Sums and Sigma Notation

Sigma notation

If m and n are integers with $m \le n$, and if f is a function defined at the integers $m, m+1, m+2, \ldots, n$, the symbol $\sum_{i=m}^{n} f(i)$ represents the sum of the values of f at those integers:

$$\sum_{i=m}^{n} f(i) = f(m) + f(m+1) + f(m+2) + \cdots + f(n).$$

The explicit sum appearing on the right side of this equation is the **expansion** of the sum represented in sigma notation on the left side.

EXAMPLE 1 $$\sum_{i=1}^{5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55.$$

The i that appears in the symbol $\sum_{i=m}^{n} f(i)$ is called an **index of summation**. To evaluate $\sum_{i=m}^{n} f(i)$, replace the index i with the integers $m, m+1, \ldots, n$, successively, and sum the results.

$$\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55.$$

The index of summation is a *dummy variable* used to represent an arbitrary point where the function is evaluated to produce a term to be included in the sum. On the other hand, the sum $\sum_{i=m}^{n} f(i)$ does depend on the two numbers m and n, called the **limits of summation**; m is the **lower limit**, and n is the **upper limit**.

EXAMPLE 2 **(Examples of sums using sigma notation)**

$$\sum_{j=1}^{20} j = 1 + 2 + 3 + \cdots + 18 + 19 + 20$$

$$\sum_{i=0}^{n} x^i = x^0 + x^1 + x^2 + \cdots + x^{n-1} + x^n$$

$$\sum_{m=1}^{n} 1 = \underbrace{1 + 1 + 1 + \cdots + 1}_{n \text{ terms}}$$

$$\sum_{k=-2}^{3} \frac{1}{k+7} = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10}$$

Sometimes we use a subscripted variable a_i to denote the ith term of a general sum instead of using the functional notation $f(i)$:

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_n.$$

In particular, an **infinite series** is such a sum with infinitely many terms:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots.$$

When adding finitely many numbers, the order in which they are added is unimportant; any order will give the same sum. If all the numbers have a common factor, then that factor can be removed from each term and multiplied after the sum is evaluated: $ca + cb = c(a + b)$. These laws of arithmetic translate into the following *linearity* rule for finite sums; if A and B are constants, then

$$\sum_{i=m}^{n}\big(Af(i) + Bg(i)\big) = A\sum_{i=m}^{n} f(i) + B\sum_{i=m}^{n} g(i).$$

Both of the sums $\sum_{j=m}^{m+n} f(j)$ and $\sum_{i=0}^{n} f(i+m)$ have the same expansion, namely, $f(m) + f(m+1) + \cdots + f(m+n)$. Therefore, the two sums are equal.

$$\sum_{j=m}^{m+n} f(j) = \sum_{i=0}^{n} f(i+m).$$

This equality can also be derived by substituting $i + m$ for j everywhere j appears on the left side, noting that $i + m = m$ reduces to $i = 0$, and $i + m = m + n$ reduces to $i = n$. It is often convenient to make such a **change of index** in a summation.

EXAMPLE 3 Express $\sum_{j=3}^{17} \sqrt{1 + j^2}$ in the form $\sum_{i=1}^{n} f(i)$.

Solution Let $j = i + 2$, Then $j = 3$ corresponds to $i = 1$ and $j = 17$ corresponds to $i = 15$. Thus,

$$\sum_{j=3}^{17} \sqrt{1 + j^2} = \sum_{i=1}^{15} \sqrt{1 + (i + 2)^2}.$$

Evaluating Sums

There is a **closed form** expression for the sum S of the first n positive integers, namely,

$$S = \sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

To see this, write the sum forwards and backwards and add the two to get

$$\begin{array}{rccccccccccc}
S = & 1 & + & 2 & + & 3 & + \cdots + & (n-1) & + & n \\
S = & n & + & (n-1) & + & (n-2) & + \cdots + & 2 & + & 1 \\
\hline
2S = & (n+1) & + & (n+1) & + & (n+1) & + \cdots + & (n+1) & + & (n+1) & = n(n+1)
\end{array}$$

The formula for S follows when we divide by 2.

It is not usually this easy to evaluate a general sum in closed form. We can only simplify $\sum_{i=m}^{n} f(i)$ for a small class of functions f.

THEOREM **Summation formulas**

(a) $$\sum_{i=1}^{n} 1 = \underbrace{1+1+1+\cdots+1}_{n \text{ terms}} = n.$$

(b) $$\sum_{i=1}^{n} i = 1+2+3+\cdots+n = \frac{n(n+1)}{2}.$$

(c) $$\sum_{i=1}^{n} i^2 = 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}.$$

(d) $$\sum_{i=1}^{n} r^{i-1} = 1+r+r^2+r^3+\cdots+r^{n-1} = \frac{r^n-1}{r-1} \quad \text{if } r \neq 1.$$

Evaluate $\sum_{k=m+1}^{n} (6k^2 - 4k + 3)$, where $1 \leq m < n$.

Solution Using the rules of summation and various summation formulas from Theorem we calculate

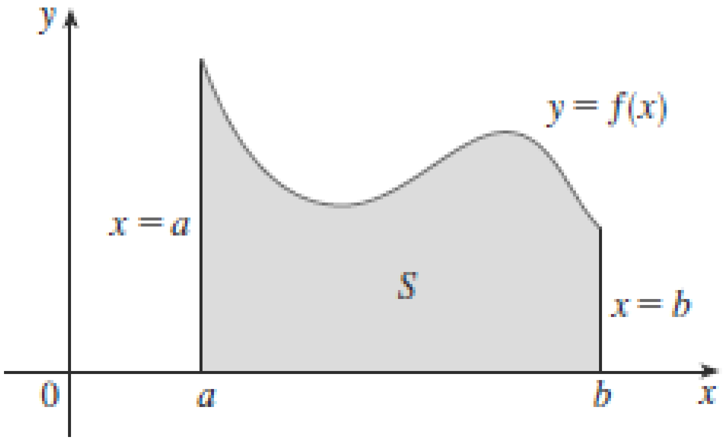
$$\begin{aligned}\sum_{k=1}^{n}(6k^2 - 4k + 3) &= 6\sum_{k=1}^{n} k^2 - 4\sum_{k=1}^{n} k + 3\sum_{k=1}^{n} 1 \\ &= 6\,\frac{n(n+1)(2n+1)}{6} - 4\,\frac{n(n+1)}{2} + 3n \\ &= 2n^3 + n^2 + 2n\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{k=m+1}^{n} (6k^2 - 4k + 3) &= \sum_{k=1}^{n}(6k^2 - 4k + 3) - \sum_{k=1}^{m}(6k^2 - 4k + 3) \\ &= 2n^3 + n^2 + 2n - 2m^3 - m^2 - 2m.\end{aligned}$$

The Area Problem

We begin by attempting to solve the *area problem:* Find the area of the region S that lies under the curve $y = f(x)$ from a to b. This means that S, is bounded by the graph of a continuous function f [where $f(x) \geqslant 0$], the vertical lines $x = a$ and $x = b$, and the x-axis.



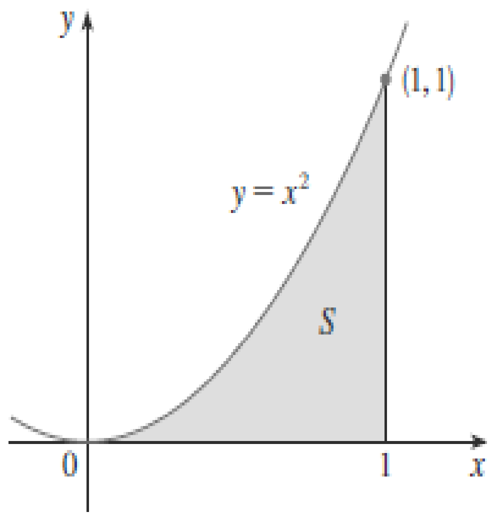
$$S = \{(x, y) \mid a \leqslant x \leqslant b, 0 \leqslant y \leqslant f(x)\}$$

FIGURE 1

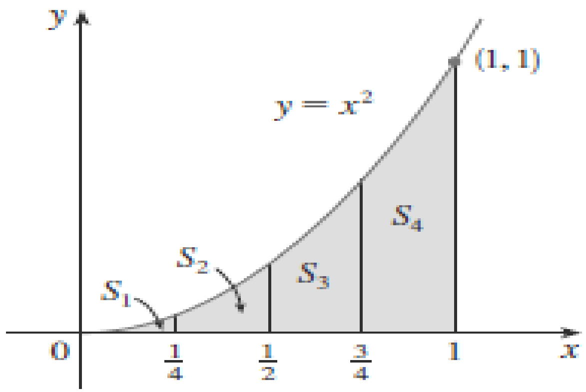
However, it isn't so easy to find the area of a region with curved sides. We all have an intuitive idea of what the area of a region is. But part of the area problem is to make this intuitive idea precise by giving an exact definition of area.

Recall that in defining a tangent we first approximated the slope of the tangent line by slopes of secant lines and then we took the limit of these approximations. We pursue a similar idea for areas. We first approximate the region S by rectangles and then we take the limit of the areas of these rectangles as we increase the number of rectangles. The following example illustrates the procedure.

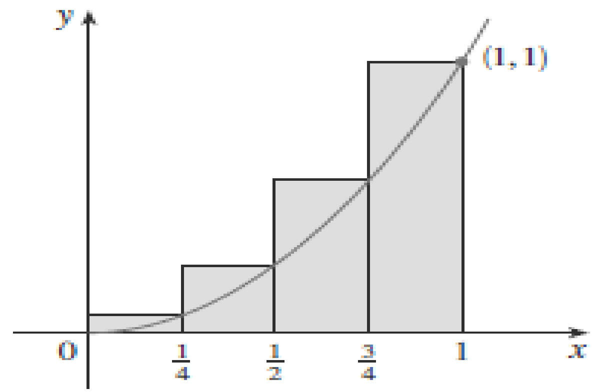
EXAMPLE 1 Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1 (the parabolic region S illustrated in Figure



SOLUTION We first notice that the area of S must be somewhere between 0 and 1 because S is contained in a square with side length 1, but we can certainly do better than that. Suppose we divide S into four strips S_1, S_2, S_3, and S_4 by drawing the vertical lines $x = \frac{1}{4}$, $x = \frac{1}{2}$, and $x = \frac{3}{4}$ as in Figure · (a).



(a)



(b)

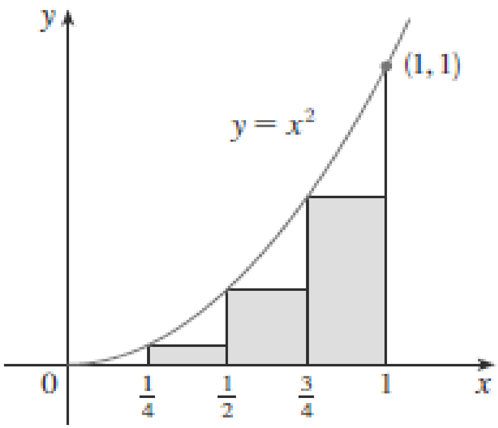
We can approximate each strip by a rectangle whose base is the same as the strip and whose height is the same as the right edge of the strip [see Figure (b)]. In other words, the heights of these rectangles are the values of the function $f(x) = x^2$ at the right end points of the subintervals $\left[0, \frac{1}{4}\right]$, $\left[\frac{1}{4}, \frac{1}{2}\right]$, $\left[\frac{1}{2}, \frac{3}{4}\right]$, and $\left[\frac{3}{4}, 1\right]$.

Each rectangle has width $\frac{1}{4}$ and the heights are $\left(\frac{1}{4}\right)^2$, $\left(\frac{1}{2}\right)^2$, $\left(\frac{3}{4}\right)^2$, and 1^2. If we let R_4 be the sum of the areas of these approximating rectangles, we get

$$R_4 = \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 + \frac{1}{4} \cdot 1^2 = \frac{15}{32} = 0.46875$$

From Figure (b) we see that the area A of S is less than R_4, so

$$A < 0.46875$$



Instead of using the rectangles in Figure (b) we could use the smaller rectangles in Figure whose heights are the values of f at the left endpoints of the subintervals. (The leftmost rectangle has collapsed because its height is 0.) The sum of the areas of these approximating rectangles is

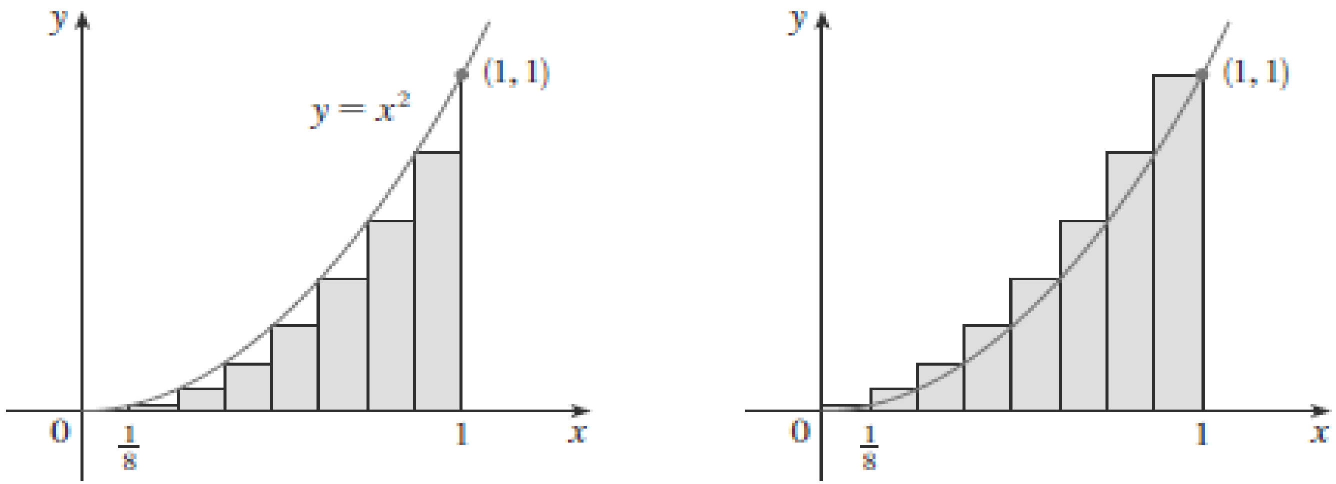
$$L_4 = \tfrac{1}{4} \cdot 0^2 + \tfrac{1}{4} \cdot \left(\tfrac{1}{4}\right)^2 + \tfrac{1}{4} \cdot \left(\tfrac{1}{2}\right)^2 + \tfrac{1}{4} \cdot \left(\tfrac{3}{4}\right)^2 = \tfrac{7}{32} = 0.21875$$

We see that the area of S is larger than L_4, so we have lower and upper estimates for A:

$$0.21875 < A < 0.46875$$

Following

We can repeat this procedure with a larger number of strips. Figure shows what happens when we divide the region S into eight strips of equal width.



(a) Using left endpoints (b) Using right endpoints

By computing the sum of the areas of the smaller rectangles (L_8) and the sum of the areas of the larger rectangles (R_8), we obtain better lower and upper estimates for A:

$$0.2734375 < A < 0.3984375$$

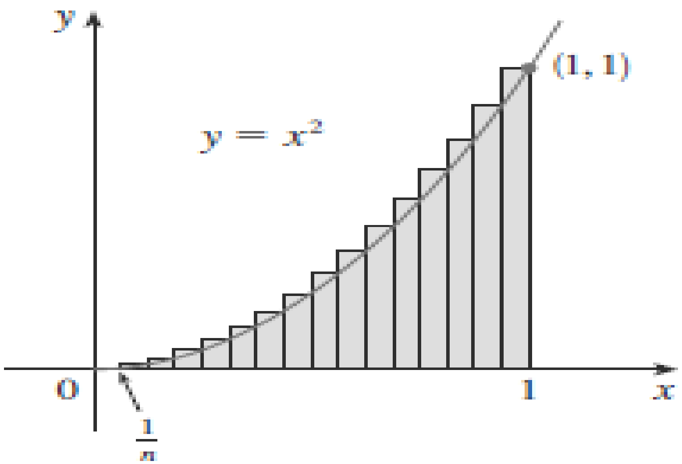
So one possible answer to the question is to say that the true area of S lies somewhere between 0.2734375 and 0.3984375.

We could obtain better estimates by increasing the number of strips.

EXAMPLE 2 For the region S in Example 1, show that the sum of the areas of the upper approximating rectangles approaches $\frac{1}{3}$, that is,

$$\lim_{n \to \infty} R_n = \frac{1}{3}$$

SOLUTION R_n is the sum of the areas of the n rectangles in Figure '. Each rectangle has width $1/n$ and the heights are the values of the function $f(x) = x^2$ at the points $1/n, 2/n, 3/n, \ldots, n/n$; that is, the heights are $(1/n)^2, (2/n)^2, (3/n)^2, \ldots, (n/n)^2$.



Thus

$$R_n = \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \frac{1}{n}\left(\frac{3}{n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{n}{n}\right)^2$$

$$= \frac{1}{n} \cdot \frac{1}{n^2}(1^2 + 2^2 + 3^2 + \cdots + n^2)$$

$$= \frac{1}{n^3}(1^2 + 2^2 + 3^2 + \cdots + n^2)$$

Here we need the formula for the sum of the squares of the first n positive integers:

$$\boxed{1} \qquad 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Putting Formula 1 into our expression for R_n, we get

$$R_n = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2}$$

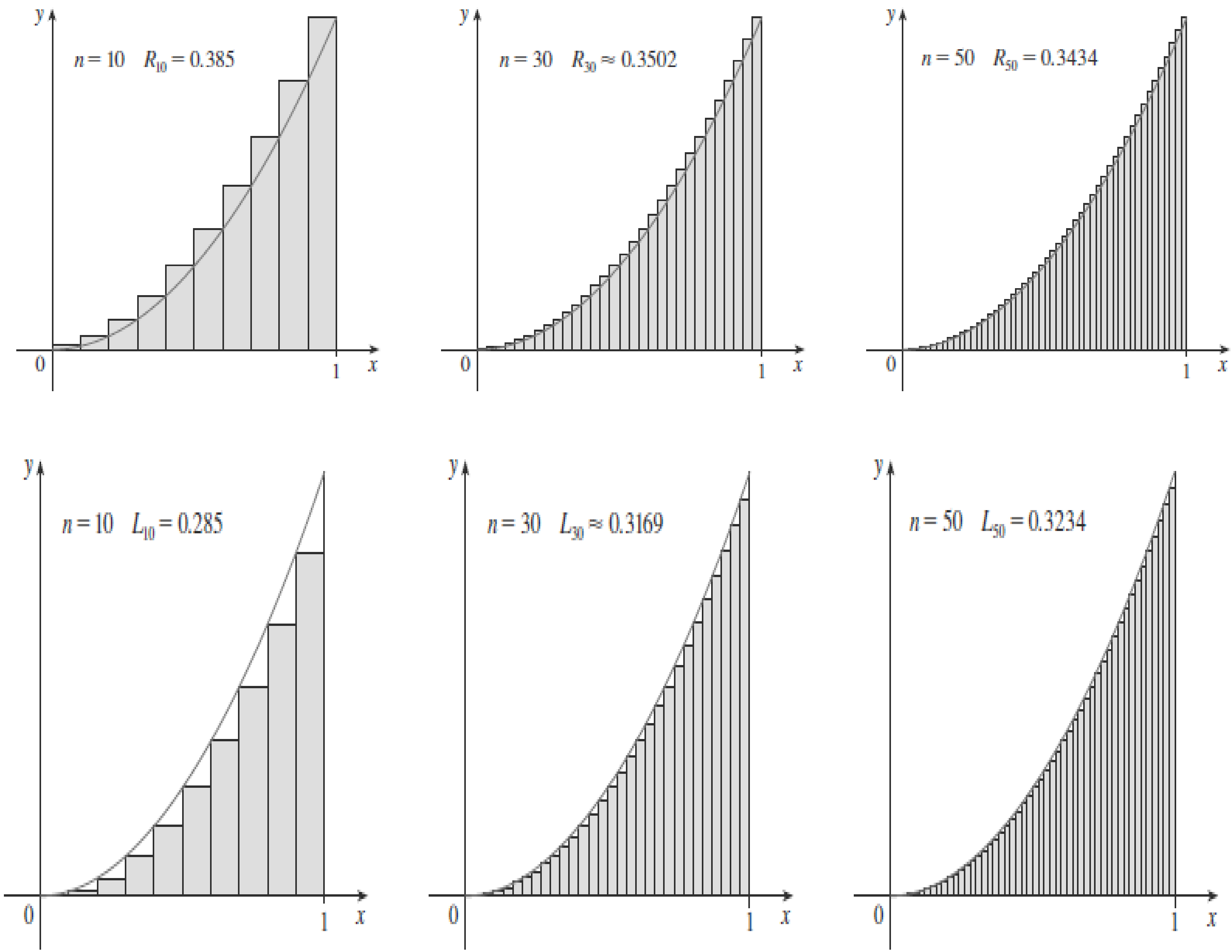
Thus, we have

$$\begin{aligned}
\lim_{n\to\infty} R_n &= \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} \\
&= \lim_{n\to\infty} \frac{1}{6}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) \\
&= \lim_{n\to\infty} \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) \\
&= \tfrac{1}{6} \cdot 1 \cdot 2 = \tfrac{1}{3}
\end{aligned}$$

It can be shown that the lower approximating sums also approach $\frac{1}{3}$, that is,

$$\lim_{n\to\infty} L_n = \frac{1}{3}$$

From Figures it appears that, as n increases, both L_n and R_n become better and better approximations to the area of S.



Therefore, we *define* the area A to be the limit of the sums of the areas of the approximating rectangles, that is,

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} L_n = \frac{1}{3}$$

Let's apply the idea of Examples 1 and 2 to the more general region S of Figure 1. We start by subdividing S into n strips $S_1, S_2, \ldots, S_n$ of equal width as in Figure . The width of the interval $[a, b]$ is $b - a$, so the width of each of the n strips is

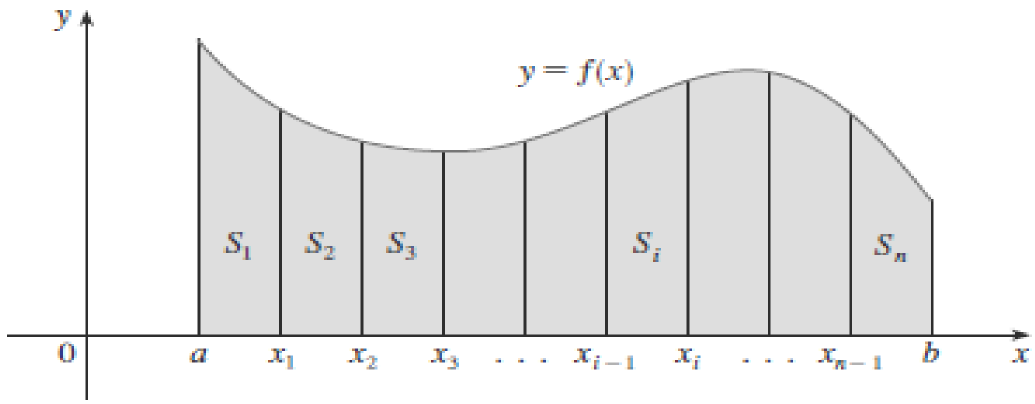
$$\Delta x = \frac{b - a}{n}$$

These strips divide the interval $[a, b]$ into n subintervals

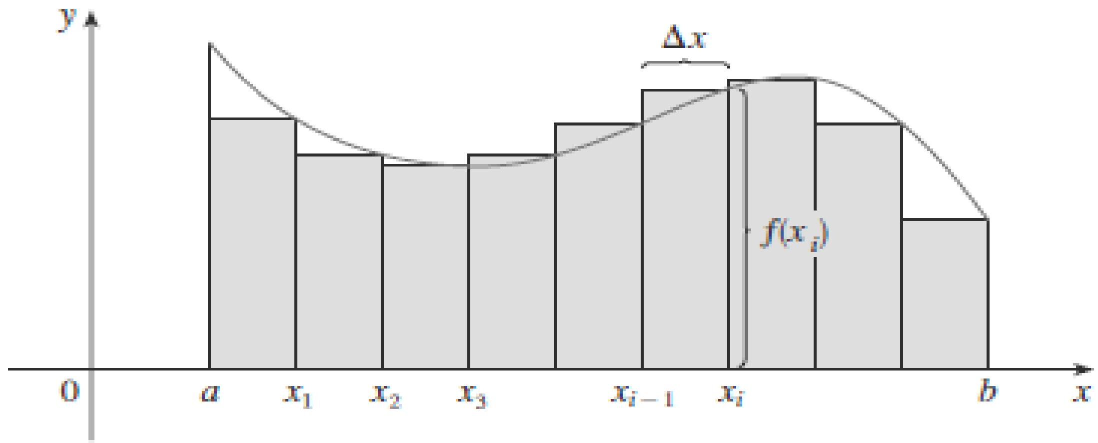
$$[x_0, x_1], \quad [x_1, x_2], \quad [x_2, x_3], \quad \ldots, \quad [x_{n-1}, x_n]$$

where $x_0 = a$ and $x_n = b$. The right endpoints of the subintervals are

$$x_1 = a + \Delta x, \quad x_2 = a + 2\,\Delta x, \quad x_3 = a + 3\,\Delta x, \quad \ldots$$

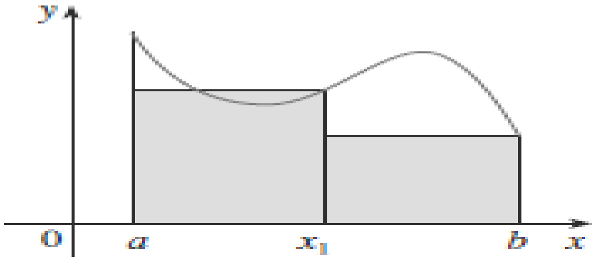


Let's approximate the ith strip S_i by a rectangle with width Δx and height $f(x_i)$, which is the value of f at the right endpoint (see Figure). Then the area of the ith rectangle

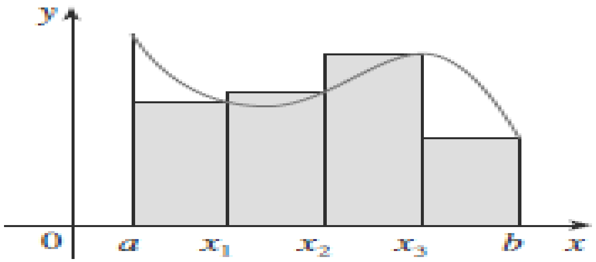


is $f(x_i)\,\Delta x$. What we think of intuitively as the area of S is approximated by the sum of the areas of these rectangles, which is

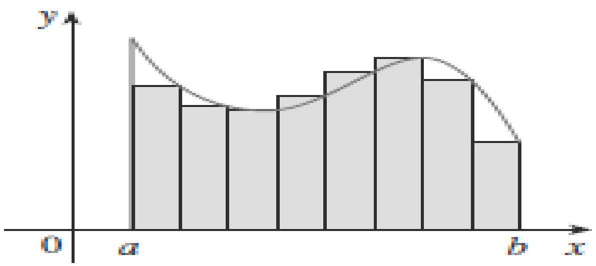
$$R_n = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$$



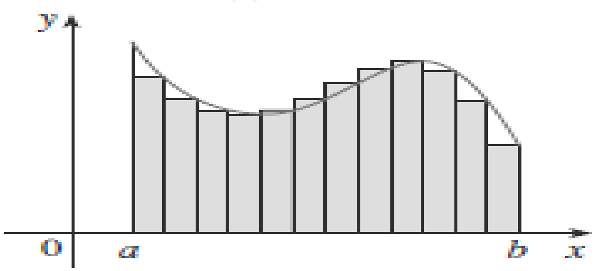
(a) $n = 2$



(b) $n = 4$



(c) $n = 8$



(d) $n = 12$

Figure shows this approximation for $n = 2, 4, 8$, and 12. Notice that this approximation appears to become better and better as the number of strips increases, that is, as $n \to \infty$. Therefore, we define the area A of the region S in the following way.

$\boxed{2}$ **Definition** The area A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

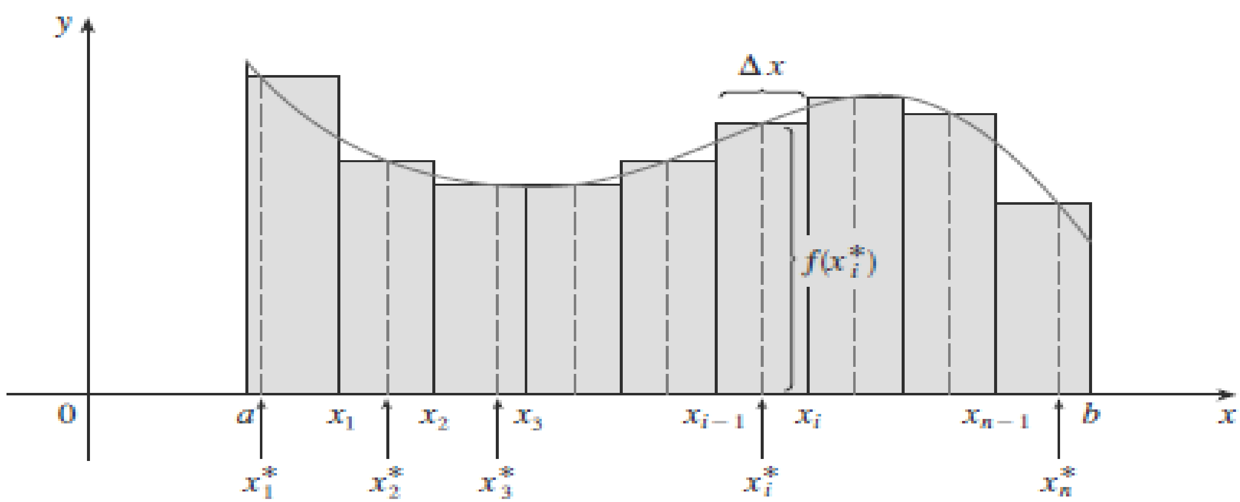
$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} [f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x]$$

It can be proved that the limit in Definition 2 always exists, since we are assuming that f is continuous. It can also be shown that we get the same value if we use left endpoints:

$$\boxed{3} \qquad A = \lim_{n \to \infty} L_n = \lim_{n \to \infty} [f(x_0)\,\Delta x + f(x_1)\,\Delta x + \cdots + f(x_{n-1})\,\Delta x]$$

In fact, instead of using left endpoints or right endpoints, we could take the height of the ith rectangle to be the value of f at *any* number x_i^* in the ith subinterval $[x_{i-1}, x_i]$. We call the numbers $x_1^*, x_2^*, \ldots, x_n^*$ the **sample points**. Figure shows approximating rectangles when the sample points are not chosen to be endpoints. So a more general expression for the area of S is

$$\boxed{4} \qquad A = \lim_{n \to \infty} [f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x]$$



We often use **sigma notation** to write sums with many terms more compactly. For instance,

$$\sum_{i=1}^{n} f(x_i)\,\Delta x = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$$

This tells us to end with $i = n$.

This tells us to add.

$$\sum_{i=m}^{n} f(x_i)\,\Delta x$$

This tells us to start with $i = m$.

So the expressions for area in Equations 2, 3, and 4 can be written as follows:

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

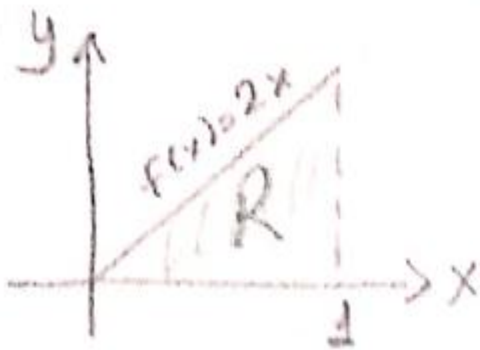
$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

We could also rewrite Formula 1 in the following way:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

1



$f(x)=2x$; R is bounded by

$f(x)=2x$, x-axis and $x=1$

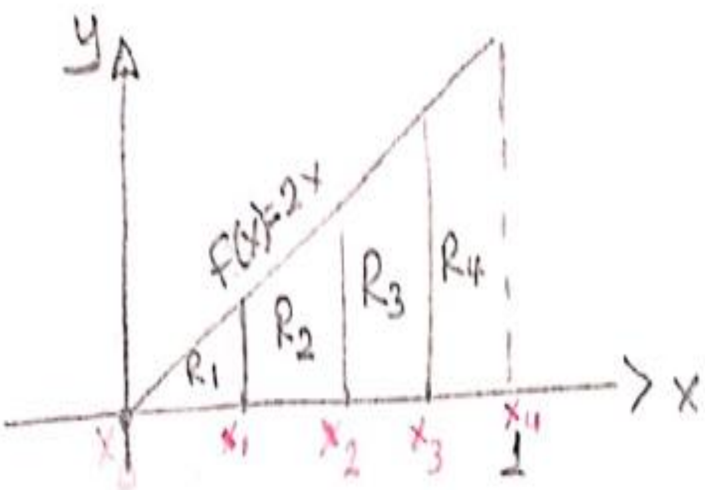
R is a right triangle

b: base

h: height

$$A=\frac{1}{2}\cdot b\cdot h=\frac{1}{2}\cdot 1\ 2=1.$$

Divide $[0,1]$ into four subintervals of equal length for $n=4$;



$x_0 = 0$

$x_1 = 1/4$

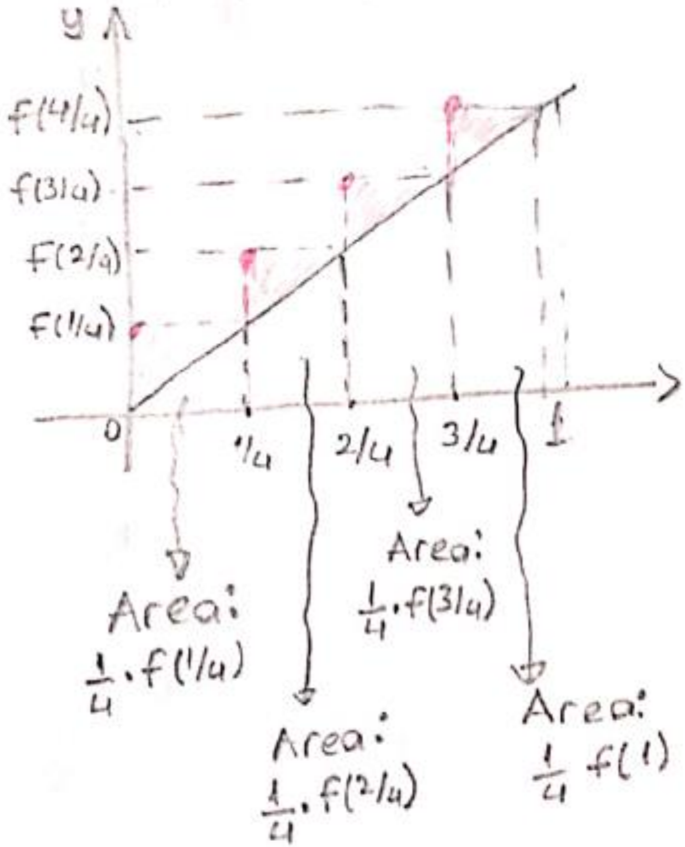
$x_2 = 2/4$

$x_3 = 3/4$

$x_4 = 4/4 = 1$

$\Delta x = \frac{1}{4}$ length of subintervals

R_1, R_2, R_3, R_4 subregions of R



$f(x) = y = 2x$

$f(1/4) = 2 \cdot 1/4$

$f(2/4) = 2 \cdot 2/4$

$f(3/4) = 2 \cdot 3/4$

$f(4/4) = 2 \cdot 4/4$

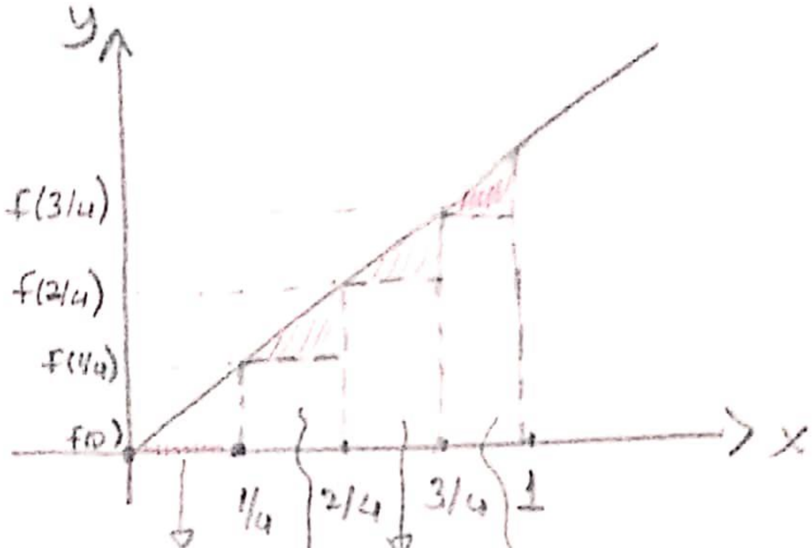
Sum of these areas:

$$\overline{S_4} = \frac{1}{4}\left[2 \cdot 1/4 + 2 \cdot 2/4 + 2 \cdot 3/4 + 2 \cdot 1\right] = \frac{5}{4}$$

$$\overline{S_4} = \sum_{i=1}^{4} f(x_i) \cdot \Delta x_i$$

$\overline{S_4} >$ area of the triangle

2



Area: $\frac{1}{4} \cdot f(0)$

Area: $\frac{1}{4} \cdot f(1/4)$

Area: $\frac{1}{4} \cdot f(2/4)$

Area: $\frac{1}{4} \cdot f(3/4)$

$f(x) = 2x$

$f(0) = 0$

$f(1/4) = 2/4$

$f(2/4) = 2 \cdot 2/4 = 1$

$f(3/4) = 2 \cdot 3/4 = 3/2$

$$\underline{S_4} = \frac{1}{4}\left[f(0) + f(1/4) + f(2/4) + f(3/4)\right]$$

$$= \frac{1}{4}\left[0 + 1/2 + 1 + 3/2\right] = 3/4$$

$$\underline{S_4} = \sum_{i=1}^{3} f(x_i)\cdot\Delta x_i$$

$\underline{S_4}$ < the area of triangle

so; $\underline{S_4} \leq A \leq \overline{S_4}$

More generally, if we divide $[0,1]$ into n subintervals of equal length Δx; then

$$\Delta x = \frac{1}{n}$$

$$\overline{S_n} = \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\cdot\Delta x = \frac{n+1}{n}$$

$$\underline{S_n} = \sum_{k=1}^{n-1} f\left(\frac{k}{n}\right)\cdot\Delta x = \frac{n-1}{n}$$

$\Rightarrow$ As n becomes larger $\underline{S_n}$ and $\overline{S_n}$ become better approximation to A.

$$\overline{S_n} = \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \cdot \Delta x \qquad f(x) = y = 2x$$

$$f\left(\frac{k}{n}\right) = 2 \cdot \frac{k}{n} \qquad \Delta x = \frac{1}{n}$$

$$= \sum_{k=1}^{n} \frac{2k}{n^2}$$

$$= \frac{2}{n^2} \sum_{k=1}^{n} k = \frac{\cancel{2}}{\cancel{n^2}_{\,n}} \cdot \frac{\cancel{n} \cdot (n+1)}{\cancel{2}} = \frac{n+1}{n}$$

$$\underline{S_n} = \sum_{k=1}^{n} f\left(\frac{k-1}{n}\right) \cdot \Delta x$$

$$= \sum_{k=1}^{n} 2\left(\frac{k-1}{n}\right) \cdot \frac{1}{n}$$

$$= \frac{2}{n^2} \sum_{k=1}^{n} (k-1)$$

$$= \frac{2}{n^2} \sum_{k=1}^{n-1} k$$

$$= \frac{\cancel{2}}{\cancel{n^2}_{\,n}} \frac{(n-1) \cdot \cancel{n}}{\cancel{2}} = \frac{n-1}{n}$$

$$\left.\begin{array}{l}\lim\limits_{n\to\infty} \underline{S_n} = \lim\limits_{n\to\infty} \dfrac{n-1}{n} = 1 \\[2em] \lim\limits_{n\to\infty} \overline{S_n} = \lim\limits_{n\to\infty} \dfrac{n+1}{n} = 1\end{array}\right\} \Rightarrow A = 1$$

The common limit of $\overline{S_n}$ and $\underline{S_n}$ as $n\to\infty$ if it exists, is called the definite integral of f over $[a,b]$ and is written

$$\int_a^b f(x)\,dx$$

a : lower limit

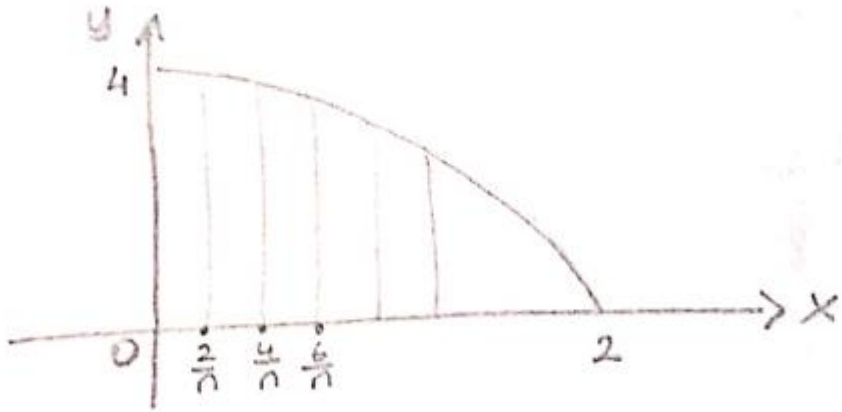
b : upper limit

That is,

$$\lim_{n\to\infty} \sum_{i=0}^{n} f(x_i)\Delta x_i = \int_a^b f(x)\,dx$$

Ex: Find the area of the region in the 1^{st} quadrant bounded by $f(x) = 4 - x^2$ and the lines $x = 0$ and $y = 0$

$$\Delta x = \frac{2}{n}$$



$$\sum_{i=1}^{n} f(x_i)\cdot\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$

$$= f\left(\frac{2}{n}\right)\cdot\frac{2}{n} + f\left(\frac{4}{n}\right)\cdot\frac{2}{n} + \cdots + f\left(\frac{2n}{n}\right)\cdot\frac{2}{n}$$

$$= \frac{2}{n}\left\{\left[\left(4 - \left(\frac{2}{n}\right)^2\right) + \left(4 - \left(\frac{2\cdot 2}{n}\right)^2\right) + \cdots + \left(4 - \left(n\cdot\frac{2}{n}\right)^2\right)\right]\right\}$$

$$= \frac{2}{n}\left[4n - \left(\frac{2}{n}\right)^2\left\{1^2 + 2^2 + \cdots + n^2\right\}\right]$$

$$= \frac{2}{n}\left[4n - \left(\frac{2}{n}\right)^2 \left\{ \frac{n(n+1)(2n+1)}{6} \right\} \right]$$

$$= \frac{2}{\cancel{n}} \cdot 4\cancel{n} - \frac{2}{\cancel{n}} \left(\frac{2}{n}\right)^2 \left\{ \frac{\cancel{n}(n+1)(2n+1)}{6} \right\}$$

$$= 8 - \frac{8}{6n^2} \cdot (2n^2 + 3n + 1)$$

$$\lim_{n \to \infty} S_n = 8 - \frac{8}{3} = \frac{16}{3}$$

Evaluate the following Definite integral by using Riemann sums.

$$\int_0^3 \left(9-x^2\right)dx$$

For the right edgeof equal length subintervals

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x, \quad \Delta x = \frac{b-a}{n}, \quad x_i = a + i\Delta x, \quad i = 0\ldots n.$$

$$\Delta x = \frac{3-0}{n} = \frac{3}{n}, \; x_i = 0 + i\frac{3}{n} = \frac{3i}{n}, \quad i = 0\ldots n. \; f(x_i) = 9 - \left(\frac{3i}{n}\right)^2$$

$$\int_0^3 \left(9-x^2\right)dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left(9-\frac{9i^2}{n^2}\right)\frac{3}{n} = \lim_{n\to\infty}\left[\frac{27}{n}\sum_{i=1}^{n}\left(1-\frac{i^2}{n^2}\right)\right] = \lim_{n\to\infty}\left[\frac{27n}{n} - \frac{27}{n^3}\sum_{i=1}^{n} i^2\right]$$

$$= \lim_{n\to\infty}\left[\frac{27n}{n} - \frac{27}{n^3}\frac{n(n+1)(2n+1)}{6}\right] = 27 - 27\left(\frac{1}{3}\right) = 18.$$

Let A be the area of the region that lies under the graph of $f(x) = e^{-x}$ between $x = 0$ and $x = 2$.

(a) Using right endpoints, find an expression for A as a limit. Do not evaluate the limit.

(b) Estimate the area by taking the sample points to be midpoints and using four subintervals and then ten subintervals.

(a) Since $a = 0$ and $b = 2$, the width of a subinterval is

$$\Delta x = \frac{2 - 0}{n} = \frac{2}{n}$$

So $x_1 = 2/n$, $x_2 = 4/n$, $x_3 = 6/n$, $x_i = 2i/n$, and $x_n = 2n/n$. The sum of the areas of the approximating rectangles is

$$\begin{aligned} R_n &= f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x \\ &= e^{-x_1}\,\Delta x + e^{-x_2}\,\Delta x + \cdots + e^{-x_n}\,\Delta x \\ &= e^{-2/n}\left(\frac{2}{n}\right) + e^{-4/n}\left(\frac{2}{n}\right) + \cdots + e^{-2n/n}\left(\frac{2}{n}\right) \end{aligned}$$

According to Definition 2, the area is

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \frac{2}{n}\left(e^{-2/n} + e^{-4/n} + e^{-6/n} + \cdots + e^{-2n/n}\right)$$

Using sigma notation we could write

$$A = \lim_{n \to \infty} \frac{2}{n} \sum_{i=1}^{n} e^{-2i/n}$$

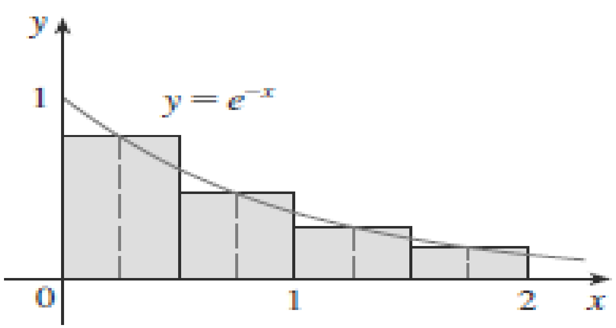
(b) With $n = 4$ the subintervals of equal width $\Delta x = 0.5$ are $[0, 0.5]$, $[0.5, 1]$, $[1, 1.5]$, and $[1.5, 2]$. The midpoints of these subintervals are $x_1^* = 0.25$, $x_2^* = 0.75$, $x_3^* = 1.25$,

and $x_4^* = 1.75$, and the sum of the areas of the four approximating rectangles is

$$\begin{aligned} M_4 &= \sum_{i=1}^{4} f(x_i^*)\,\Delta x \\ &= f(0.25)\,\Delta x + f(0.75)\,\Delta x + f(1.25)\,\Delta x + f(1.75)\,\Delta x \\ &= e^{-0.25}(0.5) + e^{-0.75}(0.5) + e^{-1.25}(0.5) + e^{-1.75}(0.5) \\ &= \tfrac{1}{2}(e^{-0.25} + e^{-0.75} + e^{-1.25} + e^{-1.75}) \approx 0.8557 \end{aligned}$$

So an estimate for the area is

$$A \approx 0.8557$$



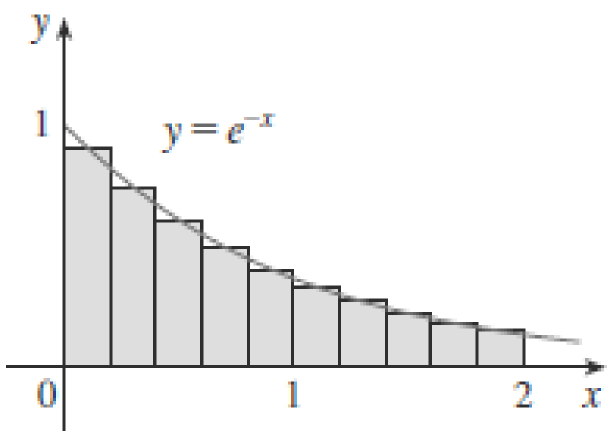
So an estimate for the area is

$$A \approx 0.8557$$

With $n = 10$ the subintervals are $[0, 0.2]$, $[0.2, 0.4]$, $\ldots$, $[1.8, 2]$ and the midpoints are $x_1^* = 0.1$, $x_2^* = 0.3$, $x_3^* = 0.5$, $\ldots$, $x_{10}^* = 1.9$. Thus

$$\begin{aligned} A \approx M_{10} &= f(0.1)\,\Delta x + f(0.3)\,\Delta x + f(0.5)\,\Delta x + \cdots + f(1.9)\,\Delta x \\ &= 0.2(e^{-0.1} + e^{-0.3} + e^{-0.5} + \cdots + e^{-1.9}) \approx 0.8632 \end{aligned}$$

From Figure it appears that this estimate is better than the estimate with $n = 4$.



Homework

Evaluate the following limit

$$A = \lim_{n \to \infty} \frac{2}{n} \sum_{i=1}^{n} e^{-2i/n}$$

The Definite Integral

1 $$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\,\Delta x = \lim_{n\to\infty}\left[f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x\right]$$

arises when we compute an area.

2 Definition of a Definite Integral If f is a continuous function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0\,(= a), x_1, x_2, \ldots, x_n\,(= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any sample points in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

Because we have assumed that f is continuous, it can be proved that the limit in Definition 2 always exists and gives the same value no matter how we choose the sample points x_i^*. If we take the sample points to be right endpoints, then $x_i^* = x_i$ and the definition of an integral becomes

3

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

If we choose the sample points to be left endpoints, then $x_i^* = x_{i-1}$ and the definition becomes

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

Alternatively, we could choose x_i^* to be the midpoint of the subinterval or any other number between x_{i-1} and x_i.

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt.$$

Upper limit of integration

The function is the integrand.

Integral sign

x is the variable of integration.

$$\underbrace{\int_a^b f(x)\,dx}_{\text{Integral of } f \text{ from } a \text{ to } b}$$

Lower limit of integration

When you find the value of the integral, you have evaluated the integral.

The various parts of the symbol $\int_a^b f(x)\,dx$ have their own names:

(i) $\int$ is called the **integral sign**; it resembles the letter S since it represents the limit of a sum.

(ii) a and b are called the **limits of integration**; a is the **lower limit**, b is the **upper limit**.

(iii) The function f is the **integrand**; x is the **variable of integration**.

(iv) dx is the **differential** of x. It replaces Δx in the Riemann sums. If an integrand depends on more than one variable, the differential tells you which one is the variable of integration.

NOTE 1 ▫ The symbol $\int$ was introduced by Leibniz and is called an **integral sign**. It is an elongated S and was chosen because an integral is a limit of sums. In the notation $\int_a^b f(x)\,dx$, $f(x)$ is called the **integrand** and a and b are called the **limits of integration**; a is the **lower limit** and b is the **upper limit**. The symbol dx has no official meaning by itself; $\int_a^b f(x)\,dx$ is all one symbol. The procedure of calculating an integral is called **integration**.

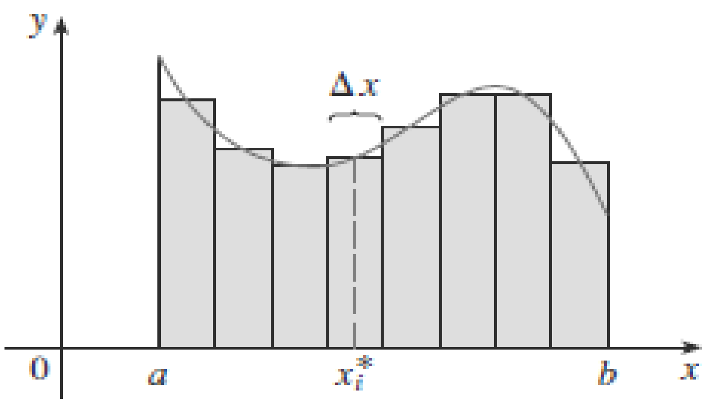
NOTE 2 ▫ The definite integral $\int_a^b f(x)\,dx$ is a number; it does not depend on x. In fact, we could use any letter in place of x without changing the value of the integral:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(r)\,dr$$

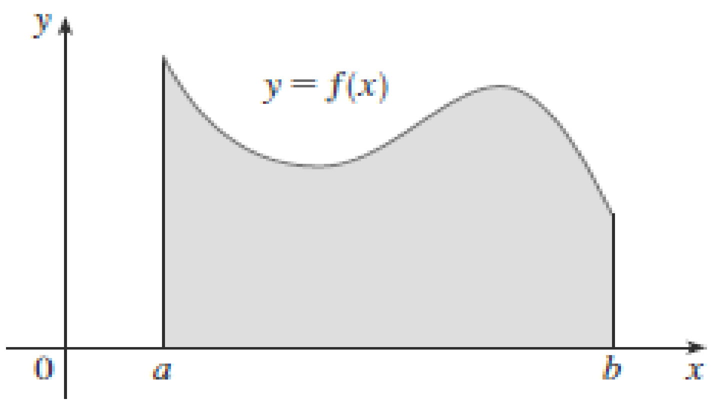
NOTE 3 ▫ The sum

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

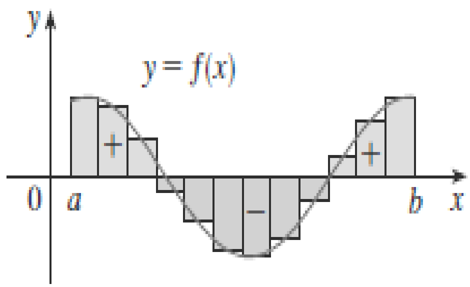
that occurs in Definition 2 is called a **Riemann sum** after the German mathematician Bernhard Riemann (1826–1866). We know that if f happens to be positive, then the Riemann sum can be interpreted as a sum of areas of approximating rectangles the definite integral $\int_a^b f(x)\,dx$ can be interpreted as the area under the curve $y = f(x)$ from a to b.



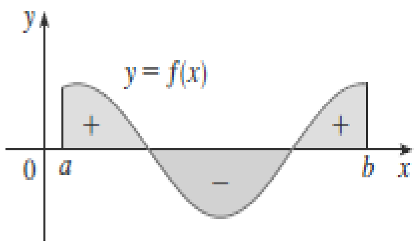
If $f(x) \geqslant 0$, the Riemann sum $\Sigma f(x_i^*)\,\Delta x$ is the sum of areas of rectangles.



If $f(x) \geqslant 0$, the integral $\int_a^b f(x)\,dx$ is the area under the curve $y = f(x)$ from a to b.



$\sum f(x_i^*)\,\Delta x$ is an approximation to the net area



$\int_a^b f(x)\,dx$ is the net area

$$\int_a^b f(x)\,dx = A_1 - A_2$$

where A_1 is the area of the region above the x-axis and below the graph of f, and A_2 is the area of the region below the x-axis and above the graph of f.

EXAMPLE 1 Express

$$\lim_{n\to\infty} \sum_{i=1}^{n} (x_i^3 + x_i \sin x_i)\,\Delta x$$

as an integral on the interval $[0, \pi]$.

SOLUTION Comparing the given limit with the limit in Definition 2, we see that they will be identical if we choose

$$f(x) = x^3 + x \sin x \qquad \text{and} \qquad x_i^* = x_i$$

(So the sample points are right endpoints and the given limit is of the form of Equation 3.) We are given that $a = 0$ and $b = \pi$.
we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} (x_i^3 + x_i \sin x_i)\,\Delta x = \int_0^{\pi} (x^3 + x \sin x)\,dx$$

LET US REMEMBER

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

The remaining formulas are simple rules for working with sigma notation:

$$\sum_{i=1}^{n} c = nc$$

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

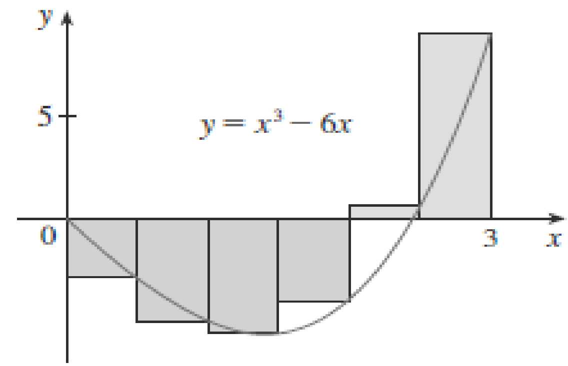
$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

EXAMPLE

(a) Evaluate the Riemann sum for $f(x) = x^3 - 6x$ taking the sample points to be right endpoints and $a = 0$, $b = 3$, and $n = 6$.

(b) Evaluate $\int_0^3 (x^3 - 6x)\,dx$.



SOLUTION

(a) With $n = 6$ the interval width is

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{6} = \frac{1}{2}$$

and the right endpoints are $x_1 = 0.5$, $x_2 = 1.0$, $x_3 = 1.5$, $x_4 = 2.0$, $x_5 = 2.5$, and $x_6 = 3.0$. So the Riemann sum is

$$\begin{aligned} R_6 &= \sum_{i=1}^{6} f(x_i)\,\Delta x \\ &= f(0.5)\,\Delta x + f(1.0)\,\Delta x + f(1.5)\,\Delta x + f(2.0)\,\Delta x + f(2.5)\,\Delta x + f(3.0)\,\Delta x \\ &= \tfrac{1}{2}(-2.875 - 5 - 5.625 - 4 + 0.625 + 9) \\ &= -3.9375 \end{aligned}$$

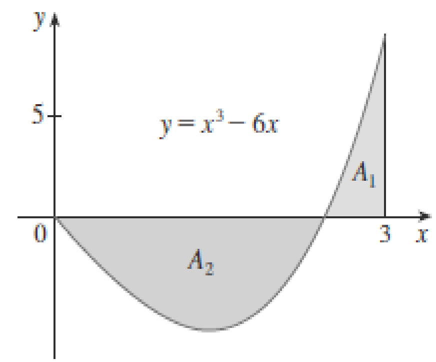
Notice that f is not a positive function and so the Riemann sum does not represent a sum of areas of rectangles. But it does represent the sum of the areas of the rectangles (above the x-axis) minus the sum of the areas of the rectangles (below the x-axis)

(b) With n subintervals we have

$$\Delta x = \frac{b-a}{n} = \frac{3}{n}$$

Thus $x_0 = 0$, $x_1 = 3/n$, $x_2 = 6/n$, $x_3 = 9/n$, and, in general, $x_i = 3i/n$. Since we are using right endpoints, we can use Equation 3:

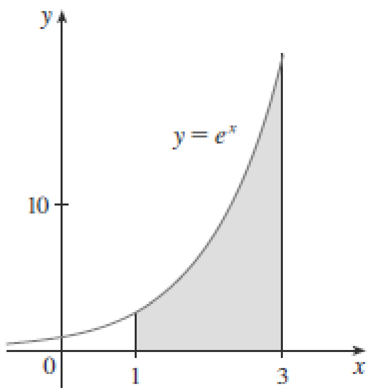
$$\begin{aligned}
\int_0^3 (x^3 - 6x)\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\frac{3}{n} \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right] \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\frac{27}{n^3} i^3 - \frac{18}{n} i\right] \\
&= \lim_{n\to\infty} \left[\frac{81}{n^4} \sum_{i=1}^{n} i^3 - \frac{54}{n^2} \sum_{i=1}^{n} i\right] \\
&= \lim_{n\to\infty} \left\{\frac{81}{n^4}\left[\frac{n(n+1)}{2}\right]^2 - \frac{54}{n^2}\,\frac{n(n+1)}{2}\right\} \\
&= \lim_{n\to\infty} \left[\frac{81}{4}\left(1+\frac{1}{n}\right)^2 - 27\left(1+\frac{1}{n}\right)\right] \\
&= \frac{81}{4} - 27 = -\frac{27}{4} = -6.75
\end{aligned}$$



$$\int_0^3 (x^3 - 6x)\,dx = A_1 - A_2 = -6.75$$

EXAMPLE

Set up an expression for $\int_1^3 e^x\,dx$ as a limit of sums.



Here we have $f(x) = e^x$, $a = 1$, $b = 3$, and

$$\Delta x = \frac{b-a}{n} = \frac{2}{n}$$

So $x_0 = 1$, $x_1 = 1 + 2/n$, $x_2 = 1 + 4/n$, $x_3 = 1 + 6/n$, and

$$x_i = 1 + \frac{2i}{n}$$

From Equation 3, we get

$$\begin{aligned}
\int_1^3 e^x\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(1 + \frac{2i}{n}\right)\frac{2}{n} \\
&= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} e^{1+2i/n}
\end{aligned}$$

Properties of the Definite Integral

When we defined the definite integral $\int_a^b f(x)\,dx$, we implicitly assumed that $a < b$. But the definition as a limit of Riemann sums makes sense even if $a > b$. Notice that if we reverse a and b, then Δx changes from $(b - a)/n$ to $(a - b)/n$. Therefore

$$\boxed{\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx}$$

If $a = b$, then $\Delta x = 0$ and so

$$\boxed{\int_a^a f(x)\,dx = 0}$$

We now develop some basic properties of integrals that will help us to evaluate integrals in a simple manner. We assume that f and g are continuous functions.

Properties of the Integral

1. $\int_a^b c\,dx = c(b-a)$, where c is any constant

2. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant

4. $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

EXAMPLE Use the properties of integrals to evaluate $\int_0^1 (4 + 3x^2)\,dx$.

SOLUTION Using Properties 2 and 3 of integrals, we have

$$\int_0^1 (4 + 3x^2)\,dx = \int_0^1 4\,dx + \int_0^1 3x^2\,dx = \int_0^1 4\,dx + 3\int_0^1 x^2\,dx$$

We know from Property 1 that

$$\int_0^1 4\,dx = 4(1-0) = 4$$

and we found that $\int_0^1 x^2\,dx = \frac{1}{3}$. So

$$\int_0^1 (4 + 3x^2)\,dx = \int_0^1 4\,dx + 3\int_0^1 x^2\,dx$$

$$= 4 + 3 \cdot \tfrac{1}{3} = 5$$

If $a \le b$ and $f(x) \le g(x)$ for $a \le x \le b$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

The **triangle inequality** for sums extends to definite integrals. If $a \le b$, then

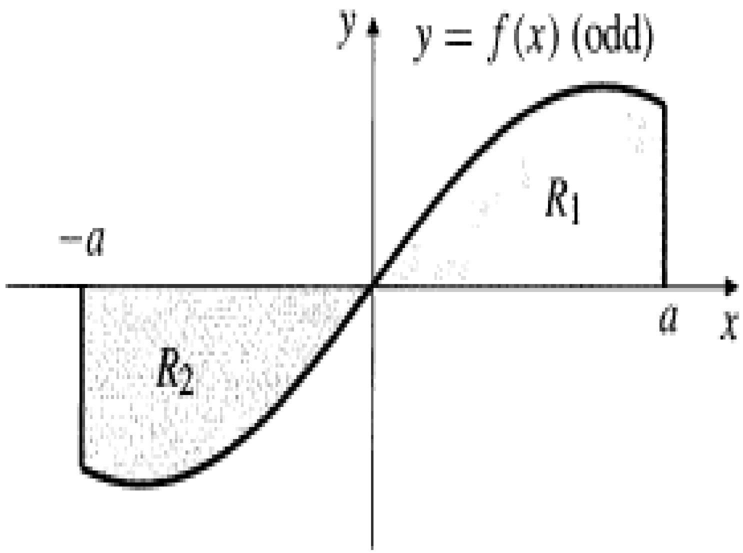
$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

The integral of an odd function over an interval symmetric about zero is zero. If f is an odd function (i.e., $f(-x) = -f(x)$), then

$$\int_{-a}^a f(x)\,dx = 0.$$

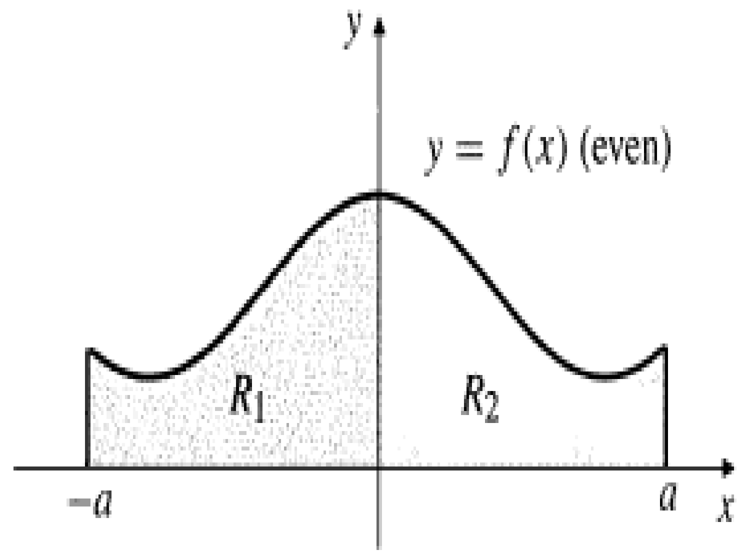
The integral of an even function over an interval symmetric about zero is twice the integral over the positive half of the interval. If f is an even function (i.e., $f(-x) = f(x)$), then

$$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx.$$



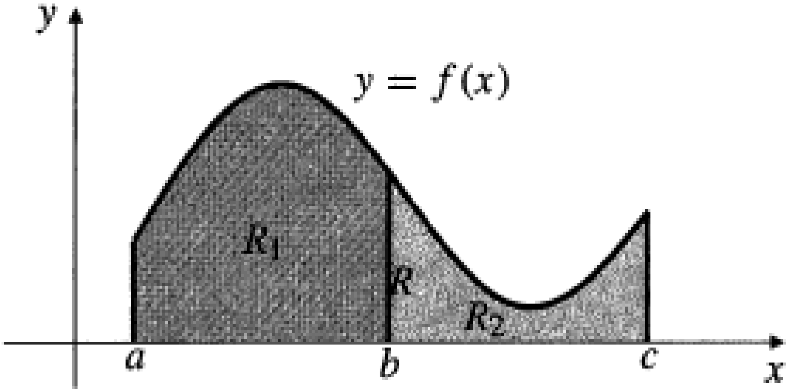
area R_1 − area R_2 = 0

$$\int_{-a}^{a} f(x)\,dx = 0$$



area R_1 + area R_2 = 2 area R_2

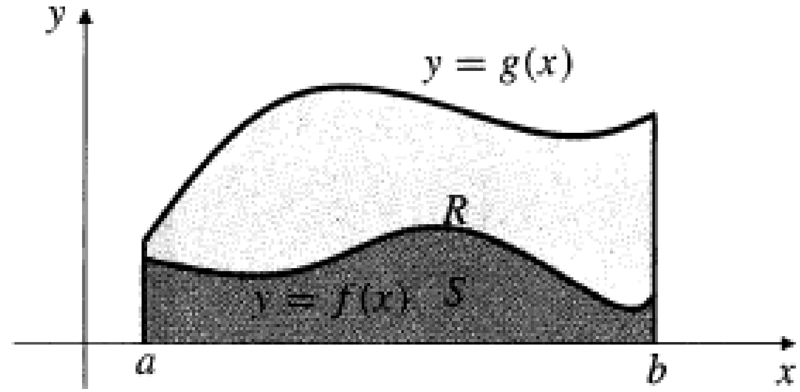
$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$



area R_1 + area R_2 = area R

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

(a)



area $S \leq$ area R

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

(b)

generalization of the triangle inequality for numbers:

$$|x+y| \leq |x| + |y|, \quad \text{or more generally,} \quad \left|\sum_{i=1}^{n} x_i\right| \leq \sum_{i=1}^{n} |x_i|\,.$$

5. $$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

EXAMPLE If it is known that $\int_0^{10} f(x)\,dx = 17$ and $\int_0^8 f(x)\,dx = 12$, find $\int_8^{10} f(x)\,dx$.

SOLUTION By Property 5, we have

$$\int_0^8 f(x)\,dx + \int_8^{10} f(x)\,dx = \int_0^{10} f(x)\,dx$$

so

$$\int_8^{10} f(x)\,dx = \int_0^{10} f(x)\,dx - \int_0^8 f(x)\,dx = 17 - 12 = 5$$

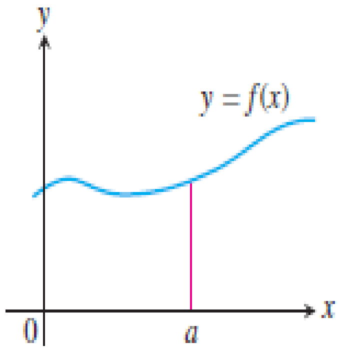
Comparison Properties of the Integral

6. If $f(x) \geqslant 0$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant 0$.

7. If $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$.

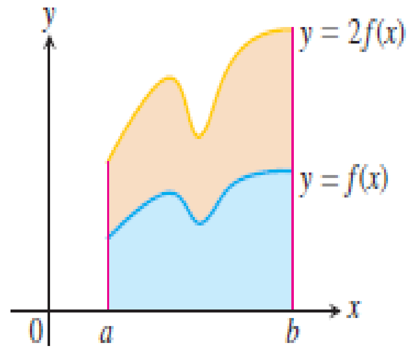
8. If $m \leqslant f(x) \leqslant M$ for $a \leqslant x \leqslant b$, then

$$m(b - a) \leqslant \int_a^b f(x)\,dx \leqslant M(b - a)$$



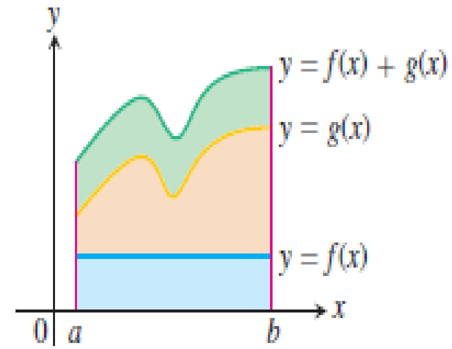
(a) *Zero Width Interval:*

$$\int_a^a f(x)\,dx = 0$$



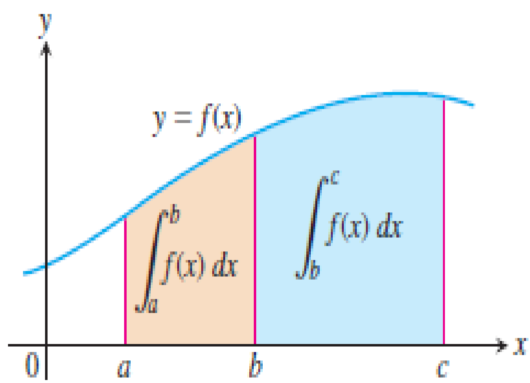
(b) *Constant Multiple:* $(k = 2)$

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$



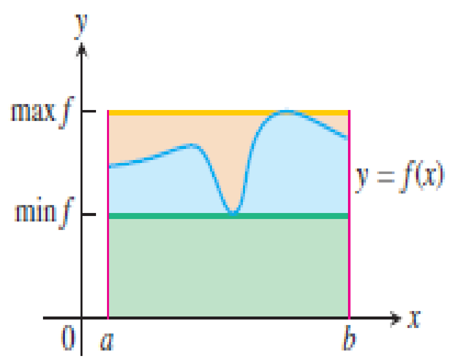
(c) *Sum: (areas add)*

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$



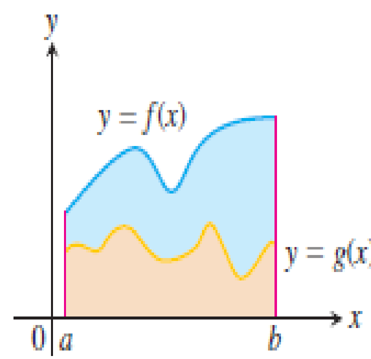
(d) *Additivity for definite integrals:*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



(e) *Max-Min Inequality:*

$$\min f \cdot (b - a) \le \int_a^b f(x)\,dx$$
$$\le \max f \cdot (b - a)$$



(f) *Domination:*

$f(x) \ge g(x)$ on $[a, b]$

$$\Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

$$\int_{-1}^{1} f(x)\,dx = 5, \quad \int_{1}^{4} f(x)\,dx = -2, \text{ and } \int_{-1}^{1} h(x)\,dx = 7.$$

Then

1. $$\int_{4}^{1} f(x)\,dx = -\int_{1}^{4} f(x)\,dx = -(-2) = 2$$

2. $$\begin{aligned}\int_{-1}^{1} [2f(x) + 3h(x)]\,dx &= 2\int_{-1}^{1} f(x)\,dx + 3\int_{-1}^{1} h(x)\,dx \\ &= 2(5) + 3(7) = 31\end{aligned}$$

3. $$\int_{-1}^{4} f(x)\,dx = \int_{-1}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx = 5 + (-2) = 3$$

(a) Express $\int_1^2 (x^2 - 1)\,dx$ as a Riemann sum with n sample points. (Take the sample points to be the *right* end points.)

(b) Evaluate the sum in the limit as $n \to \infty$.

(Some useful identities:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2.)$$

(a) To set up the n-th Riemann sum, we divide the interval $[1,2]$ into subintervals of width $\Delta x = \frac{2-1}{n} = \frac{1}{n}$. The end points of these intervals are given by $x_i = 1 + i \cdot \Delta x = 1 + \frac{i}{n}$. Hence, the Riemann sum corresponding to the choice of sample points $x_i^* = x_i$ is

$$R_n = \sum_{i=1}^{n} f(x_i) \cdot \Delta x = \sum_{i=1}^{n} (x_i^2 - 1) \cdot \Delta x = \sum_{i=1}^{n} \left[\left(1 + \frac{i}{n}\right)^2 - 1\right] \cdot \frac{1}{n}$$

$$= \sum_{i=1}^{n} \left[1 + \frac{2i}{n} + \frac{i^2}{n^2} - 1\right] \cdot \frac{1}{n} = \frac{1}{n^2} \cdot \sum_{i=1}^{n} \left[2i + \frac{i^2}{n}\right].$$

(b) To evaluate $\lim_{n\to\infty} R_n$, use the given identities to compute

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{1}{n^2} \cdot \left[2\sum_{i=1}^{n} i + \frac{1}{n}\sum_{i=1}^{n} i^2\right] = \lim_{n\to\infty} \frac{1}{n^2} \cdot \left[2\frac{n(n+1)}{2} + \frac{1}{n}\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n^2} \cdot \left[n^2 + n + \frac{2n^2+3n+1}{6}\right] = \lim_{n\to\infty} \left[1 + \frac{1}{n} + \frac{2}{6} + \frac{3}{6n} + \frac{1}{6n^2}\right] = 1 + \frac{1}{3} = \frac{4}{3}.$$

Suppose $f(x) = x^2$. Let R be the region in the xy-plane bounded by the curve $y = f(x)$ and the lines $y = 0$, $x = 2$, and $x = 3$. Find the area of R by evaluating the limit of a Riemann sum that uses the *Right Endpoint Rule*; show all reasoning.

Since $f(x)$ is continuous on the interval $[2,3]$, it is Riemann integrable. Thus we can choose the following sequence of partitions of $[2,3]$. For each n, we let

$$I_i^{(n)} = [x_{i-1}^{(n)}, x_i^{(n)}] := [2 + \frac{i-1}{n}, 2 + \frac{i}{n}]$$

$i = 1, 2, \ldots, n$. Notice that each subdivision has length $|I_i^{(n)}| = 1/n$, which goes to 0 as n goes to infinity. Hence the Riemann sum evaluated over such partitions will converge to the actual Riemann integral, which we take to be the definition of the area of R. The right endpoint of each subinterval $I_i^{(n)}$ is clearly $x_i^{(n)} = 2 + \frac{i}{n}$. By definition of Riemann integral, we have

$$\begin{aligned}
R &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^{(n)})|I_i^{(n)}| \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} (2 + \frac{i}{n})^2 \frac{1}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} (4 + \frac{4i}{n} + \frac{i^2}{n^2}) \frac{1}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n} + \sum_{i=1}^{n} \frac{4i}{n^2} + \sum_{i=1}^{n} \frac{i^2}{n^3}
\end{aligned}$$

By linearity of limit, we evaluate the limit of the three sums separately. Recall the formula for $\sum_{i=1}^{n} i$ and $\sum_{i=1}^{n} i^2$, we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n} = \lim_{n\to\infty} n\frac{4}{n} = 4$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{4i}{n^2} = \lim_{n} \frac{4}{n^2} \sum_{i=1}^{n} i = \lim_{n} \frac{4}{n^2} \frac{n(n+1)}{2} = 2$$

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \lim_{n} \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} = 1/3.$$

So the area $R = 4 + 2 + 1/3 = 19/3$.

Example Compute the Riemann sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ for the function $f(x) = \frac{1}{x}$ on $[1, 6]$ with a regular partition into $n = 5$ subintervals, and with $x_i^* = x_i$.

Solution: Note that $a = 1$, $b = 6$ and $n = 5$. Compute the following

$$\begin{aligned}
\Delta x &= \frac{b-a}{n} = \frac{6-1}{5} = 1. \\
x_i &= a + i\Delta x = 1 + i, \text{ for each } i. \\
f(x_i^*) &= f(x_i) = \frac{1}{1+i}, \text{ for each } i.
\end{aligned}$$

$$\sum_{i=1}^{5} f(x_i^*)\Delta x = \sum_{i=1}^{5} \frac{1}{i+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}.$$

Example Compute the integral $\int_0^4 x^3 dx$ by computing Riemann sums for a regular partition.

Solution: Note that $a = 0$, $b = 4$ and $f(x) = x^3$. Use a regular partition for each positive integer n. Note that when $n \to \infty$, $|P| \to 0$. Compute the following

$$\begin{aligned} \Delta x &= \frac{b-a}{n} = \frac{4-0}{n} = \frac{4}{n}. \\ x_i &= a + i\Delta x = \frac{4i}{n}, \text{ for each } i. \\ f(x_i^*) &= f(x_i) = \left(\frac{4i}{n}\right)^3 = 64\frac{i^3}{n^3}, \text{ for each } i. \end{aligned}$$

Therefore, the corresponding Riemann sum becomes (note that $\frac{1}{n^4}$ is viewed as constant with respect to the index i, and so it can be moved out of the summation sign. The last step follows from summation formulas)

$$\sum_{i=1}^{n} f(x_i^*)\Delta x = \sum_{i=1}^{n} 64\frac{i^3}{n^3}\frac{4}{n} = \frac{256}{n^4}\sum_{i=1}^{n} i^3 = \frac{256}{n^4}\frac{n^2(n+1)^2}{4}.$$

Thus the answer is

$$\int_0^4 x^3 dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} 64\frac{i^3}{n^3}\frac{4}{n} = \lim_{n\to\infty}\frac{256}{n^4}\frac{n^2(n+1)^2}{4} = 64.$$