Here ω is in $\mathrm{Hom}_{\mathbb{R}}(\Lambda^n \mathfrak{g}, C^\infty(G))$, and $d\omega$ is in $\mathrm{Hom}_{\mathbb{R}}(\Lambda^{n+1}\mathfrak{g}, C^\infty(G))$. In §3 we shall take this formula as a definition of a d operator in a more general context, at the same time changing the field from $\mathbb{R}$ to $\mathbb{C}$. For formal reasons we shall have $d^2 = 0$, and thus we can compute cohomology as ker d/image d, the quotient of the cocycles by the coboundaries.

2. Motivation from extensions

Let us briefly see how the cases $n = 1$ and $n = 2$ of (4.10) arise in considering Lie algebra extensions. Let $\mathfrak{g}$ and $\mathfrak{a}$ be finite-dimensional complex Lie algebras. A complex Lie algebra $\mathfrak{h}$ is an extension of $\mathfrak{g}$ by $\mathfrak{a}$ if there is an exact sequence of Lie algebras

$$0 \longrightarrow \mathfrak{a} \xrightarrow{\iota} \mathfrak{h} \xrightarrow{\rho} \mathfrak{g} \longrightarrow 0 . \tag{4.14}$$

(Recall that this terminology means that the maps are Lie algebra homomorphisms and that for each Lie algebra the kernel of the map going out equals the image of the map coming in. In particular, the image of $\mathfrak{a}$ must be an ideal in $\mathfrak{h}$ in order to be the kernel of ρ.)

We assume that $\mathfrak{a}$ is abelian (all brackets 0). In this case, (4.14) makes $\mathfrak{a}$ into a representation space for $\mathfrak{g}$ under the definition

$$\pi(X)Y = \iota^{-1}[\rho^{-1}(X), \iota(Y)] \quad \text{for } X \in \mathfrak{g}, \ Y \in \mathfrak{a} ; \tag{4.15}$$

here $\rho^{-1}(X)$ denotes any inverse image of X in $\mathfrak{h}$, and

ι^{-1} is defined on $[\rho^{-1}(X), \iota(Y)]$ since $\iota(\mathfrak{a})$ is an ideal in $\mathfrak{h}$. Fix the extension (4.14) and fix also a linear map $\tau : \mathfrak{g} \to \mathfrak{h}$ such that $\rho\tau = 1$. Then we can attach to this situation a map $\omega \in \mathrm{Hom}_{\mathbb{C}}(\Lambda^2\mathfrak{g}, \mathfrak{a})$ by the definition

$$\omega(X,Y) = \iota^{-1}([\tau(X), \tau(Y)] - \tau[X,Y]) . \tag{4.16}$$

(To see that the expression in parentheses is in $\iota\mathfrak{a}$, we apply the Lie algebra homomorphism ρ to it and check that we get 0; exactness of (4.14) then says that the expression is in $\mathfrak{a}$.)

Using (4.13), let us compute $d\omega$. First note that $\tau(X)$ for X in $\mathfrak{g}$ is a valid choice of $\rho^{-1}(X)$ in (4.15). Thus (4.16) gives

$$\begin{aligned}
&d\omega(Y_1, Y_2, Y_3) \\
&\quad = \pi(Y_1)(\omega(Y_2,Y_3)) - \pi(Y_2)(\omega(Y_1,Y_3)) + \pi(Y_3)(\omega(Y_1,Y_2)) \\
&\qquad - \omega([Y_1,Y_2],Y_3) + \omega([Y_1,Y_3],Y_2) - \omega([Y_2,Y_3],Y_1) \\
&\quad = \iota^{-1}[\tau(Y_1),[\tau(Y_2),\tau(Y_3)]] - \iota^{-1}[\tau(Y_2),[\tau(Y_1),\tau(Y_3)]] \\
&\qquad + \iota^{-1}[\tau(Y_3),[\tau(Y_1),\tau(Y_2)]] - \iota^{-1}[\tau(Y_1),\tau[Y_2,Y_3]] \\
&\qquad + \iota^{-1}[\tau(Y_2),\tau[Y_1,Y_3]] - \iota^{-1}[\tau(Y_3),\tau[Y_1,Y_2]] \\
&\qquad - \iota^{-1}[\tau[Y_1,Y_2],\tau(Y_3)] + \iota^{-1}[\tau[Y_1,Y_3],\tau(Y_2)] \\
&\qquad - \iota^{-1}[\tau[Y_2,Y_3],\tau(Y_1)] + \iota^{-1}\tau[[Y_1,Y_2],Y_3] \\
&\qquad - \iota^{-1}\tau[[Y_1,Y_3],Y_2] + \iota^{-1}\tau[[Y_2,Y_3],Y_1] .
\end{aligned}$$

On the right side, the first three terms add to 0 by the Jacobi identity in $\mathfrak{h}$, the next six terms cancel in pairs, and the last three terms add to 0 by the Jacobi identity in $\mathfrak{g}$. Therefore $d\omega = 0$, i.e., ω is a cocycle.

Suppose we change τ to τ'. Then $\alpha = \tau' - \tau$ has $\rho_\alpha = 0$, so that α maps $\mathfrak{g}$ into $\iota\mathfrak{a}$ and $\iota^{-1}\alpha$ is a member of $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{g},\mathfrak{a}) = \mathrm{Hom}_{\mathbb{C}}(\Lambda^1\mathfrak{g},\mathfrak{a})$. Using (4.13), we compute $d(\iota^{-1}\alpha)$ as

$$d(\iota^{-1}\alpha)(X,Y) = \pi(X)(\iota^{-1}\alpha(Y)) - \pi(Y)(\iota^{-1}\alpha(X)) - \iota^{-1}\alpha[X,Y]. \quad (4.17)$$

On the other hand, the respective ω's for τ and τ' have

$$\begin{aligned}
&\omega_{\tau'}(X,Y) - \omega_\tau(X,Y) \\
&\quad = \iota^{-1}([\tau'(X),\tau'(Y)] - \tau'[X,Y]) - \iota^{-1}([\tau(X),\tau(Y)] - \tau[X,Y]) \\
&\quad = \iota^{-1}([\alpha(X),\tau(Y)] + [\tau(X),\alpha(Y)] + 0 - \alpha[X,Y]) \\
&\quad = -\pi(Y)(\iota^{-1}\alpha(X)) + \pi(X)(\iota^{-1}\alpha(Y)) - \iota^{-1}\alpha[X,Y] && \text{by (4.15)} \\
&\quad = d(\iota^{-1}\alpha)(X,Y) && \text{by (4.17).}
\end{aligned}$$

Thus $\omega_{\tau'}$ and ω_τ differ by a coboundary.

If we take into account the expected result $d^2 = 0$, then (4.14) leads to a well defined element of the quotient of $\ker d\big|_{\mathrm{Hom}(\Lambda^2\mathfrak{g},\mathfrak{a})}$ by image $d\big|_{\mathrm{Hom}(\Lambda^1\mathfrak{g},\mathfrak{a})}$, and we shall soon call this quotient $H^2(\mathfrak{g},\mathfrak{a})$.

We can go backwards as well. Let us say that two extensions are <u>equivalent</u> if there exists a Lie algebra isomorphism σ such that

$$\begin{tikzcd} & & \mathfrak{h} \arrow[dd, "\sigma"] \arrow[dr, "\rho"] & & \\ 0 \arrow[r] & \mathfrak{a} \arrow[ur, "\iota"] \arrow[dr, "\iota'"'] & & \mathfrak{g} \arrow[r] & 0 \\ & & \mathfrak{h}' \arrow[ur, "\rho'"'] & & \end{tikzcd}$$

commutes. It is a simple matter to see that equivalent extensions lead to the same member of $H^2(\mathfrak{g},\mathfrak{a})$.

Equivalent extensions necessarily lead to the same π in (4.15). If we fix π, then the claim is that the map of equivalence classes of extensions for that π to $H^2(\mathfrak{g},\mathfrak{a})$ is one-one onto. In fact, if we are given a cocycle ω in $\ker d\big|_{\mathrm{Hom}(\Lambda^2\mathfrak{g},\mathfrak{a})}$, we introduce $\mathfrak{h}$ as a vector space. We define brackets within the subspace $\mathfrak{a}$ to be 0 and brackets between $\mathfrak{g}$ and $\mathfrak{a}$ by π. To define brackets within the subspace $\mathfrak{g}$, let τ be the injection of $\mathfrak{g}$ into $\mathfrak{h}$, and set

$$[\tau(X),\tau(Y)] = \tau[X,Y] + \iota(\omega(X,Y)),$$

where ι is the injection of $\mathfrak{a}$ into $\mathfrak{h}$. This construction allows us to form a two-sided inverse to our mapping. Hence the equivalence classes of extensions of $\mathfrak{g}$ by $\mathfrak{a}$ giving the same π are classified by $H^2(\mathfrak{g},\mathfrak{a})$.

In this context, $H^1(\mathfrak{g},\mathfrak{a})$ has an interesting interpretation, too. The starting point is the 0 element of $H^2(\mathfrak{g},\mathfrak{a})$. A representative cocycle is $\omega = 0$, and our formulas then say that the injection τ of $\mathfrak{g}$ into $\mathfrak{h}$ is a Lie algebra homomorphism. Since τ is a Lie algebra homomorphism, let us write $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{a}$ as a vector space and require that $\mathfrak{g}$ be a Lie subalgebra, $\mathfrak{a}$ be an ideal, and $\mathfrak{g}$ bracket with $\mathfrak{a}$ through π as in (4.15): $[X,Y] = \pi(X)Y$ for

$X \in \mathfrak{g}$ and $Y \in \mathfrak{a}$. We write $\mathfrak{h} = \mathfrak{g} \oplus_\pi \mathfrak{a}$ and call $\mathfrak{h}$ the <u>semidirect product</u> of $\mathfrak{g}$ and $\mathfrak{a}$ with respect to π.

Let us fix our attention on this semidirect product. Our map τ has become the inclusion $\tau : \mathfrak{g} \to \mathfrak{h}$ into the first coordinate. Let us ask about other inclusions of $\mathfrak{g}$ into $\mathfrak{h}$ that project back to the identity on $\mathfrak{g}$ and respect the bracket structure of $\mathfrak{g}$ and $\mathfrak{h}$. Such a linear map τ' must be of the form

$$\tau'(Y) = (Y, \omega(Y)), \qquad Y \in \mathfrak{g},$$

where ω is a linear function from $\mathfrak{g}$ to $\mathfrak{a}$, i.e., a member of $\mathrm{Hom}_{\mathbb{C}}(\Lambda^1\mathfrak{g}, \mathfrak{a})$. If τ' respects brackets, then

$$\tau'[Y_1, Y_2] = ([Y_1, Y_2], \omega[Y_1, Y_2])$$

is equal to

$$\begin{aligned} [\tau'(Y_1), \tau'(Y_2)] &= [(Y_1, \omega(Y_1)), (Y_2, \omega(Y_2))] \\ &= ([Y_1, Y_2], \pi(Y_1)\omega(Y_2) - \pi(Y_2)\omega(Y_1)). \end{aligned}$$

That is, ω must satisfy

$$\omega[Y_1, Y_2] = \pi(Y_1)\omega(Y_2) - \pi(Y_2)\omega(Y_1).$$

This condition is summarized by saying that ω is a <u>derivation</u> of $\mathfrak{g}$ into $\mathfrak{a}$. Referring to (4.13), we see that

$$d\omega(Y_1, Y_2) = \pi(Y_1)(\omega(Y_2)) - \pi(Y_2)(\omega(Y_1)) - \omega[Y_1, Y_2].$$

Hence τ' is a <u>Lie algebra splitting</u> (i.e., is as above and respects brackets) if and only if the corresponding ω is a cocycle.

There is a trivial kind of derivation, namely $\omega(Y) = \pi(Y)X$ for some X in $\mathfrak{a}$. Such derivations are called inner and are easily seen to be the coboundaries. Thus $H^1(\mathfrak{g},\mathfrak{a})$ classifies the Lie algebra splittings modulo those that are given in terms of inner derivations.

The identifications of H^1 and H^2 above have consequences for the structure of Lie algebras. If $\mathfrak{g}$ has a property called "semisimplicity," it is not too hard to see that $H^2(\mathfrak{g},\mathfrak{a}) = 0$. With a little supplementary argument, it follows that any finite-dimensional complex Lie algebra is a semidirect product of a semisimple Lie algebra and a solvable Lie algebra (the Levi decomposition). The same theorem and proof apply over $\mathbb{R}$ rather than $\mathbb{C}$, and a structure theorem for Lie groups drops out. Results for H^1 then furnish a uniqueness theorem.

3. Definition and examples

Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra, and let V be a $U(\mathfrak{g})$ module, possibly infinite-dimensional. (Recall that equivalently V is the representation space for a representation of $\mathfrak{g}$.) In §1, V was $C^\infty(G)$ (except that we were working with $\mathbb{R}$ as a field), and in §2, V was the abelian Lie algebra $\mathfrak{a}$. The vector space $C^n(\mathfrak{g},V)$ of n cochains is defined by

$$C^n(\mathfrak{g},V) = \mathrm{Hom}_{\mathbb{C}}(\Lambda^n\mathfrak{g},V) ,$$