- $\lim_{n\to\infty} \left(\frac{1+n^2}{3+n^2}\right)^n = \lim_{n\to\infty}\left(\frac{3+n^2}{3+n^2}+\frac{-2}{3+n^2}\right)^n = \lim_{n\to\infty}\left(1+\frac{-2}{3+n^2}\right)^n$

$\lim_{n\to\infty} n = \lim_{n\to\infty}(n^2+3)\frac{1}{n} = \lim_{n\to\infty}\left(n+\frac{3}{n}\right) = n \Rightarrow \lim_{n\to\infty}\Bigg[\underbrace{\left(1+\frac{-2}{3+n^2}\right)^{n^2+3}}_{e^{-2}}\Bigg]^{\frac{1}{n}} = e^{\lim_{n\to\infty}\frac{-2}{n}} = e^0 = 1$

$\boxed{\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e}$ $\lim_{x\to\infty} x\left(1+\frac{1}{x^2}\right) = x$ | $\lim_{x\to\infty} x\left(1-\frac{k}{x^2}\right) = x$ | $\lim_{x\to\infty}(1+x^2)\frac{1}{x} = x$ | $\lim_{x\to\infty}(k+x^2)\frac{1}{x} = x$

- $\lim_{x\to 1} x^{\frac{1}{x-1}}$ $x = (x-1)+1$ $\lim_{x\to 1}((x-1)+1)^{\frac{1}{x-1}} \Rightarrow \lim_{x\to 1}\left(1+\frac{1}{\frac{1}{x-1}}\right)^{\frac{1}{x-1}}$ $\frac{1}{x-1} = u$ $x\to 1 \Rightarrow u\to\infty$

$\lim_{u\to\infty}\left(1+\frac{1}{u}\right)^u = e$ $\lim_{x\to 1} x^{\frac{1}{x-1}} = e$

- $\lim_{x\to 2}(x-1)^{\frac{1}{x-2}}$ $x-1 = (x-2)+1$ $\lim_{x\to 2}\left(1+\frac{1}{\frac{1}{x-2}}\right)^{\frac{1}{x-2}}$ $\frac{1}{x-2} = u$ $x\to 2 \Rightarrow u\to\infty$

$\Rightarrow \lim_{u\to\infty}\left(1+\frac{1}{u}\right)^u = e$ $\lim_{x\to 2}(x-1)^{\frac{1}{x-2}} = e$

- $\lim_{x\to 3}(2x-5)^{\frac{1}{3x-9}}$ $2x-5 = 2x-6+1$
$2(x-3)+1$
$3x-9 = 3(x-3)$
$\lim_{x\to 3}\left[\left(1+\frac{2}{\frac{1}{x-3}}\right)^{\frac{1}{x-3}}\right]^{\frac{1}{3}}$ $\frac{1}{x-3} = u$ $x\to 3 \Rightarrow u\to\infty$

$= \lim_{u\to\infty}\left[\left(1+\frac{2}{u}\right)^u\right]^{\frac{1}{3}} = (e^2)^{\frac{1}{3}} = \sqrt[3]{e^2}$

- $\lim_{x\to 1}\left(\frac{3x^2-2x-1}{x^2-x}\right)$ $\frac{(3x+1)\cdot\cancel{(x-1)}}{x\cancel{(x-1)}} \Rightarrow \lim_{x\to 1}\left(\frac{3x+1}{x}\right) = 4$

- $y = x\cdot\arccos x - \sqrt{1-x^2}$, $y' = ?$ $\frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1-x^2}}$

$y' = \arccos x + x\cdot\frac{-1}{\sqrt{1-x^2}} - \frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}} \Rightarrow y' = \arccos x + \cancel{\frac{-x}{\sqrt{1-x^2}}} + \cancel{\frac{x}{\sqrt{1-x^2}}} \Rightarrow y' = \arccos x$

- $y = \tanh(\ln x)$, $y' = ?$ $\frac{d}{dx}\tanh(x) = \frac{1}{1-x^2} = \operatorname{sech}^2(x)$

$\operatorname{sech}^2(\ln x)(\ln x)' \Rightarrow \frac{\operatorname{sech}^2(\ln x)}{x}$

- $f(x) = x^5$, $(f^{-1})'(32) =$

$f(f^{-1}(x)) = x$

$\frac{d}{dx}f(f^{-1}(x)) = 1$

$f'(f^{-1}(x))\cdot\frac{d}{dx}f^{-1}(x) = 1$

$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$

$\frac{d}{dx}f^{-1}(32) = \frac{1}{f'(f^{-1}(32))} = \frac{1}{80}$

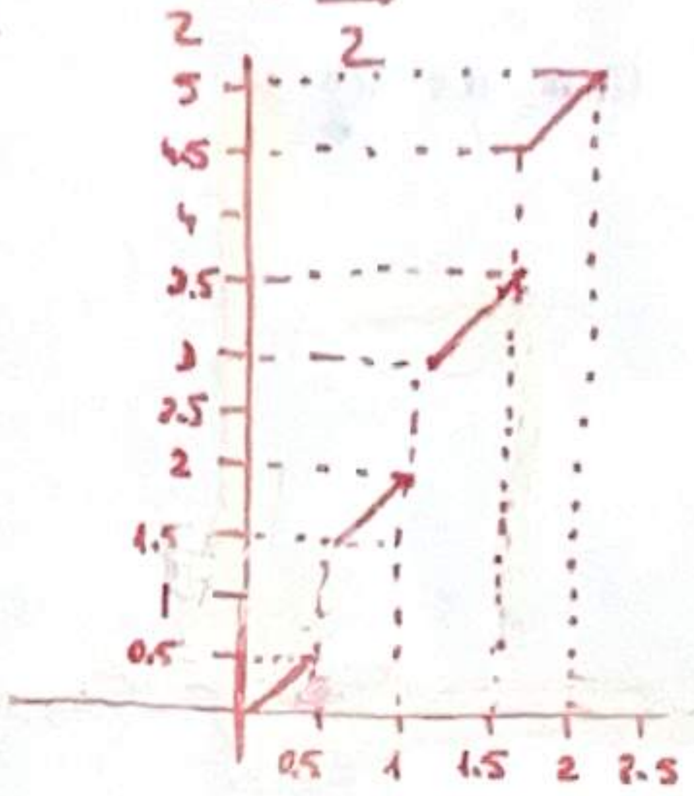
$f^{-1}(32) = 2$

$f'(2) = 80$

• $f(x) = x + [\![2x]\!]$, examine the discontinuous points of the given function on the interval $[0,2)$ and define types of discontinuties.

$n \leq 2x < n+1 \Rightarrow \frac{n}{2} \leq x < \frac{n+1}{2} \Rightarrow \frac{n+1}{2} - \frac{n}{2} = \frac{1}{2}$

$$f(x) = x + [\![2x]\!] = \begin{cases} x & ; [0, \frac{1}{2}) \\ x+1 & ; [\frac{1}{2}, 1) \\ x+2 & ; [1, \frac{3}{2}) \\ x+3 & ; [\frac{3}{2}, 2) \end{cases}$$



$$\left(\lim_{x\to\frac{1}{2}^+} f(x) = \frac{3}{2}\right) \neq \left(\lim_{x\to\frac{1}{2}^-} f(x) = \frac{1}{2}\right) ; \left(\lim_{x\to 1^+} f(x) = 3\right) \neq \left(\lim_{x\to 1^-} f(x) = 2\right) ; \left(\lim_{x\to\frac{3}{2}^+} f(x) = \frac{9}{2}\right) \neq \left(\lim_{x\to\frac{3}{2}^-} f(x) = \frac{7}{2}\right)$$

• $\lim_{x\to -2} \left(-\frac{1}{x^3}\right) = \frac{1}{8}$; $\varepsilon - \delta$ technique

for $\lim_{x\to a} f(x) = L$

$\forall \varepsilon \in \mathbb{R}^+, \exists \delta(\varepsilon) \in \mathbb{R}^+ \; |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

$|x-(-2)| < \delta \Rightarrow \left| -\frac{1}{x^3} - \frac{1}{8} \right| < \varepsilon$

$\left| \frac{1}{x^3} + \frac{1}{8} \right| < \varepsilon$

$|x+2| < \delta \Rightarrow \left| \frac{8+x^3}{8x^3} \right| < \varepsilon$

$x^3+8 = (x+2)(x^2-4x+4)$

★

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① $|x+2| < \delta < 1$ $\quad -3 < x < -1 \quad |x| < 1$ $\quad \boxed{8|x|^3 < 8}$

② $x \in (-3,-1) \quad x^2 - 2x + 4$ $\quad (-1) \rightarrow 7$ $\quad (-3) \rightarrow 19$

$7 < |x^2 - 2x + 4| < 19 \quad \boxed{|x^2 - 2x + 4| < 19}$

$\frac{|x+2| \cdot 19}{8} < \varepsilon \qquad |x+2| < \frac{8\varepsilon}{19} \qquad |x+2| < \delta < 1 \qquad \delta = \min : \left\{1, \frac{8}{19}\varepsilon\right\}$

• Use the mean value theorem to show that $\sqrt{y} - \sqrt{x} < \frac{y-x}{2\sqrt{x}}$ if $0 < x < y$

Let $f(x) = \sqrt{x}$ $\quad f'(x) = \frac{1}{2\sqrt{x}}$ $\quad 0 < x < c < y \Rightarrow f'(c) = \frac{\sqrt{y} - \sqrt{x}}{y - x}$

$\sqrt{x} < \sqrt{c} < \sqrt{y}$

$\frac{1}{2\sqrt{c}} = \frac{\sqrt{y} - \sqrt{x}}{y - x} < \frac{1}{2\sqrt{x}} \Rightarrow \sqrt{y} - \sqrt{x} < \frac{y-x}{2\sqrt{x}}$

• For the given curve by parametric equations $\begin{cases} x(t) = 6t\cos(t) \\ y(t) = 6\sqrt{3}\,t\sin(t) \end{cases}$

a) Find the equation of the tangent line at $\frac{\pi}{6}$

b) " " normal line at $\frac{\pi}{6}$

$x\left(\frac{\pi}{6}\right) = 6\cdot\frac{\pi}{6}\cdot\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}\pi \qquad y\left(\frac{\pi}{6}\right) = 6\cdot\sqrt{3}\cdot\frac{\pi}{6}\cdot\sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}\pi}{2}$

$\frac{d}{dt}\overset{=\dot{x}}{x(t)} = 6\left[\cos(t) + t\cdot(\sin(t))\right] = 6\cos t - 6t\sin t \Rightarrow 6\cdot\cos\left(\frac{\pi}{6}\right) - 6\cdot\frac{\pi}{6}\cdot\sin\left(\frac{\pi}{6}\right)$

$\Rightarrow 3\sqrt{3} - \frac{\pi}{2}$

$\frac{d}{dt}\overset{=\dot{y}}{y(t)} = 6\sqrt{3}\left(\sin(t) + t\cdot\cos(t)\right) = 6\sqrt{3}\sin t + 6\sqrt{3}t\cos t \Rightarrow 6\sqrt{3}\sin\left(\frac{\pi}{6}\right) + 6\sqrt{3}\cdot\frac{\pi}{6}\cdot\cos\left(\frac{\pi}{6}\right)$

$\Rightarrow 3\sqrt{3} + \frac{3\pi}{2}$

$y'\left(\frac{\pi}{6}\right) = \frac{\dot{y}}{\dot{x}}\Big|_{\pi/6} = \frac{3\sqrt{3} + \frac{3\pi}{2}}{3\sqrt{3} - \frac{\pi}{2}} = \frac{6\sqrt{3} + 3\pi}{6\sqrt{3} - \pi}$

$m_N \cdot m_T = -1 \qquad m_T = \frac{6\sqrt{3} + 3\pi}{6\sqrt{3} - \pi}$

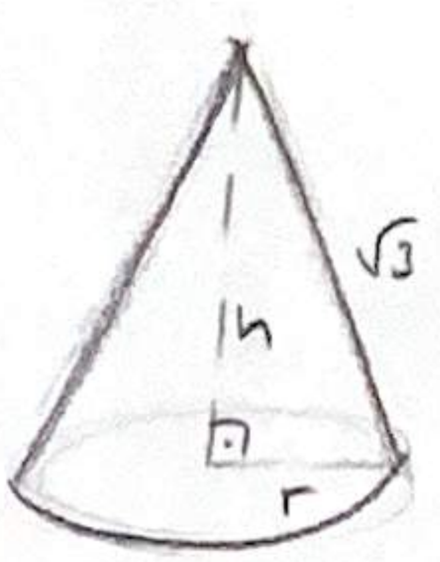
$m_N = -\left(\frac{6\sqrt{3} - \pi}{6\sqrt{3} + 3\pi}\right)$

a) $d_T: \; y - y_0 = m_T(x - x_0) \Rightarrow y - \frac{\sqrt{3}\pi}{2} = \frac{6\sqrt{3} + 3\pi}{6\sqrt{3} - \pi}\left(x - \frac{\sqrt{3}}{2}\pi\right)$

b) $d_N: \; y - y_0 = m_N(x - x_0) \Rightarrow y - \frac{\sqrt{3}}{2}\pi = -\left(\frac{6\sqrt{3} - \pi}{6\sqrt{3} + 3\pi}\right)\cdot\left(x - \frac{\sqrt{3}}{2}\pi\right)$

• A right triangle with hypotenuse of $\sqrt{3}$ is rotated about one of its legs to generate a right circular cone. Find the greatest possible volume of such a cone by determining the lengths of the legs of right triangle. ($V = \frac{1}{3}\cdot\pi\cdot r^2 h$: volume of cone, r : radius of base, h : height of cone)

because of square of r we write r^2 as h sty.



$V = \frac{1}{3}\cdot\pi\cdot r^2 h \qquad (r^2 = 3 - h^2) \qquad V = \frac{1}{3}\cdot\pi\cdot(3 - h^2)\cdot h$

$\frac{d}{dh}V(h) = \frac{1}{3}\pi\left(3 - 3h^2\right) \Rightarrow 3 = 3h^2 \Rightarrow h = \pm 1 \qquad \boxed{h = 1}$

$V'' = -2\pi h$

$V''(h=-1) = -2\pi(-1) = 2\pi > 0$ local min for $h = -1$

$V''(h=1) = -2\pi(1) = -2\pi < 0$ local max for $h = 1 \qquad \boxed{r = \sqrt{2}}$

• $\lim_{n\to\infty} (\sqrt{n^2+1} - n) = 0$ (ε-δ techique)

$\forall \varepsilon > 0, 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$ so, $x_0 > \delta(\varepsilon)$

$$|(\sqrt{n^2+1} - n) - 0| < \varepsilon$$

$$\left[(\sqrt{n^2+1} - n) \cdot \frac{(\sqrt{n^2+1} + n)}{(\sqrt{n^2+1} + n)}\right] = \frac{(n^2+1) - n^2}{\sqrt{n^2+1} + n} = \frac{1}{\sqrt{n^2+1} + n}$$

$$\left|\frac{1}{\sqrt{n^2+1} + n}\right| < \frac{1}{|\sqrt{n^2} + n|} < \frac{1}{2n} < \varepsilon \Rightarrow n > \frac{1}{2\varepsilon}$$

As there exists $\delta(\varepsilon) = \frac{1}{2\varepsilon} > 0$ for $\forall \varepsilon > 0$, the existence of limit is true.

• $\lim_{x\to 1} \frac{\tan \pi x}{1 - x^2} = \lim_{x\to 1} \frac{\frac{\sin(\pi x)}{\cos(\pi x)}}{(1+x)(1-x)} = \underbrace{\lim_{x\to 1} \frac{\sin(\pi x)}{(1-x)}}_{\pi} \cdot \underbrace{\lim_{x\to 1} \frac{1}{(1+x) \cdot \cos(\pi x)}}_{-\frac{1}{2}}$

Let $1 - x = u$ $\quad \lim_{u\to 0} \frac{\sin(\pi(1-u))}{u} = \lim_{u\to 0} \frac{\sin(\pi - \pi u)}{u}$

$$= \lim_{u\to 0} \frac{\sin(\pi u)}{u} \cdot \frac{\pi}{\pi} = \pi \cdot \underbrace{\lim_{u\to 0} \frac{\sin(\pi u)}{(\pi u)}}_{1} = \pi$$

$$\lim_{x\to 1} \frac{\tan(\pi x)}{1 - x^2} = -\frac{\pi}{2}$$

• $\lim_{x\to 119} (2x+1) = 239$ (ε-δ technique)

$\forall \varepsilon > 0$, $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$ so, there exists $x_0 > \delta(\varepsilon)$

$0 < |x - 119| < \delta \Rightarrow |(2x+1) - 239| < \varepsilon \rightarrow |2(x - 119)| < \varepsilon \quad |x - 119| < \frac{\varepsilon}{2}$

For the function $\delta(\varepsilon) = \frac{\varepsilon}{2}$ (under the condition $\forall \varepsilon > 0$) there exists limit value in the case $\delta > 0$

• $f(x) = \frac{1}{\sqrt{x+1}}$ $\quad f'(x) = ? = \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h+1}} - \frac{1}{\sqrt{x+1}}}{h} = \lim_{h\to 0} \frac{\sqrt{x+1} - \sqrt{x+h+1}}{h \cdot (\sqrt{x+h+1} \cdot \sqrt{x+1})} \cdot \frac{(\sqrt{x+1}) + \sqrt{x+h+1}}{(\sqrt{x+1} + \sqrt{x+h+1})}$

$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{x+1 - (x+h+1)}{h \cdot (\sqrt{x+h+1} \cdot (\sqrt{x+1}) \cdot (\sqrt{x+1} + \sqrt{x+h+1})}$$

while $f(x) = \frac{1}{\sqrt{x+1}}$,

$$= \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h} \cdot (\sqrt{x+h+1} \cdot \sqrt{x+1})(\sqrt{x+1} + \sqrt{x+h+1})}$$

$f'(x) = -\frac{1}{2}(x+1)^{-3/2}$

$$= -\lim_{h\to 0} \frac{1}{(\sqrt{x+h+1} \cdot \sqrt{x+1})(\sqrt{x+1} + \sqrt{x+h+1})}$$

$$\Longrightarrow -\frac{1}{(x+1) \cdot 2\sqrt{x+1}}$$

- $f(x) = \dfrac{\sin 2x}{(2-2e^{2x})\cdot 2\cos x}$ is continuous at $x=0$ $\quad f(0)=?$ $\quad \lim\limits_{x\to 0} f(x) = f(0)$

$$f(0) = \frac{\sin(2\cdot 0)}{(2-2e^{2\cdot 0})\,2\cos 0} \Rightarrow \frac{0}{0} \qquad f(x) = \frac{\cancel{2}\cdot \sin x \cdot \cancel{\cos x}}{(2-2e^{2x})\,\cancel{2\cos x}} \Rightarrow f(x) = \frac{\sin x}{x}\cdot\frac{x}{(2-2e^{2x})}$$

$$\left(x\to 0 \quad \frac{\sin x \le x \le \frac{\sin x}{\cos x}}{\sin x} \qquad 1 \le \frac{x}{\sin x} \le \frac{1}{\cos x} \qquad \cos x \le \frac{\sin x}{x} \le 1 \qquad \underbrace{\lim_{x\to 0}\cos x}_{1} \le \lim_{x\to 0}\frac{\sin x}{x} \le \underbrace{\lim_{x\to 0} 1}_{1} \right)$$

$$\lim_{x\to 0} f(x) = \underbrace{\lim_{x\to 0}\frac{\sin x}{x}}_{1}\cdot\underbrace{\lim_{x\to 0}\frac{x}{2-2e^{2x}}}_{\frac{1}{4}} \Rightarrow \lim_{x\to 0} f(x) = 1\cdot\lim_{x\to 0}\frac{1}{2}\left(\frac{x}{1-e^{2x}}\right)$$

Let $1-e^{2x} = u \qquad 1-u = e^{2x} \qquad \ln(1-u) = 2x$

$x\to 0 \Rightarrow u\to 0$

$$\lim_{u\to 0}\frac{1}{2}\left(\frac{\frac{\ln(1-u)}{2}}{u}\right) = \frac{1}{4}\lim_{u\to 0}\left(\ln(1-u)\right)^{\frac{1}{u}} = \frac{1}{4}\ln\Big(\underbrace{\lim_{u\to 0}(1-u)^{\frac{1}{u}}}_{e^{-1}}\Big) = -\frac{1}{4}\ln(e) = -\frac{1}{4}$$

$f(0) = -\frac{1}{4}$ (should get that value)

- $\lim\limits_{x\to\infty}\left(\dfrac{x^2-3}{x^2+5}\right)^x \qquad \lim\limits_{x\to\infty}\left(1+\dfrac{-8}{x^2+5}\right)^x \qquad \lim\limits_{a\to\infty}\left(1+\dfrac{m}{a}\right)^a = e^m$

$$\lim_{x\to\infty}\left[\left(1+\frac{-8}{x^2+5}\right)^{x^2+5}\right]^{\frac{x}{x^2+5}} = \Big[\underbrace{\lim_{x\to\infty}\left(1+\frac{-8}{x^2+5}\right)^{x^2+5}}_{e^{-8}}\Big]^{\frac{x}{x^2+5}} = (e^{-8})^{\underbrace{\lim_{x\to\infty}\frac{x}{x^2+5}}_{0}} = (e^{-8})^0 = 1$$

$$\lim_{x\to\infty}\frac{x}{x^2+5} = \frac{\frac{1}{x}}{1+\frac{5}{x^2}} = \frac{0}{1+0} = 0$$

- $\lim\limits_{u\to k}(u-k)^2\cdot\sin\left(\dfrac{1}{u-k}\right) \Rightarrow \lim\limits_{u\to k}(u-k)\cdot\dfrac{\sin\left(\frac{1}{u-k}\right)}{\left(\frac{1}{u-k}\right)} = \underbrace{\lim\limits_{u\to k}(u-k)}_{0}\cdot\underbrace{\lim\limits_{u\to k}\dfrac{\sin\left(\frac{1}{u-k}\right)}{\left(\frac{1}{u-k}\right)}}_{0}$

$$\lim_{u\to k}(u-k)^2\cdot\sin\left(\frac{1}{u-k}\right) = 0$$

• Let $f(x)$ be a function has inverse function. If the normal line to curve $y=f(x)$ at point $P(x_0,-1)$ is $y+2x-1=0$, find $(f^{-1})'(-1)$

$-2x+1=y$

$y-y_0=m.(x-x_0)$

$y-(-1)=-2(x-x_0)$

$y+1=-2x+2x_0 \qquad y=-2x+2x_0-1$

$-2x+2x_0-1=-2x+1 \Rightarrow x_0=1$

$m_T.m_N=-1 \qquad \boxed{m_T=\frac{1}{2}} \qquad \boxed{f'(1)=m_T=\frac{1}{2}}$

$m_T=f'(1)$

$f(f^{-1}(x))=x$

$\frac{d}{dx}f(f^{-1}(x))=1$

$f'(f^{-1}(x)).\frac{d}{dx}(f^{-1}(x))=1$

so, we are looking for:

$\frac{d}{dx}(f^{-1}(x))=\frac{1}{f'(f^{-1}(x))} \Rightarrow \frac{1}{f'(f^{-1}(-1))}$

$f^{-1}(-1)=1 \Rightarrow f(1)=-1 \qquad \Rightarrow \frac{1}{f'(\underbrace{f^{-1}(-1)}_{1})}=\frac{1}{f'(1)}=\frac{1}{\frac{1}{2}}=2$

• Check if it's differentiable at $x=1$

$$f(x)=\begin{cases}(x-1).\sin\left(\frac{1}{x-1}\right) & ; x\neq 1\\ 0 & ; x=1\end{cases}$$

$f'(x)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

$\Rightarrow \lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}=\lim_{h\to 0^-}\frac{f(1-h)-f(1)}{(1-h)-1}=\frac{((1-h)-1).\sin\left(\frac{1}{(1-h)-1}\right)-0}{-h}=\lim_{h\to 0^-}-\sin\left(\frac{1}{h}\right)$

$\Rightarrow \lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}=\lim_{h\to 0^+}\frac{f(1+h)-f(x)}{(1+h)-1}=\frac{((1+h)-1).\sin\left(\frac{1}{(1+h)-1}\right)-0}{h}=\lim_{h\to 0^+}\sin\frac{1}{h}$

The given function is not differentiable at point $x=1$ because right-hand and left-hand limit are not equal.

• for the function $f(x)=\frac{x^2-x+1}{x}$

i) domain $x\neq 0$, $D=\mathbb{R}-\{0\}$

ii) asymptotes

vertical: $\lim_{x\to 0^-}f(x)=-\infty$, $\lim_{x\to 0^+}f(x)=+\infty$ } $x=0$ vertical asymptote

horizontal: $\lim_{x\to -\infty}f(x)=-\infty$, $\lim_{x\to +\infty}f(x)=+\infty$ } there is no horizontal asymptote

oblique: $x-1+\frac{1}{x}$ oblique as.

iii) intervals on which f is increasing, decreasing, and local extreme values $f'(x)=1-\frac{1}{x^2}$ $x=1$ $x=-1$

iv) " " concave up and down, and inflection points (if any)

v) sketch the graph

iii)

x		-1		0		1	
f	+		−	‖	−		+
f'	↗		↘	‖	↘		↗

$f(x)$ is increasing on $(-\infty,-1)\cup(1,\infty)$

$f(x)$ is decreasing on $(-1,0)\cup(0,1)$

iv) $f''(x)=\frac{2}{x^3}$

x		0	
f''	−		+
f	∩		∪

But, there is no inflection point $(0\notin D)$

$f(x)$ is concave up on $(0,\infty)$

$f(x)$ is concave down on $(-\infty,0)$

v)