$ with $\alpha \in derK$ and $\beta \in \mathsf{Der}(I)$. Then define $\phi : Der(K_\vartheta) \to Der(K) \oplus Der(I)$ by

$$\phi(d) = (\alpha, \beta). \tag{12}$$

The following will check that ϕ is a Lie algebra morphism. Let $d, d' \in Der(K_\vartheta)$ and $x \in K a \in I$ such that

$$\begin{aligned} d(x + a) &= \alpha(x) + \varphi(x) + \beta(a) \\ d'(x + a) &= \alpha'(x) + \varphi'(x) + \beta'.(x), \end{aligned}$$

Then

$$\begin{aligned} dd'(x) &= d(\alpha'(x) + \varphi'(x)) \\ &= \alpha\alpha'(x) + \varphi(\alpha'(x)) + \beta'(\varphi'(x)). \end{aligned}$$

Hence, $P_K dd'(x) = \alpha\alpha'(x)$. Analogously, $P_K d'd(x) = \alpha'\alpha(x)$. So $P_K([d, d']) = [\alpha, \alpha']$. As β and β' are defined by restriction of d and d' to I, respectively, then $P_I([d, d']) = [\beta, \beta']$. Therefore,

$$\phi([d, d']) = ([\alpha, \alpha'], [\beta, \beta']) = [(\alpha, \beta), (\alpha', \beta')] = [\phi(d), \phi(d')].$$

■

The next result presents the first connection between compatible pairs and the homomorphism ϕ.

Theorem 3.7. *Let K be a Lie algebra and I a K-module. Let $\vartheta \in H^2(K, I)$ and suppose that I, as ideal of K_ϑ, it is invariant by derivations. Let $\phi : Der(K_\vartheta) \to Der(K) \oplus Der(I)$ given by $\phi(d) = (\alpha, \beta)$, defined in 12. Then $Im(\phi) \leqslant \mathsf{Comp}(K, I)$.*

Proof. Let $(\alpha, \beta) \in Im(\phi)$. Then there is $d \in Der(K_\vartheta)$ such that $\phi(d) = (\alpha, \beta)$. If $k \in K$ and $a \in I$ then

$$\begin{aligned} \beta([k, a]_\vartheta) &= d([k, a]_\vartheta) && [k, a] \in I \\ &= [d(k), a]_\vartheta + [k, d(a)]_\vartheta && d \in \mathsf{Der}(K_\vartheta) \\ &= [\alpha(k) + \varphi(k), a]_\vartheta + [k, \beta(a)]_\vartheta \\ &= [\alpha(k), a]_\vartheta + [k, \beta(a)]_\vartheta && \text{because } I \text{ is abelian} \end{aligned}$$

□

Theorem 3.8. *Let K be a Lie algebra and I a K-module. Let $\vartheta \in \mathsf{H}^2(K, I)$ and suppose that I, as ideal of K_ϑ, it is invariant by derivations. Let $\phi : \mathsf{Der}(K_\vartheta) \to \mathsf{Der}(K) \oplus \mathsf{Der}(I)$ given by $\phi(d) = (\alpha, \beta)$. Then:*

(1) $\mathsf{Im}(\phi) = \mathsf{Indu}(\mathsf{K}, \mathsf{I}, \vartheta)$
(2) $\mathsf{Ker}(\phi) \cong \mathsf{Z}^1(\mathsf{K}, \mathsf{I})$

Proof. 1) Let $(\alpha, \beta) \in \mathsf{Indu}(\mathsf{K}, \mathsf{I}, \vartheta)$. By definition

$$(\alpha, \beta) \cdot \vartheta = 0 \text{ mod } \mathsf{B}^2(K, I).$$

Then there is a linear map $T : K \to I$ such that for all $k, h \in K$ we have

$$\vartheta(\alpha(k), h) + \vartheta(k, \alpha(h)) + [k, T(h)] - [h, T(k)] = \beta(\vartheta(k, h)) + T([k, h]). \tag{13}$$

Let $k \in K$, $a \in I$ and define the linear map $(\alpha, \beta)^* : K_\vartheta \to K_\vartheta$ by

$$(\alpha, \beta)^*(k + a) = \alpha(k) + \beta(a) + T(k).$$

Let's check that $(\alpha,\beta)^*$ is a derivation of K_ϑ. Let $k+a, h+b \in K_\vartheta$. If

$$X = (\alpha,\beta)^*([k+a,h+b]_\vartheta)$$

then

$$\begin{aligned} X &= (\alpha,\beta)^*([k,h]_K + \vartheta(k,h) + [k,b] - [h,a]) \\ &= \alpha([k,h]_K) + \beta(\vartheta(k,h)) + +\beta([k,b]) - \beta([h,a]) + T([k,h]_K). \end{aligned}$$

Now, let

$$Y = [(\alpha+\beta)^*(k+a), h+b]_\vartheta + [k+a, (\alpha+\beta)^*(h+b)]_\vartheta.$$

We have

$$\begin{aligned} [(\alpha+\beta)^*(k+a), h+b]_\vartheta &= [\alpha(k) + \beta(a) + T(k), h+b]_\vartheta \\ &= [\alpha(k), h]_K + \vartheta(\alpha(k), h) + [\alpha(k), b] - [h, \beta(a) + T(k)] \end{aligned}$$

and

$$\begin{aligned} [k+a, (\alpha+\beta)^*(h+b)]_\vartheta &= [k+a, \alpha(h) + \beta(b) + T(h)] \\ &= [k, \alpha(h)]_K + \vartheta(k, \alpha(h)) + [k, \beta(b) + T(h)] - [\alpha(h), a] \end{aligned}$$

then

$$\begin{aligned} Y = \alpha([k,h]_K) &+ \vartheta(\alpha(k),h) + \vartheta(k,\alpha(h)) \\ &+ [\alpha(k),b] - [h,\beta(a)] - [h,T(k)] + [k,\beta(b)] + [k,T(h)] - [\alpha(h),a]. \end{aligned}$$

By compatible pair definition we get

$$Y = \alpha([k,h]_K) + \vartheta(\alpha(k),h) + \vartheta(k,\alpha(h)) + \beta([k,b]) - \beta([h,a]) - [h,T(k)] + [k,T(h)].$$

By equation (13)

$$Y = \alpha([k,h]_K) + \beta(\vartheta(h,k)) + T([k,h]) + \beta([k,b]) - \beta([h,a]).$$

As $X = Y$ then $(\alpha,\beta)^*$ is a derivation.

Besides, observe that $P_K(\alpha,\beta)^* = \alpha$ and $P_I(\alpha,\beta)^* = \beta$. Hence $\phi((\alpha+\beta)^*) = \alpha+\beta$, that is, $\mathsf{Indu}(\mathsf{K},\mathsf{I},\vartheta) \subseteq \mathsf{Im}(\phi)$.

Now, suppose that $(\alpha+\beta) \in \mathsf{Im}(\phi)$. Then there is $d \in \mathsf{Der}(K_\vartheta)$ such that

$$\phi(d) = (\alpha+\beta).$$

By Theorem 3.7 we have $\mathsf{Im}(\phi) \subseteq \mathsf{Comp}(K,I)$. Then it is enough show that there is a linear map $T : K \to I$ such that the equation (13) is satisfied.

For each $k+a \in K_\vartheta$ we can use the decomposition defined in (9) to write

$$d(k+a) = \alpha(k) + \varphi(k) + \beta(a).$$

By product definition in K_ϑ we get

$$\begin{aligned} [d(k+a), h+b]_\vartheta &= [\alpha(k) + \varphi(k) + \beta(a), h+b]_\vartheta \\ &= [\alpha(k), h]_K + \vartheta(\alpha(k), h) + \beta(a)] + [\alpha(k), b] - [h, \varphi(k) \end{aligned}$$

$$\begin{array}{rcl}[k+a,d(h+b)]_\vartheta & = & [k+a,\alpha(h)+\varphi(h)+\beta(b)]_\vartheta \\ & = & [k,\alpha(h)]_K+\vartheta(k,\alpha(h))+[k,\varphi(h)+\beta(b)]-[\alpha(h),a\end{array}$$

$$\begin{array}{rcl}d([k+a,h+b]_\vartheta) & = & d([k,h]_K+\vartheta(k,h)+[k,b]-[h,a]) \\ & = & \alpha([k,h]_K)+\beta(\vartheta(k,h))+\beta([k,b])-\beta([h,a])+\varphi_d([k,h])\end{array}$$

As d is a derivation then we have equality

$$d[k+a,h+b]=[d(k)+a,h+b]+[k+a,d(h)+b].$$

So,

$$\beta(\vartheta(k,h))+\varphi([k,h])=\vartheta(\alpha(k),h)+[k,\varphi(h)]-[h,\varphi(k)]+\vartheta(k,\alpha(h)).$$

Therefore $T=\varphi$ satisfies the equation (13) e $\mathsf{Im}(\phi)\subseteq\mathsf{Indu}(\mathsf{K},\mathsf{I},\vartheta)$.

2) Let $d\in\mathsf{Ker}(\phi)$. The decomposition showed in (9) provide us

$$d(k)=\varphi(k),k\in K.$$

Let $k,h\in K$. By derivation definition

$$d([k,h]_\vartheta)=[d(k),h]_\vartheta+[k,d(h)]_\vartheta. \tag{14}$$

We can use product definition in K_ϑ to write

$$d([k,h]_\vartheta)=d([k,h]_K+\vartheta(k,h)=\varphi([k,h]_K).$$

By other hand,

$$[d(k),h]_\vartheta+[k,d(h)]_\vartheta=[k,\varphi(h)]_\vartheta-[h,\varphi(k)]_\vartheta=[k,\varphi(h)]-[h,\varphi(k)].$$

Then (14) it is equal to

$$\varphi([k,h]_K)=[k,\varphi(k)]-[h,\varphi(k)],$$

and $\varphi\in\mathsf{Z}^1(\mathsf{K},\mathsf{I})$. Now define $\sigma:\mathsf{Ker}(\phi)\to\mathsf{Z}^1(\mathsf{K},\mathsf{I}),+)$ by $\sigma(d)=\varphi_d$ such that $\varphi_d(k)=d(k)$. Then $\sigma(\mathsf{Ker}(\phi))\subseteq\mathsf{Z}^1(\mathsf{K},\mathsf{I})$.

Let $d,d'\in\mathsf{Ker}(\phi)$. Then

$$\sigma(d+d')(k)=\varphi_{d+d'}(k)=(d+d')(k)=d(k)+d'(k)=\varphi(k)+\varphi'(k)=(\sigma(d)+\sigma(d'))(k).$$

So σ it is group homomorphism.

If $d,d'\in\mathsf{Ker}(\phi)$ such that $\sigma(d)=\sigma(d')$ then $\varphi_d(k)=\varphi_{d'}(k)$, for all $k\in K$ and $d=d'$. Let $T\in\mathsf{Z}^1(\mathsf{K},\mathsf{I})$ and define $d:K_\vartheta\to K_\vartheta$ by

$$d(x+a)=T(x),x\in K,a\in I.$$

d is a derivation because

$$d([k+a,h+b]_\vartheta)=d([k,b]_K+\vartheta(k,h)+[k,b]-[h,a])=T([k,h]_K)$$

and

$$[d(k+a), h+b]_\vartheta + [k+a, d(h+b)]_\vartheta = [T(k), h+b]_\vartheta + [k+a, T(h)]_\vartheta$$
$$= [k, T(h)] - [h, T(k)].$$

It follows that $\sigma(d) = T$. Therefore, σ is isomorphism □

3.4. **Compatible pairs and Jacobson Theorem.** In this section we show some examples of the use of compatible pairs.

Example 3.9. Let K and I be finite dimensional Lie algebras over an algebraically closed field $\mathbb{F}$. Suppose that K act on I by representation $\psi : K \to Der(I)$. Let $D \subseteq \mathsf{Comp}(K, I)$ be a subalgebra. By Proposition 3.3, $D \subseteq \mathsf{Der}(L)$. If D is nilpotent then L has a decomposition in generalized eigenspaces of D. This decomposition induces decompositions in K and I, because as subspaces of L they are invariants by D. Hence,

$$L = K_{\lambda_1} \oplus \cdots \oplus K_{\lambda_r} \oplus I_{\mu_1} \cdots \oplus I_{\mu_s}.$$

In particular, we have $[K_{\lambda_i}, I_{\mu_j}] \subseteq I_{\lambda_i+\mu_j}$ if $\lambda_i + \mu_j$ is eigenvalue of D in I. Otherwise $[K_{\lambda_i}, I_{\mu_j}] = 0$.

From this example we can state a result:

Proposition 3.10. *Let K and I be finite dimensional Lie algebras over an algebraically closed field $\mathbb{F}$. Suppose that K act on I by representation $\psi : K \to Der(I)$. Let $D \subseteq$ $\mathsf{Comp}(K, I)$ be a subalgebra. Suppose that 0 is not generalized eigenvalue of D. Then if either characteristic of $\mathbb{F}$ is zero or either characteristic of $\mathbb{F}$ is p and D has at most $p-1$ generalized eigenvalues the $\psi(K)$ is nilpotent.*

Proof. Let $L = K_{\lambda_1} \oplus \cdots \oplus K_{\lambda_r} \oplus I_{\mu_1} \cdots \oplus I_{\mu_s}$ the eigenspace decomposition present in Example 3.9. Suppose that 0 is not generalized eigenvalue of D. Let $E_K = \{\lambda_1, \cdots, \lambda_r\}$ and $E_I = \{\mu_1, \cdots, \mu_s\}$ be generalized eigenvalue of D in K and I, respectively. Let $k \in K_{\alpha_j}, a \in I_{\mu_i}$ then

$$\begin{cases} \psi^n(k)(a) \in I_{\mu_i+n\lambda_j} & if \quad \mu_i + n\lambda_j \in E_I \\ \psi^n(k)(a) = 0 & if \quad \mu_i + n\lambda_j \notin E_I \end{cases}$$

- If characteristic of $\mathbb{F}$ is zero then the linear functions $\mu_i+\lambda_j, \mu_i+2\lambda_j, \cdots, \mu_i+n\lambda_j \cdots$ are all distinct because $\lambda_j \neq 0$, so $\mu_i + n\lambda_j \notin E_I$ for some n and $\psi(k)^n = 0$.
- If $char(\mathbb{F}) = p$ and $s < p$ the set $\{\mu_i + \lambda_j, \mu_i + 2\lambda_j, \cdots, \mu_i + (p-1)\lambda_j, \mu_i\}$ has p distinct elements and E_I has at most $p-1$, then $\psi^n(k) = 0$ for some n with $1 \leqslant n \leqslant p$.

In both cases $\psi(k)$ is nilpotent for all $k \in K_{\lambda_j}$, $1 \leqslant j \leqslant r$. Let $S = \bigcup \psi(K_{\lambda_j})$. S is a weakly closed set such that each element is associative nilpotent then $\psi(K)$ is nilpotent. □

For our next example we need some result about traces of matrices.

Proposition 3.11. *Let $\mathbb{F}$ be a field of characteristic p. Suppose that $A \in M(n, \mathbb{F})$ with $n<p$ or $p=0$. Then A is nilpotent if, and only if, the trace of matrices A^r is zero, for $1 \leqslant r \leqslant n$.*

Proof. Let $\bar{\mathbb{F}}$ the algebraic closure of F e consider A in its Jordan normal form. This can be done because Jordan normal form is obtained from A by conjugation of matrices over $\bar{\mathbb{F}}$. But since trace and nilpotency of matrices are invariants by conjugation our results still valid for A. We will use that a matrix is nilpotent if, and only if, zero is its only eigenvalue.

A can be seen as a diagonal block matrix where each block is formed by grouping the blocks associated to same eigenvalue. Denote by A_j the block associated to eigenvalue $\lambda_t \in \overline{\mathbb{F}}$ and by n_j its order. Let $\lambda_1, \cdots, \lambda_k$ be the non-zero eigenvalues of A. Then

$$tr(A^r) = n_1\lambda_1^n + \cdots + n_k\lambda_k^n \tag{15}$$

Suppose that A is nilpotent. Then zero is the only eigenvalue of A and by equation (15) we have $tr(A^r) = 0$ for $1 \leqslant r \leqslant n$.

Conversely, suppose that $tr(A^r) = 0$ for $1 \leqslant r \leqslant n$. From equation (15) we can extract the system

$$n_1\lambda_1^r + \cdots + n_k\lambda_k^r = 0, \qquad 1 \leqslant r \leqslant k, \tag{16}$$

in the variables $n_1, \cdots, n_k$, whose matrix of coefficients is

$$C = \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_k^k \end{bmatrix}.$$

Denote by $m_i(\lambda)$ the operation that multiplies the line i of a matrix by λ and A^t the transposed matrix of A. So we can write

$$C = m_1(\lambda_1).m_2(\lambda_2)\cdots m_k(\lambda_k).V,$$

where

$$V = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 \cdots & \lambda_1^{k-1} \\ 1 & \lambda_2 & \lambda_2^2 \cdots & \lambda_2^{k-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \lambda_k & \lambda_k^2 \cdots & \lambda_k^{k-1} \end{bmatrix}$$

is the Vandermonde matrix in the variables $\lambda_1, \lambda_2, \cdots, \lambda_k$ whose determinant is $\det V = \prod_{1 \leqslant i<j \leqslant n}(\lambda_j - \lambda_i)$. As λ_i are distinct we have that $\det V$ is non-zero. Then the determinant of C is $\lambda_1.\lambda_2 \cdots \lambda_k. \det V$ and C is non-singular. Follow that the system (16) has only trivial solution. Therefore each n_j is zero. If $p = 0$ then zero is the only eigenvector of A, but if

$p \neq 0$ then $n_j = 0$ modulo p doesn't imply $n_j = 0$ and its necessary to use that each $n_j < p$ to conclude that zero is the only eigenvalue of A. □

Proposition 3.12. *Let $\mathbb{F}$ be a field of characteristic p. Let $A, B, C \in M(n, \mathbb{F})$ with $p = 0$ or $n < p$. If $[A, B] = C + \lambda B$, $\lambda \in \mathbb{F}$ and $[B, C] = 0$ then $[A, B^r] = rB^{r-1}C + \lambda r B^r$ for all $r \geqslant 1$. In particular, if $\lambda \neq 0$ and C is nilpotent then B is nilpotent.*

Proof. We proof this result by induction on r. The case $r = 1$ follow from hypotheses. Suppose that resultis valid for $(r-1)$. Then, $[A, B^{r-1}] = (r-1)B^{r-2}C + \lambda(r-1)B^{r-1}$. We can rewrite this equation as

$$\lambda(r-1)B^{r-1} = AB^{r-1} - B^{r-1}A - (r-1)B^{r-2}C.$$

Multiplying last equation to right by B we have

$$\lambda(r-1)B^r = AB^r - B^{r-1}(AB) - (r-1)B^{r-2}(CB),$$

From hypotheses we can write $AB = BA + C + \lambda B$ and $CB = BC$. Replacing them above we obtain

$$\lambda(r-1)B^r = AB^r - B^r A - B^{r-1}C - \lambda B^r - (r-1)B^{r-1}C.$$

Therefore,

$$AB^r - B^r A = \lambda r B^r + rB^{r-1}C.$$

For the second result suppose $\lambda \neq 0$ and C nilpotent with nilpotency index m. Using first part we have

$$B^r = (1/\lambda r)[A, B^r] - (1/\lambda)B^{r-1}C, \text{ for all } r \geqslant 1.$$

Observe that $(B^{r-1}C)^m = (B^{r-1})^m(C)^m = 0$, Hence,for all $r \geqslant 1$ $B^{r-1}C$ is nilpotent and has trace zero by Proposition 3.11. As trace of commutators are always zero then $tr([A, B^r]) = 0$ for all $r \geqslant 1$. Follows that $tr(B^r) = 0$ for all $r \geqslant 1$ and again by Proposition 3.11 we conclude that B is nilpotent. □

Proposition 3.13. *Let L be a Lie algebra, I an ideal of L such that L/I is nilpotent and such that $\mathsf{ad}_x^I : I \to I$ is nilpotent for all $x \in L$. Then L is nilpotent.*

Proof. As L/I is nilpotent then for each $x \in L$, $(\mathsf{ad}_{x+I}^I)^n$ is a nilpotent endomorphism in $\mathsf{End}(L/I)$, i.e., there is $n > 0$ such that $(\mathsf{ad}_x)^n(a) \in I$, for all $x \in L, a \in I$. In the other hand, ad_x^I is nilpotent, so we have a m such that $(\mathsf{ad}_x^I)^m(\mathsf{ad}_x)^n = 0$, i.e., $(\mathsf{ad}_x^I)^{m+n} = 0$. So ad_x is a nilpotent endomorphism in $\mathfrak{gl}(L)$. By Engel's theorem, L is nilpotent. □

Now we can present a similar result the proposition 3.10 but with a new proof using compatible pairs.

Theorem 3.14. *Let K and I be finite dimensional Lie algebras over a field of characteristic p such that K is nilpotent. Suppose that K act on I by representation $\psi : K \to \mathsf{Der}(I)$. Let $(\alpha, \beta) \in \mathsf{Comp}(K, I)$ such that α has no eigenvalue 0. If either $p = 0$ or $p > 0$ and dimension of I is less than p then $Tr(\psi^n(k)) = 0$, for all $k \in K$. In these two cases, $\psi(k)$ is nilpotent.*

Proof. As α has no eigenvalue 0 then it is non-singular and by Proposition 2.7 α is diagonalizable. Let $x_1, ..., x_s$ be a basis of K such that $\alpha(x_i) = \lambda_i x_i$. For all $a \in \mathfrak{gl}(I)$ denote by $[a]$ the matrix of a in this base. Then

$$[[\beta], [\psi(x_i)]] = \lambda_i[\psi(x_i)].$$

We can apply Proposition 3.12 in this last equation for $A = \beta$, $B = \psi(x_i)$, $C = 0$ and $\lambda = \lambda_i \neq 0$ to conclude that $\psi(x_i)$ is nilpotent for $1 \leqslant i \leqslant s$. Now we observe that if K is a nilpotent Lie algebra in either characteristic is 0 or characteristic p with dimension of L less than p then Lie theorem is valid. Lie theorem grants that there is a basis of I such that all matrices of representation ψ is upper triangular. Therefore, the matrices $[\psi(x_i)]$ are strictly upper triangular. Then all $\psi(k)$, for all $k \in K$, has only 0 in diagonal, because they are linear combination of $\psi(x_i)$. Hence every $\psi(k)$ is nilpotent. □

Corollary 3.15. *Let L be a solvable Lie algebra over a field $\mathbb{F}$ of characteristic p. Suppose that L has a nonsingular derivation. If either $p = 0$ or $p > 0$ and dimension of $L^{(i)}/L^{(i+1)} < p$ then L is nilpotent.*

Proof. Suppose that $L \geqslant L^{(1)} \geqslant \cdots \geqslant L^{(k)} \geqslant L^{(k+1)} = 0$ is the derived series of L. Define $L_0 = L$ and $L_i = L_{i-1}/L_{i-1}^{(k+1-i)}, 1 \leqslant i \leqslant k-1$. As each term of derived series are invariant by derivations then each L_i has a non-singular derivation. In particular, L_{k-1} is an solvable Lie algebra of derived length 2 with non-singular derivation. Then by theorem 3.14 ad_k is nilpotent for all $k \in L_{k-1}$ and by Proposition 3.13 L_{k-1} is nilpotent. By induction we have that L_i is nilpotent for every $0 \leqslant i \leqslant k-1$. Hence L is nilpotent □

4. Solvable non-nilpotent modular Lie algebras with non-singular derivations

In this section we will describe the structure of some examples of solvable non-nilpotent modular Lie algebras L with a non-singular derivation d. Based in this examples we will state some questions about this algebras. These issues will serve as a reference for further work.

Let V be a vector space and $x \in \mathfrak{gl}(V)$. Denote the action of x on V by $\mathsf{ad}_x(a) = [x, a]$, for all $a \in V$. A vector $a \in V$ is said to be x-cyclic if

$$V = \langle a, \mathsf{ad}_x(a), (\mathsf{ad}_x)^2(a), \cdots, (\mathsf{ad}_x)^{n-1}(a)\rangle.$$

The vector space V is called x-cyclic if it has a x-cyclic vector. We can present some examples of Solvable non-nilpotent modular Lie algebras with non-singular derivations using cyclic spaces.

Example 4.1. Let V be a vector space over a field $\mathbb{F}$ of characteristic $p > 0$ and $x \in \mathfrak{gl}(V)$.

(1) Let K be the vector space generated by x over $\mathbb{F}$ and I a subspace of V of dimension p. Let $\{a_1, a_2, \cdots, a_p\}$ a base of I such that $[x, a_i] = a_{i+1 mod p}$. As seen in Example 2.10, $L = K \oplus I$ has a non-singular derivation;

(2) Let K be the vector space generated by x over $\mathbb{F}$ and $I_1, I_2, \cdots, I_m$ be x-cyclic subspaces of V of dimension p such that $I_i \cap I_j = 0$ for all $i \neq j$. Let L be the Lie algebra $L = K \oplus (I_1 \oplus \cdots \oplus I_m)$. Let $\lambda, \delta_1, \cdots, \delta_m$ be non-zero elements of $\mathbb{F}$ such that $\lambda \neq s\delta_i$, for all $s \in \mathbb{F}_p$; and $\{a_{i1}, \cdots a_{ip}\}$ a base of I_i such that $[x, a_{ij}] = a_{i(j+1 mod p)}$. Define $d \in \mathfrak{gl}(L)$ by $d(x) = \lambda x$ and $d(a_{ij}) = (\delta_i + (j-1)\lambda)a_{ij}$. Then d is a non-singular derivation of L.

(3) Let K be the vector space generated by $\{x, x^2, \cdots, x^{p-1}\}$ over $\mathbb{F}$ and I a x-cyclic subspace of dimension p. Let $L = K \oplus I$. Let $\lambda, \delta \in \mathbb{F}$ both non-zero and $\lambda \neq s\delta$, for all $s \in \mathbb{F}_p$. Define the endomorphism $d \in \mathsf{End}(L)$ by $d(x^j) = j\lambda x^j$, $1 \leqslant j \leqslant p-1$ and $d(a_i) = (\delta + (i-1)\lambda)a_i$, $1 \leqslant i \leqslant rp$, d is a non-singular derivation of L_1.

All Lie algebras in Example 4.1 are solvable, non-nilpotent, with non-singular derivation and they have derived length 2. For Lie algebras over fields of characteristic $p > 3$ we could not find an example of derived length greater than 3 but in characteristic 2 we have the following example.

Example 4.2. Let L be a vector space of dimension 6 over $\mathbb{F}_4$. Let $\lambda \in \mathbb{F}_4$ such that $\lambda^2 = \lambda + 1$ and $\{a_1, a_2, a_3, a_4, a_5, a_6\}$ a basis of L over $\mathbb{F}_4$. Define the products

$$\begin{array}{llll} [a_1, a_3] = \lambda a_5 + a_6, & [a_1, a_4] = \lambda a_6, & [a_1, a_5] = \lambda^2 a_3 + a_4, & [a_3, a_5] = \lambda a_2, \\ [a_1, a_6] = \lambda^2 a_4, & [a_2, a_3] = \lambda a_6, & \text{and } [a_2, a_5] = \lambda^2 a_4. & \end{array}$$

L is a solvable non-nilpotent Lie algebra of derived length 3. The linear map $d : L \to L$ defined by $d(a_1) = a_1$, $d(a_2) = a_2$, $d(a_3) = \lambda a_3$, $d(a_4) = \lambda a_4$, $d(a_5) = \lambda^2 a_5$ and $d(a_6) = \lambda^2 a_6$ is a non-singular derivation of L.

With this examples we can state our first question:

Problem 1. Is there a solvable, non-nilpotent Lie algebra over a field of characteristic $p > 3$ with non-singular derivation and derived length greater than 2?

Suppose that the answer to Problem 1 is yes and let L be such Lie algebra. Let $I = L^{(2)}$ and $K = L/I$. As $L^{(3)} = 0$ then I is abelian and so K acts on I by adjoint representation. In this case, K is a solvable Lie algebra of derived length 2 with non-singular derivation. By Proposition 3.2, there is a cocycle $\vartheta \in \mathsf{Z}^2(K, I)$ such that $L \cong K_\vartheta$. This calculation show us that every Lie algebra that answer Problem 1 can be obtained by an extension of a solvable Lie algebra of derived length 2 with non-singular derivation. So we need to understand this Lie algebras of derived length 2 to search for an answer of Problem 1. We will study a variation of this question.

Problem 2. Let K be solvable, non-nilpotent Lie algebra over a field of characteristic $p > 3$ with non-singular derivation and derived length 2. Is there a K-module I and a cocycle $\vartheta \in \mathsf{Z}^2(K, I)$ such that K_ϑ has a non-singular derivation?

As first step to study Problem 2 we will try to describe some cases of abelian Lie algebras K acting over vector spaces. Next we present some questions that we are working.

Let $x \in \mathfrak{gl}(V)$ and I a subspace of V such that $L = \langle x \rangle \oplus I$ has a non-singular derivation.

- In Example 4.1(2) I can be decomposed in subspaces I_j such that $\langle x\rangle \oplus I_j$ has non-singular derivation. In general, can we define a smaller subspace of J of I such that $\langle x\rangle \oplus J$ has non-singular derivation?
- How the existence of non-singular derivations affect the structure of $\mathsf{Der}(L)$? Can we define some algebra structure over non-singular derivations of L?

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