Applications of Derivative

Definition

Let f be a function defined over a domain D and let $c \in D$.

- *We say f has an* **absolute (global) maximum** *on D at c if $f(c) \geq f(x)$ for all $x \in D$.*
- *We say f has an* **absolute (global) minimum** *on D at c if $f(c) \leq f(x)$ for all $x \in D$.*
- *If f has an* **absolute (global) maximum** *on D at c or an* **absolute (global) minimum** *on D at c, we say f has an* **absolute (global) extremum** *on D at c.*

The greatest value of the function $f(x)$ on the interval $[a, b]$ is simultaneously the least upper bound of the range of the function on this interval and is denoted as

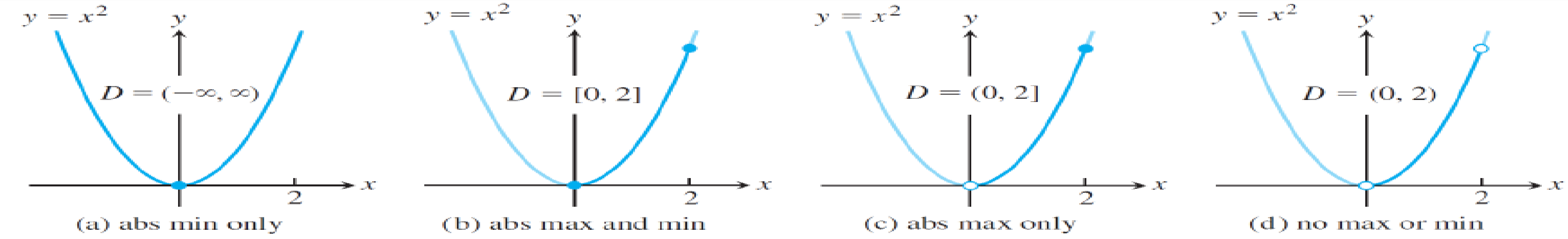
$$f(x_0) = \max_{x \in [a,b]} f(x) = \sup_{x \in [a,b]} f(x).$$

The least value of the function $f(x)$ on the interval $[a, b]$ is simultaneously the greatest lower bound of the range of the function on this interval and can be written as

$$f(x_0) = \min_{x \in [a,b]} f(x) = \inf_{x \in [a,b]} f(x).$$

These concepts characterize the behavior of a function on a finite interval, in contrast to the **local (relative) extremum**, which describes the properties of the function in a small neighborhood of a point. Therefore, the maximum and minimum values of a function on an interval are often referred as the global (absolute) maximum or, respectively, the global (absolute) minimum.

Function rule	Domain D
(a) $y = x^2$	$(-\infty, \infty)$
(b) $y = x^2$	$[0, 2]$
(c) $y = x^2$	$(0, 2]$
(d) $y = x^2$	$(0, 2)$



Graphs for Example showing absolute extrema for various domains.

Definition

- *A function f has a* **local maximum** *at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \geq f(x)$ for all $x \in I$.*
- *A function f has a* **local minimum** *at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \leq f(x)$ for all $x \in I$.*
- *A function f has a* **local extremum** *at c if f has a local maximum at c or f has a local minimum at c.*

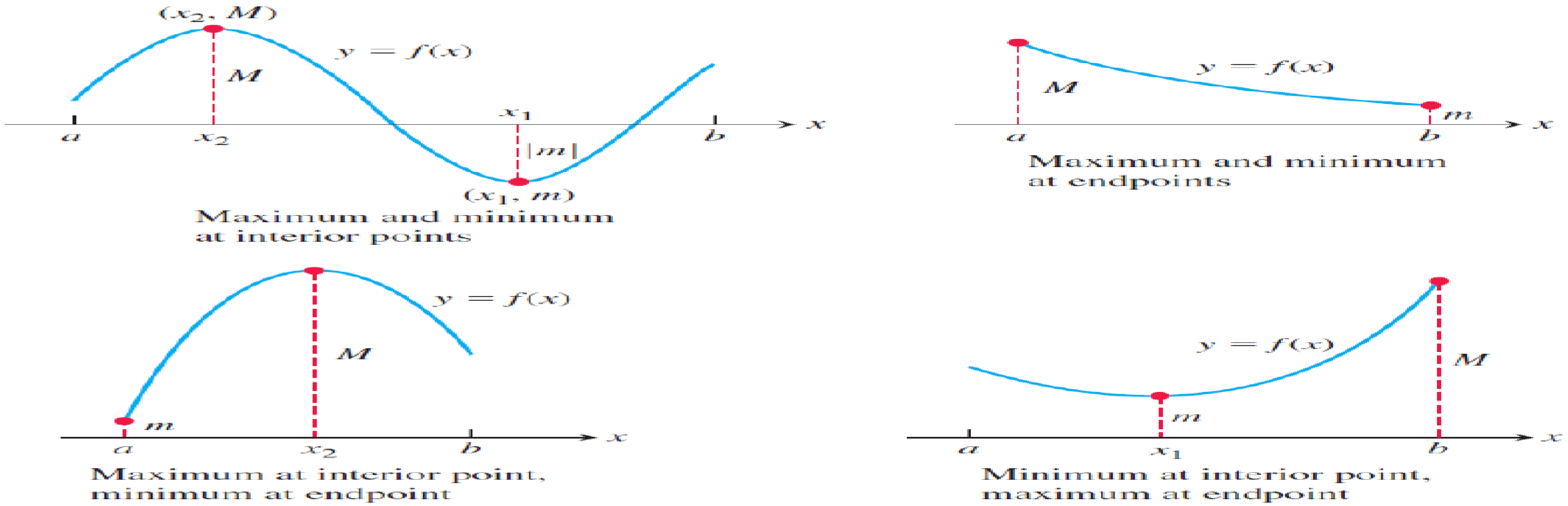
Remark

An absolute maximum is also a local maximum. Being the largest value overall, it is also the largest value in its immediate neighborhood. Hence, a list of all local maxima will automatically include the absolute maximum if there is one. *Similarly,* a list of all local minima will include the absolute minimum if there is one.

Extreme Value Theorem (Weierstrass Extreme Value Theorem)

If f is a continuous function over the closed, bounded interval $[a, b]$,

then there is a point in $[a, b]$ at which f has an absolute maximum over $[a, b]$ and there is a point in $[a, b]$ at which f has an absolute minimum over $[a, b]$.

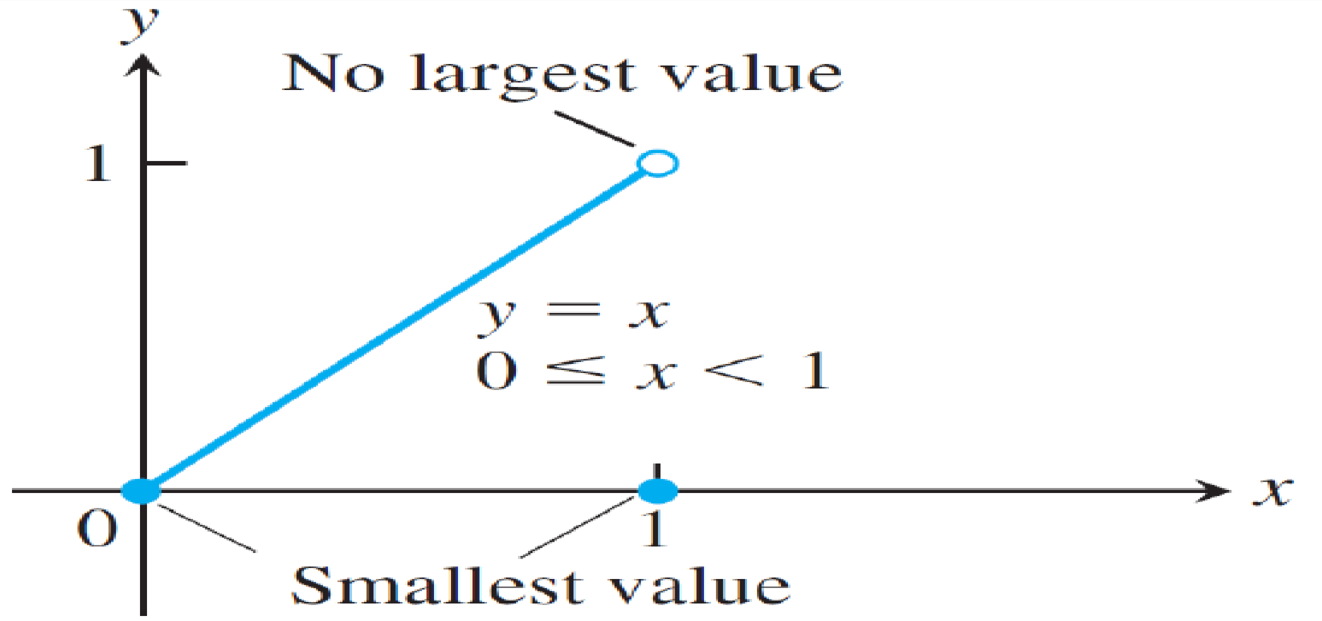


Some possibilities for a continuous function's maximum and minimum on a closed interval $[a, b]$.

Even a single point of discontinuity can keep a function from having either a maximum or minimum value on a closed interval. The function

$$y = \begin{cases} x, & 0 \leq x < 1, \\ 0, & x = 1 \end{cases}$$

is continuous at every point of $[0, 1)$ *except* $x = 1$*, yet its graph over* $[0, 1]$ *does not have a highest point.*



The function is continuous at every point of $[0, 1)$ except $x = 1$, yet its graph over $[0, 1]$ does not have a highest point.

The Extreme Value Theorem says that a continuous function on a closed interval has a maximum value and a minimum value, but it does not tell us how to find these extreme values.

Definition

Let c be an interior point in the domain of f. We say that c is a ***critical point*** *of f if $f'(c) = 0$ or $f'(c)$ is undefined.*

The value of the function at a critical point is a ***critical value****.*

Find the critical points of $f(x) = x^3 - 12x - 5$

Solution. The function f is everywhere continuous and differentiable. The first derivative

$$\begin{aligned} f'(x) &= 3x^2 - 12 = 3\left(x^2 - 4\right) \\ &= 3(x+2)(x-2) \end{aligned}$$

is zero at $x = -2$ and $x = 2$.

Finding the Maximum and Minimum Values of a Function

Theorem : Fermat's Theorem

If f has a ***local extremum*** *at c and f is* ***differentiable*** *at c, then $f'(c) = 0$.*

Proof. To prove that $f'(c)$ is zero at a local extremum, we show first that $f'(c)$ cannot be positive and second that $f'(c)$ cannot be negative. The only number that is neither positive nor negative is zero, so that is what $f'(c)$ must be.

To begin, suppose that f has a local maximum value at $x = c$ so that $f(x) - f(c) \leq 0$ for all values of x near enough to c. Since c is an interior point of f's domain, $f'(c)$ is defined by the two-sided limit

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

This means that the right-hand and left-hand limits both exist at $x = c$ and equal $f'(c)$. When we examine these limits separately, we find that

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0, \quad \text{because } (x - c) > 0 \text{ and } f(x) \leq f(c). \tag{1}$$

Similarly,

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0, \quad \text{because } (x - c) < 0 \text{ and } f(x) \leq f(c). \tag{2}$$

Together, (1) and (2) (that is $0 \leq f'(c) \leq 0$) imply $f'(c) = 0$.

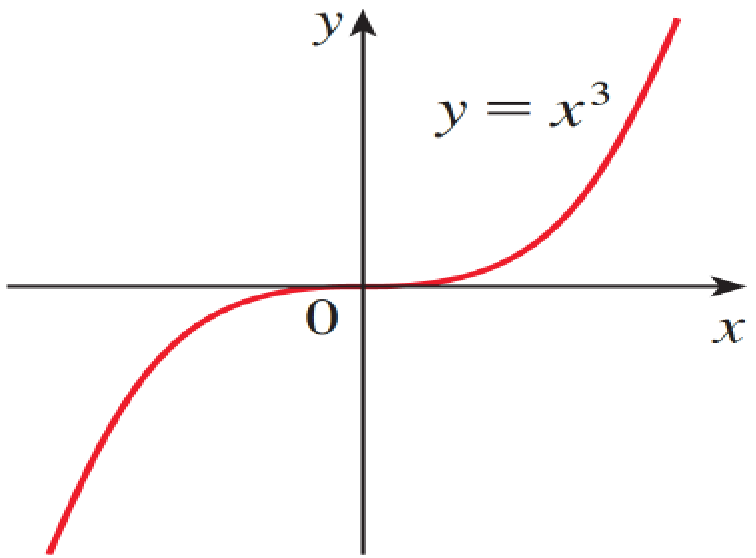
This proves the theorem for local maximum values. To prove it for local minimum values, we simply use $f(x) \geq f(c)$, which reverses the inequalities in (1) and (2).

If f has a **local maximum** *or* **minimum** *at c, then c is* **a critical number** *of f.*

The following examples caution us against reading too much into Fermat's Theorem: we can't expect to locate extreme values simply by setting $f'(x) = 0$ and solving for x.

Ex.1

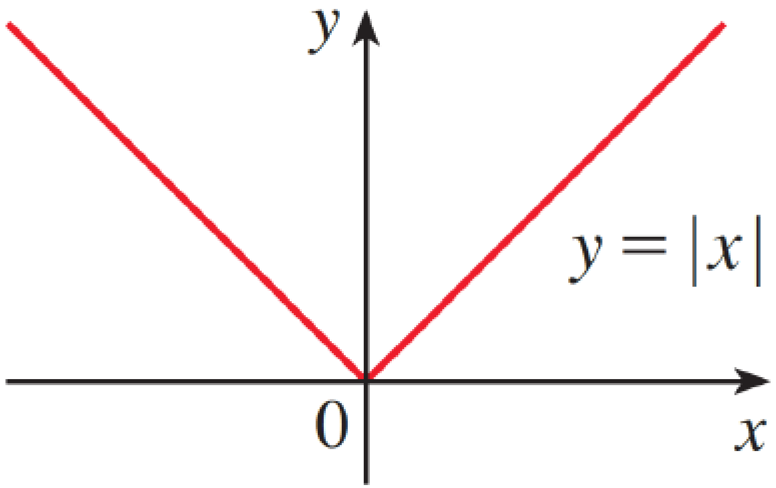
If $f(x) = x^3$, then $f'(x) = 3x^2$, so $f'(0) = 0$. But f has no maximum or minimum at 0, as you can see from its graph in fig. . (Or observe that $x^3 > 0$ for $x > 0$ but $x^3 < 0$ for $x < 0$.) The fact that $f'(0) = 0$ simply means that the curve $y = x^3$ has a horizontal tangent at $(0,0)$. Instead of having a maximum or minimum at $(0,0)$, the curve crosses its horizontal tangent there.



If $f(x) = x^3$, then $f'(0) = 0$, but f has no maximum or minimum.

Ex.2

The function $f(x) = |x|$ has its (local and absolute) minimum value at 0, but that value can't be found by setting $f'(x) = 0$ because $f'(0)$ does not exist



If $f(x) = |x|$, then $f(0) = 0$ is a minimum value, but $f'(0)$ does not exist.

Remark 3.7

Example Ex.1 ***and example*** Ex.2 ***show that we must be careful when using Fermat's Theorem.*** *Example* Ex.1 *demonstrates that* ***even when*** $f'(c) = 0$ ***there need not be a maximum or minimum at*** c. *(In other words, the converse of Fermat's Theorem is false in general.) Furthermore,* ***there may be an extreme value even when*** $f'(c)$ ***does not exist***

Fermat's Theorem does suggest that we should at least ***start*** *looking for extreme values of f at the numbers c where $f'(c) = 0$ or where $f'(c)$ does not exist. Such numbers are given a special name.*

Corollary : How to Find the Absolute Extrema of a Continuous Function f on a Finite Closed Interval

Consider a continuous function f defined over the closed interval $[a, b]$.

1. *Evaluate f at the endpoints $x = a$ and $x = b$.*
2. *Find all critical points of f that lie over the interval (a, b) and evaluate f at those critical points.*
3. *Compare all values found in (1) and (2). From Location of Absolute Extrema, the absolute extrema must occur at endpoints or critical points. Therefore, the largest of these values is the absolute maximum of f. The smallest of these values is the absolute minimum of f.*

Find the absolute maximum and minimum values of $f(x) = x^2$ on $[-2, 1]$.

Solution. The function is differentiable over its entire domain, so the only critical point is where $f'(x) = 2x = 0$, namely $x = 0$. We need to check the function's values at $x = 0$ and at the endpoints $x = -2$ and $x = 1$:

- **Critical point value:** $f(0) = 0$
- **Endpoint values:**

$$f(-2) = 4$$
$$f(1) = 1$$

The function has an absolute maximum value of 4 at $x = -2$ and an absolute minimum value of 0 at $x = 0$.

Find the absolute maximum and minimum values of $f(x) = 10x(2 - \ln x)$ *on the interval* $[1, e^2]$.

Solution. Figure suggests that f has its absolute maximum value near $x = e$ and its absolute minimum value of 0 at $x = e^2$. Let's verify this observation.

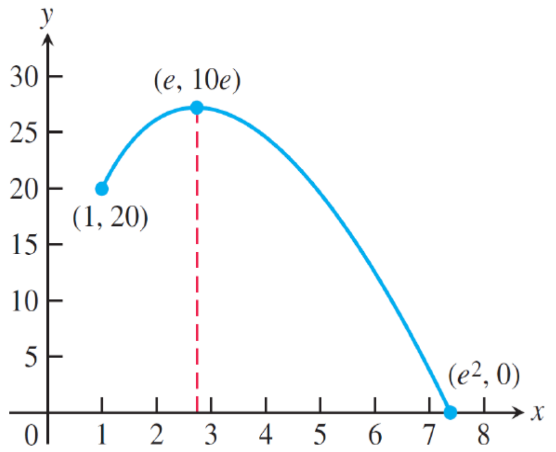
We evaluate the function at the critical points and endpoints and take the largest and smallest of the resulting values. The first derivative is

$$f'(x) = 10(2 - \ln x) - 10x\left(\frac{1}{x}\right) = 10(1 - \ln x).$$

The only critical point in the domain $[1, e^2]$ is the point $x = e$, where $\ln x = 1$. The values of f at this one critical point and at the endpoints are:

- **Critical point value:** $f(e) = 10e$
- **Endpoint values:**

$$f(1) = 10(2 - \ln 1) = 20,$$
$$f(e^2) = 10e^2(2 - 2\ln e) = 0.$$

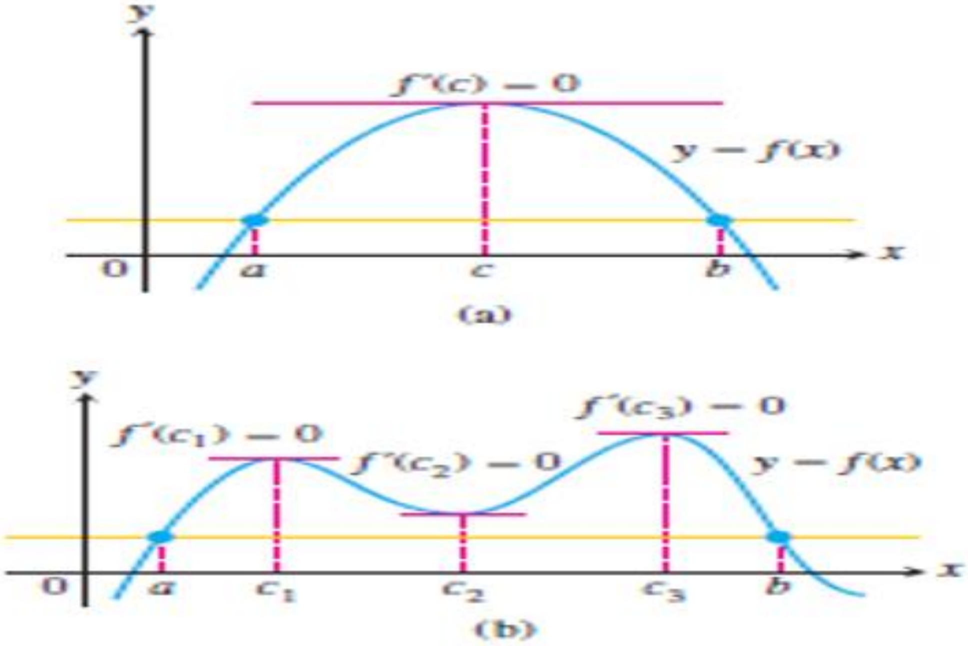


We can see from this list that the function's absolute maximum value is $10e \approx 27.2$; it occurs at the critical interior point $x = e$. The absolute minimum value is 0 and occurs at the right endpoint $x = e^2$

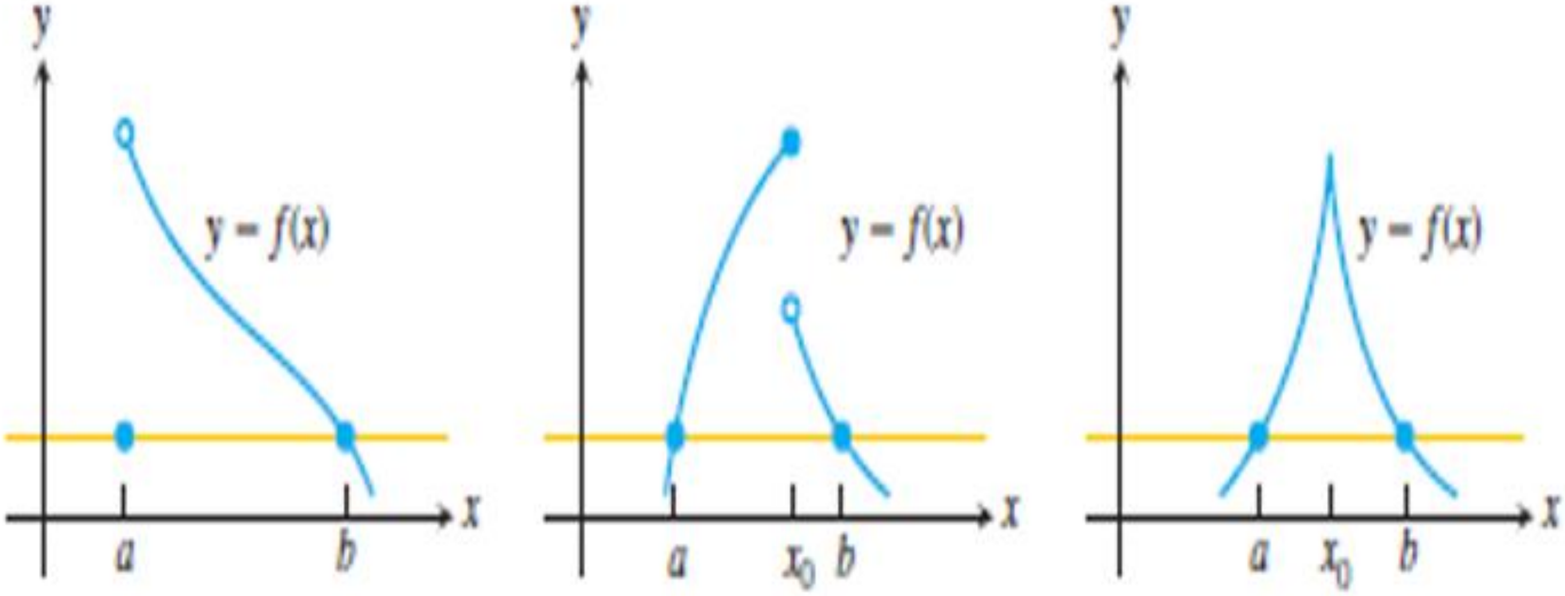
Rolle's Theorem

As suggested by its graph, if a differentiable function crosses a horizontal line at two different points, there is at least one point between them where the tangent to the graph is horizontal and the derivative is zero . We now state and prove this result.

THEOREM —Rolle's Theorem Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) at which $f'(c) = 0$.



Rolle's Theorem says that a differentiable curve has at least one horizontal tangent between any two points where it crosses a horizontal line. It may have just one (a), or it may have more (b).



(a) Discontinuous at an endpoint of $[a, b]$

(b) Discontinuous at an interior point of $[a, b]$

(c) Continuous on $[a, b]$ but not differentiable at an interior point

There may be no horizontal tangent if the hypotheses of Rolle's Theorem do not hold.

EXAMPLE Show that the equation

$$x^3 + 3x + 1 = 0$$

has exactly one real solution.

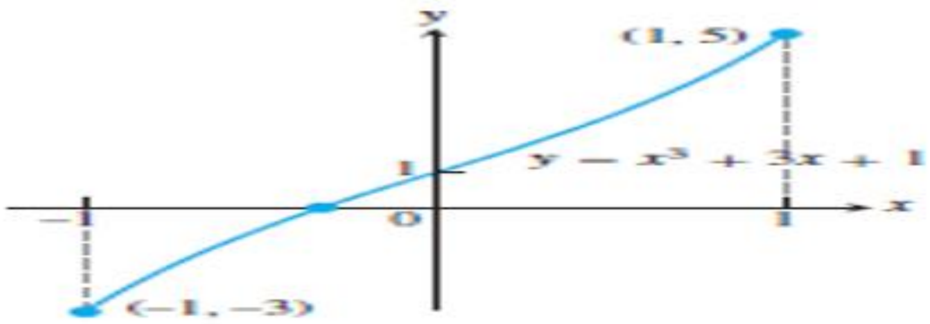
Solution We define the continuous function

$$f(x) = x^3 + 3x + 1.$$

Since $f(-1) = -3$ and $f(0) = 1$, the Intermediate Value Theorem tells us that the graph of f crosses the x-axis somewhere in the open interval $(-1, 0)$. The derivative

$$f'(x) = 3x^2 + 3$$

is never zero (because it is always positive). Now, if there were even two points $x = a$ and $x = b$ where $f(x)$ was zero, Rolle's Theorem would guarantee the existence of a point $x = c$ in between them where f' was zero. Therefore, f has no more than one zero.



The only real zero of the polynomial $y = x^3 + 3x + 1$ is the one shown here where the curve crosses the x-axis between -1 and 0

Rolle Theorem

Let f be a function that satisfies the following three hypotheses:

- *f is continuous on the closed interval $[a, b]$.*
- *f is differentiable on the open interval (a, b).*
- *$f(a) = f(b)$.*

Then there exists a number c in (a, b) such that $f'(c) = 0$.

Proof. Assume that all hypotheses of Rolle's theorem hold. There are three cases:

- **CASE I:** $f(x) = k$, (k is a real constant. Thus, $f(x) = f(a) = f(b) = k$)
- **CASE II: $f(x) > f(a)$ for some x in (a, b)**
- **CASE III: $f(x) < f(a)$ for some x in (a, b)**

CASE I: $f(x) = k$, (k is a real constant.)
Then $f'(x) = 0$ for all $x \in [a, b]$, so the number c can be taken to be any number in (a, b).
CASE II: $f(x) > f(a)$ for some x in (a, b)
By the Extreme Value Theorem, f has a maximum value somewhere in $[a, b]$. Since $f(a) = f(b)$ and $f(x)$ is not a constant function, it must attain this maximum value at a number c in the open interval (a, b). Then f has a local (relative) maximum at c and, by hypothesis 2, f is differentiable at c. Therefore $f'(c) = 0$ by Fermat's Theorem.
CASE III: $f(x) < f(a)$ for some x in (a, b)
By the Extreme Value Theorem , f has a minimum value in $[a, b]$ and, since $f(a) = f(b)$ and $f(x)$ is not a constant function, it attains this minimum value at a number c in (a, b). Again $f'(c) = 0$ by Fermat's Theorem.

Let us take a function $f(x) = x^2 - 3x$ on a closed interval $[0, 3]$ and see how we can use Rolle's theorem on this function.

Here, $a = 0$ and $b = 3$ are the end points of the interval.

Step 1: The first step is to verify that the function $f(x) = x^2 - 3x$ is continuous. Polynomials are continuous functions, so $f(x)$ is continuous.

Step 2: The function $f(x)$ must be differentiable. Since $f(x)$ is a continuous polynomial, it is differentiable. The derivative is:

$$f'(x) = 2x - 3$$

Step 3: We need to check if $f(a) = f(b)$. Evaluating the function at the end points:

$$f(0) = (0)^2 - 3(0) = 0$$
$$f(3) = (3)^2 - 3(3) = 9 - 9 = 0$$

Thus, $f(a) = f(b)$.

Step 4: We have $f'(x) = 2x - 3$. Setting this equal to zero to find c:

$$2x - 3 = 0$$
$$2x = 3$$
$$x = \frac{3}{2}$$

Hence, $f'(x) = 0$ at $x = \frac{3}{2}$.

Prove that the equation $x^3 + x - 1 = 0$ has exactly one real solution.

Solution. First we use the Intermediate Value Theorem to show that a solution exists. Let $f(x) = x^3 + x - 1$. Then $f(0) = -1 < 0$ and $f(1) = 1 > 0$. Since f is a polynomial, it is continuous, so the Intermediate Value Theorem states that there is a number c between 0 and 1 such that $f(c) = 0$. Thus the given equation has a solution.

To show that the equation has no other real solution, we use Rolle's Theorem and argue by contradiction. Suppose that it had two distinct solutions a and b ($a \neq b$). Since they are root, we have $f(a) = 0 = f(b)$. since f is a polynomial, it is differentiable on (a, b) and continuous on $[a, b]$. Thus, by Rolle's Theorem, there is a number c between a and b such that $f'(c) = 0$.

But let us take the derivative of f

$$f'(x) = 3x^2 + 1 \geqslant 1 \quad \text{for all } x$$

(since $x^2 \geqslant 0$) so $f'(x)$ can never be 0 for real X values. This gives a contradiction. Therefore the equation can't have two distinct real solutions.

THEOREM —The Mean Value Theorem Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

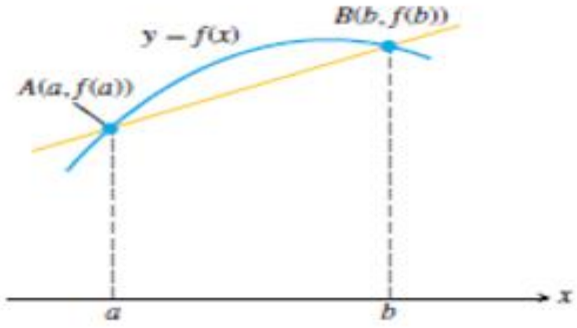
$$\frac{f(b) - f(a)}{b - a} = f'(c). \tag{1}$$

Proof We picture the graph of f and draw a line through the points $A(a, f(a))$ and $B(b, f(b))$. The line is the graph of the function

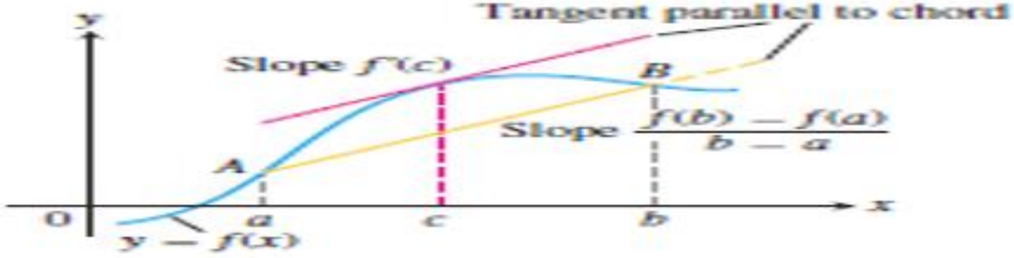
$$g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \tag{2}$$

(point-slope equation). The vertical difference between the graphs of f and g at x is

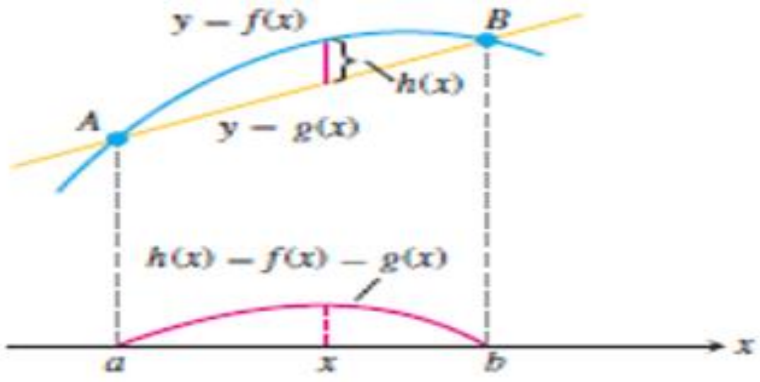
$$\begin{aligned} h(x) &= f(x) - g(x) \\ &= f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a). \end{aligned} \tag{3}$$



The graph of f and the chord AB over the interval $[a, b]$.



Geometrically, the Mean Value Theorem says that somewhere between a and b the curve has at least one tangent parallel to chord AB.



The chord AB is the graph of the function $g(x)$. The function $h(x) = f(x) - g(x)$ gives the vertical distance between the graphs of f and g at x.

The function h satisfies the hypotheses of Rolle's Theorem on $[a, b]$. It is continuous on $[a, b]$ and differentiable on (a, b) because both f and g are. Also, $h(a) = h(b) = 0$ because the graphs of f and g both pass through A and B. Therefore $h'(c) = 0$ at some point $c \in (a, b)$. This is the point we want for Equation (1).

To verify Equation (1), we differentiate both sides of Equation (3) with respect to x and then set $x = c$:

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} \qquad \text{Derivative of Eq. (3) \ldots}$$

$$h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} \qquad \text{\ldots with } x = c$$

$$0 = f'(c) - \frac{f(b) - f(a)}{b - a} \qquad h'(c) = 0$$

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \qquad \text{Rearranged}$$

which is what we set out to prove.

EXAMPLE To illustrate the Mean Value Theorem with a specific function, let's consider $f(x) = x^3 - x$, $a = 0$, $b = 2$. Since f is a polynomial, it is continuous and differentiable

for all x, so it is certainly continuous on $[0, 2]$ and differentiable on $(0, 2)$. Therefore, by the Mean Value Theorem, there is a number c in $(0, 2)$ such that

$$f(2) - f(0) = f'(c)(2 - 0)$$

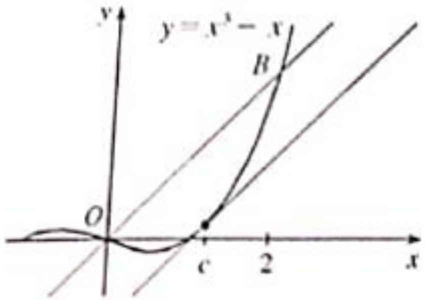
Now $f(2) = 6$, $f(0) = 0$, and $f'(x) = 3x^2 - 1$, so this equation becomes

$$6 = (3c^2 - 1)2 = 6c^2 - 2$$

which gives $c^2 = \frac{4}{3}$, that is, $c = \pm 2/\sqrt{3}$. But c must lie in $(0, 2)$, so $c = 2/\sqrt{3}$.

The tangent line at this value of c is parallel to the

secant line OB.



The main significance of the Mean Value Theorem is that it enables us to obtain information about a function from information about its derivative. The next example provides an instance of this principle.

EXAMPLE Suppose that $f(0) = -3$ and $f'(x) \leqslant 5$ for all values of x. How large can $f(2)$ possibly be?

SOLUTION We are given that f is differentiable (and therefore continuous) everywhere. In particular, we can apply the Mean Value Theorem on the interval $[0, 2]$. There exists a number c such that

$$f(2) - f(0) = f'(c)(2 - 0)$$

so

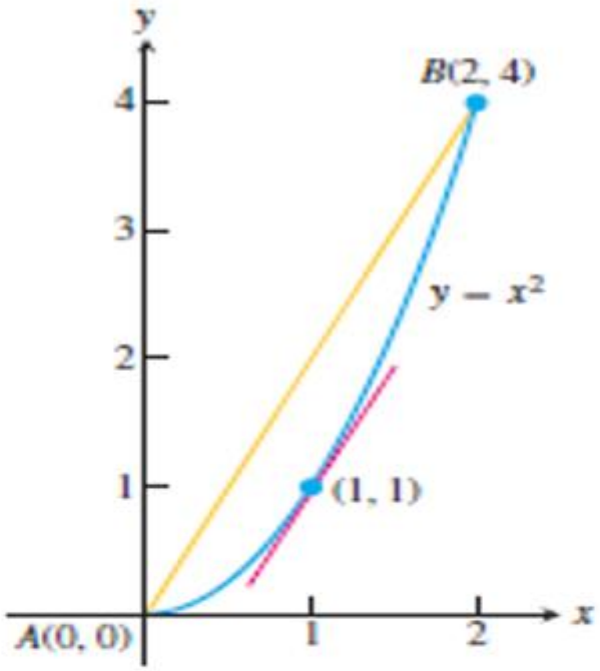
$$f(2) = f(0) + 2f'(c) = -3 + 2f'(c)$$

We are given that $f'(x) \leqslant 5$ for all x, so in particular we know that $f'(c) \leqslant 5$. Multiplying both sides of this inequality by 2, we have $2f'(c) \leqslant 10$, so

$$f(2) = -3 + 2f'(c) \leqslant -3 + 10 = 7$$

The largest possible value for $f(2)$ is 7.

EXAMPLE The function $f(x) = x^2$ is continuous for $0 \le x \le 2$ and differentiable for $0 < x < 2$. Since $f(0) = 0$ and $f(2) = 4$, the Mean Value Theorem says that at some point c in the interval, the derivative $f'(x) = 2x$ must have the value $(4 - 0)/(2 - 0) = 2$. In this case we can identify c by solving the equation $2c = 2$ to get $c = 1$. However, it is not always easy to find c algebraically, even though we know it always exists.



$c = 1$ is where the tangent is parallel to the chord.

Theorem If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

Proof Let x_1 and x_2 be any two numbers in (a, b) with $x_1 < x_2$. Since f is differentiable on (a, b), it must be differentiable on (x_1, x_2) and continuous on $[x_1, x_2]$. By applying the Mean Value Theorem to f on the interval $[x_1, x_2]$, we get a number c such that $x_1 < c < x_2$ and

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

Since $f'(x) = 0$ for all x, we have $f'(c) = 0$, and so

$$f(x_2) - f(x_1) = 0 \qquad \text{or} \qquad f(x_2) = f(x_1)$$

Therefore, f has the same value at *any* two numbers x_1 and x_2 in (a, b). This means that f is constant on (a, b).

COROLLARY 1 If $f'(x) = 0$ at each point x of an open interval (a, b), then $f(x) = C$ for all $x \in (a, b)$, where C is a constant.

COROLLARY 2 If $f'(x) = g'(x)$ at each point x in an open interval (a, b), then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$. That is, $f - g$ is a constant function on (a, b).

EXAMPLE Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passes through the point $(0, 2)$.

Solution Since the derivative of $g(x) = -\cos x$ is $g'(x) = \sin x$, we see that f and g have the same derivative. Corollary 2 then says that $f(x) = -\cos x + C$ for some

constant C. Since the graph of f passes through the point $(0, 2)$, the value of C is determined from the condition that $f(0) = 2$:

$$f(0) = -\cos(0) + C = 2, \qquad \text{so} \qquad C = 3.$$

The function is $f(x) = -\cos x + 3$.

Ex: Verify the conclusion of the Mean Value Theorem for $f(x)=\sqrt{x}$ on the interval $[a,b]$, where $0 \le a < b$

The theorem says that there must be a number c in the interval (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\frac{1}{2\sqrt{c}} = \frac{\sqrt{b}-\sqrt{a}}{b-a} = \frac{(\sqrt{b}-\sqrt{a})}{(\sqrt{b}-\sqrt{a})\,\sqrt{b}-\sqrt{a})} = \frac{1}{\sqrt{b}+\sqrt{a}}$$

$$2\sqrt{c} = \sqrt{a}+\sqrt{b} \quad \text{and} \quad c = \left(\frac{\sqrt{b}+\sqrt{a}}{2}\right)^2$$

since $a<b$, we have

$$a = \left(\frac{\sqrt{a}+\sqrt{a}}{2}\right)^2 < c < \left(\frac{\sqrt{b}+\sqrt{b}}{2}\right)^2 = b$$

so c lies in the interval (a,b)

Ex. Show that $\sin x < x$ for all $x > 0$

If $x > 2\pi$, then $\sin x \le 1 < 2\pi < x$

If $0 < x < 2\pi$, then by the mean value theorem there exists c in the open interval $(0, 2\pi)$ such that

$$\frac{\sin x}{x} = \frac{\sin x - \sin 0}{x - 0} = \frac{d}{dx} \sin x \Big|_{x=c} = \cos c < 1$$

Thus, $\sin x < x$

Ex: $f: [1,3] \to R$, $f(x) = x^2$
find the mean value of f

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{f(3)-f(1)}{3-1} = \frac{9-1}{2} = 4$$

$$2c = 4 \Rightarrow \boxed{2 = c}$$

c=2 is the point satisfying the mean value theorem

Show that the following inequality is true for $0 < x < 1$

$$\frac{\sqrt{1-x^2}}{1+x} \leq \frac{\ln(1+x)}{\arcsin x} < 1$$ by using mean value theorem.

$0 < x < 1$ için $\sqrt{\dfrac{1-x}{1+x}} \leq \dfrac{\ln(1+x)}{\arcsin x} < 1$

bağıntısını gerçekleyiniz. (ORT. DEĞ. TEO. KULLANARAK)

$f(x) = \ln x$ $\quad a = 1 \quad b = 1+x$ alınsın. $[\overset{a}{1}, \overset{b}{1+x}]$, $c \in (1, 1+x)$ $f'(c) =$

$$\frac{f(b)-f(a)}{b-a} = \frac{\ln(1+x) - \ln 1}{(1+x)-1} = \frac{\ln(1+x)}{x} = \frac{1}{c} \text{ dir.}$$

$1 < c < 1+x \Rightarrow \dfrac{1}{1+x} < \dfrac{1}{c} < \dfrac{1}{1} = 1$, dolayısıyla

$$\frac{1}{1+x} < \frac{\ln(1+x)}{x} < 1 \text{ dir}$$

1

şimdi de $f(x)=\arcsin x$ $a=0$ $b=x$ $(0<x<1)$
alalım

$$\frac{f(b)-f(a)}{b-a}=\frac{\arcsin x-\arcsin 0}{x-0}=\frac{\arcsin x}{x}=\frac{1}{\sqrt{1-c^2}}$$

$$0<c<x<1 \Rightarrow 1-0^2>1-c^2>1-x^2 \Rightarrow 1<\frac{1}{\sqrt{1-c^2}}<\frac{1}{\sqrt{1-x^2}}$$

dolayısıyla

$$1<\frac{\arcsin x}{x}<\frac{1}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}<\frac{x}{\arcsin x}<1$$

2

1 ve 2 taraf tarafa çarpalım

$$\frac{\sqrt{1-x^2}}{1+x} < \frac{\ln(1+x)}{x} \cdot \frac{x}{\arcsin x} < 1$$

$$\frac{\sqrt{1-x^2}}{1+x} < \frac{\ln(1+x)}{\arcsin x} < 1 .$$

The Generalized Mean-Value Theorem

If functions f and g are both continuous on $[a, b]$ and differentiable on (a, b), and if $g'(x) \neq 0$ for every x in (a, b), then there exists a number c in (a, b) such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

PROOF Note that $g(b) \neq g(a)$; otherwise, there would be some number in (a, b) where $g' = 0$. Hence, neither denominator above can be zero. Apply the Mean-Value Theorem to

$$h(x) = \big(f(b) - f(a)\big)\big(g(x) - g(a)\big) - \big(g(b) - g(a)\big)\big(f(x) - f(a)\big).$$

Since $h(a) = h(b) = 0$, there exists c in (a, b) such that $h'(c) = 0$. Thus,

$$\big(f(b) - f(a)\big)g'(c) - \big(g(b) - g(a)\big)f'(c) = 0,$$

and the result follows on division by the g factors.

Given the polynomial functions:

- $f(x) = 3x^2 + 4x + 5$
- $g(x) = x^2 - x + 25$

on the interval $[1, 2]$. *Verify that Cauchy's Mean Value Theorem is applicable to the functions. Find the value of* c *in* $(1, 2)$ *that satisfies the theorem.*

Solution. We observe that:

- $f(x)$ and $g(x)$ are continuous in the closed interval $[1, 2]$.
- $f(x)$ and $g(x)$ are differentiable in the open interval $(1, 2)$.
- The derivative of $g(x)$, given by $g'(x) = 2x - 1$, is not equal to zero in the interval $[1, 2]$.

Thus, Cauchy's Mean Value Theorem is applicable to $f(x)$ and $g(x)$.

Differentiating $f(x)$ and $g(x)$ with respect to x, we get:

- $f'(x) = 6x + 4$
- $g'(x) = 2x - 1$

Now, evaluating at $a = 1$ and $b = 2$, we find:

$$\begin{aligned} f(1) &= 12 \\ f(2) &= 25 \\ g(1) &= 25 \\ g(2) &= 27 \end{aligned}$$

Applying Cauchy's Mean Value Theorem:

$$\frac{f'(c)}{g'(c)} = \frac{f(2) - f(1)}{g(2) - g(1)} = \frac{25 - 12}{27 - 25} = \frac{13}{2}$$

Simplifying the equation:

$$\frac{6c + 4}{2c - 1} = \frac{13}{2}$$

Solving for c:

$$(6c + 4) \cdot 2 = 13 \cdot (2c - 1) \implies 12c + 8 = 26c - 13 \implies 14c = 21 \implies c = \frac{21}{14} = \frac{3}{2} = 1.5$$

$$c = 1.5 \in [1, 2]$$

Thus, $c = 1.5$ satisfies the conditions of Cauchy's Mean Value Theorem in the interval $[1, 2]$.