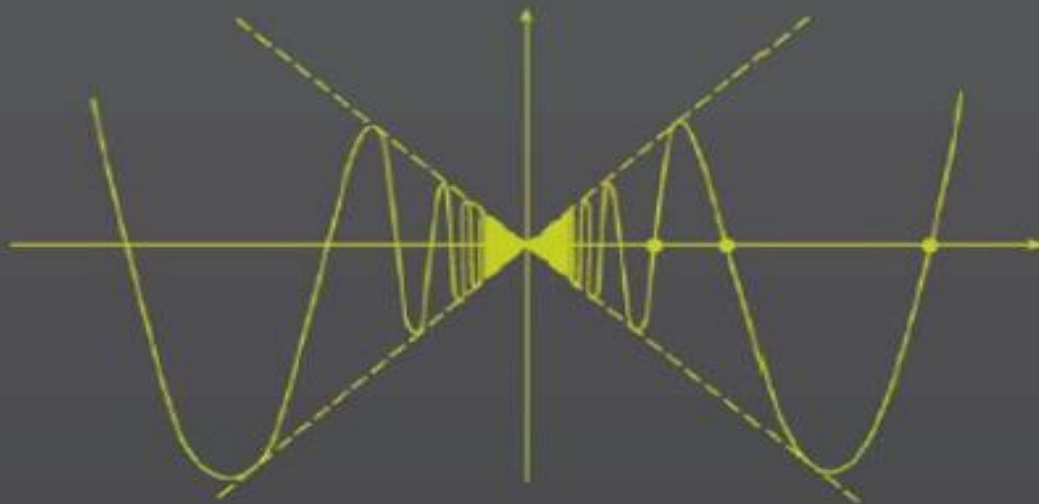
FOURTH EDITION



INTRODUCTION TO REAL ANALYSIS

Robert G. Bartle | Donald R. Sherbert

Functions; Domain and Range

DEFINITION A **function** f from a set D to a set Y is a rule that assigns a *unique* (single) element $f(x) \in Y$ to each element $x \in D$.

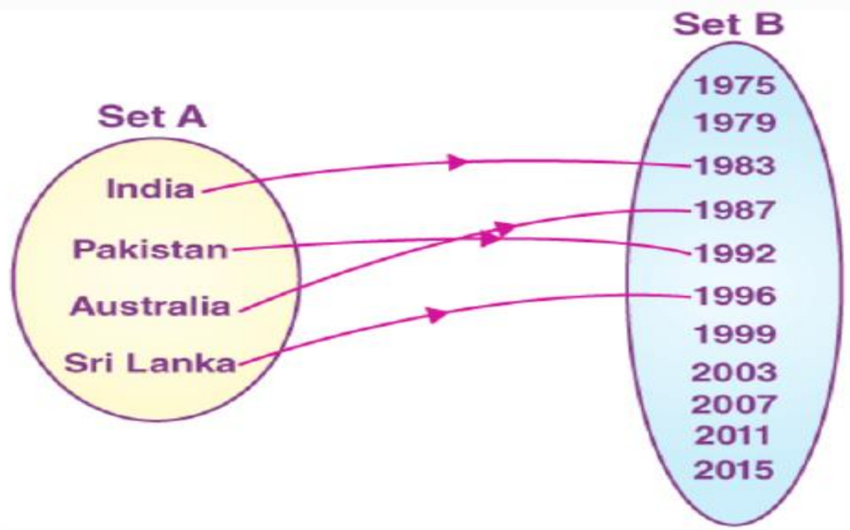
The set D of all possible input values is called the **domain** of the function. The set of all values of $f(x)$ as x varies throughout D is called the **range** of the function. The range may not include every element in the set Y. The domain and range of a function can be any sets of objects

The set D of all possible input values is called the **domain** *of the function.*

The theoretical set that contains all the outputs of the function is called the codomain. This is the set that the function maps into, but not necessarily all elements of the codomain are actual outputs.

The set of all values of $f(x)$ as x varies throughout D is called the **range** *of the function. In other words, after substituting the domain, the entire set of all possible values as outcomes of the dependent variable forms the range.*

Let us take the function $F : A \to B$ as follows:



Here, the range of the function F is $\{1983, 1987, 1992, 1996\}$. On the other hand, the whole set B is known as the codomain of the function.

EXAMPLE Let's verify the natural domains and associated ranges of some simple functions. The domains in each case are the values of x for which the formula makes sense.

Function	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

Solution The formula $y = x^2$ gives a real y-value for any real number x, so the domain is $(-\infty, \infty)$. The range of $y = x^2$ is $[0, \infty)$ because the square of any real number is nonnegative and every nonnegative number y is the square of its own square root, $y = (\sqrt{y})^2$ for $y \geq 0$.

The formula $y = 1/x$ gives a real y-value for every x except $x = 0$. For consistency in the rules of arithmetic, *we cannot divide any number by zero*. The range of $y = 1/x$, the set of reciprocals of all nonzero real numbers, is the set of all nonzero real numbers, since $y = 1/(1/y)$. That is, for $y \neq 0$ the number $x = 1/y$ is the input assigned to the output value y.

The formula $y = \sqrt{x}$ gives a real y-value only if $x \geq 0$. The range of $y = \sqrt{x}$ is $[0, \infty)$ because every nonnegative number is some number's square root (namely, it is the square root of its own square).

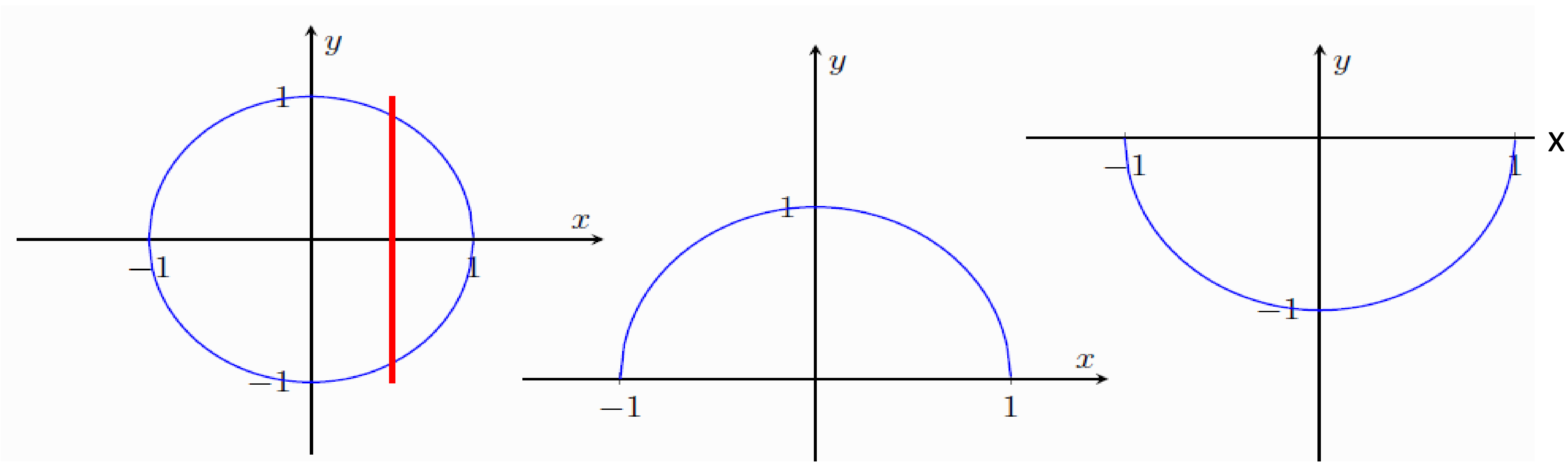
In $y = \sqrt{4 - x}$, the quantity $4 - x$ cannot be negative. That is, $4 - x \geq 0$, or $x \leq 4$. The formula gives real y-values for all $x \leq 4$. The range of $\sqrt{4 - x}$ is $[0, \infty)$, the set of all nonnegative numbers.

The formula $y = \sqrt{1 - x^2}$ gives a real y-value for every x in the closed interval from -1 to 1. Outside this domain, $1 - x^2$ is negative and its square root is not a real number. The values of $1 - x^2$ vary from 0 to 1 on the given domain, and the square roots of these values do the same. The range of $\sqrt{1 - x^2}$ is $[0, 1]$.

The Vertical Line Test for a Function

Not every curve in the coordinate plane can be the graph of a function. A function f can have only one value $f(x)$ for each x in its domain, so no vertical line can intersect the graph of a function at more than one point. If a is in the domain of the function f, then the vertical line $x = a$ will intersect the graph of f at the single point $(a, f(a))$.

A circle cannot be the graph of a function, since some vertical lines intersect the circle twice.



The circle $(x^2 + y^2 = 1)$ **is not the graph of a function**; it fails the vertical line test.

The upper semicircle is the graph of the function $f(x) = \sqrt{1 - x^2}$.

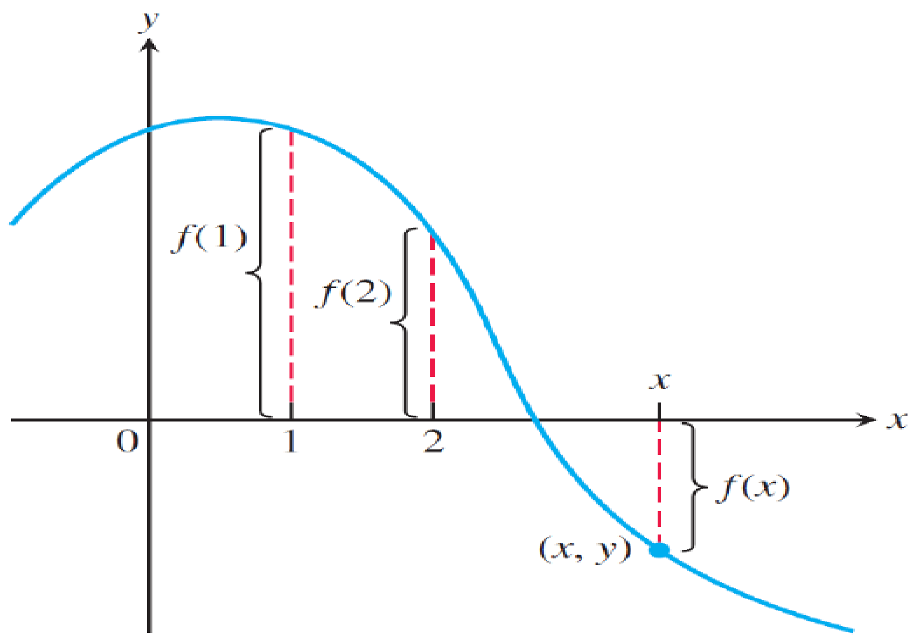
The lower semicircle is the graph of the function $g(x) = -\sqrt{1 - x^2}$.

Graphs of Functions

If f is a function with domain D, its **graph** consists of the points in the Cartesian plane whose coordinates are the input-output pairs for f. In set notation, the graph is

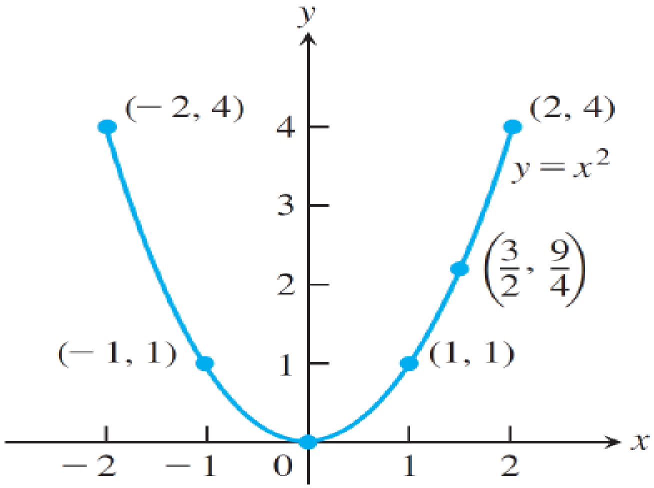
$$\{(x, f(x)) \mid x \in D\}.$$

The graph of a function f is a useful picture of its behavior. If (x, y) is a point on the graph, then $y = f(x)$ is the height of the graph above (or below) the point x. The height may be positive or negative, depending on the sign of $f(x)$



Graph the function $y = x^2$ *over the interval* $[-2, 2]$.

x	$y = x^2$
-2	4
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4



Piecewise-Defined Functions

Sometimes a function is described by using different formulas on different parts of its domain. One example is the **absolute value function**

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0, \end{cases}$$

whose graph is given in Figure . The right-hand side of the equation means that the function equals x if $x \geq 0$, and equals $-x$ if $x < 0$. Here are some other examples.

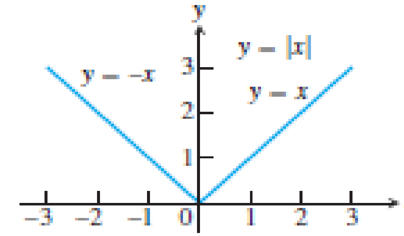


FIGURE The absolute value function has domain $(-\infty, \infty)$ and range $[0, \infty)$.

EXAMPLE The function

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

is defined on the entire real line but has values given by different formulas depending on the position of x. The values of f are given by $y = -x$ when $x < 0$, $y = x^2$ when $0 \leq x \leq 1$, and $y = 1$ when $x > 1$. The function, however, is *just one function* whose domain is the entire set of real numbers

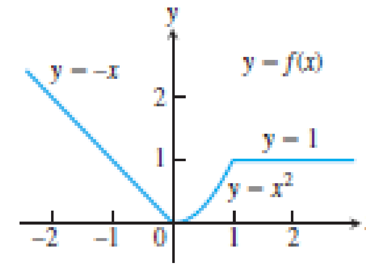
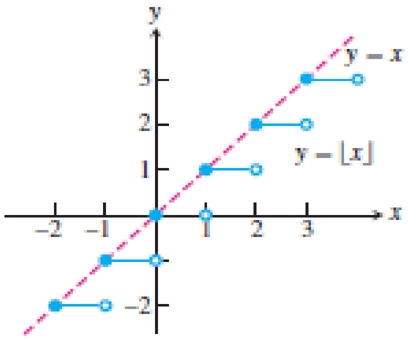


FIGURE To graph the function $y = f(x)$ shown here, we apply different formulas to different parts of its domain

EXAMPLE The function whose value at any number x is the *greatest integer less than or equal to* x is called the **greatest integer function** or the **integer floor function**. It is denoted $\lfloor x \rfloor$.

$$\lfloor 2.4 \rfloor = 2, \quad \lfloor 1.9 \rfloor = 1, \quad \lfloor 0 \rfloor = 0, \quad \lfloor -1.2 \rfloor = -2,$$
$$\lfloor 2 \rfloor = 2, \quad \lfloor 0.2 \rfloor = 0, \quad \lfloor -0.3 \rfloor = -1 \quad \lfloor -2 \rfloor = -2.$$

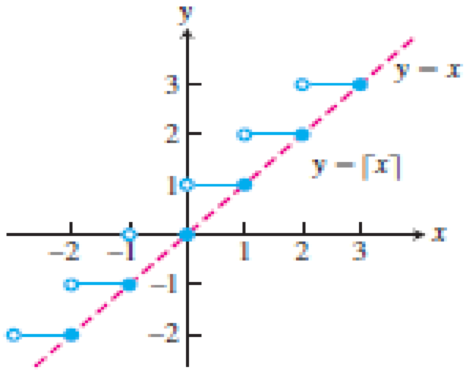


The graph of the greatest integer function $y = \lfloor x \rfloor$ lies on or below the line $y = x$, so it provides an integer floor for x

EXAMPLE The function whose value at any number x is the *smallest integer greater than or equal to* x is called the **least integer function** or the **integer ceiling function**. It is denoted $\lceil x \rceil$. Figure shows the graph. For positive values of x, this function might represent, for example, the cost of parking x hours in a parking lot which charges \$1 for each hour or part of an hour.

$$\lceil 3.2 \rceil = 4$$

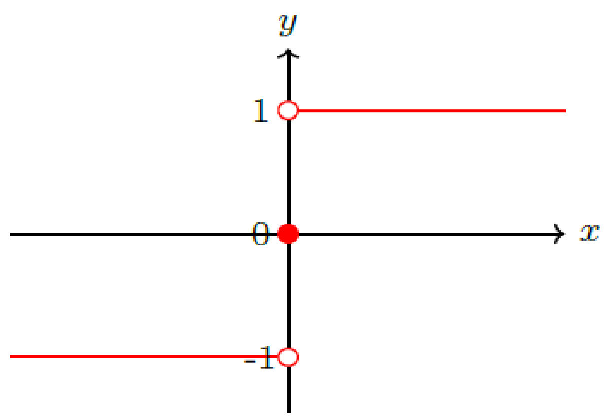
$$\lceil -1.7 \rceil = -1$$



The graph of the least integer function $y = \lceil x \rceil$ lies on or above the line $y = x$, so it provides an integer ceiling for x

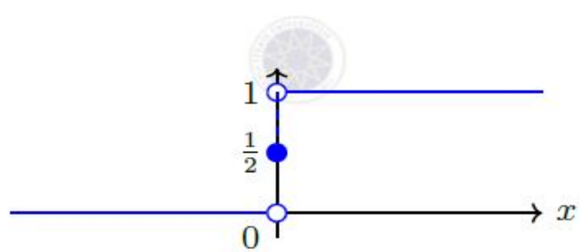
Signum (Sign) Function The signum function of a real number x is a piecewise function which is defined as follows:

$$\operatorname{sgn} x := \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$



Heaviside Step Function When defined as a piecewise constant function, the Heaviside step function is given by:

$$H(x) = \begin{cases} 0 & \text{if } x < 0, \\ \frac{1}{2} & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$



Increasing and Decreasing Functions

DEFINITIONS Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. If $f(x_2) > f(x_1)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.

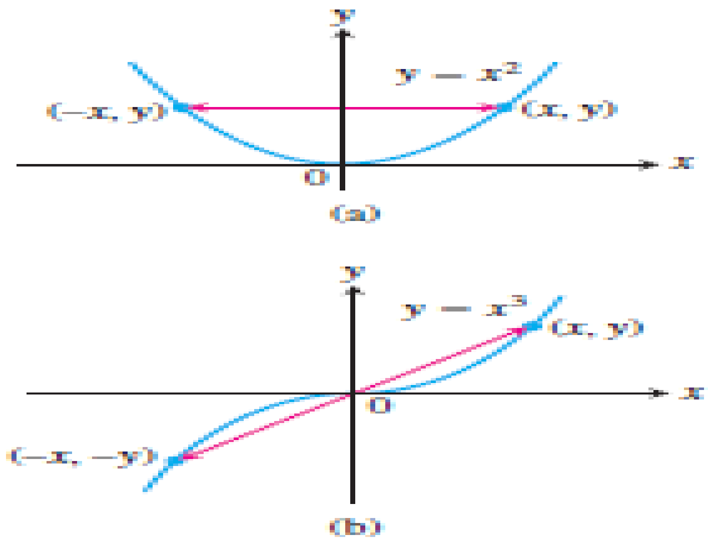
It is important to realize that the definitions of increasing and decreasing functions must be satisfied for *every* pair of points x_1 and x_2 in I with $x_1 < x_2$. Because we use the inequality $<$ to compare the function values, instead of $\leq$, it is sometimes said that f is *strictly* increasing or decreasing on I.

Even Functions and Odd Functions: Symmetry

DEFINITIONS A function $y = f(x)$ is an

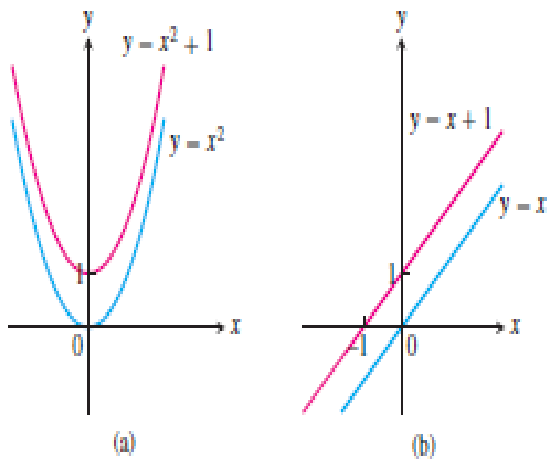
even function of x if $f(-x) = f(x)$,
odd function of x if $f(-x) = -f(x)$,

for every x in the function's domain.



(a) The graph of $y = x^2$ (an even function) is symmetric about the y-axis. (b) The graph of $y = x^3$ (an odd function) is symmetric about the origin.

$f(x) = x^2$	Even function: $(-x)^2 = x^2$ for all x; symmetry about y-axis.
$f(x) = x^2 + 1$	Even function: $(-x)^2 + 1 = x^2 + 1$ for all x; symmetry about y-axis
$f(x) = x$	Odd function: $(-x) = -x$ for all x; symmetry about the origin.
$f(x) = x + 1$	Not odd: $f(-x) = -x + 1$, but $-f(x) = -x - 1$. The two are not equal. Not even: $(-x) + 1 \neq x + 1$ for all $x \neq 0$

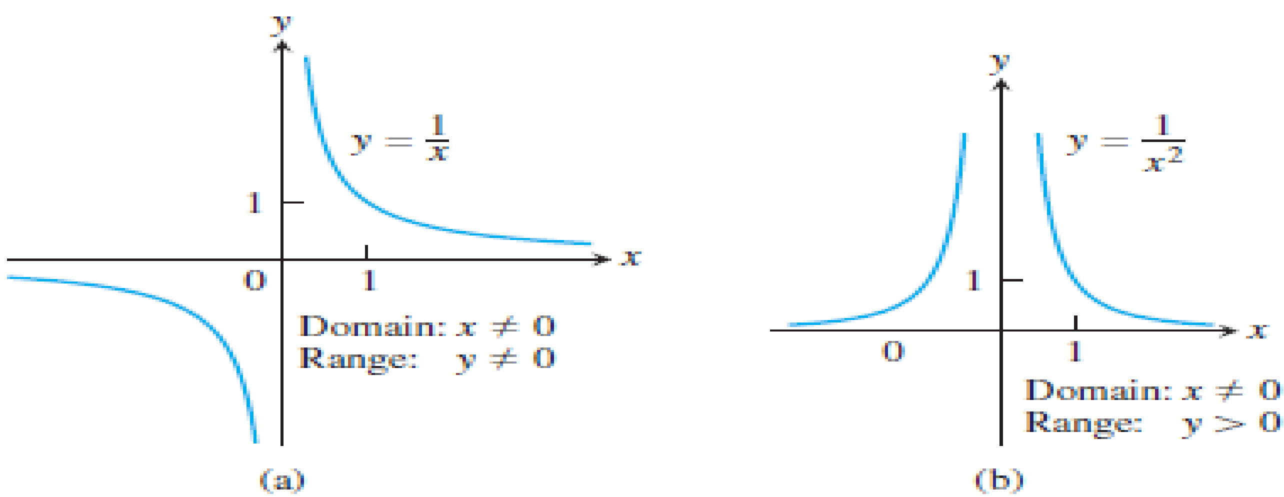


(a) When we add the constant term 1 to the function $y = x^2$, the resulting function $y = x^2 + 1$ is still even and its graph is still symmetric about the y-axis. (b) When we add the constant term 1 to the function $y = x$, the resulting function $y = x + 1$ is no longer odd. The symmetry about the origin is lost

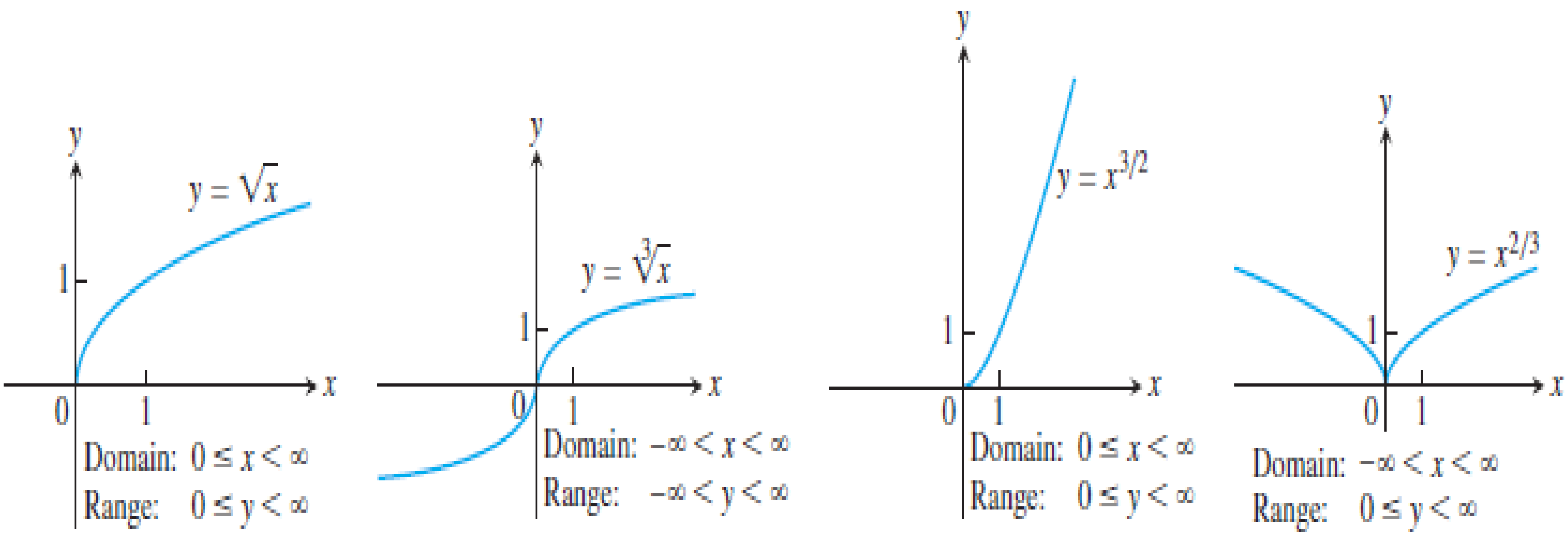
DEFINITION Two variables y and x are **proportional** (to one another) if one is always a constant multiple of the other; that is, if $y = kx$ for some nonzero constant k.

If the variable y is proportional to the reciprocal $1/x$, then sometimes it is said that y is **inversely proportional** to x (because $1/x$ is the multiplicative inverse of x).

Power Functions A function $f(x) = x^a$, where a is a constant, is called a **power function**. There are several important cases to consider.



Graphs of the power functions $f(x) = x^a$ for part (a) $a = -1$ and for part (b) $a = -2$.



Graphs of the power functions $f(x) = x^a$ for $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$, and $\frac{2}{3}$.

Definition If $\alpha \in \mathbb{R}$ and $x > 0$, the number x^α is defined to be

$$x^\alpha := e^{\alpha \ln x} = E(\alpha L(x)).$$

The function $x \mapsto x^\alpha$ for $x > 0$ is called the **power function** with exponent α.

Note If $x > 0$ and $\alpha = m/n$ where $m \in \mathbb{Z}$, $n \in \mathbb{N}$, then we defined $x^\alpha := (x^m)^{1/n}$
Hence we have $\ln x^\alpha = \alpha \ln x$, whence $x^\alpha = e^{\ln x^\alpha} = e^{\alpha \ln x}$.

We now state some properties of the power functions. Their proofs are immediate consequences of the properties of the exponential and logarithm functions

Theorem *If $\alpha \in \mathbb{R}$ and x, y belong to $(0, \infty)$, then:*

(a) $1^\alpha = 1$, (b) $x^\alpha > 0$,
(c) $(xy)^\alpha = x^\alpha y^\alpha$; (d) $(x/y)^\alpha = x^\alpha / y^\alpha$.

Theorem *If α, $\beta \in \mathbb{R}$ and $x \in (0, \infty)$, then:*

(a) $x^{\alpha+\beta} = x^\alpha x^\beta$ (b) $(x^\alpha)^\beta = x^{\alpha\beta} = (x^\beta)^\alpha$;
(c) $x^{-\alpha} = 1/x^\alpha$; (d) *if $\alpha < \beta$, then $x^\alpha < x^\beta$ for $x > 1$.*

Polynomials A function p is a **polynomial** if

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

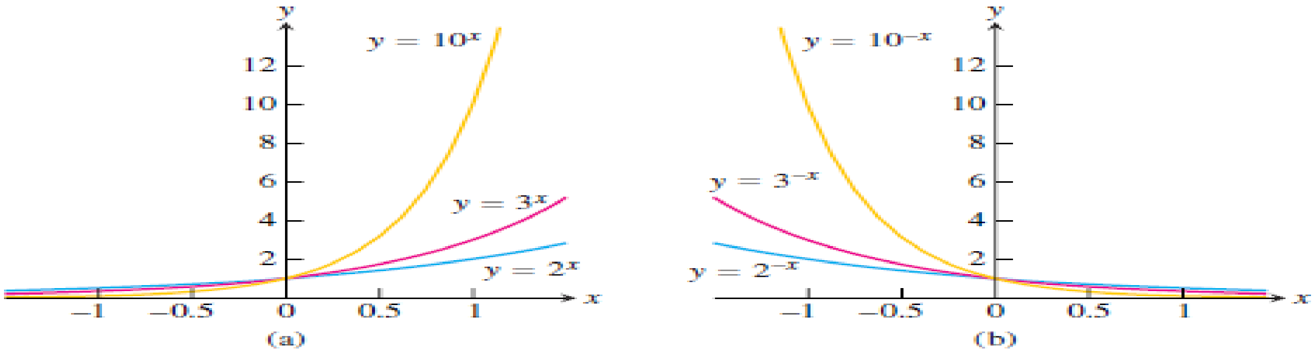
where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are real constants (called the **coefficients** of the polynomial). All polynomials have domain $(-\infty, \infty)$. If the

leading coefficient $a_n \neq 0$ and $n > 0$, then n is called the **degree** of the polynomial. Linear functions with $m \neq 0$ are polynomials of degree 1. Polynomials of degree 2, usually written as $p(x) = ax^2 + bx + c$, are called **quadratic functions**. Likewise, **cubic functions** are polynomials $p(x) = ax^3 + bx^2 + cx + d$ of degree 3.

Rational Functions A **rational function** is a quotient or ratio $f(x) = p(x)/q(x)$, where p and q are polynomials. The domain of a rational function is the set of all real x for which $q(x) \neq 0$.

Algebraic Functions Any function constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking roots) lies within the class of **algebraic functions**.

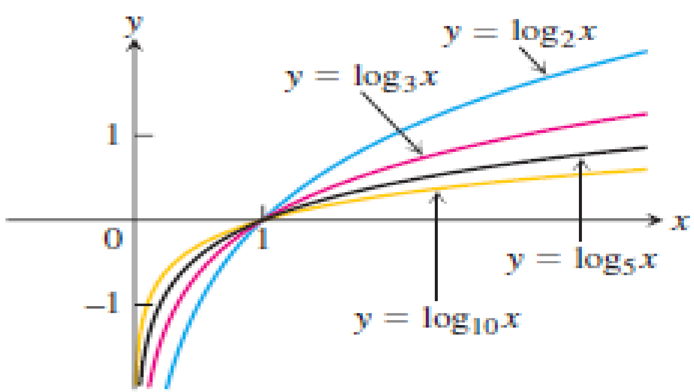
Exponential Functions Functions of the form $f(x) = a^x$, where the base $a > 0$ is a positive constant and $a \neq 1$, are called **exponential functions**. All exponential functions have domain $(-\infty, \infty)$ and range $(0, \infty)$, so an exponential function never assumes the value 0.



Graphs of exponential functions.

Logarithmic Functions These are the functions $f(x) = \log_a x$, where the base $a \neq 1$ is a positive constant. They are the *inverse functions* of the exponential functions,

the domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.



The Exponential Function

We begin by establishing the key existence result for the exponential function.

Theorem *There exists a function $E:\mathbb{R}\to\mathbb{R}$ such that:*

(i) $E'(x) = E(x)$ *for all* $x \in \mathbb{R}$.
(ii) $E(0) = 1$.

Corollary *The function E has a derivative of every order and $E^{(n)}(x) = E(x)$ for all $n \in \mathbb{N}$, $x \in \mathbb{R}$.*

Definition The unique function $E:\mathbb{R}\to\mathbb{R}$, such that $E'(x) = E(x)$ for all $x \in \mathbb{R}$ and $E(0) = 1$, is called the **exponential function**. The number $e := E(1)$ is called **Euler's number**. We will frequently write

$$\exp(x) := E(x) \qquad \text{or} \quad e^x := E(x) \qquad \text{for} \quad x \in \mathbb{R}.$$

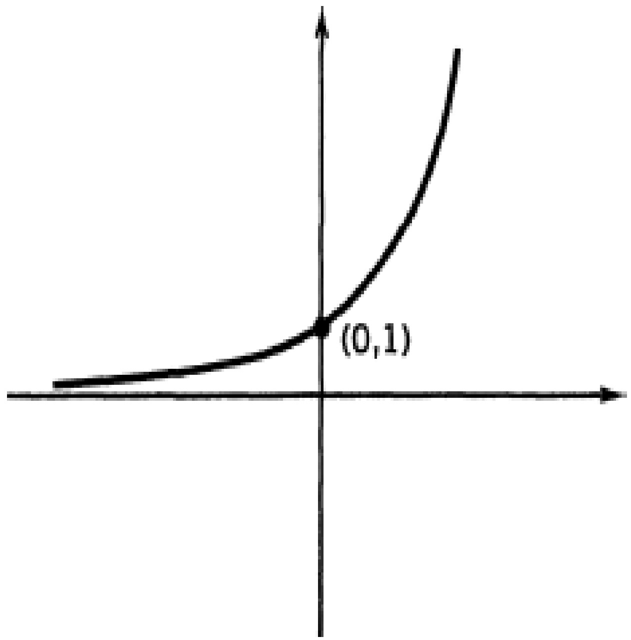
The number e can be obtained as a limit, and thereby approximated, in several different ways.

Theorem *The exponential function E is strictly increasing on* $\mathbb{R}$ *and has range equal to* $\{y \in \mathbb{R} : y > 0\}$. *Further, we have*

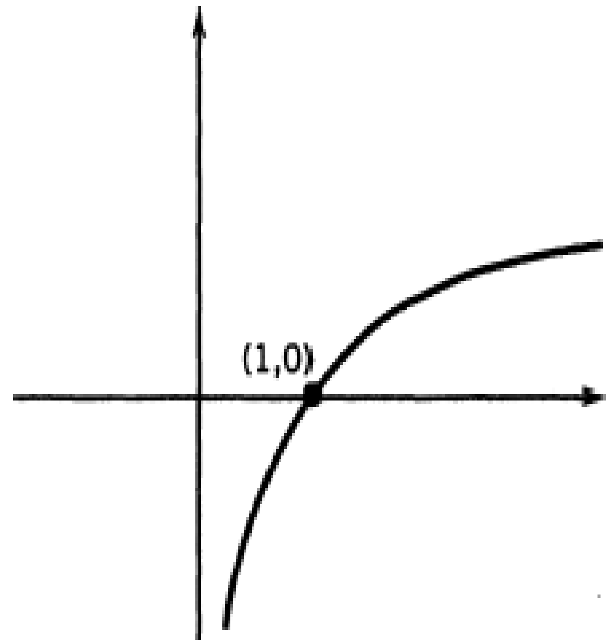
(vi) $\lim_{x \to -\infty} E(x) = 0$ and $\lim_{x \to \infty} E(x) = \infty$.

The Logarithm Function

We have seen that the exponential function E is a strictly increasing differentiable function with domain $\mathbb{R}$ and range $\{y \in \mathbb{R} : y > 0\}$.



Graph of E



Graph of L

Definition The function inverse to $E:\mathbb{R}\to\mathbb{R}$ is called the **logarithm** (or the **natural logarithm**). It will be denoted by L, or by $\ln$.

Since E and L are inverse functions, we have

$$(L\circ E)(x)=x \qquad \text{for all} \quad x\in\mathbb{R}$$

and

$$(E\circ L)(y)=y \qquad \text{for all} \quad y\in\mathbb{R},\ y>0.$$

These formulas may also be written in the form

$$\ln e^{x}=x, \qquad e^{\ln y}=y.$$

Theorem *The logarithm is a strictly increasing function L with domain $\{x \in \mathbb{R} : x > 0\}$ and range $\mathbb{R}$. The derivative of L is given by*

$$L'(x) = 1/x \quad for \quad x > 0.$$

The logarithm satisfies the functional equation

$$L(xy) = L(x) + L(y) \text{ for } x > 0, y > 0.$$

Moreover, we have

$$L(1) = 0 \quad and \quad L(e) = 1,$$

$$L(x^r) = rL(x) \quad for \quad x > 0, r \in \mathbb{Q}.$$

$$\lim_{x \to 0+} L(x) = -\infty \quad and \quad \lim_{x \to \infty} L(x) = \infty.$$

Sums, Differences, Products, and Quotients

Like numbers, functions can be added, subtracted, multiplied, and divided (except where the denominator is zero) to produce new functions. If f and g are functions, then for every x that belongs to the domains of both f and g (that is, for $x \in D(f) \cap D(g)$), we define functions $f + g$, $f - g$, and fg by the formulas

$$(f + g)(x) = f(x) + g(x).$$

$$(f - g)(x) = f(x) - g(x).$$

$$(fg)(x) = f(x)g(x).$$

Notice that the + sign on the left-hand side of the first equation represents the operation of addition of *functions*, whereas the + on the right-hand side of the equation means addition of the real numbers $f(x)$ and $g(x)$.

At any point of $D(f) \cap D(g)$ at which $g(x) \neq 0$, we can also define the function f/g by the formula

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad (\text{where } g(x) \neq 0).$$

Functions can also be multiplied by constants: If c is a real number, then the function cf is defined for all x in the domain of f by

$$(cf)(x) = cf(x).$$

EXAMPLE The functions defined by the formulas

$$f(x) = \sqrt{x} \quad \text{and} \quad g(x) = \sqrt{1 - x}$$

have domains $D(f) = [0, \infty)$ and $D(g) = (-\infty, 1]$. The points common to these domains are the points

$$[0, \infty) \cap (-\infty, 1] = [0, 1].$$

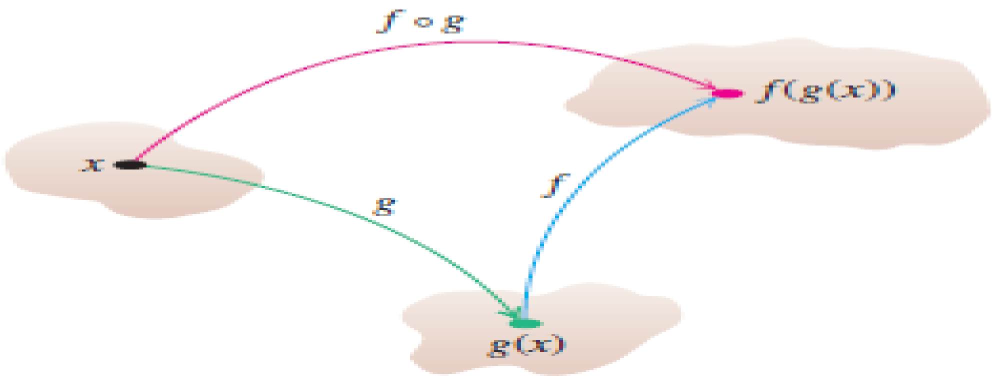
The following table summarizes the formulas and domains for the various algebraic combinations of the two functions. We also write $f \cdot g$ for the product function fg.

$f + g$	$(f + g)(x) = \sqrt{x} + \sqrt{1 - x}$	$[0, 1] = D(f) \cap D(g)$
$f - g$	$(f - g)(x) = \sqrt{x} - \sqrt{1 - x}$	$[0, 1]$
$g - f$	$(g - f)(x) = \sqrt{1 - x} - \sqrt{x}$	$[0, 1]$
$f \cdot g$	$(f \cdot g)(x) = f(x)g(x) = \sqrt{x(1 - x)}$	$[0, 1]$
f/g	$\dfrac{f}{g}(x) = \dfrac{f(x)}{g(x)} = \sqrt{\dfrac{x}{1 - x}}$	$[0, 1)$ ($x = 1$ excluded)
g/f	$\dfrac{g}{f}(x) = \dfrac{g(x)}{f(x)} = \sqrt{\dfrac{1 - x}{x}}$	$(0, 1]$ ($x = 0$ excluded)

DEFINITION If f and g are functions, the **composite** function $f \circ g$ ("f composed with g") is defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f.



EXAMPLE If $f(x) = \sqrt{x}$ and $g(x) = x + 1$, find

(a) $(f \circ g)(x)$ **(b)** $(g \circ f)(x)$ **(c)** $(f \circ f)(x)$ **(d)** $(g \circ g)(x)$.

Solution

	Composite	**Domain**
(a)	$(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x + 1}$	$[-1, \infty)$
(b)	$(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$	$[0, \infty)$
(c)	$(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{1/4}$	$[0, \infty)$
(d)	$(g \circ g)(x) = g(g(x)) = g(x) + 1 = (x + 1) + 1 = x + 2$	$(-\infty, \infty)$

To see why the domain of $f \circ g$ is $[-1, \infty)$, notice that $g(x) = x + 1$ is defined for all real x but belongs to the domain of f only if $x + 1 \geq 0$, that is to say, when $x \geq -1$.

Notice that if $f(x) = x^2$ and $g(x) = \sqrt{x}$, then $(f \circ g)(x) = \left(\sqrt{x}\right)^2 = x$. However, the domain of $f \circ g$ is $[0, \infty)$, not $(-\infty, \infty)$, since $\sqrt{x}$ requires $x \geq 0$.

Shift Formulas

Vertical Shifts

$y = f(x) + k$	Shifts the graph of f *up* k units if $k > 0$
	Shifts it *down* $\lvert k \rvert$ units if $k < 0$

Horizontal Shifts

$y = f(x + h)$	Shifts the graph of f *left* h units if $h > 0$
	Shifts it *right* $\lvert h \rvert$ units if $h < 0$

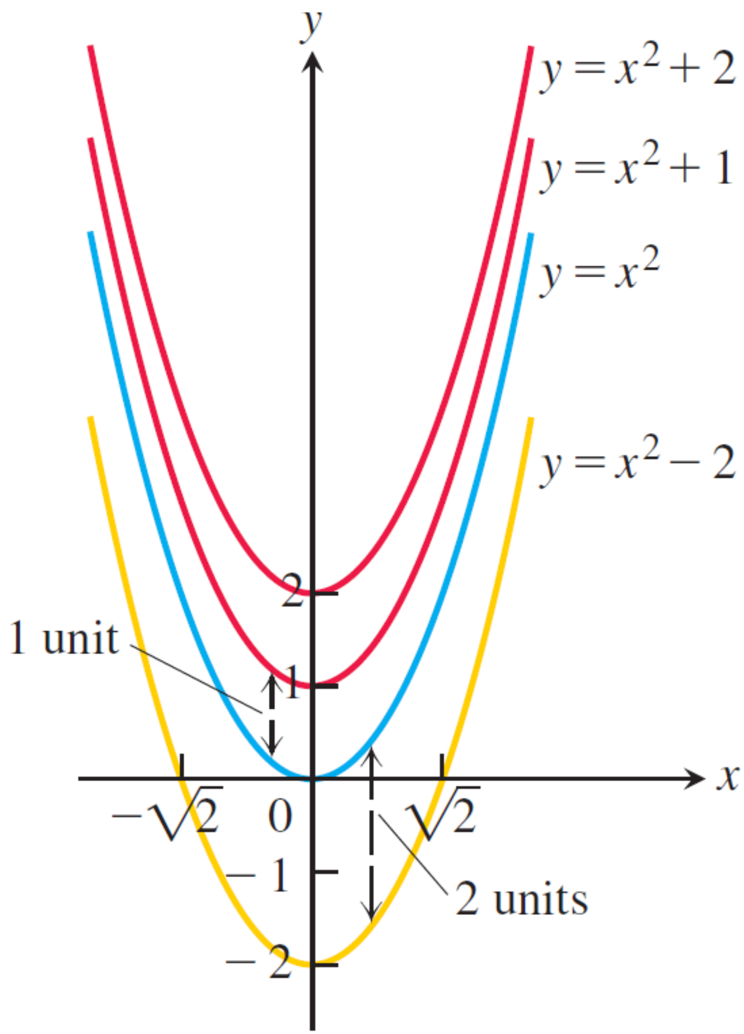
Vertical and Horizontal Scaling and Reflecting Formulas

For $c > 1$, the graph is scaled:

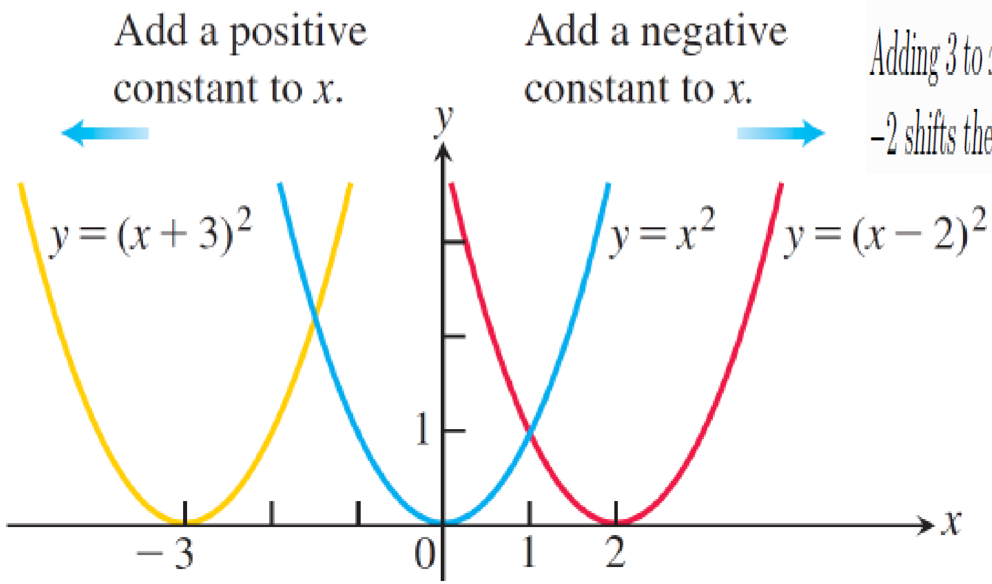
$y = cf(x)$	Stretches the graph of f vertically by a factor of c.
$y = \frac{1}{c} f(x)$	Compresses the graph of f vertically by a factor of c.
$y = f(cx)$	Compresses the graph of f horizontally by a factor of c.
$y = f(x/c)$	Stretches the graph of f horizontally by a factor of c.

For $c = -1$, the graph is reflected:

$y = -f(x)$	Reflects the graph of f across the x-axis.
$y = f(-x)$	Reflects the graph of f across the y-axis.

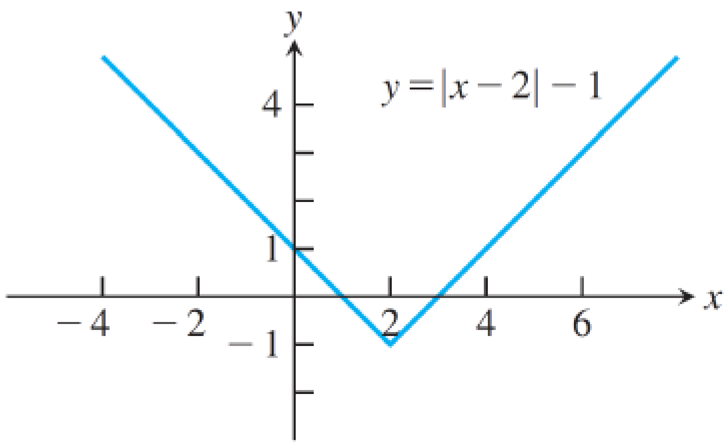


y
$y = x^2 + 2$
$y = x^2 + 1$
$y = x^2$
$y = x^2 - 2$
2
1 unit
1
x
$-\sqrt{2}$
0
$\sqrt{2}$
-1
2 units
-2



Adding 3 to x in $y=x^2$ to get $y=(x+3)^2$ shifts the graph 3 units to the left, while adding −2 shifts the graph 2 units to the right

Adding -2 *to* x *in* $y=|x|$, *and then adding* -1 *to the result, gives* $y=|x-2|-1$ *and shifts the graph 2 units to the right and 1 unit down*.

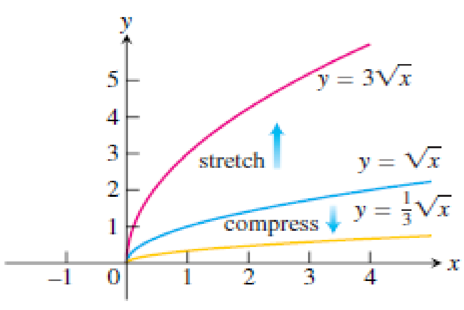


EXAMPLE Here we scale and reflect the graph of $y = \sqrt{x}$.

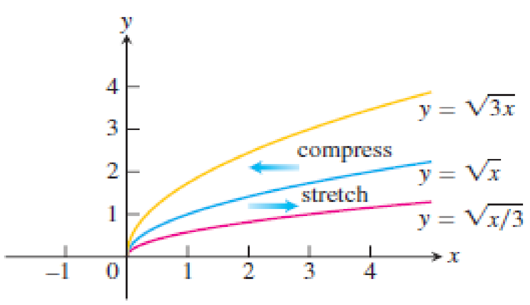
(a) **Vertical:** Multiplying the right-hand side of $y = \sqrt{x}$ by 3 to get $y = 3\sqrt{x}$ stretches the graph vertically by a factor of 3, whereas multiplying by 1/3 compresses the graph by a factor of 3

(b) **Horizontal:** The graph of $y = \sqrt{3x}$ is a horizontal compression of the graph of $y = \sqrt{x}$ by a factor of 3, and $y = \sqrt{x/3}$ is a horizontal stretching by a factor of 3 . Note that $y = \sqrt{3x} = \sqrt{3}\sqrt{x}$ so a horizontal compression *may* correspond to a vertical stretching by a different scaling factor. Likewise, a horizontal stretching may correspond to a vertical compression by a different scaling factor.

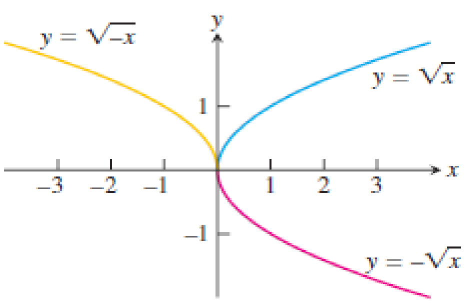
(c) **Reflection:** The graph of $y = -\sqrt{x}$ is a reflection of $y = \sqrt{x}$ across the x-axis, and $y = \sqrt{-x}$ is a reflection across the y-axis



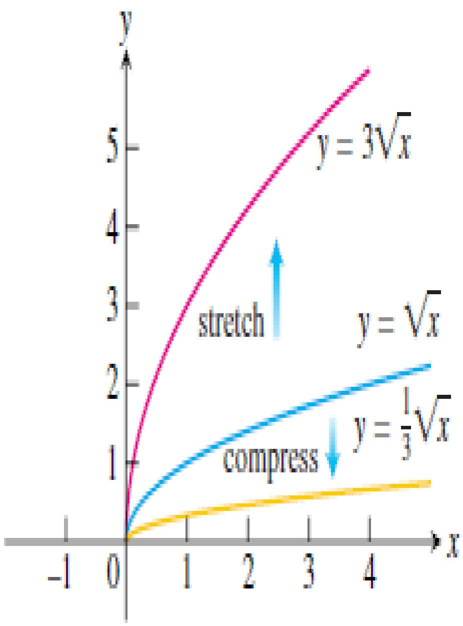
Vertically stretching and compressing the graph $y = \sqrt{x}$ by a factor of 3



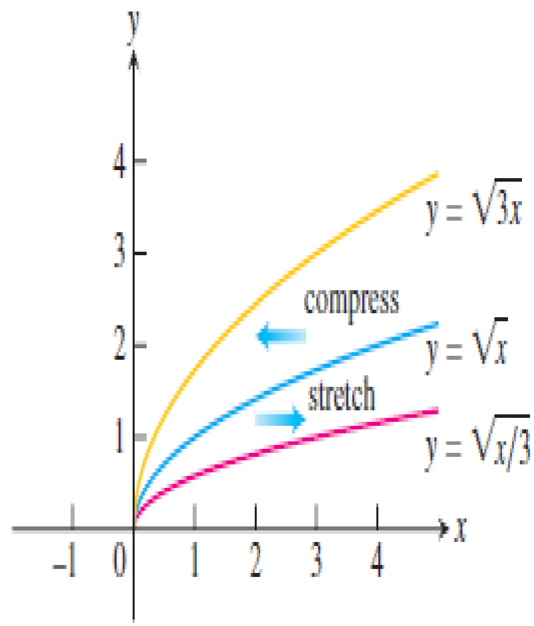
Horizontally stretching and compressing the graph $y = \sqrt{x}$ by a factor of 3



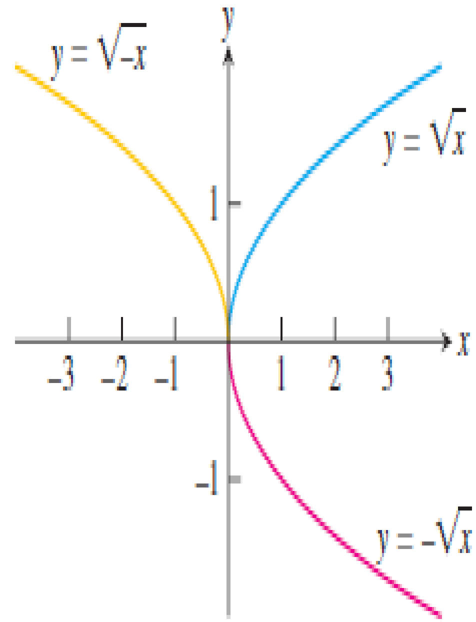
Reflections of the graph $y = \sqrt{x}$ across the coordinate axes



Vertically stretching and compressing the graph $y = \sqrt{x}$ by a factor of 3 (



Horizontally stretching and compressing the graph $y = \sqrt{x}$ by a factor of 3 (



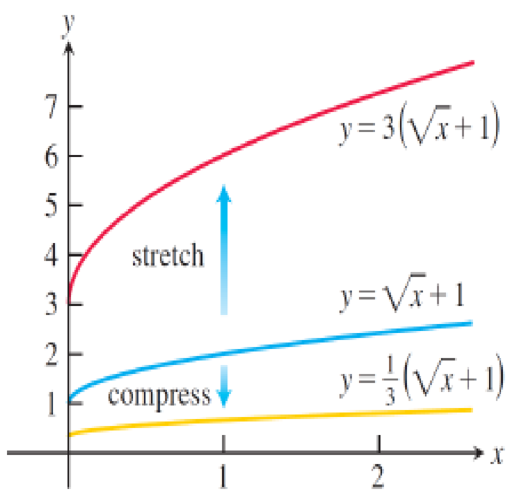
Reflections of the graph $y = \sqrt{x}$ across the coordinate axes

Here we scale and reflect the graph of $y = \sqrt{x} + 1$.

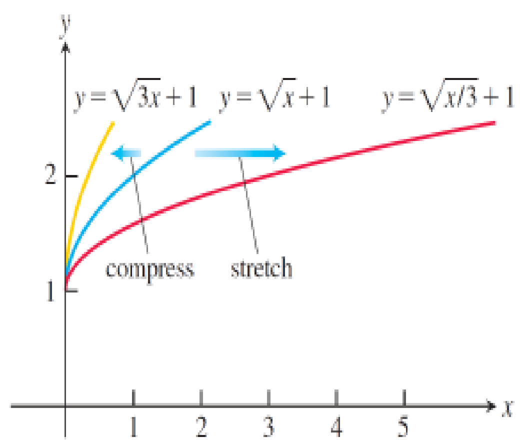
(a) **Vertical:** *Multiplying the right-hand side of* $y = \sqrt{x} + 1$ *by 3 to get* $y = 3(\sqrt{x} + 1)$ *stretches the graph vertically by a factor of 3, whereas multiplying by 1/3 compresses the graph vertically by a factor of 3*

(b) **Horizontal:** *The graph of* $y = \sqrt{3x} + 1$ *is a horizontal compression of the graph of* $y = \sqrt{x} + 1$ *by a factor of 3, and* $y = \sqrt{x/3} + 1$ *is a horizontal stretching by a factor of 3*

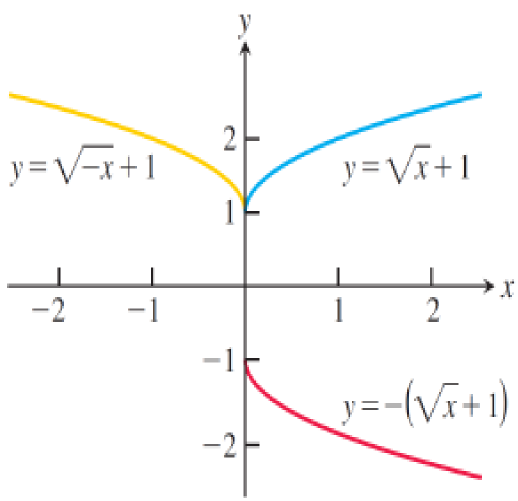
(c) **Reflection:** *The graph of* $y = -(\sqrt{x} + 1)$ *is a reflection of* $y = \sqrt{x} + 1$ *across the* x*-axis, and* $y = \sqrt{-x} + 1$ *is a reflection across the* y*-axis* .



(a) Vertical scaling of the graph.



(b) Horizontal scaling of the graph.



(c) Reflection of the graph.

Elementary function

An ***elementary function*** *is a function of a single variable (typically real or complex) that is defined as taking sums, products, roots, and compositions of finitely many polynomial, rational, trigonometric, hyperbolic, and exponential functions, and their inverses (e.g., arcsin, log, or $x^{1/n}$).*

Some examples of elementary functions

- **Constant functions:** 2, π, e, etc.
- **Rational powers of x:** x, x^2, $\sqrt{x}$ $\left(x^{\frac{1}{2}}\right)$, $x^{\frac{2}{3}}$, etc.
- **Exponential functions:** e^x, a^x.
- **Logarithms:** $\log x$, $\log_a x$.
- **Trigonometric functions:** $\sin x$, $\cos x$, $\tan x$, etc.
- **Inverse trigonometric functions:** $\arcsin x$, $\arccos x$, etc.
- **Hyperbolic functions:** $\sinh x$, $\cosh x$, etc.
- **Inverse hyperbolic functions:** $\operatorname{arcsinh} x$, $\operatorname{arccosh} x$, etc.
- All functions obtained by adding, subtracting, multiplying, or dividing a finite number of any of the previous functions.
- All functions obtained by root extraction of a polynomial with coefficients in elementary functions.

- All functions obtained by composing a finite number of any of the previously listed functions.

DEFINITION A function $f(x)$ is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for every value of x. The smallest such value of p is the **period** of f.

Periods of Trigonometric Functions

Period π: $\tan(x + \pi) = \tan x$

$\cot(x + \pi) = \cot x$

Period 2π: $\sin(x + 2\pi) = \sin x$

$\cos(x + 2\pi) = \cos x$

$\sec(x + 2\pi) = \sec x$

$\csc(x + 2\pi) = \csc x$

Special Cases of Trigonometric Functions

- Period of $\sin(ax+b)$ and $\cos(ax+b)$: $\frac{2\pi}{|a|}$.
- Period of $\tan(ax+b)$ and $\cot(ax+b)$: $\frac{\pi}{|a|}$.
- Period of $\sec(ax+b)$ and $\csc(ax+b)$: $\frac{2\pi}{|a|}$.
- If p is the period of the periodic function $f(x)$, then $af(x)+b$, where $a>0$, is also a periodic function with a period of p.
- Period of $a\sin x+b$ and $a\cos x+b$: 2π.
- Period of $a\tan x+b$ and $a\cot x+b$: π.
- Period of $a\sec x+b$ and $a\csc x+b$: 2π.

The functions $f(x)=p\tan(ax+b)+c$ and $g(x)=p\cot(ax+b)+c$ are periodic functions with period:

$$T=\frac{\pi}{|a|}.$$

$$f(x) = a \cdot \sin^n(cx + d) + b, \quad g(x) = a \cdot \cos^n(cx + d) + b$$

If n is an odd number, the periods of $f(x)$ and $g(x)$ are:

$$T_f = T_g = \frac{2\pi}{|c|}$$

If n is an even number, the periods of $f(x)$ and $g(x)$ are:

$$T_f = T_g = \frac{\pi}{|c|}$$

$$f(x) = 2\sin^2(3x) + 1$$

$$T_f = \frac{\pi}{3}$$

$$g(x) = -\cos^3\left(5x - \frac{\pi}{3}\right)$$

$$T_g = \frac{2\pi}{5}$$

$$f(x) = a \cdot \tan^n(cx + d) + b$$
$$g(x) = a \cdot \cot^n(cx + d) + b$$
$$T_f = T_g = \frac{\pi}{|c|}$$

$$f(x) = -\tan^2(4x) + 1$$
$$T_f = \frac{\pi}{4}$$
$$g(x) = 5\cot^3\left(2x - \frac{\pi}{4}\right)$$
$$T_g = \frac{\pi}{2}$$

The functions $f(x) = p\sin(ax + b) + c$ and $g(x) = p\cos(ax + b) + c$ are periodic functions with period:

$$T = \frac{2\pi}{|a|}.$$

Determine the period of the given periodic function $5\sin(2x + 3)$

Solution. The given function is:

$$5\sin(2x + 3)$$

From the function, the coefficient of x is $a = 2$.

We know that the period of the basic sine function $\sin x$ is 2π.

For the function $\sin(ax + b)$, the period is given by:

$$\text{Period} = \frac{2\pi}{|a|}$$

Substituting $a = 2$:

$$\text{Period} = \frac{2\pi}{2} = \pi$$

Thus, the period of the function $5\sin(2x + 3)$ is:

$$\boxed{\pi}$$

Period of the Sum/Difference of Periodic Functions

Let f and g be two periodic functions. The fundamental period of the function $f \pm g$ is equal to the least common multiple (LCM) of the fundamental periods of these functions.

$$T_f = \frac{a}{b}, \quad T_g = \frac{c}{d}, \quad \text{where,}$$

$$T_{f \pm g} = \text{LCM}\left(\frac{a}{b}, \frac{c}{d}\right) = \frac{\text{LCM}(a,c)}{\text{GCD}(b,d)},$$

where LCM denotes the least common multiple and GCD denotes the greatest common divisor.

$$T_f = \frac{3\pi}{4} \quad \textit{and} \quad T_g = \frac{2\pi}{5},$$

then,

$$T_{f+g} = \frac{LCM(3,2)}{GCD(4,5)}\pi$$

$$= \frac{6}{1}\pi = 6\pi.$$

Find the period of $f(x) = \tan 3x + \cos 5x$.

Solution. The given periodic function is:

$$f(x) = \tan 3x + \cos 5x$$

We know that:

- The period of $\tan x$ is π, and the period of $\cos x$ is 2π.
- The period of $\tan 3x$ is:
$$\frac{\pi}{3}$$
- The period of $\cos 5x$ is:
$$\frac{2\pi}{5}$$

To find the period of the sum $f(x) = \tan 3x + \cos 5x$, we calculate the least common multiple (LCM) of the individual periods.

The LCM of $\frac{\pi}{3}$ and $\frac{2\pi}{5}$ is given by:

$$\text{LCM}\left(\frac{\pi}{3}, \frac{2\pi}{5}\right) = \frac{\text{LCM}(\pi, 2\pi)}{\text{GCD}(3,5)} = \frac{2\pi}{1} = 2\pi$$

Therefore, the period of $f(x) = \tan 3x + \cos 5x$ is:

$$\boxed{2\pi}$$

Determine whether the function $f(x) = x + 1$ is periodic.

Solution. A function $f(x)$ is periodic if there exists a positive constant $T > 0$ such that $f(x+T) = f(x)$ for all x. For $f(x) = x + 1$,

$$f(x+T) = (x+T) + 1 = x + T + 1.$$

$$f(x+T) = f(x) \implies x + T + 1 = x + 1 \implies T = 0$$

Hence, $f(x) = x + 1$ is not periodic.

Determine the period of the function $f(x) = \sin(3x)$ and verify your result.

Solution. The general form of a sine function is $\sin(kx)$, where k affects the period of the function. The period of $\sin(x)$ is 2π, and for $f(x) = \sin(3x)$, the period is given by:

$$T = \frac{2\pi}{k},$$

where $k = 3$. Substituting $k = 3$, we get:

$$T = \frac{2\pi}{3}.$$

To verify, observe that $f(x+T) = f(x)$. Since $T = \frac{2\pi}{3}$, we have:

$$\implies f(x+T) = \sin(3x + 3T) = \sin(3x + 2\pi) = \sin(3x) = f(x) \implies f(x+T) = f(x),$$

which confirms that $T = \frac{2\pi}{3}$ is the period of $f(x)$.

Determine the period of the function $g(x) = \cos(2x) + \sin(x)$.

Solution. The period of $\cos(2x)$ is:

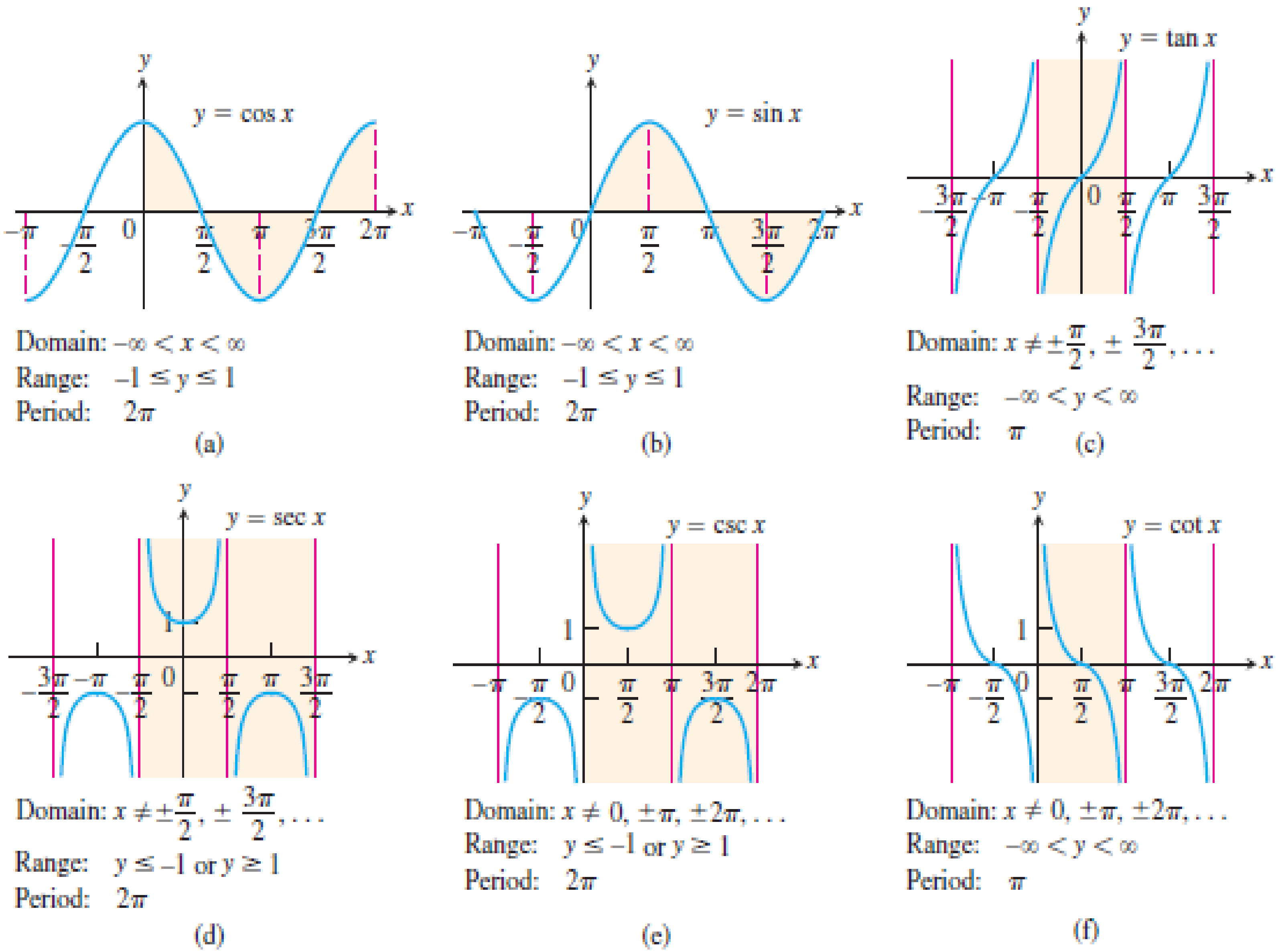
$$T_1 = \frac{2\pi}{2} = \pi,$$

and the period of $\sin(x)$ is:

$$T_2 = 2\pi.$$

To find the period of $g(x) = \cos(2x) + \sin(x)$, we calculate the least common multiple (LCM) of T_1 and T_2. Since $T_1 = \pi$ and $T_2 = 2\pi$, the LCM is 2π. Therefore, the period of $g(x)$ is:

$$T = 2\pi.$$



(a) Domain: $-\infty < x < \infty$; Range: $-1 \le y \le 1$; Period: 2π

(b) Domain: $-\infty < x < \infty$; Range: $-1 \le y \le 1$; Period: 2π

(c) Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$; Range: $-\infty < y < \infty$; Period: π

(d) Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$; Range: $y \le -1$ or $y \ge 1$; Period: 2π

(e) Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$; Range: $y \le -1$ or $y \ge 1$; Period: 2π

(f) Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$; Range: $-\infty < y < \infty$; Period: π

Trigonometric Identities

$$\cos^2\theta + \sin^2\theta = 1.$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Addition Formulas

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

Double-Angle Formulas

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\sin 2\theta = 2 \sin\theta \cos\theta$$

Half-Angle Formulas

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

The Law of Cosines

If a, b, and c are sides of a triangle ABC and if θ is the angle opposite c, then

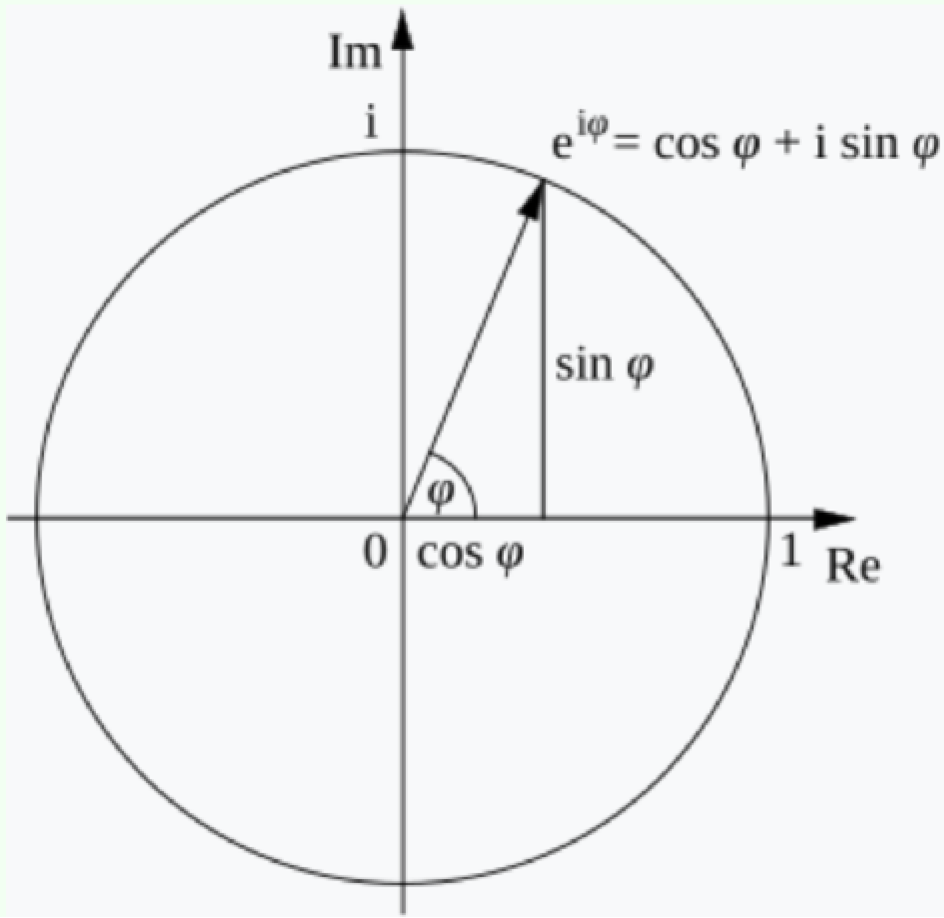
$$\boxed{c^2 = a^2 + b^2 - 2ab\cos\theta.}$$

This equation is called the **law of cosines**.

Euler's Formula states that for any real x, we have

$$e^{ix} = \cos x + i \sin x,$$

where i is the imaginary unit, $i = \sqrt{-1}$.



Euler's formula, the definitions of the trigonometric functions, and the standard identities for exponentials are sufficient to easily derive most trigonometric identities. It provides a powerful connection between analysis and trigonometry and provides an interpretation of the sine and cosine functions as weighted sums of the exponential function:

$$\cos x = Re(e^{ix}) = \frac{e^{ix} + e^{-ix}}{2},$$

$$\sin x = Im(e^{ix}) = \frac{e^{ix} - e^{-ix}}{2i}.$$

The two equations above can be derived by adding or subtracting Euler's formulas

First note that:

$$e^{-ix} = \cos(-x) + i\sin(-x).$$

Now recall that $\cos x$ is an **even function**, so $\cos(-x) = \cos x$. Furthermore, $\sin x$ is an **odd function**, so $\sin(-x) = -\sin x$.

Hence,

$$e^{-ix} = \cos(-x) + i\sin(-x),$$

$$= \cos x - i\sin x.$$

An expression for $\cos x$ is found by taking the sum of e^{ix} and e^{-ix}:

$$e^{ix} + e^{-ix} = (\cos x + i\sin x) + (\cos x - i\sin x).$$

Notice that the $i\sin x$ terms cancel, giving

$$e^{ix} + e^{-ix} = 2\cos x.$$

Dividing both sides of this expression by 2 gives the identity for $\cos x$:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}.$$

An expression for $\sin x$ is now found by taking the difference of e^{ix} and e^{-ix}:

$$e^{ix} - e^{-ix} = (\cos x + i \sin x) - (\cos x - i \sin x).$$

Notice that now the $\cos x$ terms cancel, giving

$$e^{ix} - e^{-ix} = i \sin x - (-i \sin x) = 2i \sin x.$$

Dividing both sides of this expression by $2i$ gives the identity for $\sin x$:

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

Euler's Identity

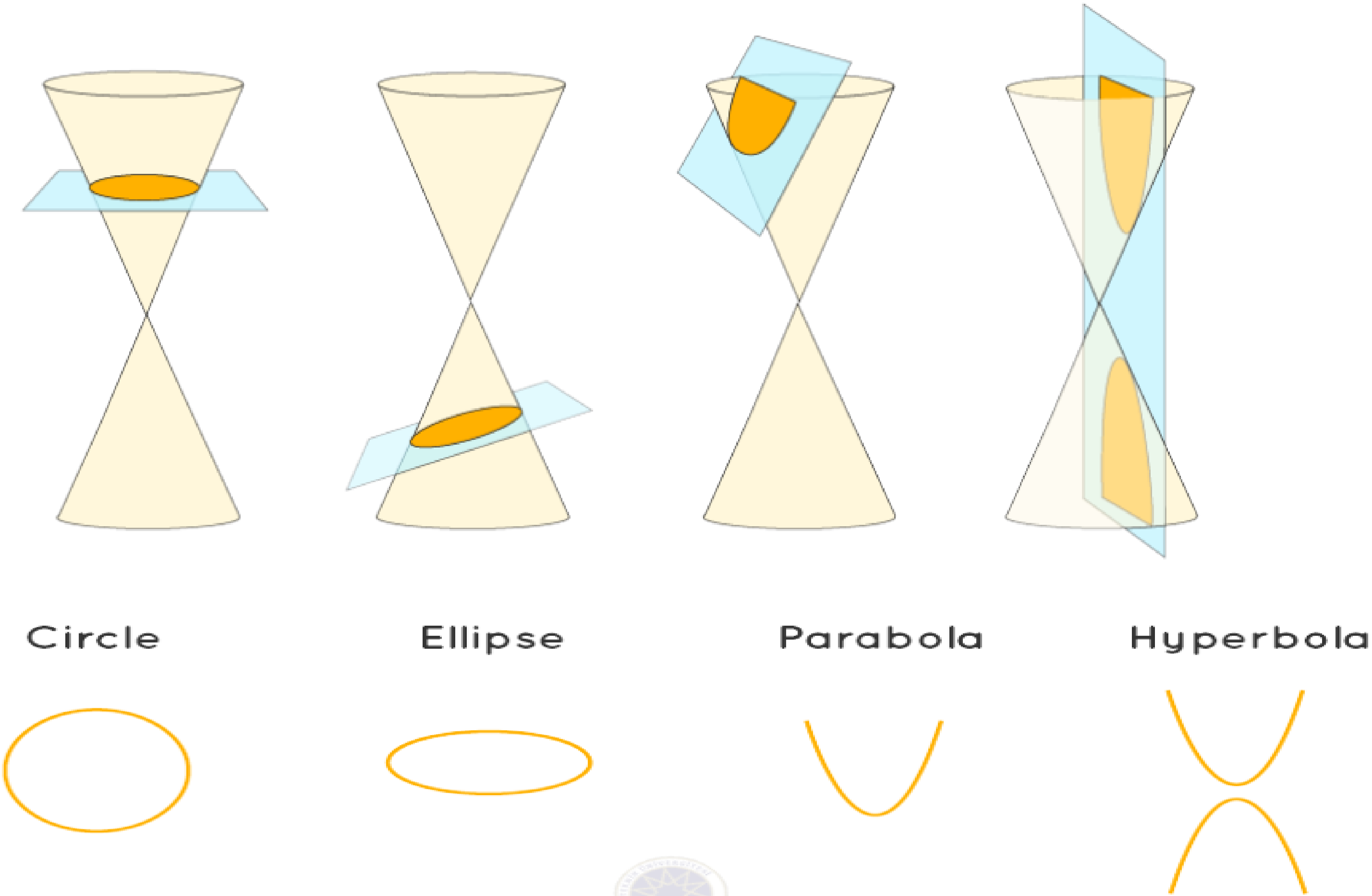
Euler's Identity is a special case of Euler's Formula, obtained from setting $x = \pi$:

$$e^{i\pi} = \cos \pi + i \sin \pi = -1,$$

since $\cos \pi = -1$ and $\sin \pi = 0$.

Hyperbolic Functions Hyperbolic functions are analogues of the ordinary trigonometric functions, but defined using the **hyperbola** rather than the **circle**. Just as the points $(cost, sint)$ form a circle with a unit radius, the points $(cosht, sinht)$ form the right half of the unit hyperbola.

The identity of the theorem $(cosh^2 x - \sinh^2 x = 1)$ also helps to provide a geometric motivation.



The hyperbolic functions are formed by taking combinations of the two exponential functions e^x and e^{-x}.

- Hyperbolic sine: the **odd part** of the exponential function, that is,

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

- Hyperbolic cosine: the **even part** of the exponential function, that is,

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

- Hyperbolic tangent:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}.$$

- Hyperbolic cotangent: for $x \neq 0$,

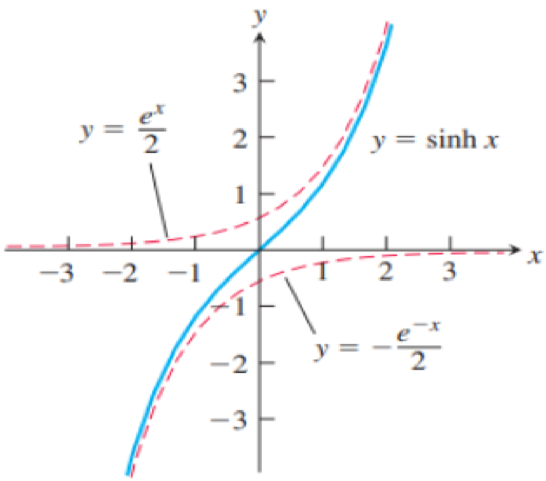
$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^{2x} + 1}{e^{2x} - 1}.$$

- Hyperbolic secant:

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

- Hyperbolic cosecant: for $x \neq 0$,

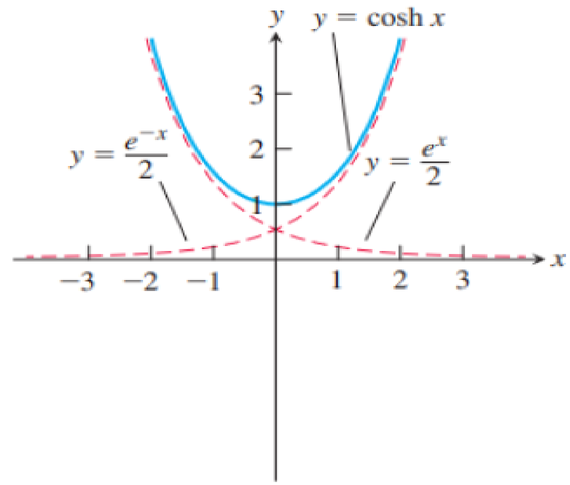
$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$



(a)

Hyperbolic sine:

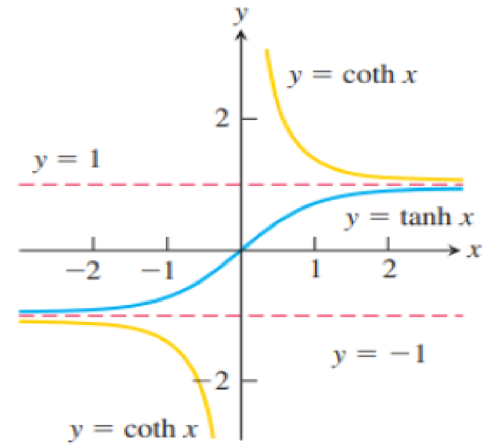
$$\sinh x = \frac{e^x - e^{-x}}{2}$$



(b)

Hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2}$$



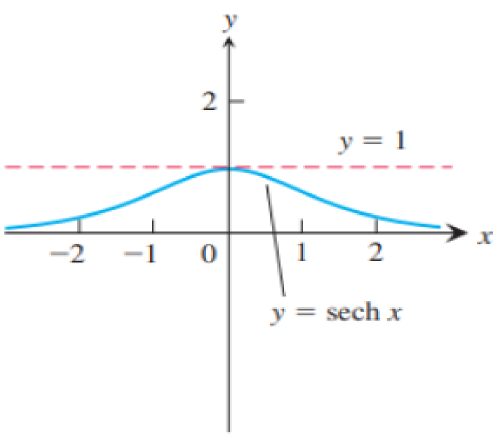
(c)

Hyperbolic tangent:

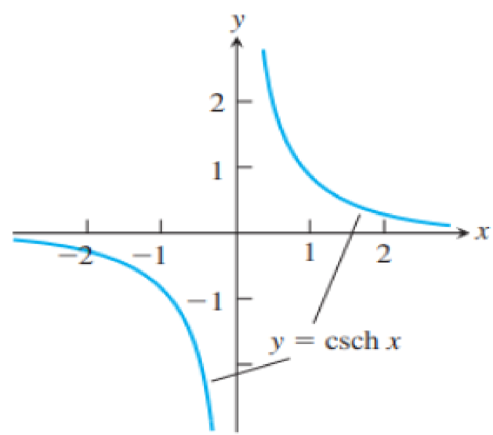
$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cotangent:

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$



(d)



(e)

Hyperbolic secant:

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

Hyperbolic cosecant:

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

$$
\begin{aligned}
\cosh^2 x - \sinh^2 x &= 1 \\
\sinh 2x &= 2 \sinh x \cosh x \\
\cosh 2x &= \cosh^2 x + \sinh^2 x \\
\cosh^2 x &= \tfrac{\cosh 2x + 1}{2} \\
\sinh^2 x &= \tfrac{\cosh 2x - 1}{2} \\
\tanh^2 x &= 1 - \operatorname{sech}^2 x \\
\coth^2 x &= 1 + \operatorname{csch}^2 x
\end{aligned}
$$

Consider a function $f : \mathbb{R} \to \mathbb{R}$, and we will write it as the sum of an even function and an odd function. Define

$$g(x) := \frac{f(x) + f(-x)}{2}$$

$$h(x) := \frac{f(x) - f(-x)}{2},$$

and we must show two things. The first is to show that $f(x) = g(x) + h(x)$. The second is to show that $g(x)$ is an even function and $h(x)$ is an odd function. Showing $f(x) = g(x) + h(x)$ is easy:

$$\begin{aligned}
g(x) + h(x) &= \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} \\
&= \frac{f(x) + f(-x) + f(x) - f(-x)}{2} \\
&= \frac{2f(x)}{2} \\
&= f(x).
\end{aligned}$$

On the other hand, showing that $g(x)$ is even and $h(x)$ is odd are more involved. Recall to show that g is even, we must show that $g(x) = g(-x)$ for each real number x. This requires using the definition of g:

$$
\begin{aligned}
g(x) &= \frac{f(x) + f(-x)}{2} \\
g(-x) &= \frac{f(-x) + f(-(-x))}{2} \\
&= \frac{f(-x) + f(x)}{2} \\
&= \frac{f(x) + f(-x)}{2} \\
&= g(x).
\end{aligned}
$$

showing that $h(x)$ is odd is basically the same procedure, except the negative signs pop up in different places.

$$
\begin{aligned}
h(x) &= \frac{f(x) - f(-x)}{2} \\
h(-x) &= \frac{f(-x) - f(-(-x))}{2} \\
&= \frac{f(-x) - f(x)}{2} \\
&= -\frac{f(x) - f(-x)}{2} \\
&= -h(x).
\end{aligned}
$$

In summary, we have shown that every real function f can be written as the sum of an even function and an odd function.

To find the even and odd parts of the function e^x, we can use the formulas for even and odd parts of any function $f(x)$.

Solution. Note that functions do not necessarily need to be even or odd. The function e^x is clearly neither, as $e^x \neq e^{-x}$ (condition for even) and $e^x \neq -e^{-x}$ (condition for odd). You can also sketch the function e^x and verify that it does not have the symmetry of an odd or even function.

The even part of e^x is given by:

$$f_e(x) = \frac{f(x) + f(-x)}{2} = \frac{e^x + e^{-x}}{2}.$$

This expression simplifies to:

$$f_e(x) = \cosh(x),$$

where $\cosh(x)$ is the hyperbolic cosine function.

The odd part of e^x is given by:

$$f_o(x) = \frac{f(x) - f(-x)}{2} = \frac{e^x - e^{-x}}{2}.$$

This expression simplifies to:

$$f_o(x) = \sinh(x),$$

where $\sinh(x)$ is the hyperbolic sine function.

Therefore, we can write e^x as the sum of its even and odd parts:

$$e^x = \cosh(x) + \sinh(x).$$

Hyperbolic functions may also be deduced from trigonometric functions with complex arguments:

- **Hyperbolic sine:**
$$\sinh x = -i \sin(ix).$$
- **Hyperbolic cosine:**
$$\cosh x = \cos(ix).$$
- **Hyperbolic tangent:**
$$\tanh x = -i \tan(ix).$$
- **Hyperbolic cotangent:**
$$\coth x = i \cot(ix).$$
- **Hyperbolic secant:**
$$\mathrm{sech} x = \sec(ix).$$
- **Hyperbolic cosecant:**
$$\mathrm{csch} x = i \csc(ix).$$

where i is the imaginary unit with $i^2 = -1$.

LET US REMEMBER

Definition Let A and B be sets. Then a **function from** A **to** B is a set f of ordered pairs in $A \times B$ such that for each $a \in A$ there exists a unique $b \in B$ with $(a, b) \in f$. (In other words, if $(a, b) \in f$ and $(a, b') \in f$, then $b = b'$.)

The set A of first elements of a function f is called the **domain** of f and is often denoted by $D(f)$. The set of all second elements in f is called the **range** of f and is often denoted by $R(f)$. Note that, although $D(f) = A$, we only have $R(f) \subseteq B$. (

The essential condition that:

$$(a, b) \in f \qquad \text{and} \qquad (a, b') \in f \qquad \text{implies that} \qquad b = b'$$

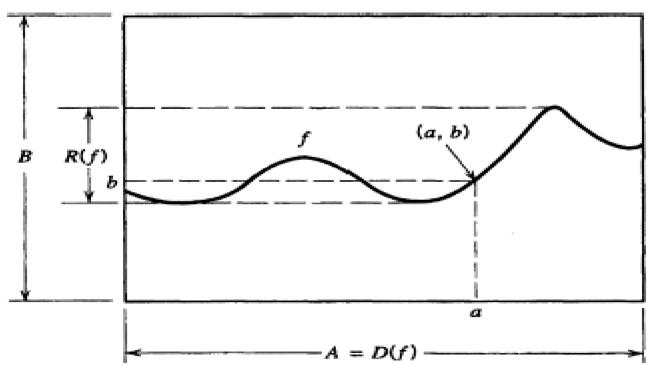
is sometimes called the *vertical line test*. In geometrical terms it says every vertical line $x = a$ with $a \in A$ intersects the graph of f exactly once.

The notation

$$f : A \to B$$

is often used to indicate that f is a function from A into B. We will also say that f is a **mapping** of A into B, or that f **maps** A into B. If (a, b) is an element in f, it is customary to write

$$b = f(a) \qquad \text{or sometimes} \qquad a \mapsto b.$$



A function as a graph

If $b = f(a)$, we often refer to b as the **value** of f at a, or as the **image** of a under f.

Definition Let $f : A \to B$ be a function from A to B.

(a) The function f is said to be **injective** (or to be **one-one**) if whenever $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. If f is an injective function, we also say that f is an **injection**.

(b) The function f is said to be **surjective** (or to map A **onto** B) if $f(A) = B$; that is, if the range $R(f) = B$. If f is a surjective function, we also say that f is a **surjection**.

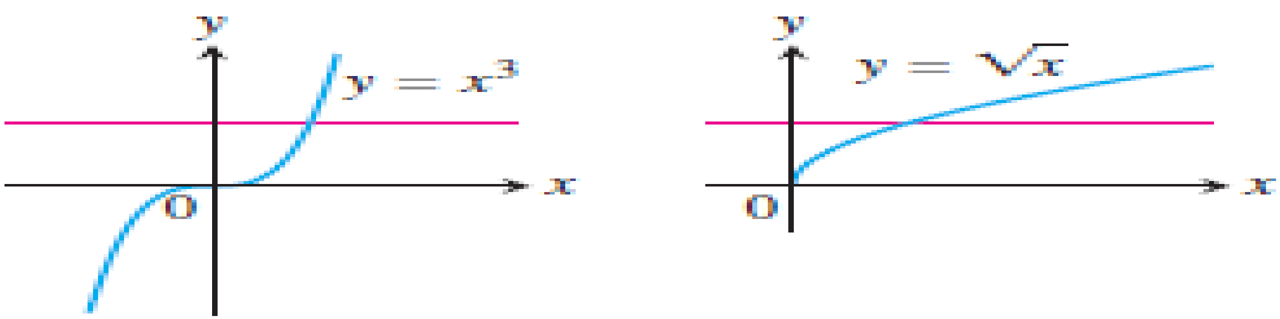
(c) If f is both injective and surjective, then f is said to be **bijective**. If f is bijective, we also say that f is a **bijection**.

One-to-One Functions

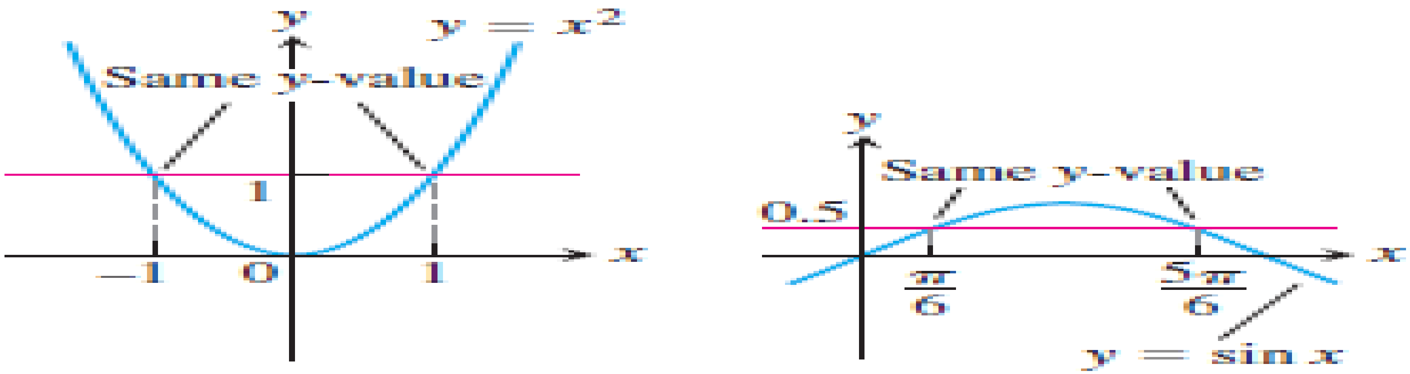
DEFINITION A function $f(x)$ is one-to-one on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

(a) $f(x) = \sqrt{x}$ is one-to-one on any domain of nonnegative numbers because $\sqrt{x_1} \neq \sqrt{x_2}$ whenever $x_1 \neq x_2$.

(b) $g(x) = \sin x$ is *not* one-to-one on the interval $[0, \pi]$ because $\sin(\pi/6) = \sin(5\pi/6)$. In fact, for each element x_1 in the subinterval $[0, \pi/2)$ there is a corresponding element x_2 in the subinterval $(\pi/2, \pi]$ satisfying $\sin x_1 = \sin x_2$, so distinct elements in the domain are assigned to the same value in the range. The sine function *is* one-to-one on $[0, \pi/2]$, however, because it is an increasing function on $[0, \pi/2]$ giving distinct outputs for distinct inputs.



(a) One-to-one: Graph meets each horizontal line at most once.



(b) Not one-to-one: Graph meets one or more horizontal lines more than once.

The Horizontal Line Test for One-to-One Functions

A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.

One-to-One/Onto Functions

Here are the definitions:

- f is one-to-one (injective) if f maps every element of A to a unique element in B. In other words no element of B are mapped to by two or more elements of A.
    - $(\forall\, a, b \in A)\ f(a) = f(b) \Rightarrow a = b$.
- f is onto (surjective)if every element of B is mapped to by some element of A. In other words, nothing is left out.
    - $(\forall b \in B)\ (\exists a \in A) f(a) = b$.
- f is one-to-one onto (bijective) if it is both one-to-one and onto. In this case the map f is also called a *one-to-one correspondence*.

Example-1

Classify the following functions $f_j : \mathbb{N} \to \mathbb{N}$ between natural numbers as one-to-one and onto.

f_j	**One-to-One?**	**Onto?**
$f_1(n) = n^2$	Yes	No
$f_2(n) = n + 3$	Yes	No
$f_3(n) = \lfloor \sqrt{n} \rfloor$	No	Yes
$f_4(n) = \begin{cases} n - 1, & \text{odd } n \\ n + 1, & \text{even } n \end{cases}$.	Yes	Yes

Reasons

- f_1 is not onto because it does not have any element n such that $f_1(n) = 3$, for instance.
- f_2 is not onto because no element n such that $f_2(n) = 0$, for instance.
- f_3 is not one-to-one since $f_3(2) = f_3(1) = 1$.

Example-2

Prove that the function $f(n) = n^2$ is one-to-one.

Proof: We wish to prove that whenever $f(m) = f(n)$ then $m = n$. Let us assume that $f(m) = f(n)$ for two numbers $m, n \in \mathbb{N}$. Therefore, $m^2 = n^2$. Which means that $m = \pm n$. Splitting cases on n, we have

- For $n \neq 0$, $-n \notin \mathbb{N}$, therefore $m = n$ for this case.
- For $n = 0$, we have $m = n = 0$. Therefore, it follows that $m = n$ for both cases.

Example-3

Prove that the function $f(n) = \lfloor \sqrt{n} \rfloor$ is onto.

Proof

Given any $m \in \mathbb{N}$, we observe that $n = m^2 \in \mathbb{N}$ is such that $f(n) = m$. Therefore, all $m \in \mathbb{N}$ are mapped onto.

LET US REMEMBER

Definition If $f : A \to B$ and $g : B \to C$, and if $R(f) \subseteq D(g) = B$, then the **composite function** $g \circ f$ (note the order!) is the function from A into C defined by

$$(g \circ f)(x) := g(f(x)) \quad \text{for all} \quad x \in A.$$

Examples (a) The order of the composition must be carefully noted. For, let f and g be the functions whose values at $x \in \mathbb{R}$ are given by

$$f(x) := 2x \qquad \text{and} \qquad g(x) := 3x^2 - 1.$$

Since $D(g) = \mathbb{R}$ and $R(f) \subseteq \mathbb{R} = D(g)$, then the domain $D(g \circ f)$ is also equal to $\mathbb{R}$, and the composite function $g \circ f$ is given by

$$(g \circ f)(x) = 3(2x)^2 - 1 = 12x^2 - 1.$$

On the other hand, the domain of the composite function $f \circ g$ is also $\mathbb{R}$, but

$$(f \circ g)(x) = 2(3x^2 - 1) = 6x^2 - 2.$$

Thus, in this case, we have $g \circ f \neq f \circ g$.
(b) In considering $g \circ f$, some care must be exercised to be sure that the range of f is contained in the domain of g. For example, if

$$f(x) := 1 - x^2 \qquad \text{and} \qquad g(x) := \sqrt{x},$$

then, since $D(g) = \{x : x \geq 0\}$, the composite function $g \circ f$ is given by the formula

$$(g \circ f)(x) = \sqrt{1 - x^2}$$

only for $x \in D(f)$ that satisfy $f(x) \geq 0$; that is, for x satisfying $-1 \leq x \leq 1$.

DEFINITION Suppose that f is a one-to-one function on a domain D with range R. The **inverse function** f^{-1} is defined by

$$f^{-1}(b) = a \text{ if } f(a) = b.$$

The domain of f^{-1} is R and the range of f^{-1} is D.

The symbol f^{-1} for the inverse of f is read "f inverse." The "-1" in f^{-1} is *not* an exponent; $f^{-1}(x)$ does not mean $1/f(x)$. Notice that the domains and ranges of f and f^{-1} are interchanged.

EXAMPLE Suppose a one-to-one function $y = f(x)$ is given by a table of values

x	1	2	3	4	5	6	7	8
$f(x)$	3	4.5	7	10.5	15	20.5	27	34.5

A table for the values of $x = f^{-1}(y)$ can then be obtained by simply interchanging the values in the columns (or rows) of the table for f:

y	3	4.5	7	10.5	15	20.5	27	34.5
$f^{-1}(y)$	1	2	3	4	5	6	7	8

$$(f^{-1} \circ f)(x) = x, \qquad \text{for all } x \text{ in the domain of } f$$
$$(f \circ f^{-1})(y) = y, \qquad \text{for all } y \text{ in the domain of } f^{-1} \text{ (or range of } f\text{)}$$

EXAMPLE Find the inverse of $y = \frac{1}{2}x + 1$, expressed as a function of x.

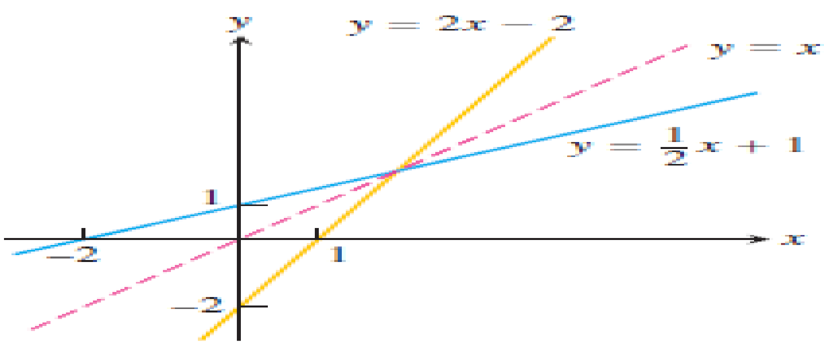
Solution

1. *Solve for x in terms of y:*
$$y = \frac{1}{2}x + 1$$
$$2y = x + 2$$
$$x = 2y - 2.$$

2. *Interchange x and y:* $y = 2x - 2$.

The inverse of the function $f(x) = (1/2)x + 1$ is the function $f^{-1}(x) = 2x - 2$. To check, we verify that both composites give the identity function:

$$f^{-1}(f(x)) = 2\left(\frac{1}{2}x + 1\right) - 2 = x + 2 - 2 = x$$
$$f(f^{-1}(x)) = \frac{1}{2}(2x - 2) + 1 = x - 1 + 1 = x.$$



Graphing $f(x) = (1/2)x + 1$ and $f^{-1}(x) = 2x - 2$ together shows the graphs' symmetry with respect to the line $y = x$ (Example 3).

EXAMPLE Find the inverse of the function $y = x^2, x \geq 0$, expressed as a function of x.

Solution We first solve for x in terms of y:

$$y = x^2$$

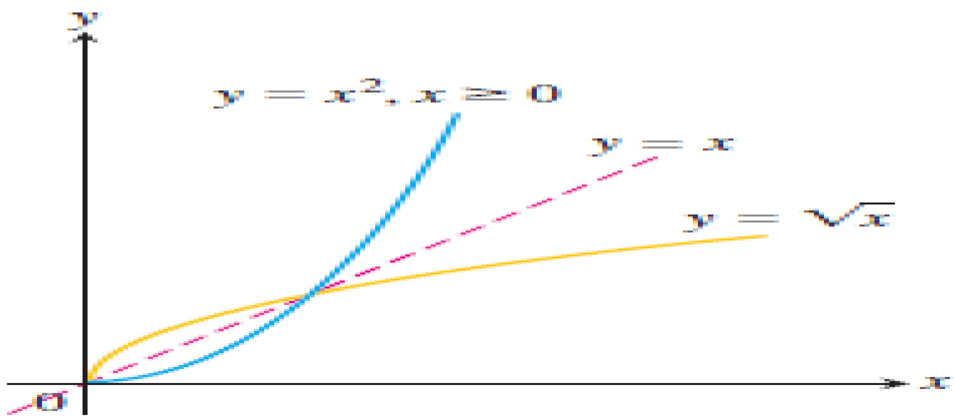
$$\sqrt{y} = \sqrt{x^2} = |x| = x \qquad |x| = x \text{ because } x \geq 0$$

We then interchange x and y, obtaining

$$y = \sqrt{x}.$$

The inverse of the function $y = x^2, x \geq 0$, is the function $y = \sqrt{x}$ (

Notice that the function $y = x^2, x \geq 0$, with domain *restricted* to the nonnegative real numbers, *is* one-to-one and has an inverse. On the other hand, the function $y = x^2$, with no domain restrictions, *is not* one-to-one and therefore has no inverse.



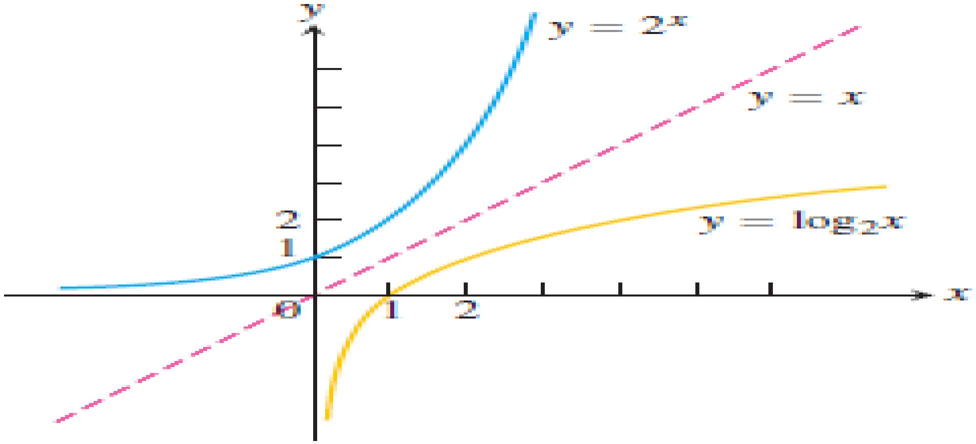
The functions $y = \sqrt{x}$ and $y = x^2, x \geq 0$, are inverses of one another

DEFINITION The logarithm function with base a, $y = \log_a x$, is the inverse of the base a exponential function $y = a^x$ ($a > 0$, $a \neq 1$).

The domain of $\log_a x$ is $(0, \infty)$, the range of a^x. The range of $\log_a x$ is $(-\infty, \infty)$, the domain of a^x. *Let us consider* $y = 2^x$. *The inverse of it* $y = \log_2 x$.

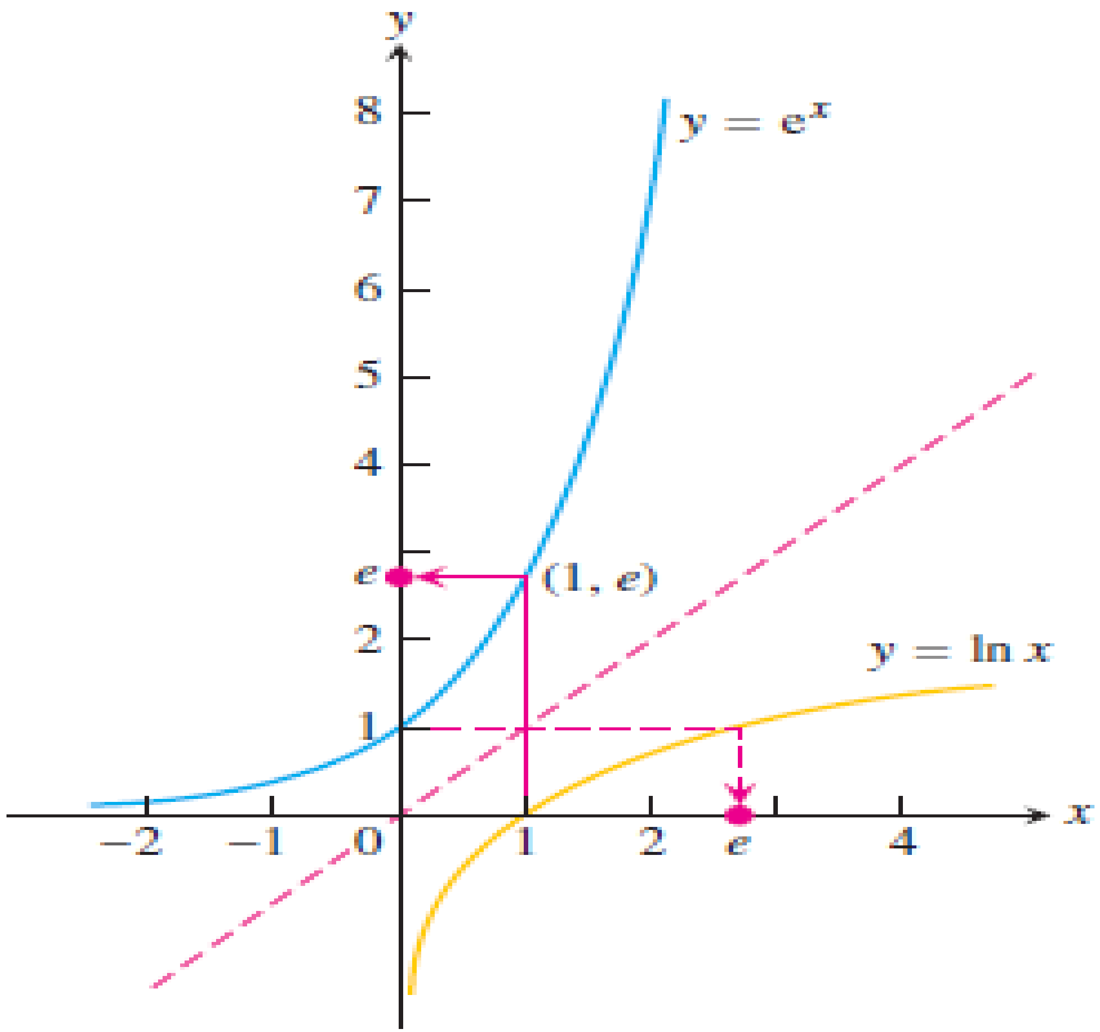
$\log_e x$ is written as $\ln x$.

$\log_{10} x$ is written as $\log x$.



$$\ln x = y \iff e^y = x.$$

$$\ln e = 1$$


y
8
7
6
5
4
e
2
1
$y = e^x$
(1, e)
$y = \ln x$
x
−2
−1
0
1
2
e
4

THEOREM —Algebraic Properties of the Natural Logarithm For any numbers $b > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1. *Product Rule:* $\ln bx = \ln b + \ln x$ $\qquad \ln 4 + \ln \sin x = \ln(4 \sin x)$

2. *Quotient Rule:* $\ln \dfrac{b}{x} = \ln b - \ln x$ $\qquad \ln\left(\frac{x+1}{2x-3}\right) = \ln(x+1) - \ln(2x-3)$

   $\ln \frac{1}{8} = -\ln 8$
   $= -\ln 2^3 = -3\ln 2$ *Power Rule*

3. *Reciprocal Rule:* $\ln \dfrac{1}{x} = -\ln x$ $\qquad$ Rule 2 with $b = 1$

4. *Power Rule:* $\ln x^r = r \ln x$

Inverse Properties for a^x and $\log_a x$

1. Base a: $a^{\log_a x} = x, \qquad \log_a a^x = x, \qquad a > 0, a \neq 1, x > 0$

2. Base e: $e^{\ln x} = x, \qquad \ln e^x = x, \qquad x > 0$

Substituting a^x for x in the equation $x = e^{\ln x}$ enables us to rewrite a^x as a power of e:

$$\begin{aligned} a^x &= e^{\ln(a^x)} && \text{Substitute } a^x \text{ for } x \text{ in } x = e^{\ln x}. \\ &= e^{x \ln a} && \text{Power Rule for logs} \\ &= e^{(\ln a)x}. && \text{Exponent rearranged} \end{aligned}$$

Every exponential function is a power of the natural exponential function.

$$a^x = e^{x \ln a}$$

That is, a^x is the same as e^x raised to the power $\ln a$: $a^x = e^{kx}$ for $k = \ln a$.

For example,

$$2^x = e^{(\ln 2)x} = e^{x \ln 2}, \qquad \text{and} \qquad 5^{-3x} = e^{(\ln 5)(-3x)} = e^{-3x \ln 5}.$$

Returning once more to the properties of a^x and $\log_a x$, we have

$$\begin{aligned} \ln x &= \ln(a^{\log_a x}) && \text{Inverse Property for } a^x \text{ and } \log_a x \\ &= (\log_a x)(\ln a). && \text{Power Rule for logarithms, with } r = \log_a x \end{aligned}$$

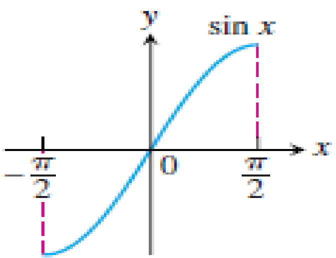
Rewriting this equation as $\log_a x = (\ln x)/(\ln a)$ shows that every logarithmic function is a constant multiple of the natural logarithm $\ln x$. This allows us to extend the algebraic properties for $\ln x$ to $\log_a x$. For instance, $\log_a bx = \log_a b + \log_a x$.

Change of Base Formula

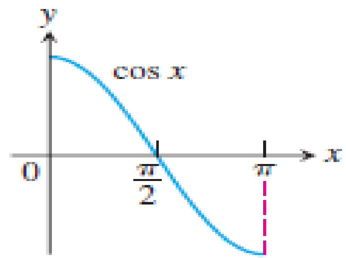
Every logarithmic function is a constant multiple of the natural logarithm.

$$\log_a x = \frac{\ln x}{\ln a} \qquad (a > 0,\ a \neq 1)$$

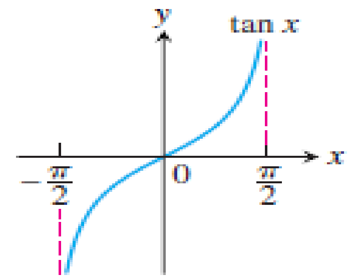
Domain restrictions that make the trigonometric functions one-to-one



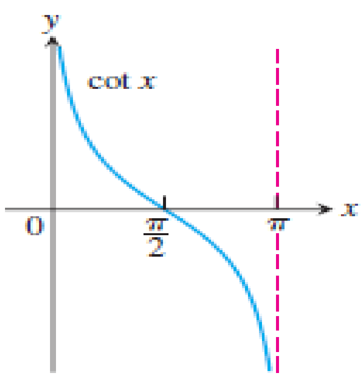
$y = \sin x$
Domain: $[-\pi/2, \pi/2]$
Range: $[-1, 1]$



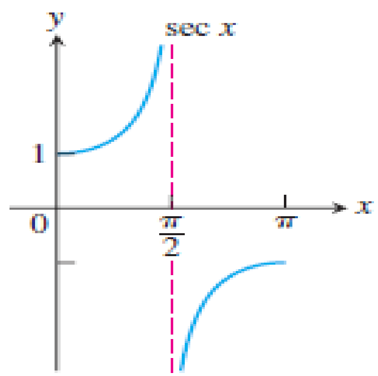
$y = \cos x$
Domain: $[0, \pi]$
Range: $[-1, 1]$



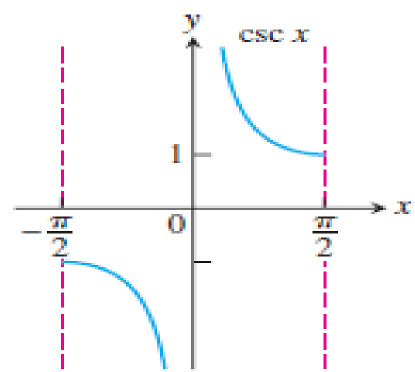
$y = \tan x$
Domain: $(-\pi/2, \pi/2)$
Range: $(-\infty, \infty)$



$y = \cot x$
Domain: $(0, \pi)$
Range: $(-\infty, \infty)$



$y = \sec x$
Domain: $[0, \pi/2) \cup (\pi/2, \pi]$
Range: $(-\infty, -1] \cup [1, \infty)$

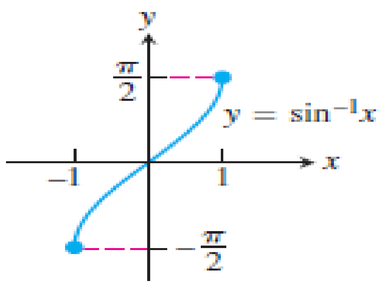


$y = \csc x$
Domain: $[-\pi/2, 0) \cup (0, \pi/2]$
Range: $(-\infty, -1] \cup [1, \infty)$

Since these restricted functions are now one-to-one, they have inverses, which we denote by

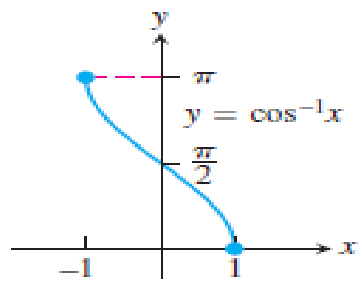
$$
\begin{array}{lll}
y = \sin^{-1} x & \text{or} & y = \arcsin x \\
y = \cos^{-1} x & \text{or} & y = \arccos x \\
y = \tan^{-1} x & \text{or} & y = \arctan x \\
y = \cot^{-1} x & \text{or} & y = \operatorname{arccot} x \\
y = \sec^{-1} x & \text{or} & y = \operatorname{arcsec} x \\
y = \csc^{-1} x & \text{or} & y = \operatorname{arccsc} x
\end{array}
$$

Domain: $-1 \le x \le 1$
Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



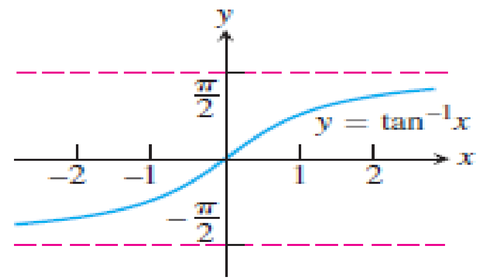
(a)

Domain: $-1 \le x \le 1$
Range: $0 \le y \le \pi$



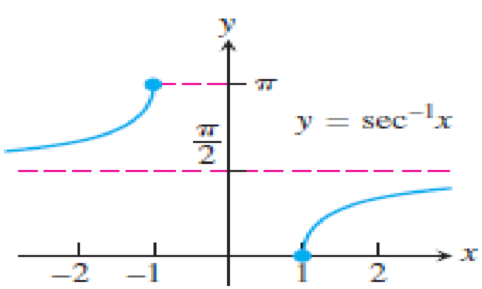
(b)

Domain: $-\infty < x < \infty$
Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$



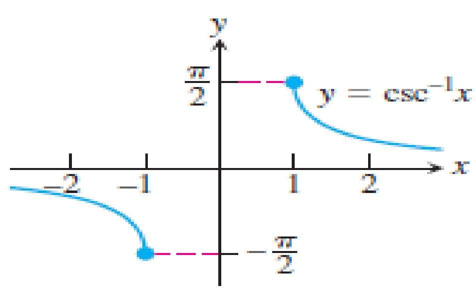
(c)

Domain: $x \le -1$ or $x \ge 1$
Range: $0 \le y \le \pi, y \ne \frac{\pi}{2}$



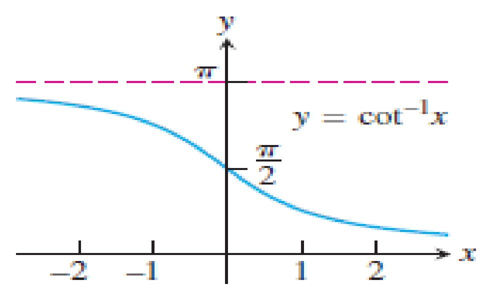
(d)

Domain: $x \le -1$ or $x \ge 1$
Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}, y \ne 0$



(e)

Domain: $-\infty < x < \infty$
Range: $0 < y < \pi$



(f)