Graph sketching

e.g.) $f(x) = x^4 - 4x^3 + 10$

Domain $\mathbb{R}$, Range $\mathbb{R}$

$(0,10)$ y-axis intercept

$f'(x) = 4x^3 - 12x^2 \Rightarrow 4x^2(x-3)$ → 0, 3 critical values

$-\infty$ 0 3 ∞

$f'(x)$	$-$	$-$	$+$
	dec.	dec.	inc.

$\boxed{(3,-17) \text{ local min}}$

$(-\infty, 3]$ dec.
$[3, \infty)$ inc.

$f''(x) = 12x^2 - 24x \Rightarrow 12x(x-2)$ → 0, 2

$-\infty$ 0 2 ∞

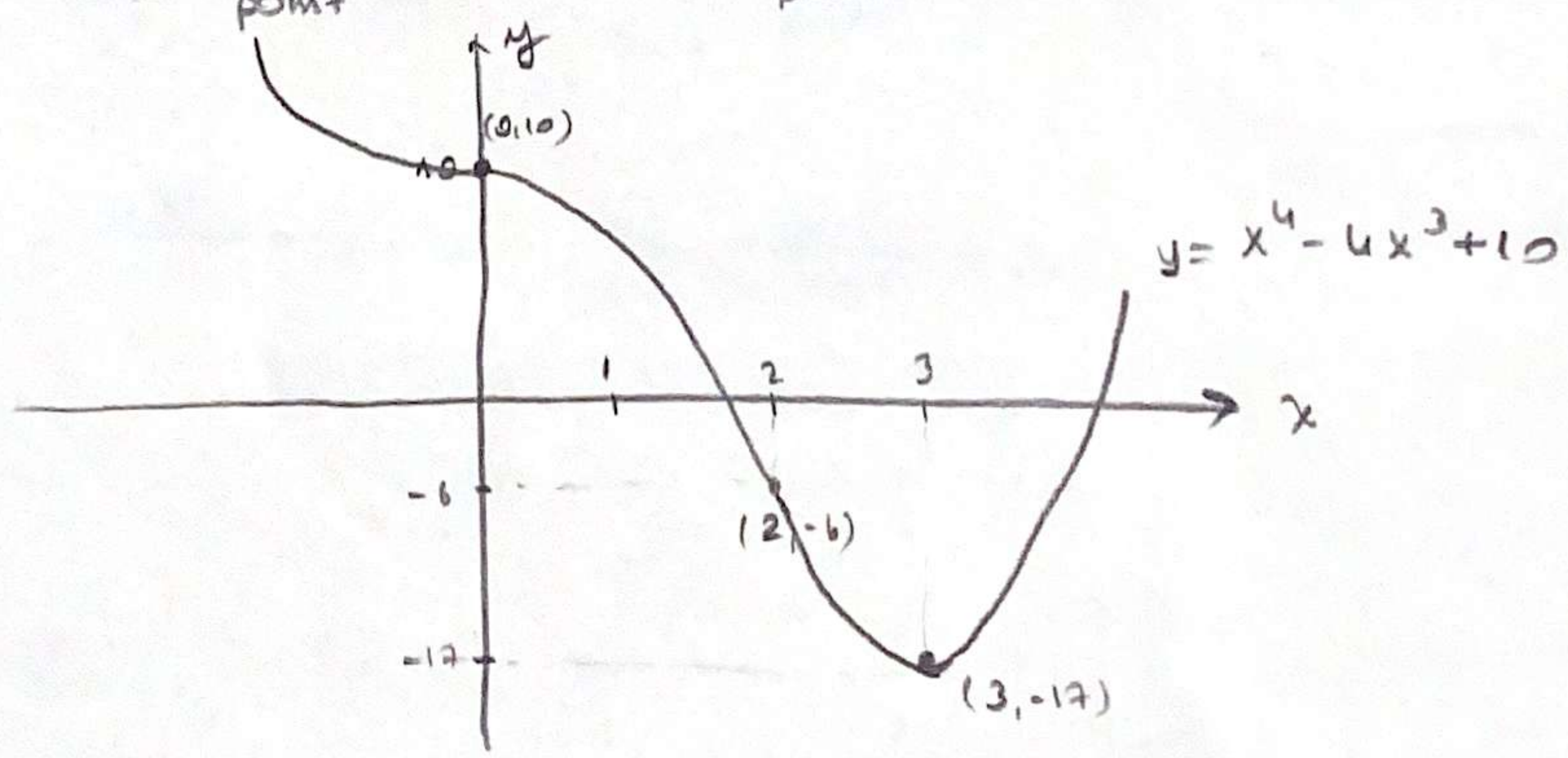
$f''(x)$	$+$	$-$	$+$
	concave up	con. down	con. up

$(-\infty, 0) \cup (2, \infty)$ concave up
$(0,2)$ concave down

$(0,10)$ and $(2,-6)$ are inflection points

$x < 0$	$0 < x < 2$	$2 < x < 3$	$x > 3$
decreasing	decreasing	decreasing	increasing
concave up	concave down	concave up	concave up

$(0,10)$ inflection point $(2,-6)$ inflection point $(3,-17)$ local min



Note: Procedure for graphing $y=f(x)$

1. Identify the domain of f and any symmetries the curve may have.
2. Find the derivatives y' and y''.
3. Find the critical points of f, if any, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes that may exist.
7. Plot key points, such as the intercepts and the points found in step 3-5, and sketch the curve together with any asymptotes that exist. $y=f(x)$

vertical asymptote

at $x=a$, if and only if at least one is satisfied;

$\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$ or both

horizontal asymptote

for $y=L$, if and only if at least one is satisfied;

$\lim_{x \to -\infty} f(x) = L$ or $\lim_{x \to \infty} f(x) = L$ or both

oblique asymptote

the straight line $y=ax+b$, if and only if at least one is satisfied,

$\lim_{x \to -\infty} (f(x) - (ax+b)) = 0$ or $\lim_{x \to \infty} (f(x) - (ax+b)) = 0$ or both

Checklist

1. Calculate $f'(x)$ and $f''(x)$

2. Determine domain

   Vertical, horizontal, oblique asymptotes

   intercepts, endpoints, "obvious" points

3. $f'(x)$
   critical points
   undefined points
   sign chart to determine whether the curve is increasing or decreasing
   maxima, minima

4. $f''(x)$
   zero points
   undefined points
   sign chart to obtain concavity
   inflection points

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eg.) $y = \dfrac{2x^2}{x^2-1}$ ★ $\boxed{\text{Domain } \mathbb{R} - \{-1, 1\}}$ $(-\infty,-1) \cup (-1,1) \cup (1,\infty)$
↳ make denominator "0"

★ $\boxed{\text{intercept} \rightarrow (0,0) \text{ origin}}$

★ checking symmetry $\dfrac{2x^2}{x^2-1} \stackrel{?}{=} \dfrac{2(-x)^2}{(-x)^2-1}$ ✓ $\boxed{\text{even function, symmetric about the } y\text{-axis}}$

★ $\dfrac{\frac{2x^2}{x^2}}{\frac{x^2}{x^2} - \frac{1}{x^2}} \Rightarrow \lim\limits_{x \to \pm\infty} \dfrac{2}{1 - \frac{1}{x^2}} = 2$ $\boxed{y = 2 \text{ horizontal asymptote}}$

★ $\lim\limits_{x \to 1^+} \dfrac{2x^2}{x^2-1} = \infty$ $\lim\limits_{x \to 1^-} \dfrac{2x^2}{x^2-1} = -\infty$

$\lim\limits_{x \to -1^+} \dfrac{2x^2}{x^2-1} = -\infty$ $\lim\limits_{x \to -1^-} \dfrac{2x^2}{x^2-1} = \infty$

$\boxed{x = 1, \; x = -1 \text{ vertical asymptotes}}$

★ $f'(x) = \dfrac{4x \cdot (x^2-1) - 2x^2 \cdot (2x)}{(x^2-1)^2} = \dfrac{-4x}{(x^2-1)^2}$

	$-\infty$	-1	0	1	∞
$f'(x)$	+	−	+	−	
$f(x)$	inc.	dec.	inc	dec	

$\boxed{\text{local min } 0,0}$

$\boxed{\begin{array}{l}(-\infty,-1) \cup (0,1) \text{ increasing} \\ (-1,0) \cup (1,\infty) \text{ decreasing}\end{array}}$

★ $f''(x) = \dfrac{(-4) \cdot (x^2-1)^2 - (-4x) \cdot 2(x^2-1) \cdot (2x)}{(x^2-1)^4}$ $\quad -(-4(x^2-1) \cdot 4x^2)$

$\dfrac{-4\,\cancel{(x^2-1)}\left[(x^2-1) - 4x^2\right]}{(x^2-1)^{\cancel{4}\,3}} \Rightarrow f''(x) = \dfrac{12x^2+4}{(x^2-1)^3}$

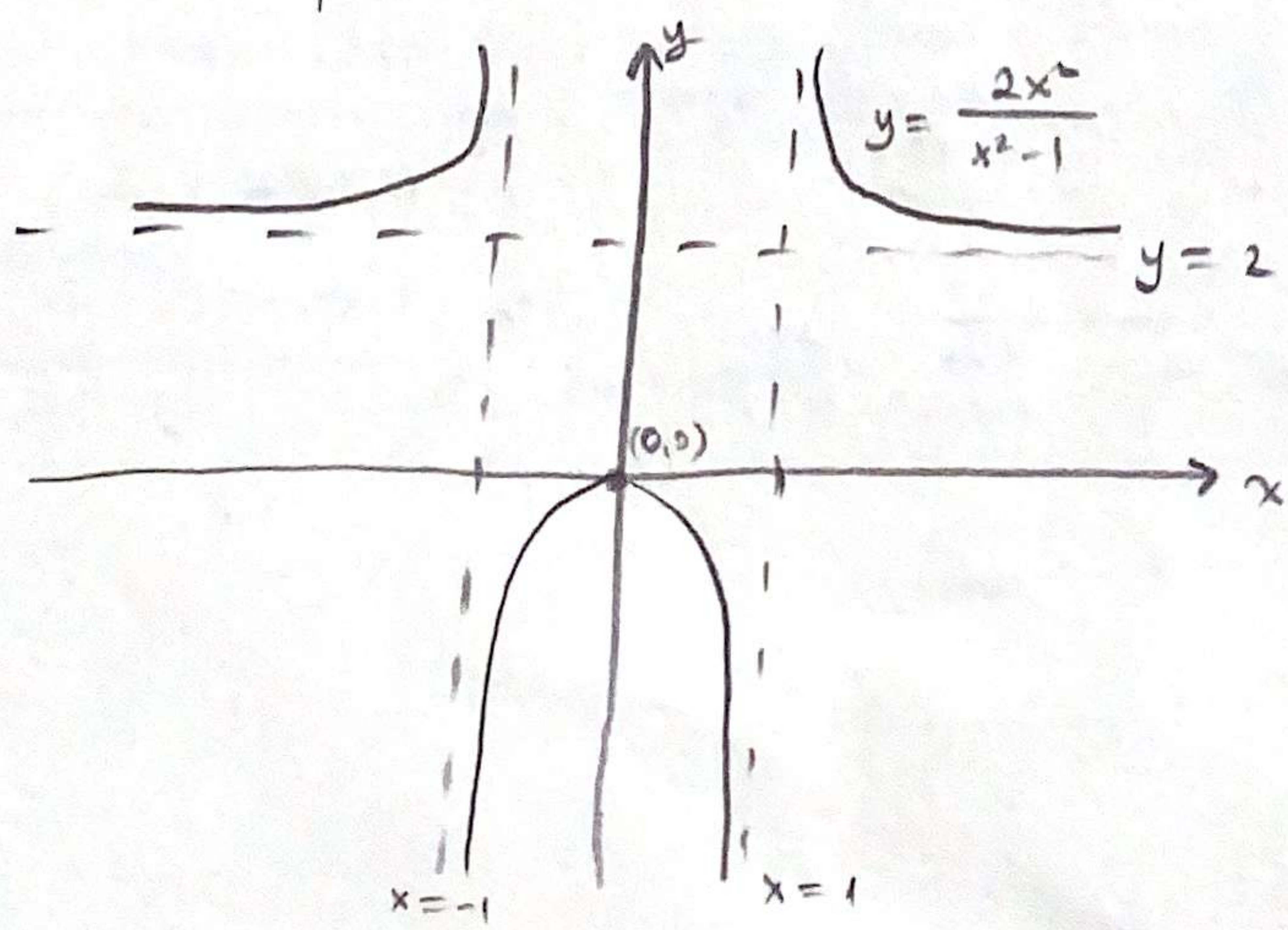
as $12x^2+4 > 0$ for all x

$f''(x) > 0$ when $(x^2-1) > 0$, $f''(x) < 0$ when $(x^2-1) < 0$

	$-\infty$	-1	1	∞
$f''(x)$	+	−	+	
$f(x)$	concave up	con. down	con. up	

$\boxed{\begin{array}{l}(-\infty,-1) \cup (1,\infty) \text{ concave up} \\ (-1,1) \text{ concave down}\end{array}}$

no inflection point $\Rightarrow$ not in the domain



e.g.) $f(x) = \frac{x^2}{\sqrt{x+1}}$ ★ $\boxed{\text{Domain } (-1, \infty)}$

$x+1>0 \quad x>-1$

★ $x=0 \quad y=0$ $\boxed{(0,0) \text{ intercept}}$

★ symmetry check $\frac{x^2}{\sqrt{x+1}} \stackrel{?}{=} \frac{(-x)^2}{\sqrt{(-x)+1}}$ X $\boxed{\text{not symmetric}}$

★ $\lim_{x\to\infty} \frac{x^2}{\sqrt{x+1}} = \lim_{x\to\infty} \frac{\cancel{x^2}\cdot 1}{\cancel{x^2}\sqrt{\frac{1}{x^3}+\frac{1}{x^4}}} = \lim_{x\to\infty} \frac{1}{\sqrt{\frac{1}{x^3}+\frac{1}{x^4}}} = \infty$

↳ there is $\boxed{\text{no horizontal asymptote}}$

★ $\lim_{x\to -1^+} \frac{x^2}{\sqrt{x+1}} = \infty \rightarrow \boxed{x=-1 \text{ vertical asymptote}}$

★ $f'(x) = \frac{2x\cdot\sqrt{x+1} - x^2\cdot\frac{1}{2\sqrt{x+1}}\cdot 1}{x+1} =$

$\frac{4x\cdot(x+1)-x^2}{2\sqrt{x+1}\cdot(x+1)} = \frac{x\cdot(3x+4)}{2\cdot(x+1)^{3/2}} \rightarrow \underbrace{0, \cancel{-\frac{4}{3}}}_{0}$ not in domain, -1 undefined

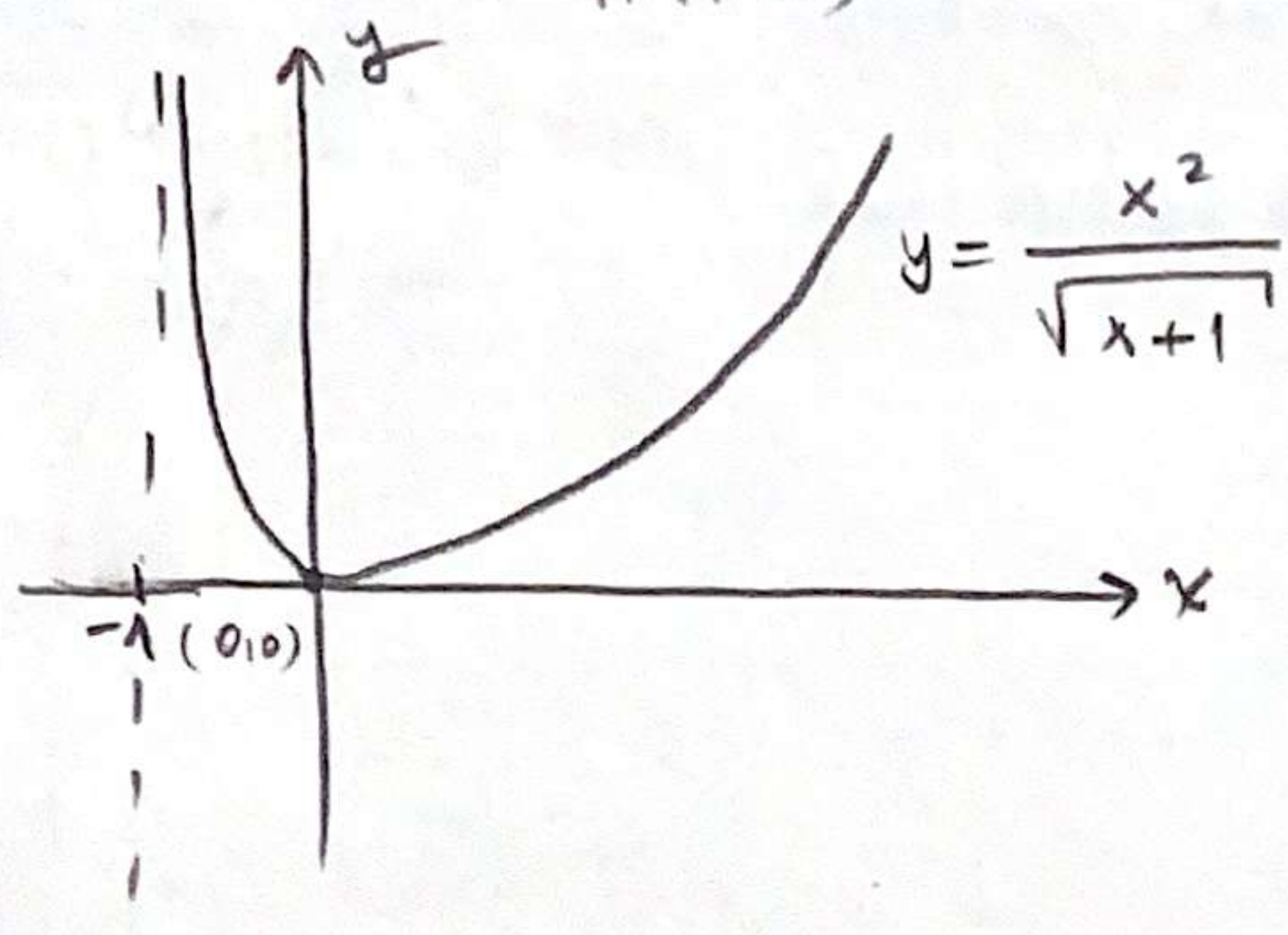
	-1	0	∞
$f'(x)$	$-$	$+$	
	dec.	inc.	

$\boxed{(0,0) \text{ local min}}$

★ $f''(x) = \frac{(6x+4)\cdot 2\cdot(x+1)^{3/2} - 3(x+1)^{\frac{1}{2}}\cdot(3x^2+4x)}{4\cdot(x+1)^3}$

$f''(x) = \frac{\cancel{(x+1)^{\frac{1}{2}}}\left[(12x^2+20x+8)-(9x^2+12x)\right]}{4\cdot\cancel{(x+1)^{\frac{1}{2}}}\cdot(x+1)^{5/2}} = \frac{3x^2+8x+8}{4\cdot(x+1)^{5/2}}$

$\boxed{f''(x)>0 \text{ for } \forall x \in (-1,\infty) \text{ concave up}}$



eg.) $f(x) = x \cdot e^x$ ★ Domain $\mathbb{R}$ ★ $\boxed{(0,0) \text{ intercept}}$

★ symmetry check $xe^x \stackrel{?}{=} (-x) \cdot e^{(-x)}$ X $\boxed{\text{not symmetric}}$

★ $\lim\limits_{x \to -\infty} x \cdot e^x = \lim\limits_{x \to -\infty} \left(\frac{x}{e^{-x}} \right) = \lim\limits_{x \to -\infty} \frac{1}{-e^{-x}} = \lim\limits_{x \to -\infty} (-e^x) = 0$

$\boxed{x\text{-axis is a horizontal asymptote}}$

★ $f'(x) = e^x + x \cdot e^x = e^x(x+1)$

e^x always positive

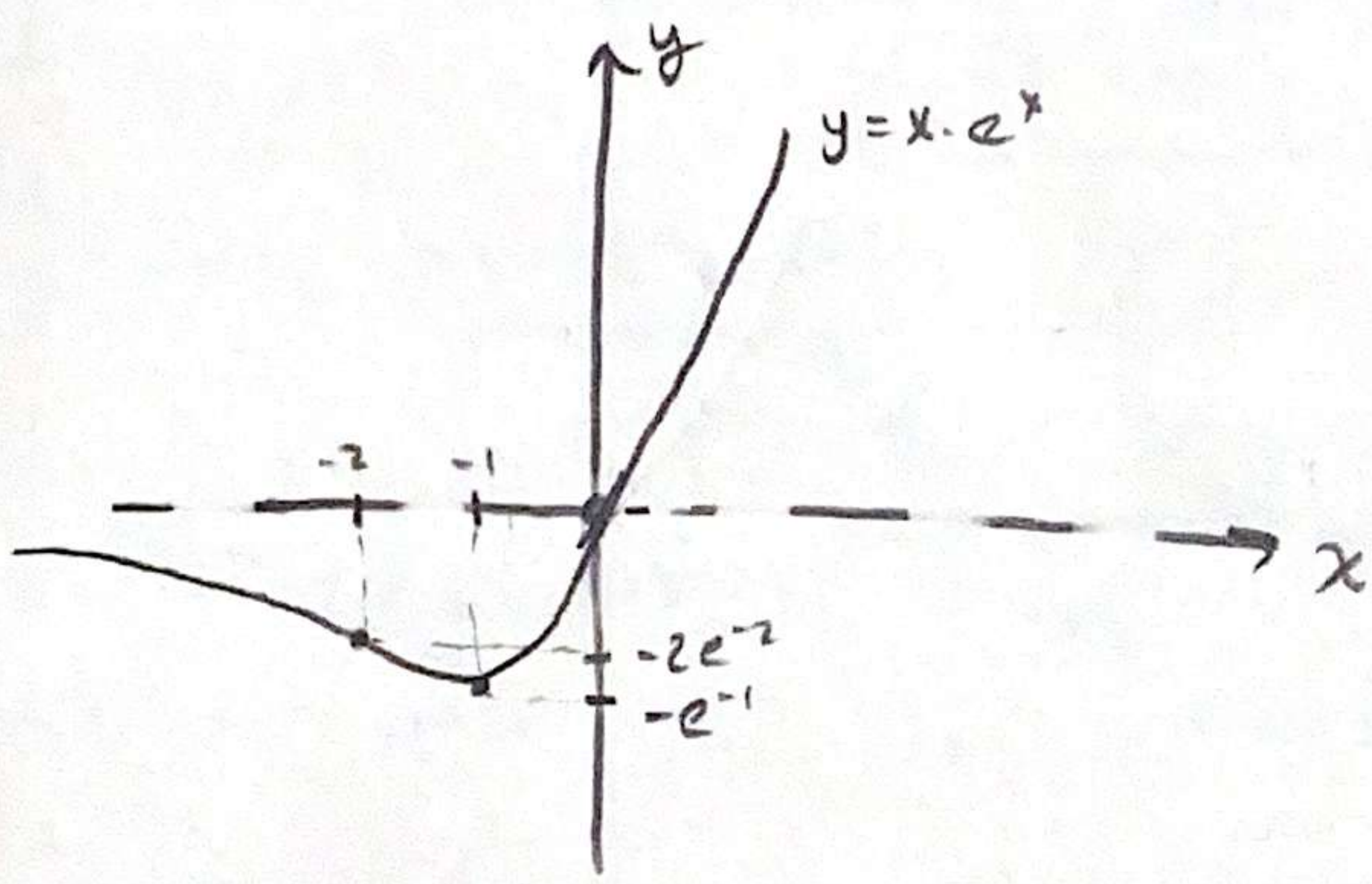
	$-\infty$		-1		∞
$f'(x)$		$-$		$+$	
		dec.		inc.	

$\boxed{(-1, -e^{-1}) \text{ local min}}$

★ $f''(x) = e^x \cdot (x+1) + e^x \Rightarrow e^x(x+2)$

	$-\infty$		-2		∞
$f''(x)$		$-$		$+$	
		con. down	•	con. up	

$\boxed{(-2, -2e^{-2}) \text{ inflection point}}$



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eg.) $f(x) = 2\cos x + \sin 2x$ ★ Domain all $\mathbb{R}$

★ $x = 0$ $y = 2$ $\boxed{(0,2) \text{ y-axis intercept}}$

$2\cos x(1 + \sin x) = 0$ $\arccos 0 = x$ $\arcsin(-1) = x$

$\frac{\pi}{2}, \frac{3\pi}{2}$ $\frac{3\pi}{2}$

$\boxed{\left(\frac{\pi}{2}, 0\right) \text{ and } \left(\frac{3\pi}{2}, 0\right) \text{ x-axis intercepts}}$

★ neither even or odd because "$2\cos x + \sin 2x \stackrel{?}{=} 2\cos(-x) + \sin(-2x)$" is not satisfied but it is a $\boxed{\text{periodic function}}$ because $f(x+2\pi) = f(x)$

★ asymptote: none

★ $f'(x) = -2\sin x + 2.\underbrace{\cos 2x}_{2(1-2\sin^2 x)}$

$= -2\sin x + 2 - 4\sin^2 x \Rightarrow -2(2\sin^2 x + \sin x - 1)$

$= -2(\sin x + 1)(2\sin x - 1)$ $\quad \arcsin(-1) = x$ $\quad \arcsin(-\frac{1}{2}) = x$

$\frac{3\pi}{2}$ $\quad \frac{\pi}{6}, \frac{5\pi}{6}$

	0	$\frac{\pi}{6}$		$\frac{5\pi}{6}$		$\frac{3\pi}{2}$		2π
$f'(x)$	+		−		+		+	
f	↗	local max	↘	local min	↗		↗	

$\left(\frac{\pi}{6}, \frac{3\sqrt{3}}{2}\right)$ local max point

$\left(\frac{5\pi}{6}, -\frac{3\sqrt{3}}{2}\right)$ local min point

★ $f''(x) = -2\cos x + 2. -\sin(2x).2$

$-2\cos x - 8\sin x.\cos x = -2\cos x(1 + 4\sin x)$

$\arccos(0) = x$ $\quad \arccos(-\frac{1}{4}) = x$

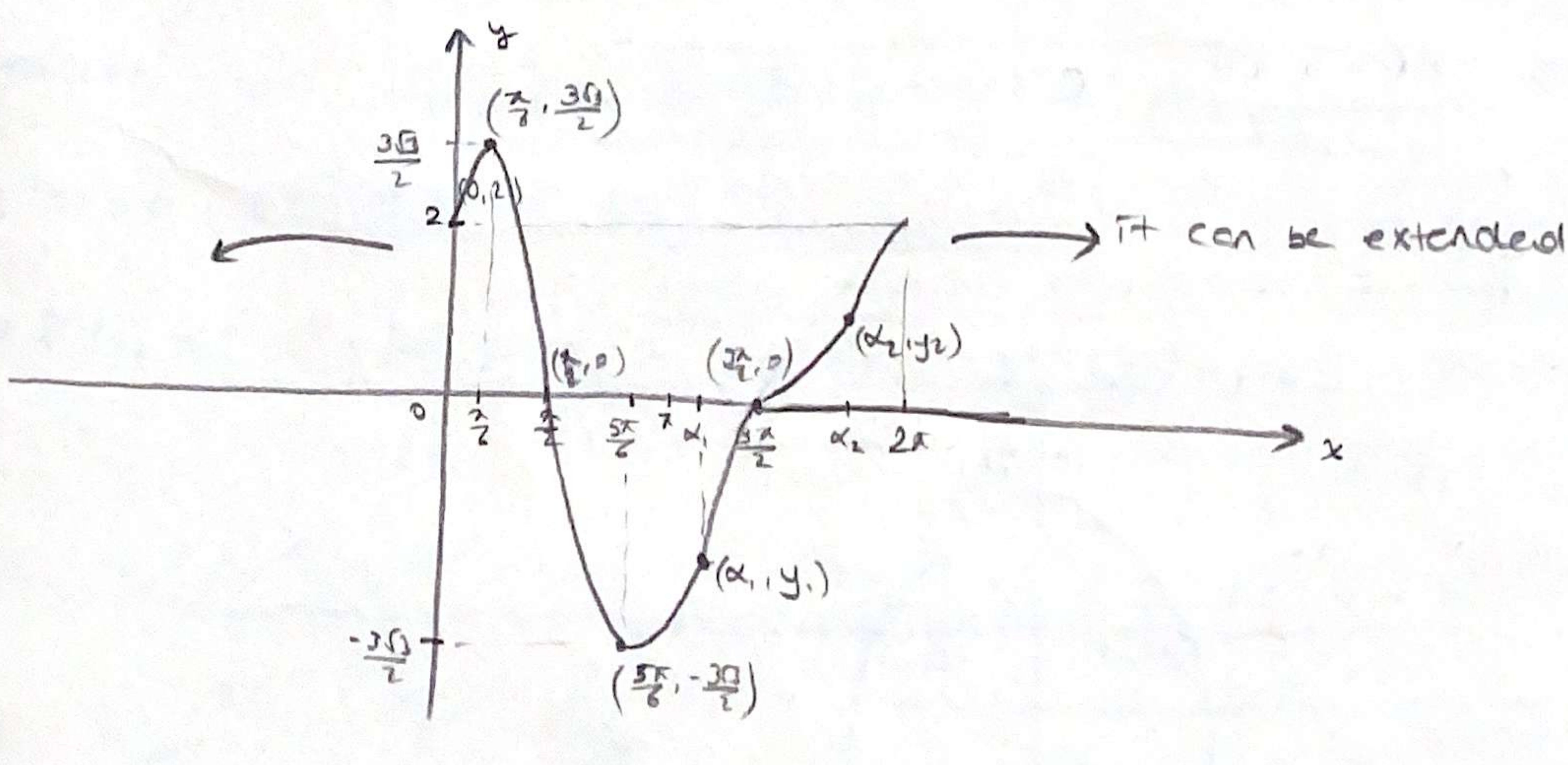
$\frac{\pi}{2}, \frac{3\pi}{2}$ $\quad \alpha_1 > \pi$ $\quad \alpha_2 < 2\pi$

	0		$\frac{\pi}{2}$		α_1		$\frac{3\pi}{2}$		α_2		2π
$f''(x)$		−		+		−		+		−	
		con. down		con up		con down		con up		con down	

$\left(\frac{\pi}{2}, 0\right)$ $\quad \left(\frac{3\pi}{2}, 0\right)$

(α_1, y_1) $\quad (\alpha_2, y_2)$

inflection points



eg.) $y = \ln(4-x^2)$ ★ Domain $4-x^2>0$ $\boxed{(-2,2)}$

★ $x=0$ $y=\ln 4$ / $y=0$ $x \Rightarrow 4-x^2=1$

$x=\pm\sqrt{3}$

$\boxed{(0, \ln 4)}$ y-axis intercept

$\boxed{(\sqrt{3},0), (-\sqrt{3},0)}$ x-axis intercepts

★ as $\ln(4-x^2) = \ln(4-(-x)^2)$ even function and

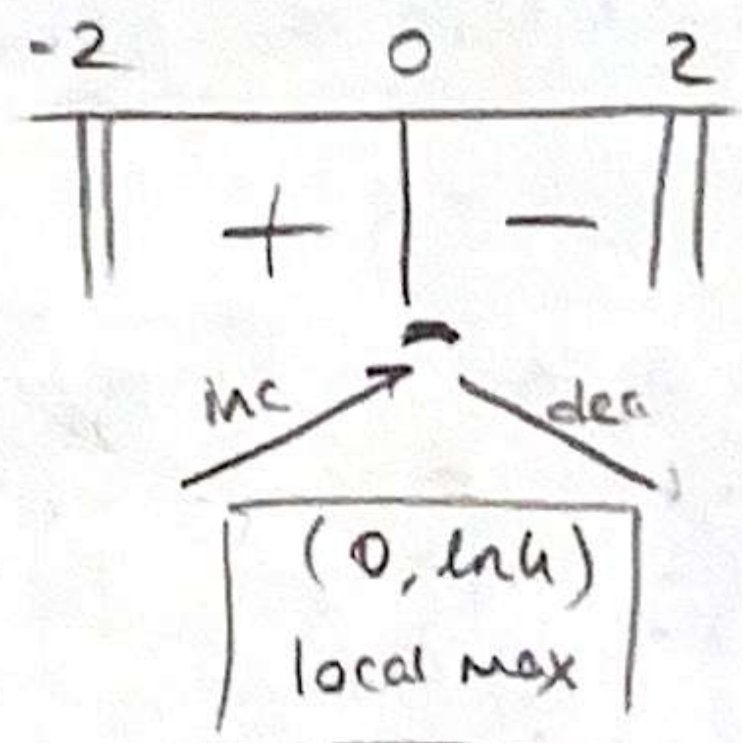
Symmetric about y-axis

★ $\lim_{x\to -2^+} \left[\ln(4-x^2)\right] = -\infty$ $\lim_{x\to 2^-} \left[\ln(4-x^2)\right] = -\infty$

$\ln(0^+)$

So $x=-2$ and $x=2$ are vertical asymptotes

★ $f'(x) = \dfrac{-2x}{4-x^2}$

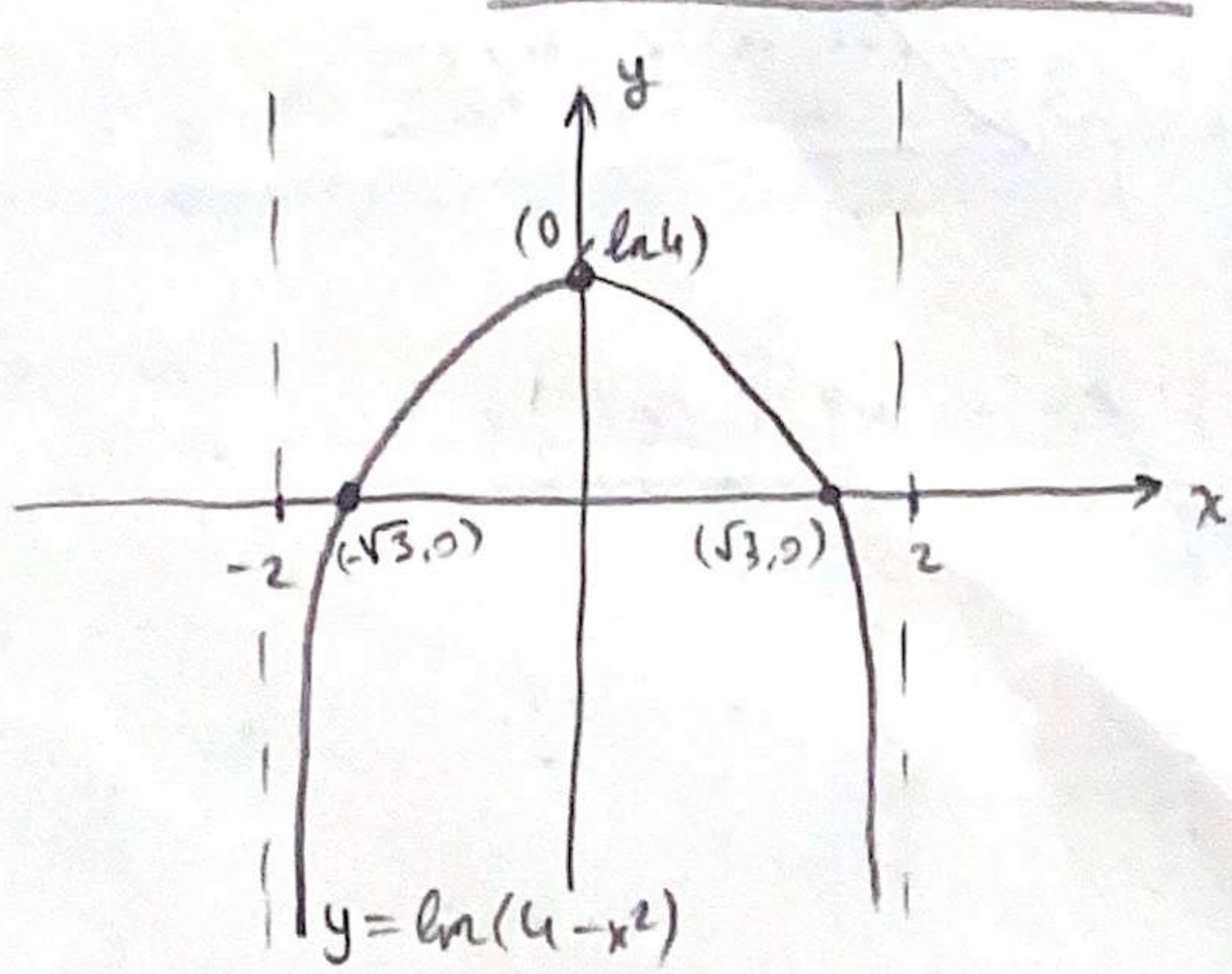


★ $f''(x) = \dfrac{(-2).(4-x^2) - (-2x).(-2x)}{(4-x^2)^2} = \dfrac{-8+2x^2-4x^2}{(4-x^2)^2}$

$\Rightarrow \dfrac{-2(4+x^2)}{(4-x^2)^2}$ $f''(x)<0$ for all $x \in (-2,2)$

So, it is concave down everywhere

no inflection point



eg.) $f(x) = \dfrac{x^3}{x^2+1}$ ★ Domain $\mathbb{R}$ ★ $(0,0)$ intercept

★ Symmetry check $\dfrac{x^3}{x^2+1} \overset{?}{=} \dfrac{(-x)^3}{(-x)^2+1}$ $f(x) = -f(-x)$ ↳ odd function

$\boxed{\text{symmetric about the origin}}$

★ as x^2+1 never 0 there is no vertical asymptote

$$\left.\begin{aligned}\lim_{x\to\infty} \frac{x^3}{x^2+1} &= \lim_{x\to\infty} \frac{x}{1+\frac{1}{x^2}} = \infty \\ \lim_{x\to-\infty} \frac{x^3}{x^2+1} &= \lim_{x\to-\infty} \frac{x}{1+\frac{1}{x^2}} = -\infty\end{aligned}\right\}$$ there is no horizontal asymptote

$$\begin{array}{r|l} \cancel{x^3} & x^2+1 \\ \hline -\cancel{x^3}\mp x & x \\ \hline -x & \end{array} \Rightarrow \frac{x\cdot(x^2+1)-x}{x^2+1} = x - \frac{x}{x^2+1}$$

$$\lim_{x\to\pm\infty} \left[\left(x - \frac{x}{x^2+1}\right) - x\right] = \lim_{x\to\pm\infty}\left(-\frac{x}{x^2+1}\right) = \lim_{x\to\pm\infty} -\frac{\frac{1}{x}}{1+\frac{1}{x^2}} = 0$$

$\boxed{\text{So, } y=x \text{ is an oblique asymptote}}$

★ $f'(x) = \dfrac{3x^2\cdot(x^2+1) - x^3(2x)}{(x^2+1)^2} = \dfrac{x^4+3x^2}{(x^2+1)^2} = \dfrac{x^2(x^2+3)}{(x^2+1)^2}$

except $x=0$, $f'(x)>0$ $\boxed{\text{increasing on } (-\infty,\infty)-\{0\}}$

$f'(0)=0$ $f'(x)$ does not change sign so, there is no local min or max.

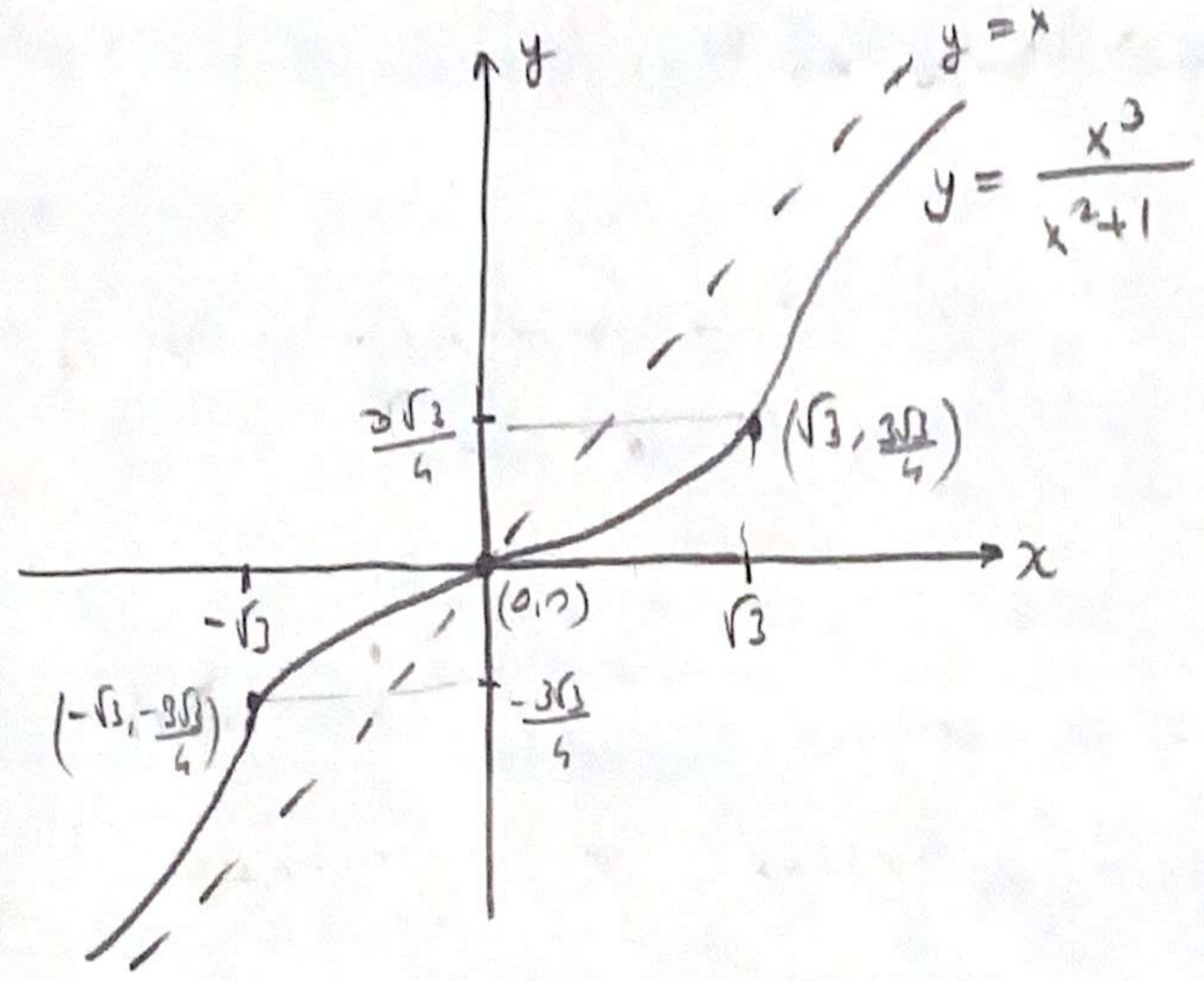
★ $f''(x) = \dfrac{(4x^3+6x)\cdot(x^2+1)^2 - (x^4+3x^2)\cdot 2(x^2+1)\cdot(2x)}{(x^2+1)^4}$

$$= \frac{\cancel{(x^2+1)}\left[(\cancel{4x^5}+\cancel{10x^3}+6x) - (\cancel{4x^5}+\cancel{12x^3})\right]}{(x^2+1)^{\cancel{4}\,3}} = \frac{-2x(x^2-3)}{(x^2+1)^3}$$

$-2x^3$ 0 $-\sqrt{3}$ $\sqrt{3}$

	$-\infty$	$-\sqrt{3}$		0		$\sqrt{3}$		∞
$f''(x)$	$+$		$-$		$+$		$-$	
	con. up		con. down		con. up		con. down	

$\boxed{\left(-\sqrt{3}, -\frac{3\sqrt{3}}{4}\right), (0,0), \left(\sqrt{3}, \frac{3\sqrt{3}}{4}\right) \quad \text{inflection points}}$



eg.) $f(x) = \frac{x^2+2x+4}{2x}$ ★ $\boxed{\text{Domain } (-\infty, 0) \cup (0, \infty)}$

★ $0 = \frac{x^2+2x+4}{2x}$ $x^2+2x+4=0$ $\Delta = 2^2 - 4.1.4$ $\Delta < 0$
↳ always positive

$\boxed{\text{there is no intercepts (neither y nor x)}}$

★ symmetry check $\frac{x^2+2x+4}{2x} \overset{?}{=} \frac{(-x)^2+2(-x)+4}{2(-x)}$

$\boxed{\text{neither odd nor even}}$

$$\begin{array}{l|l} x^2+2x+4 & 2x \\ \cline{2-2} -x^2 & \frac{x}{2}+1 \\ \hline 2x+4 & \\ -2x & \\ \hline 4 & \end{array}$$

$$\frac{2x \cdot \left(\frac{x}{2}+1\right)+4}{2x} = \left(\frac{x}{2}+1\right) + \frac{4}{2x}$$

$$\lim_{x\to\pm\infty} \left[\left[\left(\frac{x}{2}+1\right) + \frac{4}{2x}\right] - \left(\frac{x}{2}+1\right)\right] = \lim_{x\to\pm\infty} \left(\frac{4}{2x}\right) = 0$$

$\boxed{\text{So, } \left(\frac{x}{2}+1\right) \text{ is an oblique asymptote}}$

★ $f'(x) = \frac{(2x+2).(2x) - (x^2+2x+4).2}{(2x)^2} = \frac{4x^2 + \cancel{4x} - 2x^2 - \cancel{4x} - 8}{4x^2}$

$f'(x) = \frac{x^2-4}{2x^2} = \frac{(x-2)(x+2)}{2x^2}$ $\rightarrow -2, 2$
↳ 0

	$-\infty$	-2	0	2	∞
$f'(x)$	+	−	‖	−	+

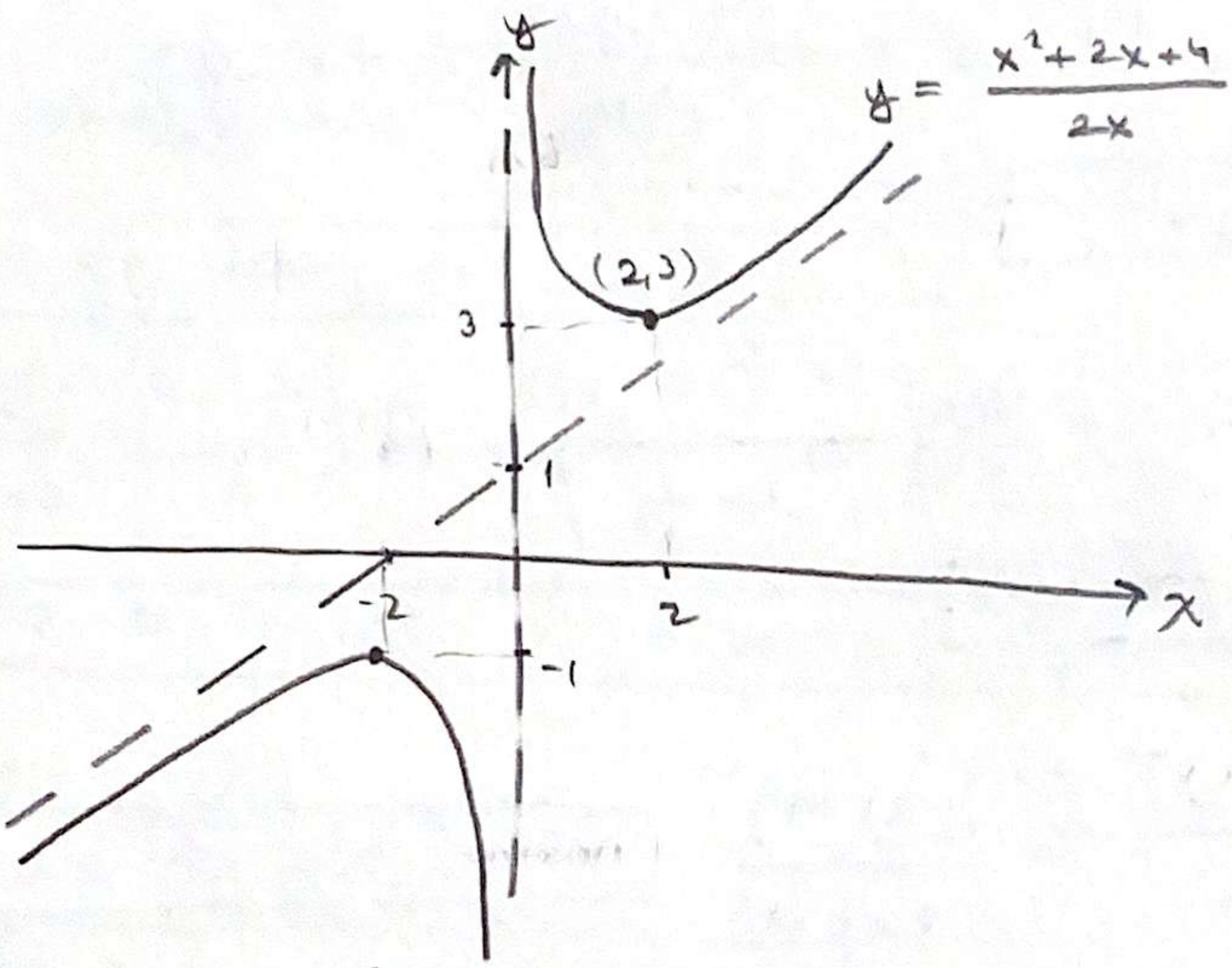
(−2, −1) (2, 3)

$\boxed{(-2,-1) \text{ local max}}$ | $\boxed{(2,3) \text{ local min}}$

★ $$f''(x) = \frac{2x \cdot (2x^2) - (x^2-4) \cdot 4x}{(2x^2)^2} = \frac{\cancel{4x^3} - \cancel{4x^3} + 16x}{4x^4} = \frac{4}{x^3}$$

	$-\infty$	0	∞
$f''(x)$	$-$	‖	$+$
	con. down		con. up

not inflection point because "0" is not in domain



★ $$\lim_{x \to 0^+} \left(\frac{x^2+2x+4}{2x} \right) = \lim_{x \to 0^+} \left(\frac{x+2+\frac{4}{x}}{2} \right) = +\infty$$

$$\lim_{x \to 0^-} \left(\frac{x^2+2x+4}{2x} \right) = \lim_{x \to 0^-} \left(\frac{x+2+\frac{4}{x}}{2} \right) = -\infty$$

$x=0$ so y-axis is vertical asymptote

eg.) $f(x) = \dfrac{(x+1)^2}{1+x^2}$ sketch the graph of the given function

★ Domain $\mathbb{R}$ $\boxed{(-\infty, \infty)}$

★ $x = 0 \quad y = 1$ $\boxed{(0,1) \text{ y-axis intercept}}$

$y = 0 \quad x = -1$ $\boxed{(-1,0) \text{ x-axis intercept}}$

★ $\dfrac{x^2+2x+1}{x^2+1}$

$$\begin{array}{l|l} x^2+2x+1 & x^2+1 \\ \cancel{-x^2 \mp 1} & \hline 1 \\ \hline 2x & \end{array}$$

$$\frac{(x^2+1)+2x}{x^2+1} = 1 + \frac{2x}{x^2+1}$$

$\boxed{\text{there is no oblique asymptote}}$

$$\lim_{x \to \pm\infty} \left(\frac{(x+1)^2}{1+x^2} \right) = \lim_{x \to \pm\infty} \left(\frac{x^2+2x+1}{x^2+1} \right) = \lim_{x \to \pm\infty} \left(\frac{1 + \frac{2}{x} + \frac{1}{x^2}}{1 + \frac{1}{x^2}} \right) = 1$$

$\boxed{y = 1 \text{ is a horizontal asymptote}}$

★ symmetry check $\dfrac{(x+1)^2}{1+x^2} \stackrel{?}{=} \dfrac{((-x)+1)^2}{1+(-x)^2}$ $\boxed{\text{neither odd nor even}}$

★ $$f'(x) = \frac{2.(x+1).1.(1+x^2) - 2x.(x+1)^2}{(1+x^2)^2} = \frac{2(x+1)\left[(1+x^2)-(x+1)\right]}{(1+x^2)^2}$$

$$f'(x) = \frac{2.(x+1)(x-1)}{(1+x^2)^2}$$

	$-\infty$ to -1	-1 to 1	1 to ∞
$f'(x)$	$+$	$-$	$+$
$f(x)$	↗	↘	↗

$\boxed{(-1,0) \text{ local max point}}$

$\boxed{(1,\ 2) \text{ local min point}}$

$\boxed{f \text{ is increasing on } (-\infty,-1) \text{ and } (1,\infty)}$

$\boxed{f \text{ is decreasing on } (-1,1)}$

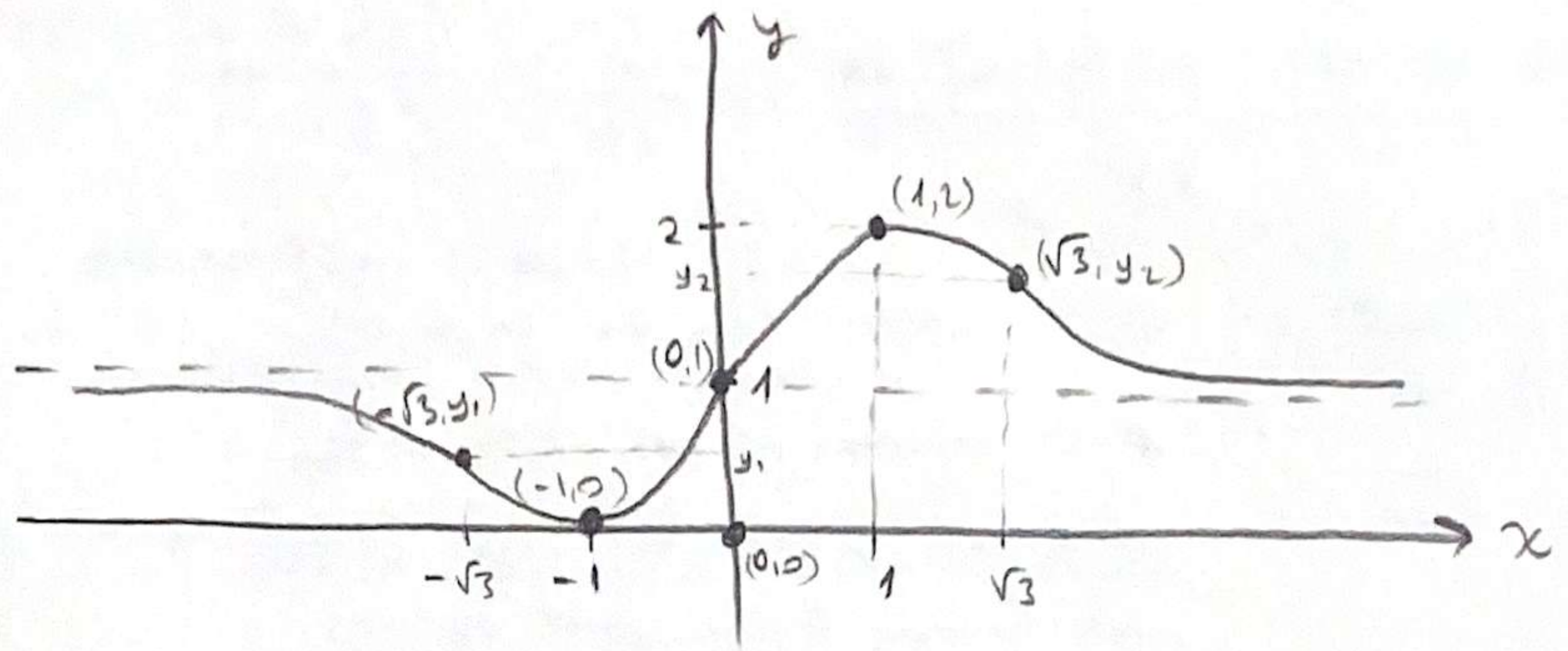
★ $$f''(x) = \frac{4x.(1+x^2)^2 - 2(1+x^2).2x.(2x^2-2)}{(1+x^2)^4}$$

$$\frac{2x.\cancel{(1+x^2)}.(2+2x^2-4x^2+4)}{(1+x^2)^{\cancel{4}\,3}} \Rightarrow \frac{-4x(x^2-3)}{(1+x^2)^3}$$

	$-\infty$ to $-\sqrt{3}$	$-\sqrt{3}$ to 0	0 to $\sqrt{3}$	$\sqrt{3}$ to ∞
$f''(x)$	$+$	$-$	$+$	$-$
	con. up	con. down	con. up	con. down

$\boxed{(-\sqrt{3}, y_1),\ (0,0),\ (\sqrt{3}, y_2) \text{ are inflection points}}$

$y_2 > y_1$



eg.) $f(x) = \dfrac{x^2+4}{2x}$

★ Domain $(-\infty, 0) \cup (0, \infty)$

★ There is any intercept

★ Symmetry check $\quad \dfrac{x^2+4}{2x} \overset{?}{=} \dfrac{(-x)^2+4}{2(-x)} \Rightarrow$ odd $[f(x) = -f(-x)]$

$\boxed{\text{symmetric about the origin } (0,0)}$

★ $\lim\limits_{x \to 0^-} \left(\dfrac{x + \frac{4}{x}}{2}\right) = -\infty \qquad \lim\limits_{x \to 0^+} \left(\dfrac{x + \frac{4}{x}}{2}\right) = +\infty$

$\boxed{x = 0 \text{ so, y-axis is a vertical asymptote}}$

$\lim\limits_{x \to +\infty} \left(\dfrac{x + \frac{4}{x}}{2}\right) = +\infty \qquad \lim\limits_{x \to -\infty} \left(\dfrac{x + \frac{4}{x}}{2}\right) = -\infty$

$\boxed{\text{there is no horizontal asymptote}}$

$$\begin{array}{r|l} x^2+4 & 2x \\ \cline{2-2} -x^2 & \frac{x}{2} \\ \hline 4 & \end{array} \qquad \frac{\left(\frac{x}{2} \cdot 2x\right) + 4}{2x} = \frac{1}{2}x + \frac{4}{2x}$$

$$\left.\begin{array}{l} \lim\limits_{x \to -\infty} \left[\left(\frac{1}{2}x + \frac{4}{2x}\right) - \frac{1}{2}x\right] = \lim\limits_{x \to -\infty} \left(\frac{4}{2x}\right) = 0 \\ \lim\limits_{x \to +\infty} \left[\left(\frac{1}{2}x + \frac{4}{2x}\right) - \frac{1}{2}x\right] = \lim\limits_{x \to +\infty} \left(\frac{4}{2x}\right) = 0 \end{array}\right\} \boxed{\tfrac{1}{2}x \text{ is an oblique asymptote}}$$

★ $f'(x) = \dfrac{2x \cdot (2x) - (x^2+4) \cdot 2}{4x^2} = \dfrac{2x^2 - 8}{4x^2} = \dfrac{x^2-4}{2x^2}$

	$-\infty$	-2	0	2	∞
$f'(x)$	+	−	−	+	
	inc.	dec.	dec.	inc.	

$\boxed{(-2, -2) \text{ local max}}$

$\boxed{(2, 2) \text{ local min}}$

f is increasing on $(-\infty, -2)$ and $(2, \infty)$

f is decreasing on $(-2, 2)$

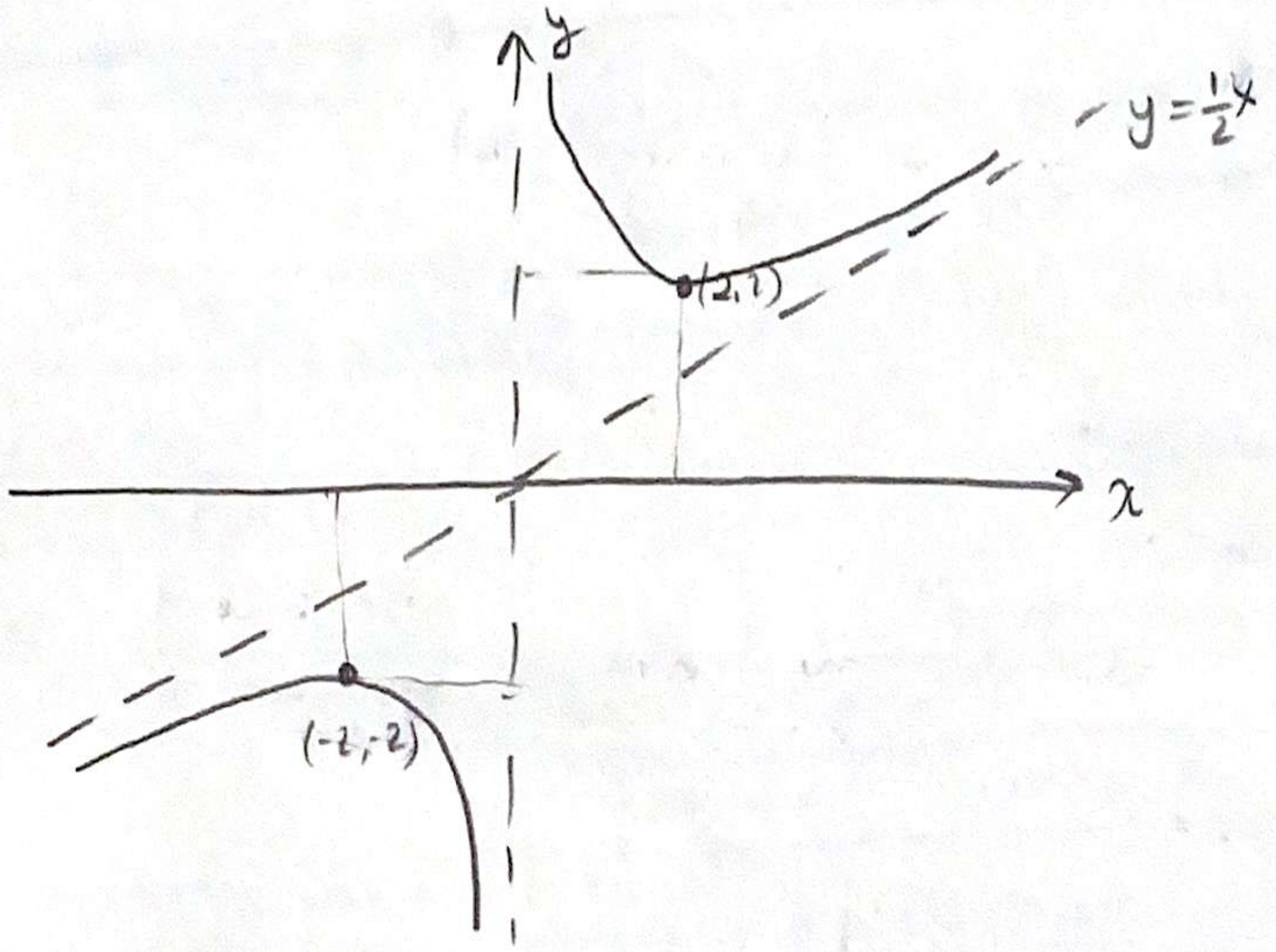
★ $f''(x) = \dfrac{2x \cdot 2x^2 - (x^2-4) \cdot 4x}{4x^4} = \dfrac{4}{x^3}$

	$-\infty$	0	∞
$f''(x)$	$-$	‖	$+$
f	con. down		con. up

f is concave down on $(-\infty, 0)$

f is concave up on $(0, \infty)$

"0" is not an inflection point because of not belonging to the domain of $f(x)$



eg.) $f(x) = \dfrac{1}{4-x^2}$ ★ Domain $(-\infty,-2) \cup (-2,2) \cup (2,\infty)$

★ $x=0$ $y = \frac{1}{4}$ $\boxed{(0, \tfrac{1}{4}) \text{ y-axis intercept}}$

There is no x-axis intercept

★ Symmetry check $\dfrac{1}{4-x^2} \stackrel{?}{=} \dfrac{1}{4-(-x)^2}$ ⟹ even function $\boxed{\text{Symmetric about y-axis}}$

★ $\lim\limits_{x \to -2^-} \left(\dfrac{1}{4-x^2}\right) = \lim\limits_{x \to -2^-} \left(\dfrac{\frac{1}{x^2}}{\frac{4}{x^2}-1}\right) = -\infty$

$\lim\limits_{x \to -2^+} \left(\dfrac{1}{4-x^2}\right) = \lim\limits_{x \to -2^+} \left(\dfrac{\frac{1}{x^2}}{\frac{4}{x^2}-1}\right) = +\infty$ $\Big) -2$ $\boxed{x=-2 \text{ is a vertical asymptote}}$

$\lim\limits_{x \to 2^-} \left(\dfrac{\frac{1}{x^2}}{\frac{4}{x^2}-1}\right) = -\infty$ $\lim\limits_{x \to 2^+} \left(\dfrac{\frac{1}{x^2}}{\frac{4}{x^2}-1}\right) = +\infty \Big) +2$

$\boxed{x=2 \text{ is a vertical asymptote}}$

$\lim\limits_{x \to \pm\infty} \left(\dfrac{\frac{1}{x^2}}{\frac{4}{x^2}-1}\right) = 0$ $\boxed{y=0 \text{ so, x-axis is a horizontal asymptote}}$

There is no oblique asymptote

$\star\ f'(x) = \dfrac{-(-2x)}{(4-x^2)^2} = \dfrac{4x}{(4-x^2)^2}$

x	$-\infty$		-2		0		2		∞
$f'(x)$		$-$	‖	$-$	|	$+$	‖	$+$	
		↘		↘		↗		↗	

$\boxed{(0, \tfrac{1}{4}) \text{ local min point}}$

f is decreasing on $(-\infty, -2) \cup (-2, 0)$

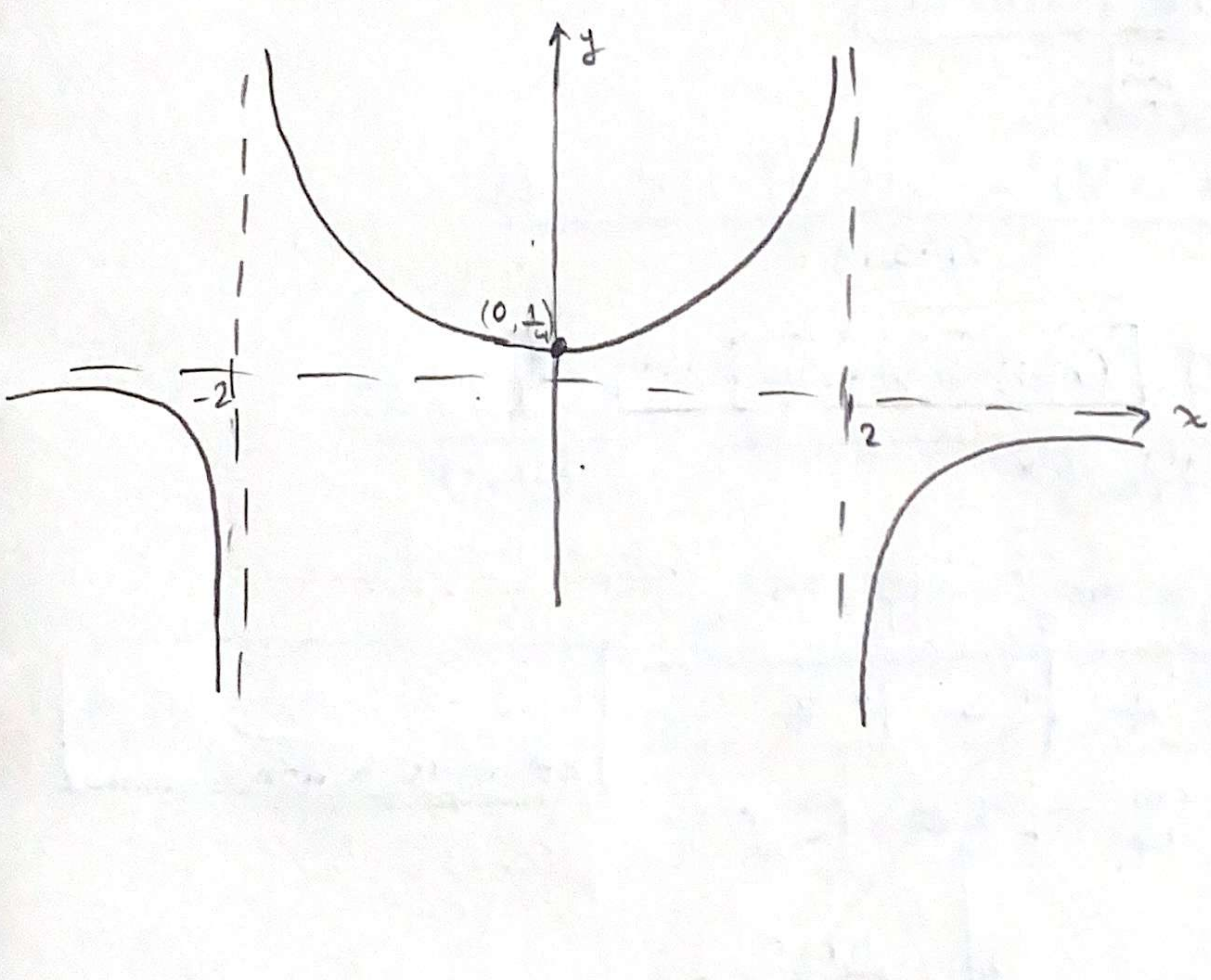
f is increasing on $(0, 2) \cup (2, \infty)$

$\star\ f''(x) = \dfrac{4 \cdot (4-x^2)^2 - 2(4-x^2) \cdot (-2x) \cdot 4x}{(4-x^2)^2}$

$= \dfrac{4 \cdot \cancel{(4-x^2)} \left[(4-x^2) + 4x^2\right]}{(4-x^2)^{\cancel{2}}} = \dfrac{4 \cdot (4 + 3x^2)}{(4-x^2)}$

x	$-\infty$		-2		2		∞
$f''(x)$		$-$	‖	$+$	‖	$-$	
		con. down		con. up		con down	

neither -2 nor 2 is a x value of an inflection point because they don't exist in the domain of the function



eg.) $f(x) = \dfrac{4x}{x^2+1}$ ★ Domain $(-\infty, \infty)$

★ $x=0$ $y=0$ $\boxed{(0,0) \text{ intercept}}$

★ symmetry check $\dfrac{4.x}{x^2+1} \overset{?}{=} \dfrac{4.(-x)}{(-x)^2+1} \Rightarrow f(x) = -f(-x)$ odd

$\boxed{\text{symmetric about the origin}}$

★ $\displaystyle\lim_{x\to\pm\infty}\left(\frac{\frac{4}{x}}{1+\frac{1}{x^2}}\right) = 0$ $\boxed{y=0 \text{ so, x-axis is a horizontal asymptote}}$

There is no vertical or oblique asymptote

★ $f'(x) = \dfrac{4.(x^2+1) - 4x.(2x)}{(x^2+1)^2} = \dfrac{4(x^2+1-2x^2)}{(x^2+1)^2} = \dfrac{-4(x^2-1)}{(x^2+1)^2}$

	$-\infty$		-1		1		∞
$f'(x)$		$-$	|	$+$	|	$-$	
f		$\searrow$		$\nearrow$		$\searrow$	

$\boxed{(1,2) \text{ local max}}$

$\boxed{(-1,-2) \text{ local min}}$

f is increasing on $(-1,1)$

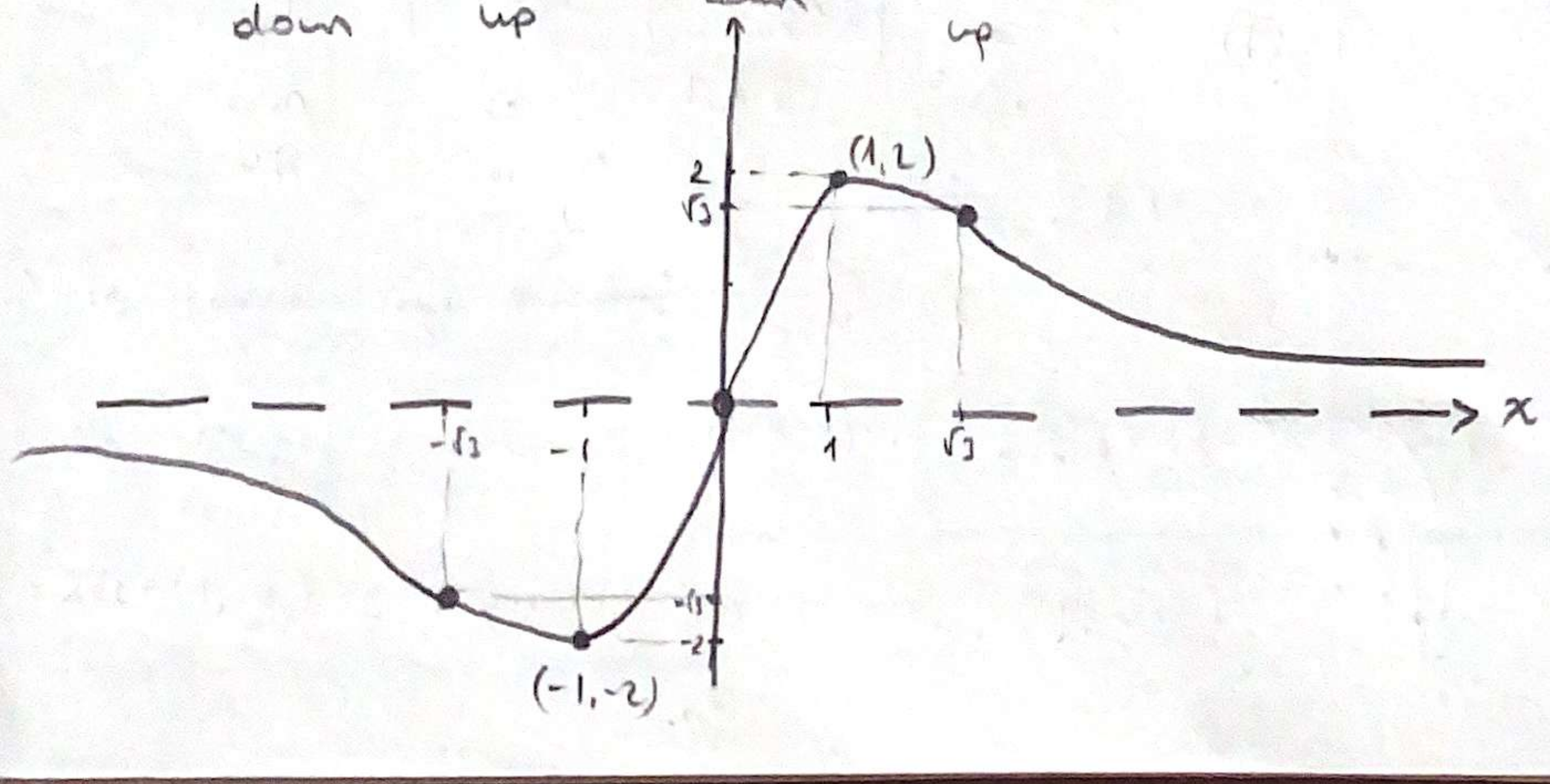
f is decreasing on $(-\infty,-1) \cup (1,\infty)$

★ $f''(x) = \dfrac{-8x.(x^2+1)^2 - 4(1-x^2).2(x^2+1).2x}{(x^2+1)^2}$

$= \dfrac{-8x\cancel{(x^2+1)}\left[(x^2+1)+2-2x^2\right]}{(x^2+1)^{\cancel{2}}} = \dfrac{8x.(x^2-3)}{(x^2+1)}$

	$-\infty$		$-\sqrt{3}$		0		$\sqrt{3}$		∞
$f''(x)$		$-$	|	$+$	|	$-$	|	$+$	
		con down		con up		con down		con up	

$\boxed{(-\sqrt{3},-\sqrt{3}),\ (0,0),\ (\sqrt{3},\sqrt{3}) \text{ are inflection points}}$



e.g.) $y = x - \frac{1}{x}$ ★ Domain $(-\infty, 0) \cup (0, \infty)$

★ $\boxed{(-1,0) \text{ and } (1,0) \text{ are x-axis intercepts}}$

There is no y-axis intercept

★ Symmetry check $x - \frac{1}{x} \stackrel{?}{=} (-x) - \frac{1}{(-x)} \Rightarrow f(x) = -f(-x)$ odd function

$\boxed{\text{Symmetric about the origin}}$

★ $\lim_{x \to 0^-} \left(x - \frac{1}{x}\right) = +\infty$ $\lim_{x \to 0^+} \left(x - \frac{1}{x}\right) = -\infty$

$\boxed{x = 0 \text{ so, y-axis is a vertical asymptote}}$

$\lim_{x \to -\infty} \left(x - \frac{1}{x}\right) = -\infty$ $\lim_{x \to +\infty} \left(x - \frac{1}{x}\right) = +\infty$

so, there is no horizontal asymptote

$\lim_{x \to \infty} \left[\left(x - \frac{1}{x}\right) - x\right] = \lim_{x \to \infty} \left(-\frac{1}{x}\right) = 0$ $\boxed{\text{so, } y = x \text{ is an oblique asymptote}}$

★ $f'(x) = 1 + \frac{1}{x^2}$

$-\infty$	0	∞
+	‖	+
↗		↗

f is increasing on $(-\infty, 0) \cup (0, \infty)$

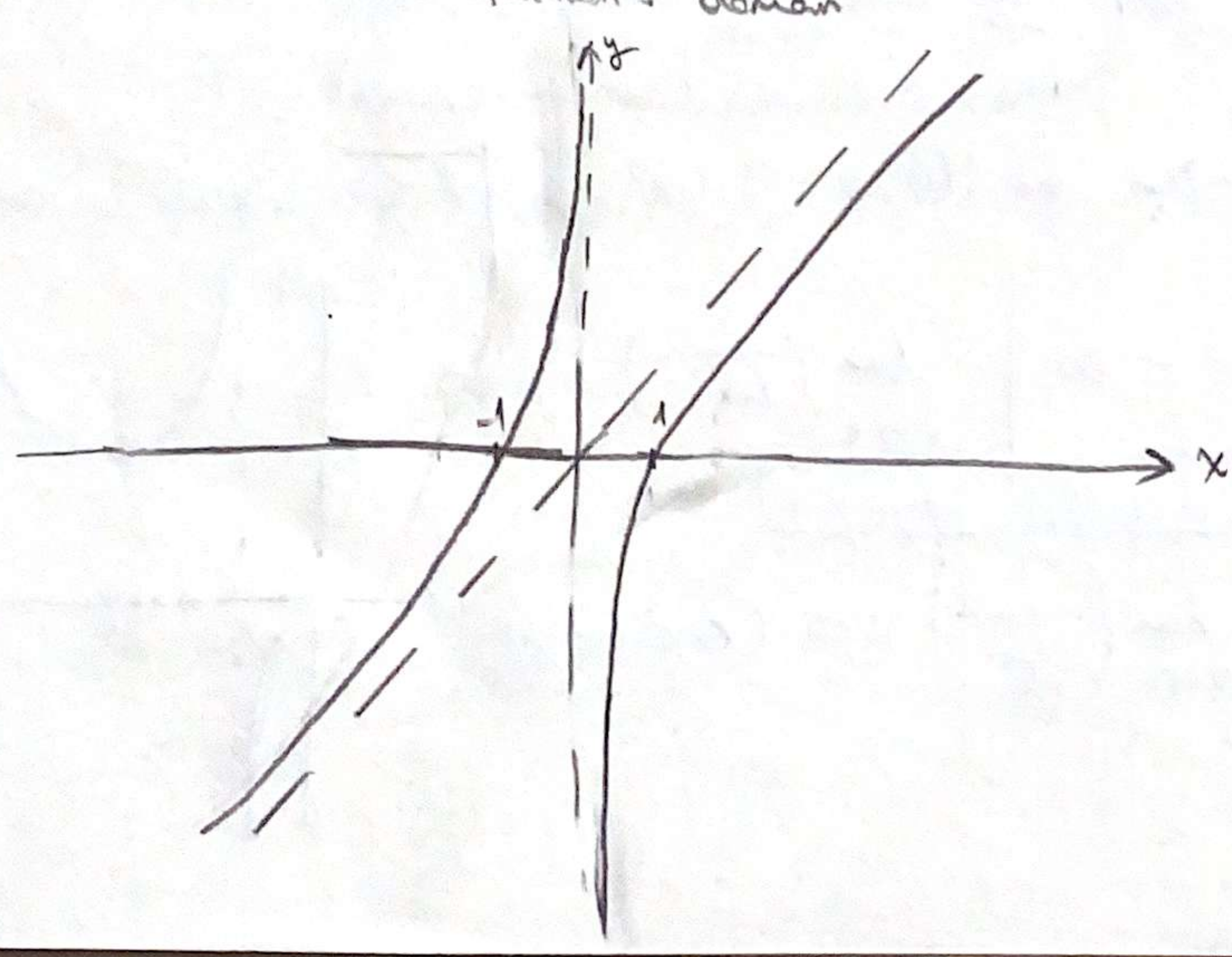
★ $f''(x) = -\frac{2}{x^3}$

$-\infty$	0	∞
+	‖	−
con. up		con. down

f is concave up on $(-\infty, 0)$

f is concave down on $(0, \infty)$

"0" is not x value of inflection point because there is not any due to the lack of "0" value in the function's domain



e.g.) $f(x) = \dfrac{x}{\ln x - 1}$ ★ $x > 0$ ($\ln x$) $\ln x - 1 \neq 0$

$\ln x \neq 1$ $x \neq e$

Domain $(0, e) \cup (e, \infty)$

★ There is no intercept ★ Not symmetric

★ $f'(x) = \dfrac{1.(\ln x - 1) - x.\left(\frac{1}{x}\right)}{(\ln x - 1)^2} = \dfrac{\ln x - 2}{(\ln x - 1)^2}$

	$-\infty$	e		e^2	∞
$f'(x)$	$-$	‖	$-$		$+$
	↘		↘	◡	↗

$\boxed{(e^2, e^2) \text{ local min}}$

f is increasing on (e^2, ∞)

f is decreasing on $(-\infty, e) \cup (e, e^2)$

★ $f''(x) = \dfrac{\left(\frac{1}{x}\right).(\ln x - 1)^2 - 2(\ln x - 1).\frac{1}{x}.(\ln x - 2)}{(\ln x - 1)^4}$

$= \dfrac{\left(\frac{1}{x}\right).\cancel{(\ln x - 1)}\left[(\ln x - 1) - 2(\ln x - 2)\right]}{(\ln x - 1)^{\cancel{4}\,3}} = \dfrac{3 - \ln x}{x.(\ln x - 1)^3}$

	0		e		e^3		∞
$f''(x)$	‖	$-$	‖	$+$		$-$	
		con down		con up		con down	

not inflection (not in domain)

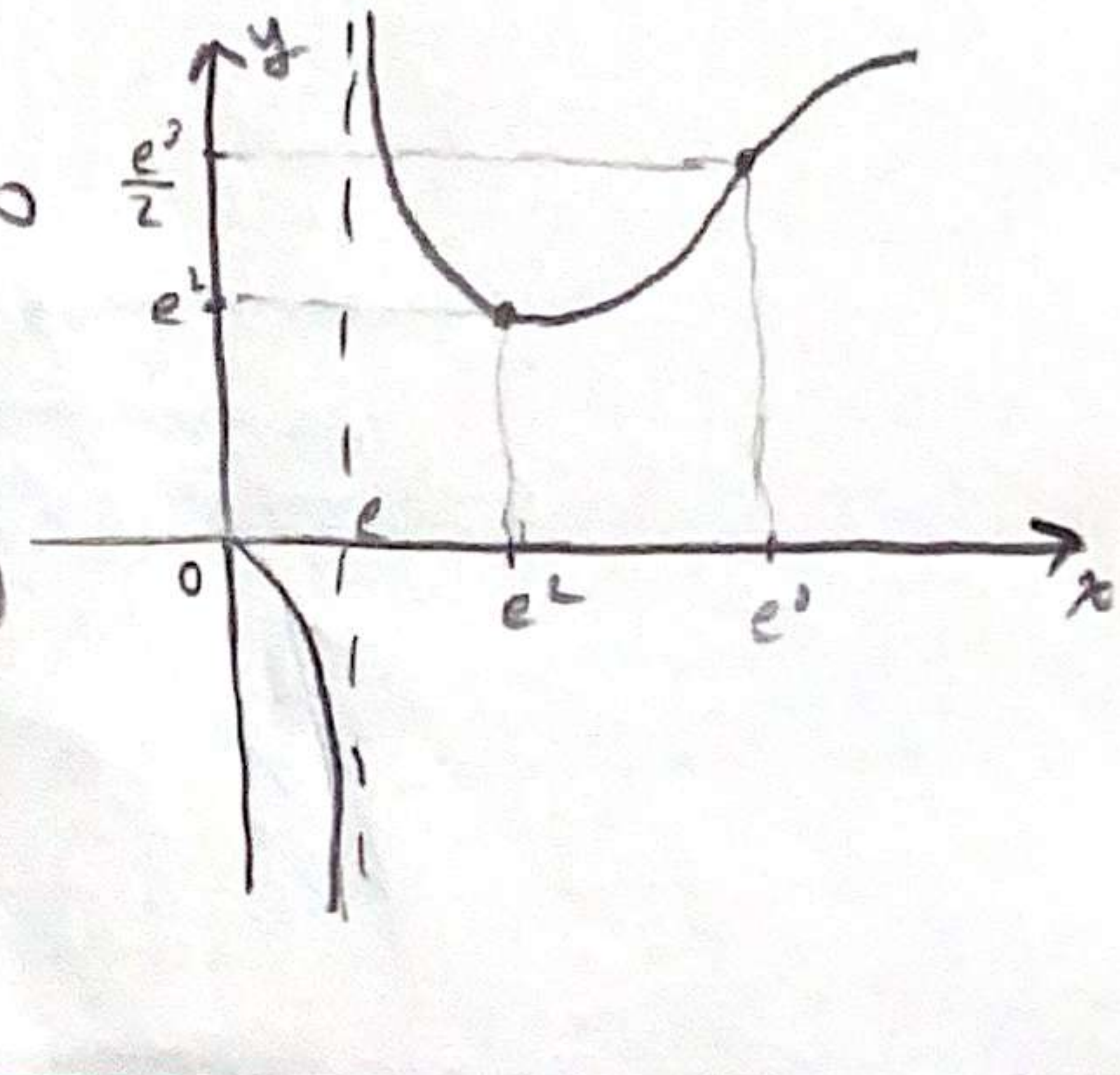
$\boxed{\left(e^3, \frac{e^3}{2}\right) \text{ is an inflection point}}$

f is concave down on $(0, e) \cup (e^3, \infty)$ | f is concave up on (e, e^3)

★ $\lim_{x \to e^-} \left(\dfrac{x}{\ln x - 1}\right) = -\infty$ | $\lim_{x \to e^+} \left(\dfrac{x}{\ln x - 1}\right) = +\infty$

$\boxed{x = e \text{ is a vertical asymptote}}$

$\lim_{x \to 0^+} \left(\dfrac{x}{\ln x - 1}\right) = \lim_{x \to 0^+} \left(\dfrac{1}{\frac{1}{x}}\right) = 0$ endpoint



eg.) $f(x) = x \cdot e^{-x}$ ★ Domain $(-\infty, \infty)$ ★ $(0,0)$ intercept

★ not symmetric

★ $\lim\limits_{x \to \pm\infty} (x \cdot e^{-x}) = \lim\limits_{x \to \pm\infty} \left(\frac{x}{e^x}\right) = \lim\limits_{x \to \pm\infty} \left(\frac{1}{e^x}\right) = 0$

$\boxed{y=0 \text{ so, x-axis is a horizontal asymptote}}$

There is no vertical or oblique asymptote

★ $f'(x) = e^{-x} + x \cdot -e^{-x} = e^{-x}(1-x)$

	$-\infty$	1	∞
$f'(x)$	$+$		$-$
	↗		↘

$\boxed{(1, 1/e) \text{ local max}}$

f is increasing on $(-\infty, 1)$

f is decreasing on $(1, \infty)$

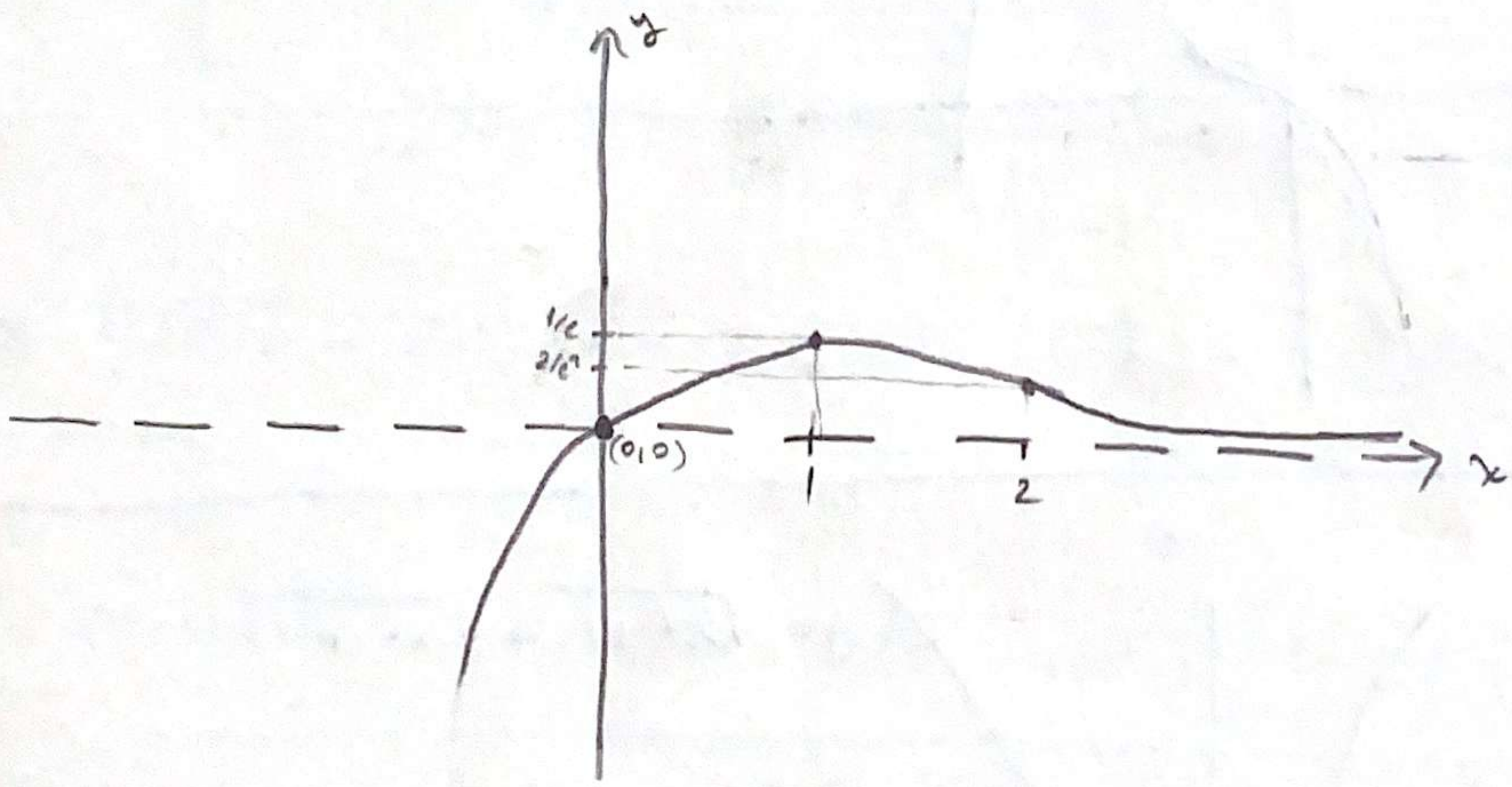
★ $f''(x) = -e^{-x} \cdot (1-x) + e^{-x} \cdot (-1) = -e^{-x}((1-x)+1) = -e^{-x}(2-x)$

	$-\infty$	2	∞
$f''(x)$	$-$		$+$
	con. down		con. up

$\boxed{(2, 2/e^2) \text{ is an inflection point}}$

f is concave down on $(-\infty, 2)$

f is concave up on $(2, \infty)$



e.g.) $f(x) = \frac{1}{5}(4x^3 - x^4) = \frac{4}{5}x^3 - \frac{1}{5}x^4$ ★ Domain $(-\infty, \infty)$

★ $(0,0)$, $(4,0)$ intercepts ★ not symmetric or periodic

★ no asymptotes

★ $f'(x) = \frac{12}{5}x^2 - \frac{4}{5}x^3 = \frac{4 \cdot x^2}{5}(3 - x)$

	$-\infty$	0		3	∞
$f'(x)$	$+$		$+$		$-$
	$\nearrow$		$\nearrow$		$\searrow$

$(3, \frac{27}{5})$ local max

f is increasing on $(-\infty, 0) \cup (0, 3)$

f is decreasing on $(3, \infty)$

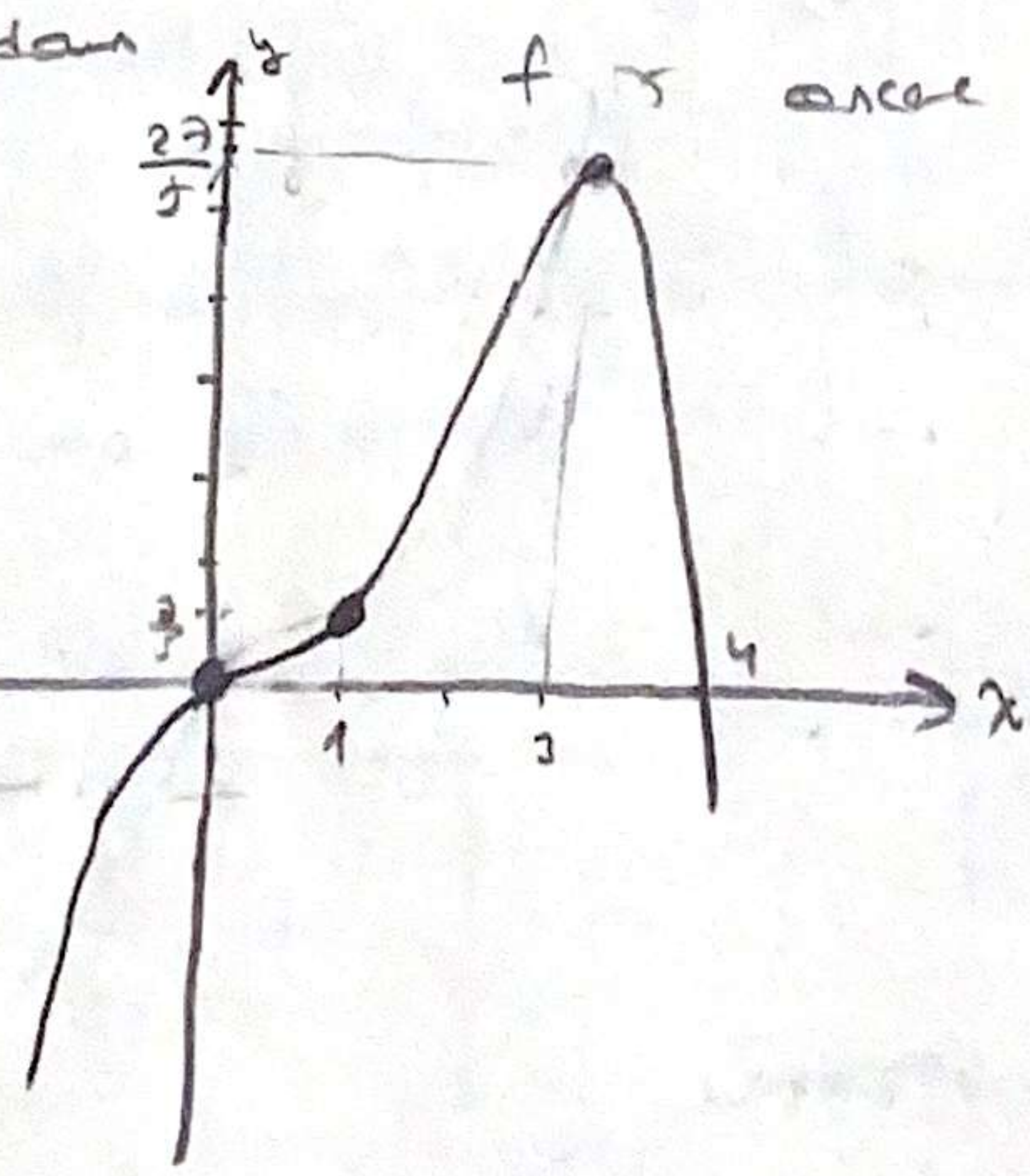
★ $f''(x) = \frac{24}{5}x - \frac{12}{5}x^2 = \frac{12}{5}x(1-x)$

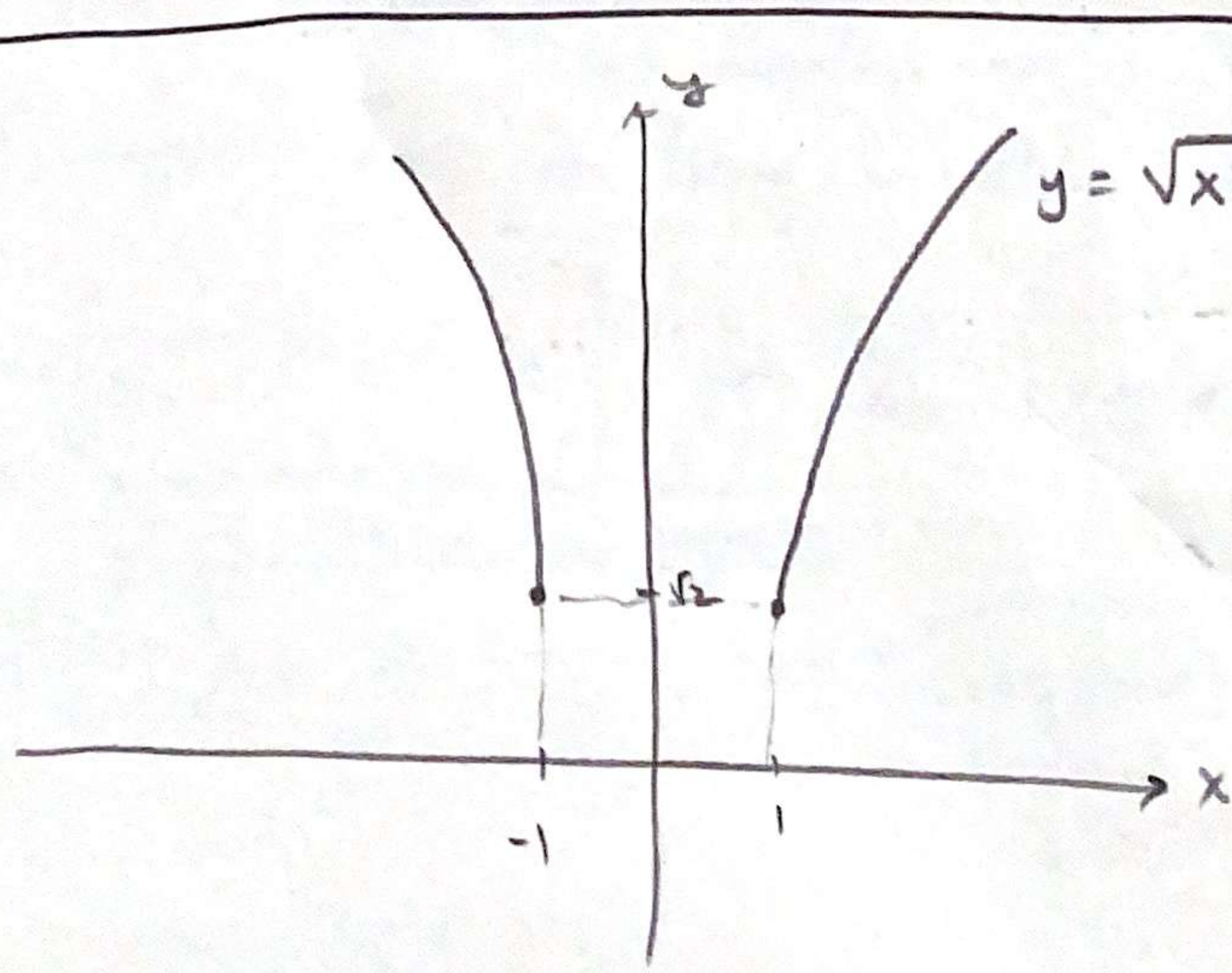
	$-\infty$	0		1	∞
$f''(x)$	$-$		$+$		$-$
	con down		con up		con. down

$(0,0)$ and $(1, \frac{3}{5})$ are inflection points

f is concave down on $(-\infty, 0) \cup (1, \infty)$

f is concave up on $(0,1)$





$y = \sqrt{x^2+1} + \sqrt{x^2-1}$

eg.) $f(x) = \sqrt{x^2+1} + \sqrt{x^2-1}$ $\qquad x^2-1 \geq 0$

	-1		1	
$+$		$-$		$+$

★ Domain $(-\infty, -1] \cup [1, \infty)$

★ no intercepts

★ $f(x) = f(-x)$ even func. → symmetric about y-axis

★ $\lim\limits_{x \to -1^-} \left(\sqrt{x^2+1} + \sqrt{x^2-1}\right) \neq \pm\infty \qquad \lim\limits_{x \to 1^+} \left(\sqrt{x^2+1} + \sqrt{x^2-1}\right) \neq \pm\infty$

there is no vertical asymptote

$\lim\limits_{x \to \pm\infty} \left(\sqrt{x^2+1} + \sqrt{x^2-1}\right) = \infty$

there is no horizontal asymptote

there is no oblique asymptote

★ $$f'(x) = \frac{1}{\cancel{2}\sqrt{x^2+1}} \cdot \cancel{2}x + \frac{1}{\cancel{2}\sqrt{x^2-1}} \cdot \cancel{2}x = \frac{x\left(\sqrt{x^2-1} + \sqrt{x^2+1}\right)}{\sqrt{(x^4-1)}}$$

	$-\infty$	0	1	∞
$f'(x)$	$-$	$+$	$+$	
f	↘	↗	↗	

"0" not in the domain

f is decreasing on $(-\infty, -1)$

f is increasing on $(1, \infty)$

★ $$f''(x) = \frac{\left[\left(\sqrt{x^2-1} + \sqrt{x^2+1}\right) + x \cdot \left(\frac{x\left(\sqrt{x^2-1} + \sqrt{x^2+1}\right)}{\sqrt{x^4-1}}\right)\right] \cdot \sqrt{x^4-1} - \frac{4x^3}{2\sqrt{x^4-1}} \cdot \frac{x\left(\sqrt{x^2-1} + \sqrt{x^2+1}\right)}{\sqrt{x^4-1}}}{(x^4-1)}$$

$$\frac{(x^2-1)\sqrt{x^2+1} + (x^2+1)\cdot\sqrt{x^2-1} + x^2\left(\sqrt{x^2-1} + \sqrt{x^2+1}\right) - \frac{2x^4\left(\sqrt{x^2-1} + \sqrt{x^2+1}\right)}{x^4-1}}{x^4-1}$$

$$= \frac{(x^2-1)\sqrt{x^2+1} + (x^2+1)\sqrt{x^2-1} + x^2\left(\sqrt{x^2-1} + \sqrt{x^2+1}\right)\left[1 - \frac{2x^2}{x^4-1}\right]}{x^4-1}$$

$x^4 - 1 - 2x^2 = (x^2-1)^2$

$f''(x) > 0$ for all $x \in (-\infty, -1] \cup [1, \infty)$ concave up e.g.) $f(x) = \dfrac{1-x^3}{x^2}$ * Domain $(-\infty, 0) \cup (0, \infty)$ * $(1,0)$ intercept

* not symmetric or periodic

* $\lim\limits_{x \to 0^-} \left(\dfrac{\frac{1}{x^2} - x}{1} \right) = \infty$ $\lim\limits_{x \to 0^+} \left(\dfrac{\frac{1}{x^2} - x}{1} \right) = \infty$

$x=0$ so, y-axis is a vertical asymptote

$\lim\limits_{x \to -\infty} \left(\dfrac{\frac{1}{x^2} - x}{1} \right) = \infty$ $\lim\limits_{x \to +\infty} \left(\dfrac{\frac{1}{x^2} - x}{1} \right) = -\infty$

there is no horizontal asymptote

$$\begin{array}{r|l} 1 - \cancel{x^3} & x^2 \\ \cancel{+x^3} & -x \\ \hline 1 & \end{array} \qquad \frac{(-x \cdot x^2) + 1}{x^2} = -x + \frac{1}{x^2}$$

$\lim\limits_{x \to \infty} \left[\left(-x + \frac{1}{x^2} \right) - (-x) \right] = \lim\limits_{x \to \infty} \left(\frac{1}{x^2} \right) = 0$

$y = -x$ is an oblique asymptote

* $f'(x) = \dfrac{-3x^2 \cdot x^2 - (1-x^3) \cdot 2x}{x^4} = \dfrac{-x^4 - 2x}{x^4} = \dfrac{-x(x^3+2)}{x^4}$

$$\begin{array}{c|c|c|c|c} & -\infty \quad & -\sqrt[3]{2} & \cancel{0} & \quad \infty \\ \hline f'(x) & - & + & - & \\ & \searrow & \nearrow & \searrow & \end{array}$$

$\left(-\sqrt[3]{2}, \frac{3}{\sqrt[3]{4}} \right)$ local min point

f is decreasing on $(-\infty, -\sqrt[3]{2})$

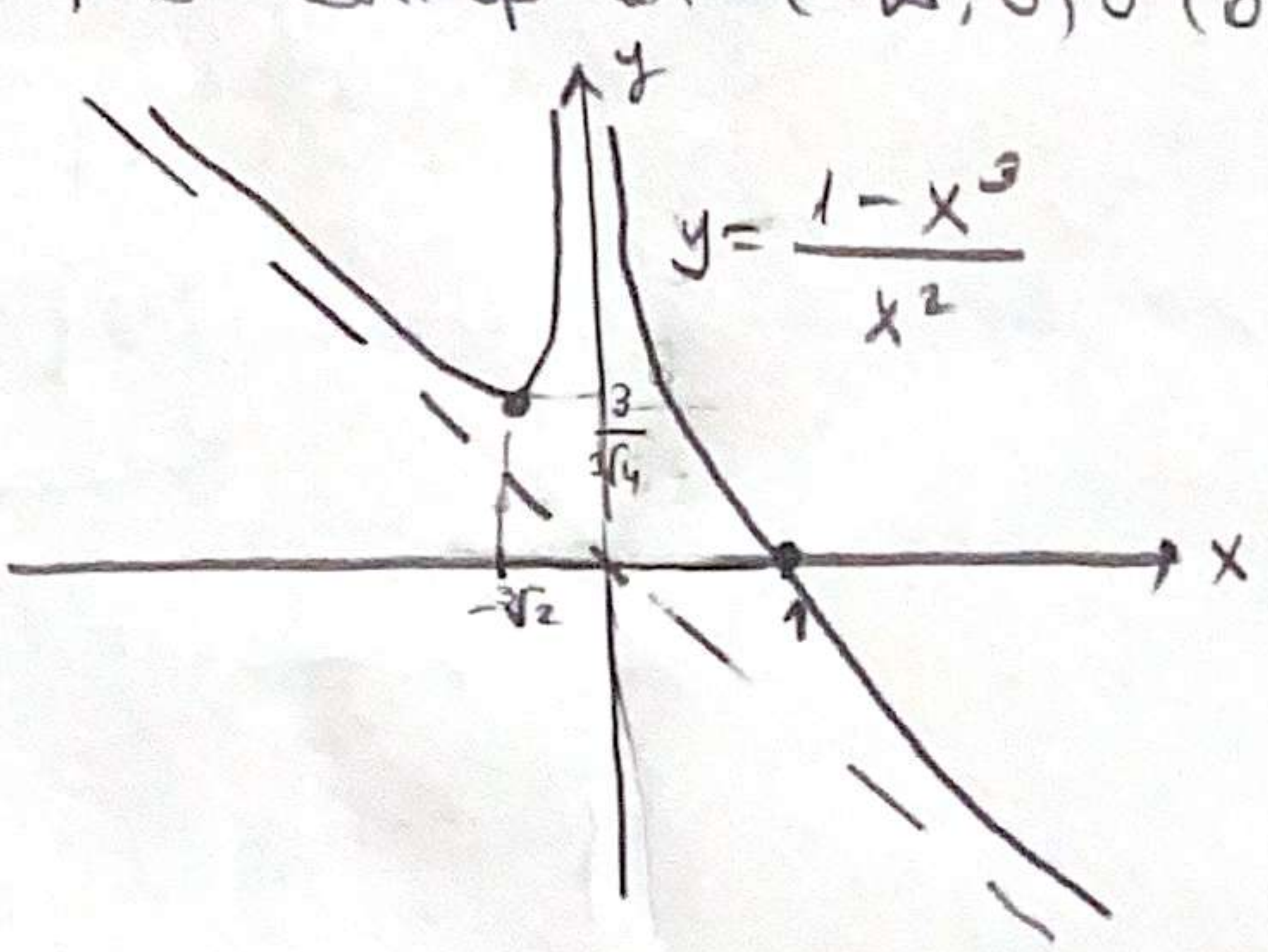
f is increasing on (

* $f''(x) = \dfrac{(-4x^3 - 2) \cdot x^4 + 4x^3(x^4 + 2x)}{x^8} = \dfrac{-\cancel{4x^7} - 2x^4 + \cancel{4x^7} + 8x^4}{x^8}$

$= \dfrac{6}{x^4}$

$$\begin{array}{c|c} & 0 \\ \hline + & + \\ \text{con up} & \text{con up} \end{array}$$

f is con. up on $(-\infty, 0) \cup (0, \infty)$



e.g.) $f(x) = \frac{\ln x}{x}$ ★ Domain $(0, \infty)$ ★ $(1,0)$ intercept

★ $\lim_{x \to 0^+} \left(\frac{\ln x}{x} \right) = \lim_{x \to 0^+} \left(\frac{1}{x} \right) = +\infty$

$x = 0$ is a vertical asymptote

$\lim_{x \to \infty} \left(\frac{1}{x} \right) = 0$ $y = 0$ is a horizontal asymptote

★ $f'(x) = \frac{\frac{1}{\cancel{x}} \cdot \cancel{x} - 1 \cdot \ln x}{x^2} = \frac{1 - \ln x^{\to e}}{x^2_{\to 0^+}}$

	0		e		∞
$f'(x)$	‖	+	|	−	
f		↗		↘	

$(e, \frac{1}{e})$ local max point

f is inc. on $(0, e)$

f is dec. on (e, ∞)

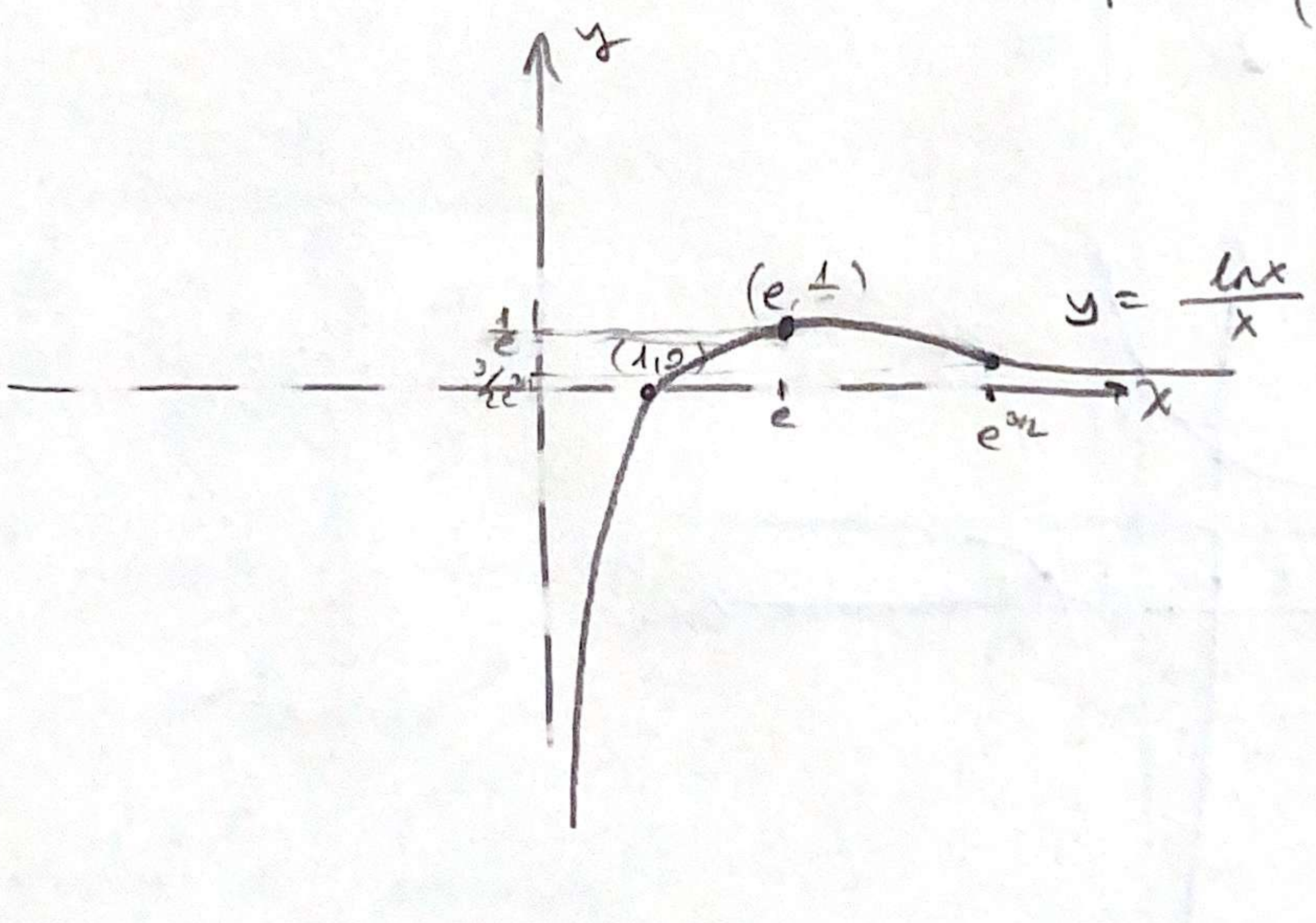
★ $f''(x) = \frac{-\frac{1}{\cancel{x}} \cdot x^{\cancel{2}} - 2x \cdot (1 - \ln x)}{x^4} = \frac{-3\cancel{x} + 2x \ln x}{x^{\cancel{4}\,3}} = \frac{(2 \ln x^{\to e^{3/2}} - 3)}{x^3_{\to 0}}$

	0		$e^{3/2}$		∞
$f''(x)$	‖	−	|	+	
f		con down		con up	

$\left(e^{3/2}, \frac{3}{2 \cdot e^{3/2}} \right)$ inflection point

f is con down on $(0, e^{3/2})$

f is con up on $(e^{3/2}, \infty)$



eg.) $f(x) = e^{\frac{1}{1-x}}$ ★ Domain $\mathbb{R} - \{1\}$

★ $\lim_{x \to \pm\infty} e^{\frac{1}{1-x}} = e^{\lim_{x\to\pm\infty} \frac{1}{1-x}} = e^{0} = 1$ $y=1$ horizontal asymptote

$x=1$ is a vertical asymptote

$g(x) = kx + b$

$k = \lim_{x \to \infty} \frac{e^{\frac{1}{1-x}}}{x} = \frac{e^{0}}{x} = 0$ $k=0$ so, there is no oblique asymp.

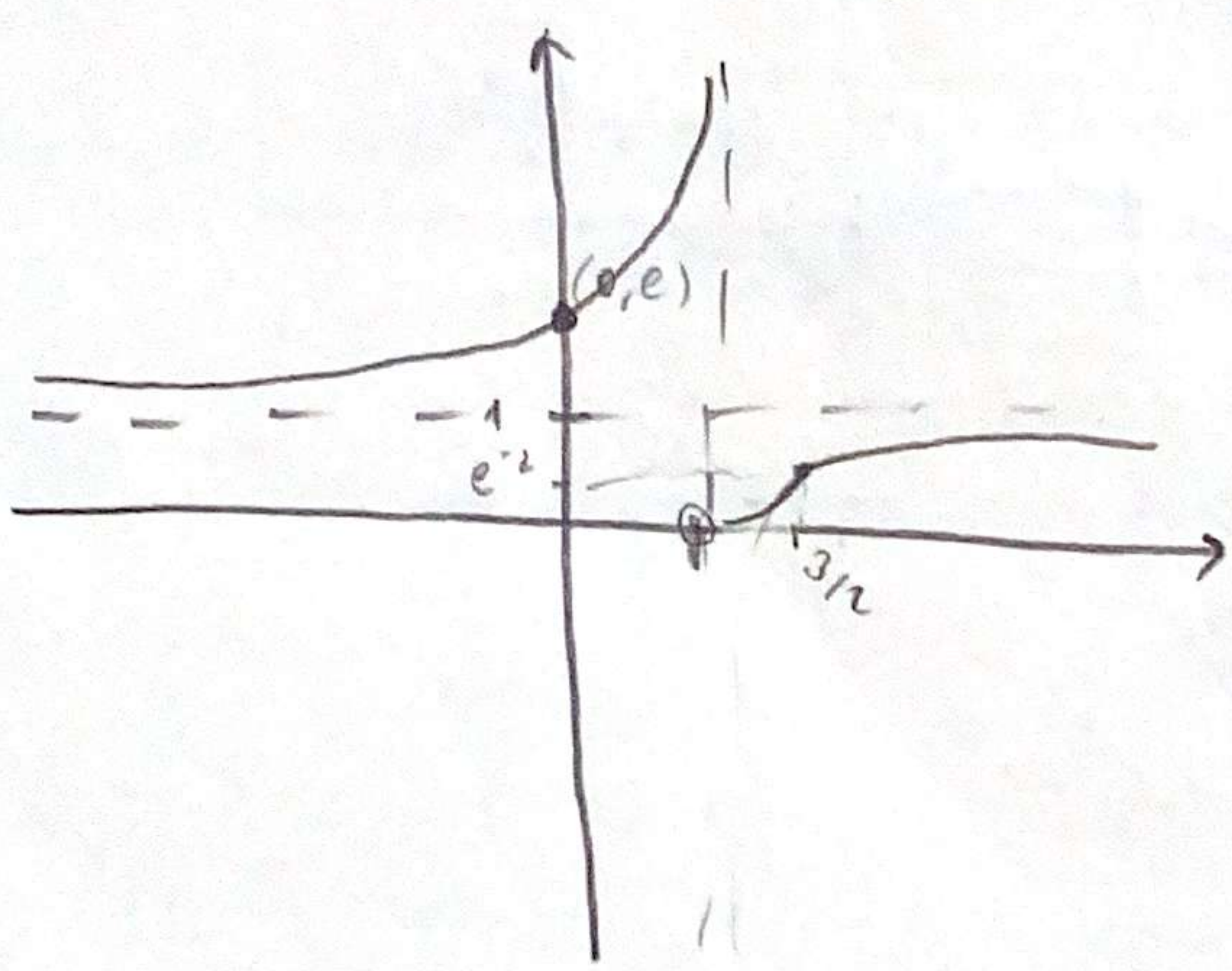
★ intercept $(0, e)$

★ $f'(x) = \frac{1}{(1-x)^2} \cdot e^{\frac{1}{1-x}}$ $f'(x) > 0$

★ $f''(x) = \left[\frac{+2}{(1-x)^3} \cdot e^{\frac{1}{1-x}} + \frac{1}{(1-x)^4} \cdot e^{\frac{1}{1-x}} \right]$

$f''(x) = e^{\frac{1}{1-x}} \frac{1}{(1-x)^3} \left[2 + \frac{1}{(1-x)} \right]$ $x = 1$, $x = 3/2$

	$-\infty$	1	$\frac{3}{2}$	∞
$f''(x)$	+	+	−	



eg.) $f(x) = \tanh(x)$ $\qquad \tanh(x) = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$

★ Domain $\mathbb{R}$

★ $\displaystyle \lim_{x \to \infty} \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} = \frac{\cancel{e^{x}}(1 - e^{-2x})}{\cancel{e^{x}}(1 + e^{-2x})} \Rightarrow y = 1$ horizontal asymptote

★ there is no vertical asymptote

★ $\displaystyle \lim_{x \to \infty} \left[\frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} - g(x) \right] = 0$ there is no such a function as $g(x)$ because there exist indeterminate "∞/∞" so, there is no oblique asymptote

★ $\displaystyle \lim_{x \to -\infty} \frac{\cancel{e^{-x}}(e^{-2x} - 1)}{\cancel{e^{-x}}(e^{-2x} + 1)} \Rightarrow y = -1$ horizontal asymptote

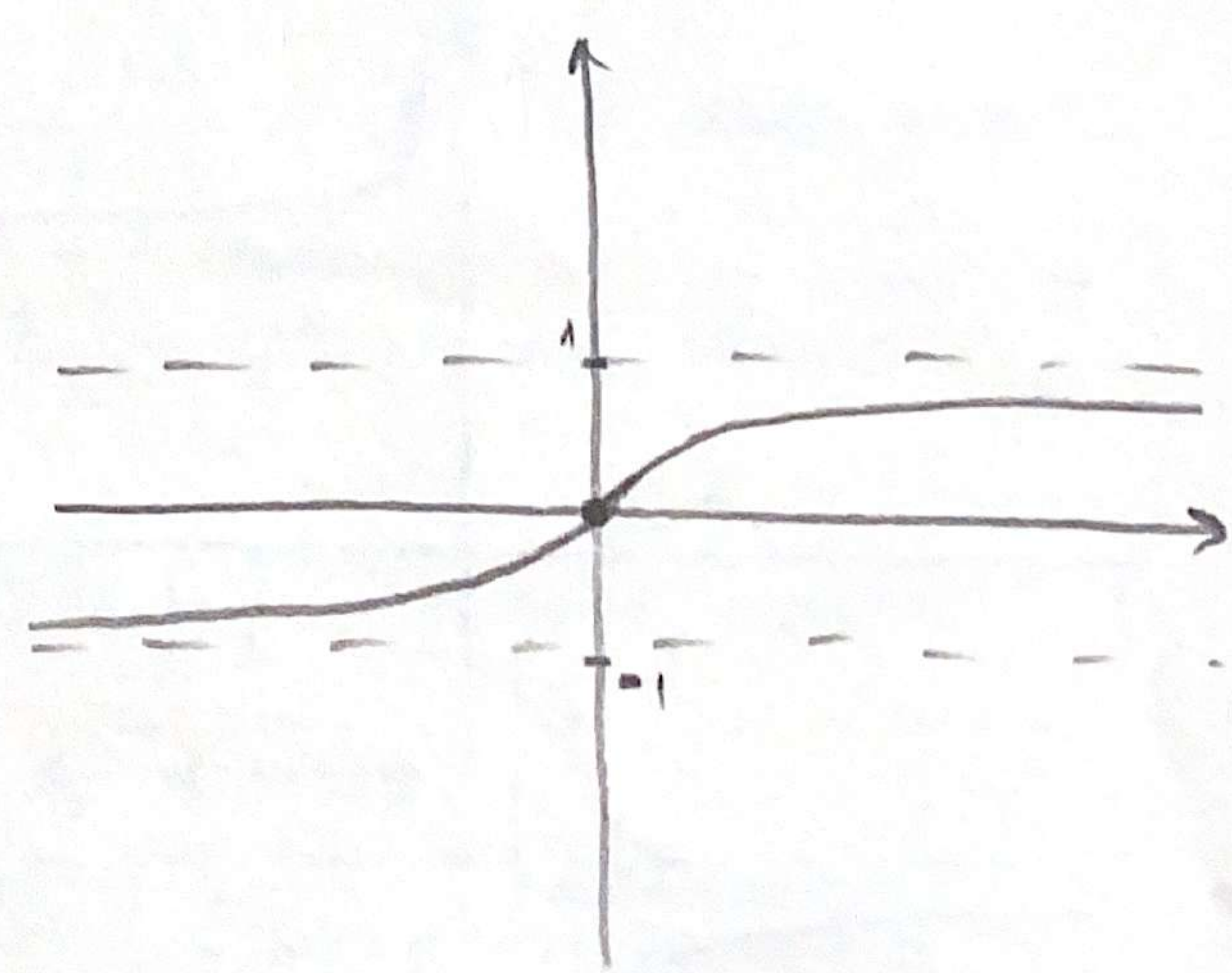
★ $x = 0$, $y = 0$ $(0,0)$ intercept

★ $\displaystyle f'(x) = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \left(\frac{1}{\left(\frac{e^{x} + e^{-x}}{2}\right)} \right)^2 \Rightarrow \frac{4}{(e^{x} + e^{-x})^2} > 0$

★ $\displaystyle f''(x) = -2 \cdot \cosh^{-3}(x) \cdot \sinh(x) = \frac{-2\sinh(x)}{\cosh^3(x)}$ $\quad \sinh(0) = 0$ ↳ inflection point

★

		0	
f'	+		+
f''	+		−
f	−		+
	↗ inc.		↗ inc.
	con. up		con. down



eg.) $f(x) = \coth(x)$ $\qquad \coth(x) = \dfrac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$

★ Domain $\mathbb{R} - \{0\}$ $\qquad$ ★ There is no intercept point

★ $\lim\limits_{x \to \pm\infty} \left(\dfrac{e^{x}+e^{-x}}{e^{x}-e^{-x}}\right) = \pm 1$ $\qquad y=-1$ and $y=1$ are horizontal asymptotes

there is neither vertical asymptote nor oblique asymptote

★ $f'(x) = [\coth(x)]' = \left(\dfrac{\cosh(x)}{\sinh(x)}\right)' = \dfrac{\sinh^2(x) - \cosh^2(x)}{\sinh^2(x)} = -\dfrac{1}{\sinh^2(x)}$

$\Rightarrow \dfrac{-1}{\left(\frac{e^{x}-e^{-x}}{2}\right)^2} = \dfrac{-4}{(e^{x}-e^{-x})^2}$ $\qquad$ so $f'(x) < 0$

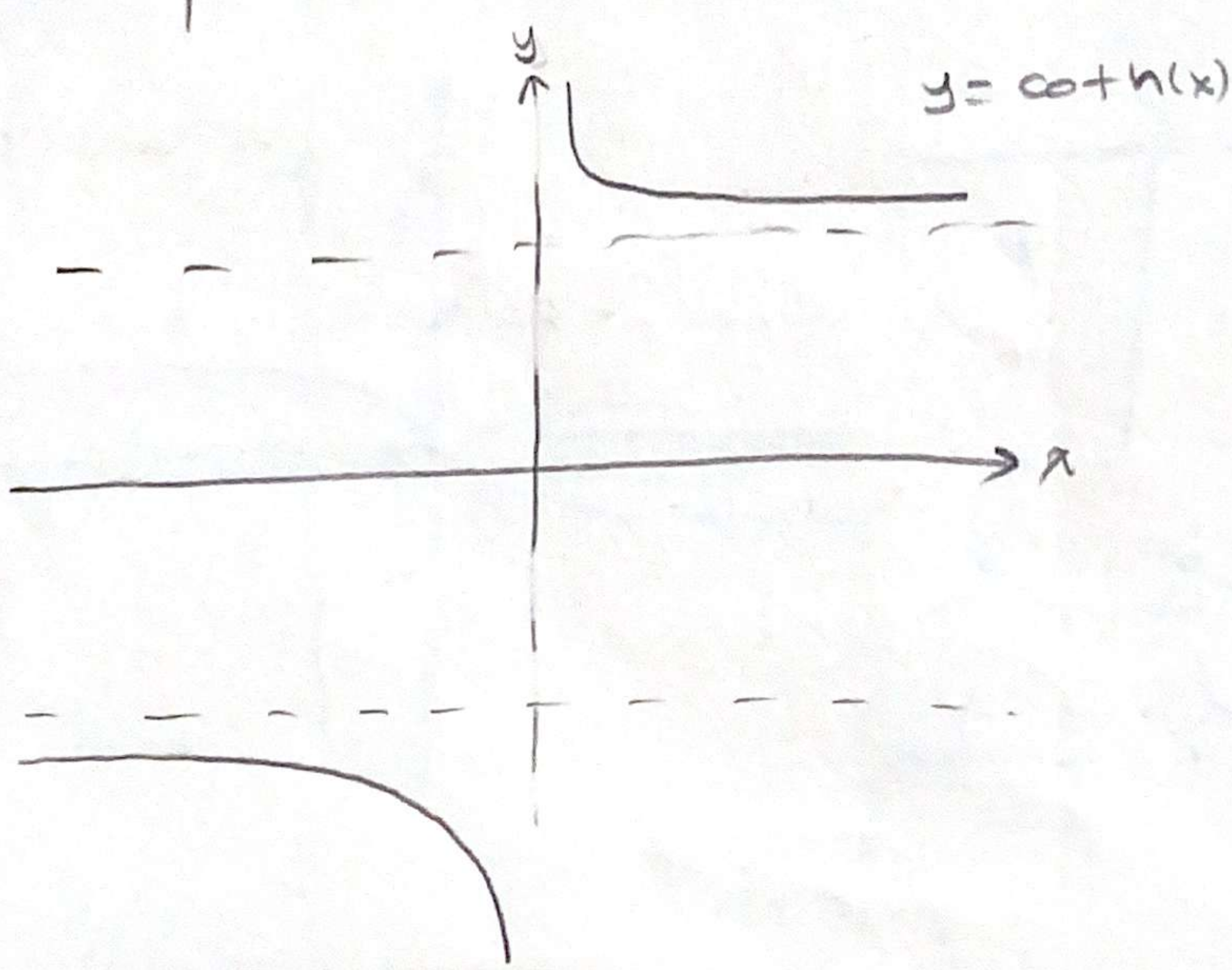
f is decreasing on all domain

★ $f''(x) = [-\sinh^{-2}(x)]' = +2 . \sinh^{-3}(x) . \cosh(x) = \dfrac{2\cosh(x)}{\sinh^3(x)}$

$x=0 \rightarrow$ inflection

		0
f'	$-$	$-$
f''	$-$	$+$
f	$-$	$+$

		0
f	$\searrow$	$\searrow$
	$\cap$	$\cup$



Parametric Equations

05.12.2025

Friday

$x = f(t)$, $y = g(t)$ $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt}$$

eg.) $\dfrac{d^2y}{dx^2} = ?$ $x = t - t^2$, $y = t - t^3$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1-3t^2}{1-2t}$$

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{\dfrac{(-6t).(1-2t) - (-2).(1-3t^2)}{(1-2t)^2}}{(1-2t)}$$

$$= \frac{-6t + 12t^2 + 2 - 6t^2}{(1-2t)^3} \Rightarrow \frac{d^2y}{dx^2} = \frac{6t^2 - 6t + 2}{(1-2t)^3}$$

eg.) $x = \dfrac{t}{t^2-1} = x(t)$, $y = \dfrac{2t}{t+1} = y(t)$

★ Domain $x(t)$ is undefined for $t=-1$ and $t=1$
$y(t)$ is undefined for $t=-1$ } $\mathbb{R} - \{-1, 1\}$

★ $x'(t) = \dfrac{1.(t^2-1) - 2t.t}{(t^2-1)^2} \Rightarrow \dfrac{-(t^2+1)}{(t^2-1)^2} \Rightarrow x'(t) < 0$ always decreasing

$y'(t) = \dfrac{2.(t+1) - 1.2t}{(t+1)^2} \Rightarrow \dfrac{2}{(t+1)^2} \Rightarrow y'(t) > 0$ always increasing

★ $\lim\limits_{t\to+1} \dfrac{t}{t^2-1} = \infty$ $\lim\limits_{t\to+1} \dfrac{2t}{t+1} = 1$ } $x \to \infty$, $y \to 1$ horizontal asymptote

↳ right and left changes the sign of infinity

$\lim\limits_{t\to-1} \dfrac{t}{t^2-1} = -\infty$ $\lim\limits_{t\to-1} \dfrac{2t}{t+1} = -\infty$ } $x \to -\infty$, $y \to -\infty$ oblique asymptote
(it may have)

$m = \frac{y}{x} = \frac{f(x)}{x}$ $\quad y = mx + n$ $\quad \lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{t\to-1} \frac{\frac{2t}{t+1}}{\frac{t}{t^2-1}} = \lim_{t\to-1} 2.(t-1) = -4$

$m = -4$ $\quad n = \lim_{x\to-\infty}\left[f(x) - mx\right]$ $\quad n = \lim_{t\to-1}\left[f(x) + 4x\right]$

$$n = \lim_{t\to-1}\left[\left(\frac{2t}{t+1}\right) + 4\left(\frac{t}{t^2-1}\right)\right] \Rightarrow \left(\frac{2t^2-2t+4t}{t^2-1}\right)$$

$$\lim_{t\to-1} \frac{2t\cancel{(t+1)}}{\cancel{(t+1)}(t-1)} \Rightarrow \frac{2t}{t-1} \Rightarrow 1$$

$y = -4x + 1 \rightarrow$ oblique asymptote

★ intercepts $\quad y = \frac{2t}{t+1} \quad x = \frac{t}{t^2-1} \quad (0,0)$

asymptote check (again)

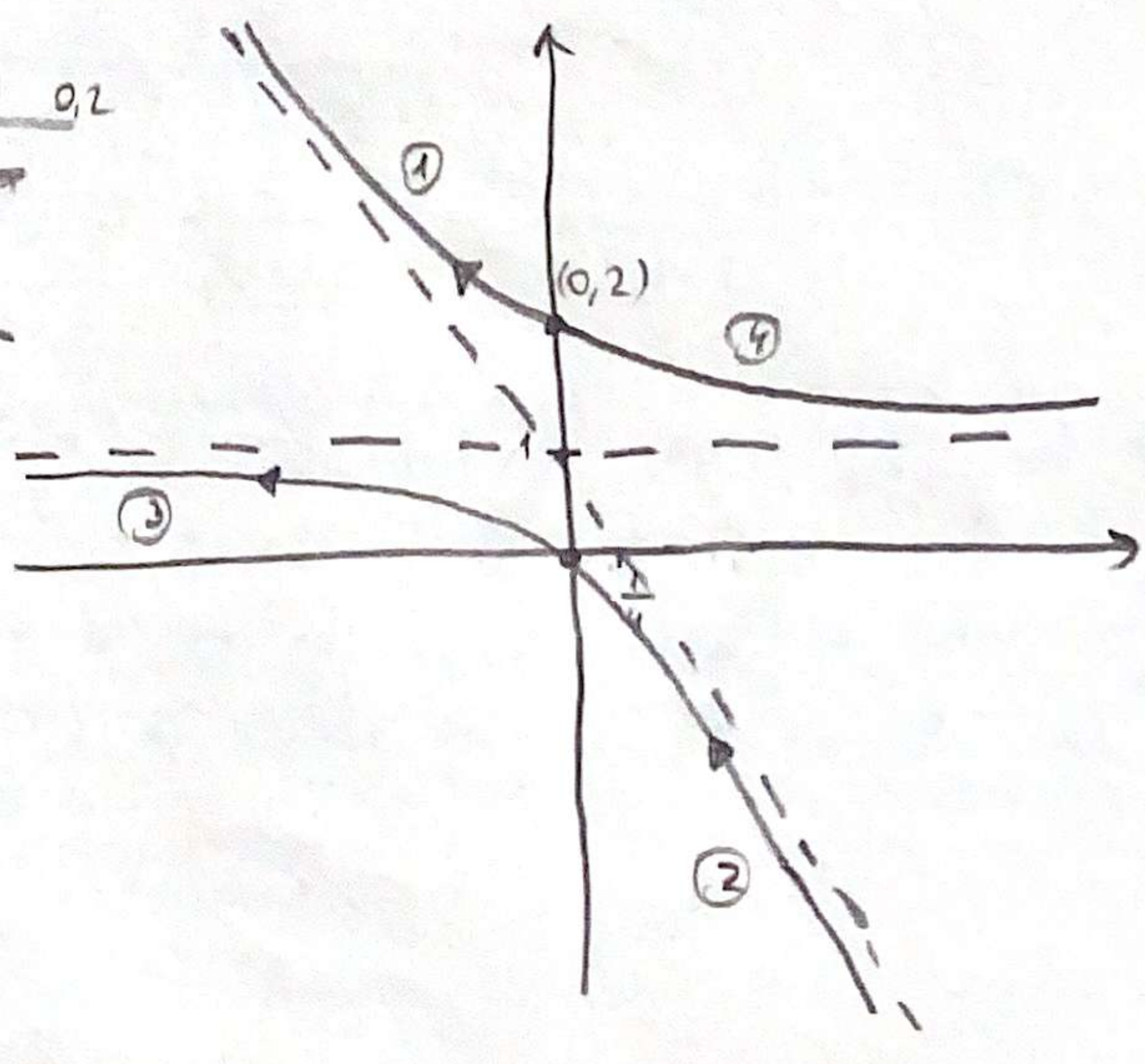
$\lim_{t\to-\infty} x = 0 \quad \lim_{t\to-\infty} y = 2 \quad t\to-\infty \begin{cases} x \to 0 \\ y \to 2 \end{cases}$

$\lim_{t\to\infty} x = 0 \quad \lim_{t\to\infty} y = 2 \quad t\to\infty \begin{cases} x \to 0 \\ y \to 2 \end{cases}$

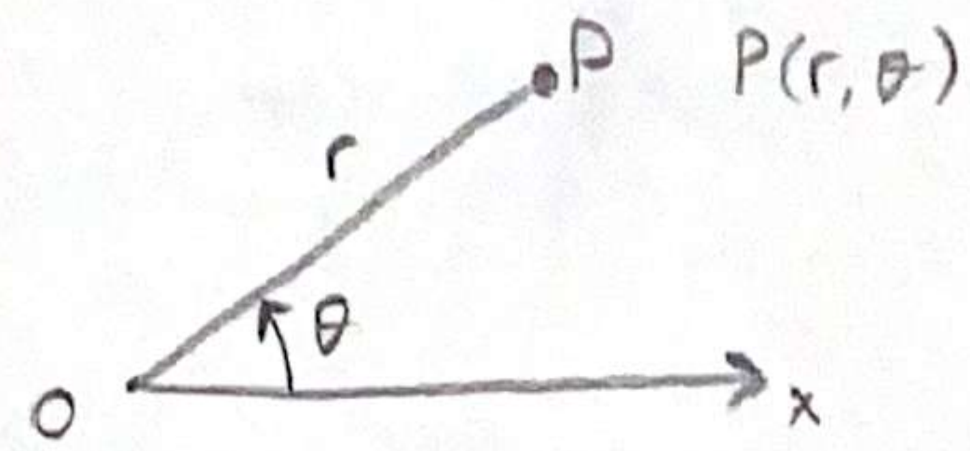
no vertical asymptote

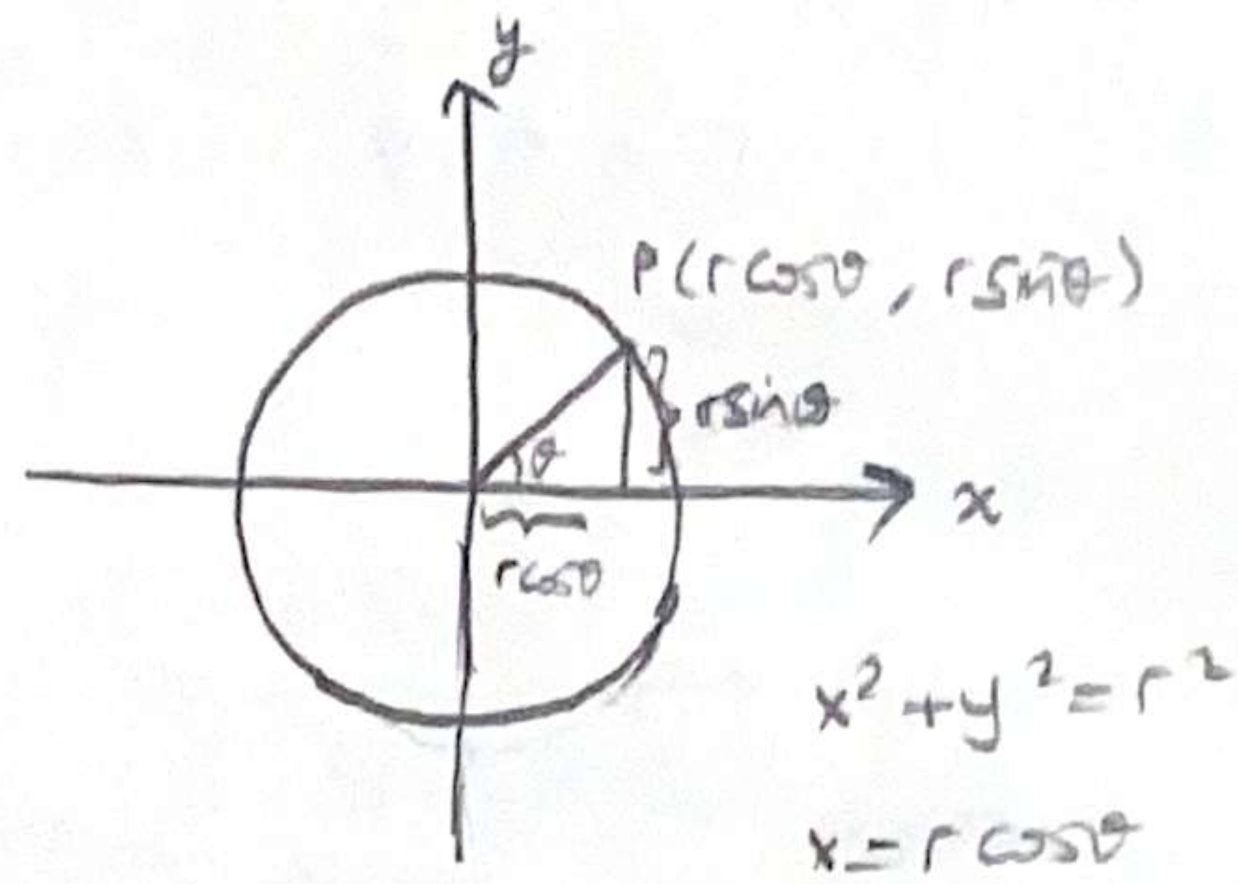
★

t	$-\infty$		-1		0		1		∞
$\dot{x}$		−		−		−		−	
$\dot{y}$		+		+		+		+	
x		↘ $-\infty$	‖	$+\infty$ ↘		↘ $-\infty$	‖	$+\infty$ ↘	0,2
y		↗ $+\infty$	‖	$-\infty$ ↗		↗	1	↗	
		①		②		③		④	



POLAR COORDINATES





$x^2 + y^2 = r^2$

$x = r\cos\theta \quad y = \sin\theta$

e.g.) $r\cos\theta = 2 \quad x = 2$

$r^2 \cdot \cos\theta \cdot \sin\theta = \underbrace{r \cdot \cos\theta}_{2} \cdot \underbrace{r\sin\theta}_{2} = 4 \qquad y = 2$

e.g.) $r^2\cos^2\theta - r^2\sin\theta = 1 \qquad (r\cos\theta)^2 - (r\sin\theta)^2 = 1 \Rightarrow x^2 - y^2 = 1$

e.g.) $x^2 + (y-3)^2 = 9 \qquad x^2 + y^2 - 6y + \cancel{9} = \cancel{9} \qquad r^2 - 6r\sin\theta = 0 \qquad r = 0, \ r = 6\sin\theta$

Symmetry test

- Replace θ with $-\theta$ $\quad (r, \theta) \to (-r, \pi - \theta), \ (r, -\theta)$ — symmetric respect to polar (x) - axis
- Replace θ with $-\theta$ and r with $-r$ $\quad (r, \theta) \to (r, \pi - \theta), \ (-r, -\theta)$ — symmetric respect to $\theta = \frac{\pi}{2}$

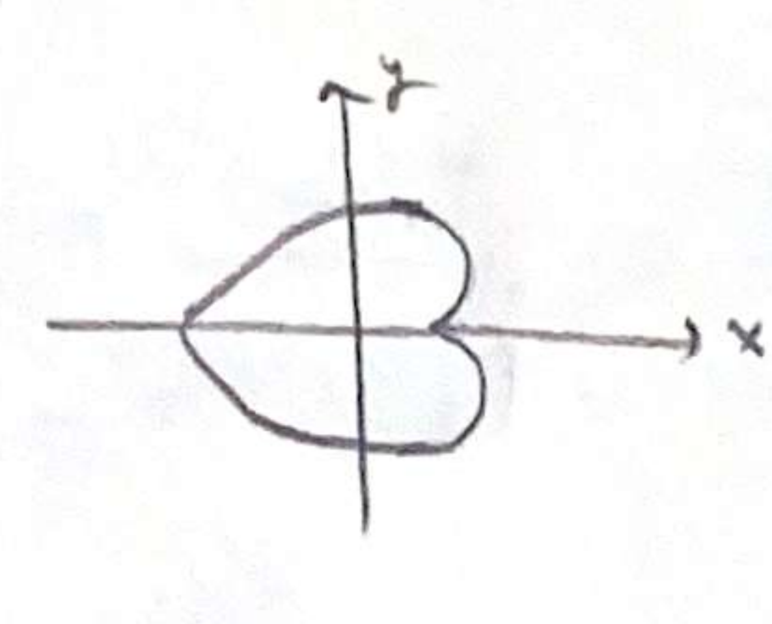
★ check all of them

- Replace r with $-r$ $\quad (r, \theta) \to (-r, \theta), \ (r, \pi + \theta)$ — symmetric about to pole (origin)

e.g.) $r = 1 - \cos\theta$ sketch the graph for $\theta \to -\theta$

$r = 1 - \cos(-\theta) \Rightarrow r = 1 - \cos\theta$ → symmetric respect to polar axis

$r' = \sin\theta \qquad \arcsin 0 = \theta \qquad \begin{pmatrix} \theta_1 = 0 \\ r = 0 \end{pmatrix} \begin{pmatrix} \theta_2 = \pi \\ r = 2 \end{pmatrix}$



θ	$1 - \cos\theta$
0	0
$\frac{\pi}{6}$	$\frac{2-\sqrt{3}}{2}$
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{3}{2}$
π	2