Applications of Derivatives

DEFINITIONS Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$

Maximum and minimum values are called extreme values of the function f. Absolute maxima or minima are also referred to as global maxima or minima.

(a) $y = x^2$ $(-\infty, \infty)$ No absolute maximum.
Absolute minimum of 0 at $x = 0$.

(b) $y = x^2$ $[0, 2]$ Absolute maximum of 4 at $x = 2$.
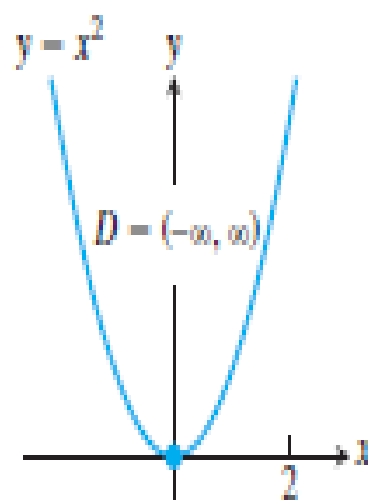Absolute minimum of 0 at $x = 0$.

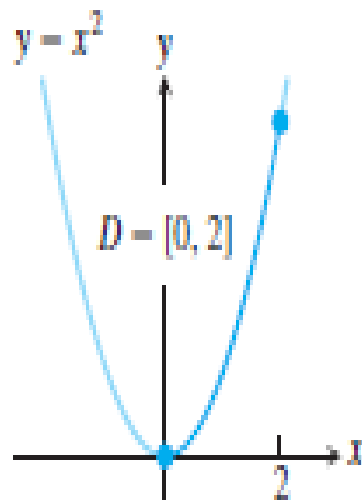**(c)** $y = x^2$ $(0, 2]$ Absolute maximum of 4 at $x = 2$.
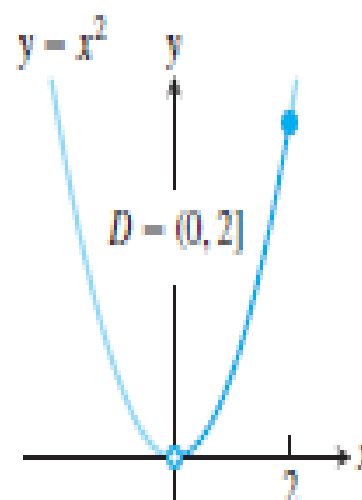No absolute minimum.

(d) $y = x^2$ $(0, 2)$ No absolute extrema.
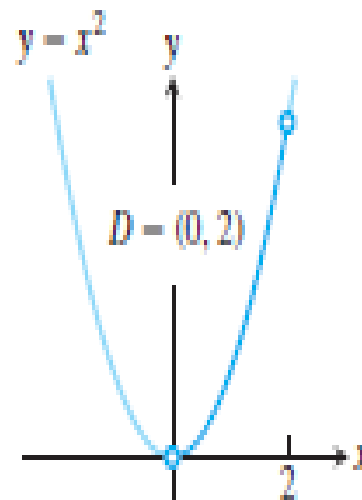

$y = x^2$ y

$D = (-\infty, \infty)$

(a) abs min only

$y = x^2$ y

$D = [0, 2]$

(b) abs max and min

$y = x^2$ y
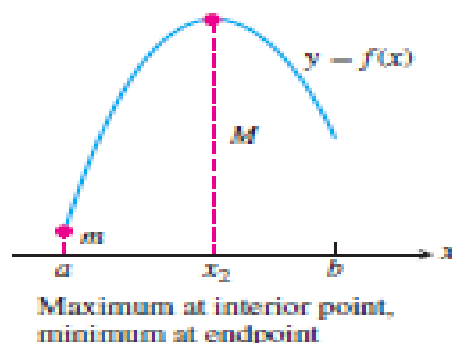
$D = (0, 2]$

(c) abs max only

$y = x^2$ y

$D = (0, 2)$
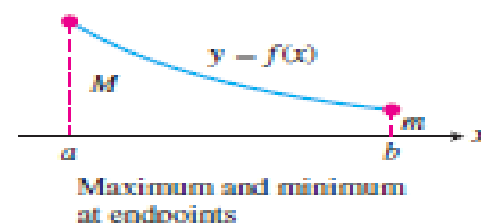
(d) no max or min

THEOREM   The Extreme Value Theorem   If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in $[a, b]$.

Maximum and minimum
at interior points

Maximum and minimum
at endpoints

Maximum at interior point,
minimum at endpoint

Minimum at interior point,
maximum at endpoint
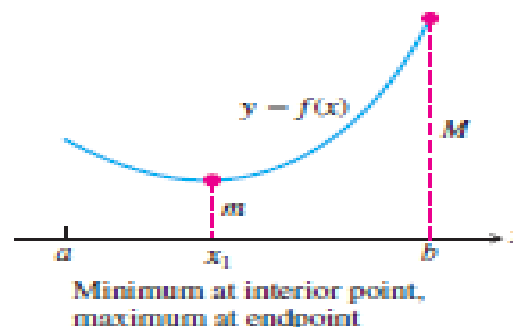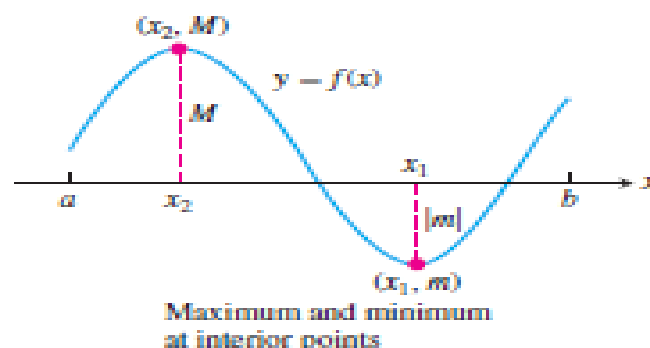
Some possibilities for a continuous function's maximum and minimum on a closed interval $[a, b]$.

DEFINITIONS   A function f has a **local maximum** value at a point c within its domain D if $f(x) \leq f(c)$ for all $x \in D$ lying in some open interval containing c.

A function f has a **local minimum** value at a point c within its domain D if $f(x) \geq f(c)$ for all $x \in D$ lying in some open interval containing c.

Absolute maximum
No greater value of f anywhere.
Also a local maximum.

Local maximum
No greater value of f nearby.

$y = f(x)$

Local minimum
No smaller value of f nearby.

Absolute minimum
No smaller value of f anywhere. Also a local minimum.

Local minimum
No smaller value of f nearby.
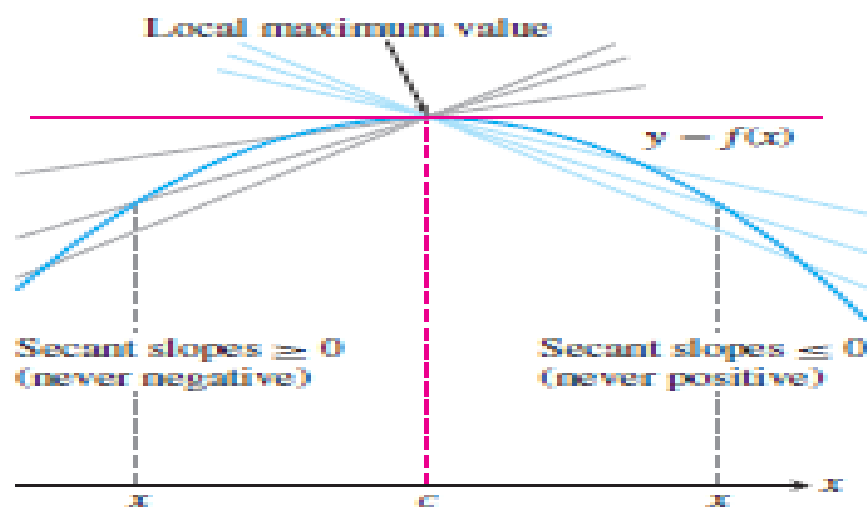
a c e d b x
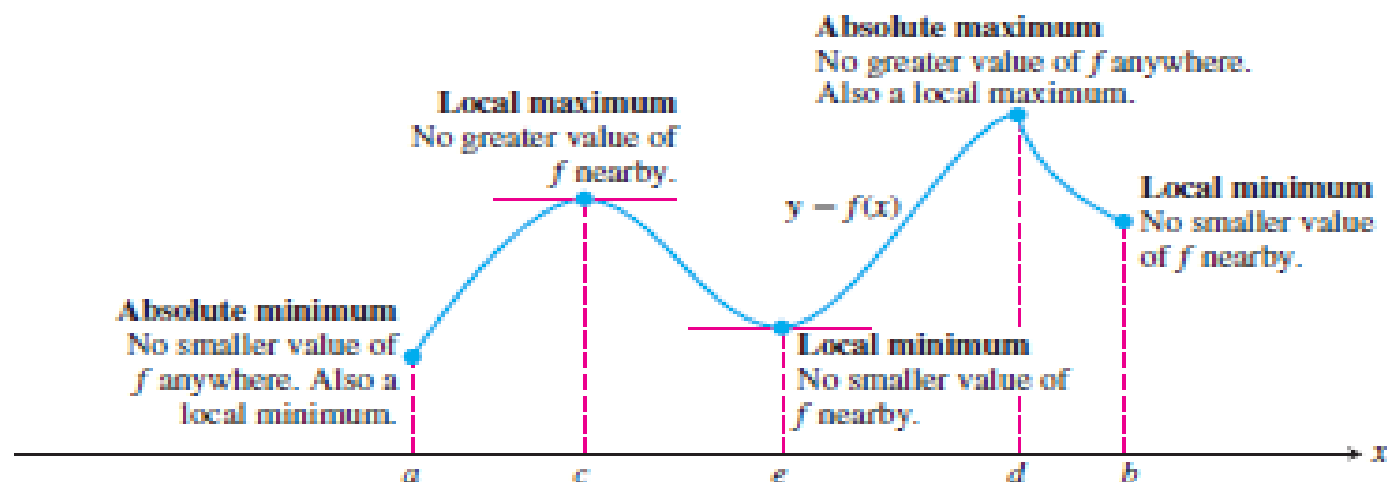
How to identify types of maxima and minima for a function with domain $a \leq x \leq b$.

An absolute maximum is also a local maximum. Being the largest value overall, it is also the largest value in its immediate neighborhood. Hence, *a list of all local maxima will automatically include the absolute maximum if there is one.* Similarly, *a list of all local minima will include the absolute minimum if there is one.*

THEOREM The First Derivative Theorem for Local Extreme Values If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0.$$

Local maximum value

$y = f(x)$

Secant slopes ≥ 0
(never negative)

Secant slopes ≤ 0
(never positive)

x

c

x

DEFINITION An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

How to Find the Absolute Extrema of a Continuous Function f on a Finite Closed Interval

1. Evaluate f at all critical points and endpoints.
2. Take the largest and smallest of these values.

EXAMPLE   Find the absolute maximum and minimum values of $f(x) = x^2$ on $[-2, 1]$.

Solution   The function is differentiable over its entire domain, so the only critical point is where $f'(x) = 2x = 0$, namely $x = 0$. We need to check the function's values at $x = 0$ and at the endpoints $x = -2$ and $x = 1$:

Critical point value: $f(0) = 0$

Endpoint values: $f(-2) = 4$

$f(1) = 1$

The function has an absolute maximum value of 4 at $x = -2$ and an absolute minimum value of 0 at $x = 0$.

EXAMPLE _ Find the absolute maximum and minimum values of $f(x) = 10x(2 - \ln x)$ on the interval $[1, e^2]$.

f has its absolute maximum value near $x = 3$ and its absolute minimum value of 0 at $x = e^2$. Let's verify this observation.

We evaluate the function at the critical points and endpoints and take the largest and smallest of the resulting values.

The first derivative is

$$f'(x) = 10(2 - \ln x) - 10x\left(\frac{1}{x}\right) = 10(1 - \ln x).$$

The only critical point in the domain $[1, e^2]$ is the point $x = e$, where $\ln x = 1$. The values of f at this one critical point and at the endpoints are

Critical point value: $f(e) = 10e$

Endpoint values: $f(1) = 10(2 - \ln 1) = 20$

$f(e^2) = 10e^2(2 - 2 \ln e) = 0.$

We can see from this list that the function's absolute maximum value is $10e \approx 27.2$; it occurs at the critical interior point $x = e$. The absolute minimum value is 0 and occurs at the right endpoint $x = e^2$.

EXAMPLE Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.

Solution We evaluate the function at the critical points and endpoints and take the largest and smallest of the resulting values.

The first derivative

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

has no zeros but is undefined at the interior point $x = 0$. The values of f at this one critical point and at the endpoints are

Critical point value: $f(0) = 0$

Endpoint values: $f(-2) = (-2)^{2/3} = \sqrt[3]{4}$

$f(3) = (3)^{2/3} = \sqrt[3]{9}.$

We can see from this list that the function's absolute maximum value is $\sqrt[3]{9} \approx 2.08$, and it occurs at the right endpoint $x = 3$. The absolute minimum value is 0, and it occurs at the interior point $x = 0$ where the graph has a cusp

Increasing Functions and Decreasing Functions

COROLLARY   Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.
If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

A function that is increasing or decreasing on an interval is said to be **monotonic** on the interval.

EXAMPLE 1   Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the intervals on which f is increasing and on which f is decreasing.

Solution   The function f is everywhere continuous and differentiable. The first derivative

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4)$$
$$= 3(x + 2)(x - 2)$$

is zero at $x = -2$ and $x = 2$. These critical points subdivide the domain of f to create nonoverlapping open intervals $(-\infty, -2), (-2, 2)$, and $(2, \infty)$ on which f' is either positive or negative. We determine the sign of f' by evaluating f' at a convenient point in each subinterval. The behavior of f is determined by then applying Corollary   to each subinterval. The results are summarized in the following table, and the graph of f is given in Figure

Interval	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
f' evaluated	$f'(-3) = 15$	$f'(0) = -12$	$f'(3) = 15$
Sign of f'	$+$	$-$	$+$
Behavior of f	increasing	decreasing	increasing

function f in the example is increasing on $-\infty < x \le -2$, decreasing on $-2 \le x \le 2$, and increasing on $2 \le x < \infty$. We do not talk about whether a function is increasing or decreasing at a single point.

First Derivative Test for Local Extrema

, at the points where f has a minimum value, $f' < 0$ immediately to the left and $f' > 0$ immediately to the right. (If the point is an endpoint, there is only one side to consider.) Thus, the function is decreasing on the left of the minimum value and it is increasing on its right. Similarly, at the points where f has a maximum value, $f' > 0$ immediately to the left and $f' < 0$ immediately to the right. Thus, the function is increasing on the left of the maximum value and decreasing on its right. In summary, at a local extreme point, the sign of $f'(x)$ changes.
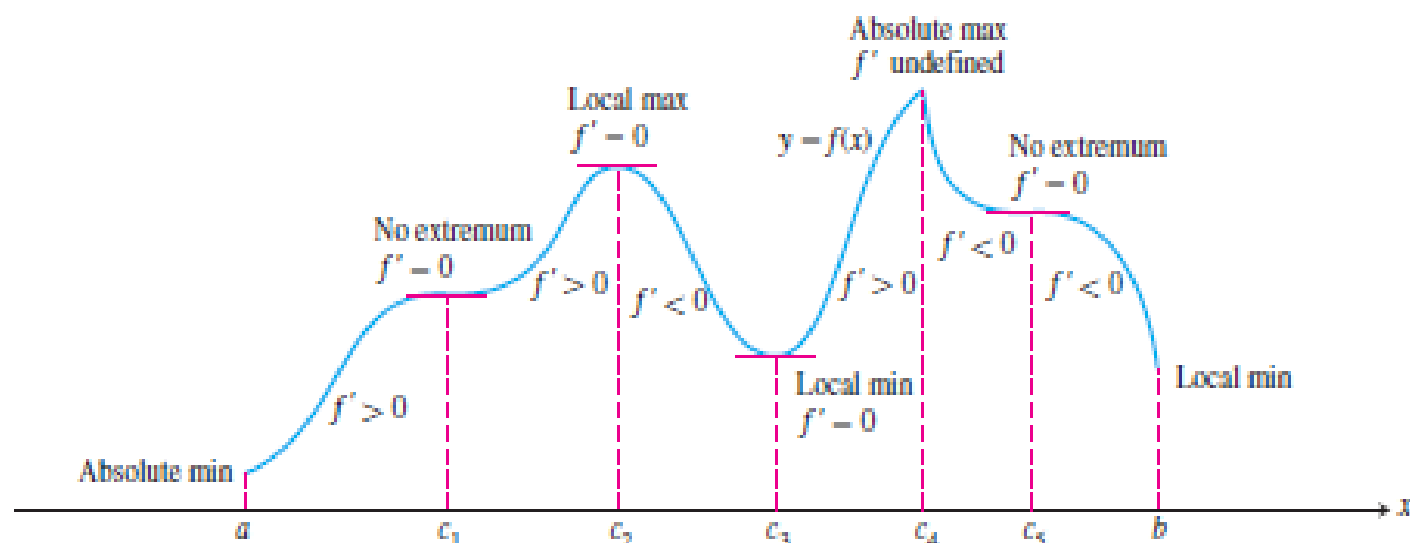
The critical points of a function locate where it is increasing and where it is decreasing. The first derivative changes sign at a critical point where a local extremum occurs.

First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across this interval from left to right,

1. if f' changes from negative to positive at c, then f has a local minimum at c;
2. if f' changes from positive to negative at c, then f has a local maximum at c;
3. if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local extremum at c.

The test for local extrema at endpoints is similar, but there is only one side to consider.

<u>Remember</u> that if f has a local maximum or minimum at c, then c must be a critical number of f (by Fermat's Theorem) but not every critical number gives rise to a maximum or a minimum so, we need a test that will tell us whether or not f has a local maximum or minimum at a critical number.

Find the critical points of

$$f(x) = x^{1/3}(x - 4) = x^{4/3} - 4x^{1/3}.$$

Identify the intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

The function f is continuous at all x since it is the product of two continuous functions, $x^{1/3}$ and $(x - 4)$. The first derivative

$$f'(x) = \frac{d}{dx}\left(x^{4/3} - 4x^{1/3}\right) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3}$$

$$= \frac{4}{3}x^{-2/3}(x - 1) = \frac{4(x - 1)}{3x^{2/3}}$$

is zero at $x = 1$ and undefined at $x = 0$. There are no endpoints in the domain, so the critical points $x = 0$ and $x = 1$ are the only places where f might have an extreme value.

The critical points partition the x-axis into intervals on which f' is either positive or negative.

Interval	$x < 0$	$0 < x < 1$	$x > 1$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing

The First Derivative Test for Local Extrema tells us that f does not have an extreme value at $x = 0$ (f' does not change sign) and that f has a local minimum at $x = 1$ (f' changes from negative to positive).

The value of the local minimum is $f(1) = 1^{1/3}(1 - 4) = -3$. This is also an absolute minimum since f is decreasing on $(-\infty, 1]$ and increasing on $[1, \infty)$.

Note that $\lim_{x \to 0} f'(x) = -\infty$, so the graph of f has a vertical tangent at the origin.

EXAMPLE   Find the critical points of
$$f(x) = (x^2 - 3)e^x.$$
Identify the intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

The function f is continuous and differentiable for all real numbers, so the critical points occur only at the zeros of f'.

Using the Derivative Product Rule, we find the derivative
$$f'(x) = (x^2 - 3) \cdot \frac{d}{dx} e^x + \frac{d}{dx}(x^2 - 3) \cdot e^x$$
$$= (x^2 - 3) \cdot e^x + (2x) \cdot e^x$$
$$= (x^2 + 2x - 3)e^x.$$

Since e^x is never zero, the first derivative is zero if and only if
$$x^2 + 2x - 3 = 0$$
$$(x + 3)(x - 1) = 0.$$

The zeros $x = -3$ and $x = 1$ partition the x-axis into intervals as follows.

Interval	$x < -3$	$-3 < x < 1$	$1 < x$
Sign of f'	$+$	$-$	$+$
Behavior of f	increasing	decreasing	increasing

We can see from the table that there is a local maximum (about 0.299) at $x = -3$ and a local minimum (about -5.437) at $x = 1$. The local minimum value is also an absolute minimum because $f(x) > 0$ for $|x| > \sqrt{3}$. There is no absolute maximum. The function increases on $(-\infty, -3)$ and $(1, \infty)$ and decreases on $(-3, 1)$.

For $|x| > \sqrt{3}$, we have $x^2 - 3 > 0$ and $e^x > 0$, so $f(x) > 0$.
Therefore negative values occur only for $|x| < \sqrt{3}$, and the smallest value is $-2e$ at $x = 1$.
⇒ **Absolute minimum** at $x = 1$, with $f(1) = -2e$.

Ex: Find the local maximum and minimum values of the function

$$g(x) = x + 2\sin x \quad , \quad 0 \le x \le 2\pi$$

To find the critical numbers of g)

$$g'(x) = 1 + 2\cos x$$

So, $g'(x) = 0$ when $\cos x = -\dfrac{1}{2}$; $x = \dfrac{2\pi}{3}$ $x = \dfrac{4\pi}{3}$

Interval	$g'(x)$	g
$0 < x < \dfrac{2\pi}{3}$	$+$	increasing
$\dfrac{2\pi}{3} < x < \dfrac{4\pi}{3}$	$-$	decreasing
$\dfrac{4\pi}{3} < x < 2\pi$	$+$	increasing

Because $g'(x)$ changes from positive to negative at $\frac{\pi}{3}$ there is a local max at $\frac{2\pi}{3}$ and the local maximum value is

$$g(2\pi/3) = \frac{2\pi}{3} + 2 \sin \frac{2\pi}{3} \cong 3.83$$

Likewise, $g'(x)$ changes from neg. to pos. at $\frac{4\pi}{3}$ and so

$$g(4\pi/3) = \frac{4\pi}{3} + 2 \sin \frac{4\pi}{3} \cong 2.46 \quad \text{is a local min. value.}$$

Definition: If the graph of f lies above all of its tangents on an interval I, then it is called concave upward on I.

If the graph of f lies below all of its tangents on I, it is called concave downword on I.
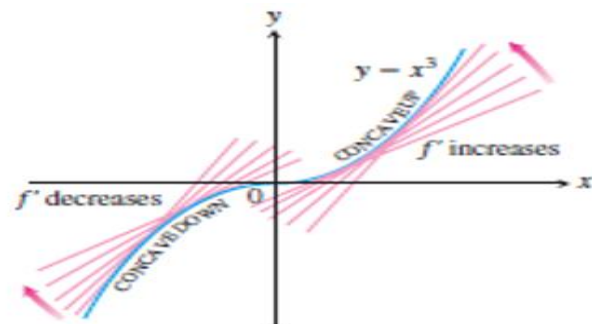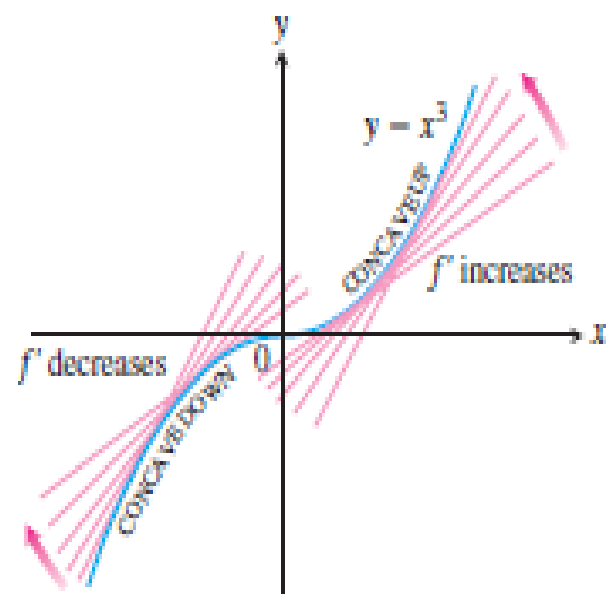

Concavity Test

a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upword on I

b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downword on I.

Concavity

DEFINITION   The graph of a differentiable function $y = f(x)$ is
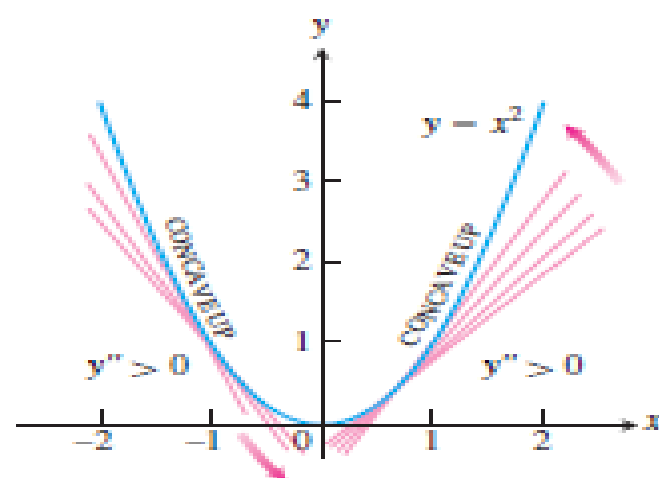
(a) **concave up** on an open interval I if f' is increasing on I;

(b) **concave down** on an open interval I if f' is decreasing on I.

The Second Derivative Test for Concavity

Let $y = f(x)$ be twice-differentiable on an interval I.

1. If $f'' > 0$ on I, the graph of f over I is concave up.

2. If $f'' < 0$ on I, the graph of f over I is concave down.

The graph of $f(x) = x^2$
is concave up on every interval

DEFINITION   A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.
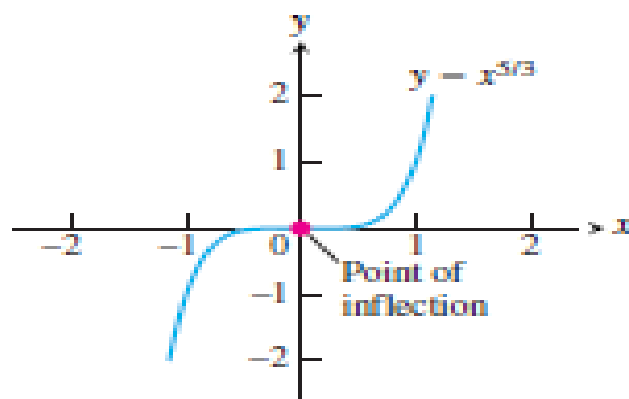
At a point of inflection $(c, f(c))$, either $f''(c) = 0$ or $f''(c)$ fails to exist.

EXAMPLE   The graph of $f(x) = x^{5/3}$ has a horizontal tangent at the origin because $f'(x) = (5/3)x^{2/3} = 0$ when $x = 0$. However, the second derivative

$$f''(x) = \frac{d}{dx}\left(\frac{5}{3}x^{2/3}\right) = \frac{10}{9}x^{-1/3}$$

fails to exist at $x = 0$. Nevertheless, $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$, so the second derivative changes sign at $x = 0$ and there is a point of inflection at the origin.

Here is an example showing that an inflection point need not occur even though both derivatives exist and $f'' = 0$.

The graph of $f(x) = x^{5/3}$ has a horizontal tangent at the origin where the concavity changes, although f'' does not exist at $x = 0$

EXAMPLE   The curve $y = x^4$ has no inflection point at $x = 0$. Even though the second derivative $y'' = 12x^2$ is zero there, it does not change sign.

As our final illustration, we show a situation in which a point of inflection occurs at a vertical tangent to the curve where neither the first nor the second derivative exists.
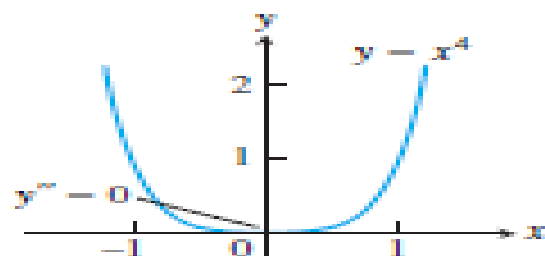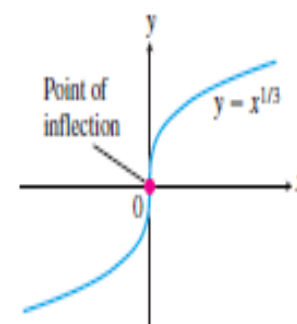

The graph of $y = x^4$ has no inflection point at the origin, even though $y'' = 0$ there (Example 4).

EXAMPLE   The graph of $y = x^{1/3}$ has a point of inflection at the origin because the second derivative is positive for $x < 0$ and negative for $x > 0$:

$$y'' = \frac{d^2}{dx^2}\left(x^{1/3}\right) = \frac{d}{dx}\left(\frac{1}{3}x^{-2/3}\right) = -\frac{2}{9}x^{-5/3}.$$

However, both $y' = x^{-2/3}/3$ and y'' fail to exist at $x = 0$, and there is a vertical tangent there

A point of inflection where y' and y'' fail to exist (Example 5).

THEOREM —Second Derivative Test for Local Extrema   Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.

$f' = 0, f'' < 0$
$\Rightarrow$ local max

$f' = 0, f'' > 0$
$\Rightarrow$ local min
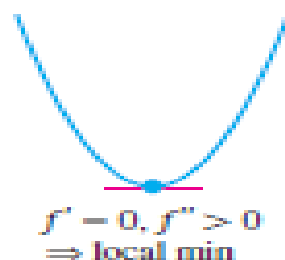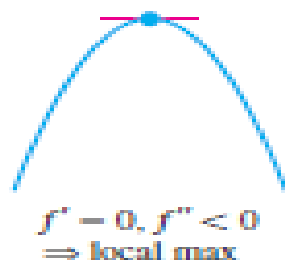
Ex: Discuss the curve $y = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima

$f(x) = x^4 - 4x^3$

$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$

$f''(x) = 12x^2 - 24x = 12x(x - 2)$

To find the critical numbers we set $f'(x) = 0$ and obtain $x = 0$, $x = 3$

To use the second derivative test,

$f''(0) = 0$, $f''(3) = 36$

Since $f'(3) = 0$ and $f''(3) > 0$; $f(3) = -27$ is a local minimum

Since $f''(0) = 0$, the second derivative test gives no information about the critical number 0.

But, since $f'(x) < 0$ for $x < 0$ and also for $0 < x < 3$, the first derivative test tells us that f does not have a local maximum or minimum at 0.

$f'(x) = 4x^2(x-3)$

for $x < 0$, $f'(x) < 0$

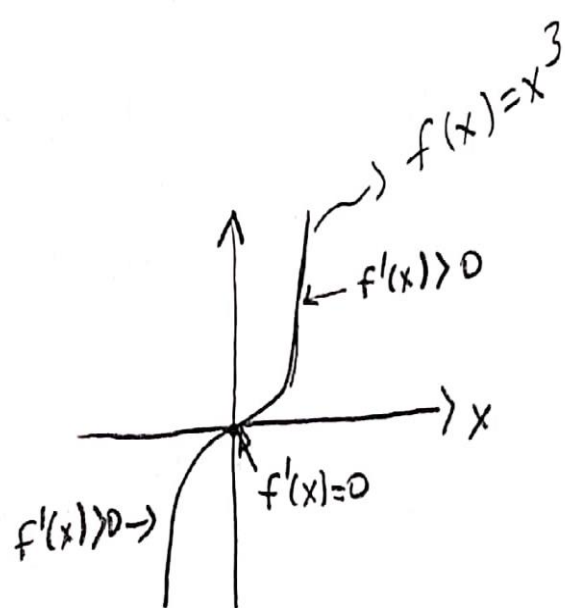for $0 < x < 3$, $f'(x) < 0$

} there is no max or min at 0

Since $f''(x) = 0$, when $x = 0$ or 2

Interval	$f''(x) = 12x(x-2)$	Concovity
$(-\infty, 0)$	$+$	Upword
$(0, 2)$	$-$	downword
$(2, \infty)$	$+$	Upword

The point $(0,0)$ is on <u>inflection point</u> since the curve chonges from concove upword to concove downward there.

Also, $(2, -16)$ is on <u>inflection point</u> since the curve chonges from concove downword to concove upword there.

$f(x) = x^3$

$f(x) = x^3$

$) f(x) = x^3$

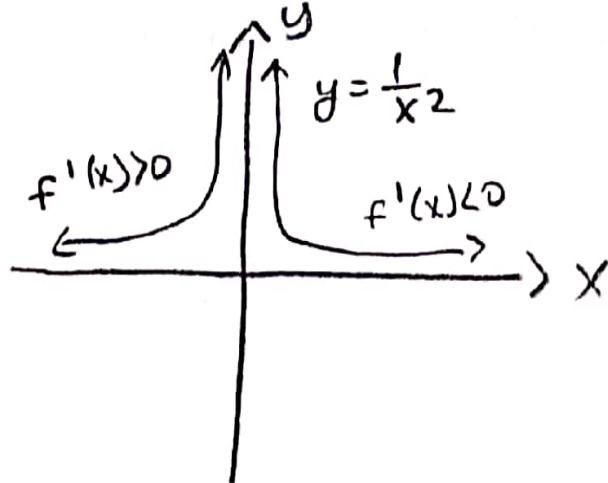$\leftarrow f'(x) > 0$

$\rightarrow x$

$f'(x) = 0$

$f'(x) > 0 \rightarrow$

zero is the critical value but does not give a relative extremum.

$f'(x) = 3x^2$. Since $f'(0) = 0$, 0 is the critical point. But, if $x < 0$, then $3x^2 > 0$ and if $x > 0$, then $3x^2 > 0$. Since $f'(x)$ does not change sign at 0, there is no relative extremum at 0.

$$f(x) = \frac{1}{x^2}$$

$$f'(x) = -\frac{2}{x^3}$$

Although $f'(x)$ does not exist at 0, 0 is not a critical value, because 0 is not in the domain of f. Thus, a relative extremum cannot occur at 0.

Determine where the given function is concave up
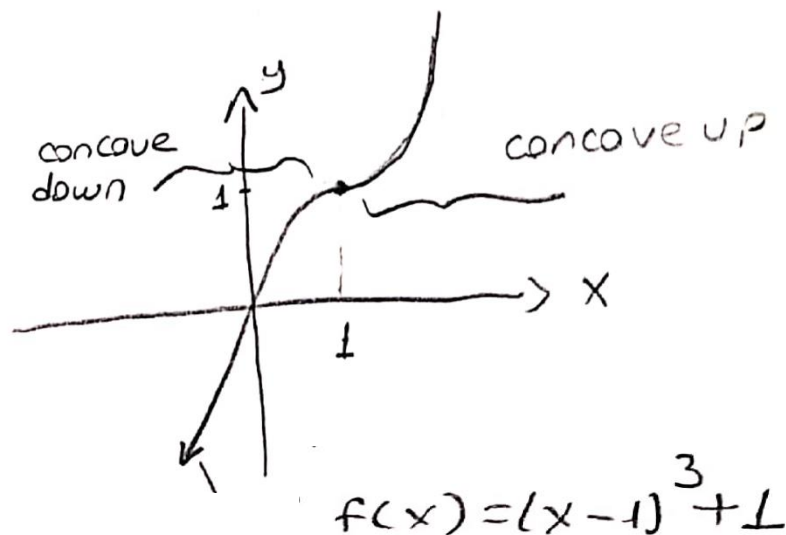and where it is concove down

$$y = f(x) = (x-1)^3 + 1$$

we must examine the signs of y''.

$$y' = 3(x-1)^2$$
$$y'' = 6(x-1)$$

Thus, f is concave up when $6(x-1) > 0$, that is, when $x > 1$. And f is concove down when $6(x-1) < 0$; that is, when $x < 1$.

concove
down

concove up

$$f(x) = (x-1)^3 + 1$$

Definition: A function f has an inflection point at a iff f is continuous at a and f changes concavity at a.

Ex: Discuss concavity and find all inflection points of $y = f(x) = \frac{1}{x}$

$f(x) = x^{-1}$ for $x \neq 0$,

$f'(x) = -x^{-2}$

$f''(x) = 2x^{-3}$ for $x \neq 0$

$f''(x)$ is never 0 but it is not defined when $x = 0$. Since f is not continuous at 0, we conclude that 0 is not a candidate for an inflection point. Thus, the given function has no inflection point. However, 0 must be considered in an analysis of concavity.

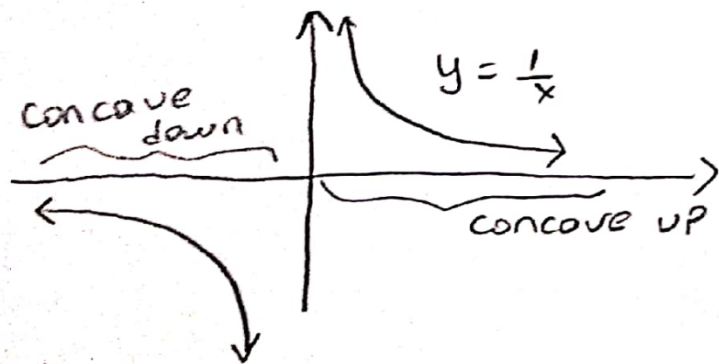However, 0 must be considered in an analysis of concavity.

$\frac{1}{x^3}$	$-\infty$	0	∞
$f''(x)$	$-$	$+$	
	$-$	$+$	
$f(x)$	$\cap$	$\cup$	

If $x > 0$ then $f''(x) > 0$

If $x < 0$ then $f''(x) < 0$.

Thus, f is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$

Although Concavity changes around $x = 0$, there is no inflection point there because f is not continuous at 0 (nor is it even defined there)

Concave down

$y = \frac{1}{x}$

Concave up

(0 is not in the domain of f and cannot correspond to an inflection point)