Lie Algebra Extensions

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1 Definitions

In this section we define Lie algebra extensions e some cohomology groups.

Definition 1. Let K,H and L be Lie algebras. L is an extension of K by H if there is a exact sequence of Lie algebras,

$$0 \to H \xrightarrow{i} L \xrightarrow{s} K \to 0.$$

- if there is an ideal S of L such that $L = S \oplus Ker(s)$ then the extension is **trivial**;
- if there is a subalgebra S of L such that $L = S \oplus Ker(s)$ then the extension is **split**;
- if $ker(s)$ is contained in the center of L, denoted by $Z(L)$, then L is a **central** extension.

Definition 2. Let K and I be algebras. We say that K act on I if there is a algebra morphism $\psi : K \to Der(I)$. In this case, the action will be denoted by

$$[a, k] := \psi(k)(a), k \in K, a \in I.$$

When brackets represents the adjoint representation we also use $ad^K : K \to Der(I)$ to $ad^K_k(a) = [a, k]$, for all $k \in K$ and $a \in I$. When the domain of representation is clear we just use $ad_k(a) = [a, k]$.

Example 1. Let L be a Lie algebra with an abelian ideal I and $K = L/I$. Let $ad^L : L \to L$ be the adjoint representation of L. Define the Lie algebra representation $ad^K : K \to Der(I)$ by $ad^K_{x+I}(a) = [a, x] = ad^L_x(a)$ for all $x \in L$ and $a \in I$. This is well defined because I is abelian. Then K acts on I. In this case, we say that the action is induced by adjoint representation.

Definition 3. Let K be an algebra and I a K-module. Denote by $C^2(K, I)$ the vector space of alternating bilinear maps $\theta : K \times K \to I$.

- If $\theta \in C^2(K, I)$ has the property

  $$\theta(x, [y, z]) + \theta(y, [z, x]) + \theta(z, [x, y]) = [\theta(y, z), x] + [\theta(z, x), y] + [\theta(x, y), z],$$

  for all $x, y, z \in K$, then θ are said **cocycle** and the vector space of coclycles is denoted by $Z^2(K, I)$;

- A cocycle θ are said a **coboundary** if there is a linear map $T : K \to I$ such that

$$\theta(k, h) = T([h, k]) + [T(h), k] - [T(k), h],$$

for all $k, h \in K$. The set of coboundaries are denoted by $B^2(K, I)$.

- Let $H^2(K, I) = Z^2(K, I)/B^2(K, I)$ be the quotient space of cocycles by coboundaries.

- The first cohomology group of K and I is definied by

$$Z^1(K, I) = \{\nu \in Hom(K, I) \mid \nu([k, h]_K) = [\nu(k), h] - [\nu(h), k], \text{ for all } k, h \in K\}.$$

Next we present some results of Lie algebra extensions and cohomology. Their proofs can be seen, for example, in [2] section 2 of chapter 4.

Proposition 1. *Let K be a Lie algebra and I a K-module. Let $\theta \in Z^2(K, I)$ and $\nu \in B^2(K, I)$. Define the algebra $K_\theta = K \oplus I$ with product*

$$[x + a, y + b]_\theta = [x, y]_K + \theta(x, y) + [a, y] - [b, x], \text{ para } x, y \in K \text{ e } a, b \in I. \tag{1}$$

Then,

1. *K_θ is a Lie algebra extension of K by I;*
2. *K_θ is isomorphic to $K_{\theta+\nu}$;*
3. *K_ν is a split extension of K by I.*

Proposition 2. *Let L be a Lie algebra and I an abelian ideal of L. If $K = L/I$ then there is $\theta \in Z^2(K, I)$ such that $L \cong K_\theta$.*

2 Compatible Pairs

Definition 4. Let K and I be Lie algebras such that K act on I. Define the set $Comp(K, I)$, of the **compatible pairs**, as the elements $(\alpha, \beta) \in Der(K) \oplus Der(I)$ with the property

$$\beta([a, k]) = [\beta(a), k] + [a, \alpha(k)], \text{ for all } k \in K, a \in I. \tag{2}$$

We can write equation (2) in other forms. Let $\psi : K \to Der(I)$ the representation that defines the action of K on I. Then $\psi(k)(a) = [a, k]$ for all $k \in K$ and $a \in I$. So $(\alpha, \beta) \in Comp(K, I)$ means

$$\beta\psi(k) = \psi(k)\beta + \psi(\alpha(k)), \text{ for all } k \in K.$$

Using commutator, this is equivalent to

$$[\psi(k), \beta] = -\psi(\alpha(k)), \text{ for all } k \in K. \tag{3}$$

Let $ad : Der(I) \to Der(I)$ be the adjoint representation. Then (3) can be rewrite as

$$ad_\beta\psi(k) = -\psi(\alpha(k)), \text{ for all } k \in K.$$

Therefore, $(\alpha,\beta)\in Comp(K,I)$ if the follow diagram commutes

$$\begin{tikzcd} K \arrow[r, "\psi"] \arrow[d, "-\alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & Der(I) \arrow[d, "ad_\beta"] \\ K \arrow[r, "\psi"] & Der(I). \end{tikzcd}$$

Proposition 3. *Let K and I be Lie algebras such that K act on I. Then $Comp(K,I)$ is a subalgebra of $Der(K)\oplus Der(I)$.*

Proof: Let $(\alpha,\beta),(\alpha',\beta')\in Comp(K,I)$ and suppose that the action is given by representation $\psi: K\to Der(I)$ such that $\psi(k)(a)=[a,k]$, for all $k\in K$ and $a\in I$.

First we check that $Comp(K,I)$ is a vector subspace using equation (3). If $\lambda\in\mathbb{F}$ and $k\in K$ then

$$\begin{array}{rcl} [\psi(k),\beta+\lambda\beta'] & = & [\psi(k),\beta]+\lambda[\psi(k),\beta'] \\ & = & -\psi(k)(\alpha)-\lambda\psi(k)(\alpha') \\ & = & \psi(k)(\alpha+\lambda\alpha'). \end{array}$$

So $(\alpha,\beta)+\lambda(\alpha',\beta')\in Comp(K,I)$.

Using compatible pair definition we have

$$\beta'\psi(k)=\psi(k)\beta'+\psi(\alpha'(k)).$$

Then

$$\begin{array}{rcl} \beta\beta'\psi(k) & = & \beta\psi(k)\beta'+\beta\psi(\alpha'(k)) \\ & = & \psi(k)\beta\beta'+\psi(\alpha(k))\beta'+\psi(\alpha'(k))\beta+\psi(\alpha'\alpha(k)) \end{array}$$

In the same way

$$\beta'\beta\psi(k)=\psi(k)\beta'\beta+\psi(\alpha'(k))\beta+\psi(\alpha(k))\beta'+\psi(\alpha\alpha'(k)).$$

Then

$$[\beta,\beta']\psi(k)=\psi(k)(\beta\beta'-\beta'\beta)+\psi((\alpha\alpha'-\alpha'\alpha)(k))=\psi(k)[\beta,\beta']+\psi([\alpha,\alpha'](k)).$$

Hence $[(\alpha,\beta),(\alpha',\beta')]\in Comp(K,I)$.

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Proposition 4. *Let K and I be Lie algebras such that K act on I. Let L be the semidirect sum $L=K\oplus_\psi I$. For each $(\alpha,\beta)\in Comp(K,I)$ define $(\alpha,\beta): L\to L$ by $(\alpha,\beta)(k,a)=\alpha(k)+\beta(a)$ for all $k\in K$ and $a\in I$. Then $(\alpha,\beta)\in Der(L)$.*

Proof: Let $a,a'\in I$ and $k,k'\in K$. Then

$$\begin{array}{rcl} (\alpha,\beta)[k+a,k'+a'] & = & (\alpha,\beta)([a,a']_I+[a,k']-[a',k]+[k,k']) \\ & = & \alpha([k,k'])+\beta([a,a']+[a,k']-[a',k]) \\ & = & [\alpha(k),k']+[k,\alpha(k')]+[\beta(a),a']+[a,\beta(a')] \\ & + & [\beta(a),k']+[a,\alpha(k')]-[\beta(a'),k]-[a',\alpha(k)] \\ & = & [(\alpha,\beta)(k+a),k'+a']+[k+a,(\alpha,\beta)(a')] \end{array}$$

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Definition 5. Let K and I be vector spaces. Let $(\alpha, \beta) \in \mathfrak{gl}(K) \oplus \mathfrak{gl}(I)$ and $T \in Hom(K, \mathfrak{gl}(I))$. Let $ad : Der(I) \to Der(I)$ be the adjoint representation of I. We define the action of $\mathfrak{gl}(K) \oplus \mathfrak{gl}(I)$ on $Hom(K, \mathfrak{gl}(I)))$ by

$$(\alpha, \beta) \cdot T = ad_\beta T + T\alpha. \tag{4}$$

To proof this is an action observe that $(\alpha, \beta) \cdot T$ is a linear map because is linear combination of composition and sums of linear maps. Let's check that it preserves Lie brackets.

Let $(\alpha, \beta), (\alpha', \beta') \in \mathfrak{gl}(K) \oplus \mathfrak{gl}(I)$ and $k \in K$. By definition

$$(\alpha', \beta') \cdot T = ad_{\beta'} T + T\alpha'.$$

So

$$(\alpha, \beta) \cdot ((\alpha', \beta') \cdot T) = ad_\beta ad_{\beta'} T + ad_{\beta'} T\alpha + ad_\beta T\alpha' + T\alpha'\alpha.$$

In the same way,

$$(\alpha', \beta') \cdot ((\alpha, \beta) \cdot T) = ad_{\beta'} ad_\beta T + ad_\beta T\alpha' + ad_{\beta'} T\alpha + T\alpha\alpha'.$$

Hence,

$$\begin{array}{rcl} (\alpha, \beta) \cdot ((\alpha', \beta') \cdot T) - (\alpha', \beta') \cdot ((\alpha, \beta) \cdot T) & = & ad_\beta ad_{\beta'} T - ad_{\beta'} ad_\beta T + T\alpha\alpha' - T\alpha'\alpha \\ & = & [ad_\beta, ad_{\beta'}]T + T[\alpha, \alpha']. \end{array}$$

Therefore,

$$[(\alpha, \beta), (\alpha', \beta')] \cdot T = ([\alpha, \alpha'], [\beta, \beta']) \cdot T.$$

A particular case of this action is when $Der(K) \oplus Der(I)$ act on Lie algebra representation of K on I. This action is well defined: if $\psi : K \to Der(I)$ and $k \in K$ then $ad_\beta T(k) + T\alpha(k)$ is a derivation of I because $ad_\beta T(k), T\alpha(k) \in Der(I)$. If we calculate the annihilator we get:

$$\begin{array}{rcll} Ann(\psi) & = & \{(\alpha, \beta) \in Der(K) \oplus Der(I) \mid (\alpha, \beta) \cdot \psi = 0\} & \\ & = & \{(\alpha, \beta) \in Der(K) \oplus Der(I) \mid ad_\beta \psi + \psi\alpha = 0\} & \\ & = & Comp(K, I), & \text{by}(3). \end{array}$$

We just proof the follow proposition:

Proposition 5. *Let K and I be Lie algebras such that K act on I by representation $\psi : K \to Der(I)$. If $Der(K) \oplus Der(I)$ act on $Hom(K, Der(I))$ as in (4) then $Comp(K, I) = Ann(\psi)$.*

3 Nilpotent Subalgebras of Compatible Pairs

Proposition 6. *Let V be a finite dimensional vector space over a algebraically closed field $\mathbb{F}$ and $D \subseteq \mathfrak{gl}(V)$ a nilpotent linear algebra. Then V has a unique decomposition $V = V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_n}$ into D-modules such that*

$$V_{\lambda_i} = \{v \in V \mid \text{ for all } d \in D \text{ there is an } m > 0 \text{ such that } (d - \lambda(k))^m v = 0\},$$

where $\lambda_i : D \to \mathbb{F}, 1 \leq i \leq n$. The space V_{λ_i} is called a generalized eigenspace of D with eigenvalue λ_i.

Proof: A proof of this fact can be found in chapter 3 of [1], for example.

Let K and I be Lie algebras such that K act on I. If $D \subseteq Comp(K,I)$ is nilpotent subalgebra then by **Proposition 4**, D can be seen as subalgebra of derivations of semidirect sum $L = K \oplus I$. The decomposition of L in eigenspaces of D induces decompositions in K and I, because as subspaces of L they are invariants by D. So each nilpotent subalgebra of $Comp(K,I)$ induces unique decompositions in generalized eigenspaces of in K and I.

Proposition 7. *Let K and I be Lie algebras such that K act on I. If $(\alpha,\beta) \in Comp(K,I)$ then*

$$(\beta-(\lambda+\mu))^n[a,k] = \sum_{i=0}^{n} \binom{n}{i} [(\beta-\lambda)^{n-i}(a), (\alpha-\mu)^i(k)] \text{ for all } a \in I, k \in K \text{ and } \lambda, \mu \in \mathbb{F} \quad (5)$$

Proof: Suppose that $\mathbb{F}$ is the base field. We will proof this result by induction on n. If $n = 1$ the result follow by compatible pair definition. Suppose that the result is valid for $n > 0$. then

$$\begin{aligned}(\beta - (\lambda + \mu))^{n+1}[a,k] &= (\beta - (\lambda + \mu)) \sum_{i=0}^{n} \binom{n}{i} [(\beta - \lambda)^{n-i}(a), (\alpha - \mu)^i(k)] \\ &= \sum_{i=0}^{n} \binom{n}{i} \big([(\beta - \lambda)^{n+1-i}(a), (\alpha - \mu)^i(k)] + [(\beta - \lambda)^{n+1-i}(a), (\alpha - \mu)^{i+1}(k)]\big) \\ &= \sum_{i=0}^{n+1} \binom{n+1}{i} [(\beta - \lambda)^{n+1-i}(a), \text{ for all } a \in I, k \in K \text{ and } \lambda, \mu \in \mathbb{F} \quad (6)\end{aligned}$$

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Proposition 8. *Let K and I be Lie algebras over an algebraically closed field such that K act on I. Let D be a nilpotent subalgebra of $Comp(K,I)$. If $\lambda, \mu : D \to \mathbb{F}$ are generalized eigenvalues of D, respectively, then $[I_\mu, K_\lambda] \subseteq I_{\lambda+\mu}$ if $\lambda + \mu$ is a generalized eigenvalue of D. Otherwise $[I_\mu, K_\lambda] = 0$.*

Proof: Let $a \in I_\mu$, $k \in K_\lambda$ and $d \in D$ then by **Proposition 3**

$$(\beta - (\lambda(d) + \mu(d))I)^n[a,k] = \sum_{i=0}^{n} \binom{n}{i} [(\beta - \lambda(d)I)^{n-i}(a), (\alpha - \mu(d)I)^i(k)].$$

And for n big enough this is 0. So, $[I_{\mu_i}, K_{\lambda_j}] \subset I_{\mu_i+\lambda_j}$ if $\mu_i + \lambda_j$ is generalized eigenvalue of D, otherwise $[I_{\mu_i}, K_{\lambda_j}] = 0$ is nonsingular and it follows that $(\beta - (\lambda + \mu)) = 0$.

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Proposition 9. *Suppose R is a finite dimensional algebra over a field $\mathbb{F}$. If S is a multiplicatively closed subset each of whose elements is a sum of nilpotent elements then S is nilpotent.*

Proof: [3], **Proposition 2.6.32** pg 178.

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Theorem 1. *Let K and I be finite dimensional Lie algebras over an algebraically closed field $\mathbb{F}$ such that K act on I by representation $\psi : K \to Der(I)$. Let D a nilpotent subalgebra of $Comp(K, I)$ such that zero is not generalized eigenvalue of D in K. So if or $char(\mathbb{F}) = 0$ or $char(\mathbb{F}) = p$ and dimension of I is smaller than p then ψ is a nilpotent representation.*

Proof: Let $\lambda_1, \cdots, \lambda_r$ and $\mu_1, \cdots, \mu_s \in \mathbb{F}$ be generalized eigenvalue of D in K and I, respectively. If $a \in I_{\mu_i}$ and $k \in K_{\alpha_j}$ then by **Proposition 3**, we have $(\psi^n(a) \in I_{\mu_i + n\lambda_j}$, with $\lambda_j \neq 0$, if $\mu_i + n\lambda_j$ is generalized eigenvalue of D in I and $(ad_k)^n = 0$ otherwise. If $char(\mathbb{F}) = 0$ then $(ad_k)^n = 0$ for some n because thet set of eigenvalues of D is finite; if $char(\mathbb{F}) = p$ the set $\{\mu_i + \lambda_j, \mu_i + 2\lambda_j, \cdots, \mu_i + (p-1)\lambda_j, \mu_i\}$ has p distinct elements and D has at most $p - 1$ eigenvalues in I then $\psi^n = 0$ for some $1 \leq n \leq p$. In both cases ψ is nilpotent for all $k \in K_{\lambda_j}$, $1 \leq j \leq r$. Hence every element of $\psi(K) : K \to Der(I)$ can be written as sum of nilpotent elements. Therefore, by **Proposition 9**, ψ is nilpotent.

4 Derivations of K_θ

Definition 6. Let K and I be vector spaces. Given $(\alpha, \beta) \in \mathfrak{gl}(K) \oplus \mathfrak{gl}(I)$, $\theta \in C^2(K, I)$ and $h, k \in K$, define the action of $\mathfrak{gl}(K) \oplus \mathfrak{gl}(I)$ on $C^2(K, I)$ by

$$(\alpha, \beta) \cdot \theta(h, k) = \beta(\theta(h, k)) - \theta(\alpha(k), h) - \theta(k, \alpha(h)). \tag{7}$$

Proposition 10. *Let K be a Lie algebra and I a K-module. Considere the action of $Comp(K, I)$ on $C^2(K, I)$ defined in (7). Then the vector spaces $Z^2(K, I)$ and $B^2(K, I)$ are invariants by this action.*

Proof: Let $k, h, l \in K$, $(\alpha, \beta) \in Comp(K, I)$ and $\theta \in Z^2(K, I)$. By definition

$$\begin{aligned}(\alpha, \beta) \cdot \theta(k, [h, l]) &= \beta(\theta(k, [h, l])) - \theta(\alpha(k), [h, l]) - \theta(k, \alpha([h, l])) \\ &= \beta(\theta(k, [h, l])) - \theta(\alpha(k), [h, l]) - \theta(k, [\alpha(h), l]) - \theta(k, [h, \alpha(l)]).\end{aligned}$$

If

$$X = (\alpha, \beta) \cdot \theta(k, [h, l]) + (\alpha, \beta) \cdot \theta(h, [l, k]) + (\alpha, \beta) \cdot \theta(l, [k, h]),$$

then

$$\begin{aligned}X = \beta(\theta(k, [h, l])) + \beta(\theta(h, [l, k])) + \beta(\theta(l, [k, h])) \\ - \theta(\alpha(k), [h, l]) - \theta(\alpha(h), [l, k]) - \theta(\alpha(l), [k, h]) \\ - \theta(k, [\alpha(h), l]) - \theta(h, [\alpha(l), k]) - \theta(l, [\alpha(k), h]) \\ - \theta(k, [h, \alpha(l)]) - \theta(h, [l, \alpha(k)]) - \theta(l, [k, \alpha(h)])\end{aligned}$$

Using coclyce definition

$$\begin{aligned}X = \beta([\theta(k, h), l]) + \beta([\theta(h, l), k]) + \beta([\theta(l, k), h]) \\ - [\theta(\alpha(k), h), l] - [\theta(\alpha(h), l), k] - [\theta(\alpha(l), k), h] \\ - [\theta(k, \alpha(h)), l] - [\theta(h, \alpha(l)), k] - [\theta(l, \alpha(k)), h] \\ - [\theta(k, h), \alpha(l)] - [\theta(h, l), \alpha(k)] - [\theta(l, k), \alpha(h)].\end{aligned}$$

(α, β) is a compatible pair then we can replace in X the equalities

$$\beta([\theta(k,h),l]) = [\beta(\theta(k,h)),l] + [\theta(k,h)),\alpha(l)];$$
$$\beta([\theta(h,l),k]) = [\beta(\theta(h,l)),k] + [\theta(h,l)),\alpha(k)];$$
$$\beta([\theta(l,k),h]) = [\beta(\theta(l,k)),h] + [\theta(l,k)),\alpha(h)].$$

Hence

$$\begin{aligned} X = [\beta(\theta(k,h)),l] &+ [\beta(\theta(h,l)),k] + [\beta(\theta(l,k)),h] \\ &- [\theta(\alpha(k),h),l] - [\theta(\alpha(h),l),k] - [\theta(\alpha(l),k),h] \\ &- [\theta(k,\alpha(h)),l] - [\theta(h,\alpha(l)),k] - [\theta(l,\alpha(k)),h]. \end{aligned}$$

Again, by action definition we obtain

$$X = [(\alpha,\beta)\cdot\theta(h,l),k] + [(\alpha,\beta)\cdot\theta(l,k),h] + [(\alpha,\beta)\cdot\theta(k,h),l].$$

So $(\alpha,\beta)\cdot\theta \in Z^2(K,I)$.

Now suppose that $\theta \in B^2(K,I)$. Then there is a linear map $\nu : K \to I$ such that

$$\theta(k,h) = \nu([k,h]) - [\nu(k),h] - [k,\nu(h)]. \tag{8}$$

Let $Y = (\alpha,\beta)\cdot\theta(k,h)$. By (8) we have

$$Y = (\alpha,\beta)\cdot(\nu([k,h]) - (\alpha,\beta)\cdot([\nu(k),h]) - (\alpha,\beta)\cdot([k,\nu(h)]).$$

Using action difinition we have

$$\begin{aligned} Y = \beta(\nu([k,h])) &- \nu([\alpha(k),h]) - \nu([k,\alpha(h)]) \\ &- \beta([\nu(k),h]) + [\nu(\alpha(k)),h] + [\nu(k),\alpha(h)] \\ &- \beta([k,\nu(h)]) + [\alpha(k),\nu(h)] + [k,\nu(\alpha(h))], \end{aligned}$$

we can use that α is a derivation and (α,β) is a compatible pair to conclude

$$\begin{aligned} Y = \beta\nu([k,h]) &- \nu\alpha([k,h]) \\ &- [\beta\nu(k),h] - [\nu(k),\alpha(h)] + [\nu\alpha(k),h] + [\nu(k),\alpha(h)] \\ &- [\beta(k),\nu(h)] - [k,\beta\nu(h)] + [\beta(k),\nu(h)] + [k,\nu\alpha(h)], \end{aligned}$$

Hence,

$$Y = (\beta\nu - \nu\alpha)[k,h] - [(\beta\nu - \nu\alpha)(k),h] + [k,(\beta\nu - \nu\alpha)(h)].$$

If $T = \beta\nu - \nu\alpha : K \to I$ then

$$(\alpha,\beta)\cdot\theta(k,h) = T([k,h]) - [T(k),h] - [k,T(h)].$$

Therefore, $(\alpha,\beta)\cdot\theta \in B^2(K,I)$.

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This result allow us to define an action of $Comp(K,I)$ on $H^2(K,I)$: let $\theta \in Z^2(K,I)$ and $(\alpha,\beta) \in Comp(K,I)$. Define the action

$$(\alpha,\beta)\cdot(\theta + B^2(K,I)) = ((\alpha,\beta)\cdot\theta) + B^2(B,I). \tag{9}$$

This is well defined by **Proposition 10**.

Definition 7. Let K be a Lie algebra and I a K-module. Let $\theta \in Z^2(K,I)$ and consider the action of $Comp(K,I)$ on $H^2(K,I)$ defined in (9). Define the set of induced pairs of $Comp(K,I)$ by

$$Indu(K,I,\theta) = Ann_{Comp(K,I)}(\theta + B^2(K,I)).$$

Let K be a Lie algebra and I a K-module. Let $\theta \in H^2(K,I)$ and suppose that I, as ideal of K_θ, it is invariant by derivation $d \in Der(K_\theta)$. Let $P_K : K_\theta \to K$ and $P_I : K_\theta \to I$ the natural projection of K_θ on K and K_θ on I, respectively. Define the maps

- $\alpha : K \to K$ by $\alpha(k) = P_K d(k)$, for all $k \in K$;
- $\beta : I \to I$ by $\beta(a) = d(a)$, for all $a \in I$;
- $\varphi : K \to I$ by $\varphi(k) = P_I d(k)$, for all $k \in K$.

Then,

$$d(x+a) = \alpha(x) + \varphi(x) + \beta(a) \text{ for all } a \in I \text{ and } x \in K. \tag{10}$$

Hence, $\beta \in Der(I)$, $\alpha \in Der(K)$ and $\varphi \in Hom(K,I)$.

We can see that β is a derivation of I bacause it is restriction of d to I.

Let $x,y \in K$. By product definition on K_θ we have

$$d([x,y]_\theta) = d([x,y]_K + \theta(x,y)).$$

By decomposition showed in (10)

$$d([x,y]_\theta) = \alpha([x,y]_K) + \varphi([x,y]_K) + \beta(\theta(x,y)).$$

We can calculate

$$[d(x),y]_\theta + [x,d(y)]_\theta = [\alpha(x)+\varphi(x),y] + [x,\alpha(y)+\varphi(y)], \tag{11}$$

and use product definition of K_θ to get

$$\begin{aligned}[d(x),y]_\theta + [x,d(y)]_\theta = [\alpha(x),y]_K + [x,\alpha(y)]_K + \theta(\alpha(x),y)\\ + \theta(y,\alpha(x)) + [\varphi(x),\alpha(y)] - [\varphi(y),\alpha(x)].\end{aligned} \tag{12}$$

Comparing the components of K in (11) and (12) we have

$$\alpha([x,y]_K) = [\alpha(x),y]_K + [x,\alpha(y)]_K.$$

So $\alpha \in Der(K)$.

Proposition 11. *Let K be a Lie algebra and I a K-module. Let $\theta \in H^2(K,I)$ and suppose that I, as ideal of K_θ, it is invariant by derivations. From the decomposition showed in (10) define $\phi : Der(K_\theta) \to Der(K) \oplus Der(I)$ by $\phi(d) = (\alpha, \beta)$. Then ϕ is a Lie algebra morphism.*

Proof: Let $d, d' \in Der(K_\theta)$ and $x \in K$ such that

$$\begin{array}{rcl} d(x+a) & = & \alpha(x) + \varphi(x) + \beta(a) \\ d'(x+a) & = & \alpha'(x) + \varphi'(x) + \beta'(x), \end{array}$$

for all $x \in K$ and $a \in I$. Then,

$$\begin{array}{rcl} dd'(x) & = & d(\alpha'(x) + \varphi'(x)) \\ & = & \alpha\alpha'(x) + \varphi(\alpha'(x)) + \beta'(\varphi'(x)). \end{array}$$

Hence, $P_K dd'(x) = \alpha\alpha'(x)$. Analogously, $P_K d'd(x) = \alpha'\alpha'(x)$. So $P_K([d,d']) = [\alpha, \alpha']$. As β and β' are defined by d and d' restriction to I then $P_I([d,d']) = [\beta, \beta']$. Therefore,

$$\phi([d,d']) = ([\alpha,\alpha'],[\beta,\beta']) = [(\alpha,\beta),(\alpha',\beta')] = [\phi(d),\phi(d')].$$

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Theorem 2. *Let K be a Lie algebra and I a K-module. Let $\theta \in H^2(K,I)$ and suppose that I, as ideal of K_θ, it is invariant by derivations. Let $\phi : Der(K_\theta) \to Der(K) \oplus Der(I)$ given by $\phi(d) = (\alpha,\beta)$, defined in **Proposition 11**. Then $Im(\phi) \leq Comp(K,I)$.*

Proof: Let $(\alpha,\beta) \in Im(\phi)$. Then there is $d \in Der(K_\theta)$ such that $\phi(d) = (\alpha,\beta)$. If $k \in K$ and $a \in I$ then

$$\begin{array}{rcll} \beta([a,k]_\theta) & = & d([a,k]_\theta) & [a,k] \in I \\ & = & [d(a),k]_\theta + [a,d(k)]_\theta & d \in Der(K_\theta) \\ & = & [\beta(a),k]_\theta + [a,\alpha(k)+\varphi(k)]_\theta & \\ & = & [\beta(a),k]_\theta + [a,\alpha(k)]_\theta & \text{because } I \text{ is abelian} \end{array}$$

■

Theorem 3. *Let K be a Lie algebra and I a K-module. Let $\theta \in H^2(K,I)$ and suppose that I, as ideal of K_θ, it is invariant by derivations. Let $\phi : Der(K_\theta) \to Der(K) \oplus Der(I)$ given by $\phi(d) = (\alpha,\beta)$. Then:*

1. *$Im(\phi) = Indu(K,I,\theta)$*
2. *$ker(\phi) \cong Z^1(K,I)$*

Proof: 1) Let $(\alpha,\beta) \in Indu(K,I,\theta)$. By definition

$$(\alpha,\beta)\cdot\theta = 0 \text{ mod } B^2(K,I).$$

Then, there is a linear map $T : K \to I$ such that for all $k,h \in K$ we have

$$\theta(\alpha(k),h) + \theta(k,\alpha(h)) + [T(k),h] - [T(h),k] = \beta(\theta(k,h)) + T([k,h]). \tag{13}$$

Let $k \in K$, $a \in I$ and define the linear map $(\alpha,\beta)^* : K_\theta \to K_\theta$ by

$$(\alpha,\beta)^*(k+a) = \alpha(k) + \beta(a) + T(k).$$

Let's check that $(\alpha,\beta)^*$ is a derivation of K_θ. Let $k+a, h+b \in K_\theta$. If

$$X = (\alpha,\beta)^*([k+a,h+b]_\theta)$$

then

$$\begin{aligned} X &= (\alpha,\beta)^*([k,h]_K + \theta(k,h) + [a,h] - [b,k]) \\ &= \alpha([k,h]_K) + \beta(\theta(k,h)) + \beta([a,h]) - \beta([b,k]) + T([k,h]_K). \end{aligned}$$

Now, let

$$Y = [(\alpha+\beta)^*(k+a), h+b]_\theta + [k+a, (\alpha+\beta)^*(h+b)]_\theta.$$

We have

$$\begin{aligned} [(\alpha+\beta)^*(k+a), h+b]_\theta &= [\alpha(k) + \beta(a) + T(k), h+b]_\theta \\ &= [\alpha(k), h]_K + \theta(\alpha(k), h) + [\beta(a) + T(k), h] - [b, \alpha(k)] \end{aligned}$$

and

$$\begin{aligned} [k+a, (\alpha+\beta)^*(h+b)]_\theta &= [k+a, \alpha(h) + \beta(b) + T(h)] \\ &= [k, \alpha(h)]_K + \theta(k, \alpha(h)) + [a, \alpha(h)] - [\beta(b) + T(h), k] \end{aligned}$$

then

$$\begin{aligned} Y = \alpha([k,h]_K) &+ \theta(\alpha(k),h) + \theta(k,\alpha(h)) \\ &+ [T(k),h] - [T(h),k] + [\beta(a),h]) + [a,\alpha(h)] - [\beta(b),k] - [b,\alpha(k)]. \end{aligned}$$

By compatible pair definition we get

$$Y = \alpha([k,h]_K) + \theta(\alpha(k),h) + \theta(k,\alpha(h)) + \beta([a,h]) - \beta([b,k]) + [T(k),h] - [T(h),k].$$

By equation (13)

$$Y = \alpha([k,h]_K) + \beta(\theta(h,k)) + T([k,h]) + \beta([a,h]) - \beta([b,k]).$$

As $X = Y$ then $(\alpha,\beta)^*$ is a derivation.

Besides, observe that $P_K(\alpha,\beta)^* = \alpha$ and $P_I(\alpha,\beta)^* = \beta$. Hence $\phi((\alpha+\beta)^*) = \alpha+\beta$, that is, $Indu(K,I,\theta) \subseteq Im(\phi)$.

Now, suppose that $(\alpha+\beta) \in Im(\phi)$. Then there is $d \in Der(K_\theta)$ such that

$$\phi(d) = (\alpha+\beta).$$

By **Theorem 2** we have $Im(\phi) \subseteq Comp(K,I)$. Then it is enough show that there is a linear map $T : K \to I$ such that the equation 13 is satisfied.

For each $k+a \in K_\theta$ we can use the decomposition defined in (10) to write

$$d(k+a) = \alpha(k) + \varphi(k) + \beta(a).$$

By product definition in K_θ we get

$$\begin{aligned}[d(k+a), h+b]_\theta &= [\alpha(k)+\varphi(k)+\beta(a), h+b]_\theta \\ &= [\alpha(k), h]_K + \theta(\alpha(k), h) + [\varphi(k)+\beta(a), h] - [b, \alpha(k)]\end{aligned}$$

$$\begin{aligned}[k+a, d(h+b)]_\theta &= [k+a, \alpha(h)+\varphi(h)+\beta(b)]_\theta \\ &= [k, \alpha(h)]_K + \theta(k, \alpha(h)) + [a, \alpha(h)] - [\varphi(h)+\beta(b), k]\end{aligned}$$

$$\begin{aligned}d([k+a, h+b]_\theta) &= d([k,h]_K + \theta(k,h) + [a,h] - [b,k]) \\ &= \alpha([k,h]_K) + \beta(\theta(k,h)) + \beta([a,h]) - \beta([b,k]) + \varphi_d([k,h])\end{aligned}$$

As d is a derivation then we have equality

$$d[k+a, h+b] = [d(k)+a, h+b] = [k+a, d(h)+b].$$

SO,

$$\beta(\theta(k,h)) + \varphi([k,h]) = \theta(\alpha(k), h) + [\varphi(k), h] + \theta(k, \alpha(h)) - [\varphi(h), k].$$

Therefore $T = \varphi$ satisfies the equation (13) e $Im(\phi) \subseteq Indu(K, I, \theta)$.

2) Let $d \in ker(\phi)$. The decomposition showed in (10) provide us

$$d(k) = \varphi(k), k \in K.$$

Let $k, h \in K$. By derivation definition

$$d([k,h]_\theta) = [d(k), h]_\theta + [k, d(h)]_\theta. \tag{14}$$

By product definition in K_θ we can write (14) as

$$d([k,h]_K + \theta(k,h)) = [\varphi(k), h]_\theta + [k, \varphi(h)]_\theta.$$

Because $d \in Ker(\phi)$ then (14) it is equal to

$$\varphi([k,h]_K) = [\varphi(k), h]_K - [\varphi(h), k]_K.$$

Hence, $\varphi \in Z^1(K, I)$. Now define $\sigma : ker(\phi) \to (Z^1(K, I), +)$ by $\sigma(d) = \varphi_d$ such that $\varphi_d(k) = d(k)$. Then $\sigma(Ker(\phi)) \subseteq Z^1(K, I)$.

Let $d, d' \in ker(\phi)$. Then

$$\sigma(d+d')(k) = \varphi_{d+d'}(k) = (d+d')(k) = d(k) + d'(k) = \varphi(k) + \varphi'(k) = (\sigma(d) + \sigma(d'))(k).$$

So σ it is group homomorfism.

If $d, d' \in Ker(\phi)$ such that $\sigma(d) = \sigma(d')$ then $\varphi_d(k) = \varphi_{d'}(k)$, for all $k \in K$ and $d = d'$. Let $T \in Z^1(K, I)$ and define $d : K_\theta \to K_\theta$ by

$$d(x+a) = T(x), x \in K, a \in I.$$

d is a derivation because

$$d([k+a, h+b]_\theta) = T([k,h]_K)$$

and

$$[d(k+a), h+b]_\theta + [k+a, d(h+b)]_\theta = [T(k), h]_K + [k + T(h)]_K.$$

It follows that $\sigma(d) = T$. Therefore, σ is isomorphism

$\blacksquare$

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