Question: $f(x) = \begin{cases} (x-1)\cdot \sin\left(\frac{1}{x-1}\right) & , x \neq 1 \\ 0 & , x = 1 \end{cases}$ Is differentiable at $x=1$

To talk about differentiability at given point function has to be continuous, then right-handed derivative and left-handed derivative should same.

① Continuity [for $(x=1)$]

$f(1) = \lim_{x\to1^+} f(x) = \lim_{x\to1^-} f(x)$

$f(1)=0$

a) $\lim_{x\to1^+} \dfrac{\sin\left(\frac{1}{x-1}\right)}{\left(\frac{1}{x-1}\right)} = \lim_{u\to+\infty} \dfrac{\sin u}{u} = 0$

let $\frac{1}{x-1} = u$

↳ from sandwich theorem

$-1 \leq \sin a \leq 1$

$-\frac{1}{a} \leq \frac{\sin a}{a} \leq \frac{1}{a}$

$\underbrace{\lim_{a\to+\infty} \cancel{-\frac{1}{a}}}_{0} \leq \lim_{a\to+\infty} \frac{\sin a}{a} \leq \underbrace{\lim_{a\to+\infty} \cancel{\frac{1}{a}}}_{0}$

b) $\lim_{u\to-\infty} \dfrac{\sin u}{u} = 0$

$\underbrace{\lim_{b\to-\infty} \cancel{-\frac{1}{b}}}_{0} \leq \lim_{b\to-\infty} \frac{\sin b}{b} \leq \underbrace{\lim_{b\to-\infty} \cancel{\frac{1}{b}}}_{0}$

$f(1) = \lim_{x\to1^+} f(x) = \lim_{x\to1^-} f(x) = 0$ so, function is continuous at given $(x=1)$ number

② Derivatives

$f_+'(1)$

$\lim_{h\to0}\left(\dfrac{f(1+h)-f(1)}{h}\right) = f_+'(1) = \lim_{h\to0} \dfrac{((\cancel{1}+h)-\cancel{1})\cdot \sin\left(\frac{1}{((h+1)-1)}\right)}{\cancel{h}} \Rightarrow \lim_{h\to0}\left(\sin\left(\frac{1}{h}\right)\right) = +\infty$

$f_-'(1)$

$\lim_{m\to0} \dfrac{f(1-m)-f(1)}{-m} = f_-'(1) = \lim_{m\to0} \dfrac{((1-\cancel{m})-1)\cdot \sin\left(\frac{1}{((1-m)-1)}\right)}{-\cancel{m}} = \lim_{m\to0}\left(-\sin\left(\frac{1}{m}\right)\right) = -\infty$

$f_+'(1) \neq f_-'(1) \Rightarrow$ left and right handed derivatives are not same so, function is not differentiable at $x=1$.

Question: Let $f(x)$ be a function that has an inverse function $f^{-1}(x)$. If the normal line to the curve $y=f(x)$ at the point $P(x_0,-1)$ is $y+2x-1=0$, find $(f^{-1})'(-1)$

$y=-2x+1 \qquad \dfrac{-1}{f'(x_0)} = -2 \qquad x_0 = 1,\ f'(1) = \frac{1}{2}$

$(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$

$(f^{-1})'(-1) = \dfrac{1}{1/2} = 2$

Question: For the function $f(x) = \dfrac{x^2 - x + 1}{x}$

i) domain ii) asymptotes iii) increasing, decreasing and local extrema iv) concavity and inflection p.

v) sketching

i) $f(x) = \dfrac{x^2 - x + 1 \rightarrow \text{numerator}}{x \rightarrow \text{denominator}}$ $(-1)^2 - 4.1.1 = \Delta \quad \Delta < 0$ numerator is always positive and defined for $\mathbb{R}$

denominator is not defined for $x = 0$

$D_f : (-\infty, 0) \cup (0, \infty)$

ii) $\lim\limits_{x \to 0^-} \dfrac{x^2 - x + 1}{x} = \lim\limits_{x \to 0^-} \dfrac{x - 1 + \frac{1}{x}}{1} = \underline{\underline{-\infty}}$, $\lim\limits_{x \to 0^-} \dfrac{x - 1 + \frac{1}{x}}{1} = \underline{\underline{+\infty}}$

So, $x = 0$ namely y-axis is a vertical asymptote

$\lim\limits_{x \to -\infty} \dfrac{x - 1 + \frac{1}{x}}{1} = \underline{\underline{-\infty}}$, $\lim\limits_{x \to +\infty} \dfrac{x - 1 + \frac{1}{x}}{1} = \underline{\underline{+\infty}}$

So, there is no horizontal asymptote

$$\begin{array}{r|l} x^2 - x + 1 & x \\ \hline -x^2 & x - 1 \\ \hline -x + 1 & \\ +x & \\ \hline 1 & \end{array}$$

$\dfrac{x^2 - x + 1}{x} = \dfrac{x(x-1) + 1}{x} = \underbrace{(x-1)}_{\text{oblique asy.}} + \dfrac{1}{x}$ $y = x - 1$ is an oblique asymptote

$\underbrace{y = mx + n}_{\text{oblique asy}}$ $m = \lim\limits_{x \to \pm\infty} \dfrac{f(x)}{x}$, $n = \lim\limits_{x \to \pm\infty} (f(x) - mx)$

iii) $f'(x) = \dfrac{(2x-1).x - (x^2 - x + 1).1}{x^2} = \dfrac{x^2 - 1}{x^2}$

		-1		0		1	
f'(x)	+		−	‖	−		+
f(x)	↗		↘	‖	↘		↗
		(−1,−3) local max				(1,1) local min	

f is increasing on $(-\infty, -1) \cup (1, \infty)$

f is decreasing on $(-1, 0) \cup (0, 1)$

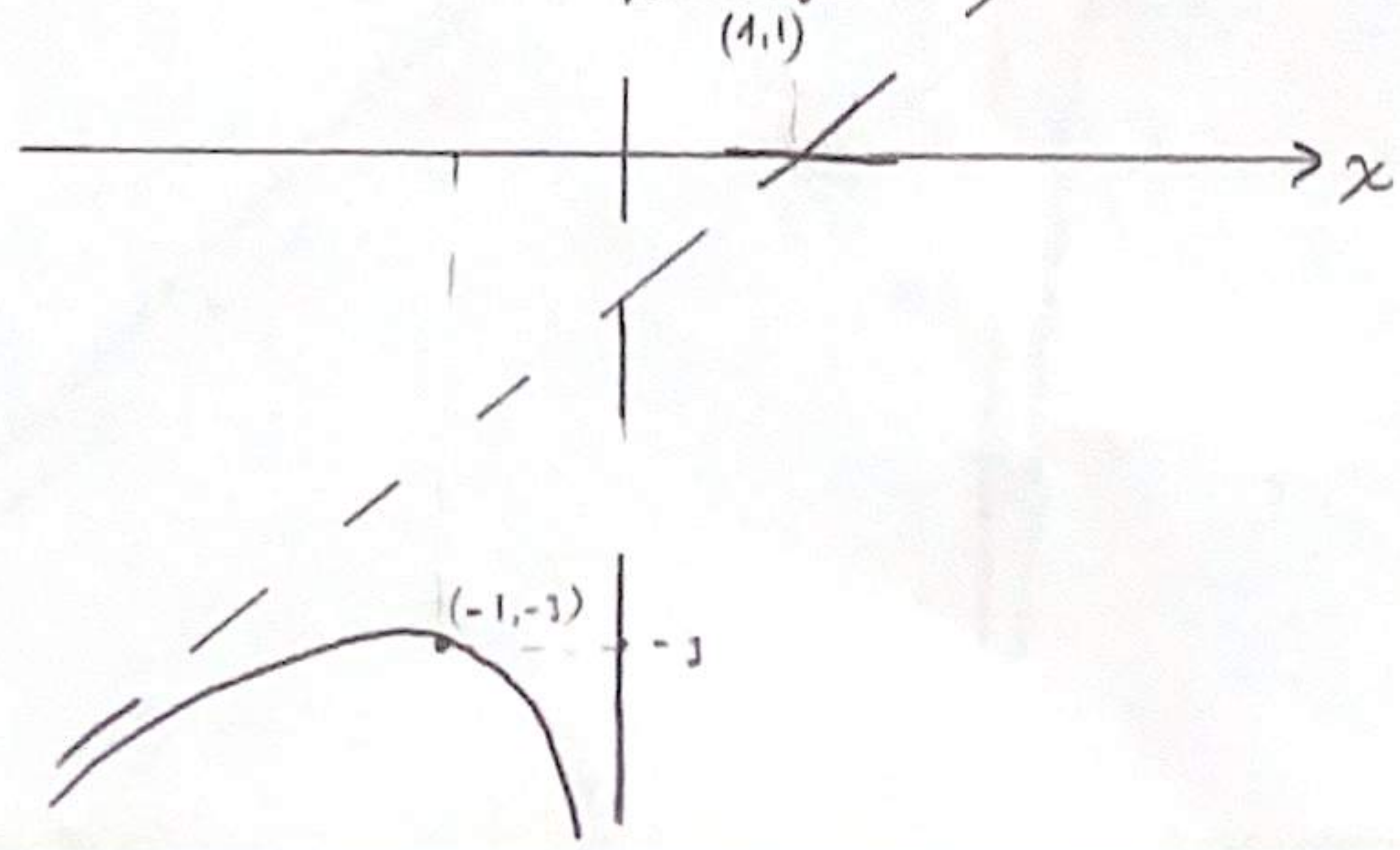
iv) $f''(x) = \dfrac{2x.x^2 - 2x.(x^2 - 1)}{x^4} = \dfrac{2}{x^3}$,

		0	
f''(x)	−	‖	+
f(x)	∩		∪

f is concave up on $(0, \infty)$

f is concave down on $(-\infty, 0)$

v)



Question: $f(x) = \sinh(x)$, find the slope of the tangent $y = f^{-1}(x)$ at the point $P(0,0)$

$f(f^{-1}(x)) = x$ $\qquad y = \sinh(x) \quad y' = \cosh(x)$

$f'(f^{-1}(x)) \cdot \frac{d}{dx} f^{-1}(x) = 1 \qquad \frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))} \qquad (f^{-1}(0))' = \frac{1}{1} = 1$

Question: For the function $f(x) = \frac{x^2-1}{x}$

i) domain ii) asymptotes iii) decreasing - increasing iv) concavity v) sketching

i) $D_f: (-\infty, 0) \cup (0, \infty)$ intercepts $(-1,0)$ and $(1,0)$

ii) $\lim\limits_{x\to 0^+} \frac{x^2-1}{x} = \lim\limits_{x\to 0^+} \frac{x - \frac{1}{x}}{1} = -\infty$, $\lim\limits_{x\to 0^-} \frac{x^2-1}{x} = \lim\limits_{x\to 0^-} \frac{x-\frac{1}{x}}{1} = +\infty$ $x=0$ namely y-axis is a vertical asymptote

$\lim\limits_{x\to +\infty} \frac{x-\frac{1}{x}}{1} = +\infty$, $\lim\limits_{x\to -\infty} \frac{x-\frac{1}{x}}{1} = -\infty$ so there is no horizontal asymptote

$$\begin{array}{r|l} x^2-1 & x \\ \hline -x^2 & x \\ \hline -1 & \end{array}$$

$\frac{(x \cdot x)-1}{x} = x - \frac{1}{x}$ $\quad y = x$ is an oblique asymptote

or, $\lim\limits_{x\to \pm\infty} \frac{f(x)}{x} = m$ and $\lim\limits_{x\to\pm\infty} (f(x) - mx) = n$ and $y = mx + n$ is an oblique asymptote

$\frac{\frac{x^2-1}{x}}{x} = 1 - \frac{1}{x^2} \quad m = 1 \qquad \frac{x^2-1}{x} - x \Rightarrow n = 0$

iii) $f'(x) = \frac{2x \cdot x - 1 \cdot (x^2-1)}{x^2} = \frac{x^2+1}{x^2}$

		0	
$f'(x)$	+	‖	+
$f(x)$	↗		↗

f is increasing on $(-\infty, 0) \cup (0, \infty)$

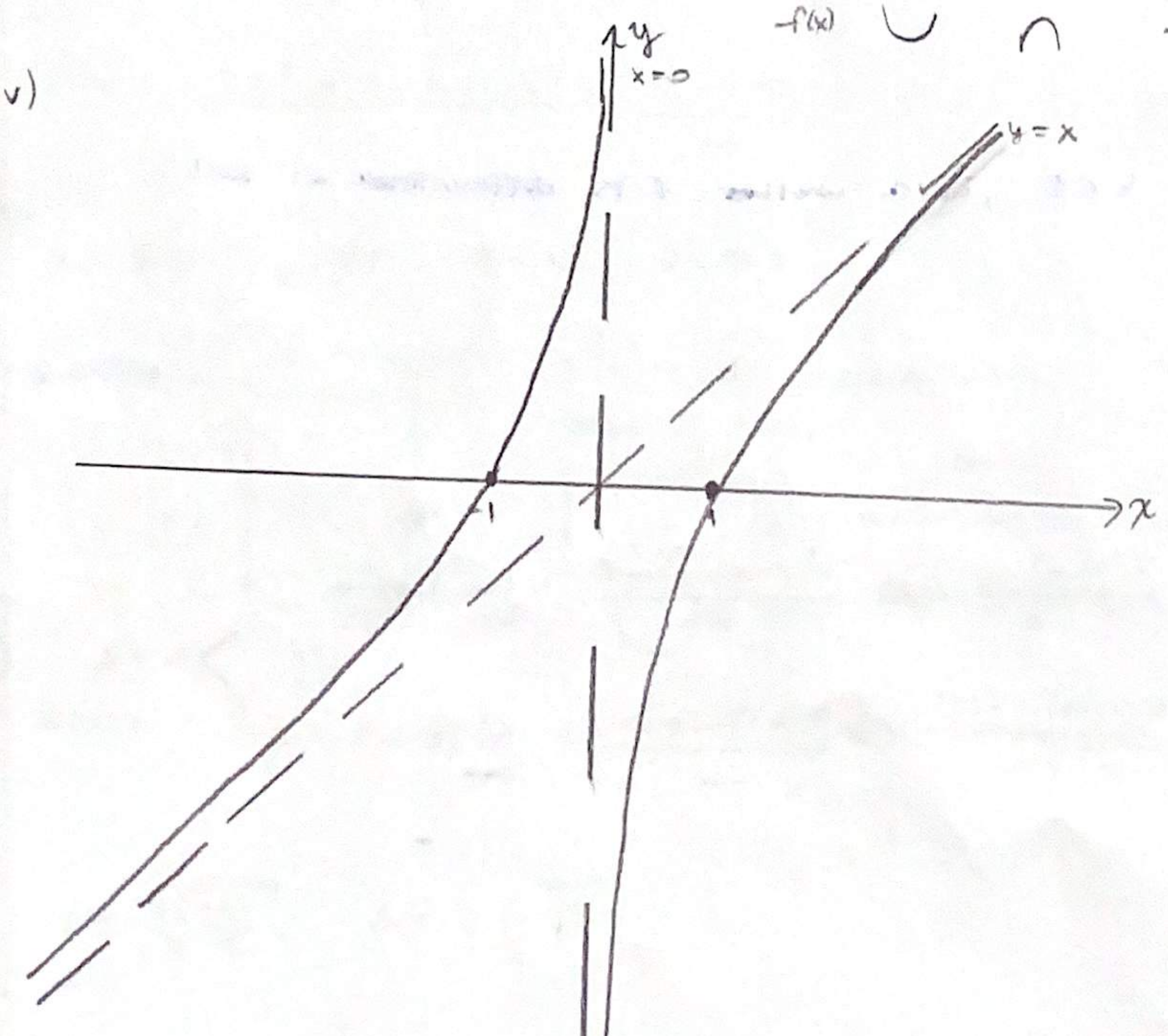
iv) $f''(x) = \frac{2x \cdot x^2 - 2x \cdot (x^2+1)}{x^4} = \frac{-2}{x^3}$

		0	
$f''(x)$	+	‖	−
$f(x)$	∪		∩

f is concave up on $(-\infty, 0)$

f is concave down on $(0, \infty)$

v)



Question: Find the tangent and normal lines at $t=\frac{\pi}{6}$

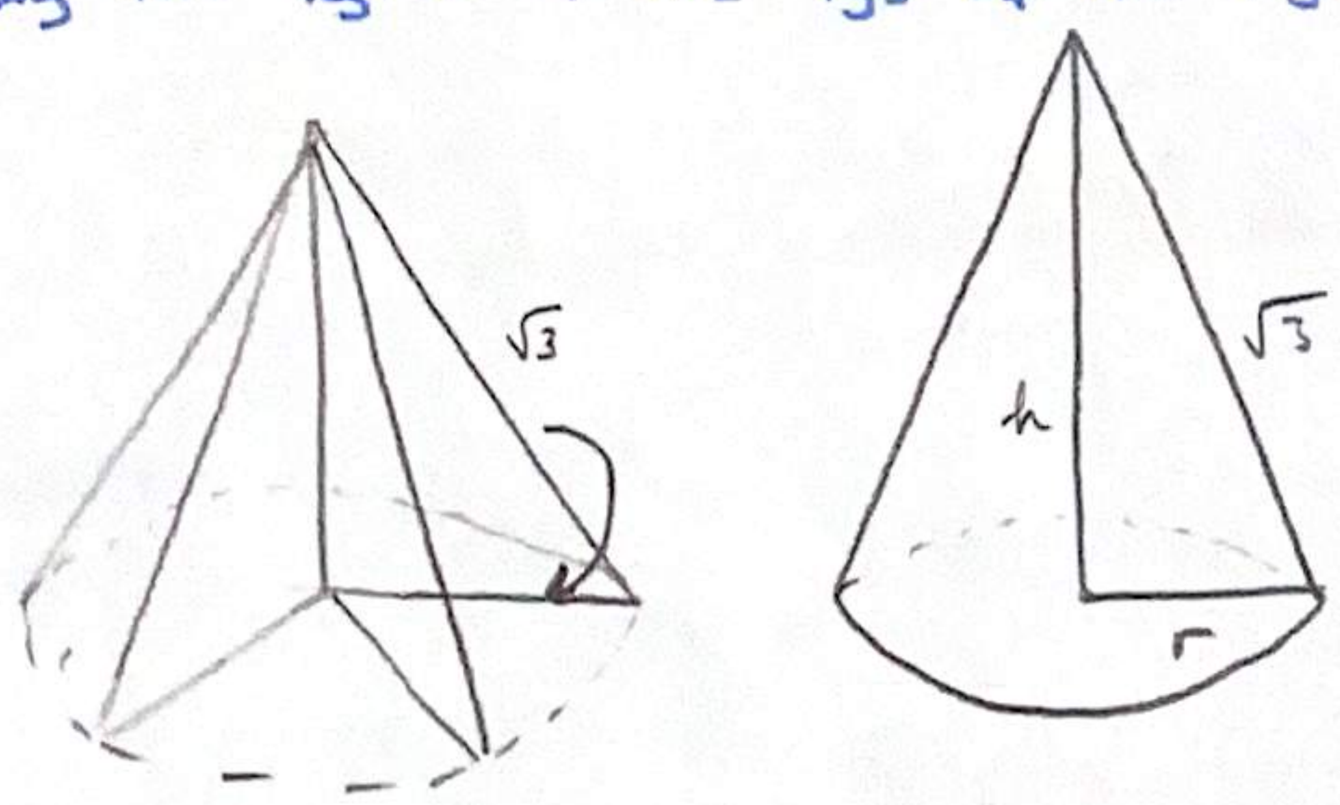
$$\begin{cases} x(t) = 6t\cos(t) \\ y(t) = 6\sqrt{3}t\sin(t) \end{cases} \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6\sqrt{3}.\sin(t) + 6\sqrt{3}t.\cos(t)}{6.\cos(t) + 6t.(-\sin(t))} = \frac{\sqrt{3}\,(\sin(t)+t.\cos(t))}{\cos(t)-t.\sin(t)}\Bigg|_{t=\frac{\pi}{6}}$$

$$\Rightarrow \frac{\sqrt{3}\left(\frac{1}{2}+\frac{\pi}{6}\frac{\sqrt{3}}{2}\right)}{\frac{\sqrt{3}}{2}-\frac{\pi}{6}.\frac{1}{2}} = \frac{\sqrt{3}\left(1+\frac{\sqrt{3}\pi}{6}\right)}{\sqrt{3}-\frac{\pi}{6}} = \frac{6\sqrt{3}+3\pi}{6\sqrt{3}-\pi} = m_T \qquad -\frac{6\sqrt{3}-\pi}{6\sqrt{3}+3\pi} = m_N$$

$$x\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}\pi}{2} \qquad y\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}\pi}{2}$$

$$d_T: \left(y-\frac{\sqrt{3}\pi}{2}\right) = \frac{6\sqrt{3}+3\pi}{6\sqrt{3}-\pi}\left(x-\frac{\sqrt{3}\pi}{2}\right) \qquad d_N: \left(y-\frac{\sqrt{3}\pi}{2}\right) = \left(-\frac{6\sqrt{3}-\pi}{6\sqrt{3}+3\pi}\right).\left(x-\frac{\sqrt{3}\pi}{2}\right)$$

Question: A right triangle with hypotenuse of $\sqrt{3}$ is rotated about one of its legs to generate a right circular cone. Find the greatest possible volume of such a cone by determining the lengths of the legs of the right triangle $\left(V=\frac{1}{3}\pi r^2 h\right)$



$$V = \frac{1}{3}\pi.r^2.h \qquad (\sqrt{3})^2 = h^2+r^2$$

$$3-h^2 = r^2$$

$$V = \frac{1}{3}\pi\,(3-h^2).h \qquad V = \frac{\pi}{3}(3h-h^3)$$

$$V' = \frac{\pi}{3}(3-3h^2) \qquad h = \pm 1$$

		-1		1	
V'	$-$		$+$		$-$
V	$\searrow$		$\nearrow$	local max	$\searrow$

$h=1 \qquad r=\sqrt{2}$

Question: $f(x) = \begin{cases} x^2-1, & x\le 1 \\ k(x-1), & x>1 \end{cases}$ $k\in\mathbb{R}$, Find k for f is differentiable at $x=1$

a) continuity (no needed)

$f(1) = 0 \qquad f(1) = \lim_{x\to 1^-}(x^2-1) = \lim_{x\to 1^+} k(x-1) = 0$

b) derivative

definition

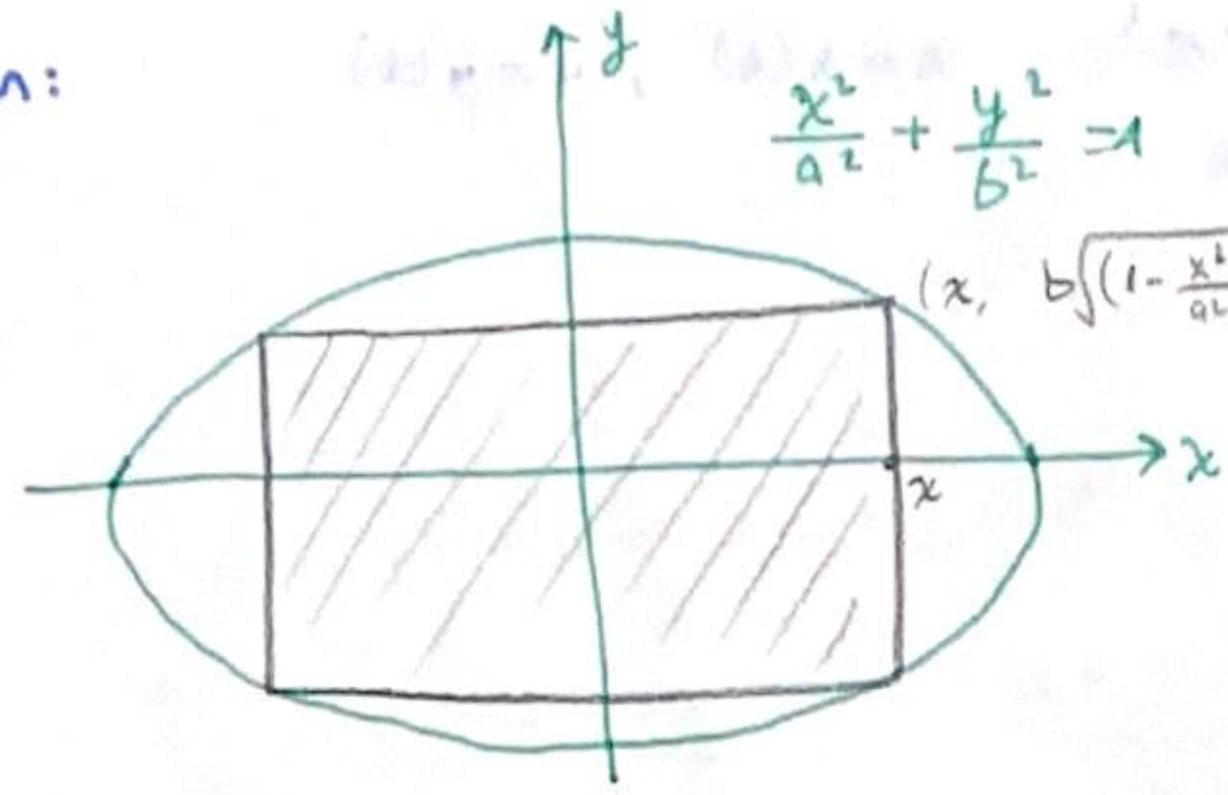
$$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} \qquad \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} \Rightarrow \frac{k((1+h)-1)-0}{h} = \lim_{h\to 0} k = k$$

$$\lim_{h\to 0}\frac{f(1-h)-f(1)}{-h} \Rightarrow \frac{(1-h)^2-1-0}{-h} = \lim_{h\to 0}(2-h) = 2$$

$\Big\} \; k=2$

Question:

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ max rectangle area, lengths

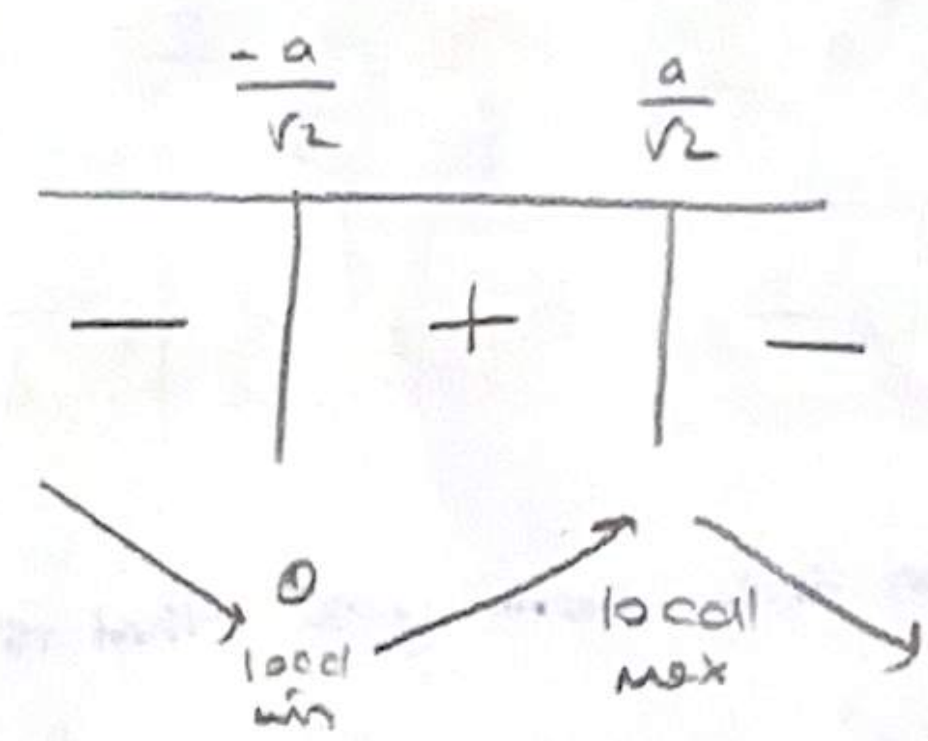


$A = 2x \cdot 2b\sqrt{1 - \frac{x^2}{a^2}}$

$A = 4\frac{b}{a}\sqrt{a^2x^2 - x^4}$

$A' = \frac{\cancel{2}4\frac{b}{a}}{\cancel{2}\sqrt{a^2x^2 - x^4}} \cdot (2a^2x - 4x^3)$

$A' = \frac{4b}{a} \cdot \frac{x^{\not 2}(a^2 - 2x^2)}{x\sqrt{a^2 - x^2}}$ $\to \pm\frac{a}{\sqrt{2}}$, $\to \pm a$



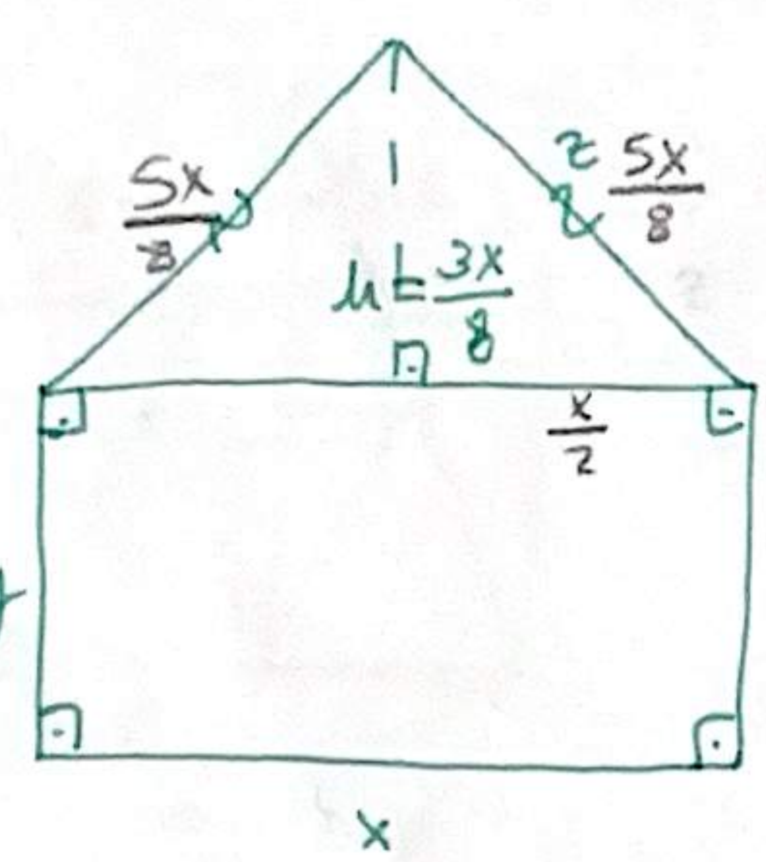
$x = \pm\frac{a}{\sqrt{2}}$ $\quad y = \frac{b}{\sqrt{2}}$

① $x = -\frac{a}{\sqrt{2}}$

$2x \cdot 2y = A$ $\quad \frac{2a}{\sqrt{2}} \cdot \frac{2b}{\sqrt{2}} = 2ab$ br^2

Question:

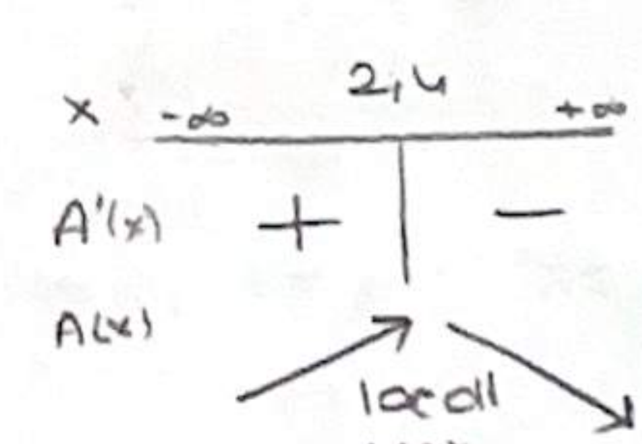
Max area, perimeter $= 9$



$\frac{\overset{9}{\cancel{18}}x}{\underset{4}{\cancel{8}}} + 2y = 9$ $\quad y = \frac{36 - 9x}{8}$

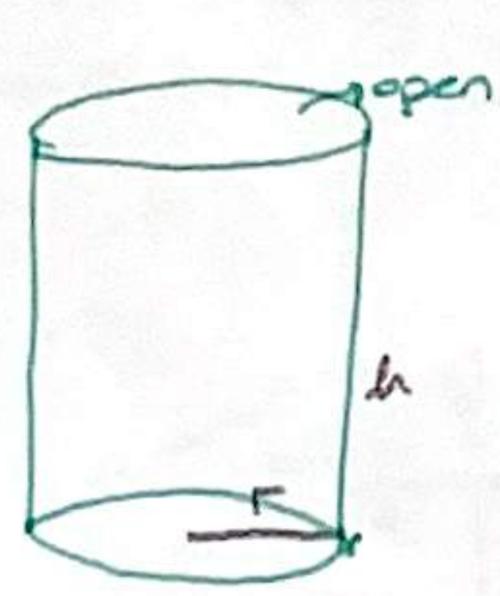
$y = \frac{36-9x}{8}$ $\quad A(x) = \underbrace{\frac{3x}{8} \cdot \frac{x}{2}}_{\text{triangle}} + \underbrace{x \cdot \frac{36-9x}{8}}_{\text{rectangle}}$

$A(x) = \frac{3x^2 + 72x - 18x^2}{16} = \frac{72x - 15x^2}{16}$ $\quad A'(x) = \frac{72 - 30x}{16} \to \frac{72}{30}$



$x = 2,4$ $\quad y = 1,8$ $\quad z = 1,5$ $\quad h = 0,9$

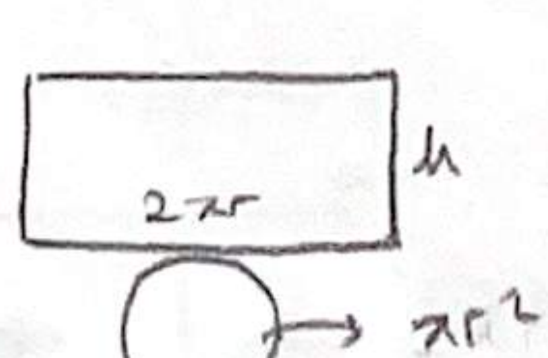
Question:



base is more expensive $(k \to 2k)$

volume is $300\ m^3$

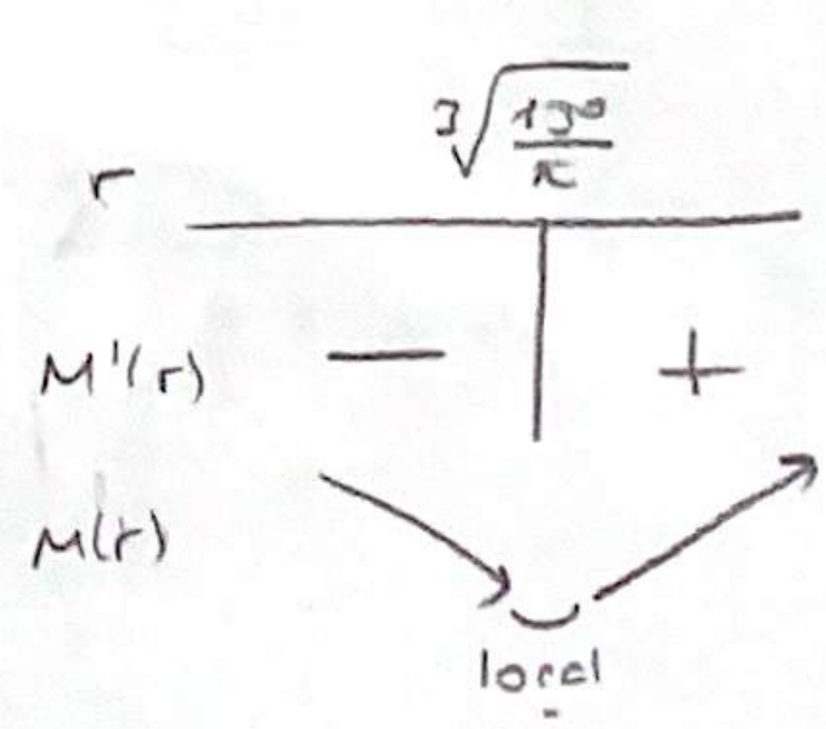
$\pi r^2 h = 300$ $\quad h = \frac{300}{\pi r^2}$



$M = \underbrace{h \cdot 2\pi r \cdot k}_{\text{rectangle}} + \underbrace{\pi r^2 \cdot 2k}_{\text{circle}}$

$M(r) = \frac{600k}{r} + 2\pi r^2 k$

$M'(x) = -\frac{600k}{r^2} + 4\pi r k$ $\Rightarrow 4\pi r^3 \cancel{k} = \overset{150}{\cancel{600}}\cancel{k}$ $\quad r = \sqrt[3]{\frac{150}{\pi}}$



$r = \sqrt[3]{\frac{150}{\pi}}$ $\quad h = 2\sqrt[3]{\frac{150}{\pi}}$

Question: For the curve given by $\begin{cases} t^2\sin(x) + x^3 = e^t \\ \sin(y) = t\cdot\sin(t) - 2t \end{cases}$ $x = x(t)$, $y = y(t)$

Find the normal line at $t=0$

$t=0 \quad x(0)=1 \quad y(0)=0$

$2t\cdot\sin(x) + t^2\cdot\cos(x)\cdot x'(t) + 3x^2\cdot x'(t) = e^t$ $\qquad \cos(y)\cdot y' = \sin(t) + t\cdot\cos(t) - 2$

$3x^2\cdot x'(0) = 1 \qquad x'(0) = \frac{1}{3}$ $\qquad y'(0) = -2$

$$\frac{dy}{dx} = \frac{dy(t)/dt}{dx(t)/dt} = \frac{-2}{1/3} = -6 = m_T \qquad m_N\cdot m_T = -1$$

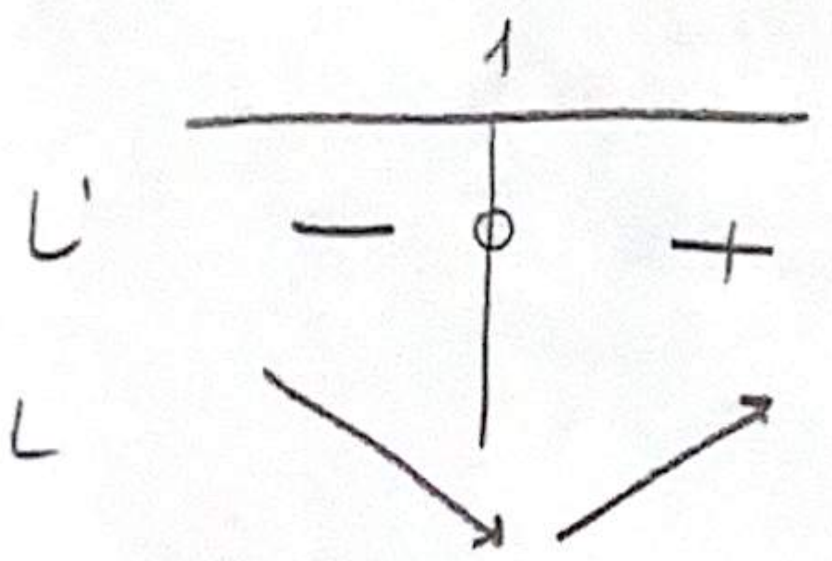
$\frac{1}{6} \quad -6$

$(y - y_0) = m_N(x - x_0) \Rightarrow (y - 0) = \frac{1}{6}(x-1) \Rightarrow x - 6y - 1 = 0$

Question: Find a point $P(a,b)$ on the curve $y=x^2$ that is closest to point $Q(3,0)$. Also find the distance $L = |PQ|$. $\qquad b=a^2$

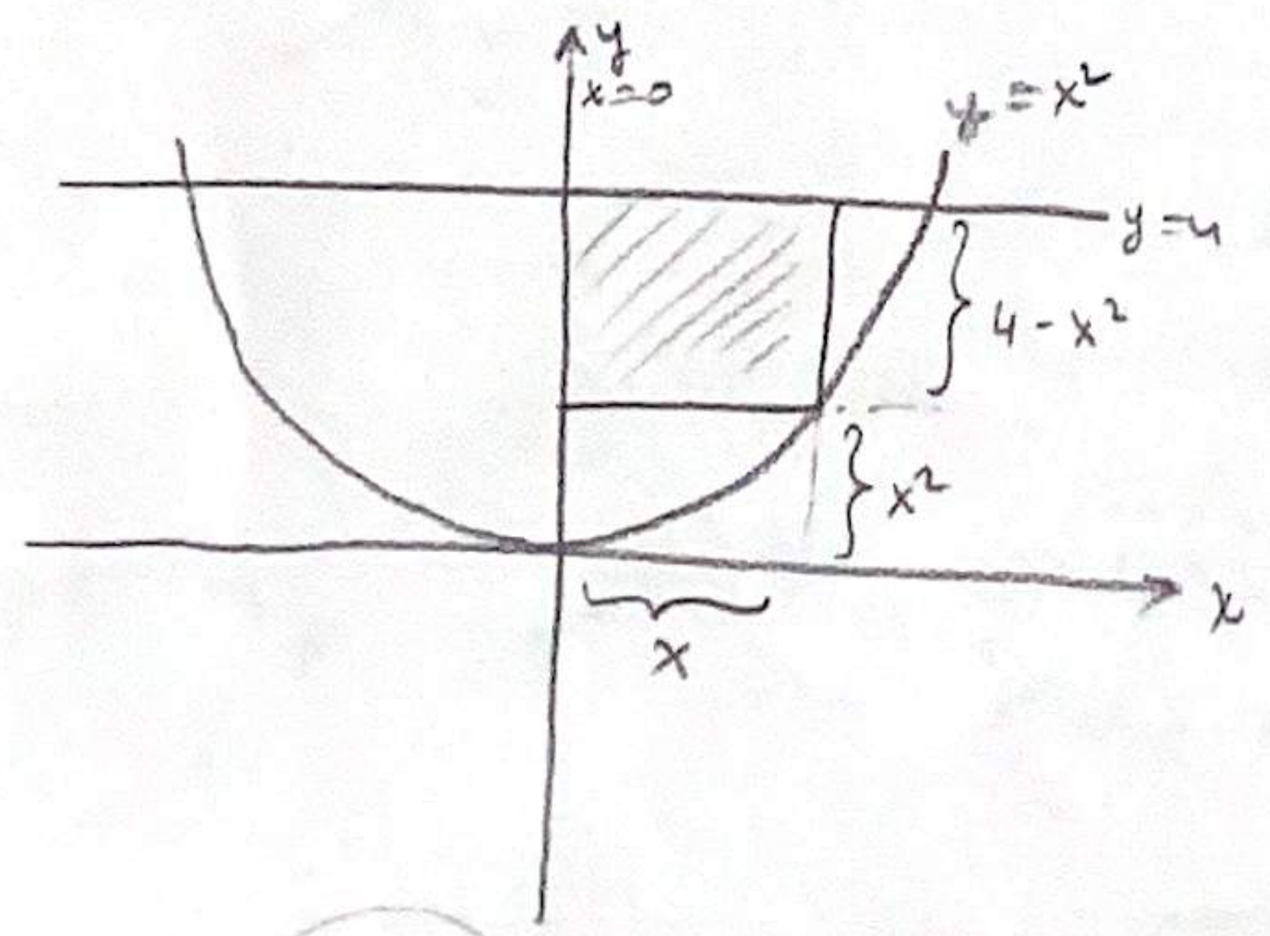
$$L = \sqrt{(3-a)^2 + (0-b)^2} = \sqrt{(3-a)^2 + (0-a^2)^2} = \sqrt{9-6a+a^2+a^4}$$

$$L' = \frac{4a^3+2a-6}{2\sqrt{9-6a+a^2+a^4}} \Rightarrow \frac{2a^3+a-3}{\sqrt{9-6a+a^2+a^4}} = \frac{2(a^3-1)+a-1}{\sqrt{9-6a+a^2+a^4}} = \frac{\overset{\to a=1}{(a-1)}\left[2a^2+2a+3\right]}{\sqrt{9-6a+a^2+a^4}}$$



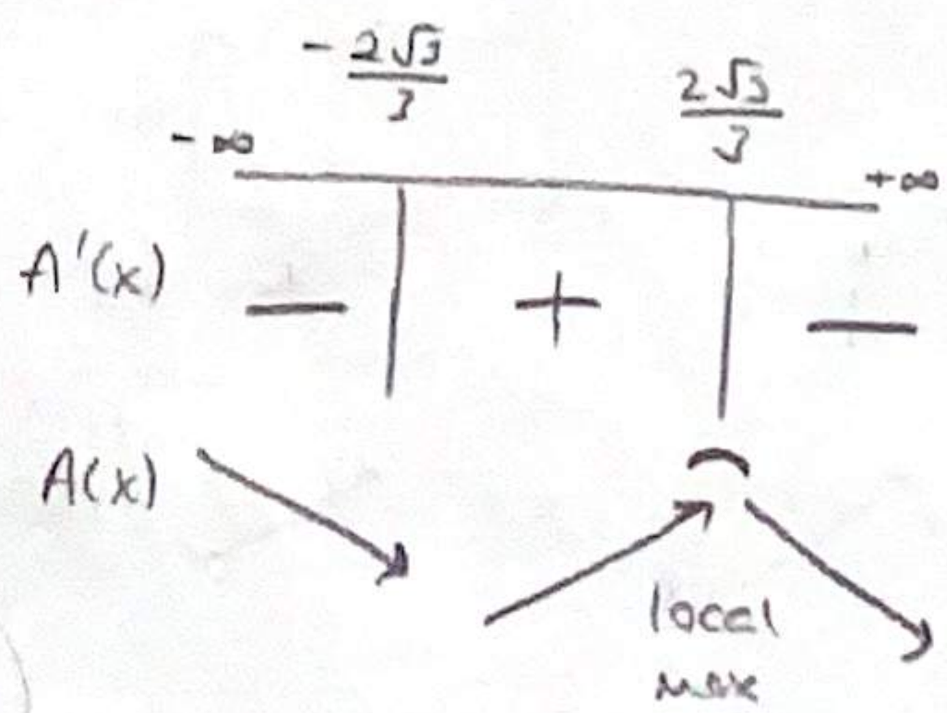
$(1, 1)$ local min point $\quad L = \sqrt{5}$

Question: $y = x^2$; $x=0$, $y=4$ Max area rectangle



$A = x(4-x^2) \qquad A = 4x - x^3 \qquad A' = 4 - 3x^2$

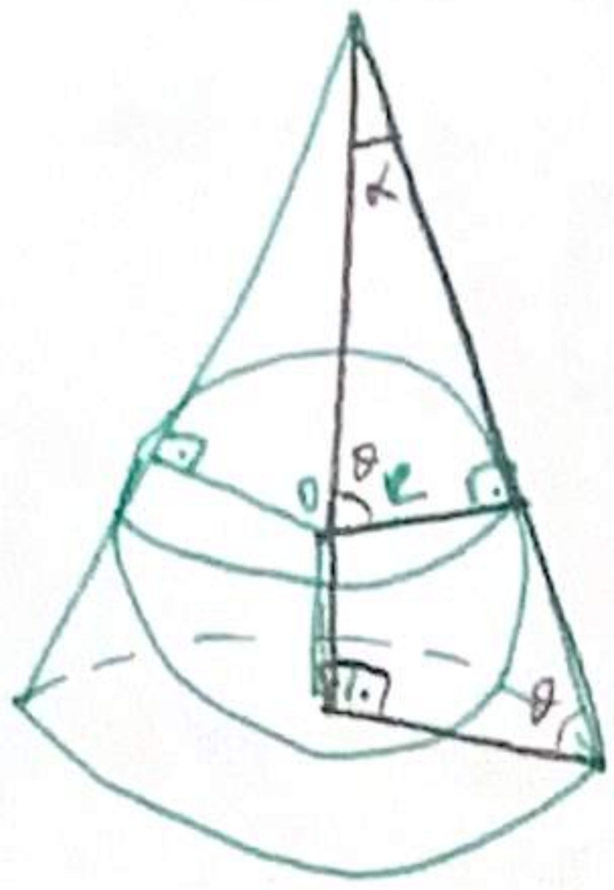
$A' = 0 \qquad x = \pm\frac{2\sqrt{3}}{3}$

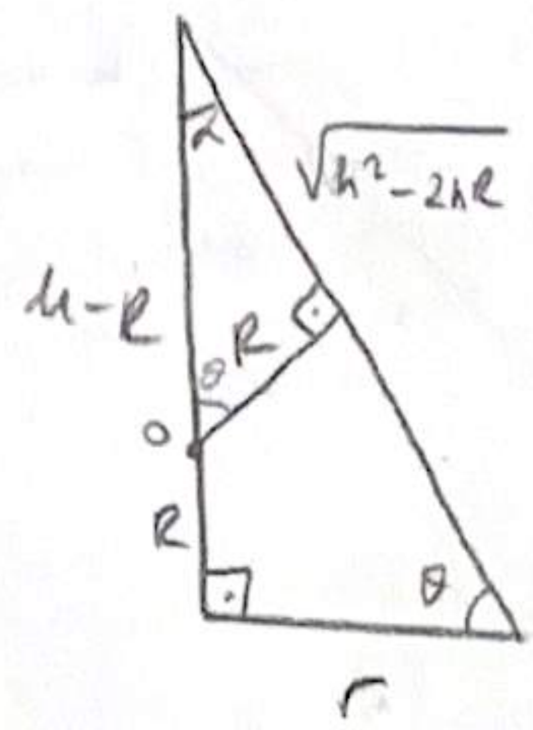


$x = \frac{2\sqrt{3}}{3}$ (length(1)) $\qquad 4 - x^2 = \frac{8}{3}$ (length(2)) $\qquad \frac{2\sqrt{3}}{3}\cdot\frac{8}{3} = \frac{16\sqrt{3}}{9}$ (Area)

Question:

Max volume for cone





$$\frac{r}{h} = \frac{R}{\sqrt{h^2-2hR}} \qquad r = \frac{R \cdot h}{\sqrt{h^2-2hR}}$$

$$V(h) = \frac{1}{3}\pi \cdot \frac{R^2 h^2}{(h^2-2hR)} \cdot h = \frac{1}{3}\pi \frac{R^2h^2}{h-2R} \qquad V'(h) = \frac{1}{3}\pi R^2 \left(\frac{2h(h-2R)-h^2}{(h-2R)^2}\right)$$

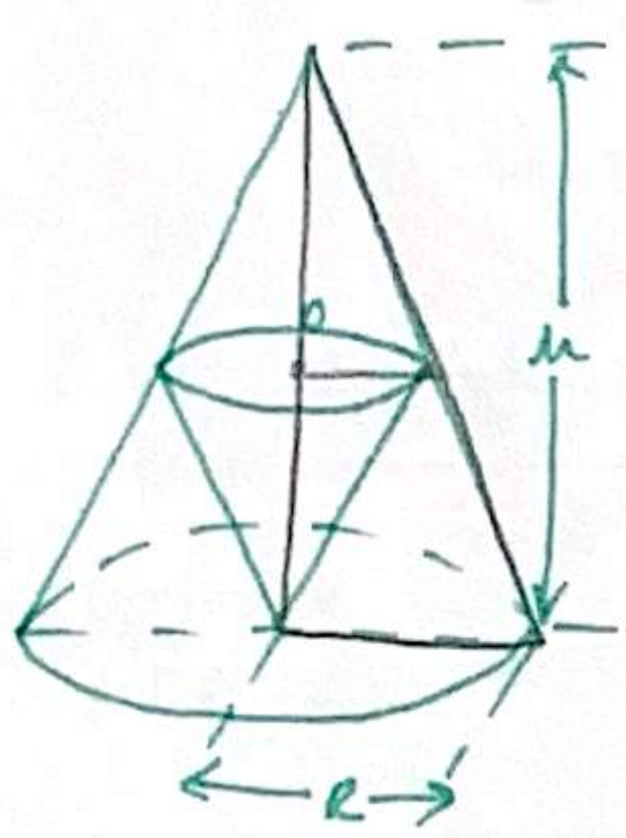
$$V'(h) \Rightarrow \frac{\pi R^2 (h^2-4hR)}{3(h-2R)^2} = 0 \qquad h^2-4hR=0 \qquad h(h-4R)=0 \qquad h_1=0 \quad h_2=4R$$

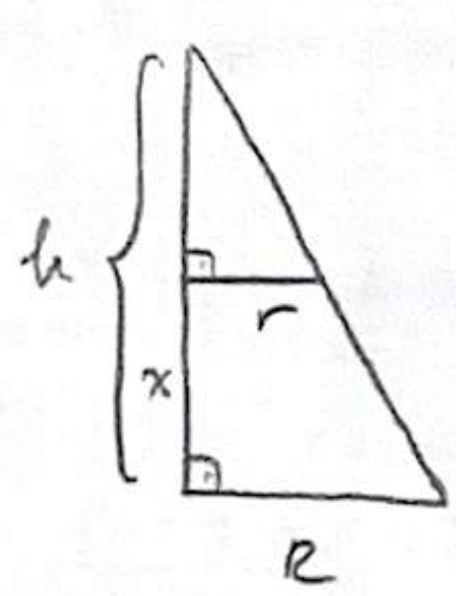
h		0		2R		4R	
$V'(h)$	+		−	‖	−		+
$V(h)$	↗		↘		↘	local min	↗

$$h=4R \qquad r=\sqrt{2}R \qquad V(h) = \frac{1}{3}\pi\, 2 \cdot R^2 \cdot 4R = \frac{8\pi R^3}{3}$$

Question:

Max volumed inversed cone





$$\frac{h-x}{r} = \frac{h}{R} \qquad h-x = \frac{hr}{R} \qquad h-\frac{hr}{R} = x$$

$$V(r) = \frac{1}{3}\pi \cdot r^2 \cdot \underbrace{h\left(1-\frac{r}{R}\right)}_{x} \qquad V'(r) \Rightarrow \frac{1}{3}\pi h\left(2r-\frac{3r^2}{R}\right) = 0 \qquad \frac{2rR-3r^2}{R}=0$$

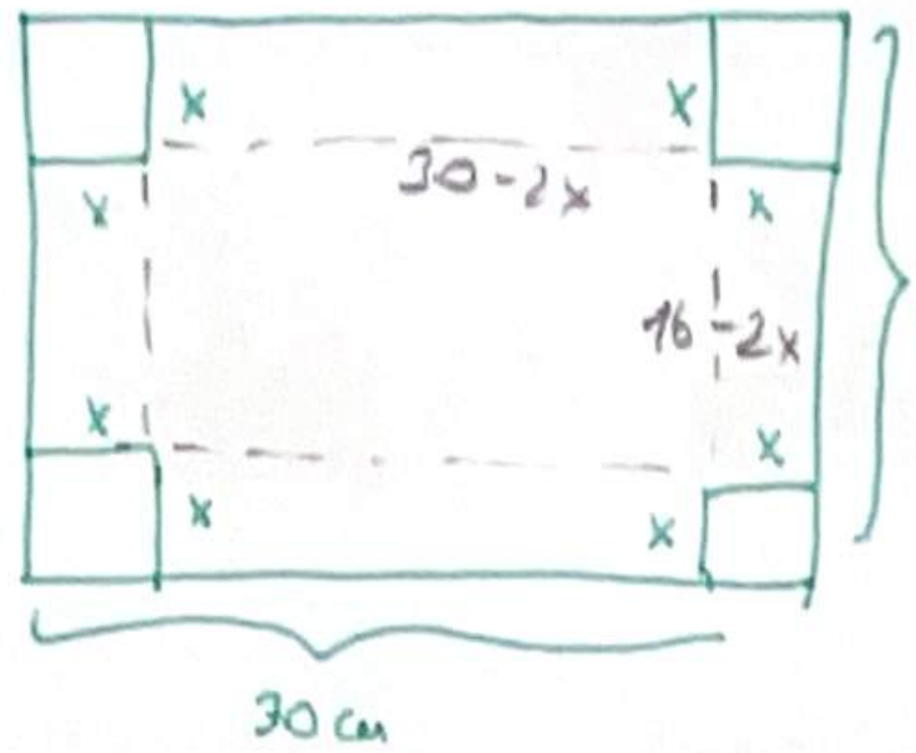
$$r(2R-3r)=0 \qquad r_1=0 \quad r_2=\frac{2R}{3}$$

r		0		$2R/3$	
$V'(r)$	−		+		−
$V(r)$	↘		↗	local max	↘

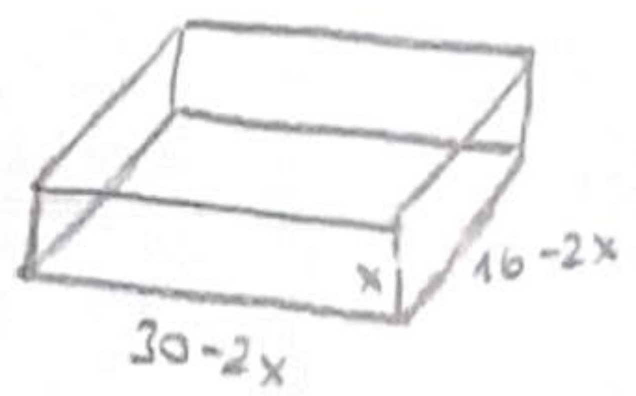
$$r=\frac{2R}{3} \qquad x=\frac{h}{3} \qquad V(2R/3) = \frac{1}{3}\pi \frac{4R^2}{9}\frac{h}{3}$$

$$V_{max} = \frac{4\pi R^2 h}{81}$$

Question:



Max volume



$V(x) = (30-2x).(16-2x).x = 4.(15-x).(8x-x^2) = 480x - 92x^2 + 4x^3$

$V'(x) = 480 - 184x + 12x^2 = 0 \qquad 3x^2 - 46x + 120 = 0$

$x \quad -12$
$3x \quad -10$

$(x-12)(3x-10)$

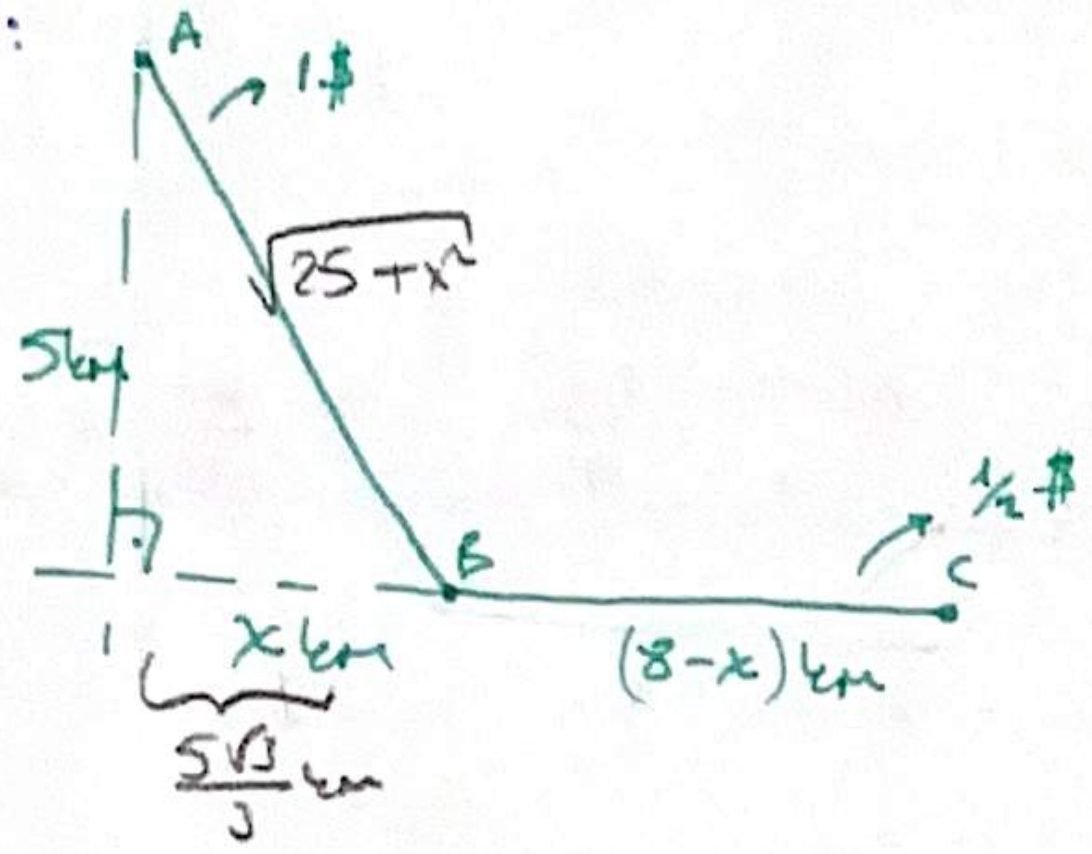
x	$-\infty$	$\frac{10}{3}$	12	∞
$V'(x)$	+		−	+
$V(x)$	↗	local max	↘	↗

$x = \frac{10}{3}$

$V(10/3) = 4\left(15 - \frac{10}{3}\right)\left(\frac{80}{3} - \frac{100}{9}\right)$

$4 \cdot \frac{35}{3} \cdot \frac{140}{9} = \frac{19600}{27}\ cm^3$

Question:



Min cost, find B point

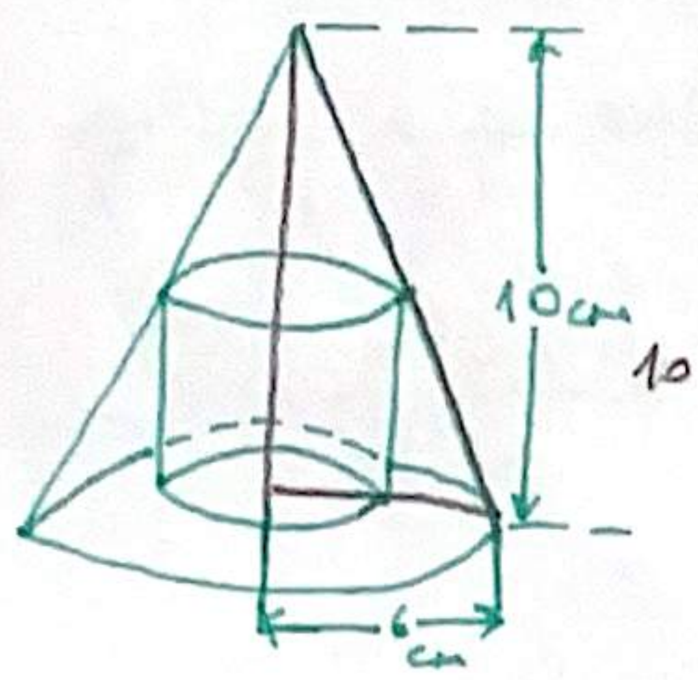
$C(x) = 1 \cdot \sqrt{25+x^2} + \frac{1}{2} \cdot (8-x)$

$C'(x) = \frac{2x}{2\sqrt{25+x^2}} - \frac{1}{2} \Rightarrow \frac{2x - \sqrt{25+x^2}}{2\sqrt{25+x^2}} = 0$

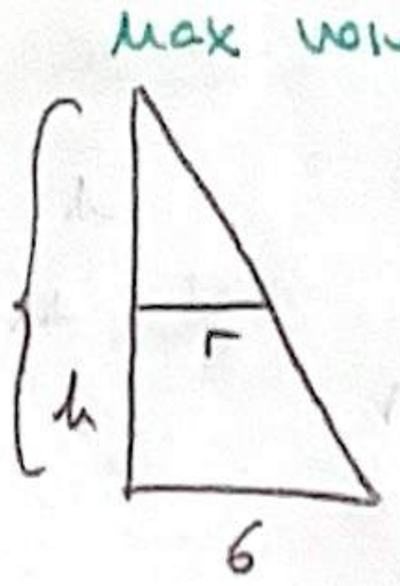
$4x^2 = 25 + x^2 \Rightarrow x = \pm\frac{5\sqrt{3}}{3}$

x	$-\frac{5\sqrt{3}}{3}$		$\frac{5\sqrt{3}}{3}$	
$C'(x)$	+		−	+
$C(x)$	↗		↘ local min	↗

Question:



Max volumed cylinder inside given cone



$\frac{5}{3}\ \cancel{\frac{10}{6}} = \frac{10-h}{r} \qquad 5r = 30 - 3h \qquad h = \frac{30-5r}{3}$

$V(r) = \pi r^2\left(\frac{30-5r}{3}\right) = 10\pi r^2 - \frac{5\pi r^3}{3}$

$V'(r) \Rightarrow 20\pi r - 5\pi r^2 = 0 \qquad 5\pi r(4-r) = 0 \qquad r_1 = 0 \quad r_2 = 4$

r		0		4	
$V'(r)$	−		+		−
$V(r)$	↘		↗	local max	↘

$r = 4 \qquad h = \frac{10}{3} \qquad \pi \cdot 16 \cdot \frac{10}{3} = \frac{160\pi}{3}$

Question: check the continuity of the given function

$$f(x) = \frac{(x-3).\arcsin(x)}{\underset{0}{x}.\left(1-e^{\frac{1}{\underset{1}{x-1}}}\right)}$$

$D[\arcsin x] : [-1,1]$

and possible discontinuous x values are 0 and 1

for $x=0$

$\lim_{x\to 0} f(x) = f(0)$

$$\lim_{x\to 0^-} \frac{(x-3)\ \arcsin(x)}{x\left(1-e^{\frac{1}{x-1}}\right)} = \frac{\arcsin(x) + (x-3)(\cos(\arcsin(x)))}{\left(1-e^{\frac{1}{x-1}}\right) + \frac{1}{(x-1)^2}.x.e^{\frac{1}{x-1}}} = \frac{-3e}{e-1}$$

because of existence of 0/0 indeterminity it is applicable l'hospital

$f(0) = -\frac{3e}{e-1}$

$\lim_{x\to 0^+} = \frac{-3}{e-1}$ so, function is continuous at $x=0$

for $x=1$

$\lim_{x\to 1} f(x) = f(1)$

$$\lim_{x\to 1^-} \frac{(x-3)\ \arcsin(x)}{x.\left(1-e^{\frac{1}{x-1}}\right)} = \frac{-2.(\pi/2)}{1.\ 1} = -\pi$$

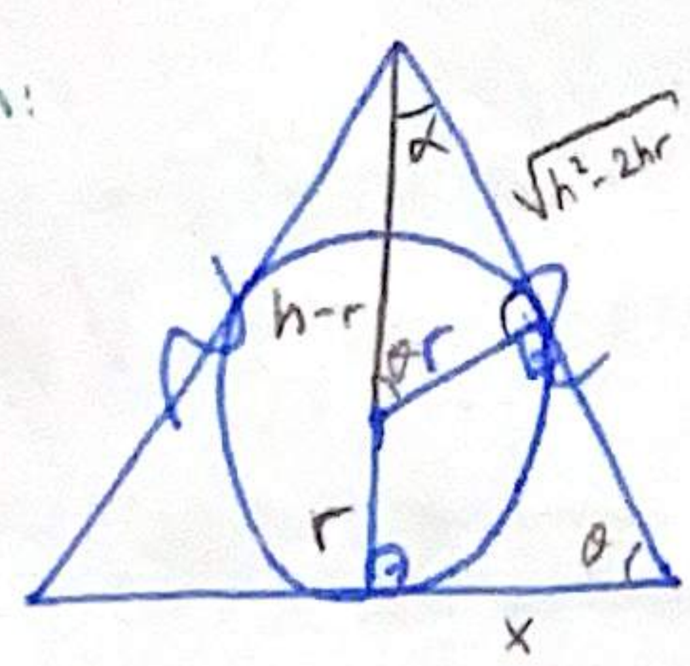
as the function is undefined at the right side of 1 so, limit value is $-\pi$ and then if we set $f(1)$ as $-\pi$ we can say that function is continuous at $x=1$

Question: $f(x) = \sinh(x) + e^x + 1$, for $f^{-1}(x)$ find tangent line at $P(0,2)$

$(f^{-1})' = \frac{1}{f'(f^{-1}(x))}$ $f'(x) = \cosh(x) + e^x$ $f^{-1}(2) = 0$

$(f^{-1}(2))' = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(0)} = \frac{1}{2}$ $(y-0) = \frac{1}{2}(x-2)$ $y = \frac{x-2}{2}$

Question:



min area for the isosceles triangle

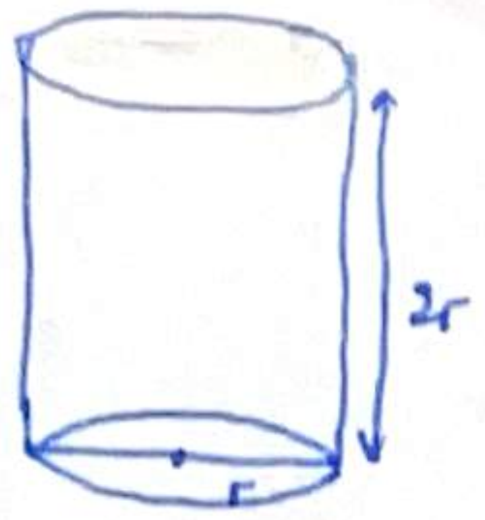
$\tan\alpha = \frac{x}{h} = \frac{r}{\sqrt{h^2-2hr}}$ $x = \frac{hr}{\sqrt{h^2-2hr}}$ $A(h) = \frac{h^2 r}{\sqrt{h^2-2hr}}$

$$A'(h) = \frac{2hr\sqrt{h^2-2hr} - \frac{h^2 r\,(\cancel{2h-2r})^{(h-r)}}{\cancel{2}\sqrt{h^2-2hr}}}{(h^2-2hr)} = \frac{2hr(h^2-2hr) - h^2r(h-r)}{(h^2-2hr)\sqrt{h^2-2hr}}$$

$$\Rightarrow \frac{2h^4 r \overset{-3h^2r^2}{\cancel{-4h^2r^2}} - h^3 r \cancel{+h^2r^2}}{h(h-2r)\sqrt{h^2-2hr}} = \frac{\cancel{h}.\overset{0}{hr}(2h^2-3r-h)}{\cancel{h}(h-2r)\sqrt{h^2-2hr}}$$

0 2r 0 2r

Question:



Linear approximation for $r = 3{,}01$ cm (volume of cylinder).

$V = \pi r^2 (2r)$ $\quad V = 2\pi r^3 \quad V' = 6\pi r^2$

$V(r) \approx L(r) = V(r) + V'(r)(r-a)$ for $a = 3$

$L(3) = V(3) + V'(3)(3{,}01 - 3)$

$L(3) = 54\pi + 54\pi(0{,}01) \qquad L(3) = 54{,}54\pi$ cm^3

Question: $y = \dfrac{e^x}{x^2-1}$ ⊛ $D: (-\infty,-1) \cup (-1,1) \cup (1,\infty)$ ⊛ $(0,-1)$ x intercept

⊛ $\lim_{x\to -1^-} \dfrac{e^x}{x^2-1} = +\infty \qquad \lim_{x\to -1^+} \dfrac{e^x}{x^2-1} = -\infty$ $\quad\Big|\quad$ $\lim_{x\to 1^-} \dfrac{e^x}{x^2-1} = -\infty \qquad \lim_{x\to 1^+} \dfrac{e^x}{x^2-1} = +\infty$

$x = -1$ (vertical asymptote) $\qquad$ $x = 1$ vertical asymptote

⊛ $\lim_{x\to -\infty} \dfrac{e^x}{x^2-1} = \lim_{x\to -\infty} \dfrac{\frac{e^x}{x^2}}{1-\frac{1}{x^2}} = 0 \qquad \lim_{x\to +\infty} \dfrac{\frac{e^x}{x^2}}{1-\frac{1}{x^2}} = 0$ $\qquad$ there is no oblique asymptote

$y=0$ namely x-axis is a horizontal asymptote

⊛ $y' = \dfrac{e^x.(x^2-1) - 2x.e^x}{(x^2-1)^2} \Rightarrow y' = \dfrac{e^x(x^2-2x-1)}{(x^2-1)^2}$ $\quad$ $\to 1+\sqrt{2}, 1-\sqrt{2}$ $\quad$ $\to -1, 1$

x	$-\infty$	-1	$1-\sqrt{2}$	1	$1+\sqrt{2}$	$+\infty$
y'	+	+	−	−	+	
y	↗	↗ local min ↘		↘ local max ↗		

y is increasing on $(-\infty,-1) \cup (-1, 1-\sqrt{2}) \cup (1+\sqrt{2}, \infty)$

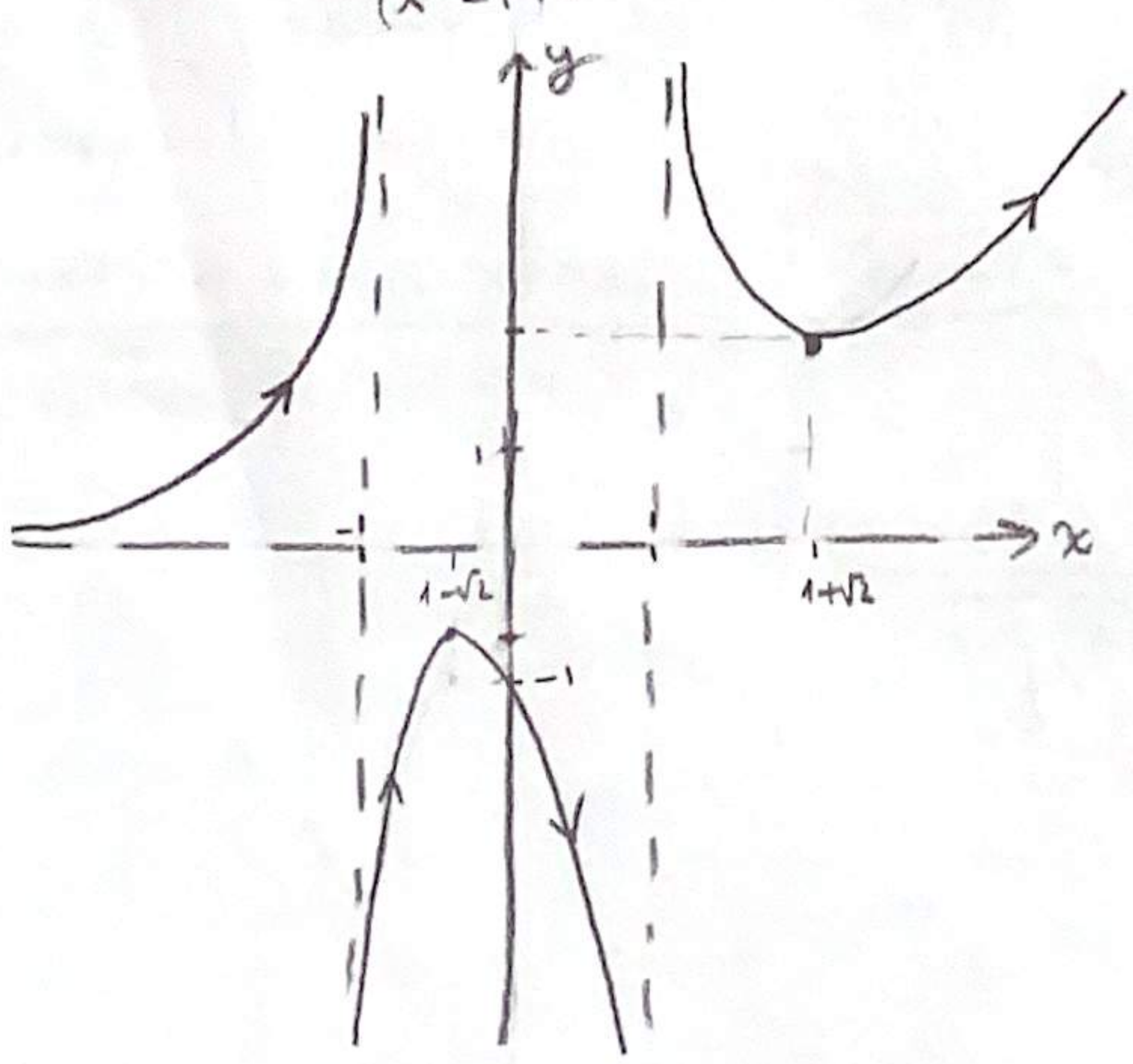
y is decreasing on $(1-\sqrt{2}, 1) \cup (1, 1+\sqrt{2})$

local max point $\left(1+\sqrt{2}, \dfrac{e^{1+\sqrt{2}}}{2+2\sqrt{2}}\right)$ $\qquad$ local min point $\left(1-\sqrt{2}, \dfrac{e^{1-\sqrt{2}}}{2-2\sqrt{2}}\right)$

⊛ $y'' = \dfrac{[e^x(x^2-2x-1) + e^x.(2x-2)].(x^2-1)^2 - 2(x^2-1).2x.e^x(x^2-2x-1)}{(x^2-1)^4}$

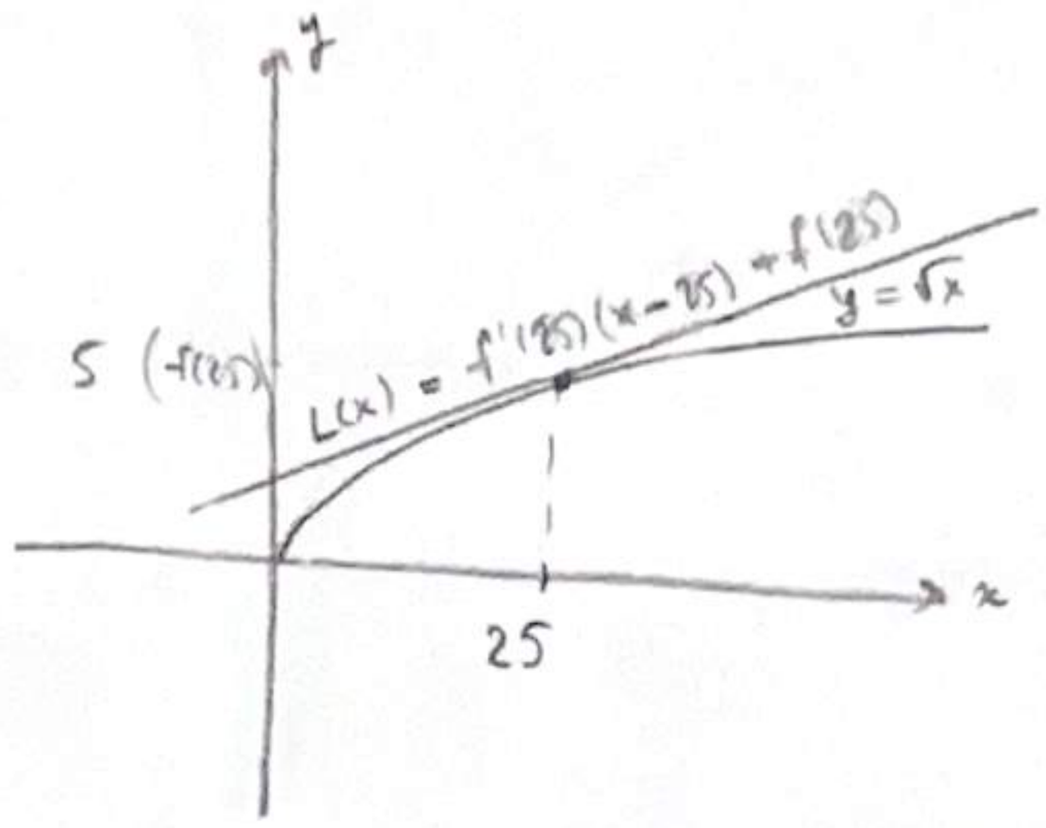
$\dfrac{e^x(x^2-3).(x^2-1)^2 - 4x(x^2-1).e^x(x^2-2x-1)}{(x^2-1)^4} = \dfrac{e^x(x^2-1)\left((x^2-1)(x^2-3) - 4x.(x^2-2x-1)\right)}{(x^2-1)^4}$ $\quad$ $x^4 - 4x^2 + 3 - 4x^3 + 8x^2 + 4x$

$y'' = \dfrac{e^x(x^2-1).(x^4-4x^3+4x^2+4x+3)}{(x^2-1)^4}$ $\quad$ second derivative (?)



Question: $\sqrt{26}$, linear approximation

let $f(x) = \sqrt{x}$



$f(x) \approx L(x) = f'(25)(x-25) + f(25)$

$f'(x) = \dfrac{1}{2\sqrt{x}} \Rightarrow f'(25) = \dfrac{1}{10}$

$L(26) = \dfrac{1}{10}(26-25) + 5 \Rightarrow \sqrt{26} \approx 5,1$

Question: $\sqrt[4]{626} \approx 5,002$

$f(x) \approx L(x) = f'(x)(x-a) + f(a)$

let $f(x) = \sqrt[4]{x}$ central point $(a) = 625$

$f'(x) = \dfrac{1}{4\sqrt[4]{x^3}}$ $f(625) = 5$ $f'(625) = 500$

$f(626) \approx L(626) = \dfrac{1}{500}(626-625) + 5$

Question: $(1{,}05)^{1{,}05}$

let $f(x) = x^x$ $f(x) \approx L(x) = f'(a)(x-a) + f(a)$ central point $(a) = 1$

$\ln[f(x)] = x\ln x \Rightarrow \dfrac{1}{f(x)_{x^x}} \cdot f'(x) = \ln x + x \cdot \dfrac{1}{x} \Rightarrow f'(x) = x^x(\ln x + 1)$

$f(1) = 1$ $f'(1) = 1$ $L(1{,}05) = 1.(1{,}05-1) + 1 \Rightarrow (1{,}05)^{1{,}05} \approx 1{,}05$

Question: $\sin 29°$

let $f(x) = \sin x$ $f(x) \approx L(x) = f'(a)(x-a) + f(a)$ central point $(a) = 30°$

$\boxed{1° = \dfrac{\pi}{180}}$

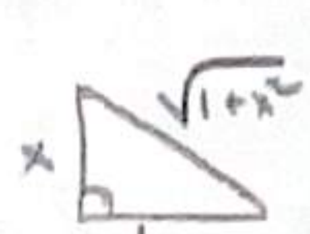
$f'(x) = \cos x$ $f(30) = \dfrac{1}{2}$ $f'(30) = \dfrac{\sqrt{3}}{2}$

$f(29) \approx L(29) = \dfrac{\sqrt{3}}{2}(29° - 30°) + \dfrac{1}{2} \Rightarrow \sin 29° \approx \dfrac{1}{2} - \dfrac{\sqrt{3}\pi}{180}$

Question! $\arctan(0.98)$

let $f(x) = \arctan x$ $f(x) \approx L(x) = f'(a)(x-a) + f(a)$ central point $(a) = 1$

$\dfrac{d}{dx}(\arctan x) = \dfrac{1}{\sec^2(\arctan x)}$



$f'(x) = \dfrac{1}{1+x^2} \Rightarrow f'(1) = \dfrac{1}{2}$ $f(1) = \dfrac{\pi}{4}$

$f(0.98) \approx L(0.98) = \dfrac{1}{2}(0.98 - 1) + \dfrac{\pi}{4} \Rightarrow \arctan(0.98) \approx \dfrac{\pi}{4} - 0{,}01$

Question: $\sqrt{1.02}$ Let $f(x) = \sqrt{1+x}$ $f'(x) = \dfrac{1}{2\sqrt{1+x}}$ $a=0$ $f(0)=1$ $f'(0)=\frac{1}{2}$

$f(1.02) \approx L(1.02) = 1 + \frac{1}{2}(1.02-1) \Rightarrow \sqrt{1.02} \approx 1.01$

or let $g(x) = \sqrt{x}$ $g'(x) = \dfrac{1}{2\sqrt{x}}$ $a=1$ $g(1)=1$ $g'(1)=\frac{1}{2}$

Question: $f(x) = \cos x$; at $\frac{\pi}{2}$ find linearization

$f'(x) = -\cos x$ $f(\frac{\pi}{2}) = 0$ $f'(\frac{\pi}{2}) = -1$ $\cos(1.75) \approx L(1.75) = 0 + (-1)(1.75 - \frac{\pi}{2})$

$\cos(1.75) \approx \frac{\pi}{2} - 1.75 \approx 0.18$

Question: $f(x) = x^3 + x$; $f^{-1}(10.1)$ $\underbrace{f(x)}_{10} = x(x^2+1)$ $x=2$ $\dfrac{d(f^{-1}(x))}{dx} = \dfrac{1}{f'(f^{-1}(x))}$

$f'(x) = 3x^2+1$ $f'(2) = 13$

$f^{-1}(10.1) \approx L(10.1) = \underbrace{f^{-1}(10)}_{2} + (f^{-1}(10))'(10.1-10)$

$2 + \frac{1}{13}(0.1)$

$$\begin{array}{r|l} 1000 & 13 \\ -910 & 0.00769 \\ 900 & \\ -780 & \\ 1200 & \\ -1170 & \\ 30\ldots & \end{array}$$

$f^{-1}(10.1) \approx 2.00769$

Question: $(1.02)^{1/3}$ $f(x) = \sqrt[3]{x}$ $a=1$ $f'(x) = \dfrac{1}{3\sqrt[3]{x^2}}$ $f(1)=1$ $f'(1) = \frac{1}{3}$

$f(1.02) \approx L(1.02) = 1 + \frac{1}{3}(1.02-1)$

$$\begin{array}{r|l} 2000 & 300 \\ & 0.006 \end{array}$$

$(1.02)^{1/3} \approx 1.0067$

Question: $t>0$ and $F(t) = \displaystyle\int_0^{\sqrt{t}} \cos(x^2)\,dx$, $F'(\frac{\pi}{4}) = ?$

$F'(x) = \cos(\sqrt{t})^2 \cdot \dfrac{1}{2\sqrt{t}} - \cancel{\cos(0^2)\cdot 0}^{\,0} \Rightarrow F'(x) = \dfrac{\cos t}{2\sqrt{t}} = \dfrac{\frac{\sqrt{2}}{2}}{2\sqrt{\frac{\pi}{4}}} = \dfrac{\sqrt{2\pi}}{2\pi}$

Question: $F(x) = \dfrac{1}{x} \underbrace{\displaystyle\int_1^x [2t - F'(t)]\,dt}_{\varphi(t)} \Rightarrow F'(1) = ?$

$F'(x) = \left(-\frac{1}{x^2}\right) \cdot \left(\displaystyle\int_1^x (2t - F'(t))\,dt\right) + \frac{1}{x}\left((2x - F'(x)).1 - (2.1 - F'(1)).0\right)$

$F'(1) = (-1)\cancel{\left(\displaystyle\int_1^1 ((2t-F'(t))\,dt\right)}^{\,0} + 1\,(2-F'(1)).1 - 0$ $F'(1) = 2 - F'(1)$ $F'(1) = 1$

Question: $F(x) = \dfrac{1}{x}\left(\displaystyle\int_1^x [e^{1-\sqrt{t}} - F'(2-t)]\,dt\right)$, $F'(1) = ?$

$F'(x) = \left(-\frac{1}{x^2}\right)\left(\displaystyle\int_1^x (e^{1-\sqrt{t}} - F'(2-t))\right) + \frac{1}{x}\left((e^{1-\sqrt{x}} - F'(2-x)).1 - \cancel{(e^{1-\sqrt{1}} - F'(2-1)).0}\right)$

$F'(1) = 0 + 1.(e^0 - F'(1) - 0)$ $F'(1) = 1/2$

Question: $f(x) = \dfrac{x^2+1}{x^2+x+1}$, find the absolute min. and max. at $[-2,2]$

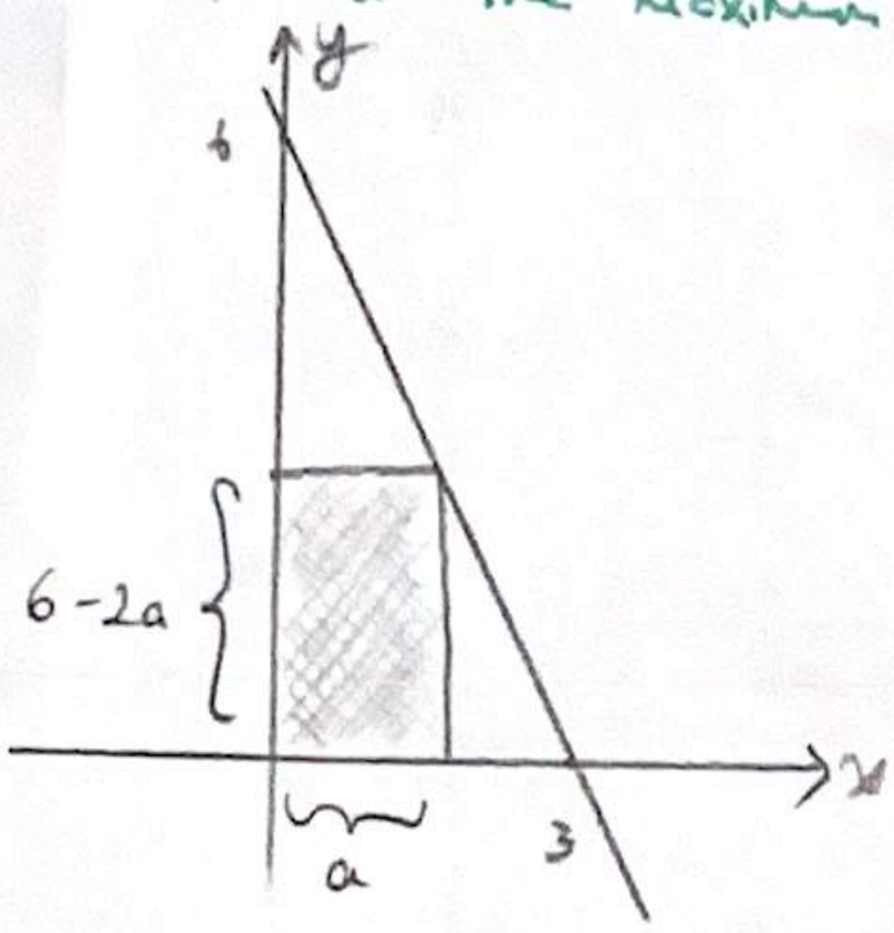
$f'(x) = \dfrac{2x\cdot(x^2+x+1) - (x^2+1)\cdot(2x+1)}{(x^2+x+1)^2} = \dfrac{(x^2-1)}{(x^2+x+1)^2}$ $x_1 = 1$, $x_2 = -1$

$f(1) = \frac{2}{3}$, $f(-1) = 2$ / $f(-2) = \frac{5}{3}$, $f(2) = \frac{5}{7}$

$f(1) = \frac{2}{3}$ $(1, \frac{2}{3}) \Rightarrow \underset{x\in[-2,2]}{\text{Min } f(x)}$

$f(-1) = 2$ $(-1, 2) \Rightarrow \underset{x\in[-2,2]}{\text{Max } f(x)}$

Question: Find the maximum rectangle's area between x-axis , y-axis and $2x+y=6$



$A = a\cdot(6-2a)$ $A' = 6-4a$ $a = \frac{3}{2}$

$6a - 2a^2$

$A = \frac{3}{2}\cdot 3 = 9/2$ br²

Question: Linearization $L(x) - f(a) = f'(a)(x-a)$

i) $\sqrt{37} = ?$ $6{,}08\overline{3}$ ii) $\sqrt{9.2} = ?$ $3.0\overline{3}$ iii) $\ln(1.02) = ?$

$f(x) = \sqrt{x}$ $f'(x) = \dfrac{1}{2\sqrt{x}}$

$a = 36$

$L(37) = 6 + \frac{1}{12}\cdot(37-36) = 6{,}08\overline{3}$

$100 \div 12$: -96, 40 → $0{,}08\overline{3}$

$\sqrt{37} \approx 6{,}08\overline{3}$

$g(x) = \ln x$ $g'(x) = \dfrac{1}{x}$ $a = 1$

$g(1) = 0$ $g'(1) = 1$

$L(1.02) = 0 + 1\cdot(1.02 - 1)$

$\ln(1.02) \approx 0{,}02$

$L(9.2) = 3 + \frac{1}{6}(9.2 - 9) = 3 + \frac{1}{30}$

$100 \div 30$: -90, 10 → $0{,}0\overline{3}$

$\Rightarrow \sqrt{9.2} \approx 3{,}0\overline{3}$

iv) $(2.007)^5$ $L(x) - f(a) = f'(a)(x-a)$

$f(x) = x^5$ $f'(x) = 5x^4$ $a = 2$

$L(2.007) = 32 + 80\,(2.007 - 2)$

$32 + 8\cancel{0}\cdot\frac{7}{100\cancel{0}}$ $\Rightarrow 32.56$

Question: $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \dfrac{96i + 20n}{8in + n^2} = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \underbrace{\dfrac{96i+20n}{8i+n}}_{}\cdot\dfrac{1}{n}$

$f(x_i)$ Δx

$\dfrac{96\frac{i}{n} + 20}{8\frac{i}{n} + 1}$ $\dfrac{96x + 20}{8x + 1}$

$\Delta x = \dfrac{b-a}{n}$ $x_i = a + \Delta x\, i$ $a = 0$ $b = 1$

$\frac{1}{n}$

$\boxed{\text{Hint: interval } [0,4]}$

$\lim\limits_{n\to\infty} \frac{4}{n} \sum\limits_{i=1}^{n} \underbrace{\dfrac{24i + 5n}{8i + n}}_{} \Rightarrow \int_0^4 \left(\dfrac{6x+5}{2x+1}\right) dx$

$\underbrace{\dfrac{24\frac{i}{n} + 5}{8\frac{i}{n} + 1}}_{}$

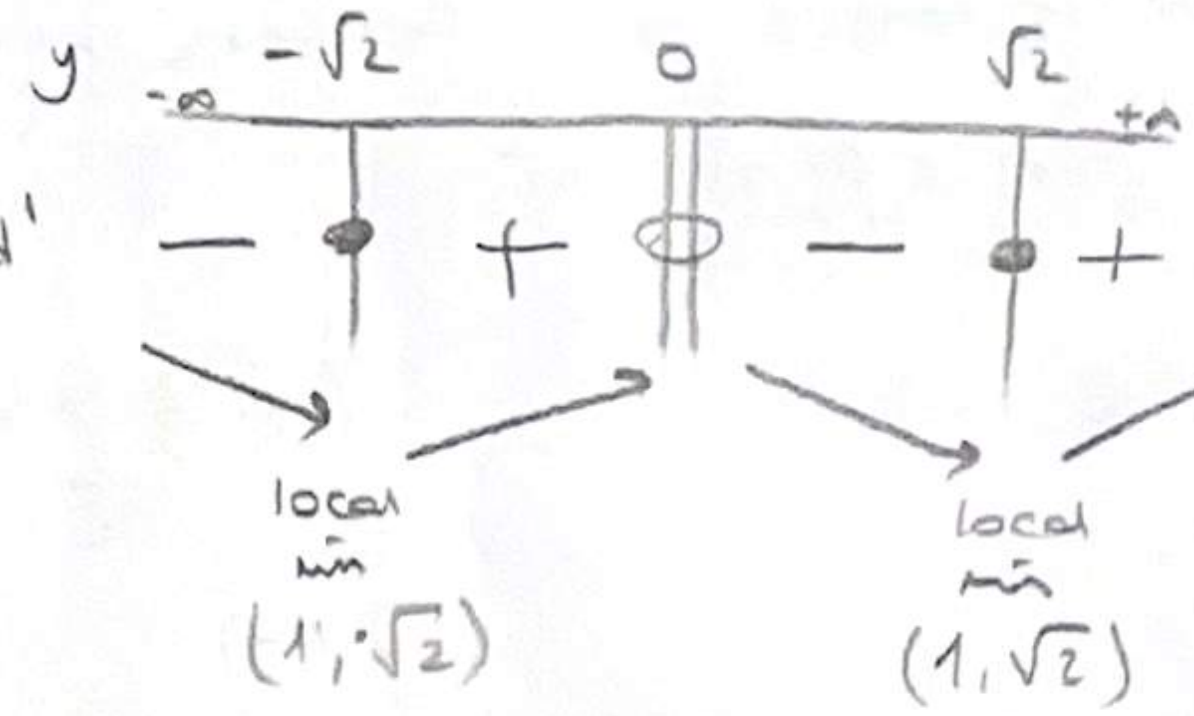
$\dfrac{6(\frac{4i}{n}) + 5}{2(\frac{4i}{n}) + 1}$

Question: $x \cdot y^2 = 2$, find point(s) closest to the origin

$$\left(\frac{2}{y^2}, y\right) \qquad d = \sqrt{(y-0)^2 + \left(\frac{2}{y^2}-0\right)^2} = \sqrt{y^2 + \frac{4}{y^4}} = \sqrt{\frac{y^6+4}{y^4}}$$

$$d' = \frac{1}{2\sqrt{\frac{y^6+4}{y^4}}} \cdot \frac{6 \cdot y^5 \cdot y^4 - 4 \cdot y^3 (y^6+4)}{y^8} = \frac{2y^9 - 16y^3}{2 \cdot y^8 \sqrt{\frac{y^6+4}{y^4}}} = \frac{(y^6 - 8)}{y^3\sqrt{y^6+4}}$$

$$\Rightarrow \frac{\overset{\to \mp\sqrt{2}}{(y^2-2)}\,(y^4+2y^2+4)}{\underset{\to 0}{y^3}\sqrt{y^6+4}}$$



Question: $\int_0^2 (x^2 + 2x + 3)\, dx$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x$$

$b = 2 \quad a = 0 \quad \Delta x = \frac{2}{n}$

$x_i = a + \Delta x \cdot i \quad \boxed{x_i = \frac{2i}{n}}$

$$\lim_{n\to\infty} \left(\sum_{i=1}^{n} \left[\left(\frac{2i}{n}\right)^2 + 2\left(\frac{2i}{n}\right) + 3 \right] \cdot \frac{2}{n} \right) = \lim_{n\to\infty} \left[\sum_{i=1}^{n} \left(\frac{8i^2}{n^3} + \frac{8i}{n^2} + \frac{6}{n} \right) \right]$$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{8}{n^3} \underbrace{\sum_{i=1}^{n} i^2}_{} + \frac{8}{n^2} \underbrace{\sum_{i=1}^{n} i}_{} + \frac{6}{n} \underbrace{\sum_{i=1}^{n} 1}_{n} \right) = \lim_{n\to\infty} \left[\frac{4}{3} \cdot \left(2 + \cancel{\frac{3}{n}} + \cancel{\frac{1}{n^2}}\right) + 4 \cdot \left(1 + \cancel{\frac{1}{n}}\right) + 6 \right]$$

$\frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \qquad \frac{8}{n^2} \cdot \frac{n(n+1)}{2}$

$\frac{4}{3} \cdot \frac{2n^2+3n+1}{n^2} \qquad 4\left(1+\frac{1}{n}\right)$

$\frac{4}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$

$$\frac{8}{3} + 4 + 6 = \frac{38}{3}$$

Question: $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left[8\left(1 + \frac{i}{n}\right)^3 + 3\left(1 + \frac{i}{n}\right)^2 \right] \qquad \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \int_a^b f(x)\,dx$

$\frac{b-a}{n} = \Delta x$

$a + \Delta x \cdot i = x_i$
$1 + \frac{1}{n} \cdot i$

$a = 1 \quad b = 2$

$f(x) = 8x^3 + 3x^2$

$$\int_1^2 (8x^3 + 3x^2)\,dx$$

Question: $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi^2 i}{n^2} \cos^2\left(\frac{\pi i}{n}\right) \qquad \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \int_a^b f(x)\,dx$

$$\boxed{\int_0^\pi (x \cos^2 x)\,dx} \qquad \underbrace{\frac{\pi}{n}}_{\Delta x} \cdot \underbrace{\cos^2\left(\frac{\pi i}{n}\right) \cdot \frac{\pi i}{n}}_{f(x_i)}$$

$\frac{b-a}{n} = \Delta x = \frac{\pi}{n}$

$x_i = \cancel{a} + \Delta x \cdot i \quad \boxed{x_i = \frac{\pi i}{n}} \quad \boxed{b = \pi} \quad \boxed{a = 0}$

$f(x) = x \cdot \cos^2 x$

Question: $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\frac{i}{n^2}}{\left(\frac{2i}{n}+1\right)^3} \qquad \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \int_a^b f(x)\,dx$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\frac{1}{n}}_{\Delta x} \cdot \frac{\frac{i}{n}}{\left(\frac{2i}{n}+1\right)^3} \qquad \int_0^1 \frac{x}{(2x+1)^3}\,dx$$

$x_i = \underset{0}{\cancel{a}} + \underset{\frac{1}{n}}{\Delta x} \cdot i$

$b = 1 \quad a = 0 \quad f(x) = \frac{x}{(2x+1)^3}$

Question: $x = \tan 2\theta$ $\quad y = -1 + \sec 2\theta$ $\quad \left.\dfrac{d^2y}{dx^2}\right|_{\theta = \frac{\pi}{6}} = ?$

$$(\tan 2\theta)' = \left(\frac{\sin 2\theta}{\cos 2\theta}\right)' = \frac{\overbrace{\cos 2\theta \cdot 2 \cdot \cos 2\theta + \sin 2\theta \cdot 2 \sin 2\theta}^{2(\cos^2 2\theta + \sin^2 2\theta)}}{\cos^2 2\theta}$$

$$= 2 \sec^2 2\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cancel{2} \cdot \sin 2\theta \cdot \cancel{\sec^2 2\theta}}{\cancel{2 \sec^2 2\theta}} = \sin 2\theta$$

$$(-1 + \sec 2\theta)' = \left(-1 + \frac{1}{\cos 2\theta}\right)' = 0 - \frac{-\sin 2\theta \cdot 2}{(\cos^2 2\theta)}$$

$$= 2 \sin 2\theta \cdot \sec^2 2\theta$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\cancel{2} \cdot \cos 2\theta}{\cancel{2} \sec^2 2\theta} = \cos^3 2\theta \Rightarrow \cos^3\left(2 \cdot \frac{\pi}{6}\right) = \frac{1}{8}$$

Question: $\lim\limits_{x \to 0^+} (6^x - 2^x + 1)^{1/x} = L$ $\quad \to 1^\infty$

$$\exp\left[\lim_{x \to 0^+}\left(\frac{\ln(6^x - 2^x + 1)}{x}\right)\right] = L$$

Note: $(\ln y)' = (\ln a^x)' \Rightarrow \dfrac{y'}{y} = \ln a \Rightarrow y' = y \cdot \ln a$ $\quad (6^x)' = 6^x \ln 6 \quad (2^x)' = 2^x \ln 2$

$$\Rightarrow \exp\left[\lim_{x \to 0^+}\left(\frac{\frac{6^x \ln 6 - 2^x \ln 2 + 0}{6^x - 2^x + 1}}{1}\right)\right] = \exp\left(\lim_{x \to 0^+} (\ln 6 - \ln 2)\right) = \exp\left(\ln\left(\frac{6}{2}\right)\right) = e^{\ln 3} = 3$$

Question: $f(x) = x^5 - 2x^2 + x$, find a root on $[0,1]$ by using Rolle Theorem

for the continuous $f: [a,b] \to \mathbb{R}$ function and f is differentiable for $\forall x \in (a,b)$. If $f(a) = f(b)$, there should exists at least one c point satisfies: $f'(c) = 0$.

$a = 0$, $b = 1$ $\quad f(0) = 0 \quad f(1) = 0$

$$f'(x) = 5x^4 - 4x + 1 \qquad f''(x) = 20x^3 - 4 = 4(5x^3 - 1) = 4(\sqrt[3]{5}x - 1)\underbrace{(\sqrt[3]{25}x^2 + \sqrt[3]{5}x + 1)}_{\text{always positive}}$$

$$\hookrightarrow \frac{1}{\sqrt[3]{5}}$$

	$-\infty$	$\frac{1}{\sqrt[3]{5}}$	∞
$f''(x)$	$-$		$+$

Because of the sign change there should exists at least one root that satisfies $f'(x) = 0$

Question: $\displaystyle\int_{-3}^{-1} (1 - x^3)\,dx \Rightarrow \int_a^b f(x_i)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x$

$b = -1 \quad a = -3$

$f(x) = 1 - x^3 \qquad \Delta x = \dfrac{(-1)-(-3)}{n} = \dfrac{2}{n}$

$x_i = a + \Delta x\, i = -3 + \dfrac{2}{n} \cdot i$

$$\lim_{n \to \infty} \sum_{i=1}^{n} \left[1 - \left(-3 + \frac{2i}{n}\right)^3\right] \cdot \frac{2}{n}$$

$$\binom{3}{0}\left(\frac{2i}{n}\right)^3(-3)^0 + \binom{3}{1}\left(\frac{2i}{n}\right)^2(-3)^1 + \binom{3}{2}\left(\frac{2i}{n}\right)^1(-3)^2 + \binom{3}{3}\left(\frac{2i}{n}\right)^0(-3)^3$$

$$= \frac{-16i^3}{n^4} + \frac{72i^2}{n^3} + \frac{-108i}{n^2} + \frac{54}{n}$$

$$\lim_{n \to \infty}\left(\frac{-16}{n^4}\sum_{i=1}^{n} i^3 + \frac{72}{n^3}\sum_{i=1}^{n} i^2 + \frac{-108}{n^2}\sum_{i=1}^{n} i + \frac{54}{n}\sum_{i=1}^{n} 1\right)$$

$$\lim_{n \to \infty}\left(\frac{-16}{n^4}\left[\frac{n(n+1)}{2}\right]^2 + \frac{72}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{-108}{n^2}\left(\frac{n \cdot (n+1)}{2}\right) + \frac{54}{n} n\right)$$

$$\lim_{n \to \infty}\left(\frac{-4(n^2 + 2n + 1)}{n^2} + \frac{12(n+1)(2n+1)}{n^2} + \frac{-54(n+1)}{n} + 54\right) = 22$$

$$-4\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) \qquad \frac{12\left(\frac{2n^2}{n^2} + \frac{3n}{n^2} + \frac{1}{n^2}\right)}{1} \qquad -54 \qquad 54$$

$$-4 \qquad 24$$

Question : Sketch $f(x) = \dfrac{x^2-5}{x^2-4}$

i) Domain : $(-\infty,-2) \cup (-2,2) \cup (2,\infty)$. ii) $(-\sqrt{5},0)$ and $(\sqrt{5},0)$ are x-inter.

$(0, 5/4)$ is y-intercept

iii) $\lim\limits_{x\to -2^-} \left(\dfrac{x^2-5}{x^2-4}\right) = -\infty$ $\lim\limits_{x\to -2^+} \left(\dfrac{x^2-5}{x^2-4}\right) = +\infty$ $x=-2$ is a vertical asymptote

$\lim\limits_{x\to 2^-} \left(\dfrac{x^2-5}{x^2-4}\right) = +\infty$ $\lim\limits_{x\to 2^+} \left(\dfrac{x^2-5}{x^2-4}\right) = -\infty$ $x=2$ is a vertical asymptote

$\lim\limits_{x\to -\infty} \left(\dfrac{x^2-5}{x^2-4}\right) = 1$ $\lim\limits_{x\to +\infty} \left(\dfrac{x^2-5}{x^2-4}\right) = 1$ $y=1$ is a horizontal asymptote

$$\begin{array}{r|l} x^2-5 & x^2-4 \\ \hline -x^2+4 & 1 \\ \hline -1 & \end{array} \Rightarrow 1 + \frac{-1}{x^2-4} \qquad \lim_{x\to\pm\infty}\left(\left(1+\frac{-1}{x^2-4}\right)-1\right) = \lim_{x\to\pm\infty}\left(\frac{-1}{x^2-4}\right) = 0$$

iv) $$f'(x) = \frac{2x.(x^2-4) - 2x.(x^2-5)}{(x^2-4)^2} = \frac{2x(x^2-4-x^2+5)}{(x^2-4)^2} = \frac{\overset{\to 0}{2x}}{\underbrace{(x^2-4)^2}_{-2,+2}}$$

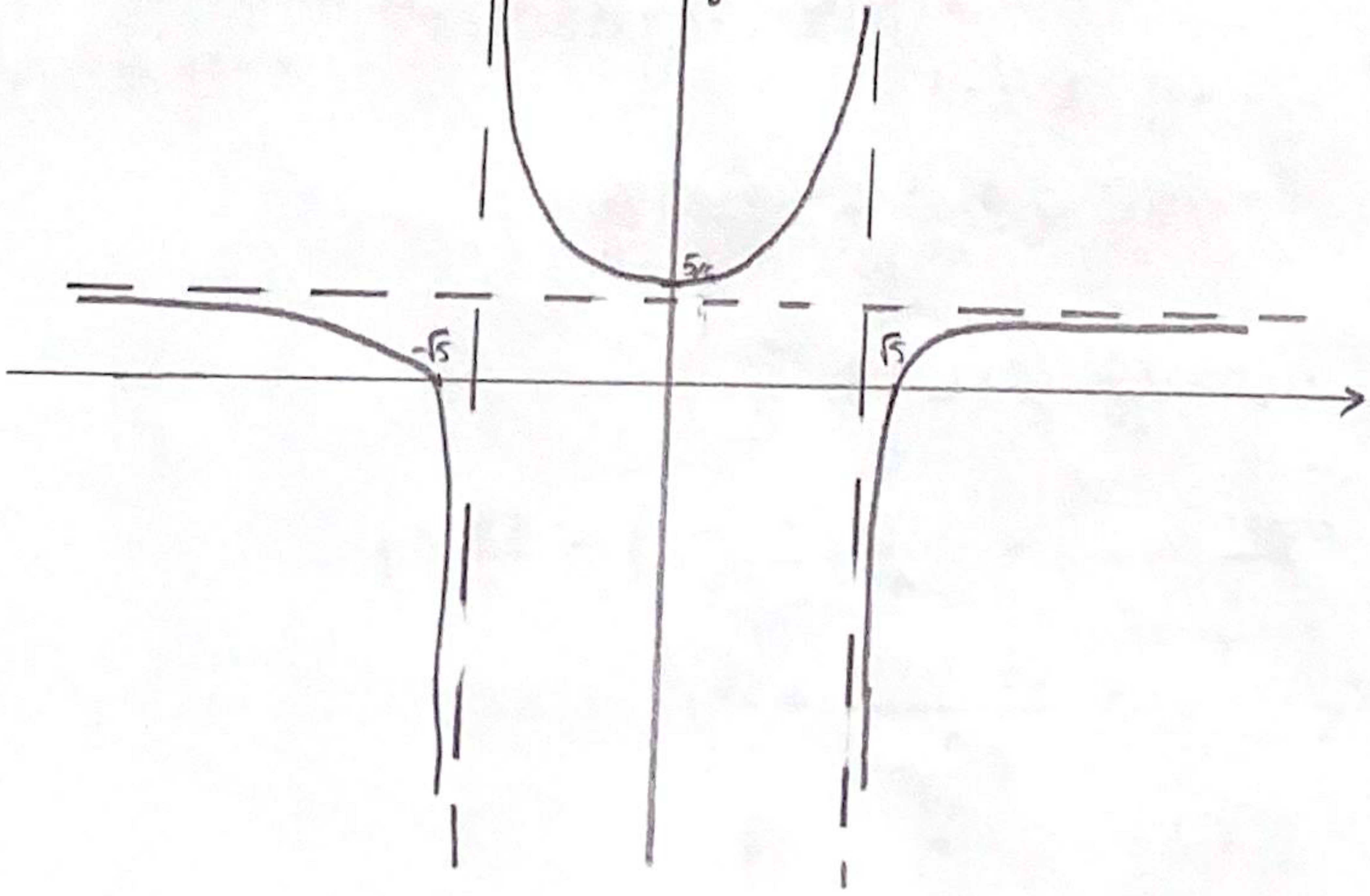
	$-\infty$		-2		0		2		$+\infty$
$f'(x)$		$-$	‖	$-$		$+$	‖	$+$	
$f(x)$		↘		↘	local min	↗		↗	

- f is increasing on $(0,2) \cup (2,\infty)$
- f is decreasing on $(-\infty,-2) \cup (-2,0)$
- $(0, \frac{5}{4})$ is a local minimum (extremum)

v) $$f''(x) = \frac{\overset{2(x^2-4)}{2}.\overset{(x^2-4)}{(x^2-4)^2} - 2x.2(x^2-4).2x}{(x^2-4)^4} = \frac{2(x^2-4)\left[(x^2-4)-4x^2\right]}{(x^2-4)^4} = \frac{-2.(x^2-4)(3x^2+4)}{(x^2-4)^4}$$

	$-\infty$		-2		2		∞
$f''(x)$		$-$	‖	$+$	‖	$-$	
$f(x)$		∩ con. down		∪ concave up		∩ con. down	

- f is concave upward on $(-2,2)$
- f is concave downward on $(-\infty,-2) \cup (2,\infty)$
- There no exists inflection point since the values change sign of second derivative are not in domain



Question: $\int_0^x f(t)\,dt = x + \int_x^0 \frac{f(t)}{1+t^2}\,dt \Rightarrow \int_0^x \frac{(2+t^2)\cdot f(t)}{1+t^2}\,dt = \underbrace{x}_{g(x)}$ $\quad g(x) = x$

$$\left(\frac{(2+x^2)\cdot f(x)}{(1+x^2)}\cdot 1 - 0\right) = g'(x) \Rightarrow f(x) = \frac{1+x^2}{2+x^2}$$

$$\left(\int_{a(t)}^{b(t)} f(x,t)\,dx\right)'_t = \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t}\,dx + f(b(t),t)\cdot b'(t) - f(a(t),t)\cdot a'(t)$$

Question: $F(t) = \int_0^{t^2} \arcsin\left(\frac{x}{t^2}\right) dx$, $F'(t) = \pi t$ $\quad (\arcsin(a))' = \frac{1}{\sqrt{1-a^2}}$

$$F'(t) = \int_0^{t^2} \underbrace{\frac{\partial\left(\arcsin\left(\frac{x}{t^2}\right)\right)}{\partial t}}\,dx + \left(\underbrace{\arcsin\left(\frac{t^2}{t^2}\right)\cdot 2t}_{\pi t} - \cancel{\arcsin\left(\frac{0}{t^2}\right)\cdot 0}\right)$$

$$\int_0^{t^2}\left(\frac{\frac{-2x}{t^3}}{\sqrt{1-\left(\frac{x}{t^2}\right)^2}}\right)dx \Rightarrow \int_0^{t^2} \frac{-2x}{t^{\cancel{3}}\frac{\sqrt{t^4-x^2}}{\cancel{t}}}\,dx \Rightarrow \int_0^{t^2} \frac{-2x}{t^2\sqrt{t^4-x^2}}\,dx$$

$t^4 - x^2 = u$

$-2x\,dx = du$

$$\Rightarrow \frac{1}{t^2}\int_0^{t^2} \frac{1}{\sqrt{u}}\,du \Rightarrow \frac{1}{t^2}\left(\frac{\sqrt{u}}{2}\Big|_0^{t^2}\right) = 0$$

Question: $F(u) = \int_u^{u^2} \frac{\sin(ut)}{t}\,dt \Rightarrow F'(u)$

$$F'(u) = \int_u^{u^2} \frac{\partial \sin(t,u)}{\partial u}\,dt + \left(\frac{\sin(u\cdot u^2)}{u^2}\cdot 2u - \frac{\sin(u\cdot u)}{u}\cdot 1\right)$$

$$\frac{2\sin(u^3) - \sin(u^2)}{u}$$

$$\int_u^{u^2} \frac{\cancel{t}\cdot\cos(ut)}{\cancel{t}}\,dt$$

$ut = a$

$u\,dt = da$

$$\int_u^{u^2} \cos(a)\,\frac{da}{u} \Rightarrow \frac{1}{u}\left(\sin(ut)\Big|_u^{u^2}\right) = \frac{\sin(u^3) - \sin(u^2)}{u}$$

$$F'(u) = \frac{3\sin(u^3) - 2\sin(u^2)}{u}$$

Question : $r = 1 - \cos\theta$, sketch $\quad r^2 = x^2 + y^2$

$x = (1-\cos\theta)\cos\theta \Rightarrow x = \cos\theta - \cos^2\theta \qquad y = (1-\cos\theta).\sin\theta \Rightarrow y = \sin\theta - \sin\theta.\cos\theta$

1) x-axis $(r,\theta) \longrightarrow (r,-\theta)$ or $(-r, \pi-\theta)$ ✓

2) y-axis $(r,\theta) \longrightarrow (r, \pi-\theta)$ or $(-r,-\theta)$ ✗

3) origin $(r.\theta) \longrightarrow (-r,\theta)$ or $(r, \pi+\theta)$ ✗

i) $r(\theta) = r(-\theta)$ is symmetric with respect to x-axis

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2

ii) $r = 1 - \frac{x}{r} \Rightarrow r^2 - r + x = 0 \qquad x^2+y^2 - \sqrt{x^2+y^2} + x = 0$ (cartesian)

iii) $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta.\sin\theta + \cos\theta(1-\cos\theta)}{-\sin\theta - 2\cos\theta.(-\sin\theta)} = \frac{\cos\theta - \cos 2\theta}{-\sin\theta(1-2\cos\theta)}\Big|_{\theta=\frac{\pi}{2}} = -1$

iv) $\frac{dy}{d\theta} = 0 \qquad \frac{dx}{d\theta} = 0$

$\cos\theta - \cos 2\theta = 0$

$-(2\cos^2\theta - \cos\theta - 1) = 0$

$2\cos\theta \quad 1$

$\cos\theta \quad -1$

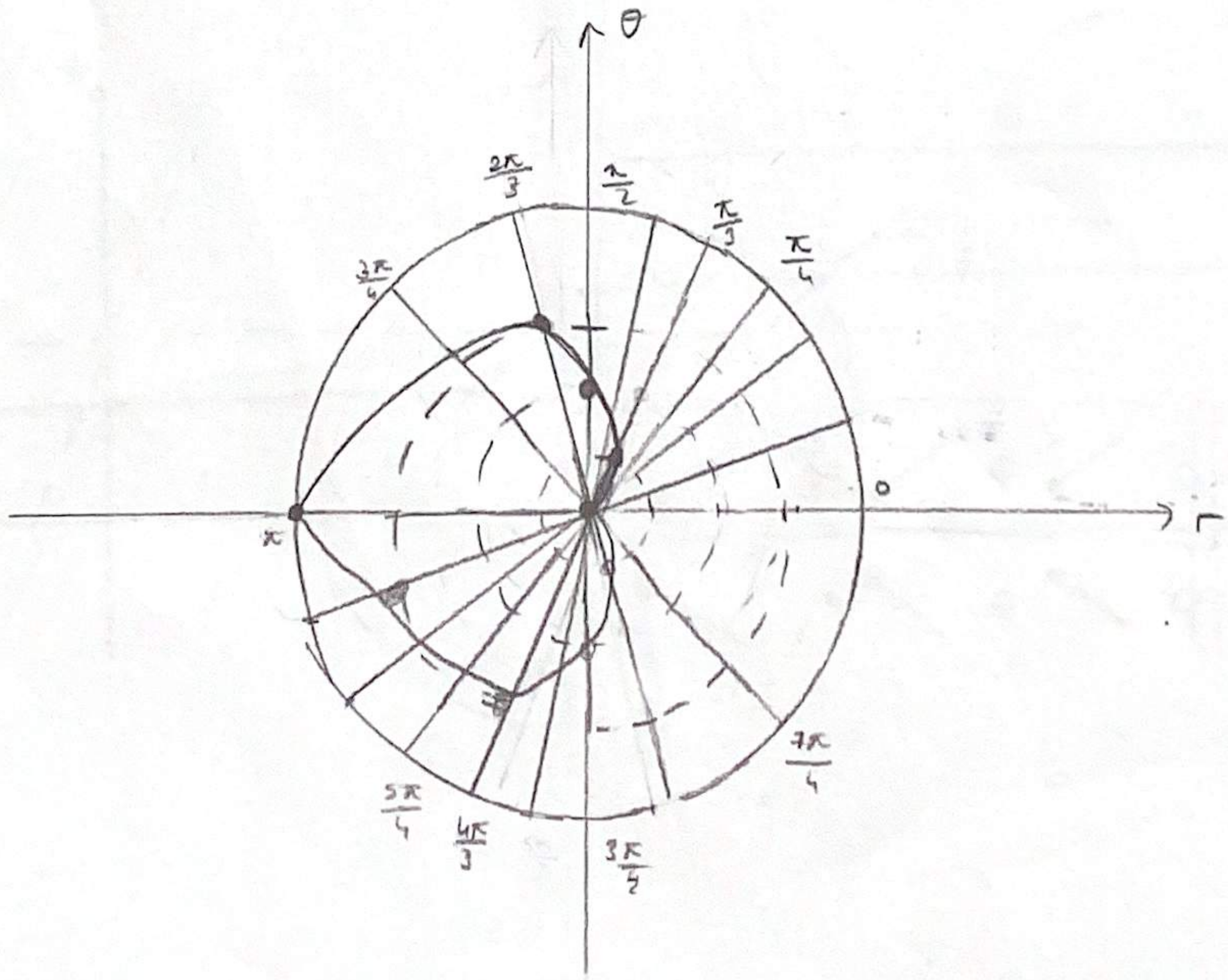
$-(2\cos\theta + 1)(\cos\theta - 1)$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3} \qquad \theta = 0, 2\pi$

$-\sin\theta(1-2\cos\theta) = 0 \qquad \theta = 0, \pi, 2\pi \quad \theta = \frac{\pi}{3}, \frac{5\pi}{3}$

vertical $\Rightarrow (r,\theta)$: $\left(\frac{1}{2}, \frac{\pi}{3}\right), \left(\frac{1}{2}, \frac{5\pi}{3}\right), (2, \pi)$

horizontal $\Rightarrow (r\theta)$: $\left(\frac{3}{2}, \frac{2\pi}{3}\right), \left(\frac{3}{2}, \frac{4\pi}{3}\right)$



Question: $x = \dfrac{t}{t^2-1}$, $y = \dfrac{2t}{t+1}$; sketch the graph

① $x(t)$ is defined where $t^2-1=0$ so $t \neq \mp 1$ $y(t)$ is undefined at $t=-1$

Domain: $\mathbb{R} - \{-1, 1\}$ / $(0,0)$ and $(0,2)$ are intercepts

② $\dot{x} = \dfrac{dx}{dt} = \dfrac{1.(t^2-1) - 2t.t}{(t^2-1)^2} = \dfrac{-(t^2+1)}{(t^2-1)^2} < 0$ x is decreasing

$\dot{y} = \dfrac{dy}{dt} = \dfrac{2(t+1) - 1.2t}{(t+1)^2} = \dfrac{2}{(t+1)^2} > 0$ y is increasing

③ $\lim\limits_{t \to -1} \dfrac{t}{t^2-1} = -\infty$ $\lim\limits_{t \to -1} \dfrac{2t}{t+1} = -\infty$ } may have oblique asymptote ★

$\lim\limits_{t \to 1} \dfrac{t}{t^2-1} = \infty$ $\lim\limits_{t \to 1} \dfrac{2t}{t+1} = 1$ } $y=1$ is a horizontal asymptote

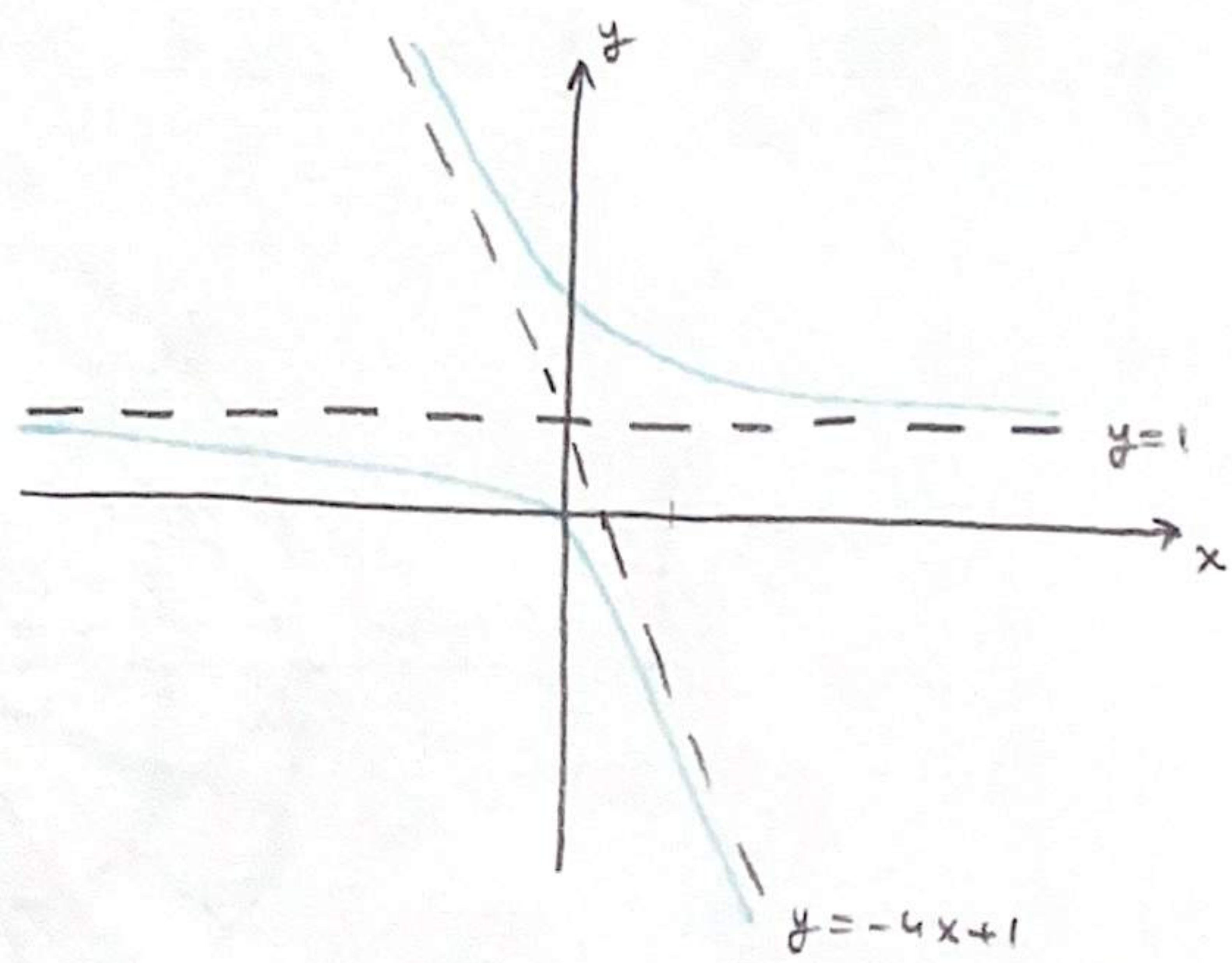
★ $m = \lim\limits_{x \to -\infty} \dfrac{f(x)}{x} = \lim\limits_{t \to -1} \dfrac{\frac{2t}{t+1}}{\frac{t}{t^2-1}} = \lim\limits_{t \to -1} 2(t-1) = -4$

$n = \lim\limits_{x \to -\infty} (f(x) - mx) = \lim\limits_{t \to -1} \left[\dfrac{2t}{t+1} + 4\left(\dfrac{t}{t^2-1}\right)\right] = \lim\limits_{t \to -1} \left[\dfrac{2t\cancel{(t+1)}}{(t-1)\cancel{(t+1)}}\right] = 1$ } $y=-4x+1$ is an oblique asymptote

④ $\lim\limits_{t \to -\infty} \dfrac{t}{t^2-1} = 0$, $\lim\limits_{t \to -\infty} \dfrac{2t}{t+1} = 2$ $\lim\limits_{t \to +\infty} \dfrac{t}{t^2-1} = 0$, $\lim\limits_{t \to +\infty} \dfrac{2t}{t+1} = 2$

⑤

t	$-\infty$		-1		0		1		$+\infty$
$\dot{x}$		$-$		$-$		$-$		$-$	
$\dot{y}$		$+$		$+$		$+$		$+$	
x	0	$\searrow$	$-\infty \mid +\infty$	$\searrow$	0	$\searrow$	$-\infty \mid +\infty$	$\searrow$	0
y	2	$\nearrow$	$+\infty \mid -\infty$	$\nearrow$	0	$\nearrow$	1	$\nearrow$	2



Question: $r = 4\cos(2\theta)$ $x = r.\cos\theta$ $y = r.\sin\theta$ $r^2 = x^2 + y^2$

1) x - axis $(r, \theta) \longrightarrow \underline{(r, -\theta)}$ or $(-r, \pi - \theta)$ ✓

2) y - axis $(r, \theta) \longrightarrow \underline{(r, \pi - \theta)}$ or $(-r, -\theta)$ ✓

3) origin $(r, \theta) \longrightarrow (-r, \theta)$ or $(r, \pi + \theta)$ ✗

i) $r(\theta) = r(-\theta)$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{4\pi}{6}$	$\frac{\pi}{2}$
r	4	2	0	-2	-4

ii) $r = 4(\cos^2\theta - \sin^2\theta) \Rightarrow r = 4\left(\frac{x^2}{r^2} - \frac{y^2}{r^2}\right) \Rightarrow r^3 = 4(x^2 - y^2)$

$(x^2+y^2)^3 - 4(x^2-y^2) = 0$ (cartesian)

iii) $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-8.\sin 2\theta . \sin\theta + 4.\cos 2\theta . \cos\theta}{-8 \sin 2\theta . \cos\theta + 4\cos 2\theta . (-\sin\theta)} = \frac{2\sin 2\theta . \sin\theta - \cos 2\theta . \cos\theta}{2\sin 2\theta . \cos\theta + \cos 2\theta . \sin\theta}$

$\sin 2\theta = 2\sin\theta . \cos\theta \Rightarrow \frac{\cos\theta(2\sin^2\theta - (1 - 2\sin^2\theta))}{\sin\theta(2\cos^2\theta + (1 - 2\sin^2\theta))} = \frac{\cos\theta\,(2\sin\theta - 1)(2\sin\theta + 1)}{\sin\theta}$

↳ $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$ $\frac{dy}{dx} = 1$

iii) $\frac{dx}{d\theta} = \sin\theta$ $(\theta = 0, \pi, 2\pi)$ (r, θ) : $(4, 0), (4, \pi), (4, 2\pi)$ vertical

$\frac{dy}{d\theta} = \cos\theta(4\sin^2\theta - 1)$ $(\theta = \frac{\pi}{2}, \frac{3\pi}{2})$ (r, θ) : $(-4, \frac{\pi}{2}), (-4, \frac{3\pi}{2})$ horizontal

$(\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4})$ (r, θ) : $(0, \frac{\pi}{4}), (0, \frac{\pi}{4}), (0, \frac{5\pi}{4}), (0, \frac{7\pi}{4})$