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Finite-Dimensional Simple Lie Algebras with a Nonsingular Derivation

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We consider the known finite-dimensional simple Lie algebras of characteristic $p > 3$ and determine all finite-dimensional simple Lie algebras over an algebraically closed field of characteristic $p > 7$ admitting a nonsingular derivation. We also show that the $\mathbb{Z}/(p^n - 1)\mathbb{Z}$-graded simple Lie algebras which admit a nonsingular derivation must be of Hamiltonian type.

INTRODUCTION

In recent studies on pro-p groups of finite coclass, Shalev and Zelmanov [SZ, S] encountered finite-dimensional Lie algebras L over a field F of characteristic $p > 0$ which admit a derivation $D \in \operatorname{Der} L$ such that $Dx \neq 0$ for every nonzero $x \in L$. Viewed as a linear transformation $D: L \to L$ on L, the derivation D is nonsingular. In the papers of Shalev and Zelmanov the derivations of finite order ($D^n = I$ for some positive integer n) play a

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special role. Such a derivation is analogous to a fixed-point-free automorphism of finite order. It is well known (see [HB]) that a Lie algebra L admitting a fixed-point-free automorphism of order n necessarily is solvable and has derived length $r = r(n)$ which depends only on n. The situation is quite different for nonsingular derivations of order n. Indeed, the existence of such a derivation does not preclude an algebra from being simple.

EXAMPLE. Let $\operatorname{char} F = p = 2$ and let $L = W(1; m)$ be the Zassenhaus simple Lie algebra of dimension $2^m - 1$,

$$L = \langle e_a | \alpha \in GF(2^m), \alpha \neq 0 \rangle,$$

with a basis labelled by the nonzero elements of the finite field $GF(2^m)$ and with multiplication given by $[e_\alpha, e_\beta] = (\beta - \alpha)e_{\alpha+\beta}$. Then the map D: $e_\alpha \to \alpha e_\alpha$ is a nonsingular derivation of order $2^m - 1$.

N. Jacobson [J1] was the first to investigate Lie algebras L with a nonsingular derivation D in the case when $\operatorname{char} F = 0$ or when L is a restricted Lie algebra of characteristic $p > 0$. The Lie algebra L decomposes into generalized eigenspaces $L = \oplus\sum_\alpha L_\alpha$ relative to D. Since $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$ and 0 is not an eigenvalue, adx is nilpotent for all $x \in L_\alpha$ when $\operatorname{char} F = 0$. Jacobson's result on weakly closed sets of nilpotent transformations [J2, Chap. 2] and Engel's theorem imply that L is nilpotent. When $\operatorname{char} F = p > 0$ and L is restricted, the semisimple part D_s of D is also nonsingular, and L decomposes into eigenspaces $L = \oplus\sum_\alpha L_\alpha$ relative to D_s. Since L is restricted, for $x_\alpha \in L_\alpha$ we have $x_\alpha^{[p]} = \sum_\beta y_\beta$, where $y_\beta \in L_\beta$. But then the calculation, $\sum_\beta [y_\beta, z_\gamma] = ad(x_\alpha^{[p]})(z_\gamma) = (adx_\alpha)^p(z_\gamma) \in L_\gamma$, shows that $(adx_\alpha)^p = 0$ for all $x_\alpha \in L_\alpha$ and all α. Once again, we can conclude that L is nilpotent.

In this paper we consider the known finite-dimensional simple Lie algebras of characteristic $p > 3$ and determine those which admit a nonsingular derivation. The classical Lie algebras (the analogues of the complex simple Lie algebras) are restricted, and so by the preceding paragraph, they do not admit a nonsingular derivation. Thus, we focus our attention on the nonclassical simple Lie algebras (necessarily on the nonrestricted ones), which are either of Cartan type or are Melikyan algebras over fields of characteristic 5. The main result of this paper is a proof of the following theorem:

THEOREM. *Assume that L is a finite-dimensional simple Lie algebra of Cartan type or a Melikyan algebra over an algebraically closed field of* *characteristic $p > 3$. If L admits a nonsingular derivation, then L is isomorphic to:*

(1) *a special Lie algebra $S(m; \mathbf{n}, \omega)$, with $m > 2$ and*

$$\omega = \left(1 + x_1^{(p^{n_1}-1)} \dots x_m^{(p^{n_m}-1)}\right) dx_1 \wedge \cdots \wedge dx_m, \qquad or$$

(2) *a Hamiltonian Lie algebra $H(m; \mathbf{n}, \omega)$, with $m = 2s$ and*

$$\omega = \sum_{i=1}^{s} dx_i \wedge dx_{i+s} + \sum_{i,j=1}^{m} \alpha_{i,j} x_i^{(p^{n_i}-1)} x_j^{(p^{n_j}-1)} dx_i \wedge dx_j,$$

$\alpha_{i,j} \in F$, $\det(\alpha_{i,j}) \neq 0$.

According to the recent classification [SW] of the modular simple Lie algebras of characteristic $p > 7$, any nonclassical simple Lie algebra is of Cartan type. Consequently, we have determined all simple Lie algebras of characteristic $p > 7$ which admit a nonsingular derivation. Moreover, the detailed analysis used in proving this result allows us to distinguish between the special and Hamiltonian Lie algebras by the behavior of their nonsingular derivations.

COROLLARY. *Suppose that L is a finite-dimensional simple Lie algebra over an algebraically closed field of characteristic $p > 7$ with a nonsingular derivation D such that $\dim L_\alpha > 1$ for some eigenvalue α of D. Then $L \cong S(m; \mathbf{n}, \omega)$, where ω is as in (1) of the theorem.*

As a consequence of these results, we prove in Section 4 that any $\mathbb{Z}/(p^n - 1)\mathbb{Z}$-graded simple Lie algebra $L = \sum_i L_i$, $\dim L_i = 1$, with a nonsingular derivation D such that $D(L_i) \subseteq L_{i+1}$ necessarily is a Hamiltonian algebra $H(m; \mathbf{n}, \omega)$ as in (2) of the theorem. It was communicated to the authors by A. Shalev and E. Zelmanov that such algebras are of particular interest in the theory of pro-p groups of finite coclass.

To make the exposition more self-contained, we have included a short discussion of the simple Lie algebras of Cartan type and the Melikyan algebras in Section 1. The connection between nonsingular derivations and tori of maximal dimension in $\operatorname{Der} L$ is established in Section 2. Section 3 is devoted to a proof of the main theorem. The special and Hamiltonian Lie algebras of Cartan type are studied in detail in the final section.

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1. Simple Lie Algebras of Characteristic $p > 3$

Here we give a brief description of the known simple Lie algebras over an algebraically closed field F of characteristic $p > 3$ and their derivation algebras. Further details can be found in [Ka, KS, W1, SF, Ku4]). Our notation is chosen largely to conform to [SF].

1.1. *Nonclassical Simple Lie Algebras of Characteristic $p > 3$*

Let $\mathbb{N}^m$ denote the set of all m-tuples of nonnegative integers, and let $A(m)$ be the divided power algebra in m indeterminates over an algebraically closed field F: $A(m) = \langle x^{(a)} | a \in \mathbb{N}^m \rangle$ with

$$x^{(a)}x^{(b)} = \binom{a+b}{a} x^{(a+b)},$$

where for $a = (a_1, \ldots, a_m)$ and $b = (b_1, \ldots, b_m)$, $\binom{a+b}{a} = \prod_{i=1}^m \binom{a_i + b_i}{a_i}$. When $\varepsilon_i = (\delta_{i,1}, \ldots, \delta_{i,m})$, where $\delta_{i,j}$ is the Kronecker delta, we denote $x^{(\varepsilon_i)}$ by x_i. For each m-tuple $\mathbf{n} = (n_1, \ldots, n_m)$ of positive integers in $\mathbb{N}^m$, let

$$A(m; \mathbf{n}) = \left\langle x^{(a)} | a = (a_1, \ldots, a_m) \in \mathbb{N}^m,\ a_i < p^{n_i} \right\rangle.$$

Then $A(m; \mathbf{n})$ is a subalgebra of $A(m)$ of dimension p^n, where $n = \|\mathbf{n}\| = n_1 + \cdots + n_m$. The grading of $A(m)$ and $A(m; \mathbf{n})$ gotten by assigning $\deg x_i = 1$ for each i is called the *standard grading*, and it gives a corresponding filtration $A(m)_{(0)} \supset A(m)_{(1)} \supset \ldots$ and $A(m; \mathbf{n})_{(0)} \supset A(m; \mathbf{n})_{(1)} \supset \ldots$ of the algebras. The algebra $A(m)$ possesses certain mappings $y \to y^{(j)}$ for $j = 0, 1, 2, \ldots$ of $A(m)_{(1)}$ into $A(m)$ termed divided power maps with the properties

$$\begin{aligned} y^{(0)} &= 1 \\ (\alpha y)^{(j)} &= \alpha^j y^{(j)} \\ (y+z)^{(j)} &= \sum_{i=0}^{j} y^{(i)} z^{(j-i)} \\ (x^{(a)})^{(j)} &= \frac{(ja)!}{(a!)^j j!} x^{(ja)} \end{aligned}$$

for all $y, z \in A(m)_{(1)}$ and $a \in \mathbb{N}^m$, $a \neq 0$.

Following Kostrikin and Šafarevič [KS], we say that a derivation D of $A(m)$ is *special* if $Dy^{(j)} = y^{(j-1)} Dy$ for all $y \in A(m)_{(1)}$ and $j \geq 1$. The Lie algebra $W(m)$ of special derivations of $A(m)$ is a free module over $A(m)$ with basis $\{\partial_i | i = 1, \ldots, m\}$, $\partial_i x^{(a)} = x^{(a-\varepsilon_i)}$. The *general Lie algebra of Cartan type* $W(m; \mathbf{n})$ is the stabilizer of $A(m; \mathbf{n})$ in $W(m)$. Thus, $W(m; \mathbf{n})$ is a free module over $A(m; \mathbf{n})$ with the same basis $\{\partial_i | i = 1, \ldots, m\}$. The Lie algebra $W(m; \mathbf{n})$ is simple unless $m = 1$ and the characteristic is 2 (see the example in the Introduction).

Let $\Omega(m)$ denote the complex of differential forms over $W(m)$, and let $\Omega(m; \mathbf{n})$ be the subcomplex of forms whose values on $W(m; \mathbf{n})$ are contained in $A(m; \mathbf{n})$. Then $\Omega(m)$ (resp. $\Omega(m; \mathbf{n})$) is the exterior algebra on $\{dx_1, \ldots, dx_m\}$ over $A(m)$ (resp. $A(m; \mathbf{n})$). The grading on $A(m)$ extends to the standard grading on $W(m), W(m, \mathbf{n}), \Omega(m), \Omega(m; \mathbf{n})$ gotten by setting $\deg dx_i = -\deg x_i = -1 = \deg \partial_i$, and this grading gives rise to an associated filtration (the standard filtration) of the algebra. Denote by $\hat{\Omega}(m)$ the completion of $\Omega(m)$ relative to the standard filtration and let

$$\begin{aligned}
\omega_0 &= dx_1 \wedge \cdots \wedge dx_m \\
\omega_1 &= (\exp x_i)\, dx_1 \wedge \cdots \wedge dx_m, \qquad i = 1, \ldots, m \qquad\qquad (1.1)\\
\omega_2 &= (1 - e)\, dx_1 \wedge \cdots \wedge dx_m, \qquad e = \prod_{i=1}^{m} x_i^{(p^{n_i} - 1)}.
\end{aligned}$$

The exponential of an element $y \in A(m)$ is given relative to the divided power maps: $\exp y = \sum_{j \geq 0} y^{(j)}$ and $\exp y \in \hat{A}(m)$, the completion of $A(m)$ relative to the standard filtration.

Let ω denote one of the forms $\omega_0, \omega_1, \omega_2$ in (1.1) and for $m > 2$ define

$$\begin{aligned}
\tilde{S}(m; \mathbf{n}, \omega) &= \{D \in W(m; \mathbf{n}) | D\omega = 0\} \\
C\tilde{S}(m; \mathbf{n}, \omega) &= \{D \in W(m; \mathbf{n}) | D\omega = c\omega, c \in F\}.
\end{aligned}$$

The *special simple Lie algebra of Cartan type* $S(m; \mathbf{n}, \omega)$ is the second derived algebra of $\tilde{S}(m; \mathbf{n}, \omega)$. Note that $\tilde{S}(m; \mathbf{n}, \omega)$ consists of all the special derivations $D \in W(m; \mathbf{n})$ whose divergence $\operatorname{div}(\phi D) = 0$, where $\operatorname{div}(f_1\partial_1 + \cdots + f_m\partial_m) = \sum_{i=1}^{m} \partial_i f_i$, and

$$\phi = \begin{cases} 1 & \text{if } \omega = \omega_0 \\ \exp x_i & \text{if } \omega = \omega_1 \\ 1 + e & \text{if } \omega = \omega_2. \end{cases}$$

Now it has been shown in [Ki1],

$$\dim S(m;\mathbf{n},\omega) = \begin{cases} (m-1)(p^n-1) & \text{if } \omega = \omega_0 \\ (m-1)p^n & \text{if } \omega = \omega_1 \\ (m-1)(p^n-1) & \text{if } \omega = \omega_2, \end{cases}$$

for $n = n_1 + \cdots + n_m$. Moreover, according to [KS],

$$\tilde{S}(m;\mathbf{n},\omega_0) = S(m;\mathbf{n},\omega_0) + \langle e_i\partial_i | i = 1,\ldots,m\rangle, \tag{1.2}$$

where $e_i = (\prod_{j\neq i} x_j^{(p^{n_j}-1)})$, and the codimension of $\tilde{S}(m;\mathbf{n},\omega_0)$ in $C\tilde{S}(m;\mathbf{n},\omega_0)$ is 1.

The standard grading of $W(m;\mathbf{n})$ induces a grading in $C\tilde{S}(m;\mathbf{n},\omega_0)$ and, hence, in $S(m;\mathbf{n},\omega_0)$, which is called standard as well, and the standard filtration of $W(m;\mathbf{n})$ induces the standard filtration of the algebras $C\tilde{S}(m;\mathbf{n},\omega)$ and $S(m;\mathbf{n},\omega)$ for any ω.

By results of [Ka, T, W3, Ku2] the Lie algebras $L = S(m;\mathbf{n},\omega_j)$ for $j = 0,1,2$ are all pairwise nonisomorphic simple filtered Lie algebras such that

$$S(m;\mathbf{n},\omega_0) \subseteq grL \subseteq C\tilde{S}(m;\mathbf{n},\omega_0).$$

Furthermore by [Ki1],

$$\begin{aligned} \tilde{S}(m;\mathbf{n},\omega_1) &= S(m;\mathbf{n},\omega_1) \\ \operatorname{codim}_{C\tilde{S}(m;\mathbf{n},\omega_1)}\tilde{S}(m;\mathbf{n},\omega_1) &= 1 \\ \tilde{S}(m;\mathbf{n},\omega_2) &= S(m;\mathbf{n},\omega_2) + \langle e_i\partial_i | i = 1,\ldots,m\rangle \\ C\tilde{S}(m;\mathbf{n},\omega_1) &= \tilde{S}(m;\mathbf{n},\omega_2), \end{aligned} \tag{1.3}$$

where the last equality follows from the fact that if $d\omega = 0$, then $D\omega = d\omega$ for any $D \in W(m;\mathbf{n})$, and $\omega_2 \neq d\omega$ in $\Omega(m;\mathbf{n})^m$.

For the class of Hamiltonian algebras, let $m = 2s$ and assume that $\omega = \sum_{i,j=1}^{s} \omega_{i,j}\, dx_i \wedge dx_j$ is a 2-form in $\hat{\Omega}(m)^2$ such that $d\omega = 0$ and $\omega(0)$ is a nondegenerate skew-symmetric form on $\hat{W}(m)/\hat{W}(m)_{(0)}$, where $\hat{W}(m) = \hat{W}(m)_{(-1)} \supset \hat{W}(m)_{(0)} \supset \ldots$ is the standard filtration. The infinite Hamiltonian algebra $\hat{H}(m)$ consists of all derivations $D \in \hat{W}(m)$ such that $D\omega = 0$. The intersection $\hat{H}(m) \cap W(m;\mathbf{n})$ depends on ω, and so it will be denoted $\tilde{H}(m;\mathbf{n},\omega)$. Correspondingly, we have

$$\begin{aligned} \tilde{H}(m;\mathbf{n},\omega) &= \{D \in W(m;\mathbf{n}) | D\omega = 0\} \\ C\tilde{H}(m;\mathbf{n},\omega) &= \{D \in W(m;\mathbf{n}) | D\omega = c\omega,\ c \in F\}. \end{aligned}$$

To obtain the simple Lie algebras of Cartan type H, the form ω cannot be chosen arbitrarily. It was shown in [Ka, Sk1, Sk3, BGOSW] that to construct pairwise nonisomorphic Lie algebras of type H one should take

$$\omega_0 = \sum_{i=1}^{s} dx_i \wedge dx_{i+s},$$

$$\omega_1 = d\left[\exp x_i(x_1\,dx_{s+1} + \cdots + x_s\,dx_{2s})\right], \qquad i = 1,\ldots,2s,$$

$$\omega_2 = \omega_0 + \sum_{i,j=1}^{m} \alpha_{i,j} x_i^{(p^{n_i}-1)} x_j^{(p^{n_j}-1)}\,dx_i \wedge dx_j.$$

Here $(\alpha_{i,j})$ is a skew-symmetric matrix with entries in F. The exact description of the matrix $(\alpha_{i,j})$ is given in [Sk1, Sk3], but we will not require that. In general, a *simple Lie algebra of Cartan type H* is

$$H(m;\mathbf{n},\omega) = \tilde{H}(m;\mathbf{n},\omega)^{(2)},$$

where ω is either ω_0, ω_1, or of type ω_2, and by [Ki2] the dimension of $H(m;\mathbf{n},\omega)$ is given by

$$\dim H(m;\mathbf{n},\omega) = \begin{cases} p^n - 2 & \text{if } \omega = \omega_0 \\ p^n & \text{if } \omega = \omega_1 \text{ and } s+1 \not\equiv 0 \bmod p \\ p^n - 1 & \text{if } \omega = \omega_1 \text{ and } s+1 \equiv 0 \bmod p \\ p^n - 1 & \text{if } \omega = \omega_2 \text{ and } \det(\alpha_{i,j}) \neq 0 \\ p^n - 2 & \text{if } \omega = \omega_2 \text{ and } \det(\alpha_{i,j}) = 0, \end{cases}$$

$m = 2s$, and $n = n_1 + \cdots + n_m = \|n\|$.

The forms $\omega = \omega_0, \omega_1, \omega_2$ can be rewritten as $\omega = \exp v \sum_{i,j} \omega_{i,j}\,dx_i \wedge dx_j$, where $v = x_k$ or 0 for some k and $\omega_{i,j} \in A(m;\mathbf{n})$. Then for any $D \in \tilde{H}(m;\mathbf{n},\omega)$, we have

$$D = D_f = \sum_{i,j=1}^{m} h_i \partial_i, \qquad h_i = \sum_{j=1}^{m} \bar{\omega}_{j,i}(\partial_j f + f \partial_j v),$$

where $(\bar{\omega}_{i,j}) = (\omega_{i,j})^{-1}$, $f \in A(m;\mathbf{n})$ if $v \neq 0$, and $f \in \tilde{A}(m;\mathbf{n}) = A(m;\mathbf{n}) + \langle x_i^{(p^{n_i})} | i = 1,\ldots,m \rangle$ if $v = 0$. By identifying D_f with f we have

$$\tilde{H}(m;\mathbf{n},\omega) = \begin{cases} A(m;\mathbf{n}) & \text{if } \omega = \omega_1 \\ \tilde{A}(m;\mathbf{n})/\langle 1 \rangle & \text{if } \omega = \omega_0, \omega_2, \end{cases}$$

where in all cases the product is the Poisson bracket,

$$\{f, g\} = \sum_{i,j=1}^{m} \bar{\omega}_{i,j}(\partial_i f + f\partial_i v)(\partial_j g + g\partial_j v).$$

Then by [Ki2],

$$\operatorname{codim}_{C\tilde{H}(m;\mathbf{n},\omega)} \tilde{H}(m;\mathbf{n},\omega) = \begin{cases} 1 & \text{if } \omega = \omega_0 \text{ or } \omega_1 \\ 0 & \text{if } \omega = \omega_2, \end{cases}$$

and

$$H(m;\mathbf{n},\omega)$$

$$= \begin{cases} \langle x^{(a)} \in A(m;\mathbf{n})/\langle 1\rangle | x^{(a)} \neq e\rangle & \text{if } \omega = \omega_0, \\ \tilde{H}(m;\mathbf{n},\omega) = A(m;\mathbf{n}) & \text{if } \omega = \omega_1,\ s+1 \not\equiv 0(p), \\ \tilde{H}(m;\mathbf{n},\omega)^{(1)} = \langle x^{(a)} \in A(m;\mathbf{n})/\langle 1\rangle | x^{(a)} \neq e\rangle & \text{if } \omega = \omega_1,\ s+1 \equiv 0(p), \\ A(m;\mathbf{n})/\langle 1\rangle & \text{if } \omega = \omega_2,\ \det(\alpha_{i,j}) \neq 0, \\ \text{an ideal of codimension one in } A(m;\mathbf{n})/\langle 1\rangle & \text{if } \omega = \omega_2,\ \det(\alpha_{i,j}) = 0, \end{cases} \tag{1.4}$$

where as above $e = \prod_{i=1}^{m} x_i^{(p^{n_i}-1)}$. Kirillov's paper contains a precise description of $H(m;\mathbf{n},\omega)$ in the last case, but we will not require that here.

The Lie algebra $H(m;\mathbf{n},\omega_0)$ is a homogeneous subalgebra in the graded Lie algebra $W(m;\mathbf{n})$ relative to the standard grading. All the Lie algebras $H(m;\mathbf{n},\omega)$ have the standard filtration induced from the standard filtration on $W(m;\mathbf{n})$. According to [Ka] (see [Ku2] for an explicit description), the filtered Lie algebras $L = H(m;\mathbf{n},\omega)$ constructed above are simple Lie algebras such that

$$H(m;\mathbf{n},\omega_0) \subseteq grL \subseteq C\tilde{H}(m;\mathbf{n},\omega_0).$$

Assume now that $m = 2s+1$ and $\omega = dx_m + \sum_{i=1}^{s} x_i\, dx_{i+s} - x_{i+s}\, dx_i$. Then a *simple Lie algebra of Cartan type K* is $\tilde{K}(m;\mathbf{n})^{(1)}$, where

$$\tilde{K}(m;\mathbf{n}) = \{D \in W(m;\mathbf{n}) | D\omega = g\omega,\ g \in A(m;\mathbf{n})\}.$$

By results in [KS],

$$\tilde{K}(m;\mathbf{n}) = \left\{ D_f \in W(m;\mathbf{n}) | D_f = \sum_{i=1}^{s} (\partial_{i+s} f + x_i \partial_m f)\partial_i \right.$$

$$\left. + \sum_{i=s+1}^{2s} (-\partial_{i-s} f + x_i \partial_m f)\partial_i + \left(2f - \sum_{i=1}^{m-1} x_i \partial_i f\right)\partial_m,\ f \in A(m;\mathbf{n}) \right\}.$$

If D_f is identified with f, then we obtain the Jacobi bracket on $\tilde{K}(m;\mathbf{n}) = A(m;\mathbf{n})$,

$$[f,g] = \partial_m g\, \Delta f - \partial_m f\, \Delta g - \sum_{i=1}^{s} (\partial_i f \partial_{i+s} g - \partial_{i+s} f \partial_i g),$$

where $\Delta f = 2f - \sum_{i=1}^{2s} x_i \partial_i f$. Then

$$K(m;\mathbf{n}) = \begin{cases} \tilde{K}(m;\mathbf{n}) = A(m;\mathbf{n}) & \text{if } m+3 \not\equiv 0 \bmod p \\ \langle x^{(a)} \in A(m;\mathbf{n}) | x^{(a)} \neq e \rangle & \text{if } m+3 \equiv 0 \bmod p. \end{cases}$$

The Lie algebra $K(m;\mathbf{n})$ is not a homogeneous subalgebra of $W(m;\mathbf{n})$ relative to the standard grading. However, $K(m;\mathbf{n})$ becomes homogeneous if $W(m;\mathbf{n})$ is given the grading of type $(1,\ldots,1,2)$, which means $\deg x_i = -\deg \partial_i = 1$ for $i = 1,\ldots,m-1$, and $\deg x_m = -\deg \partial_m = 2$. This is referred to as the standard grading for $K(m;\mathbf{n})$. According to [Ku3], $K(m;\mathbf{n})$ is the unique simple filtered Lie algebra L such that

$$K(m;\mathbf{n}) \subseteq grL \subseteq \tilde{K}(m;\mathbf{n}).$$

The main classification result [SW] states that any simple Lie algebra over an algebraically closed field of characteristic $p > 7$ is isomorphic to either a classical Lie algebra or a Lie algebra of Cartan type $W(m;\mathbf{n})$, $S(m;\mathbf{n},\omega)$, $H(m;\mathbf{n},\omega)$, $K(m;\mathbf{n})$, for $\omega = \omega_0, \omega_1, \omega_2$ as described above.

At present only one series of exceptional simple Lie algebras is known for $p > 3$. They are the Melikyan algebras $\mathscr{G}(\mathbf{n})$ with $\mathbf{n} = (n_1, n_2)$ of characteristic $p = 5$ [M]. According to [Ku3], the Melikyan algebras $\mathscr{G} = \mathscr{G}(\mathbf{n})$ possess a $\mathbb{Z}/3\mathbb{Z}$-grading, $\mathscr{G} = \mathscr{G}_{\mathbf{0}} \oplus \mathscr{G}_{\mathbf{1}} \oplus \mathscr{G}_2$, which may be described as

$$\mathscr{G}_0 = W(2;\mathbf{n}), \qquad \mathscr{G}_1 = A(2;\mathbf{n}), \qquad \mathscr{G}_2 = \widetilde{W(2;\mathbf{n})},$$

where $\widetilde{W(2;\mathbf{n})}$ is a second copy of the space $W(2;\mathbf{n})$. Multiplication in $\mathscr{G}_0$ is the same as in $W(2;\mathbf{n})$, and the other products are specified by

$$\begin{aligned}
&[D,f] = Df - 2f \operatorname{div} D && D \in \mathscr{G}_0,\ f \in \mathscr{G}_1\\
&\left[D_1,\tilde{D}_2\right] = \left[\widetilde{D_1,D_2}\right] + 2\operatorname{div} D_1 \tilde{D}_2 && D_1 \in \mathscr{G}_0,\ \tilde{D}_2 \in \mathscr{G}_2\\
&[f,g] = 2(g\partial_2 f - f\partial_2 g)\tilde{\partial}_1 - 2(g\partial_1 f - f\partial_1 g)\tilde{\partial}_2 && f,g \in \mathscr{G}_1\\
&\left[f,\tilde{D}\right] = fD && f \in \mathscr{G}_1,\ \tilde{D} \in \mathscr{G}_2,
\end{aligned}$$

and for $\tilde{D}_1 = f_1\tilde{\partial}_1 + f_2\tilde{\partial}_2$ and $\tilde{D}_2 = g_1\tilde{\partial}_1 + g_2\tilde{\partial}_2$ in $\mathscr{G}_2$,

$$\left[\tilde{D}_1,\tilde{D}_2\right] = f_1 g_2 - f_2 g_1 \in \mathscr{G}_1.$$

1.2. *Derivations of Nonclassical Simple Lie Algebras*

If $L = W(m;\mathbf{n})$, then $\operatorname{Der} L \cong \overline{W(m;\mathbf{n})}$, where $\overline{W(m;\mathbf{n})}$ denotes the p-closure of $W(m;\mathbf{n})$ in $\operatorname{Der} A(m;\mathbf{n}) = W(n;\mathbf{1})$, where $n = n_1 + \cdots + n_m$. Moreover, according to [Wi2]

$$\overline{W(m;\mathbf{n})} = W(m;\mathbf{n}) + \left\langle \partial_i^{p^{k_i}} \mid 1 \le k_i \le n_i - 1,\ i = 1,\ldots,m \right\rangle.$$

In particular,

$$\operatorname{codim}_{\overline{W(m;\mathbf{n})}} W(m;\mathbf{n}) = n - m.$$

It is known [Ka, Sk2, Ku2]) that for a simple Lie algebra L of Cartan type

$$\operatorname{Der} L \cong N_{\overline{W(m;\mathbf{n})}}(L).$$

From that it may be deduced that for simple Lie algebras of type S or H, $\operatorname{Der} L = \overline{L} + \overline{C\tilde{L}} = \overline{C\tilde{L}}$, where the line means the p-closure of L or $C\tilde{L}$ in $\operatorname{Der} A(m;\mathbf{n})$. We now consider the simple algebras individually and tabulate the results. First for algebras of type S,

$$\operatorname{Der} L = \begin{cases}
C\tilde{S}(m;\mathbf{n},\omega_0) \oplus \left\langle \partial_i^{p^{k_i}} \mid 1 \le k_i \le n_i - 1,\ i = 1,\ldots,m \right\rangle & \text{if } L = S(m;\mathbf{n},\omega_0)\\
CS(m;\mathbf{n},\omega_1) \oplus \left\langle \partial_i^{p^{k_i}} \mid 1 \le k_i \le n_i - 1,\ i = 1,\ldots,m \right\rangle & \text{if } L = S(m;\mathbf{n},\omega_1)\\
\tilde{S}(m;\mathbf{n},\omega_2) \oplus \left\langle \tilde{\partial}_i^{p^{k_i}} \mid 1 \le k_i \le n_i - 1,\ i = 1,\ldots,m \right\rangle & \text{if } L = S(m;\mathbf{n},\omega_2),
\end{cases}$$

where $\tilde{S}(m; \mathbf{n}, \omega_1) = S(m; \mathbf{n}, \omega_1)$, $\tilde{S}(m; \mathbf{n}, \omega_2) = S(m; \mathbf{n}, \omega_2) \oplus \langle e_i | i = 1, \dots, m\rangle$, and $e_i = (\prod_{j \neq i} x_j^{(p^{n_j}-1)})\partial_i$ as in (1.3), and $\bar{\partial}_i = (1+e)\partial_i$.

For algebras of type $L = H(m; \mathbf{n}, \omega_0)$,

$$\operatorname{Der} L = C\tilde{H}(m; \mathbf{n}, \omega_0) \oplus \langle \partial_i^{p^{k_i}} | 1 \le k_i \le n_i - 1, i = 1, \dots, m\rangle$$

$$\tilde{H}(m; \mathbf{n}, \omega_0) = H(m; \mathbf{n}, \omega_0) + \left\langle D_{x_i^{(p^{n_i})}} | i = 1, \dots, m\right\rangle$$

$$\operatorname{codim}_{C\tilde{L}} \tilde{L} = 1$$

$$\bar{L} = L \oplus \left\langle \partial_i^{p^{k_i}} | 1 \le k_i \le n_i - 1, i = 1, \dots, m\right\rangle. \tag{1.5}$$

When $m = 2s$ and $L = H(m; \mathbf{n}, \omega_1)$,

$$\tilde{L} = \begin{cases} L & \text{if } s + 1 \not\equiv 0 \bmod p, \\ L + \langle D_e \rangle,\ e = \prod_{i=1}^{m} x_i^{(p^{n_i}-1)} & \text{if } s + 1 \equiv 0 \bmod p, \end{cases} \tag{1.6}$$

so

$$\operatorname{codim}_{C\tilde{L}} \tilde{L} \le 1, \qquad \operatorname{codim}_{\operatorname{Der} L} \bar{L} \le 2.$$

In addition, we will need an estimate of a $\operatorname{codim}_{\operatorname{Der} L} L$. Now according to [Ku2, Lemma 0.2],

$$\dim \bar{L}/(C\tilde{L})_{(0)} \cap \bar{L} = n,$$

where $(C\tilde{L})_{(0)}$ is the 0-term in the standard filtration of $C\tilde{L}$. Then since

$$\dim \bar{L}/L_{(0)} = \dim \bar{L}/(C\tilde{L})_{(0)} \cap \bar{L} + \dim (C\tilde{L})_{(0)} \cap \bar{L}/L_{(0)} \le n + 2,$$

we have

$$\dim \bar{L}/L = \dim \bar{L}/L_{(0)} - \dim L/L_{(0)} \le n + 2 - m,$$
$$\operatorname{codim}_{\operatorname{Der} L} L \le n - m + 4.$$

For algebras of type $L = H(m; \mathbf{n}, \omega_2)$,

$$C\tilde{H}(m; \mathbf{n}, \omega_2) = \tilde{H}(m; \mathbf{n}, \omega_2) = H(m; \mathbf{n}, \omega_2) + \left\langle D_{x_i^{(p^{n_i})}} | i = 1, \dots, m\right\rangle,$$

$$\operatorname{Der} L = \bar{L} + \tilde{H}(m; \mathbf{n}, \omega_2).$$

Now when $m = 2s + 1$ and $L = \tilde{K}(m;\mathbf{n})^{(1)} = K(m;\mathbf{n})$ then

$$\begin{aligned}
\operatorname{Der} L &= N_{\overline{W(m;\mathbf{n})}}\left(\tilde{K}(m;\mathbf{n})^{(1)}\right)\\
&= \overline{\tilde{K}(m;\mathbf{n})}\\
&= K(m;\mathbf{n}) \oplus \left\langle D_1^{p^k} \mid 1 \le k \le n_m - 1\right\rangle\\
&\qquad \oplus \left\langle D_{x_i}^{p^{l_i}} \mid 1 \le l_i \le n_{i+s} - 1,\ i = 1,\dots,s\right\rangle\\
&\qquad \oplus \left\langle D_{x_i}^{p^{l_i}} \mid 1 \le l_i \le n_{i-s} - 1,\ i = s+1,\dots,2s\right\rangle.
\end{aligned}$$

Finally, if $L = \mathscr{G}(\mathbf{n})$ for $\mathbf{n} = (n_1, n_2)$ is a Melikyan algebra, then according to [Ku3],

$$\operatorname{Der} L = \overline{L} = \mathscr{G}_0 \oplus \mathscr{G}_1 \oplus \mathscr{G}_2,$$

where

$$\overline{\mathscr{G}}_0 = \overline{W(2;\mathbf{n})},$$
$$\operatorname{codim}_{\overline{L}} L = n - 2, \qquad n = n_1 + n_2.$$

2. Tori and Nonsingular Derivations

In what follows we will regard a simple Lie algebra L as a subalgebra of the p-algebra $\operatorname{Der} L$.

Proposition 2.1. *Let L be a finite-dimensional simple Lie algebra over an algebraically closed field F of characteristic $p > 0$. If L has a nonsingular derivation, then there exists a nonsingular derivation D of L such that*

(i) *D is a regular semisimple element in* $\operatorname{Der} L$,

(ii) *the Cartan subalgebra $H = \mathscr{C}_{\operatorname{Der} L}(D)$ (the centralizer of D in* $\operatorname{Der} L$*) contains a torus of maximal dimension,*

(iii) $H \cap L = (0)$.

Proof. By Premet's theorem [P, Theorem 2(i)] there exists a nonempty Zariski-open subset $\mathscr{O}$ of $\operatorname{Der} L$ with the property that for all $x \in \mathscr{O}$, the nilspace $(\operatorname{Der} L)_x^0 = \{y \in \operatorname{Der} L \mid (ad x)^m(y) = 0 \text{ for some } m\}$ is a Cartan subalgebra of $\operatorname{Der} L$ containing a torus of maximal dimension. Let $\mathscr{O}'$ be the set of all nonsingular derivations of L. Then $\mathscr{O}'$ is also a nonempty Zariski-open subset in $\operatorname{Der} L$, and $\mathscr{O} \cap \mathscr{O}' \neq (0)$. Consider an arbitrary nonzero $x \in \mathscr{O} \cap \mathscr{O}'$, and let $x = x_s + x_n$ be its Jordan–Chevalley decomposition; i.e., x_s is semisimple, x_n is nilpotent, and $[x_s, x_n] = 0$. Clearly, all the assertions are fulfilled with $D = x_s$. ∎

We will require some information on tori in $\overline{W(m;\mathbf{n})}$.

LEMMA 2.2. *Assume T is a torus in $\overline{W(m;\mathbf{n})}$ with $\dim T = k$. Let Φ (resp. Φ') be the set of all weights in $A(m;\mathbf{n})$ (resp. $W(m;\mathbf{n})$) with respect to T. Then*

(i) *$\Phi = \Phi' = T^*_{F_p}$, where $T^*_{F_p}$ is the space over F_p which is dual to $\langle D \in T | D^p = D\rangle_{F_p}$.*

(ii) *$\dim A(m;\mathbf{n})_\mu = p^{n-k}$, and $\dim W(m;\mathbf{n})_\mu = mp^{n-k}$ for any $\mu \in T^*_{F_p}$.*

Proof. Let $\hat{T}$ be a maximal torus in $\operatorname{Der} A(m;\mathbf{n}) = W(n;\mathbf{1})$ $(n = n_1 + \cdots + n_m)$ containing T, and let $\hat{\Phi}$ be the set of all weights relative to $\hat{T}$ in $A(m;\mathbf{n}) = A(n;\underline{1})$. We may use results of Demuškin [D] for $W(n;\underline{1})$ to conclude that $\hat{\Phi} = \hat{T}^*_{F_p}$, and $\dim A(m;\mathbf{n}) = \dim A(n;\underline{1})_\nu = 1$ for any $\nu \in \hat{T}^*_{F_p}$. Evidently, the restriction $\eta\colon \nu \to \nu|_T$ is surjective, and $|\operatorname{Ker}\eta| = p^{n-k}$. Therefore, $\Phi = T^*_{F_p}$ as asserted, and $\dim A(m;\mathbf{n})_\mu = p^{n-k}$ for any $\mu \in T^*_{F_p}$.

Choose derivations $D_{\beta_1}, \ldots, D_{\beta_m}$ in $W(m;\mathbf{n})$ such that each D_{β_i} is a weight vector relative to T, and $\{D_{\beta_1}, \ldots, D_{\beta_m}\}$ is a basis for the free $A(m;\mathbf{n})$-module $W(m;\mathbf{n})$. Then each $\beta_i \in T^*_{F_p}$ and $W(m;\mathbf{n}) = \sum_{i=1}^m (\oplus \sum_\mu A(m;\mathbf{n})_\mu) D_{\beta_i}$. Thus, $\Phi = \Phi' = T^*_{F_p}$ and $\dim W(m;\mathbf{n})_\mu = mp^{n-k}$ as claimed. ∎

3. THE PROOF OF THE THEOREM

Suppose that L is a finite-dimensional simple Lie algebra of Cartan type over an algebraically closed field F of characteristic $p > 3$ or that $\operatorname{char} F = 5$ and L is a Melikyan algebra. Assume that L admits a nonsingular derivation D. In light of Proposition 2.1 we may assume that D is contained in a torus T of $\operatorname{Der} L$, T is generated by p-powers of D, and $\dim T = k$. We will treat the various kinds of algebras separately.

3.1. $L = W(m;\mathbf{n})$. If T is a torus in $\operatorname{Der} L = \overline{W(m;\mathbf{n})}$ (see Section 1.2), then by Lemma 2.2, $\dim \mathscr{C}_L(T) = \dim W(m;\mathbf{n})_0 = mp^{n-k}$. Consequently, $W(m;\mathbf{n})$ does not admit a nonsingular derivation.

3.2. $L = S(m;\mathbf{n},\omega)$, $\omega = \omega_0, \omega_1, \omega_2$. Without loss of generality, we may assume that $\omega_1 = \exp x_1\, dx_1 \wedge \cdots \wedge dx_m$. Now consider the 1-

cocycles $\lambda_0, \lambda_1, \lambda_2$ on $W(m; \mathbf{n})$ with values in $A(m; \mathbf{n})$ given by

$$\lambda_0 = \operatorname{div}, \qquad \lambda_1 = dx_1 + \operatorname{div}, \qquad \lambda_2 = de + \operatorname{div},$$
$$\lambda_i([D_1, D_2]) = D_1\lambda_i(D_2) - D_2\lambda_i(D_1), \qquad i = 0, 1, 2.$$

Since $\lambda_i(C\tilde{S}(m; \mathbf{n}, \omega_i)) \subseteq F \subseteq A(m; \mathbf{n})$, λ_i is a morphism of $C\tilde{S}(m; \mathbf{n}, \omega_i)$-modules. Obviously, λ_i can be extended to a morphism of modules over $\overline{C\tilde{S}(m; \mathbf{n}, \omega_i)} = \operatorname{Der} S(m; \mathbf{n}, \omega_i)$ and $\operatorname{Ker} \lambda_i = \tilde{S}(m; \mathbf{n}, \omega_i)$. In particular, λ_i is a morphism of T-modules. Thus,

$$(3.1) \qquad \dim \mathscr{C}_{\tilde{S}(m; \mathbf{n}, \omega_i)}(T) = \dim \mathscr{C}_{W(m; \mathbf{n})}(T) - \dim \mathscr{C}_{A(m; \mathbf{n})}(T)$$
$$= mp^{n-k} - p^{n-k} = (m-1)p^{n-k}.$$

Taking into account the fact that $\operatorname{codim}_{\tilde{S}(m; \mathbf{n}, \omega_i)} S(m; \mathbf{n}, \omega_i) \leq m$ we conclude that if $k < n$, then $\mathscr{C}_{S(m; \mathbf{n}, \omega_i)}(T) \neq (0)$. Hence, we may assume for such a torus T that $\dim T = n = n_1 + \cdots + n_m$.

(a) $\omega = \omega_0$. Let $T_1 = \overline{\tilde{S}(m; \mathbf{n}, \omega_0)} \cap T$. Since the codimension of $\overline{\tilde{S}(m; \mathbf{n}, \omega_0)}$ in $\operatorname{Der} L$ is equal to one, $\dim T_1 \geq n - 1$. We argue that $T_1 \subseteq \overline{S(m; \mathbf{n}, \omega_0)}$. Indeed, if

$$t = \bar{y} + \sum_i \alpha_i e_i \partial_i, \qquad \text{where } e_i = \left(\prod_{j \neq i} x_j^{(p^{n_j} - 1)}\right), \ \alpha_i \in F,$$

and t is a toral element of T_1, then, since $[\bar{y}, e_i] \in L$, $[e_i, e_j] = 0$, and $e_i^p = 0$, we have $t = t^p \in \bar{L}$. Now $\operatorname{codim}_{\bar{L}} L = n - m$ and $\dim T_1 \geq n - 1$. Hence $T_1 \cap L \neq (0)$. But, since $T_1 \cap L \subseteq \mathscr{C}_L(T)$, we arrive at a contradiction. Consequently, $L = S(m; \mathbf{n}, \omega_0)$ has no nonsingular derivations.

(b) $\omega = \omega_1$. Since $\tilde{S}(m; \mathbf{n}, \omega_1) = S(m; \mathbf{n}, \omega_1)$ (see Section 1.2), it follows from Eq. (3.1) that $\mathscr{C}_{S(m; \mathbf{n}, \omega_1)}(T) \neq (0)$, contrary to assumption. Hence, the algebras of type $S(m; \mathbf{n}, \omega_1)$ do not possess nonsingular derivations.

3.3. $L = H(m; \mathbf{n}, \omega)$, $m = 2s$, and $\omega = \omega_0, \omega_1, \omega_2$. Let $P(m; \mathbf{n}, \omega)$ denote the corresponding Poisson Lie algebra; i.e., $P(m; \mathbf{n}, \omega) = A(m; \mathbf{n})$ with the Poisson bracket,

$$(3.2) \qquad \{f, g\} = \sum_{i,j=1}^{m} \bar{\omega}_{i,j}(\partial_i f + f\partial_i v)(\partial_j g + g\partial_j v),$$

where v as in Section 1.1. Note that by (1.4)

(3.3)

$$H(m;\mathbf{n},\omega) \cong \begin{cases} P(m;\mathbf{n},\omega)^{(1)} + \langle 1\rangle/\langle 1\rangle, & \omega = \omega_0, \omega_2 \\ P(m;\mathbf{n},\omega) & \omega = \omega_1,\ s+1 \not\equiv 0 \bmod p \\ P(m;\mathbf{n},\omega)^{(1)} & \omega = \omega_1,\ s+1 \equiv 0 \bmod p. \end{cases}$$

For $\omega = \omega_0, \omega_2$ we consider the Lie algebras

$$\tilde{P}(m;\mathbf{n},\omega) = P(m;\mathbf{n},\omega) + \left\langle x_i^{(p^{n_i})} \middle| i = 1,\dots,m \right\rangle$$

with the same product as in (3.2), but in these cases $v = 0$. Then

$$\tilde{H}(m;\mathbf{n},\omega) \cong \begin{cases} \tilde{P}(m;\mathbf{n},\omega)/\langle 1\rangle, & \omega = \omega_0, \omega_2 \\ P(m;\mathbf{n},\omega) & \omega = \omega_1. \end{cases}$$

Since 1 is central in $P(m;\mathbf{n},\omega)$ for $\omega = \omega_1, \omega_2$, we have in all cases that $P(m;\mathbf{n},\omega)$ is a module over $\tilde{H}(m;\mathbf{n},\omega)$. Moreover, it is a module over $\overline{H(m;\mathbf{n},\omega)} + \tilde{H}(m;\mathbf{n},\omega)$. When $\omega = \omega_0$, the action of the degree derivation z on $P(m;\mathbf{n},\omega_0)$ may be defined by

$$\{z, x^{(a)}\} = (\|a\| - 2)x^{(a)}.$$

Now the Poisson bracket (3.2) is related to the usual multiplication in the algebra $A(m;\mathbf{n})$ by

$$\{f, gh\} = g\{f,h\} + D_f(g)h. \tag{3.4}$$

This relation remains valid if $f \in \tilde{P}(m;\mathbf{n},\omega)$ as well. Formula (3.4) follows directly from the theory of truncated coinduced modules [Ku2] or it can be verified by a straightforward calculation.

The action of $H(m;\mathbf{n},\omega)$ on $P(m;\mathbf{n},\omega)$ extends naturally to one of $\overline{H(m;\mathbf{n},\omega)}$ (for instance, by using (3.4)) and so it may be extended to an action of $\overline{H(m;\mathbf{n},\omega)} + C\tilde{H}(m;\mathbf{n},\omega) = \operatorname{Der} L$. Denote the action of $D \in \operatorname{Der} L$ on $h \in P(m;\mathbf{n},\omega)$ by $\{D, h\}$. Then formula (3.4) is true for $D \in \operatorname{Der} L$ also, where

$$\{D, gh\} = g\{D,h\} + D(g)h.$$

Now assume that the torus $T \subset \operatorname{Der} H(m;\mathbf{n},\omega)$ is as above and let Φ_P denote the set of all weights of $P(m;\mathbf{n},\omega)$ with respect to T. Choose an invertible element u in $A(m;\mathbf{n})$ such that u is a root vector for T in $P(m;\mathbf{n},\omega)$. Any element $f \in P(m;\mathbf{n},\omega)$ may be written as gu for some g.

Then as in Lemma 2.2, we can conclude that $\Phi_P = T^*_{F_p}$ and $\dim P(m;\mathbf{n},\omega)_\mu = p^{n-k}$ for any $\mu \in \Phi_P$. Thus, if $k < n$ then it follows from (3.3) that $\mathscr{C}_{H(m;\mathbf{n},\omega)}(T) \neq (0)$. As a result, we may suppose that $\dim T = n$. We now consider the various cases individually.

(a) $\omega = \omega_0$. Assume that $T_1 = T \cap (\overline{H(m;\mathbf{n},\omega_0)} + \tilde{H}(m;\mathbf{n},\omega_0))$. Since the codimension of $\overline{H(m;\mathbf{n},\omega_0)} + \tilde{H}(m;\mathbf{n},\omega_0)$ in $\operatorname{Der} L$ is one, $\dim T_1 \geq n-1$. As we have done for $S(m;\mathbf{n},\omega_0)$, we may consider $T_1 \subset \overline{H(m;\mathbf{n},\omega_0)} = \bar{L}$. Since $\operatorname{codim}_{\bar{L}} L = n-m$, it follows that $T_1 \cap H(m;\mathbf{n},\omega_0) \neq (0)$. However, $T_1 \cap H(m;\mathbf{n},\omega_0) \subseteq \mathscr{C}_{H(m;\mathbf{n},\omega_0)}(T)$, a contradiction. Thus the Lie algebra $H(m;\mathbf{n},\omega_0)$ does not admit a nonsingular derivation.

(b) $\omega = \omega_1$. Suppose first that $s+1 \not\equiv 0 \bmod p$. Then $H(m;\mathbf{n},\omega_1) = P(m;\mathbf{n},\omega_1)$. Since $\Phi_P = T^*_{F_p}$ in this case, $0 \in \Phi_P$ and $\mathscr{C}_L(T) \neq (0)$. As a result, there can be no nonsingular derivations in this case.

Suppose now that $s+1 \equiv 0 \bmod p$. Then according to our calculations in Section 1.2, $\operatorname{codim}_{\operatorname{Der} L} L \leq n-m+4$. Hence $\dim(T \cap L) \geq n-(n-m+4) = m-4 > 0$, since $m = 2s$ and $s = kp-1$, where $k > 0$ and $p > 3$. Thus, the algebra $L = H(m;\mathbf{n},\omega_1)$ with $s+1 \equiv 0 \bmod p$ has no nonsingular derivations.

(c) $\omega = \omega_2, (\alpha_{i,j}) \neq 0$, and $\det(\alpha_{i,j}) = 0$. In this case $\dim L = p^m - 2$ and

$$P(m;\mathbf{n},\omega_2) = \left(P(m;\mathbf{n},\omega_2)^{(1)} + \langle 1 \rangle\right) \oplus \langle f_\sigma \rangle,$$

where f_σ is some root vector with respect to the torus T. Since $\operatorname{Der} L = \bar{L} + \tilde{H}(m;\mathbf{n},\omega_2)$ by (1.6), $\{t,g\} \in P(m;\mathbf{n},\omega_2)^{(1)}$ for any $t \in T$, and $g \in P(m;\mathbf{n},\omega_2)$. Indeed,

$$t = \sum_i (adg_i)^{p^{k_i}} + \sum_j \beta_j adx_j^{(p^{n_j})}, \qquad \beta_j \in F,\ g_i \in P(m;\mathbf{n},\omega_2)^{(1)},$$

and $x = y + \gamma e$, where $e = \prod_{i=1}^m x_i^{(p^{n_i}-1)}$ and $y \in P(m;\mathbf{n},\omega_2)^{(1)} + \langle 1 \rangle$. Therefore, $\{t,x\} = \{t,y\} + \gamma\{t,e\}$. Now $\{t,y\} \in P(m;\mathbf{n},\omega_2)^{(1)}$ because $P(m;\mathbf{n},\omega_2)^{(1)}$ is a characteristic ideal in $P(m;\mathbf{n},\omega_2)$. Moreover, since $(adg_i)^{p^{k_i}}(e) \in P(m;\mathbf{n},\omega_2)^{(1)}$ and $adx_j^{(p^{n_j})}(e) = 0$, we have $\{t,e\} \in P(m;\mathbf{n},\omega_2)^{(1)}$. Thus, $\{t,f_\sigma\} = 0$ for any $t \in T$. This means that $\mathscr{C}_{P(m;\mathbf{n},\omega_2)}(T) = \langle 1, f_\sigma \rangle$, which is impossible since $\dim P(m;\mathbf{n},\omega_2)_\mu = 1$ for any $\mu \in T^*_{F_p}$. We have reached a contradiction, so the algebras $H(m;\mathbf{n},\omega_2)$ have no nonsingular derivations.

3.4. $L = K(m;\mathbf{n})$, $m = 2s+1$. We identify the space $K(m;\mathbf{n})$ with $A(m;\mathbf{n})$, where the Lie product in $K(m;\mathbf{n})$ is then given by

$$[f,g] = \partial_m g\,\Delta f - \partial_m f\,\Delta g - \sum_{i=1}^{s}(\partial_i f \partial_{i+s} g - \partial_{i+s} f \partial_i g), \tag{3.5}$$

where $\Delta f = 2f - \sum_{i=1}^{2s} x_i \partial_i f$. The usual associative product on $A(m;\mathbf{n})$ behaves as follows, relative to this Lie multiplication,

$$[f, gh] = g[f,h] + D_f(g)h. \tag{3.6}$$

This may be deduced from the theory of truncated coinduced modules (see [Ku2]) or it may be verified by direct calculation. As we have noted in Section 1.2, $\operatorname{Der} K(m;\mathbf{n})^{(1)} = \overline{K(m;\mathbf{n})}$, and so (3.6) is true for any $D \in \operatorname{Der} L$.

Now similar to the argument in Section 3.3, the set Φ_K of all roots of $K(m;\mathbf{n})$ with respect to T coincides with $T^*_{F_p}$, and $\dim K(m;\mathbf{n})_\mu = p^{n-k}$, where $k = \dim T$. Consequently, we may assume that $k = n$, as otherwise $\mathscr{C}_L(T) \neq (0)$.

Assume that $m + 3 \equiv 0 \bmod p$. Since by (1.8) $\operatorname{codim}_{\overline{K(m;\mathbf{n})}} K(m;\mathbf{n})^{(1)} \leq n - m - 1$, it follows that $\dim T \cap L \geq n - (n - m + 1) = m - 1 > 0$. Hence the algebra $K(m;\mathbf{n})$ does not admit a nonsingular derivation when $m + 3 \equiv 0 \bmod p$. When $m + 3 \not\equiv 0 \bmod p$, we have $\mathscr{C}_L(T) = \dim K(m;\mathbf{n})_0 = 1$, so this algebra does not possess nonsingular derivations either.

3.5 $L = \mathscr{S}(\mathbf{n})$, $\mathbf{n} = (n_1, n_2)$, *a Melikyan algebra*. We know from Section 1.2 that in this case $\operatorname{Der} L = \overline{\mathscr{S}_0} \oplus \mathscr{S}_1 \oplus \mathscr{S}_2$, where $\overline{\mathscr{S}_0} = \overline{W(2;\mathbf{n})}$. According to [Wa], $\overline{\mathscr{S}_0}$ has a torus of dimension $n = n_1 + n_2$. Therefore, by Lemma 2.1, we can assume that the torus T satisfies $\dim T \geq n$. Since $\operatorname{codim}_{\operatorname{Der} L} L = n - 2$, $\dim T \cap L \geq n - (n-2) \geq 2$. As a result, $\mathscr{C}_L(T) \neq (0)$, and $\mathscr{S}(\mathbf{n})$ has no nonsingular derivations.

We postpone the treatment of the Lie algebras $S(m;\mathbf{n},\omega_2)$ and $H(m;\mathbf{n},\omega_2)$ with $\det(\alpha_{i,j}) \neq 0$ until the next section.

4. Nonsingular Derivations of Simple Lie Algebras

In this final section we will exhibit a nonsingular derivation for each of the algebras $S(m;\mathbf{n},\omega_2)$ and will show for any nonsingular derivation D of $S(m;\mathbf{n},\omega_2)$ that the eigenspaces relative to D must have dimension $m-1$. This enables us to characterize the simple Lie algebras L with a

$\mathbb{Z}/(p^n - 1)\mathbb{Z}$-grading $L = \sum_i L_i$, $\dim L_i = 1$, which admit a nonsingular derivation D such that $D(L_i) \subseteq L_{i+1}$ as Lie algebras of type $H(m; \mathbf{n}, \omega_2)$ with $\det(\alpha_{i,j}) \neq 0$.

4.1 $L = S(m; \mathbf{n}, \omega_2)$. Since we may assume that $\omega_2 = (1 + e)\, dx_1 \wedge \cdots \wedge dx_m$, it follows that $S(m; \mathbf{n}, \omega_2) = (1 - e)S(m; \mathbf{n}, \omega_0)$. Then using the formula

$$(fD)^p = f^p D^p + \big((fD)^{p-1} f\big) D \qquad \text{for all } f \in A(m; \mathbf{n}),\ D \in \operatorname{Der} A(m; \mathbf{n})$$

(see, for instance, [Ku1]), we obtain

$$((1 + e)\partial_i)^p = \partial_i^p + x^{(\delta - (p-1)\varepsilon_i)}\partial_i,$$

where $\delta = (p^{n_i} - 1, \ldots, p^{n_m} - 1)$. Repeated applications of the well-known formula for p-powers (see [J2, p. 187]) gives

$$((1 + e)\partial_i)^{p^k} = \partial_i^{p^k} + x^{(\nu_{k,i})}\partial_i, \tag{4.2}$$

where $\nu_{k,i} = \delta - (p^k - 1)\varepsilon_i$. In particular,

$$((1 + e)\partial_i)^{p^{n_i}} = e_i\partial_i = \Big(\sum_{j \neq i} x_j^{(p^{n_j} - 1)}\Big)\partial_i. \tag{4.3}$$

For simplicity we denote $x_j^{(p^{n_j}-1)}$ by $\bar{x}_j$. Now we define

(4.4)

$$D_1 = (1 + e)\partial_1 + \bar{x}_1\partial_2 + \bar{x}_1\bar{x}_2\partial_3 + \cdots + \bar{x}_1 \cdots \bar{x}_{m-1}\partial_m,$$

$$D_i = (1 + e)\partial_i + \bar{x}_i\partial_{i+1} + \bar{x}_i\bar{x}_{i+1}\partial_{i+2} + \cdots + \bar{x}_i \cdots \bar{x}_{m-1}\partial_m$$
$$+ \bar{x}_i \cdots \bar{x}_{m-1}\bar{x}_m\partial_1 + \cdots + \bar{x}_i \cdots \bar{x}_{m-1}\bar{x}_m\bar{x}_1 \cdots \bar{x}_{i-2}\partial_{i-1},$$

for $i = 2, \ldots, m$.

LEMMA 4.5. (i) *For* $i = 1, \ldots, m - 1$, $D_i^{p^{n_i}} = D_{i+1}$, *and* $D_m^{p^{n_m}} = D_1$.

(ii) $T = \langle D_i^{p^{k_i}} \mid i = 1, \ldots, m$ *and* $0 \leq k_i \leq n_i - 1\rangle$ *is a torus in* $\operatorname{Der} L$ *of dimension* p^n, $n = n_1 + \cdots + n_m$.

Proof. (i) The derivations D_i are obtained from D_1 by permuting the subscripts. Thus, it suffices to prove that $D_1^{p^{n_1}} = D_2$. To accomplish this, consider $D_1' \in W(2; (n_1, n_2))$ defined by $D_1' = (1 + e)\partial_1 + x_1^{(p^{n_1}-1)}\partial_2$. Then successive applications of the formula for p-powers give

$$(D_1')^p = \partial_1^p + x_1^{(p^{n_1}-p)}\bar{x}_2\partial_1 + x_1^{(p^{n_1}-p)}\partial_2$$

$$\vdots$$

$$(D_1')^{p^{n_1-1}} = \partial_1^{p^{n_1-1}} + x_1^{(p^{n_1}-p^{n_1-1})}\bar{x}_2\partial_1 + x_1^{(p^{n_1}-p^{n_1-1})}\partial_2,$$

$$\begin{aligned}(D_1')^{p^{n_1}} &= \left(\partial_1^{p^{n_1-1}} + x_1^{(p^{n_1}-p^{n_1-1})}\bar{x}_2\partial_1\right)^p \\ &\quad - ad\left(\partial_1^{p^{n_1-1}} + x_1^{(p^{n_1}-p^{n_1-1})}\bar{x}_2\partial_1 + x_1^{(p^{n_1}-p^{n_1-1})}\partial_2\right)^{p-1}\left(x_1^{(p^{n_1}-p^{n_1-1})}\partial_2\right) \\ &= \bar{x}_2\partial_1 - \left[x_1^{(p^{n_1}-1)}\partial_2, \partial_1^{p^{n_1-1}} + x_1^{(p^{n_1}-p^{n_1-1})}\bar{x}_2\partial_1\right] \\ &= \bar{x}_2\partial_1 + \partial_2 + x_1^{(p^{n_1-1}-1)}x_1^{(p^{n_1}-p^{n_1-1})}\bar{x}_2\partial_2 \\ &= (1+e)\partial_2 + \bar{x}_2\partial_1.\end{aligned}$$

Let $\partial_2' = \partial_2 + \bar{x}_2\partial_3 + \cdots + \bar{x}_2 \cdots \bar{x}_{m-1}\partial_m$. The algebra $A' = A(m-1; (n_2, \ldots, n_m))$ in indeterminates $x_2, \ldots, x_m$ does not have any nontrivial ideals which are invariant under ∂_2'. By results in [Ku1], $A(m-1; (n_2, \ldots, n_m))$ may be identified with $A(1; n_2 + \cdots + n_m)$, relative to a new indeterminate x_2'. Moreover, $A'\partial_2'$ corresponds to $W(1; n_2 + \cdots + n_m)$ under this identification. Thus, we can write D_1 as

$$D_1 = (1+e)\partial_1 + \bar{x}_1\partial_2' \in W(2; (n_1, n_2 + \cdots + n_m)).$$

Putting this all together, we have

$$D_1^{p^{n_1}} = (1+e)\partial_2' + \bar{x}_2'\partial_1 = (1+e)\partial_2' + \bar{x}_2 \ldots \bar{x}_m\partial_1 = D_2,$$

as claimed.

(ii) It follows from part (i) that D_1 is a semisimple element in $\bar{S}(m; \mathbf{n}, \omega_2)$. Since T is the p-subalgebra generated by D_1, it follows that T is a torus in $W(m; \mathbf{n})$. The projection of T onto $\langle \partial_i^{k_i} | 0 \le k_i \le n_i - 1, i = 1, \ldots, m\rangle$ modulo $W(m; \mathbf{n})_{(0)}$ is surjective, so $\dim T \ge n$. However, $T \subseteq \operatorname{Der} A(m; \mathbf{n}) = W(n; \mathbf{1})$, which has maximal toral dimension n (see [D]). Consequently we have the desired conclusion $\dim T = n$. ∎

LEMMA 4.6. *D_1 (as in 4.4) is a nonsingular derivation of $S(m; \mathbf{n}, \omega_2)$ and $D_1^{p^n-1} = I$.*

Proof. If $[D_1, D] = 0$ for some $D \in \tilde{S}(m; \mathbf{n}, \omega_2)$, then $[D_i, D] = [D_1^{p^{n_1+\cdots+n_{i-1}}}, D] = 0$ for $i = 2, \ldots, m$. Since $\langle D_1, \ldots, D_m \rangle \oplus \tilde{S}(m; \mathbf{n}, \omega_2)_{(0)} = \tilde{S}(m; \mathbf{n}, \omega_2)$, we conclude that $D \notin \tilde{S}(m; \mathbf{n}, \omega_2)_{(0)}$, the 0-term of the standard filtration. Then

$$D = \alpha_1 D_1 + \cdots + \alpha_m D_m + D_0, \qquad \text{where } D_0 \in \tilde{S}(m; \mathbf{n}, \omega_2)_{(0)}, \alpha_i \in F.$$

Hence, $[D_i, D - \alpha_1 D_1 + \cdots + \alpha_m D_m] = [D_i, D_0] = 0$ for all $i = 1, \ldots, m$. Therefore (see (4.4)), $D_0 = 0$ and $D = \alpha_1 D_1 + \cdots + \alpha_m D_m$. It remains to observe that $\langle D_1, \ldots, D_m \rangle \cap S(m; \mathbf{n}, \omega_2) = (0)$. Since $D_1^{p^n} = D_1$ and D_1 is nonsingular, $D_1^{p^n - 1} = I$. ∎

LEMMA 4.7. *Let D be a nonsingular derivation of $L = S(m; \mathbf{n}, \omega_2)$. Then* $\dim L_\alpha = m - 1$ *for any eigenvalue α of D.*

Proof. Let T be the torus of $\operatorname{Der} L \subseteq \overline{W(m; \mathbf{n})}$ generated by the semisimple part D_s of D. Note that D_s is also a nonsingular derivation. From what has been proved in Section 3.2, it follows that $\dim T = n = n_1 + \cdots + n_m$. Now the mapping

$$\lambda_2 \colon W(m; \mathbf{n}) \to A(m; \mathbf{n}), \qquad \lambda_2(D) = -De + \operatorname{div} D$$

is a morphism of T-modules and $\operatorname{Ker} \lambda_2 = \tilde{S}(m; \mathbf{n}, \omega_2)$. According to Lemma 2.2, $\dim W(m; \mathbf{n})_\mu = m$ for any $\mu \in T_{F_p}^*$ and $\dim A(m; \mathbf{n})_\mu = 1$. If $\lambda_2(D) = c \in F$, then $D \in C\tilde{S}(m; \mathbf{n}, \omega_2)$. However, $C\tilde{S}(m; \mathbf{n}, \omega_2) = \tilde{S}(m; \mathbf{n}, \omega_2)$ forces $D \in \tilde{S}(m; \mathbf{n}, \omega_2)$ and $c = 0$. Thus, $(\operatorname{Im} \lambda_2)_0 = (0)$, and

$$\operatorname{Im} \lambda_2 = \sum_{\mu \neq 0} A(m; \mathbf{n})_\mu, \qquad \mu \in T_{F_p}^*,$$

which implies that $\dim S(m; \mathbf{n}, \omega_2)_\mu = \dim W(m; \mathbf{n})_\mu - \dim A(m; \mathbf{n})_\mu = m - 1$ for all $\mu \neq 0$. Since $\operatorname{codim}_{\tilde{S}(m; \mathbf{n}, \omega_2)} S(m; \mathbf{n}, \omega_2) = n - m$, it follows that $\dim T \cap \tilde{S}(m; \mathbf{n}, \omega_2) = m$.

Let $W(m; \mathbf{n})_0$ denote the zero weight space in $W(m; \mathbf{n})$ with respect to T. Since $T \cap \tilde{S}(m; \mathbf{n}, \omega_2) \subseteq W(m; \mathbf{n})_0$ and $\dim W(m; \mathbf{n})_0 = m$, it must be that $T \cap \tilde{S}(m; \mathbf{n}, \omega_2) = W(m; \mathbf{n})_0$. We know that $T \cap S(m; \mathbf{n}, \omega_2) = (0)$ and $\operatorname{codim}_{\tilde{S}(m; \mathbf{n}, \omega_2)} S(m; \mathbf{n}, \omega_2) = n - m$. Thus, $S(m; \mathbf{n}, \omega_2)_\mu = \tilde{S}(m; \mathbf{n}, \omega_2)_\mu$ for any $\mu \neq 0$ in $T_{F_p}^*$. Since T is generated by D_s, we have for α an eigenvalue of D_s that $S(m; \mathbf{n}, \omega_2)_\alpha = S(m; \mathbf{n}, \omega_2)_\mu$ for some $\mu \neq 0$ in $T_{F_p}^*$. ∎

Using the decomposition of $S(m; \mathbf{n}, \omega_2)$ with respect to the torus T generated by D_1 (as in (4.4)) and taking into account that $W(m; \mathbf{n})_\mu = A(m; \mathbf{n})_\mu T_1$, where $T_1 = \langle D_1, \ldots, D_m \rangle$, we can write

$$S(m; \mathbf{n}, \omega_2) = \sum_{\nu \neq 0} A(m; \mathbf{n})_\nu T_{1, \nu},$$

where ν runs over a subgroup Λ of T_1^* consisting of the restrictions

$\nu = \mu|_{T_1}$, $\mu \in T_{F_p}^*$, and where $T_{1,\nu} = \operatorname{Ker} \nu$. We can identify the elements of $A(m; \mathbf{n})_\nu T_{1,\nu}$ with pairs (ν, t_ν), where $t_\nu \in \operatorname{Ker} \nu$. Then

$$[(\nu, t_\nu), (\lambda, t_\lambda)] = (\nu + \lambda, \nu(t_\lambda)t_\nu - \lambda(t_\nu)t_\lambda).$$

This is the basis and multiplication of the algebra $S(m; \mathbf{n}, \omega_2)$ presented in [SW].

4.2 $L = H(m; \mathbf{n}, \omega_2)$, $\det(\alpha_{i,j}) \neq 0$. To study nonsingular derivations for these algebras, we first consider the Lie algebras of V. M. Galkin (private communication), which are the simplest examples of Block algebras [B]. The Block algebras of dimension $p^n - 1$ are of type $H(m; \mathbf{n}, \omega_2)$ with $\det(\alpha_{i,j}) \neq 0$.

Let $l = q^{2m}$, where $q = p^n$. Then $GF(q) \subseteq GF(l)$, and $[GF(l): GF(q)] = 2m$. Let $f(\ ,\ )$ denote a $GF(q)$-bilinear form on $GF(l)$ with values in F, which is nondegenerate and skew-symmetric. The *Galkin algebra* G has basis $\{T_\alpha | \alpha \in GF(l),\ \alpha \neq 0\}$ such that

$$[T_\alpha, T_\beta] = f(\alpha, \beta) T_{\alpha+\beta}. \tag{4.8}$$

It is not difficult to check that G is a simple Lie algebra, and $DT_\alpha = \alpha T_\alpha$ is a nonsingular derivation. Clearly, $D^{p^{2mn}-1} = I$, and $\dim G_\alpha = 1$ for each eigenspace of D.

Remark. Galkin considered more general Lie algebras $G = \langle T_v | v \in V \rangle$ with product (4.5), where V is a vector space over some field F of arbitrary characteristic.

We conclude by proving the following result which characterizes Lie algebras L with a $\mathbb{Z}/(p^n - 1)\mathbb{Z}$-grading, $L = \sum_i L_i$, $\dim L_i = 1$, which admit a nonsingular derivation D such that $D(L_i) \subseteq L_{i+1}$ for each i. Shalev had constructed examples of such algebras in characteristics 3 and 5 and verified that they are Lie by computer calculations.

THEOREM 4.9. *Suppose that $p > 7$ and that L is a simple Lie algebra with a $\mathbb{Z}/(p^n - 1)\mathbb{Z}$-grading, $L = \sum_i L_i$, $\dim L_i = 1$, which admits a nonsingular derivation D with $D(L_i) \subseteq L_{i+1}$ for each i. Then L is of Cartan type $H(m; \mathbf{n}, \omega_2)$ with $\det(\alpha_{i,j}) \neq 0$, and L admits a periodic derivation $D' = \lambda^{-1} D$ for some $\lambda \in F$.*

Proof. Let L be a Lie algebra L with a $\mathbb{Z}/(p^n - 1)\mathbb{Z}$-grading, $L = \sum_i L_i$ such that $D(L_i) \subseteq L_{i+1}$, and $\dim L_i = 1$. Then L has dimension $p^n - 1$ and $D^{p^n-1} = \xi I$ for some $\xi \in F$. Take λ such that $\lambda^{p^n-1} = \xi$. The derivation $D' = \lambda^{-1} D$ is periodic, $(D')^{p^n-1} = I$, and $\dim L_\alpha = 1$ for any eigenvalue α of D'. Moreover, L must be of type $H(m; \mathbf{n}, \omega_2)$ with $\det(\alpha_{i,j}) \neq 0$ by the results of Section 3 and Lemma 4.3. ∎

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