Question: $\lim\limits_{x\to 0^+} x.(\ln x)^2 \qquad \lim\limits_{x\to 0^+}\left[\dfrac{(\ln x)^2}{\left(\frac{1}{x}\right)}\right] \rightarrow \infty/\infty$ ind. form

$\Rightarrow \lim\limits_{x\to 0^+} \dfrac{2\ln x \cdot \frac{1}{x}}{-\frac{1}{x^2}} = \lim\limits_{x\to 0^+}\left(-\dfrac{2\ln x}{\frac{1}{x}}\right)$

$\Rightarrow -\lim\limits_{x\to 0^+} \dfrac{\frac{2}{x}}{-\frac{1}{x^2}} = \lim\limits_{x\to 0^+}(2x) = 0$

Question: $\lim\limits_{x\to\infty} x.e^{\left(\sqrt{x^2+1}-x^2\right)} = L \qquad \exp\left(\lim\limits_{x\to\infty}\left[\ln\left(x.e^{\sqrt{x^2+1}-x^2}\right)\right]\right) = L \qquad \exp\left[\lim\limits_{x\to\infty}\left(\ln x + \sqrt{x^2+1}-x^2\right)\right]$

$\Rightarrow \exp\left[\left(\lim\limits_{x\to\infty}\dfrac{\ln x}{x} + \lim\limits_{x\to\infty}\left(\sqrt{1+\frac{1}{x^2}}-x\right)\right)\cdot\lim\limits_{x\to\infty} x\right] \Rightarrow \exp \lim\limits_{x\to\infty}\dfrac{\left(\sqrt{x^2+1}-x^2\right)\left(\sqrt{x^2+1}+x^2\right)}{\left(\sqrt{x^2+1}+x^2\right)}$

$\Rightarrow \exp\left[\lim\limits_{x\to\infty}\left(\dfrac{x^2+1-x^4}{\sqrt{x^2+1}+x^2}\right)\right] = \exp\left[\lim\limits_{x\to\infty}\left(\dfrac{x^4\left(\frac{1}{x^2}+\frac{1}{x^4}-1\right)}{x^2\left(\sqrt{\frac{1}{x^2}+\frac{1}{x^4}}+\frac{1}{x^2}\right)}\right)\right] = \exp(-\infty) \Rightarrow e^{-\infty} = 0$

(5) $\lim\limits_{x\to\infty}\left(\dfrac{x}{e^{x^2+\sqrt{x^2+1}}}\right) \rightarrow \infty/\infty \Rightarrow \lim\limits_{x\to\infty}\left(\dfrac{1}{e^{x^2+\sqrt{x^2+1}}\cdot\left(2x+\frac{2x}{2\sqrt{x^2+1}}\right)}\right) = 0$

Question: $\lim\limits_{x\to 0}\left(\dfrac{e^x.\sin(x^2)}{1+x^2-\cos x}\right) \rightarrow 0/0$ indeterminate form $= 2/3$

$\Rightarrow \lim\limits_{x\to 0}\left(\dfrac{e^x\sin x^2 + e^x\cos x^2.2x}{2x+\sin x.1}\right) = \lim\limits_{x\to 0}\left(\dfrac{e^x(\sin x^2+\cos x^2.2x)}{2x+\sin x}\right) \rightarrow 0/0$

$\Rightarrow \lim\limits_{x\to 0}\left(\dfrac{e^x.1(\sin x^2+\cos x^2.2x)+e^x(\cos x^2.2x-\sin x^2.2x.2x+2.\cos x^2)}{2+\cos x}\right) = \dfrac{2}{3}$

Question: $\lim\limits_{x\to 1^+}\left(\dfrac{x}{\ln x}-\dfrac{1}{x^2-x}\right) = \lim\limits_{x\to 1^+}\left(\dfrac{x^3-x^2-\ln x}{\ln x(x^2-x)}\right) \rightarrow \infty/\infty$

$\Rightarrow \lim\limits_{x\to 1^+}\left[\dfrac{3x^2-2x-\frac{1}{x}}{\frac{1}{x}.(x^2-x)+\ln x(2x-1)}\right] = \lim\limits_{x\to 1^+}\left(\dfrac{3x^3-2x^2-1}{x^2-x+\ln x(2x^2-x)}\right) \rightarrow 0/0$

$\Rightarrow \lim\limits_{x\to 1^+}\dfrac{9x^2-4x}{(2x-1)+\frac{1}{x}.(2x^2-x)+\ln x.(4x-1)} = \lim\limits_{x\to 1^+}\dfrac{9x^2-4x}{(4x-2)+\underbrace{\ln x(4x-1)}_{0}} = \dfrac{5}{2}$

$= \lim\limits_{x\to 1^+}\left(\dfrac{x}{\ln x}-\dfrac{1}{x^2-x}\right) = \dfrac{5}{2}$

Question: $\lim_{x\to 0} \dfrac{\ln(1+2x) - 2\ln(1+x) + 2\ln(1-x) - \ln(1-2x)}{x^3}$

$$L.H.1)\quad \frac{2\ln'(1+2x) - 2\ln'(1+x) - 2\ln'(1-x) + 2\ln'(1-2x)}{3x^2}$$

$$L.H.2)\quad \frac{4\ln''(1+2x) - 2\ln''(1+x) + 2\ln''(1-x) - 4\ln''(1-2x)}{6x}$$

$$L.H.3)\quad \frac{8\ln'''(1+2x) - 2\ln'''(1+x) - 2\ln'''(1-x) + 8\ln'''(1-2x)}{6} \Rightarrow \lim_{x\to 0} \frac{12\ln'''(1)}{6} = 2\ln'''(1)$$

Question: $[0,3]$ $\quad f(x) = \dfrac{x^2-3x+1}{2x+1}$ $\quad$ Defined and continuous on $[0,3]$ $[a,b]$

$$\frac{(2x-3)\cdot(2x+1) - 2(x^2-3x+1)}{(2x+1)^2} \Rightarrow \frac{2x^2+2x-5}{(2x+1)^2} \quad \text{differentiable on } (0,3)$$

$$f'(c) = \frac{f(b)-f(a)}{b-a} \qquad \frac{2c^2+2c-5}{(2c+1)^2} = \frac{\frac{1}{7}-1}{3-0} = -\frac{2}{7}$$

$14c^2 + 14c - 35 = -8c^2 - 8c - 2 \qquad 22c^2 + 22c - 33 \Rightarrow 2c^2 + 2c - 3 = 0$

$4 - 4\cdot 2\cdot(-3) = 28 \qquad \dfrac{-2+2\sqrt{7}}{4} = \dfrac{\sqrt{7}-1}{2} \qquad \cancel{\dfrac{-2-2\sqrt{7}}{4} = \dfrac{-\sqrt{7}-1}{2}}$ not in interval

---

Question: $f(x) = \begin{cases} x^2-1, & x \le 1 \\ k(x-1), & x > 1 \end{cases}$ Find k, f is differentiable at $x=1$

continuity $\lim_{x\to 1^+} [k(x-1)] = \lim_{x\to 1^-} (x^2-1) = f(1) = 0$, any k

differentiability $f_+'(1) = f_-'(1)$ should be satisfied

$f_-'(1)$

$$\lim_{h\to 0^-} \left(\frac{f(x+h)-f(x)}{h}\right) \Rightarrow \lim_{h\to 0^-} \left(\frac{f(1+h)-f(1)}{h}\right) \Rightarrow \lim_{h\to 0^-} \frac{[(1+h)^2-1]-0}{h} \Rightarrow \lim_{h\to 0^-} \frac{h^2+2h}{h} \Rightarrow \lim_{h\to 0^-} \frac{h(h+2)}{h} = 2$$

$f_+'(1)$

$$\lim_{h\to 0^+} \left(\frac{f(x+h)-f(x)}{h}\right) \Rightarrow \lim_{h\to 0^+} \left(\frac{f(1+h)-f(1)}{h}\right) \Rightarrow \lim_{h\to 0^+} \frac{k[(1+h)-1]-0}{h} = k$$

$f_-'(1) = f_+'(1) = 2 = k \qquad$ so, $k = 2$

Question: $f(x) = x\,e^{\frac{x}{x-1}}$, find the oblique asymptote

$\lim_{x\to\infty} \dfrac{f(x)}{x} = m_1 \quad \lim_{x\to\infty} (f(x) - m_1 x) = n_1 \quad y = m_1 x + n_1 \quad \lim_{x\to\infty} \dfrac{x\,e^{\left(1+\frac{1}{x-1}\right)}}{x} = e \quad \lim_{x\to\infty} x\cdot e\cdot e^{\frac{1}{x-1}} - e\,x = e$

$$\frac{ex\left(e^{\frac{1}{x-1}}-1\right)}{[illegible]\ \frac{1}{x-1}}$$

$\boxed{ex+e}$

$\lim_{x\to -\infty} \dfrac{f(x)}{x} = m_2 \quad \lim_{x\to -\infty} [f(x) - m_2 x] = n_2 \quad y = m_2 x + n_2$

Question: $f(x) = \sinh(x)$, find the slope of tangent line and find, $y=f^{-1}(x)$ at point $P(0,0)$

$f(x) = \frac{e^x - e^{-x}}{2}$ $\quad f'(x) = \frac{e^x + e^{-x}}{2} = [\sinh(x)]' = \cosh(x)$ $\quad f'(0) = 1$ $\quad \boxed{m_T = 1}$

$[\operatorname{arcsinh}(x)]' = \frac{1}{(\sinh)'(\operatorname{arcsinh}(x))} \Rightarrow \frac{1}{\cosh(\operatorname{arcsinh}(0))} = 1$

Question: $\underbrace{\lim_{x\to 0^+} (2-e^{\sqrt{x}})^{\frac{2}{x}}}$ 1^∞ indeterminate form

$\lim_{x\to 0^+} (2-e^{\sqrt{x}})^{\frac{2}{x}} = L$ $\quad \exp\left[\lim_{x\to 0^+}\left(\frac{2}{x} \ln(2-e^{\sqrt{x}})\right)\right] = L$ $\quad \exp\left[2\left(\underbrace{\lim_{x\to 0^+}\left(\frac{\ln(2-e^{\sqrt{x}})}{x}\right)}_{0/0 \text{ indeterminate form}}\right)\right] = L$

L'Hospital

$= \exp\left[2\left(\lim_{x\to 0^+}\left(\frac{\frac{1}{2-e^{\sqrt{x}}} \cdot -e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}}{1}\right)\right)\right] \Rightarrow \exp\left[\lim_{x\to 0^+}\left(\frac{e^{\sqrt{x}}}{(e^{\sqrt{x}}-2)\sqrt{x}}\right)\right] \Rightarrow \exp(-\infty)$

$= e^{-\infty} = 0$

Question: $f(x)$'s normal line equation at $P(x_0,-1)$ $y+2x-1=0$, find $(f^{-1})'(-1)$

$y = -2x+1$ $\quad P(1,-1)$

$(y-(-1)) = m_N(x-x_0)$
$y+1 = -2x+2x_0 \quad x_0 = 1$ $\quad m_N = -2$ $\quad m_T \cdot m_N = -1$, $\frac{1}{2} \cdot -2$ $\quad f'(1) = \frac{1}{2}$ $\quad (f^{-1})'(-1) = \frac{1}{f'(\underbrace{f^{-1}(-1)}_{1})} = \frac{1}{1/2} = 2$

$f'(1) = \frac{1}{2}$

Question: $f(6) = -2$ $\quad f'(x) \leq 10$ $\quad$ differentiable at $(6,15)$ continuous at $[6,15]$

max of $f(15)$

$f'(c) = \frac{f(15)-f(6)}{15-6} \Rightarrow \frac{f(15)+2}{9} \leq 10$ $\quad f(15) \leq 88$ $\quad \boxed{f(15) = 88}$

Question: $A(t) = 8t + e^{-3t}$ on $[-2,3]$ MVT $\quad A'(t) = 8-3 \cdot e^{-3t}$

$\overbrace{8-3 \cdot e^{-3c}}^{A'(c)} = \frac{A(3)-A(-2)}{3-(-2)}$ $\quad A(3) = 24+e^{-9}$, $\quad A(-2) = e^{6}-16$

$8-3 \cdot e^{-3c} = \frac{24+e^{-9}-e^{6}+16}{5} \Rightarrow 40-15 \cdot e^{-3c} = 40+e^{-9}-e^{6}$

$\frac{e^{6}-e^{-9}}{\sqrt[3]{15}} = \sqrt[3]{e^{-9}} \cdot e^{-c}$ $\quad c = -\ln\left((e^{2}-e^{-3})/\sqrt[3]{15}\right)$

$\lim_{x\to 0^+} (1-\cos x)^x = L \qquad \exp\left[\lim_{x\to 0^+} x \ln(1-\cos x)\right] = L$

$\Rightarrow \exp\left[\lim_{x\to 0^+} \frac{\ln(1-\cos x)}{1/x}\right] \Rightarrow \exp\left[\lim_{x\to 0^+} \frac{\frac{\sin x}{1-\cos x}}{-1/x^2}\right]$

$\Rightarrow \exp\left[\lim_{x\to 0^+} \left(\frac{-x^2 \sin x}{1-\cos x}\right)\right] \Rightarrow \exp\left[\lim_{x\to 0^+} \left(\frac{2x \sin x + x^2 \cos x}{-\sin x}\right)\right]$

$\lim_{x\to 0^+} (1-\cos x)^x = 1$

$\lim_{x\to 0^+} (1-e^{\sin x}) \ln x \Rightarrow \lim_{x\to 0^+} \left[\frac{1-e^{\sin x}}{\left(\frac{1}{\ln x}\right)}\right]$

$\Rightarrow \lim_{x\to 0^+} \frac{-(e^{\sin x}\cos x)}{-\left(\frac{1}{\ln x}\right)^2 \frac{1}{x}} \Rightarrow \lim_{x\to 0^+} \frac{e^{\sin x}\cos x}{\frac{1}{(\ln x)^2 x}}$

$\Rightarrow \underbrace{\lim_{x\to 0^+} e^{\sin x}\cos x}_{1} \cdot \underbrace{\lim_{x\to 0^+} \frac{(\ln x)^2}{\left(\frac{1}{x}\right)}}_{0} \qquad \lim_{x\to 0^+} \frac{2 \ln x \cdot \frac{1}{x}}{-\frac{1}{x^2}} = -\lim_{x\to 0^+} \frac{2\ln x}{\frac{1}{x}}$

$\lim_{x\to 0^+} \left[(1-e^{\sin x}) \ln x\right] = 0$

$\lim_{x\to\infty} \frac{x-\sin x}{x+\sin x} \Rightarrow \lim_{x\to\infty} \frac{1-\cos x}{1+\cos x}$

$\lim_{x\to\infty} \left(\frac{1-\frac{\sin x}{x}}{1+\frac{\sin x}{x}}\right) = 1$

$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$

$\lim_{x\to\infty} -\frac{1}{x} \le \lim_{x\to\infty} \frac{\sin x}{x} \le \lim_{x\to\infty} \frac{1}{x}$

Question: $\lim_{x \to \infty} 2^{\sqrt{x^2-1}-x}$ $\qquad 2^{\lim_{x\to\infty} \frac{(\sqrt{x^2-1}-x)(\sqrt{x^2-1}+x)}{(\sqrt{x^2-1}+x)}}$

we multiply both numerator and denominator with the conjugate of the numerator in order to get rid of $\infty - \infty$ indeterminate form

$$2^{\lim_{x\to\infty} \frac{x^2-1-x^2}{\sqrt{x^2-1}+x}} = 2^{\frac{-1}{\lim_{x\to\infty}\sqrt{x^2-1}+x}}$$

while x approaches ∞ $\sqrt{x^2-1}+x$ approaches ∞ too so $-1/\infty$ approaches 0

$\Rightarrow 2^0 = 1$

Question: $\lim_{x\to\infty} \frac{x \cdot \ln^2 x}{x^2+e^x}$ as there exists ∞/∞ indeterminate form so, it can be applied l'hospital in order to change and get rid of the indeterminate form

1st l'hospital

$$\lim_{x\to\infty} \frac{1 \cdot \ln^2 x + x \cdot 2\ln x \cdot \frac{1}{x}}{2x + e^x \cdot 1} = \lim_{x\to\infty} \frac{\ln x(\ln x + 2)}{2x+e^x} \qquad \frac{\infty}{\infty} \text{ indeterminate form}$$

2nd l'hospital

$$\lim_{x\to\infty} \frac{\frac{1}{x}(\ln x+2) + \ln x\left(\frac{1}{x}\right)}{2+e^x \cdot 1} = \lim_{x\to\infty} \frac{2\ln x\left(\frac{1}{x}\right) + \frac{2}{x}}{2+e^x} = 2\left(\lim_{x\to\infty} \frac{\ln x + 1}{(2+e^x)\,x}\right) \quad \frac{\infty}{\infty} \text{ ind. form}$$

3rd l'hospital

$$2\lim_{x\to\infty} \frac{\frac{1}{x}}{e^x \cdot 1 \cdot x + (2+e^x) \cdot 1} = 2\lim_{x\to\infty} \frac{1}{[2+e^x(x+1)]x} \quad \text{as } x\to\infty,\ [2+e^x(x+1)]x \to \infty$$

$$\Rightarrow \lim_{x\to\infty} \frac{x \cdot \ln^2 x}{x^2+e^x} = 0$$

Question: $\lim_{x\to\infty} (e^x-1)^{1/x} = L$ $\qquad \exp\left(\lim_{x\to\infty}\left[\frac{\ln(e^x-1)}{x}\right]\right) = L$

As $\ln(e^x-1)$ and x approach infinity while x approaches infinity so there exists ∞/∞ ind. form. Then it is applicable, l'hospital

(3)

$$\Rightarrow \exp\left(\lim_{x\to\infty} \frac{\frac{1}{e^x-1} \cdot e^x \cdot 1}{1}\right) = \exp\left[\lim_{x\to\infty}\left(\frac{e^x}{e^x-1}\right)\right]$$

$$\Rightarrow^{(5)} \exp\left[\lim_{x\to\infty}\left(\frac{e^x}{e^x}\right)\right] = \exp\left(\lim_{x\to\infty} 1\right) \Rightarrow e^1 = L$$

$$\Rightarrow \lim_{x\to\infty} (e^x-1)^{1/x} = e$$

Question: $(x^2 \cdot \cos h(x)) \frac{x}{x-x}$ $\ldots$ $\left[\, 2x \ldots \ln\left(\frac{2^x + \cos h(x)}{x} \right) \right]$ [illegible]

Question: By using M.V.T find endpoints for $\sqrt{83}$ $\qquad \sqrt{81} < \sqrt{83} < \sqrt{100}$

Let $f(x) = \sqrt{x}$ $\qquad$ Conditions: 1) $f(x)$ must be continuous on $[81, a]$ $\qquad 9 < \sqrt{83} < 10$

$a > 83$ $\qquad$ $f(x)$'s domain is $[0, \infty)$ ✓

2) $f(x)$ must be differentiable on $(81, a)$

$f'(x) = \frac{1}{2\sqrt{x}}$ } defined on $(0, \infty)$ ✓

① ✗

$f'(83) = \frac{f(a) - f(81)}{a - 81}$ $\qquad f'(83) = \frac{1}{2\sqrt{83}} = \frac{\sqrt{a} - 9}{a - 81} \Rightarrow \frac{1}{2\sqrt{83}} = \frac{1}{\sqrt{a} + 9}$

$2\sqrt{83} = 9 + \sqrt{a}$

② ✓ $\exists c \in (81, 83)$: $f'(c) = \frac{\sqrt{83} - \sqrt{81}}{83 - 81}$

$\frac{1}{2\sqrt{100}} < \frac{1}{2\sqrt{x_0}} = \frac{\sqrt{83} - 9}{2} < \frac{1}{2\sqrt{81}}$ $\qquad \frac{1}{10} \ldots < \frac{1}{2\sqrt{x_0}} < \frac{\sqrt{83}}{2} - 9$

$9{,}111\ldots$

Question: for $0 < a < b$ prove that by M.V.T $\frac{b-a}{1+b^2} < \arctan b - \arctan a < \frac{b-a}{1+a^2}$

Let $f(x) = \arctan x$

M.V.T $\quad 0 < a < c < b$ $\qquad f'(c) = \frac{f(b) - f(a)}{b - a}$ There exists at least one c while $f(x)$ satisfies following conditions

1) $f(x)$ must be continuous on $[a, b]$ ✓

2) $f(x)$ must be differentiable on (a, b) $\quad f'(x) = \frac{1}{1+x^2}$ ✓

$f'(c) = \frac{\arctan b - \arctan a}{b - a}$ $\qquad \frac{1}{1+b^2} < \frac{1}{1+c^2} < \frac{1}{1+a^2}$ (because of the interval) $0 < a < c < b$

$\frac{1}{1+c^2} = \frac{\arctan b - \arctan a}{b - a} \Rightarrow \frac{b-a}{1+c^2} = \tan b - \tan a$

$\frac{b-a}{1+b^2} < \frac{b-a}{1+c^2} = \tan a - \tan b < \frac{b-a}{1+a^2}$

Question: $f(x) = \begin{cases} 1 + 2x^2 & , x \leq 2 \\ 7 + 3x - x^2 & , x > 2 \end{cases}$ $\qquad$ M.V.T check on $[0, 4]$

1) Continuity check: $\lim_{x \to 2^-} (1 + 2x^2) = \lim_{x \to 2^+} (7 + 3x - x^2) = f(2) = 9$ ✓

2) Differentiability check: $f_-'(x) = 4x$, $= 8$ $\qquad f_+'(x) = 3 - 2x$, $= -1$ $\qquad f_-'(2) \neq f_+'(2)$ ✗

Question: $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$ $|\tan a - \tan b| \leq 4|a-b|$

let $f(x) = \tan x$ MVT says that $f'(c) = \frac{f(a)-f(b)}{a-b}$ $-\frac{\pi}{3} < b < c < a < \frac{\pi}{3}$

- Conditions: $\tan x$ is continuous on $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$ ✓

$\tan x$ is differentiable on $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$ $\sec^2 x = \frac{1}{\cos^2 x}$ ✓

$|\sec^2 x| = \left|\frac{\tan a - \tan b}{a-b}\right|$ $|\sec^2 x|\,|a-b| = |\tan a - \tan b|$

$|\tan a - \tan b| \leq |a-b|\,\left|\sec^2\left(\frac{\pi}{3}\right)\right| = 4|a-b|$

Question: $f: [1,4] \to \mathbb{R}^+$, $f(x) = \sqrt{x^2-x}$ MVT

Conditions

1) is continuous on $[1,4]$ $x^2 - x \geq 0$ $x(x-1) \geq 0$ Domain $\mathbb{R} - (0,1)$ ✓

2) is differentiable on $(1,4)$ $\frac{1}{2\sqrt{x^2-x}} \cdot (2x-1)$ → undefined at $x=0$ and $x=1$ (not in domain) ✓

$f(4) = 2\sqrt{3}$ $f(1) = 0$

MVT $1 < c < 4$ $f'(c) = \frac{f(4)-f(1)}{4-1} \Rightarrow \frac{2x_0-1}{2\sqrt{x_0^2-x_0}} = \frac{2\sqrt{3}-0}{3}$

$6x_0 - 3 = 4\sqrt{3(x_0^2-x_0)} \Rightarrow 36x_0^2 + 9 - 36x_0 = 16 \cdot 3(x_0^2 - x_0)$

$12x_0^2 - 12x_0 - 9 = 0$ $4x_0^2 - 4x_0 - 3 = 0$

$\Delta = 16 - 4(-3)4 = 64$ $\frac{4 \mp 8}{8} = \frac{3}{2}, \cancel{-\frac{1}{2}}$

$\boxed{c = 3/2}$ $\boxed{x_0 = 3/2 \in (1,4)}$

Question: $f(x) = \begin{cases} x^3 + x, & x \leq 1 \\ 4x-2, & x > 1 \end{cases}$ $[-2,2]$ MVT $x_0 = ?$

Conditions:

1) continuity on $[-2,2]$ ✓

$\lim_{x \to 1^-}(x^3+x) = \lim_{x \to 1^+}(4x-2) = f(1) = 2$

2.) Differentiability on $(-2,2)$ ✓

$\underline{f'_-(1)} = 3x^2+1 = \underline{f'_+(1)} = 4 = 4$

M.V.T says that there exists at least one c that satisfies the following condition

$-2 < c < 2$ $f'(c) = \frac{f(2)-f(-2)}{2-(-2)}$ $f(-2) = -10$ $f(1) = 2$ $f(2) = 6$

$-2 < c_1 \leq 1$ $3c_1^2 + 1 = \frac{f(1)-f(-2)}{1-(-2)} \Rightarrow 3c_1^2+1 = 4 \Rightarrow 3(c_1^2-1) = 0$ $c_1 = \pm 1$

$1 < c_2 < 2$ $4 = \frac{f(2)-f(1)}{2-1} \Rightarrow 4 = 4$ $x_0 \in (1,2)$

$x_0 = [1,2) \cup \{-1\}$

Question:
$$\lim_{x\to 0^+}\left(\frac{2^x+\cosh(x)}{2}\right)^{\frac{2}{\sin(x)}} = L \qquad \exp\left(\lim_{x\to 0^+}\underbrace{\left[\frac{2}{\sin(x)}\cdot\ln\left(\frac{2^x+\cosh(x)}{2}\right)\right]}_{0/0 \text{ indeterminate form}}\right) = L$$

$$\sinh(x) = \frac{e^x-e^{-x}}{2} \qquad \cosh(x) = \frac{e^x+e^{-x}}{2}$$

$$= \exp\left[2\cdot\lim_{x\to 0^+}\left(\frac{\ln\left(\frac{2^x+\cosh(x)}{2}\right)}{\sin(x)}\right)\right] \Rightarrow \exp\left(2\cdot\lim_{x\to 0^+}\frac{\left(\frac{2}{2^x+\cosh(x)}\right)\cdot\frac{(2^x\ln 2+\sinh(x))}{2}}{\cos(x)}\right) = \exp(\ln 2) = 2$$

Question: $0<x<y \qquad \sqrt{y}-\sqrt{x} < \frac{y-x}{2\sqrt{x}} \qquad \frac{\sqrt{y}-\sqrt{x}}{y-x} < \frac{1}{2\sqrt{x}} \qquad f(x)=\sqrt{x}$

$$f'(c) = \frac{\sqrt{y}-\sqrt{x}}{y-x} \qquad \frac{1}{2\sqrt{c}} = \frac{\sqrt{y}-\sqrt{x}}{y-x} \qquad \text{as } 0<x<c<y$$

$$\frac{1}{2\sqrt{y}} < \frac{1}{2\sqrt{c}} < \frac{1}{2\sqrt{x}}$$

$$\frac{\sqrt{y}-\sqrt{x}}{y-x} < \frac{1}{2\sqrt{x}} \qquad \boxed{\frac{1}{2\sqrt{c}} < \frac{1}{2\sqrt{x}}}$$

Question: $\ln\sqrt[3]{x^2+y^2} + y^2 = \frac{x^3}{2-x} + \arctan(x-y) - \arccos(x) + 4x^3y^2e^{x+y} + \cosh(x) + \frac{3\pi}{2}$

① ② ③ ④ ⑤ ⑥ ⑦ ⑧

tangent and normal line equations at P(0,1)

① $\frac{1}{\sqrt[3]{x^2+y^2}}\cdot\frac{(2x+2y\cdot y')}{3\sqrt[3]{(x^2+y^2)^2}}$ ④ $\frac{(1-y')}{\sec^2(\arctan(x-y))}$ ⑦ $\sinh(x)$

② $2y\cdot y'$ ⑤ $\frac{1}{\sin(\arccos(x))}$ ⑧ 0

③ $\frac{3x^2(2-x)-x^3(-1)}{(2-x)^2}$ ⑥ $4\left[3x^2(y^2e^{x+y}) + x^3(2y\cdot y'\cdot e^{x+y} + y^2\cdot e^{x+y}(1+y'))\right]$

$$\frac{2}{3}\frac{(x+yy')}{(x^2+y^2)} + 2y\cdot y' = \frac{2x^2(3-x)}{(2-x)^2} + \frac{1-y'}{\sec^2(\arctan(x-y))} + \frac{1}{\sin(\arccos(x))} + ⑥ + \sinh(x)$$

$$\frac{2}{3}y' + 2y' = \frac{1-y'}{2} + 1 \qquad \frac{4y'}{6} + \frac{12y'}{6} = \frac{3-3y'}{6} + \frac{6}{6} \Rightarrow \boxed{\frac{9}{15} = y'}$$

$$(y-1) = \frac{9}{15}(x-0) \Rightarrow \boxed{y_T = \frac{9}{15}x+1} \qquad m_T\cdot m_N = -1 \quad \frac{9}{15}\ \ \frac{-15}{9}$$

$$(y-1) = \frac{-15}{9}(x-0) \Rightarrow \boxed{y_N = \frac{-15}{9}x+1}$$

Question: constant func. $f(x) = \arcsin\left(\frac{2x}{x^2+1}\right) + 2\arctan(x)$; $x > 1$

$f(g^{-1}(x)) = \longrightarrow g'(g^{-1}(x))\,(g^{-1}(x))' =$ $\qquad g^{-1}(x) = \frac{1}{g'(g^{-1}(x))}$

$\left[\arcsin\left(\frac{2x}{x^2+1}\right)\right]' = \frac{1}{\cos\left(\arcsin\left(\frac{2x}{x^2+1}\right)\right)} \cdot \frac{2(x^2+1) - 2x \cdot 2x}{(x^2+1)^2}$ $\qquad (2\arctan(x))' = \frac{2}{\sec^2(\arctan(x))}$

$\underbrace{\frac{\cancel{x^2+1}}{x^2-1} \cdot \frac{2(1-x^2)}{(x^2+1)^{\cancel{2}}}}$ $\qquad$  $\qquad \sec\theta = \sqrt{1+x^2}$ $\qquad \frac{2}{1+x^2}$

$\frac{-2}{1+x^2} + \frac{2}{1+x^2} = 0$

Question: $g(x) = \sqrt{\sin\left(\frac{ax+b}{cx+d}\right)}$ $\quad a, b, c, d \left(x \neq -\frac{c}{d}\right)$ are constants, find $g'(x)$

$\frac{ax+b}{cx+d} = u(x)$ $\qquad g(x) = \sqrt{\sin(u(x))}$ $\qquad g'(x) = \lim_{\Delta x \to 0} \frac{\sqrt{\sin(u(x+\Delta x))} - \sqrt{\sin(u(x))}}{\Delta x} \cdot \frac{\sqrt{\sin(u(x+\Delta x))} + \sqrt{\sin(u(x))}}{\sqrt{\sin(u(x+\Delta x))} + \sqrt{\sin(u(x))}}$

$g'(x) = \lim_{\Delta x \to 0} \frac{\sin u(x+\Delta x) - \sin(u(x))}{\Delta x\left(\sqrt{\sin(u(x+\Delta x))} + \sqrt{\sin(u(x))}\right)}$

$A = 2\alpha \quad B = 2\theta$

$\sin A - \sin B = \sin 2\alpha - \sin 2\theta = 2(\sin\alpha\cos\alpha - \sin\theta\cos\theta)$

$\cos(\alpha+\theta) = \cos\alpha\cos\theta - \sin\alpha\sin\theta \qquad \sin(\alpha-\theta) = \sin\alpha\cos\theta - \sin\theta\cos\alpha$

$\cos\alpha\sin\alpha\,\underline{\cos^2\theta} - \sin\theta\cos\theta\,\underline{\cos^2\alpha} - \underline{\sin^2\alpha}\,\sin\theta\cos\theta + \underline{\sin^2\theta}\cos\alpha\sin\alpha \Rightarrow 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

$A = u(x+\Delta x) \qquad B = u(x)$

$g'(x) = \lim_{\Delta x \to 0} \frac{2\cos\left(\frac{u(x+\Delta x)+u(x)}{2}\right) \cdot \sin\left(\frac{u(x+\Delta x)-u(x)}{2}\right)}{\Delta x\left(\sqrt{\sin(u(x+\Delta x))} + \sqrt{\sin(u(x))}\right)} = \lim_{\Delta x \to 0} \frac{\cos\left(\frac{u(x+\Delta x)+u(x)}{2}\right)}{\Delta x\left(\sqrt{\sin(u(x+\Delta x))} + \sqrt{\sin(u(x))}\right)} \cdot \lim_{\Delta x \to 0} \underbrace{\frac{\sin\left(\frac{u(x+\Delta x)-u(x)}{2}\right)}{\left(\frac{u(x+\Delta x)-u(x)}{2}\right)}}_{1} \cdot u(x+\Delta x) - u(x)$

$\Rightarrow \underbrace{\lim_{\Delta x \to 0}\left(\frac{u(x+\Delta x)-u(x)}{\Delta x}\right)} \cdot \lim_{\Delta x \to 0} \frac{\cos\left(\frac{u(x+\Delta x)+u(x)}{2}\right)}{\sqrt{\sin(u(x+\Delta x))} + \sqrt{\sin(u(x))}} \cdot 1 \to \frac{\cos(u(x))}{2\sqrt{\sin(u(x))}}$

$\frac{ax + a\Delta x + b}{cx + c\Delta x + d} - \frac{ax+b}{cx+d}$ $\qquad \cancel{acx^2} + \cancel{adx} + ac\Delta x\cdot x + ad\Delta x + \cancel{bcx} + \cancel{bd} - \cancel{acx^2} - \cancel{bcx} - ac\Delta x\cdot x - bc\Delta x - \cancel{adx} - \cancel{bd}$

$(cx+d)(cx+c\Delta x+d)$

$\lim_{\Delta x \to 0}\left(\frac{\frac{\cancel{\Delta x}(ad-bc)}{(cx+d)(cx+c\Delta x+d)}}{\cancel{\Delta x}}\right) = \frac{ad-bc}{(cx+d)^2}$

$\Rightarrow g'(x) = \left[\sqrt{\sin\left(\frac{ax+b}{cx+d}\right)}\right]' = \frac{ad-bc}{(cx+d)^2} \cdot \frac{\cos\left(\frac{ax+b}{cx+d}\right)}{2\sqrt{\sin\left(\frac{ax+b}{cx+d}\right)}}$

Question: $y=\dfrac{x\cdot e^x}{x^2+e^x}$ asymptotes? $D(f)=\mathbb{R}$

① Vertical asymptote: The denominator x^2+e^x is strictly positive for all $x\in\mathbb{R}$, hence the function is defined everywhere and its denominator does not approach zero at any finite point. Therefore, the given function $\left(y=f(x)=\dfrac{x\cdot e^x}{x^2+e^x}\right)$ $\boxed{\text{admits no vertical asymptote}}$ because vertical asymptote existence condition is while x approaches a defined value like a from right-hand or left-hand, function can approach unclear positive or negative infinity at least one side must satisfy that condition.

② Horizontal asymptote: for $y=L$, if and only if at least one is satisfied:

$\lim\limits_{x\to-\infty} f(x)=L$ or $\lim\limits_{x\to+\infty} f(x)=L$ or both

$\lim\limits_{x\to-\infty}\left(\dfrac{x\cdot e^x}{x^2+e^x}\right)$ as $x\to-\infty$ numerator $(x\cdot e^x)\to 0$ and denominator $(x^2+e^x)\to+\infty$

$$\lim_{x\to-\infty}\left(\frac{\frac{e^x}{x}}{1+\frac{e^x}{x^2}}\right) \overset{\text{L'Hospital}}{\Rightarrow} \lim_{x\to-\infty}\left(\frac{\frac{e^x x-e^x}{x^2}}{\frac{e^x x^2-2x e^x}{x^4}}\right)=0 \qquad \boxed{y=0 \text{ is a horizontal asymptote}}$$

$$\lim_{x\to+\infty}\left(\frac{x\cdot e^x}{x^2+e^x}\right) \Rightarrow \lim_{x\to+\infty}\left(\frac{\frac{e^x}{x}}{1+\frac{e^x}{x^2}}\right) \Rightarrow \lim_{x\to+\infty}\left(\frac{\frac{e^x x-e^x}{x^2}}{\frac{e^x x^2-2x e^x}{x^4}}\right)=0$$

③ Oblique asymptote: $y=mx+n$ $\quad m=\lim\limits_{x\to+\infty}\left(\dfrac{f(x)}{x}\right)$, $n=\lim\limits_{x\to+\infty}(f(x)-mx)$

$$m)\ \frac{\frac{x\cdot e^x}{x^2+e^x}}{x} \Rightarrow m=\lim_{x\to\infty}\underbrace{\left(\frac{e^x}{x^2+e^x}\right)}_{\frac{\infty}{\infty}} \Rightarrow m=\lim_{x\to\infty}\underbrace{\left(\frac{e^x}{2x+e^x}\right)}_{\frac{\infty}{\infty}} \Rightarrow m=\lim_{x\to\infty}\left(\frac{e^x}{2+e^x}\right) \Rightarrow m=\lim\frac{e^x}{e^x}$$

$\boxed{m=1}$

$$n)\ n=\lim_{x\to\infty}\left(\frac{x\cdot e^x}{x^2+e^x}-x\right) \Rightarrow n=\lim_{x\to\infty}\underbrace{\left(\frac{-x^3}{x^2+e^x}\right)}_{\frac{-\infty}{\infty}} \Rightarrow n=\lim_{x\to\infty}\underbrace{\left(\frac{-3x^2}{2x+e^x}\right)}_{\frac{-\infty}{\infty}} \Rightarrow n=\lim_{x\to\infty}\underbrace{\left(\frac{-6x}{2+e^x}\right)}_{\frac{-\infty}{\infty}} \Rightarrow$$

$$n=\lim_{x\to\infty}\left(\frac{-6}{e^x}\right) \Rightarrow n=0 \qquad \boxed{y=x \text{ is an oblique asymptote}}$$

Question: MVT $0<x<1$ $\quad \dfrac{\sqrt{1-x^2}}{1+x} < \dfrac{\ln(1+x)}{\arcsin x} < 1$ $\quad$ Let $f(x)=\ln(x)$

$[1, 1+x]$

$$f'(c)=\frac{1}{c}=\frac{\ln(1+x)-\ln 1}{(1+x)-1} \Rightarrow \frac{1}{c}=\frac{\ln(1+x)}{x} \;\Big|\; 1<c<1+x \quad \frac{1}{1+x}<\frac{1}{c}<1 \quad \boxed{\frac{1}{1+x}<\frac{\ln(1+x)}{x}<1}$$

$g(x)=\arcsin(x)$ $\quad \overset{a}{0}<c<\overset{b}{x}<1 \quad 1-0^2>1-c^2>1-x^2$

$$g'(c)=\frac{g(b)-g(a)}{b-a}=\frac{\arcsin x}{x}=\frac{1}{\sqrt{1-c^2}} \qquad 1<\frac{1}{\sqrt{1-c^2}}<\frac{1}{\sqrt{1-x^2}} \qquad 1<\frac{\arcsin x}{x}<\frac{1}{\sqrt{1-x^2}} \quad \boxed{\sqrt{1-x^2}<\frac{x}{\arcsin x}<1}$$

Question: normal line equation at $t=0$

$$\begin{cases} t^2 \cdot \sin(x) + x^3 = e^t & t=0 \quad x=1 \\ \sin(y) = t \cdot \sin(t) - 2t & t=0 \quad y=0 \end{cases}$$

$2t \cdot \sin(x) + t^2 \cos(x) \cdot x' + 3x^2 \cdot x' = e^t$ $\boxed{x' = \frac{1}{3}}$

$\cos(y) \cdot y' = \sin(t) + t \cdot \cos(t) - 2$ $\boxed{y' = -2}$

$\frac{dy}{dx} = -6 \quad m_t = -6 \quad m_n = \frac{1}{6}$

$(y-0) = \frac{1}{6}(x-1)$

$\boxed{x - 6y - 1 = 0}$ d_N

Question: $f(x) = \frac{x^2-1}{x}$

i) Domain ii) Asymptotes iii) Increasing - decreasing, extremum iv) concave up / down, inflections

i)

x	$-\infty$	-1	0	1	∞
$f(x)$	$-$	$+$	$-$	$+$	

$f(-1) = 0 \quad (-1, 0)$
$f(1) = 0 \quad (1, 0)$ } x-axis intercepts

$D_f : (-\infty, \infty) - \{0\}$

ii) $\lim_{x \to 0^+} \frac{x^2-1}{x} = -\infty \quad \lim_{x \to 0^-} \frac{x^2-1}{x} = \infty$ $x=0$ (y-axis) is vertical asymptote

$\lim_{x \to \infty} \frac{x^2-1}{x} \Rightarrow \lim_{x \to \infty} \frac{x - \frac{1}{x}}{1} = \infty \quad \lim_{x \to -\infty} = -\infty$ there is no horizontal asymptote

$\lim_{x \to \mp\infty} \left[\left(x - \frac{1}{x}\right) - x\right] = 0$ $y=x$ is an oblique asymptote

iii) $\frac{2x \cdot x - (x^2-1)}{x^2} = \frac{x^2+1}{x^2}$

	0	
$+$		$+$
↗		↗

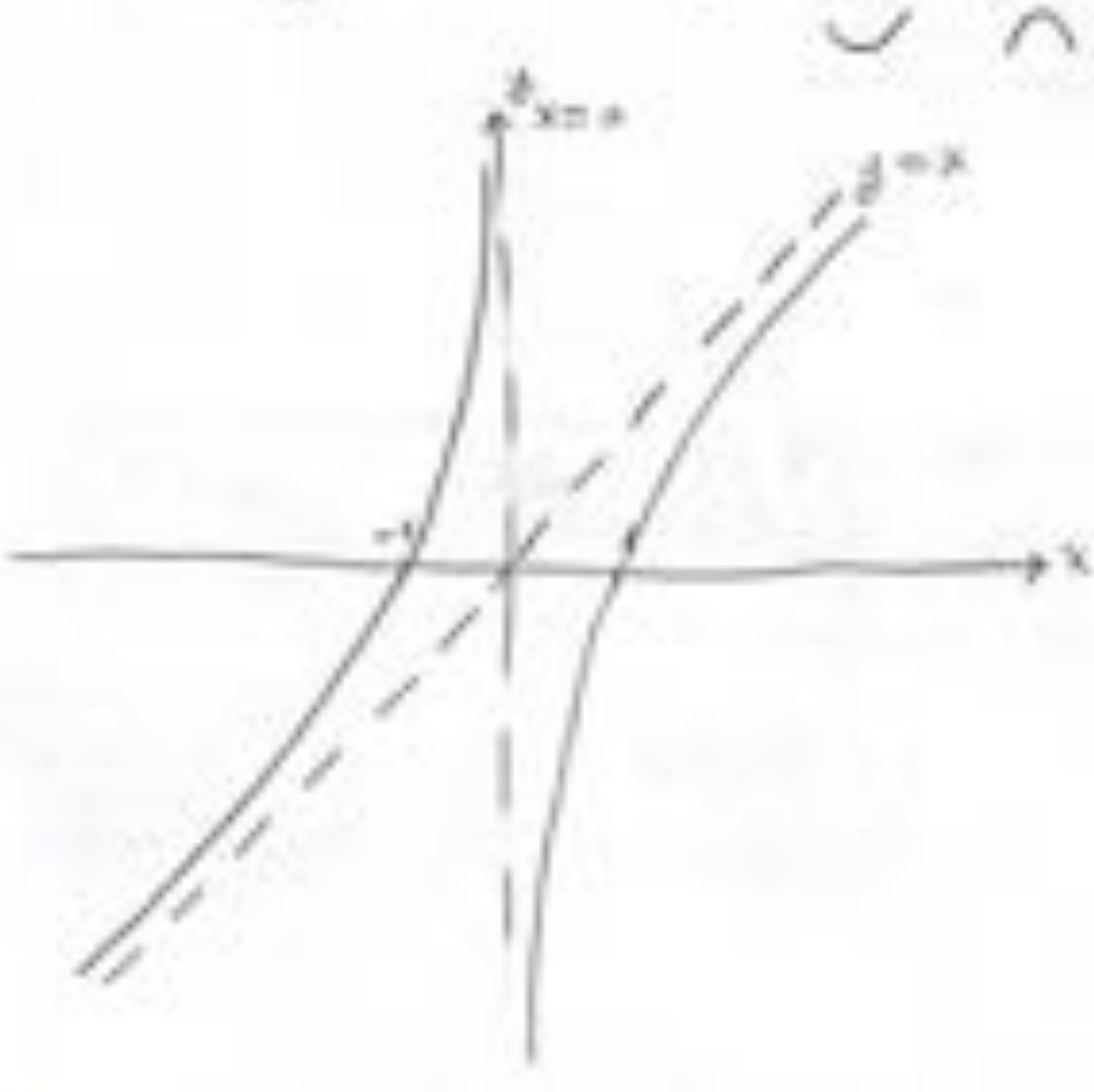
$f'(x)$ always positive so, $f(x)$ is always increasing

no extremum

iv) $\frac{2x \cdot x^2 - 2x(x^2+1)}{x^4} = -\frac{2}{x^3}$

	0	
$+$		$-$
∪		∩

$(-\infty, 0)$ concave up
$(0, \infty)$ concave down



Question: sketch $f(x) = \dfrac{x^2+4}{2x}$ * Domain: $\mathbb{R} - \{0\}$

* $f(x) = -f(-x)$ $\boxed{\text{symmetric about the origin}}$

* Intercepts: there is no intercept

* $\displaystyle\lim_{x\to 0^+}\left(\frac{x^2+4}{2x}\right) = \lim_{x\to 0^+}\left(\frac{x+\frac{4}{x}}{2}\right) = \infty \qquad \lim_{x\to 0^-}\left(\frac{x+\frac{4}{x}}{2}\right) = -\infty$ $\boxed{x=0\ (y\text{-axis})\ \text{vertical asymptote}}$

$\displaystyle\lim_{x\to\infty}\left(\frac{x+\frac{4}{x}}{2}\right) = \infty \qquad \lim_{x\to -\infty}\left(\frac{x+\frac{4}{x}}{2}\right) = -\infty$ there is no horizontal asymptote.

$y = mx+n$: $\displaystyle\lim_{x\to\infty}\left(\frac{\frac{x^2+4}{2x}}{x}\right) = \lim_{x\to\infty}\left(\frac{x^2+4}{2x^2}\right) = \lim_{x\to\infty}\left(\frac{1}{2}\left(1+\frac{4}{x^2}\right)\right) = \frac{1}{2} \qquad \lim_{x\to\infty}\left(\frac{x^2+4}{2x} - \frac{x}{2}\right) = n = 0$

$\boxed{y = \frac{1}{2}x \text{ is an oblique asymptote}}$

$y = mx+n$: $\displaystyle\lim_{x\to-\infty}\left(\frac{\frac{x^2+4}{2x}}{x}\right) = \lim_{x\to-\infty}\left(\frac{1}{2}\left(1+\frac{4}{x^2}\right)\right) = \frac{1}{2} \qquad \lim_{x\to-\infty}\left(\frac{x^2+4}{2x} - \frac{x}{2}\right) = 0$

* $f'(x) = \dfrac{2x\cdot 2x - 2(x^2+4)}{4x^2} = \dfrac{2x^2-8}{4x^2} = \dfrac{x^2-4}{2x^2}$

	$-\infty$	-2	0	2	∞
$f'(x)$	$-$	$+$	‖	$-$	$+$
$f(x)$	↘ local max $(-2,-2)$	↗	‖	↘ local min $(2,2)$	↗

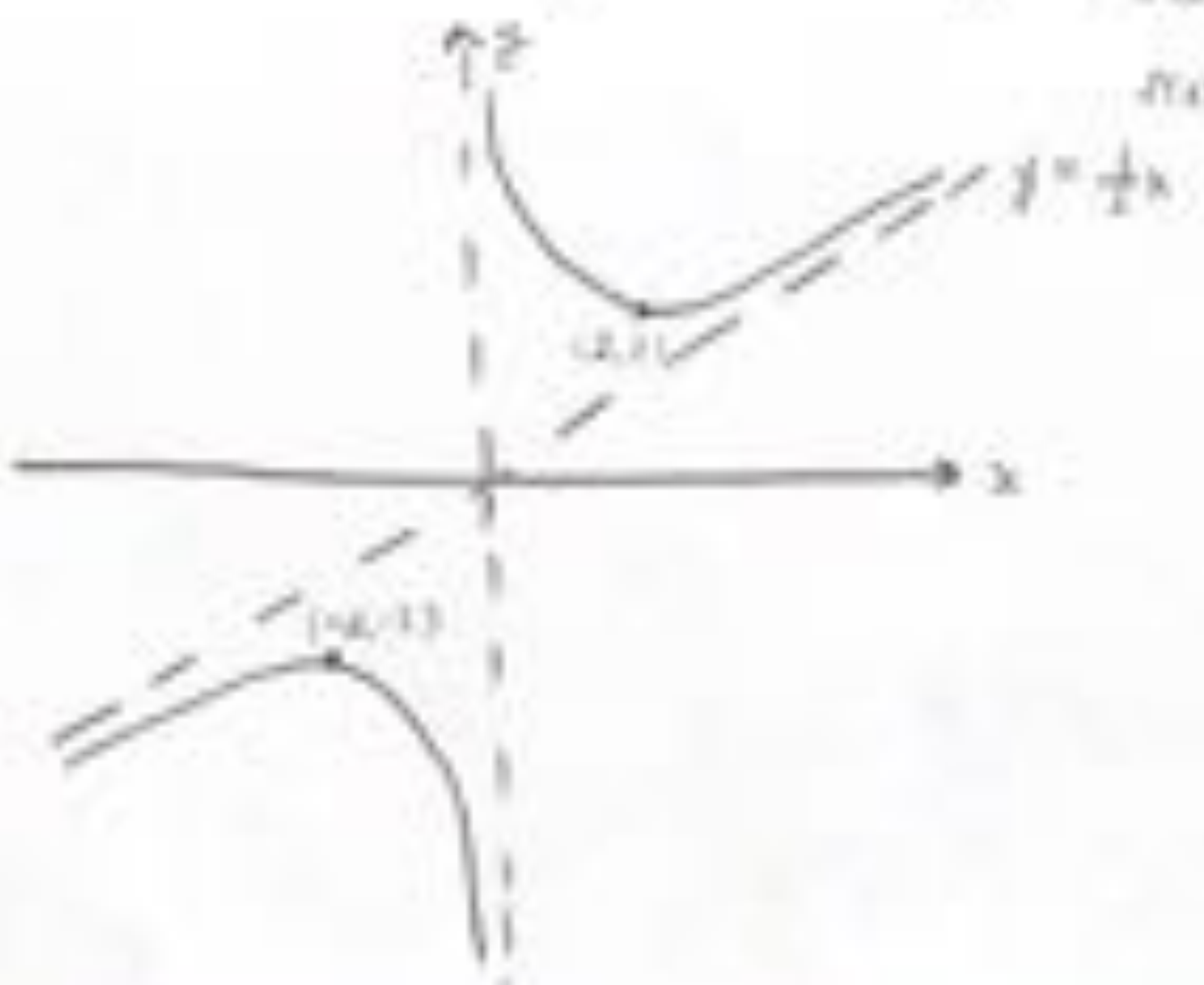
Increasing $(-2,0) \cup (2,\infty)$

Decreasing $(-\infty,-2) \cup (0,2)$

* $f''(x) = \dfrac{2x\cdot 2x - 2(x^2-4)}{4x^2} = \dfrac{x^2+4}{2x}$

	$-\infty$	0	∞
$f''(x)$	$+$	‖	$+$
$f(x)$	∪	‖	∪

Concave up $(-\infty,0) \cup (0,\infty)$



Question: tangent line, $r = 3 + 8\sin\theta$ at $\theta = \frac{\pi}{6}$ $\quad x = r\cos\theta \quad y = r\sin\theta \quad r = f(\theta) = 3 + 8\sin\theta$

$r' = f'(\theta) = 8\cos\theta$

$\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}\Bigg|_{\theta=\frac{\pi}{6}} \qquad \dfrac{4\sqrt{3}\cdot\frac{1}{2} + 7\cdot\frac{\sqrt{3}}{2}}{4\sqrt{3}\cdot\frac{\sqrt{3}}{2} - 7\cdot\frac{1}{2}} = \dfrac{11\sqrt{3}}{5}$ $\qquad x = \frac{7\sqrt{3}}{2} \quad y = \frac{7}{2}$

$\left(y - \frac{7}{2}\right) = \frac{11\sqrt{3}}{5}\left(x - \frac{7\sqrt{3}}{2}\right)$

Question: horizontal and vertical tangents of the curve $r = 1+\sin\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta \cdot \sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta \cdot \cos\theta - (1+\sin\theta)\sin\theta} \quad 2\sin\theta\cos\theta + \cos\theta$$

$y = r \cdot \sin\theta$

$(1+\sin\theta)\sin\theta$

$x = r \cdot \cos\theta$

$(1+\sin\theta)\cos\theta$

$$\underbrace{\cos^2\theta}_{1-\sin^2\theta} - \sin\theta - \sin^2\theta$$

$$-(2\sin^2\theta + \sin\theta - 1)$$

$2\sin\theta \quad -1$

$\sin\theta \quad +1$

$(1-2\sin\theta)(\sin\theta+1)$

$$\Rightarrow \frac{\cos\theta\,(2\sin\theta+1)}{(1-2\sin\theta)(\sin\theta+1)} \longrightarrow \cos\theta\,(2\sin\theta+1) = 0$$

$\cos\theta = 0 \quad \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$

$\sin\theta = -\frac{1}{2} \quad \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

$\sin\theta = \frac{1}{2} \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$\sin\theta = -1 \quad \theta = \frac{3\pi}{2}$

Horizontal : $dy = 0 \Rightarrow \left(\frac{\pi}{2}, 2\right), \left(\frac{7\pi}{6}, \frac{1}{2}\right), \left(\frac{11\pi}{6}, \frac{1}{2}\right)$

Vertical : $dx = 0 \Rightarrow \left(\frac{\pi}{6}, \frac{3}{2}\right), \left(\frac{5\pi}{6}, \frac{3}{2}\right), \left(\frac{3\pi}{2}, 0\right) \rightarrow \boxed{r=0 \text{ (the pole)}}$