Parametric Equations

Definition: If x and y are given functions

$$\left.\begin{array}{l} x = f(t) \\ y = g(t) \end{array}\right\}$$ over an interval I of t-values, then

the set of points $(x,y) = (f(t), g(t))$ defined by these equations is a parametric curve. These equations are parametric equations for the curve.

$$\left.\begin{array}{l} x = f(t) \\ y = g(t) \end{array}\right\}$$ is given

Parametric formula for dy/dx

if all three derivatives exist and $dx/dt \neq 0$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(y') = \frac{dy'/dt}{dx/dt}$$

If the equations $x=f(t)$, $y=g(t)$ define y as a twice-differentiable function of x, where $dx/dt \neq 0$ and $y' = dy/dx$ then at any point

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

Find d^2y/dx^2 as a function of t if

$x = t - t^2$

$y = t - t^3$.

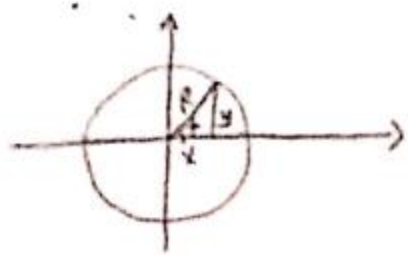
1. $y' = dy/dx = \frac{dy/dt}{dx/dt} = \frac{1-3t^2}{1-2t}$

2. $\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{1-3t^2}{1-2t}\right) = \frac{2-6t+6t^2}{(1-2t)^2}$

3. $\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{2-6t+6t^2}{(1-2t)^3}$

Parametric Curves

$x = x(t)$
$y = y(t)$



$x^2 + y^2 = R^2$

$\sin t = \frac{y}{R}$ $\quad \cos t = \frac{x}{R}$

$$\left.\begin{aligned} x &= R\cos t \\ y &= R\sin t \end{aligned}\right\} \text{parametric}$$

Ex:

$$\left.\begin{aligned} x &= \frac{t}{t^2-1} = x(t) \\ y &= \frac{2t}{t+1} = y(t) \end{aligned}\right\} \text{sketch the graph}$$

① $t^2 - 1 = 0 \Rightarrow t = \mp 1$
x is undefined for $t = \mp 1$

$t + 1 = 0 \Rightarrow t = -1$
y is undefined for $t = -1$

$(-\infty \,; t \neq \mp 1 \,; +\infty) = R - \{\mp 1\}$

(2) $\dot{x} = \dfrac{dx}{dt} = \dfrac{t^2-1-2t^2}{(t^2-1)^2} = \dfrac{-(t^2+1)}{(t^2-1)^2} < 0$ $\dot{x}$ is decreasing

$\dot{y} = \dfrac{dy}{dt} = \dfrac{2t+2-2t}{(t+1)^2} = \dfrac{2}{(t+1)^2} > 0$ $\dot{y}$ is increasing

(3) $\lim\limits_{t \to +1} x = \infty$

$\lim\limits_{t \to 1} y = 1$

$t \to 1 \begin{cases} x \to \infty \\ y \to 1 \end{cases}$ $y=1$ is horizontal asymptote.

$\lim\limits_{t \to -1} x = -\infty$

$\lim\limits_{t \to -1} y = -\infty$

$t \to -1 \left\{\begin{matrix} x \to -\infty \\ y \to -\infty \end{matrix}\right\}$ It may have oblique asymptote.

$y = mx + n$

$$m = \lim_{x \to -\infty} \frac{f(x)}{x} = \lim_{t \to -1} \frac{\frac{2t}{t+1}}{\frac{t}{t^2-1}} = \lim_{t \to -1} 2(t-1) = -4$$

$$n = \lim_{x \to -\infty} [f(x) - mx] = \lim_{\substack{x \to -\infty \\ (t \to -1)}} \left[\frac{2t}{t+1} + 4\left(\frac{t}{t^2-1}\right)\right]$$

$$= \lim_{t \to -1} \left(\frac{2t(t-1) + 4t}{t^2-1}\right) = \lim_{t \to -1} \left(\frac{2t^2 - 2t + 4t}{t^2-1}\right)$$

$$= \lim_{t \to -1} \frac{2t\cancel{(t+1)}}{(t-1)\cancel{(t+1)}} = \frac{-2}{-2} = 1 \Rightarrow n = 1$$

$\boxed{y = -4x + 1}$ oblique asymptote.

4 $\lim_{t \to -\infty} x = 0$

$\lim_{t \to -\infty} y = 2$

$t \to -\infty \begin{cases} x \to 0 \\ y \to 2 \end{cases}$ there is no asymp.

$\lim_{t \to +\infty} x = 0$

$\lim_{t \to +\infty} y = 2$

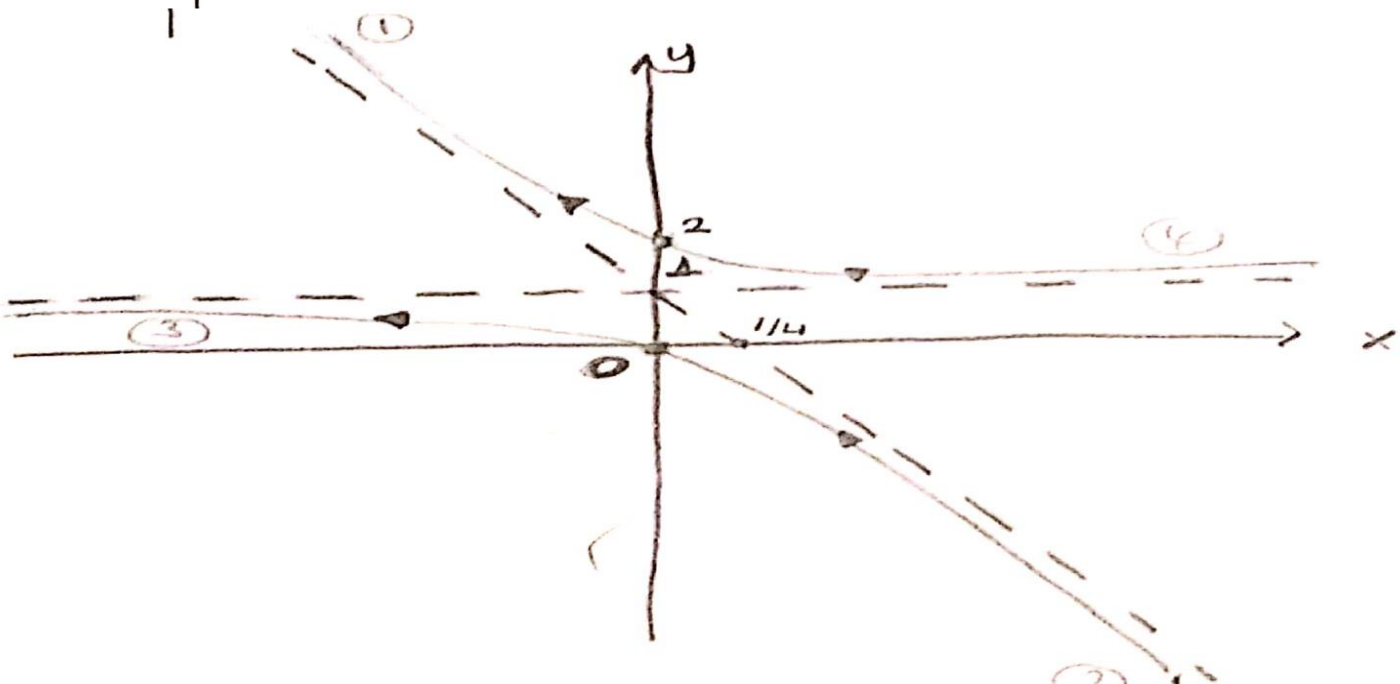
$t \to +\infty \begin{cases} x \to 0 \\ y \to 2 \end{cases}$ there is no asymp.

if $x = 0 \Rightarrow t = 0 \to y = 0$

$t \to 0 \begin{cases} x \to 0 \\ y \to 0 \end{cases}$

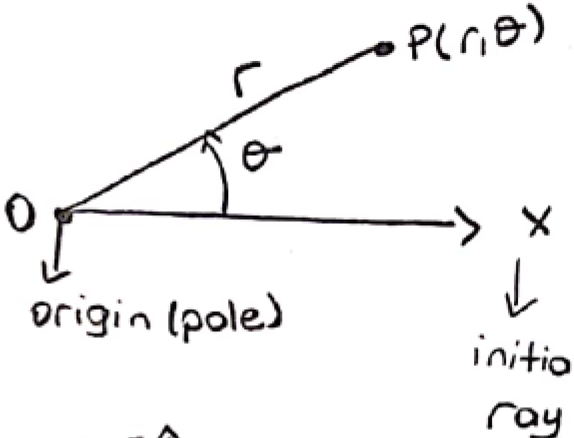
⑤

t	$-\infty$		-1		0		1		$+\infty$
$\dot{x}$		$-$		$-$		$-$		$-$	
$\dot{y}$		$+$		$+$		$+$		$+$	
x	0	① $\searrow$	$-\infty \,\|\, +\infty$	$\searrow$	0	② $\searrow$	$-\infty \,\|\, +\infty$	③ ④ $\searrow$	0
y	2	① $\nearrow$	$+\infty \,\|\, -\infty$	$\nearrow$	0	② $\nearrow$	③ 1	$\nearrow$ ④	2



POLAR COORDINATES

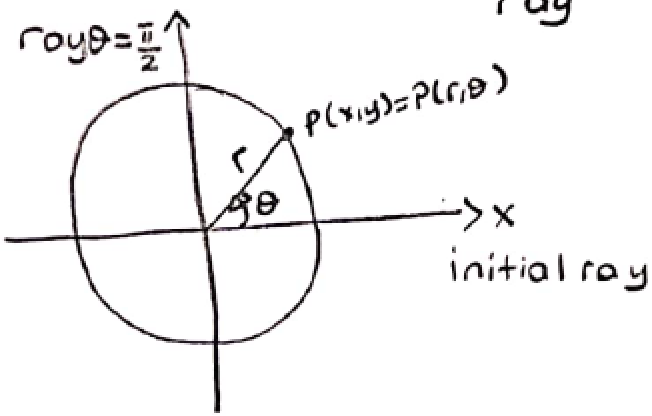
To define polar coordinates for the plane, we start with an origin, called the pole, and an initial ray.



$P(r, \theta)$

r: directed distance from O to P

θ: Directed angle from initial ray to OP



$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \quad x^2+y^2=r^2$$

$$\tan\theta = \frac{y}{x}$$

Polor Equation	cortesian equivolent
$r\cos\theta = 2$	$x = 2$
$r^2\cos\theta\sin\theta = 4$	$x \cdot y = 4$
$r^2\cos^2\theta - r^2\sin^2\theta = 1$	$x^2 - y^2 = 1$

Ex: Find the polar equation for the circle

$$x^2 + (y-3)^2 = 9$$

$$x^2 + y^2 - 6y + 9 = 9$$

$$x^2 + y^2 - 6y = 0$$

$$r^2 - 6r\sin\theta = 0$$

$r = 0$ or $r = 6\sin\theta$

SLOPE OF CURVE $r=f(\theta)$

$$\left.\frac{dy}{dx}\right|_{(r,\theta)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

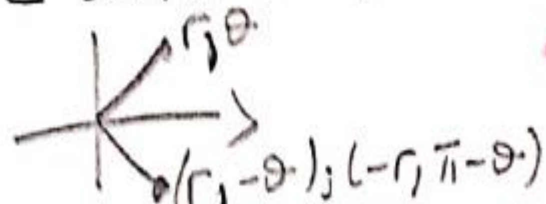
provided $dx/d\theta \neq 0$ at (r,θ)

POLAR CURVES

For polar equation; angle measurements for θ, and then corresponding value can be determined.

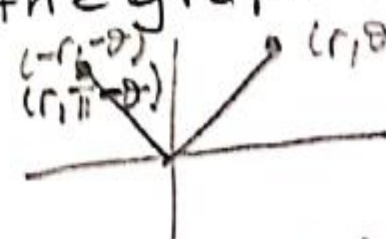
Symmetry test for Polar coordinates:

1. Replace θ with $-\theta$. If an equivalent equation results, the graph is symmetric with respect to the polar axis. (x-axis)



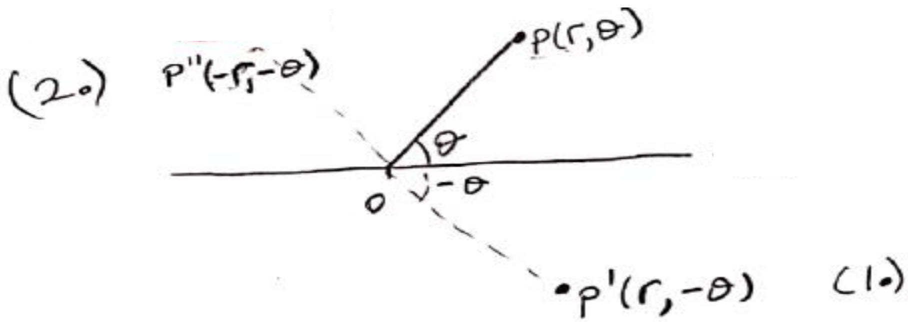
$(r,\theta) \to (-r, \pi-\theta), (r, -\theta)$

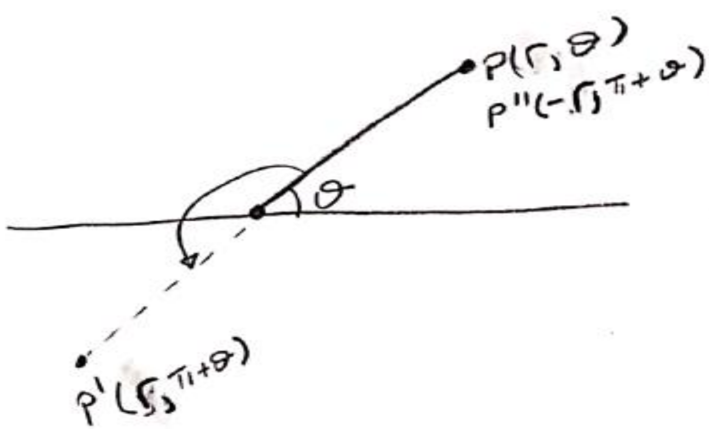
2. Replace θ with $-\theta$ and r with $-r$. If an equivalent equation results, the graph is symmetric with respect to $\theta = \frac{\pi}{2}$



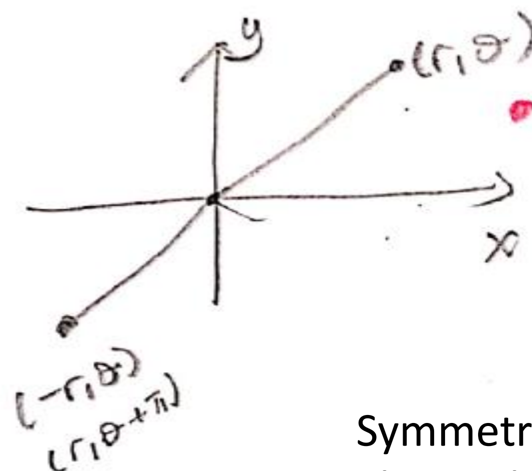
$(r,\theta) \to (r, \pi-\theta)$ or $(-r, -\theta)$

3. Replace r with $-r$. If an equivalent equation results, the graph is symmetric with respect to the pole.





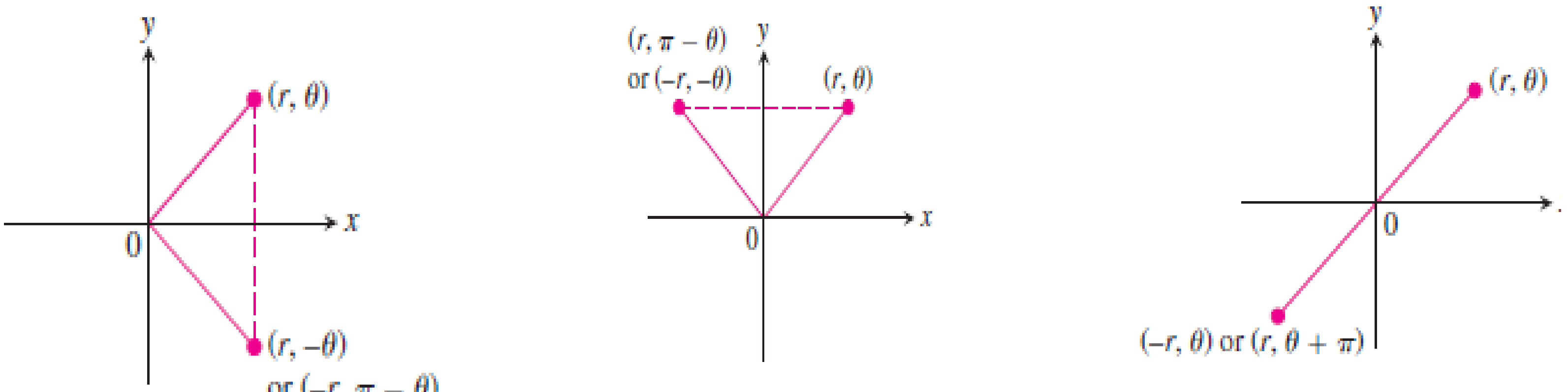
(3.)



Symmetry about the origin

Symmetry Tests for Polar Graphs

1. *Symmetry about the x-axis:* If the point (r, θ) lies on the graph, then the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph
2. *Symmetry about the y-axis:* If the point (r, θ) lies on the graph, then the point $(r, \pi - \theta)$ or $(-r, -\theta)$ lies on the graph
3. *Symmetry about the origin:* If the point (r, θ) lies on the graph, then the point $(-r, \theta)$ or $(r, \theta + \pi)$ lies on the graph



(a) About the *x*-axis (b) About the *y*-axis (c) About the origin we consider symmetry with respect to the line $\theta = \frac{\pi}{2}$ (y-axis). We replace (r, θ) with $(-r, -\theta)$ to determine if the new equation is equivalent to the original equation. For example, suppose we are given the equation $r = 2\sin\theta$;

$$\begin{aligned}
r &= 2\sin\theta \\
-r &= 2\sin(-\theta) && \text{Replace } (r,\theta) \text{ with } (-r,-\theta). \\
-r &= -2\sin\theta && \text{Identity: } \sin(-\theta) = -\sin\theta. \\
r &= 2\sin\theta && \text{Multiply both sides by } -1.
\end{aligned}$$

This equation exhibits symmetry with respect to the line $\theta = \frac{\pi}{2}$.

, we consider symmetry with respect to the polar axis (x -axis). We replace (r, θ) with $(r, -\theta)$ or $(-r, \pi - \theta)$ to determine equivalency between the tested equation and the original. For example, suppose we are given the equation $r = 1 - 2\cos\theta$.

$$\begin{aligned}
r &= 1 - 2\cos\theta \\
r &= 1 - 2\cos(-\theta) && \text{Replace } (r,\theta) \text{ with } (r,-\theta). \\
r &= 1 - 2\cos\theta && \text{Even/Odd identity}
\end{aligned}$$

The graph of this equation exhibits symmetry with respect to the polar axis.

Ex: $r = 1 - \cos\theta$ sketch the graph

for $\theta \longrightarrow -\theta$

$1 - \cos(-\theta) = 1 - \cos\theta = r$

So, it is symmetric respect to x-axis.

$(r, -\theta)$.

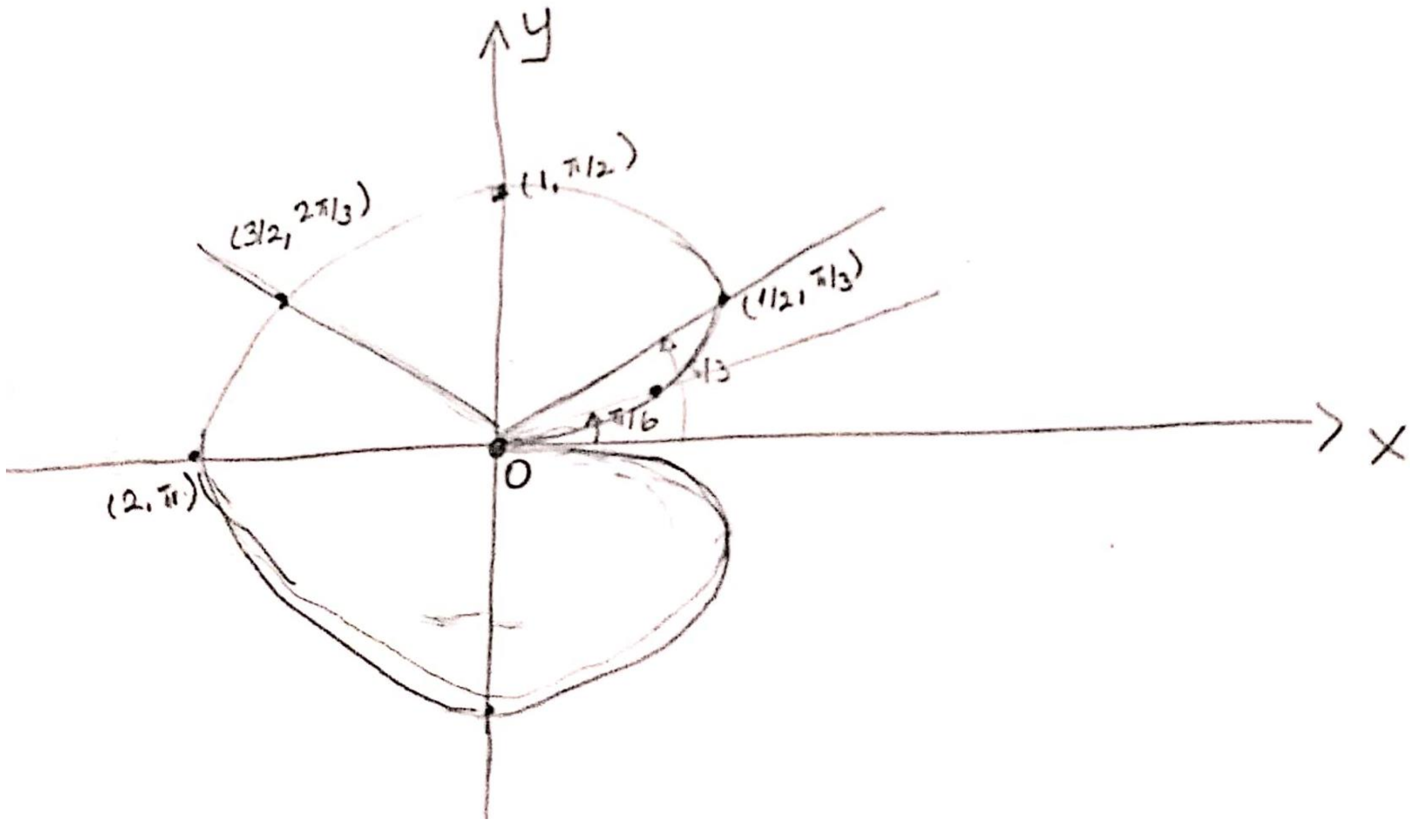
$r' = \sin\theta = 0$

$\theta = 0 \quad r = 0$

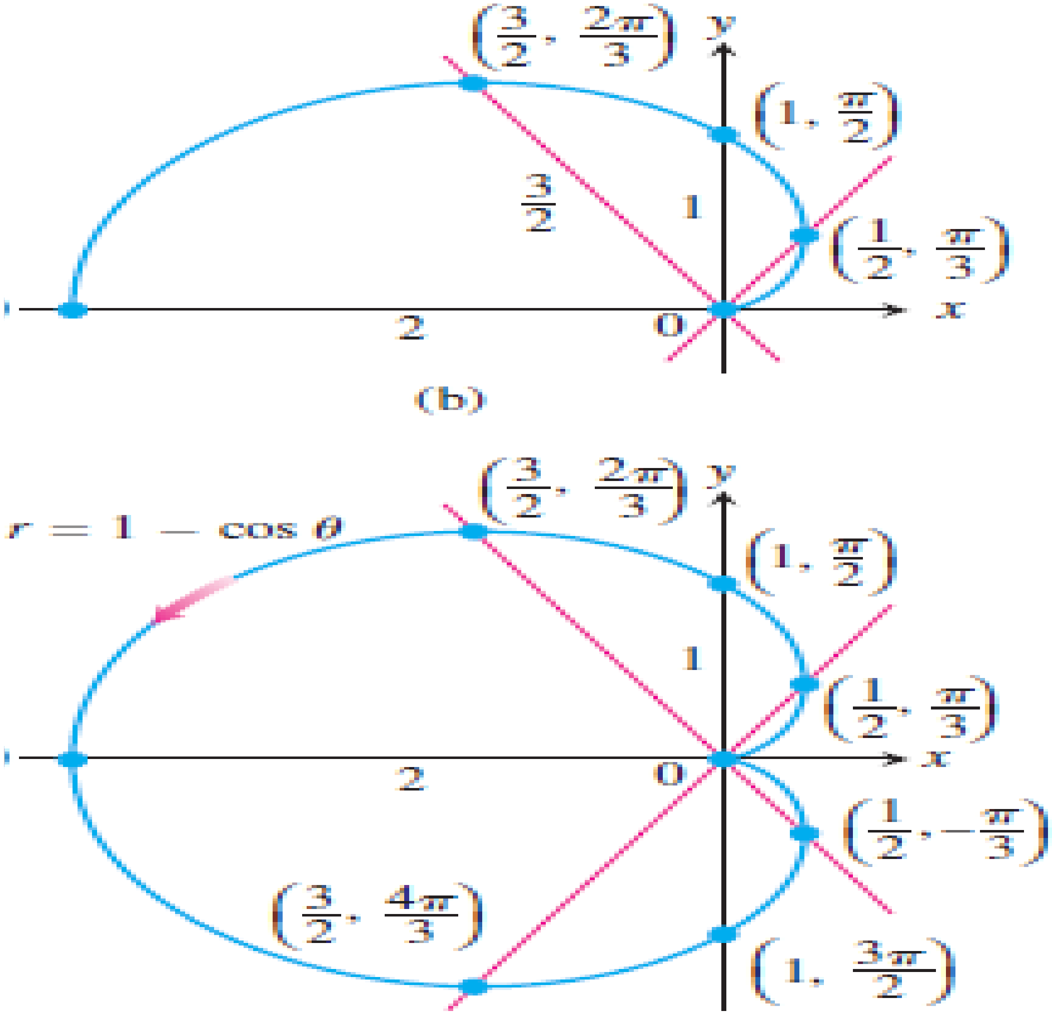
$\theta = \pi \quad r = 2$

θ	$r = 1 - \cos\theta$
0	0
$\frac{\pi}{6}$	$\frac{2-\sqrt{3}}{2}$
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{3}{2}$
π	2

(r, θ)



y
(1, π/2)
(3/2, 2π/3)
(1/2, π/3)
π/3
π/6
x
O
(2, π)



arrow shows the direction of increasing θ.

Ex: $r = 1 + \sin\theta$

$\theta \longrightarrow -\theta$

$1 + \sin(-\theta) = 1 - \sin\theta$

It is not symmetric respect to x-axis.

$-r = 1 + \sin(-\theta)$

$-r = 1 - \sin\theta$

$r = -1 + \sin\theta$?

but

$r = 1 + \sin(\pi - \theta)$

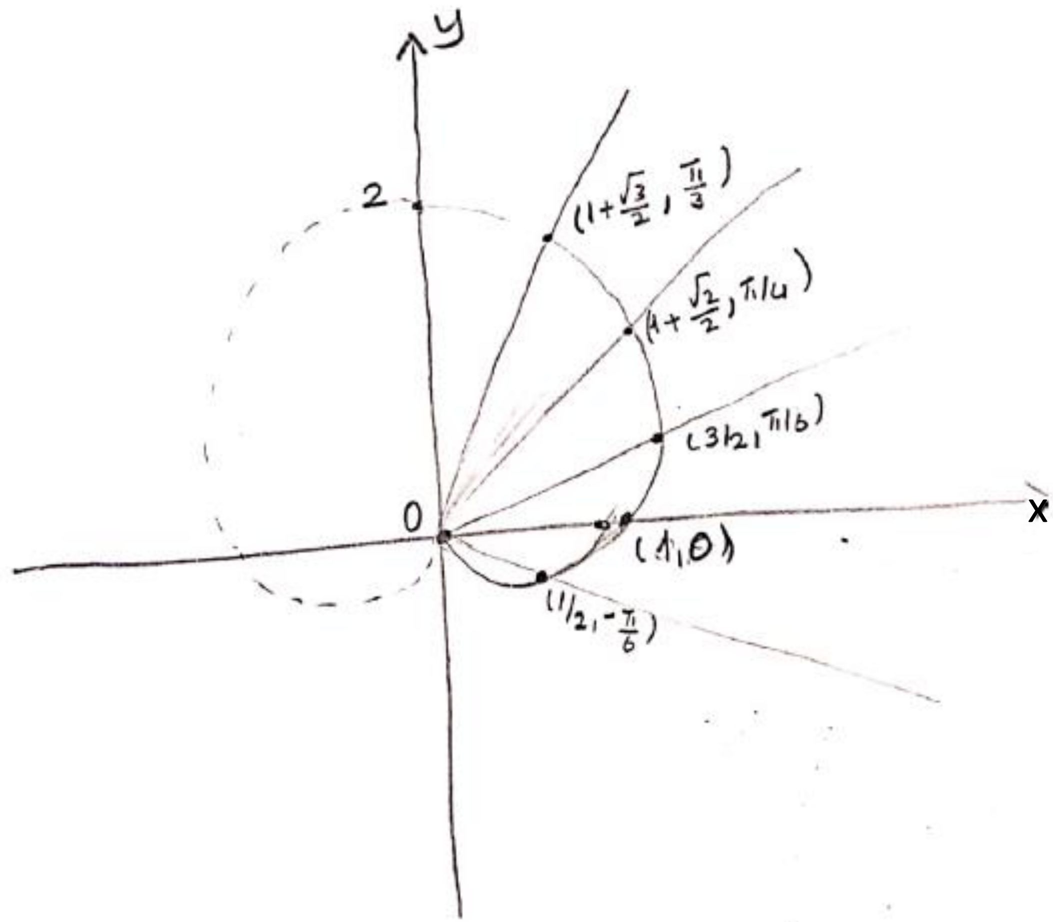
$r = 1 + \sin\pi\cos\theta - \cos\pi\sin\theta$

$r = 1 + \sin\theta$ ✓

It is symmetric respect to y-axis.

$\theta = \frac{\pi}{2}$

θ	$r = 1 + \sin\theta$
0	1
$\frac{\pi}{6}$	$\frac{3}{2}$
$\frac{\pi}{4}$	$1 + \frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$1 + \frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	2
$-\frac{\pi}{2}\left(\frac{3\pi}{2}\right)$	0
$-\frac{\pi}{6}$	$\frac{1}{2}$
$-\frac{\pi}{4}$	$1 - \frac{\sqrt{2}}{2}$



$$r' = \cos\theta = 0$$
$$\theta = \frac{\pi}{2} \quad r = 2$$

Tangents With Polar Coordinates

We now need to discuss some calculus topics in terms of polar coordinates.

We will start with finding tangent lines to polar curves. In this case we are going to assume that the equation is in the form $r = f(\theta)$. With the equation in this form we can actually use the equation for the derivative $\frac{dy}{dx}$ we derived when we looked at **tangent lines with parametric equations.** To do this however requires us to come up with a set of parametric equations to represent the curve. This is actually pretty easy to do.

From our work in the previous section we have the following set of conversion equations for going from polar coordinates to Cartesian coordinates.

$$x = r\cos\theta \qquad y = r\sin\theta$$

Now, we'll use the fact that we're assuming that the equation is in the form $r = f(\theta)$. Substituting this into these equations gives the following set of parametric equations (with q as the parameter) for the curve.

$$x = f(\theta)\cos\theta \qquad y = f(\theta)\sin\theta$$

Now, we will need the following derivatives.

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta$$
$$= \frac{dr}{d\theta}\cos\theta - r\sin\theta$$

$$\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$$
$$= \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

The derivative $\frac{dy}{dx}$ is then,

Derivative with Polar Coordinates

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Determine the equation of the tangent line to $r = 3 + 8\sin\theta$ at $\theta = \dfrac{\pi}{6}$.

We'll first need the following derivative.

$$\frac{dr}{d\theta} = 8\cos\theta$$

The formula for the derivative $\frac{dy}{dx}$ becomes,

$$\frac{dy}{dx} = \frac{8\cos\theta\sin\theta + (3 + 8\sin\theta)\cos\theta}{8\cos^2\theta - (3 + 8\sin\theta)\sin\theta} = \frac{16\cos\theta\sin\theta + 3\cos\theta}{8\cos^2\theta - 3\sin\theta - 8\sin^2\theta}$$

The slope of the tangent line is,

$$m = \left.\frac{dy}{dx}\right|_{\theta = \frac{\pi}{6}} = \frac{4\sqrt{3} + \frac{3\sqrt{3}}{2}}{4 - \frac{3}{2}} = \frac{11\sqrt{3}}{5}$$

Now, at $\theta = \frac{\pi}{6}$ we have $r = 7$. We'll need to get the corresponding x-y coordinates so we can get the tangent line.

$$x = 7\cos\left(\frac{\pi}{6}\right) = \frac{7\sqrt{3}}{2} \qquad y = 7\sin\left(\frac{\pi}{6}\right) = \frac{7}{2}$$

The tangent line is then,

$$y = \frac{7}{2} + \frac{11\sqrt{3}}{5}\left(x - \frac{7\sqrt{3}}{2}\right)$$

Examine the horizontal and vertical tangents of the curve

$$\rho = 1+\sin\theta$$

$$\frac{dy}{dx} = \frac{\frac{d\rho}{d\theta}\sin\theta + \rho\cos\theta}{\frac{d\rho}{d\theta}\cos\theta - \rho\sin\theta}$$

$$\rho = 1+\sin\theta$$

$$= \frac{\cos\theta.\sin\theta \qquad \cos\theta}{\cos\theta.\cos\theta - (1+\sin\theta)\sin\theta}$$

$$= \frac{\cos\theta\,(1+2\sin\theta)}{\underbrace{\cos^2\theta}_{1-\sin^2\theta} - \sin\theta - \sin^2\theta}$$

$$= -\frac{\cos\theta\,(1+2\sin\theta)}{(\sin\theta+1)(2\sin\theta-1)}$$

1° $\cos\theta \cdot (1+2\sin\theta)=0$

$\theta = \frac{\pi}{2};\ \frac{3\pi}{2};\ \frac{7\pi}{6};\ \frac{11\pi}{6}$

$(\sin\theta+1)\cdot(2\sin\theta-1)=0$

$\theta = \frac{3\pi}{2},\ \frac{\pi}{6};\ \frac{5\pi}{6}$

There is a horizontal tangent at

2° $\left(\frac{\pi}{2}, 2\right), \left(\frac{7\pi}{6}, \frac{1}{2}\right), \left(\frac{11\pi}{6}, \frac{1}{2}\right)$

There is a vertical tangent at

$\left(\frac{\pi}{6}, \frac{3}{2}\right), \left(\frac{5\pi}{6}, \frac{3}{2}\right)$ and

at $\theta = \frac{3\pi}{2}$

a) The equation $r = 1 - \cos\theta$ of the curve is given in polar form

Sketch the graph.
Determine the equation of the curve in cartesian coordinates.

Find the slope of the tangent line of the curve at $\theta = \dfrac{\pi}{2}$.
Find the points on the curve where the tangent line is vertical.

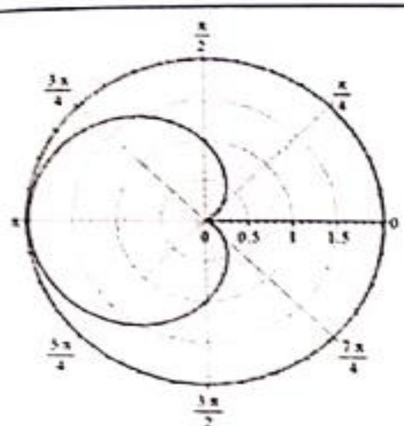
Find the points on the curve where the tangent line is horizontal.

because

is symmetric recpect to (x-axis) .

i) $r(\theta) = r(-\theta)$, $r = 1 - \cos\theta$

$r = 1 - \cos\theta$



	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2

ii)

$$\left.\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned}\right\} \quad , r = 1-\cos\theta \Rightarrow r = 1-\frac{x}{r}$$

$$r^2 - r + x = 0$$

$$\boxed{x^2 + y^2 - \sqrt{x^2+y^2} + x = 0}$$

iii)

$$\left.\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned}\right\} \Rightarrow \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(1-\cos\theta)(1+2\cos\theta)}{\sin\theta(2\cos\theta-1)}, \quad \boxed{\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{2}} = -1}$$

Vertical tangent

iv)

$$\frac{dy}{d\theta} \neq 0\,, \quad \frac{dx}{d\theta} = 0$$

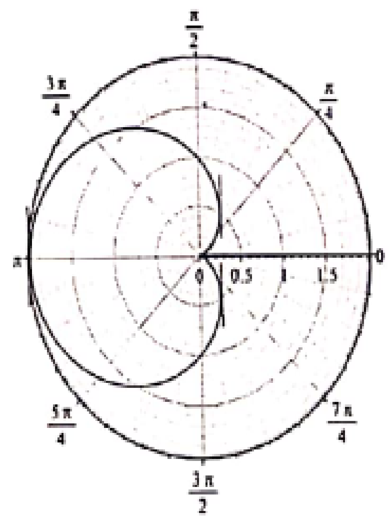
$$\left.\begin{aligned} \frac{dy}{d\theta} &= (1-\cos\theta)(1+2\cos\theta) \\ \frac{dx}{d\theta} &= \sin\theta(2\cos\theta-1) \end{aligned}\right\}$$

$$\frac{dy}{d\theta} = 0 \Rightarrow \theta = 0,\ \theta = \frac{2\pi}{3},\ \theta = \frac{4\pi}{3}, \theta = 2\pi$$

$$\frac{dx}{d\theta} = 0 \Rightarrow \theta = 0,\ \theta = \frac{\pi}{3},\ \theta = \pi,\ \theta = \frac{5\pi}{3},\ \theta = 2\pi$$

$$(r,\theta): \boxed{\left(\frac{1}{2},\frac{\pi}{3}\right)}, \boxed{\left(\frac{1}{2},\frac{5\pi}{3}\right)}, \boxed{(2,\pi)}$$

Vertical tangent points



Horizontal tangent

v)

$$\frac{dy}{d\theta}=0\,,\quad \frac{dx}{d\theta}\neq 0$$

Horizontal tangent points

$$(r,\theta):\boxed{\left(\frac{3}{2},\frac{2\pi}{3}\right)},\ \boxed{\left(\frac{3}{2},\frac{4\pi}{3}\right)}$$

For the Pole point

$$\theta=0$$

$$\frac{dy}{d\theta}=0 \text{ ve } \frac{dx}{d\theta}=0 \text{ dır.}$$