L'Hospital's Rules

Theorem *Let f and g be defined on $[a, b]$, let $f(a) = g(a) = 0$, and let $g(x) \neq 0$ for $a < x < b$. If f and g are differentiable at a and if $g'(a) \neq 0$, then the limit of f/g at a exists and is equal to $f'(a)/g'(a)$. Thus*

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Proof. Since $f(a) = g(a) = 0$, we can write the quotient $f(x)/g(x)$ for $a < x < b$ as follows:

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{\dfrac{f(x) - f(a)}{x - a}}{\dfrac{g(x) - g(a)}{x - a}}.$$

Applying Theorem , we obtain

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to a+} \dfrac{f(x) - f(a)}{x - a}}{\lim\limits_{x \to a+} \dfrac{g(x) - g(a)}{x - a}} = \frac{f'(a)}{g'(a)}.$$

THEOREM — L'Hôpital's Rule Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

L'Hospital's Rule, I *Let* $-\infty \le a < b \le \infty$ *and let* f, g *be differentiable on* (a,b) *such that* $g'(x) \neq 0$ *for all* $x \in (a,b)$. *Suppose that*

$$\lim_{x \to a+} f(x) = 0 = \lim_{x \to a+} g(x).$$

(a) *If* $\displaystyle\lim_{x \to a+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$, *then* $\displaystyle\lim_{x \to a+} \frac{f(x)}{g(x)} = L$.

(b) *If* $\displaystyle\lim_{x \to a+} \frac{f'(x)}{g'(x)} = L \in \{-\infty, \infty\}$, *then* $\displaystyle\lim_{x \to a+} \frac{f(x)}{g(x)} = L$.

Using L'Hôpital's Rule

To find

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

by l'Hôpital's Rule, continue to differentiate f and g, so long as we still get the form $0/0$ at $x = a$. But as soon as one or the other of these derivatives is different from zero at $x = a$ we stop differentiating. L'Hôpital's Rule does not apply when either the numerator or denominator has a finite nonzero limit.

EXAMPLE The following limits involve 0/0 indeterminate forms, so we apply l'Hôpital's Rule. In some cases, it must be applied repeatedly.

(a) $\displaystyle \lim_{x\to 0}\frac{3x-\sin x}{x}=\lim_{x\to 0}\frac{3-\cos x}{1}=\left.\frac{3-\cos x}{1}\right|_{x=0}=2$

(b) $\displaystyle \lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\lim_{x\to 0}\frac{\frac{1}{2\sqrt{1+x}}}{1}=\frac{1}{2}$

(c) $\displaystyle \lim_{x\to 0}\frac{\sqrt{1+x}-1-x/2}{x^2}$ $\quad \frac{0}{0}$

$\displaystyle =\lim_{x\to 0}\frac{(1/2)(1+x)^{-1/2}-1/2}{2x}$ $\quad$ Still $\frac{0}{0}$; differentiate again.

$\displaystyle =\lim_{x\to 0}\frac{-(1/4)(1+x)^{-3/2}}{2}=-\frac{1}{8}$ $\quad$ Not $\frac{0}{0}$; limit is found.

(d) $\lim_{x\to 0} \dfrac{x - \sin x}{x^3}$ $\qquad \frac{0}{0}$

$$= \lim_{x\to 0} \frac{1 - \cos x}{3x^2} \qquad \text{Still } \tfrac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sin x}{6x} \qquad \text{Still } \tfrac{0}{0}$$

$$= \lim_{x\to 0} \frac{\cos x}{6} = \frac{1}{6} \qquad \text{Not } \tfrac{0}{0}\text{; limit is found.}$$

Here is a summary of the procedure we followed in Example 1.

EXAMPLE Be careful to apply l'Hôpital's Rule correctly:

$$\lim_{x\to 0} \frac{1 - \cos x}{x + x^2} \qquad \tfrac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sin x}{1 + 2x} = \frac{0}{1} = 0. \qquad \text{Not } \tfrac{0}{0}\text{; limit is found.}$$

Up to now the calculation is correct, but if we continue to differentiate in an attempt to apply l'Hôpital's Rule once more, we get

$$\lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2},$$

which is not the correct limit. L'Hôpital's Rule can only be applied to limits that give indeterminate forms, and $0/1$ is not an indeterminate form.

L'Hôpital's Rule applies to one-sided limits as well.

EXAMPLE In this example the one-sided limits are different.

(a) $\lim_{x \to 0^+} \frac{\sin x}{x^2}$ $\qquad \frac{0}{0}$

$$= \lim_{x \to 0^+} \frac{\cos x}{2x} = \infty \qquad \text{Positive for } x > 0$$

(b) $\lim_{x \to 0^-} \frac{\sin x}{x^2}$ $\qquad \frac{0}{0}$

$$= \lim_{x \to 0^-} \frac{\cos x}{2x} = -\infty \qquad \text{Negative for } x < 0$$

THEOREM —Cauchy's Mean Value Theorem Suppose functions f and g are continuous on $[a, b]$ and differentiable throughout (a, b) and also suppose $g'(x) \neq 0$ throughout (a, b). Then there exists a number c in (a, b) at which

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof We apply the Mean Value Theorem First we use it to show that $g(a) \neq g(b)$. For if $g(b)$ did equal $g(a)$, then the Mean Value Theorem would give

$$g'(c) = \frac{g(b) - g(a)}{b - a} = 0$$

for some c between a and b, which cannot happen because $g'(x) \neq 0$ in (a, b).

We next apply the Mean Value Theorem to the function

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}[g(x) - g(a)].$$

This function is continuous and differentiable where f and g are, and $F(b) = F(a) = 0$. Therefore, there is a number c between a and b for which $F'(c) = 0$. When expressed in terms of f and g, this equation becomes

$$F'(c) = f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)}[g'(c)] = 0$$

so that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

The proof of l'Hôpital's Rule is based on Cauchy's Mean Value Theorem, an extension of the Mean Value Theorem that involves two functions instead of one. We prove Cauchy's Theorem first and then show how it leads to l'Hôpital's Rule.

Examples (a) We have

$$\lim_{x \to 0+} \frac{\sin x}{\sqrt{x}} = \lim_{x \to 0+} \left[\frac{\cos x}{1/(2\sqrt{x})} \right] = \lim_{x \to 0+} 2\sqrt{x} \cos x = 0.$$

$f(x) := \sin x$ and $g(x) := \sqrt{x}$ are differentiable on $(0, \infty)$ and both approach 0 as $x \to 0+$. Moreover, $g'(x) \neq 0$ on $(0, \infty)$,

(b) We have $\displaystyle\lim_{x \to 0} \left[\frac{1 - \cos x}{x^2} \right] = \lim_{x \to 0} \frac{\sin x}{2x}$.

The quotient in the second limit is again indeterminate in the form $0/0$. However, the hypotheses are again satisfied so that a second application of L'Hospital's Rule is permissible. Hence, we obtain

$$\lim_{x \to 0} \left[\frac{1 - \cos x}{x^2} \right] = \lim_{x \to 0} \frac{\sin x}{2x} = \lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}.$$

(c) We have $\displaystyle\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{e^x}{1} = 1$.

Similarly, two applications of L'Hospital's Rule give us

$$\lim_{x \to 0} \left[\frac{e^x - 1 - x}{x^2} \right] = \lim_{x \to 0} \frac{e^x - 1}{2x} = \lim_{x \to 0} \frac{e^x}{2} = \frac{1}{2}.$$

(d) We have $\displaystyle\lim_{x \to 1} \left[\frac{\ln x}{x - 1} \right] = \lim_{x \to 1} \frac{(1/x)}{1} = 1$.

L'Hospital's Rule, II

This rule is very similar to the first one, except that it treats the case where the denominator becomes infinite as $x \to a+$. Again we will consider only right-hand limits, but it is possible that $a = -\infty$. Left-hand limits and two-sided limits are handled similarly.

L'Hospital's Rule, II *Let* $-\infty \le a < b \le \infty$ *and let* f, g *be differentiable on* (a, b) *such that* $g'(x) \neq 0$ *for all* $x \in (a, b)$. *Suppose that*

$$\lim_{x \to a+} g(x) = \pm\infty.$$

(a) *If* $\displaystyle\lim_{x \to a+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$, *then* $\displaystyle\lim_{x \to a+} \frac{f(x)}{g(x)} = L$.

(b) *If* $\displaystyle\lim_{x \to a+} \frac{f'(x)}{g'(x)} = L \in \{-\infty, \infty\}$, *then* $\displaystyle\lim_{x \to a+} \frac{f(x)}{g(x)} = L$.

Examples (a) We consider $\lim_{x\to\infty} \frac{\ln x}{x}$.

Here $f(x) := \ln x$ and $g(x) := x$ on the interval $(0, \infty)$. If we apply the left-hand version , we obtain $\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$.

(b) We consider $\lim_{x\to\infty} e^{-x}x^2$.

Here we take $f(x) := x^2$ and $g(x) := e^x$ on $\mathbb{R}$. We obtain

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0.$$

(c) We consider $\lim_{x\to 0+} \frac{\ln \sin x}{\ln x}$.

Here we take $f(x) := \ln \sin x$ and $g(x) := \ln x$ on $(0, \pi)$. we obtain

$$\lim_{x\to 0+} \frac{\ln \sin x}{\ln x} = \lim_{x\to 0+} \frac{\cos x/\sin x}{1/x} = \lim_{x\to 0+} \left[\frac{x}{\sin x}\right] \cdot [\cos x].$$

Since $\lim_{x\to 0+} [x/\sin x] = 1$ and $\lim_{x\to 0+} \cos x = 1$, we conclude that the limit under consideration equals 1.

(d) Consider $\lim_{x\to\infty} \frac{x - \sin x}{x + \sin x}$. This has indeterminate form ∞/∞. An application of L'Hospital's Rule gives us

$$\lim_{x\to\infty} \frac{1 - \cos x}{1 + \cos x},$$

which is useless because this limit does not exist. (Why not?) However, if we rewrite the original limit, we get directly that

$$\lim_{x\to\infty} \frac{1 - \dfrac{\sin x}{x}}{1 + \dfrac{\sin x}{x}} = \frac{1 - 0}{1 + 0} = 1.$$

Indeterminate Forms ∞/∞, $\infty \cdot 0$, $\infty - \infty$

Sometimes when we try to evaluate a limit as $x \to a$ by substituting $x = a$ we get an indeterminant form like ∞/∞, $\infty \cdot 0$, or $\infty - \infty$, instead of $0/0$. We first consider the form ∞/∞.

In more advanced treatments of calculus it is proved that l'Hôpital's Rule applies to the indeterminate form ∞/∞ as well as to $0/0$. If $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ as $x \to a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists. In the notation $x \to a$, a may be either finite or infinite. Moreover, $x \to a$ may be replaced by the one-sided limits $x \to a^+$ or $x \to a^-$.

EXAMPLE Find the limits of these ∞/∞ forms:

(a) $\lim_{x\to\pi/2} \frac{\sec x}{1+\tan x}$ (b) $\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}}$ (c) $\lim_{x\to\infty} \frac{e^x}{x^2}$.

Solution

(a) The numerator and denominator are discontinuous at $x = \pi/2$, so we investigate the one-sided limits there. To apply l'Hôpital's Rule, we can choose I to be any open interval with $x = \pi/2$ as an endpoint.

$$\lim_{x\to(\pi/2)^-} \frac{\sec x}{1+\tan x} \qquad \frac{\infty}{\infty} \text{ from the left}$$

$$= \lim_{x\to(\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x\to(\pi/2)^-} \sin x = 1$$

The right-hand limit is 1 also, with $(-\infty)/(-\infty)$ as the indeterminate form. Therefore, the two-sided limit is equal to 1.

(b) $$\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{1/\sqrt{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0 \qquad \frac{1/x}{1/\sqrt{x}} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$$

(c) $$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

Next we turn our attention to the indeterminate forms $\infty \cdot 0$ and $\infty - \infty$. Sometimes these forms can be handled by using algebra to convert them to a $0/0$ or ∞/∞ form. Here again we do not mean to suggest that $\infty \cdot 0$ or $\infty - \infty$ is a number. They are only notations for functional behaviors when considering limits. Here are examples of how we might work with these indeterminate forms.

EXAMPLE Find the limits of these $\infty \cdot 0$ forms:

(a) $\lim_{x\to\infty}\left(x \sin \frac{1}{x}\right)$ **(b)** $\lim_{x\to 0^+} \sqrt{x}\ln x$

Solution

(a) $$\lim_{x\to\infty}\left(x \sin \frac{1}{x}\right) = \lim_{h\to 0^+}\left(\frac{1}{h}\sin h\right) = \lim_{h\to 0^+}\frac{\sin h}{h} = 1$$ $\infty \cdot 0$; Let $h = 1/x$.

(b) $$\lim_{x\to 0^+}\sqrt{x}\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/\sqrt{x}}$$ $\infty \cdot 0$ converted to ∞/∞

$$= \lim_{x\to 0^+}\frac{1/x}{-1/2x^{3/2}}$$ l'Hôpital's Rule

$$= \lim_{x\to 0^+}\left(-2\sqrt{x}\right) = 0$$

EXAMPLE Find the limit of this $\infty - \infty$ form:

$$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right).$$

Solution If $x \to 0^+$, then $\sin x \to 0^+$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to \infty - \infty.$$

Similarly, if $x \to 0^-$, then $\sin x \to 0^-$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to -\infty - (-\infty) = -\infty + \infty.$$

Neither form reveals what happens in the limit. To find out, we first combine the fractions:

$$\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x} \qquad \text{Common denominator is } x \sin x.$$

Then we apply l'Hôpital's Rule to the result:

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) &= \lim_{x\to 0}\frac{x - \sin x}{x \sin x} && \frac{0}{0}\\
&= \lim_{x\to 0}\frac{1 - \cos x}{\sin x + x\cos x} && \text{Still } \frac{0}{0}\\
&= \lim_{x\to 0}\frac{\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0.
\end{aligned}$$

Indeterminate Powers

Limits that lead to the indeterminate forms 1^∞, 0^0, and ∞^0 can sometimes be handled by first taking the logarithm of the function. We use l'Hôpital's Rule to find the limit of the logarithm expression and then exponentiate the result to find the original function limit.

If $\lim_{x \to a} \ln f(x) = L$, then

$$\lim_{x \to a} f(x) = \lim_{x \to a} e^{\ln f(x)} = e^L.$$

Here a may be either finite or infinite.

EXAMPLE Apply l'Hôpital's Rule to show that $\lim_{x \to 0^+} (1 + x)^{1/x} = e$.

Solution The limit leads to the indeterminate form 1^∞. We let $f(x) = (1 + x)^{1/x}$ and find $\lim_{x \to 0^+} \ln f(x)$. Since

$$\ln f(x) = \ln (1 + x)^{1/x} = \frac{1}{x} \ln (1 + x),$$

l'Hôpital's Rule now applies to give

$$\lim_{x \to 0^+} \ln f(x) = \lim_{x \to 0^+} \frac{\ln (1 + x)}{x} \qquad \frac{0}{0}$$

$$= \lim_{x \to 0^+} \frac{\frac{1}{1 + x}}{1}$$

$$= \frac{1}{1} = 1.$$

Therefore, $\lim_{x \to 0^+} (1 + x)^{1/x} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} e^{\ln f(x)} = e^1 = e.$

EXAMPLE Find $\lim_{x\to\infty} x^{1/x}$.

Solution The limit leads to the indeterminate form ∞^0. We let $f(x) = x^{1/x}$ and find $\lim_{x\to\infty} \ln f(x)$. Since

$$\ln f(x) = \ln x^{1/x} = \frac{\ln x}{x},$$

l'Hôpital's Rule gives

$$\begin{aligned}
\lim_{x\to\infty} \ln f(x) &= \lim_{x\to\infty} \frac{\ln x}{x} \qquad \frac{\infty}{\infty} \\
&= \lim_{x\to\infty} \frac{1/x}{1} \\
&= \frac{0}{1} = 0.
\end{aligned}$$

Therefore $\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln f(x)} = e^0 = 1.$

Examples (a) Let $I := (0, \pi/2)$ and consider

$$\lim_{x \to 0+} \left(\frac{1}{x} - \frac{1}{\sin x} \right),$$

which has the indeterminate form $\infty - \infty$. We have

$$\begin{aligned}
\lim_{x \to 0+} \left(\frac{1}{x} - \frac{1}{\sin x} \right) &= \lim_{x \to 0+} \frac{\sin x - x}{x \sin x} = \lim_{x \to 0+} \frac{\cos x - 1}{\sin x + x \cos x} \\
&= \lim_{x \to 0+} \frac{-\sin x}{2 \cos x - x \sin x} = \frac{0}{2} = 0.
\end{aligned}$$

(b) Let $I := (0, \infty)$ and consider $\lim\limits_{x \to 0+} x \ln x$, which has the indeterminate form $0 \cdot (-\infty)$. We have

$$\lim_{x \to 0+} x \ln x = \lim_{x \to 0+} \frac{\ln x}{1/x} = \lim_{x \to 0+} \frac{1/x}{-1/x^2} = \lim_{x \to 0+} (-x) = 0.$$

(c) Let $I := (0, \infty)$ and consider $\lim\limits_{x \to 0+} x^x$, which has the indeterminate form 0^0.

We recall from calculus (that $x^x = e^{x \ln x}$. It follows from part (b) and the continuity of the function $y \mapsto e^y$ at $y = 0$ that $\lim\limits_{x \to 0+} x^x = e^0 = 1$.

(d) Let $I := (1, \infty)$ and consider $\lim\limits_{x\to\infty} (1+1/x)^x$, which has the indeterminate form 1^∞. We note that

$$(1+1/x)^x = e^{x\ln(1+1/x)}.$$

Moreover, we have

$$\begin{aligned}
\lim_{x\to\infty} x\ln(1+1/x) &= \lim_{x\to\infty} \frac{\ln(1+1/x)}{1/x} \\
&= \lim_{x\to\infty} \frac{(1+1/x)^{-1}(-x^{-2})}{-x^{-2}} = \lim_{x\to\infty} \frac{1}{1+1/x} = 1.
\end{aligned}$$

Since $y \mapsto e^y$ is continuous at $y=1$, we infer that $\lim\limits_{x\to\infty} (1+1/x)^x = e$.

(e) Let $I := (0, \infty)$ and consider $\lim\limits_{x\to 0+} (1+1/x)^x$, which has the indeterminate form ∞^0. In view of formula), we consider

$$\lim_{x\to 0+} x\ln(1+1/x) = \lim_{x\to 0+} \frac{\ln(1+1/x)}{1/x} = \lim_{x\to 0+} \frac{1}{1+1/x} = 0.$$

Therefore we have $\lim\limits_{x\to 0+} (1+1/x)^x = e^0 = 1$.

Evaluate the limit

$$\lim_{x\to 1^+} x^{\frac{1}{x-1}}.$$

Solution Since $\lim_{x\to 1^+} x = 1$ and $\lim_{x\to 1^+} \frac{1}{x-1} = \infty$, we have an indeterminate form of the type 1^∞.

Define the expression whose limit we are taking as a function.

$$y = x^{\frac{1}{x-1}}$$

Take the natural logarithm of both sides and then the limit.

$$\ln y = \frac{1}{x-1} \ln x$$

Thus,

$$\lim_{x\to 1^+} \ln y = \lim_{x\to 1^+} \frac{\ln x}{x-1}$$

Since $\lim_{x\to 1^+} \ln x = 0$ and $\lim_{x\to 1^+}(x-1) = 0$, this is an indeterminate form $\frac{0}{0}$, so we can apply L'Hôpital's Rule.

$$\lim_{x\to 1^+} \frac{\ln x}{x-1} = \lim_{x\to 1^+} \frac{(\ln x)'}{(x-1)'} = \lim_{x\to 1^+} \frac{\frac{1}{x}}{1} = \lim_{x\to 1^+} \frac{1}{x} = 1$$

Thus, the limit of the natural logarithm of the function is 1.

$$\lim_{x\to 1^+} \ln y = 1$$

As x approaches 1, the expression $\ln y$ approaches 1, so y approaches $e^1 = e$.
Therefore, the limit of the given expression is e.

$$\lim_{x\to 1^+} y = \lim_{x\to 1^+} x^{\frac{1}{x-1}} = e$$

Calculate $\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x}$.

SOLUTION First notice that as $x \to 0^+$, we have $1 + \sin 4x \to 1$ and $\cot x \to \infty$, so the given limit is indeterminate. Let

$$y = (1 + \sin 4x)^{\cot x}$$

Then $$\ln y = \ln[(1 + \sin 4x)^{\cot x}] = \cot x \ln(1 + \sin 4x)$$

so l'Hospital's Rule gives

$$\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} \frac{\ln(1 + \sin 4x)}{\tan x} = \lim_{x \to 0^+} \frac{\dfrac{4 \cos 4x}{1 + \sin 4x}}{\sec^2 x} = 4$$

So far we have computed the limit of $\ln y$, but what we want is the limit of y. To find this

we use the fact that $y = e^{\ln y}$:

$$\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x} = \lim_{x \to 0^+} y = \lim_{x \to 0^+} e^{\ln y} = e^4$$

Evaluate $\lim_{x\to\infty}\left(1+\sin\frac{3}{x}\right)^x$.

Solution This indeterminate form is of type 1^∞. Let $y=\left(1+\sin\frac{3}{x}\right)^x$. Then, taking ln of both sides,

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}x\ln\left(1+\sin\frac{3}{x}\right)\qquad[\infty\cdot 0]$$

$$=\lim_{x\to\infty}\frac{\ln\left(1+\sin\frac{3}{x}\right)}{\frac{1}{x}}\qquad\left[\frac{0}{0}\right]$$

$$=\lim_{x\to\infty}\frac{\frac{1}{1+\sin\frac{3}{x}}\left(\cos\frac{3}{x}\right)\left(-\frac{3}{x^2}\right)}{-\frac{1}{x^2}}=\lim_{x\to\infty}\frac{3\cos\frac{3}{x}}{1+\sin\frac{3}{x}}=3.$$

Hence $\lim_{x\to\infty}\left(1+\sin\frac{3}{x}\right)^x=e^3$.

l'Hôpital's Rule cannot be used to evaluate $\lim_{x \to 1+} x/(\ln x)$ because this is not an indeterminate form. The denominator approaches 0 as $x \to 1+$, but the numerator does not approach 0. Since $\ln x > 0$ for $x > 1$, we have, directly,

$$\lim_{x \to 1+} \frac{x}{\ln x} = \infty.$$

(Had we tried to apply l'Hôpital's Rule, we would have been led to the erroneous answer $\lim_{x \to 1+}(1/(1/x)) = 1$.)

Evaluate the limit:

$$\lim_{x \to \infty} \left(\sqrt{x^2 - x} - x \right)$$

Solution. As $x \to \infty$, both $\sqrt{x^2 - x}$ and x approach infinity, resulting in an indeterminate form of type $\infty - \infty$.

To resolve this indeterminate form, we rationalize the expression by multiplying the numerator and the denominator by the conjugate.

$$\lim_{x \to \infty} \left(\sqrt{x^2 - x} - x \right) = \lim_{x \to \infty} \frac{\left(\sqrt{x^2 - x} - x \right) \left(\sqrt{x^2 - x} + x \right)}{\sqrt{x^2 - x} + x}$$

This simplifies to:

$$= \lim_{x \to \infty} \frac{x^2 - x - x^2}{\sqrt{x^2 - x} + x}$$

$$= \lim_{x \to \infty} \frac{-x}{\sqrt{x^2 - x} + x}$$

Now, factor x out of the terms in the denominator:

$$= \lim_{x \to \infty} \frac{-x}{x\sqrt{1 - \frac{1}{x}} + x}$$

Simplify by canceling x from the numerator and the denominator:

$$= \lim_{x \to \infty} \frac{-1}{\sqrt{1 - \frac{1}{x}} + 1}$$

Simplify by canceling x from the numerator and the denominator:

$$= \lim_{x\to\infty} \frac{-1}{\sqrt{1-\frac{1}{x}}+1}$$

As $x \to \infty$, the term $\frac{1}{x}$ approaches zero:

$$= \frac{-1}{\sqrt{1-0}+1}$$

$$= \frac{-1}{1+1} = \frac{-1}{2}$$

Therefore, the solution is:

$$\lim_{x\to\infty} \left(\sqrt{x^2-x}-x\right) = -\frac{1}{2}$$

$$\lim_{x \to 0} \frac{5-\sqrt{x+25}}{x} = ?$$

$$\lim_{x \to 0} \frac{5-\sqrt{x+25}}{x} = \lim_{x \to 0} \frac{(5-\sqrt{x+25})(5+\sqrt{x+25})}{x(5+\sqrt{x+25})} = \lim_{x \to 0} \frac{25-(x+25)}{x(5+\sqrt{x+25})} = -\frac{1}{10}$$

Calculate

$$\lim_{x\to\infty} \frac{x^3+3x+5}{2x^3-6x+1}.$$

Solution. Substituting $x \to \infty$ shows that this is of the form $\frac{\infty}{\infty}$. Divide the numerator and denominator by x^3 (the highest degree in this expression). Thus, we obtain

$$\lim_{x\to\infty} \frac{x^3+3x+5}{2x^3-6x+1} = \lim_{x\to\infty} \frac{\frac{x^3+3x+5}{x^3}}{\frac{2x^3-6x+1}{x^3}} = \lim_{x\to\infty} \frac{\frac{x^3}{x^3}+\frac{3x}{x^3}+\frac{5}{x^3}}{\frac{2x^3}{x^3}-\frac{6x}{x^3}+\frac{1}{x^3}}$$

$$= \lim_{x\to\infty} \frac{1+\frac{3}{x^2}+\frac{5}{x^3}}{2-\frac{6}{x^2}+\frac{1}{x^3}} = \frac{\lim_{x\to\infty}\left(1+\frac{3}{x^2}+\frac{5}{x^3}\right)}{\lim_{x\to\infty}\left(2-\frac{6}{x^2}+\frac{1}{x^3}\right)}$$

$$= \frac{\lim_{x\to\infty} 1+\lim_{x\to\infty}\frac{3}{x^2}+\lim_{x\to\infty}\frac{5}{x^3}}{\lim_{x\to\infty} 2-\lim_{x\to\infty}\frac{6}{x^2}+\lim_{x\to\infty}\frac{1}{x^3}} = \frac{1+0+0}{2-0-0} = \frac{1}{2}.$$

Calculate

$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-\sqrt{x^2-1}\right)$$

Solution. If $x \to \infty$, then

$$\lim_{x\to\infty}\sqrt{x^2+1}=\infty \text{ and } \lim_{x\to\infty}\sqrt{x^2-1}=\infty$$

Thus, we deal here with an indeterminate form of type $\infty - \infty$. Multiply this expression (both the numerator and the denominator) by the corresponding conjugate expression.

$$L=\lim_{x\to\infty}\left(\sqrt{x^2+1}-\sqrt{x^2-1}\right)=\lim_{x\to\infty}\frac{\left(\sqrt{x^2+1}\right)^2-\left(\sqrt{x^2-1}\right)^2}{\left(\sqrt{x^2+1}+\sqrt{x^2-1}\right)}=$$

$$\lim_{x\to\infty}\frac{x^2+1-(x^2-1)}{\left(\sqrt{x^2+1}+\sqrt{x^2-1}\right)}=\lim_{x\to\infty}=\lim_{x\to\infty}\frac{\cancel{x^2}+1-\cancel{x^2}+1}{\left(\sqrt{x^2+1}+\sqrt{x^2-1}\right)}=\lim_{x\to\infty}\frac{2}{\left(\sqrt{x^2+1}+\sqrt{x^2-1}\right)}.$$

By using the product and the sum rules for limits, we obtain

$$L=\lim_{x\to\infty}\frac{2}{\left(\sqrt{x^2+1}+\sqrt{x^2-1}\right)}=\frac{\lim_{x\to\infty}2}{\lim_{x\to\infty}\sqrt{x^2+1}+\lim_{x\to\infty}\sqrt{x^2-1}}\sim\frac{2}{\infty+\infty}\sim\frac{2}{\infty}=0$$

EXAMPLE Find the tangent to the curve

$$x = \sec t, \qquad y = \tan t, \qquad -\frac{\pi}{2} < t < \frac{\pi}{2},$$

at the point $(\sqrt{2}, 1)$, where $t = \pi/4$

Solution The slope of the curve at t is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sec^2 t}{\sec t \tan t} = \frac{\sec t}{\tan t}.$$

Setting t equal to $\pi/4$ gives

$$\left.\frac{dy}{dx}\right|_{t=\pi/4} = \frac{\sec(\pi/4)}{\tan(\pi/4)} = \frac{\sqrt{2}}{1} = \sqrt{2}.$$

The tangent line is

$$\begin{aligned} y - 1 &= \sqrt{2}\,(x - \sqrt{2}) \\ y &= \sqrt{2}\,x - 2 + 1 \\ y &= \sqrt{2}\,x - 1. \end{aligned}$$

EXAMPLE Find d^2y/dx^2 as a function of t if $x = t - t^2, y = t - t^3$.

Solution

1. Express $y' = dy/dx$ in terms of t.

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 - 3t^2}{1 - 2t}$$

2. Differentiate y' with respect to t.

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{1 - 3t^2}{1 - 2t}\right) = \frac{2 - 6t + 6t^2}{(1 - 2t)^2} \qquad \text{Derivative Quotient Rule}$$

3. Divide dy'/dt by dx/dt.

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{(2 - 6t + 6t^2)/(1 - 2t)^2}{1 - 2t} = \frac{2 - 6t + 6t^2}{(1 - 2t)^3} \qquad \text{Eq. (2)}$$