Question: $\lim\limits_{x\to 0^+} x.(\ln x)^2$ $\quad \lim\limits_{x\to 0^+}\left[\frac{(\ln x)^2}{\left(\frac{1}{x}\right)}\right] \to \infty/\infty$ ind. form

$$\Rightarrow \lim_{x\to 0^+} \frac{2.\ln x.\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+}\left(-\frac{2\ln x}{\frac{1}{x}}\right)$$

$$\Rightarrow -\lim_{x\to 0^+}\frac{\frac{2}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+}(2x) = 0$$

Question: $\lim\limits_{x\to\infty}\left(x.e^{\sqrt{x^2-1}-x^2}\right) = L \quad \exp\left(\lim\limits_{x\to\infty}\left[\ln\left(x.e^{\sqrt{x^2-1}-x^2}\right)\right]\right) = L \quad \exp\left[\lim\limits_{x\to\infty}\left(\ln x + \sqrt{x^2-1}-x^2\right)\right]$

$$\Rightarrow \exp\left[\left(\underbrace{\lim_{x\to\infty}\frac{\ln x}{x}}_{0} + \lim_{x\to\infty}\left(\sqrt{1-\frac{1}{x^2}}-x\right)\right).\lim_{x\to\infty} x\right] \Rightarrow \exp.\lim_{x\to\infty}\frac{\left(\sqrt{x^2-1}-x^2\right)\left(\sqrt{x^2-1}+x^2\right)}{\left(\sqrt{x^2-1}+x^2\right)}$$

$$\Rightarrow \exp\left[\lim_{x\to\infty}\left(\frac{x^2-1-x^4}{\sqrt{x^2-1}+x^2}\right)\right] = \exp.\left[\lim_{x\to\infty}\left(\frac{\cancel{x^4}\left(\frac{1}{x^2}-\frac{1}{x^4}-1\right)}{\cancel{x^4}\left(\sqrt{\frac{1}{x^{14}}-\frac{1}{x^{16}}}+\frac{1}{x^2}\right)}\right)\right] = \exp(-\infty) \Rightarrow e^{-\infty} = 0$$

② $\lim\limits_{x\to\infty}\left(\frac{x}{e^{x^2-\sqrt{x^2-1}}}\right) \to \infty/\infty \quad \Rightarrow \lim\limits_{x\to\infty}\left(\frac{1}{\underbrace{e^{x^2-\sqrt{x^2-1}}}_{\to\infty}\underbrace{\left(2x-\frac{2x}{2\sqrt{x^2-1}}\right)}_{\to\infty}}\right) = 0$

Question: $\lim\limits_{x\to 0}\left(\frac{e^x.\sin(x^2)}{1+x^2-\cos x}\right) = 2/3 \quad \to 0/0$ indeterminate form

$$\Rightarrow \lim_{x\to 0}\left(\frac{e^x.1.\sin x^2 + e^x.\cos x^2.2x}{2x+\sin x.1}\right) = \lim_{x\to 0}\left(\frac{e^x(\sin x^2+\cos x^2.2x)}{2x+\sin x}\right) \to 0/0$$

$$\Rightarrow \lim_{x\to 0}\left(\frac{e^x.1(\sin x^2+\cos x^2.2x) + e^x(\cos x^2.2x - \sin x^2.2x.2x + 2.\cos x^2)}{2+\cos x}\right) = \frac{2}{3}$$

Question: $\lim\limits_{x\to 1^+}\left(\frac{x}{\ln x}-\frac{1}{x^2-x}\right) = \lim\limits_{x\to 1^+}\left(\frac{x^3-x^2-\ln x}{\ln x(x^2-x)}\right) \to \infty/\infty$

$$\Rightarrow \lim_{x\to 1^+}\left[\frac{3x^2-2x-\frac{1}{x}}{\frac{1}{x}.(x^2-x)+\ln x.(2x-1)}\right] = \lim_{x\to 1^+}\left(\frac{3x^3-2x^2-1}{x^2-x+\ln x(2x^2-x)}\right) \to 0/0$$

$$\Rightarrow \lim_{x\to 1^+}\frac{9x^2-4x}{(2x-1)+\frac{1}{x}.(2x^2-x)+\ln x.(4x-1)} = \lim_{x\to 1^+}\frac{9x^2-4x}{(4x-2)+\underbrace{\ln x(4x-1)}_{0}} = \frac{5}{2}$$

$$= \lim_{x\to 1^+}\left(\frac{x}{\ln x}-\frac{1}{x^2-x}\right) = \frac{5}{2}$$

Question: $\lim_{x\to 0} \dfrac{h(t+2x)-2h(t+x)+2h(t-x)-h(t-2x)}{x^3}$

LH) $\dfrac{2h'(t+2x)-2h'(t+x)-2h'(t-x)+2h'(t-2x)}{3x^2}$

LH2) $\dfrac{4h''(t+2x)-2h''(t+x)+2h''(t-x)-4h''(t-2x)}{6x}$

LH3) $\dfrac{8h'''(t+2x)-2h'''(t+x)-2h'''(t-x)+8h'''(t-2x)}{6} \Rightarrow \lim_{x\to 0} \dfrac{\overset{2}{\cancel{12}}h'''(t)}{\cancel{6}} \Rightarrow 2h'''(t)$

Question: $[0,3]$ $\quad f(x)=\dfrac{x^2-3x+1}{2x+1}$ Defined and continuous on $[0,3]$

$[a,b]$

$\dfrac{(2x-3).(2x+1)-2(x^2-3x+1)}{(2x+1)^2} \Rightarrow \dfrac{2x^2+2x-5}{(2x+1)^2}$ differentiable on $(0,3)$

$f'(c)=\dfrac{f(b)-f(a)}{b-a}$ $\quad \dfrac{2c^2+2c-5}{(2c+1)^2}=\dfrac{\frac{1}{7}-1}{3-0}=\dfrac{-2}{7}$

$14c^2+14c-35=-8c^2-8c-2$ $\quad 22c^2+22c-33 \Rightarrow 2c^2+2c-3=0$

$4-4.2.(-3)=28$ $\quad \dfrac{-2+2\sqrt{7}}{4}=\dfrac{\sqrt{7}-1}{2}$ $\quad \cancel{\dfrac{-2-2\sqrt{7}}{4}=\dfrac{-\sqrt{7}-1}{2}}$ not in interval

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Question: $f(x)=\begin{cases} x^2-1, & x\le 1 \\ k(x-1), & x>1 \end{cases}$ Find k, f is differentiable at $x=1$

continuity $\quad \lim_{x\to 1^+}[k(x-1)]=\lim_{x\to 1^-}(x^2-1)=f(1)=0$ any k

differentiability $\quad f_+'(1)=f_-'(1)$ should be satisfied

$f_-'(1)$

$\lim_{h\to 0^-}\left(\dfrac{f(x+h)-f(x)}{h}\right) \Rightarrow \lim_{h\to 0^-}\left(\dfrac{f(1+h)-f(1)}{h}\right) \Rightarrow \lim_{h\to 0^-}\dfrac{((1+h)^2-1)-0}{h} \Rightarrow \lim_{h\to 0^-}\dfrac{h^2+2h}{h} \Rightarrow \lim_{h\to 0^-}\dfrac{\cancel{h}(h+2)}{\cancel{h}}=2$

$f_+'(1)$

$\lim_{h\to 0^+}\left(\dfrac{f(x+h)-f(x)}{h}\right) \Rightarrow \lim_{h\to 0^+}\left(\dfrac{f(1+h)-f(1)}{h}\right) \Rightarrow \lim_{h\to 0^+}\dfrac{k[(1+h)-1]-0}{h}=k$

$f_-'(1)=f_+'(1)=2=k$ $\quad$ so, k is 2

Question: $f(x)=x.e^{\frac{x}{x-1}}$, find the oblique asymptote

$\lim_{x\to\infty}\dfrac{f(x)}{x}=m_1$ $\quad \lim_{x\to\infty}(f(x)-m_1x)=n_1$ $\quad y=m_1x+n_1$ $\quad \lim_{x\to\infty}\dfrac{\cancel{x}.e^{\left(1+\frac{1}{x-1}\right)}}{\cancel{x}}=e$ $\quad \lim_{x\to\infty} x.e.e^{\frac{1}{x-1}}-e.x=e$

$\dfrac{ex\left(e^{\frac{1}{x-1}}-1\right)}{(x-1)\cdot\frac{1}{x-1}}$ $\quad 1$

$\boxed{ex+e}$

$\lim_{x\to-\infty}\dfrac{f(x)}{x}=m_2$ $\quad \lim_{x\to-\infty}(f(x)-m_2x)=n_2$ $\quad y=m_2x+n_2$

Question: $f(x) = \sinh(x)$, find the slope of tangent line and find $y=f^{-1}(x)$ at point $P(0,0)$

$f(x) = \dfrac{e^x - e^{-x}}{2}$ $\quad f'(x) = \dfrac{e^x + e^x}{2} = [\sinh(x)]' = \cosh(x)$ $\quad f'(0) = 1$ $\quad \boxed{m_T = 1}$

$[\operatorname{arcsinh}(x)]' = \dfrac{1}{(\sinh)'(\operatorname{arcsinh}(x))} \Rightarrow \dfrac{1}{\cosh(\operatorname{arcsinh}(0))} = 1$

Question: $\lim\limits_{x \to 0^+} \underbrace{(2 - e^{\sqrt{x}})^{\frac{2}{x}}}_{} \to 1^{\infty}$ indeterminate form

$\lim\limits_{x\to0^+} (2 - e^{\sqrt{x}})^{\frac{2}{x}} = L$ $\quad \exp\left[\lim\limits_{x\to0^+}\left(\frac{2}{x} \cdot \ln(2 - e^{\sqrt{x}})\right)\right] = L$ $\quad \exp\left[2\left(\underbrace{\lim\limits_{x\to0^+}\left(\frac{\ln(2-e^{\sqrt{x}})}{x}\right)}_{0/0 \text{ indeterminate form}}\right)\right] = L$

1st l'hospital

$\Rightarrow \exp\left[2\left(\lim\limits_{x\to0^+}\left(\dfrac{\frac{1}{2-e^{\sqrt{x}}} \cdot -e^{\sqrt{x}} \frac{1}{2\sqrt{x}}}{1}\right)\right)\right] \Rightarrow \exp\left[\lim\limits_{x\to0^+}\left(\dfrac{e^{\sqrt{x}}}{(e^{\sqrt{x}} - 2)\sqrt{x}}\right)\right] \Rightarrow \exp(-\infty)$

$= e^{-\infty} = 0$

Question: $f(x)$'s normal line equation at $P(x_0,-1)$ $\quad y+2x-1=0$, find $(f^{-1})'(-1)$

$y = -2x+1$ $\quad P(1,-1)$

$(y-(-1)) = m_N(x - x_0)$

$y+1 = -2x + 2x_0$ $\quad x_0 = 1$

$m_N = -2$ $\quad m_T \cdot m_N = -1$, $\frac{1}{2} \cdot -2$ $\quad f'(1) = \frac{1}{2}$

$(f^{-1})'(-1) = \dfrac{1}{f'(\underbrace{f^{-1}(-1)}_{1})} = \dfrac{1}{1/2} = 2$ $\quad f'(1) = \frac{1}{2}$

Question: $f(6) = -2$ $\quad f'(x) \leq 10$ $\quad$ differentiable at $(6,15)$ $\quad$ continuous at $[6,15]$

max of $f(15)$

$f'(c) = \dfrac{f(15) - f(6)}{15-6} \Rightarrow \dfrac{f(15) - (-2)}{9} \leq 10$ $\quad f(15) \leq 88$ $\quad \boxed{f(15) = 88}$

Question: $A(t) = 8t + e^{-3t}$ on $[-2,3]$ MVT $\quad A'(t) = 8 - 3 \cdot e^{-3t}$

$\overbrace{8 - 3 \cdot e^{-3c}}^{A'(c)} = \dfrac{A(3) - A(-2)}{3-(-2)}$ $\quad A(3) = 24 + e^{-9}$

$A(-2) = e^6 - 16$

$8 - 3 \cdot e^{-3c} = \dfrac{24 + e^{-9} - e^6 + 16}{5} \Rightarrow \cancel{40} - 15 \cdot e^{-3c} = \cancel{40} + e^{-9} - e^6$

$\dfrac{e^2 - e^{-3}}{\sqrt[3]{15}} = \sqrt[3]{15} \cdot e^{-c}$ $\quad c = -\ln\left((e^2 - e^{-3})/\sqrt[3]{15}\right)$

Question: $\lim_{x\to 0^+} (1-\cos x)^x = L \qquad \exp\left[\lim_{x\to 0^+} \left(x \cdot \ln(1-\cos x)\right)\right] = L$

Let's arrange the limit in order to have a L'Hospital applicable form

① $\Rightarrow \exp\left[\lim_{x\to 0^+} \underbrace{\left(\frac{\ln(1-\cos x)}{1/x}\right)}_{0/\infty \text{ not form}}\right] \Rightarrow \exp\left[\lim_{x\to 0^+}\left(\frac{\frac{1}{1-\cos x}\cdot \sin x}{-1/x^2}\right)\right]$

$\Rightarrow \exp\left[\lim_{x\to 0^+}\left(\frac{x^2 \sin x}{\cos x - 1}\right)\right] \Rightarrow \exp\left[\lim_{x\to 0^+}\left(\frac{2x \sin x + x^2 \cos x}{-\sin x}\right)\right] \Rightarrow \exp\left[\lim_{x\to 0^+}\left(-2x - \frac{x^2 \cos x}{\sin x}\right)\right]$ ②

$\Rightarrow \exp\left[\lim_{x\to 0^+} \frac{2\sin x + 2x\cos x + 2x\cos x - x^2 \sin x}{-\cos x}\right] = \exp 0 \Rightarrow e^0 = 1 = L$

$\lim_{x\to 0^+} (1-\cos x)^x = 1$

② $\exp\left[\underbrace{\lim_{x\to 0^+} -2x}_{0} - \underbrace{\lim_{x\to 0^+}\left(\frac{x}{\sin x}\right)}_{1} \cdot \underbrace{(x \cdot \cos x)}_{0}\right] = \exp 0 = e^0 = 1$

Question: $\lim_{x\to 0^+} (1-e^{\sin x}) \cdot \ln x \Rightarrow \lim_{x\to 0^+}\left[\frac{1-e^{\sin x}}{\left(\frac{1}{\ln x}\right)}\right]$ → 0/0 not form

$\Rightarrow \lim_{x\to 0^+} \frac{-(e^{\sin x} \cdot \cos x)}{-\left(\frac{1}{\ln x}\right)^2 \cdot \frac{1}{x}} \Rightarrow \lim_{x\to 0^+} \frac{e^{\sin x}\cos x}{\frac{1}{(\ln x)^2 \cdot x}}$

$\Rightarrow \underbrace{\lim_{x\to 0^+} e^{\sin x} \cdot \cos x}_{1} \cdot \underbrace{\lim_{x\to 0^+} \frac{(\ln x)^2}{\left(\frac{1}{x}\right)}}_{0} \to \lim_{x\to 0^+} \frac{2 \cdot \ln x \cdot \frac{1}{x}}{-\frac{1}{x^2}} = -\lim_{x\to 0^+} \frac{2\ln x}{\frac{1}{x}}$

$\Rightarrow -\lim_{x\to 0^+} \frac{\frac{2}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} (2x) = 0 \quad \Big\} \quad \lim_{x\to 0^+}\left[(1-e^{\sin x}) \ln x\right] = 0$

Question: $\lim_{x\to\infty} \frac{x - \sin x}{x + \sin x} \to \frac{\infty}{\infty} \Rightarrow \lim_{x\to\infty} \frac{1-\cos x}{1+\cos x} \cdot \frac{(1+\cos x)}{(1+\cos x)} \Rightarrow \lim_{x\to\infty} \frac{1-\cos^2 x}{(1+\cos x)^2} = \lim_{x\to\infty} \frac{\sin^2 x}{4\cos^4 \frac{x}{2}}$

$[1+(2\cos^2\frac{x}{2} - 1)]^2$

$= \lim_{x\to\infty} \frac{4\sin^2\frac{x}{2}\cos^2\frac{x}{2}}{4\cos^4\frac{x}{2}} = \lim_{x\to\infty} \tan^2\frac{x}{2}$

$\lim_{x\to\infty}\left(\frac{1-\frac{\sin x}{x}}{1+\frac{\sin x}{x}}\right) = 1$, with $\frac{\sin x}{x} \to 0$

$\frac{-1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$

$\lim_{x\to\infty} -\frac{1}{x} \le \lim_{x\to\infty} \frac{\sin x}{x} \le \lim_{x\to\infty} \frac{1}{x}$

Question: $\lim_{x \to \infty} 2^{\sqrt{x^2-1} - x}$ $\quad 2^{\lim_{x\to\infty} \frac{(\sqrt{x^2-1} - x).(\sqrt{x^2-1}+x)}{(\sqrt{x^2-1}+x)}}$

We multiple both numerator and denominator with the conjugate of the numerator in order to get rid of $\infty - \infty$ indeterminate form

$$2^{\lim_{x\to\infty} \frac{\cancel{x^2}-1 - \cancel{x^2}}{\sqrt{x^2-1}+x}} = 2^{\frac{-1}{\lim_{x\to\infty} \sqrt{x^2-1}+x}}$$

while x approches ∞ $\sqrt{x^2-1}+x$ approches ∞ too so $-1/\infty$ approches 0

$\Rightarrow 2^0 = 1$

Question: $\lim_{x\to\infty} \frac{x.\ln^2 x}{x^2+e^x}$ as there exists ∞/∞ indeterminate form so, it can be applied l'hospital in order to arrange until get rid of the indeterminate form

1st l'hospital

$$\lim_{x\to\infty} \frac{1.\ln^2 x + x.2\ln x.\frac{1}{x}}{2x+e^x.1} = \lim_{x\to\infty} \frac{\ln x(\ln x+2)}{2x+e^x} \quad \frac{\infty}{\infty} \text{ indeterminate form}$$

2nd l'hospital

$$\lim_{x\to\infty} \frac{\frac{1}{x}.(\ln x+2) + \ln x(\frac{1}{x})}{2+e^x.1} = \lim_{x\to\infty} \frac{2\ln x(\frac{1}{x}) + \frac{2}{x}}{2+e^x} = 2\left(\lim_{x\to\infty} \frac{\ln x+2}{(2+e^x).x}\right) \quad \frac{\infty}{\infty} \text{ ind. form}$$

3rd l'hospital

$$2\lim_{x\to\infty} \frac{\frac{1}{x}}{e^x.1.x+(2+e^x).1} = 2\lim_{x\to\infty} \frac{1}{[2+e^x(x+1)]x}$$ as $x\to\infty$, $[2+e^x(x+1)]x \to \infty$

$$\Rightarrow \lim_{x\to\infty} \frac{x.\ln^2 x}{x^2+e^x} = 0$$

Question: $\lim_{x\to\infty} (e^x-1)^{1/x} = L$ $\quad \exp.\left(\lim_{x\to\infty}\left[\frac{\ln(e^x-1)}{x}\right]\right) = L$

As $\ln(e^x-1)$ and x approches infinity while x approches infinity So, there exists ∞/∞ ind. form Then it is applicable, l'hospital.

1

$$\Rightarrow \exp.\left(\lim_{x\to\infty} \frac{\frac{1}{e^x-1}.e^x.1}{1}\right) = \exp.\left[\lim_{x\to\infty}\left(\frac{e^x}{e^x-1}\right)\right]$$

$\Rightarrow$ 2

$$\exp.\left[\lim_{x\to\infty}\left(\frac{e^x}{e^x}\right)\right] = \exp\left(\lim_{x\to\infty} 1\right) \Rightarrow e^1 = L$$

$$\Rightarrow \lim_{x\to\infty} (e^x-1)^{\frac{1}{x}} = e$$

Question: $(2^x + \cosh(x))\frac{2}{\ldots}$ ... $\left[2x\ldots \ln\left(\frac{2^x+\cosh(x)}{\ldots}\right)\right]$

Question: By using M.V.T find endpoints for $\sqrt{83}$ $\qquad \sqrt{81} < \sqrt{83} < \sqrt{100}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 9 < \sqrt{83} < 10$

Let $f(x) = \sqrt{x}$ $\qquad$ Conditions; 1) $f(x)$ must be continuous on $[9, a]$

$a > 9$ $\qquad$ $f(x)$'s domain is $[0, \infty)$ ✓

2) $f(x)$ must be differentiable on $(9, a)$

$f'(x) = \frac{1}{2\sqrt{x}}$ } defined on $(0, \infty)$ ✓

① ✗

$$f'(83) = \frac{f(a) - f(81)}{a - 81}$$

$$f'(83) = \frac{1}{2\sqrt{83}} = \frac{\sqrt{a} - 9}{a - 81} \Rightarrow \frac{1}{2\sqrt{83}} = \frac{1}{\sqrt{a} + 9}$$

$$2\sqrt{83} - 9 = \sqrt{a}$$

② ✓ $\exists c \in (81, 83)$: $f'(c) = \dfrac{\sqrt{83} - \sqrt{81}}{83 - 81}$

$$\frac{1}{2\sqrt{100}} < \frac{1}{2\sqrt{x_0}} = \frac{\sqrt{83} - 9}{2} < \frac{1}{2\sqrt{81}}$$

$$\frac{1}{10} + 9 < \frac{1}{2\sqrt{x_0}} < \frac{1}{9} + 9$$

$9,111\ldots$

Question: for $0 < a < b$ prove that by M.V.T $\dfrac{b-a}{1+b^2} < \arctan b - \arctan a < \dfrac{b-a}{1+a^2}$

Let $f(x) = \arctan x$

M.V.T $\quad 0 < a < c < b \quad f'(c) = \dfrac{f(b) - f(a)}{b - a}$ $\quad$ There exists at least one c while f(x) satisfies following conditions

1) $f(x)$ must be continuous on $[a, b]$ ✓

2) $f(x)$ must be differentiable on (a, b) $\quad f'(x) = \dfrac{1}{1+x^2}$ ✓

$$f'(c) = \frac{\arctan b - \arctan a}{b - a} \qquad \frac{1}{1+b^2} < \frac{1}{1+c^2} < \frac{1}{1+a^2}$$

(because of the interval) $0 < a < c < b$

$$\frac{1}{1+c^2} = \frac{\arctan b - \arctan a}{b - a} \Rightarrow \frac{b-a}{1+c^2} = \tan b - \tan a$$

$$\frac{b-a}{1+b^2} < \frac{b-a}{1+c^2} = \tan a - \tan b < \frac{b-a}{1+a^2}$$

Question: $f(x) = \begin{cases} 1 + 2x^2 & , x \leq 2 \\ 7 + 3x - x^2 & , x > 2 \end{cases}$ $\qquad$ M.V.T check on $[0, 4]$

1) Continuity check: $\lim\limits_{x \to 2^-} (1 + 2x^2) = \lim\limits_{x \to 2^+} (7 + 3x - x^2) = f(2) = 9$ ✓

2) Differentiability check: $f_-'(x) = \underset{2}{4x} = 8 \qquad f_+'(x) = \underset{2}{3 - 2x} = -1 \qquad f_-'(2) \neq f_+'(2)$ ✗

Question: $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$ $|\tan a - \tan b| \leq 4|a-b|$

Let $f(x) = \tan x$ M.V.T says that $f'(c) = \frac{f(a)-f(b)}{a-b}$ $-\frac{\pi}{3} < b < c < a < \frac{\pi}{3}$

- Conditions: $\tan x$ is continuous on $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$ ✓

$\tan x$ is differentiable on $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$ $\sec^2 x = \frac{1}{\cos^2 x}$ ✓

$|\sec^2 x| = \left|\frac{\tan a - \tan b}{a-b}\right|$ $|\sec^2 x| \cdot |a-b| = |\tan a - \tan b|$

$|\tan a - \tan b| \leq |a-b| \cdot \left|\sec^2\left(\frac{\pi}{3}\right)\right| = 4|a-b|$

Question: $f: [1,4] \to \mathbb{R}^+$, $f(x) = \sqrt{x^2 - x}$ MVT

Conditions

1) is continuous on $[1,4]$ $x^2 - x \geq 0$ $x(x-1) \geq 0$

0	1	
+	−	+

Domain $\mathbb{R} - (0,1)$ ✓

2) is differentiable on $(1,4)$ $\frac{1}{2\sqrt{x^2-x}} \cdot (2x-1)$ → undefined at $x=0$ and $x=1$ (not in domain) ✓

$f(4) = 2\sqrt{3}$ $f(1) = 0$

MVT $1 < c < 4$ $f'(c) = \frac{f(4)-f(1)}{4-1} \Rightarrow \frac{2x_0 - 1}{2\sqrt{x_0^2 - x_0}} = \frac{2\sqrt{3} - 0}{3}$

$6x_0 - 3 = 4\sqrt{3(x_0^2 - x_0)} \Rightarrow 36x_0^2 + 9 - 36x_0 = 16 \cdot 3(x_0^2 - x_0)$

$12x_0^2 - 12x_0 - 9 = 0$ $4x_0^2 - 4x_0 - 3 = 0$

$\boxed{c = 3/2}$ $\boxed{x_0 = 3/2 \in (1,4)}$ $\Delta = 16 - 4(-3) \cdot 4 = 64$ $\frac{4 \pm 8}{8} = \underset{x_{01}}{\frac{3}{2}}, \underset{x_{02}}{\cancel{-\frac{1}{2}}}$

Question: $f(x) = \begin{cases} x^3 + x, & x \leq 1 \\ 4x - 2, & x > 1 \end{cases}$ $[-2, 2]$ M.V.T. $x_0 = ?$

Conditions:

1) Continuity on $[-2,2]$ ✓

$\lim_{x \to 1^-}(x^3 + x) = \lim_{x \to 1^+}(4x - 2) = f(1) = 2$

2) Differentiability on $(-2,2)$ ✓

$f'_-(1) = 3x^2 + 1 = f'_+(1) = 4 = 4$

M.V.T says that there exists at least one c that satisfies the following condition:

$-2 < c < 2$ $f'(c) = \frac{f(2) - f(-2)}{2-(-2)}$ $f(-2) = -10$ $f(1) = 2$ $f(2) = 6$

$-2 < c_1 \leq 1$ $3x_0^2 + 1 = \frac{f(1) - f(-2)}{1-(-2)} \Rightarrow 3x_0^2 + 1 = 4 \Rightarrow 3(x_0^2 - 1) = 0$ $x_0 = \pm 1$

$1 < c_2 < 2$ $4 = \frac{f(2) - f(1)}{2-1} \Rightarrow 4 = 4$ $x_0 \in (1,2)$

$x_0 = [1,2) \cup \{-1\}$

Question: $\lim_{x\to 0^+} \left(\frac{2^x+\cosh(x)}{2}\right)^{\frac{2}{\sin(x)}} = L$ $\exp\left(\lim_{x\to 0^+}\underbrace{\left[\frac{2}{\sin(x)}\cdot \ln\left(\frac{2^x+\cosh(x)}{2}\right)\right]}_{0/0 \text{ indeterminate form}}\right) = L$

$\sinh(x) = \frac{e^x-e^{-x}}{2}$ $\cosh(x) = \frac{e^x+e^{-x}}{2}$

$$= \exp\left[2\cdot\lim_{x\to 0^+}\left(\frac{\ln\left(\frac{2^x+\cosh(x)}{2}\right)}{\sin(x)}\right)\right] \Rightarrow \exp\left(2\cdot\lim_{x\to 0^+}\frac{\left(\frac{2}{2^x+\cosh(x)}\right)\cdot\frac{(2^x\ln 2+\sinh(x))}{2}}{\cos(x)}\right) = \exp(\ln 2) = 2$$

Question: $0 < x < y$ $\sqrt{y}-\sqrt{x} < \frac{y-x}{2\sqrt{x}}$ $\frac{\sqrt{y}-\sqrt{x}}{y-x} < \frac{1}{2\sqrt{x}}$ $f(x) = \sqrt{x}$

$f'(c) = \frac{\sqrt{y}-\sqrt{x}}{y-x}$ $\frac{1}{2\sqrt{c}} = \frac{\sqrt{y}-\sqrt{x}}{y-x}$ as $0 < x < c < y$

$\frac{1}{2\sqrt{y}} < \frac{1}{2\sqrt{c}} < \frac{1}{2\sqrt{x}}$

$\frac{\sqrt{y}-\sqrt{x}}{y-x} < \frac{1}{2\sqrt{x}}$ $\boxed{\frac{1}{2\sqrt{c}} < \frac{1}{2\sqrt{x}}}$

Question: $\ln\sqrt[3]{x^2+y^2} + y^2 = \frac{x^3}{2-x} + \arctan(x-y) + \arccos(x) + 4x^3y^2e^{x+y} + \cosh(x) + \frac{3\pi}{4}$

① ② ③ ④ ⑤ ⑥ ⑦ ⑧

tangent and normal line equations at $P(0,1)$

① $\frac{1}{\sqrt[3]{x^2+y^2}}\cdot\frac{(2x+2y\cdot y')}{3\sqrt[3]{(x^2+y^2)^2}}$ ④ $\frac{(1-y')}{\sec^2(\arctan(x-y))}$ ⑦ $\sinh(x)$

⑧ 0

② $2y\cdot y'$ ⑤ $\frac{1}{+\sin(\arccos(x))}$

③ $\frac{3x^2(2-x)-x^3(-1)}{(2-x)^2}$ ⑥ $4\left[3x^2(y^2e^{x+y}) + x^3\left(2y\cdot y'\cdot e^{x+y} + y^2\cdot e^{x+y}(1+y')\right)\right]$

$$\frac{2}{3}\frac{(x+y\cdot y')}{(x^2+y^2)} + 2y\cdot y' = \frac{2x^2(3-x)}{(2-x)^2} + \frac{1-y'}{\sec^2(\arctan(x-y))} + \frac{1}{\sin(\arccos(x))} + ⑥ + \sinh(x)$$

$$\frac{2}{3}y' + 2y' = \frac{1-y'}{2} + 1 \qquad \frac{4y'}{6} + \frac{12y'}{6} = \frac{3-3y'}{6} + \frac{6}{6} \Rightarrow \boxed{\frac{9}{19} = y'}$$

$(y-1) = \frac{9}{19}(x-0) \Rightarrow \boxed{y_T = \frac{9}{19}x+1}$

$M_T\cdot M_N = -1$

$\frac{9}{19}\ \ \frac{-19}{9}$

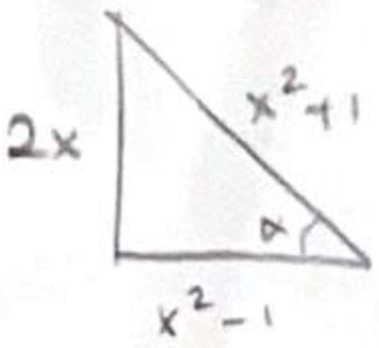
$(y-1) = \frac{19}{9}(x-0) \Rightarrow \boxed{y_N = -\frac{19}{9}x+1}$

Question: constant func. $f(x) = \arcsin\left(\frac{2x}{x^2+1}\right) + 2\arctan(x)$; $x>1$

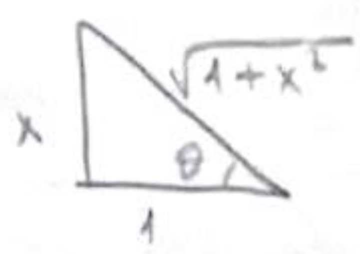
$g(g^{-1}(x)) = \longrightarrow g'(g^{-1}(x)) \cdot (g^{-1}(x))' = \qquad (g^{-1}(x))' = \frac{1}{g'(g^{-1}(x))}$

$\left[\arcsin\left(\frac{2x}{x^2+1}\right)\right]' = \frac{1}{\cos\left(\arcsin\left(\frac{2x}{x^2+1}\right)\right)} \cdot \frac{2(x^2+1) - 2x \cdot 2x}{(x^2+1)^2}$

$(2\arctan(x))' = \frac{2}{\sec^2(\arctan(x))}$



$\frac{\cancel{x^2+1}}{x^2-1} \cdot \frac{\overset{-2}{2(1-x^2)}}{(x^2+1)^{\cancel{2}}}$



$\sec\theta = \sqrt{1+x^2}$

$\frac{2}{1+x^2}$

$\frac{-2}{1+x^2} + \frac{2}{1+x^2} = 0$

Question: $g(x) = \sqrt{\sin\left(\frac{ax+b}{cx+d}\right)}$ a, b, c, d $\left(x \neq -\frac{c}{d}\right)$ are constants, find $g'(x)$

$\frac{ax+b}{cx+d} = u(x) \qquad g(x) = \sqrt{\sin(u(x))}$

$g'(x) = \lim_{\Delta x \to 0} \frac{\sqrt{\sin(u(x+\Delta x))} - \sqrt{\sin(u(x))}}{\Delta x} \cdot \frac{\sqrt{\sin(u(x+\Delta x))} + \sqrt{\sin(u(x))}}{\sqrt{\sin(u(x+\Delta x))} + \sqrt{\sin(u(x))}}$

$g'(x) = \lim_{\Delta x \to 0} \frac{\sin u(x+\Delta x) - \sin(u(x))}{\Delta x\left(\sqrt{\sin(u(x+\Delta x))} + \sqrt{\sin(u(x))}\right)}$

$A = 2\alpha \quad B = 2\theta$

$\sin A - \sin B = \sin 2\alpha - \sin 2\theta = 2 \cdot (\sin\alpha \cdot \cos\alpha - \sin\theta \cdot \cos\theta)$

$\cos(\alpha+\theta) = \cos\alpha \cdot \cos\theta - \sin\alpha \cdot \sin\theta \qquad \sin(\alpha-\theta) = \sin\alpha \cdot \cos\theta - \sin\theta \cdot \cos\alpha$

$\cos\alpha \cdot \sin\alpha \cdot \underline{\cos^2\theta} - \sin\theta \cdot \cos\theta \cdot \underline{\cos^2\alpha} - \underline{\sin^2\alpha} \cdot \sin\theta \cdot \cos\theta + \underline{\sin^2\theta} \cdot \cos\alpha \cdot \sin\alpha \Rightarrow 2 \cdot \cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$

$A = u(x+\Delta x) \qquad B = u(x)$

$g'(x) = \lim_{\Delta x \to 0} \frac{2\cos\left(\frac{u(x+\Delta x)+u(x)}{2}\right) \cdot \sin\left(\frac{u(x+\Delta x)-u(x)}{2}\right)}{\Delta x\left(\sqrt{\sin(u(x+\Delta x))} + \sqrt{\sin(u(x))}\right)} \Rightarrow \lim_{\Delta x \to 0} \frac{\cos\left(\frac{u(x+\Delta x)+u(x)}{2}\right)}{\Delta x\left(\sqrt{\sin(u(x+\Delta x))} + \sqrt{\sin(u(x))}\right)} \cdot \lim_{\Delta x \to 0} \underbrace{\frac{\sin\left(\frac{u(x+\Delta x)-u(x)}{2}\right)}{\left(\frac{u(x+\Delta x)-u(x)}{2}\right)}}_{1} \cdot u(x+\Delta x) - u(x)$

$\Rightarrow \underbrace{\lim_{\Delta x \to 0}\left(\frac{u(x+\Delta x)-u(x)}{\Delta x}\right)}$ $\cdot \lim_{\Delta x \to 0} \frac{\cos\left(\frac{u(x+\Delta x)+u(x)}{2}\right)}{\sqrt{\sin(u(x+\Delta x))} + \sqrt{\sin(u(x))}} \cdot 1 \rightarrow \frac{\cos(u(x))}{2\sqrt{\sin(u(x))}}$

$\frac{ax + a\Delta x + b}{\underset{(cx+d)}{cx + c\Delta x + d}} - \frac{ax+b}{\underset{(cx+c\Delta x+d)}{cx+d}}$

$\cancel{acx^2} + \cancel{adx} + \cancel{acx\Delta x} + ad\Delta x + \cancel{bcx} + \cancel{bd} - \cancel{acx^2} - \cancel{bcx} - \cancel{acx\Delta x} - bc\Delta x - \cancel{adx} - \cancel{bd}$

$\lim_{\Delta x \to 0}\left(\frac{\frac{\cancel{\Delta x}(ad-bc)}{(cx+d)(cx+c\Delta x+d)}}{\cancel{\Delta x}}\right) = \frac{ad-bc}{(cx+d)^2}$

$\Rightarrow g'(x) = \left[\sqrt{\sin\left(\frac{ax+b}{cx+d}\right)}\right]' = \frac{ad-bc}{(cx+d)^2} \cdot \frac{\cos\left(\frac{ax+b}{cx+d}\right)}{2\sqrt{\sin\left(\frac{ax+b}{cx+d}\right)}}$

Question: $y = \frac{x.e^x}{x^2+e^x}$ asymptotes ? $D(f) = \mathbb{R}$

① vertical asymptote : The denominator x^2+e^x is strictly positive for all $x \in \mathbb{R}$, hence the function is defined everywhere and its denominator does not approach zero at any finite point. Therefore, the given function $\left(y = f(x) = \frac{x.e^x}{x^2+e^x}\right)$ $\boxed{\text{admits no vertical asymptotes}}$ Because vertical asymptote existence condition is while x approaches a defined value like a from right-hand or left-hand, makes the function approach whether positive or negative infinity at least one side must satisfy that condition.

② Horizontal asymptote: for $y=L$, if and only if at least one is satisfied:

$\lim_{x\to-\infty} f(x) = L$ or $\lim_{x\to+\infty} f(x) = L$ or both

$\lim_{x\to-\infty}\left(\frac{x.e^x}{x^2+e^x}\right)$ as $x \to -\infty$ numerator $(x.e^x) \to 0$ and denominator $(x^2+e^x) \to +\infty$

$$\lim_{x\to-\infty}\left(\frac{\frac{e^x}{x}}{1+\frac{e^x}{x^2}}\right) \overset{\text{l'hospital}}{\Rightarrow} \lim_{x\to-\infty}\left(\frac{\frac{e^x.1-e^x}{x^2}}{\frac{e^x x^2 - 2x.e^x}{x^4}}\right) = 0$$

$\boxed{y = 0 \text{ is a horizontal asymptote}}$

$$\lim_{x\to+\infty}\left(\frac{x.e^x}{x^2+e^x}\right) \Rightarrow \lim_{x\to+\infty}\left(\frac{\frac{e^x}{x}}{1+\frac{e^x}{x^2}}\right) \Rightarrow \lim_{x\to+\infty}\left(\frac{\frac{e^x 1-e^x}{x^2}}{\frac{e^x x^2-2xe^x}{x^4}}\right) = 0$$

③ Oblique asymptote: $y = mx+n$ $m = \lim_{x\to+\infty}\left(\frac{f(x)}{x}\right)$, $n = \lim_{x\to+\infty}(f(x)-mx)$

m) $$\frac{\frac{x\ e^x}{x^2+e^x}}{x} \Rightarrow m = \lim_{x\to\infty}\left(\underbrace{\frac{e^x}{x^2+e^x}}_{\frac{\infty}{\infty}}\right) \Rightarrow m = \lim_{x\to\infty}\left(\underbrace{\frac{e^x}{2x+e^x}}_{\frac{\infty}{\infty}}\right) \Rightarrow m = \lim_{x\to\infty}\left(\frac{e^x}{2+e^x}\right) \Rightarrow m = \lim_{x\to\infty}\left(\frac{e^x}{e^x}\right)$$

$\boxed{m=1}$

n) $$n = \lim_{x\to\infty}\left(\frac{x.e^x}{x^2+e^x} - x\right) \Rightarrow n = \lim_{x\to\infty}\left(\underbrace{\frac{-x^3}{x^2+e^x}}_{\frac{-\infty}{\infty}}\right) \Rightarrow n = \lim_{x\to\infty}\left(\underbrace{\frac{-3x^2}{2x+e^x}}_{\frac{-\infty}{\infty}}\right) \Rightarrow n = \lim_{x\to\infty}\left(\underbrace{\frac{-6x}{2+e^x}}_{\frac{-\infty}{\infty}}\right) \Rightarrow$$

$n = \lim_{x\to\infty}\left(\frac{-6}{e^x}\right) \Rightarrow n=0$ $\boxed{y = x \text{ is an oblique asymptote}}$

Question: MVT $0<x<1$ $\frac{\sqrt{1-x^2}}{1+x} \leq \frac{\ln(1+x)}{\arcsin x} < 1$ Let $f(x) = \ln(x)$

$[1, 1+x]$

$f'(c) = \frac{1}{c} = \frac{\ln(1+x) - \ln 1}{(1+x)-1} \Rightarrow \frac{1}{c} = \frac{\ln(1+x)}{x}$ | $1<c<1+x$ $\frac{1}{1+x} < \frac{1}{c} < 1$ $\boxed{\frac{1}{1+x} < \frac{\ln(1+x)}{x} < 1}$

$g(x) = \arcsin(x)$ $\overset{a}{0} < c < \overset{b}{x} < 1$ $1-0^2 > 1-c^2 > 1-x^2$

$g(c)$

$\frac{g(b)-g(a)}{b-a} = \frac{\arcsin x}{x} = \frac{1}{\sqrt{1-c^2}}$ $1 < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-x^2}}$ $1 < \frac{\arcsin x}{x} < \frac{1}{\sqrt{1-x^2}}$ $\boxed{\sqrt{1-x^2} < \frac{x}{\arcsin x} < 1}$

Question: normal line equation at $t=0$

$$\begin{cases} t^2 \cdot \sin(x) + x^3 = e^t & t=0 \quad x=1 \\ \sin(y) = t \cdot \sin(t) - 2t & t=0 \quad y=0 \end{cases}$$

$2t \cdot \sin(x) + t^2 \cdot \cos(x) \cdot x' + 3x^2 \cdot x' = e^t$ $\quad \boxed{x' = \frac{1}{3}}$

$\cos(y) \cdot y' = \sin(t) + t \cdot \cos(t) - 2$ $\quad \boxed{y' = -2}$

$\frac{dy}{dx} = -6 \qquad m_T = -6 \qquad m_N = \frac{1}{6}$

$(y-0) = \frac{1}{6}(x-1)$

$\boxed{x - 6y - 1 = 0}$ d_N

Question: $f(x) = \frac{x^2 - 1}{x}$

i) Domain   ii) Asymptotes   iii) Increasing - decreasing, extremum   iv) Concave up / down, inflections

i)

x	$-\infty$	-1	0	1	∞
$f(x)$	$-$	$+$	‖	$-$	$+$

$f(-1) = 0 \quad (-1,0)$
$f(1) = 0 \quad (1,0)$ } x-axis intercepts

$D_f : (-\infty, \infty) - \{0\}$

ii) $\lim_{x \to 0^+} \frac{x^2-1}{x} = -\infty \qquad \lim_{x \to 0^-} \frac{x^2-1}{x} = \infty$ $\qquad x=0$ (y-axis) is vertical asymptote

$\lim_{x \to +\infty} \frac{x^2-1}{x} \Rightarrow \lim_{x \to +\infty} \frac{x - \frac{1}{x}}{1} = +\infty \qquad \lim_{x \to -\infty} = -\infty$ $\quad$ there is no horizontal asymptote

$\lim_{x \to \pm\infty} \left[\left(x - \frac{1}{x} \right) - x \right] = 0$ $\quad y = x$ is an oblique asymptote

iii) $\frac{2x \cdot x - (x^2 - 1)}{x^2} = \frac{x^2 + 1}{x^2}$

		0	
$f'(x)$	$+$	‖	$+$
	↗		↗

$f'(x)$ always positive so, $f(x)$ is always increasing

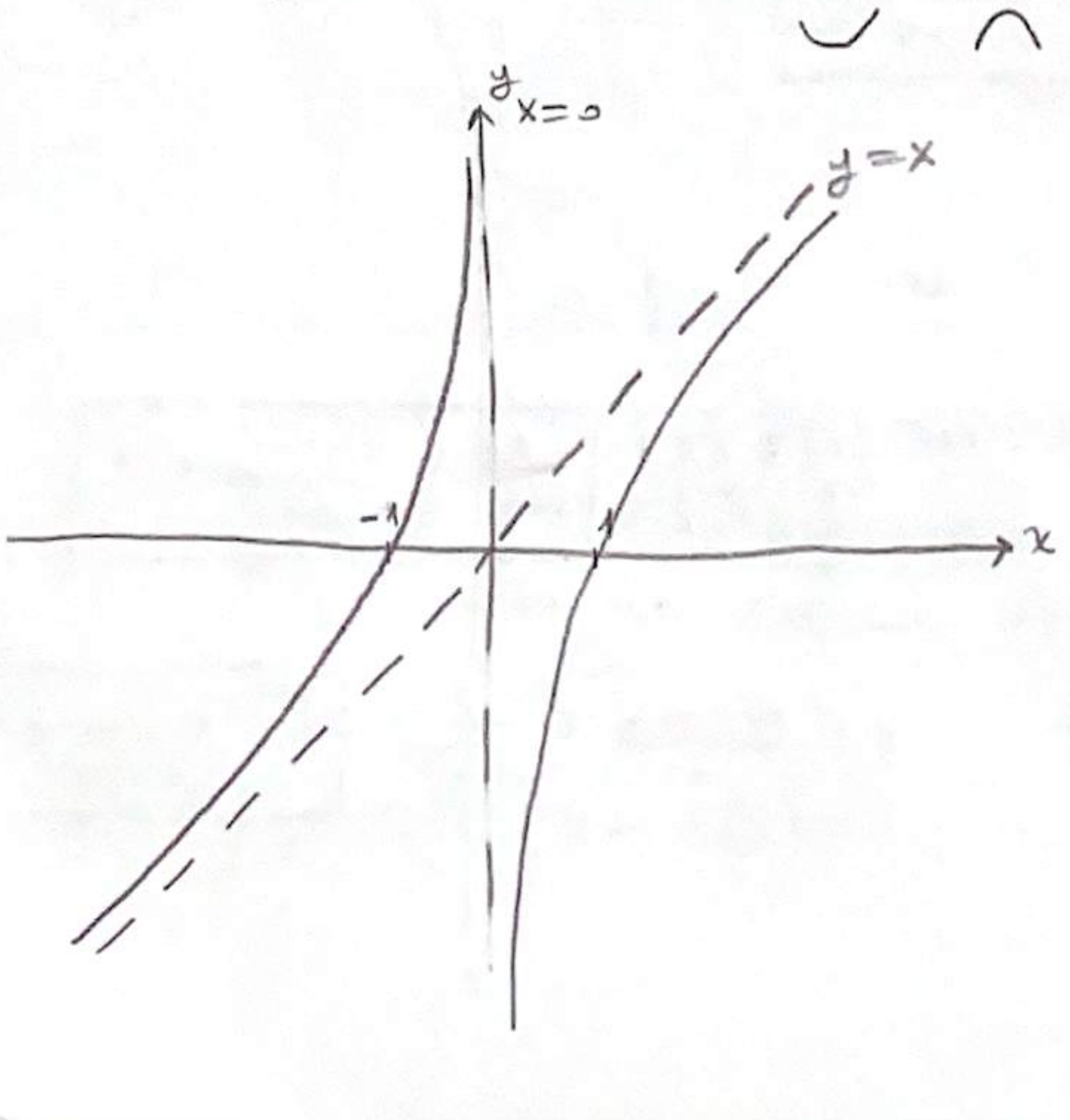
no extremum

iv) $\frac{2x \cdot x^2 - 2x(x^2+1)}{x^4} = \frac{-2x}{x^4}$

		0	
$f''(x)$	$+$	‖	$-$
	∪		∩

$(-\infty, 0)$ concave up

$(0, \infty)$ concave down



Question: sketch $f(x) = \dfrac{x^2+4}{2x}$ * Domain: $\mathbb{R} - \{0\}$

* $f(x) = -f(-x)$ $\boxed{\text{symmetric about the origin}}$

* Intercepts: there is no intercept

* $\lim\limits_{x\to 0^+} \left(\dfrac{x^2+4}{2x}\right) = \lim\limits_{x\to 0^+}\left(\dfrac{x+\frac{4}{x}}{2}\right) = \infty$ $\lim\limits_{x\to 0^-}\left(\dfrac{x+\frac{4}{x}}{2}\right) = -\infty$ $\boxed{x=0 \ (y\text{-axis}) \text{ vertical asymptote}}$

$\lim\limits_{x\to +\infty}\left(\dfrac{x+\frac{4}{x}}{2}\right) = \infty$ $\lim\limits_{x\to -\infty}\left(\dfrac{x+\frac{4}{x}}{2}\right) = -\infty$ there is no horizontal asymptote

$y = m_1x+n_1$ $\lim\limits_{x\to\infty}\left(\dfrac{\frac{x^2+4}{2x}}{x}\right) = \lim\limits_{x\to\infty}\left(\dfrac{x^2+4}{2x^2}\right) = \lim\limits_{x\to\infty}\left(\dfrac{1}{2}\left(1+\dfrac{4}{x^2}\right)\right) = \dfrac{1}{2}$ $\lim\limits_{x\to\infty}\left(\dfrac{x^2+4}{2x} - \dfrac{x}{2}\right) = n_1 = 0$

$\boxed{y = \tfrac{1}{2}x \text{ is an oblique asymptote}}$

$y = m_2x+n_2$ $\lim\limits_{x\to -\infty}\left(\dfrac{\frac{x^2+4}{2x}}{x}\right) = \lim\limits_{x\to -\infty}\left(\dfrac{1}{2}\left(1+\dfrac{4}{x^2}\right)\right) = \dfrac{1}{2}$ $\lim\limits_{x\to -\infty}\left(\dfrac{x^2+4}{2x} - \dfrac{x}{2}\right) = 0$

* $f'(x) = \dfrac{2x \cdot 2x - 2(x^2+4)}{4x^2} = \dfrac{2x^2-8}{4x} = \dfrac{x^2-4}{2x}$

	$-\infty$	-2	0	2	∞
$f'(x)$	$-$	$+$	‖	$-$	$+$
$f(x)$	↘	↗	‖	↘	↗

local min $(-2,-2)$ local min $(2,2)$

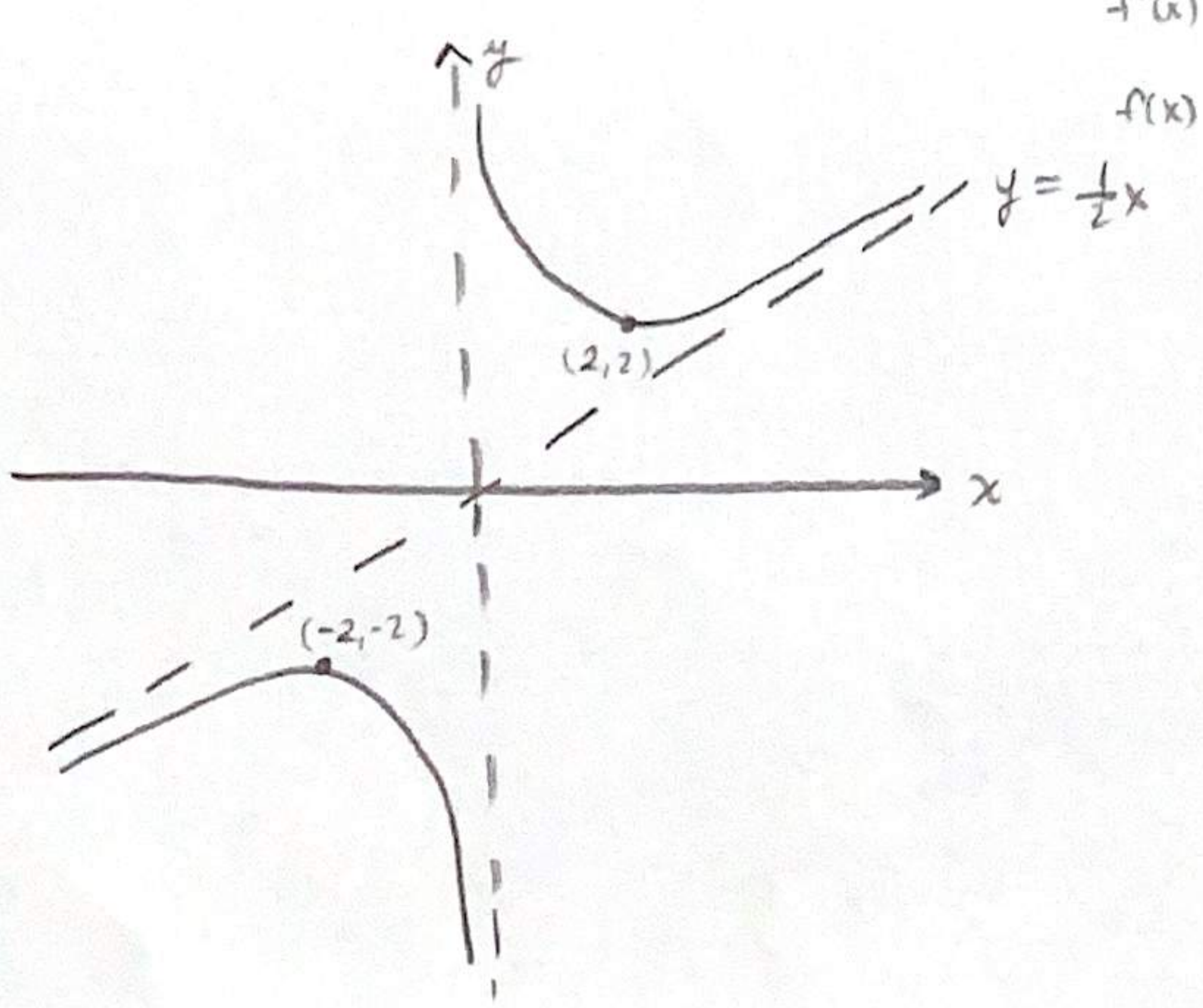
Increasing: $(-2,0) \cup (2,\infty)$

Decreasing: $(-\infty,-2) \cup (0,2)$

* $f''(x) = \dfrac{2x \cdot 2x - 2(x^2-4)}{4x^2} = \dfrac{x^2+4}{2x}$

	$-\infty$	0	∞
$f''(x)$	$+$	‖	$+$
$f(x)$	∪	‖	∪

Concave up: $(-\infty,0) \cup (0,\infty)$



Question: tangent line, $r = 3+8\sin\theta$ at $\theta = \frac{\pi}{6}$ $x = r\cdot\cos\theta$ $y = r\cdot\sin\theta$ $r = f(\theta) = 3+8\sin\theta$

$r' = f'(\theta) = 8\cos\theta$

$\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{f'(\theta)\cdot\sin\theta + f(\theta)\cdot\cos\theta}{f'(\theta)\cdot\cos\theta - f(\theta)\cdot\sin\theta}\Bigg|_{\theta=\frac{\pi}{6}}$ $\dfrac{4\sqrt{3}\cdot\frac{1}{2} + 7\cdot\frac{\sqrt{3}}{2}}{4\sqrt{3}\cdot\frac{\sqrt{3}}{2} - 7\cdot\frac{1}{2}} = \dfrac{11\sqrt{3}}{5}$ $x = \dfrac{7\sqrt{3}}{2}$ $y = \dfrac{7}{2}$

$\left(y - \dfrac{7}{2}\right) = \dfrac{11\sqrt{3}}{5}\left(x - \dfrac{7\sqrt{3}}{2}\right)$

Question: horizontal and vertical tangents of the curve $r = 1+\sin\theta$

$y = r \cdot \sin\theta$

$(1+\sin\theta)\cdot\sin\theta$

$x = r\cdot\cos\theta$

$(1+\sin\theta)\cos\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta\cdot\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cdot\cos\theta - (1+\sin\theta)\sin\theta}$$

$2\sin\theta\cos\theta + \cos\theta$

$\underbrace{\cos^2\theta}_{1-\sin^2\theta} - \sin\theta - \sin^2\theta$

$-(2\sin^2\theta + \sin\theta - 1)$

$2\sin\theta \quad -1$

$\sin\theta \quad +1$

$(1-2\sin\theta)(\sin\theta+1)$

$$\Rightarrow \frac{\cos\theta\,(2\sin\theta+1)}{(1-2\sin\theta)(\sin\theta+1)} \longrightarrow \cos\theta\,(2\sin\theta+1) = 0$$

$\cos\theta = 0 \quad \theta = \frac{\pi}{2}, \cancel{\frac{3\pi}{2}}, \frac{7\pi}{6}, \frac{11\pi}{6}$

$\sin\theta = -\frac{1}{2} \quad \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

$\sin\theta = \frac{1}{2} \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$\sin\theta = -1 \quad \theta = \frac{3\pi}{2}$

Horizontal : $N = 0 \quad D \neq 0 \Rightarrow \left(\frac{\pi}{2}, 2\right), \left(\frac{7\pi}{6}, \frac{1}{2}\right), \left(\frac{11\pi}{6}, \frac{1}{2}\right)$

Vertical : $N \neq 0 \quad D = 0 \Rightarrow \left(\frac{\pi}{6}, \frac{3}{2}\right), \left(\frac{5\pi}{6}, \frac{3}{2}\right), \left(\frac{3\pi}{2}, 0\right)$ $\boxed{r = 0 \text{ ([illegible] point)}}$