THEOREM —**Integrability of Continuous Functions** If a function f is continuous over the interval $[a, b]$, or if f has at most finitely many jump discontinuities there, then the definite integral $\int_a^b f(x)\,dx$ exists and f is integrable over $[a, b]$.

Properties of Definite Integrals

1. **Order of Integration:** $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ A Definition

2. **Zero Width Interval:** $\displaystyle\int_a^a f(x)\,dx = 0$ A Definition when $f(a)$ exists

3. **Constant Multiple:** $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ Any constant k

4. **Sum and Difference:** $\displaystyle\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. **Additivity:** $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

6. **Max-Min Inequality:** If f has maximum value max f and minimum value min f on $[a, b]$, then

$$\min f \cdot (b - a) \le \int_a^b f(x)\,dx \le \max f \cdot (b - a).$$

7. **Domination:** $f(x) \ge g(x)$ on $[a, b] \implies \displaystyle\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

$f(x) \ge 0$ on $[a, b] \implies \displaystyle\int_a^b f(x)\,dx \ge 0$ (Special Case)

Show that the value of $\int_0^1 \sqrt{1 + \cos x}\, dx$ is less than or equal to $\sqrt{2}$.

Solution The Max-Min Inequality for definite integrals (Rule 6) says that $\min f \cdot (b - a)$ is a *lower bound* for the value of $\int_a^b f(x)\, dx$ and that $\max f \cdot (b - a)$ is an *upper bound*. The maximum value of $\sqrt{1 + \cos x}$ on $[0, 1]$ is $\sqrt{1 + 1} = \sqrt{2}$, so

$$\int_0^1 \sqrt{1 + \cos x}\, dx \le \sqrt{2} \cdot (1 - 0) = \sqrt{2}.$$

Area Under the Graph of a Nonnegative Function

We now return to the problem that started   that of defining what we mean by the *area* of a region having a curved boundary.   we approximated the area under the graph of a nonnegative continuous function using several types of finite sums of areas of rectangles capturing the region—upper sums, lower sums, and sums using the midpoints of each subinterval—all being cases of Riemann sums constructed in special ways. Theorem   guarantees that all of these Riemann sums converge to a single definite integral as the norm of the partitions approaches zero and the number of subintervals goes to infinity. As a result, we can   *define* the area under the graph of a nonnegative integrable function to be the value of that definite integral.

Average Value of a Continuous Function Revisited

we introduced informally the average value of a nonnegative continuous function f over an interval $[a, b]$, leading us to define this average as the area under the graph of $y = f(x)$ divided by $b - a$. In integral notation we write this as

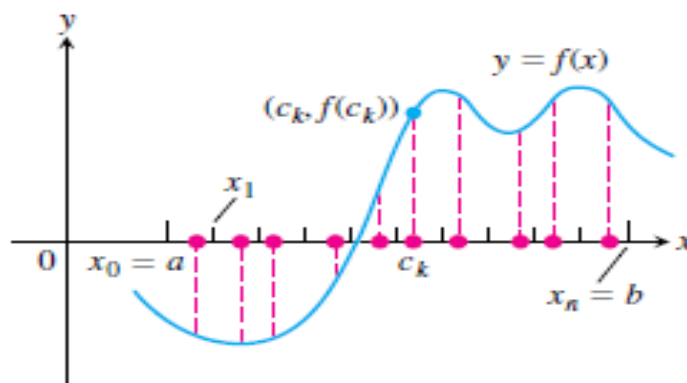$$\text{Average} = \frac{1}{b - a} \int_a^b f(x)\, dx.$$

We can use this formula to give a precise definition of the average value of any continuous (or integrable) function, whether positive, negative, or both.

Alternatively, we can use the following reasoning. We start with the idea from arithmetic that the average of n numbers is their sum divided by n.

We divide $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$ and evaluate f at a point c_k in each   . The average of the n sampled values is

$$\frac{f(c_1) + f(c_2) + \cdots + f(c_n)}{n} = \frac{1}{n} \sum_{k=1}^{n} f(c_k)$$

$$= \frac{\Delta x}{b - a} \sum_{k=1}^{n} f(c_k) \qquad \Delta x = \frac{b - a}{n}, \text{ so } \frac{1}{n} = \frac{\Delta x}{b - a}$$

$$= \frac{1}{b - a} \sum_{k=1}^{n} f(c_k) \, \Delta x \qquad \text{Constant Multiple Rule}$$

The average is obtained by dividing a Riemann sum for f on $[a, b]$ by $(b - a)$. As we increase the size of the sample and let the norm of the partition approach zero, the average approaches $(1/(b - a)) \int_a^b f(x) \, dx$. Both points of view lead us to the following definition.

DEFINITION   If f is integrable on $[a, b]$, then its **average value** on $[a, b]$, also called its **mean**, is

$$\text{av}(f) = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

Find the average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$.

Solution   We recognize $f(x) = \sqrt{4 - x^2}$ as a function whose graph is the upper semi-circle of radius 2 centered at the origin

The area between the semicircle and the x-axis from -2 to 2 can be computed using the geometry formula
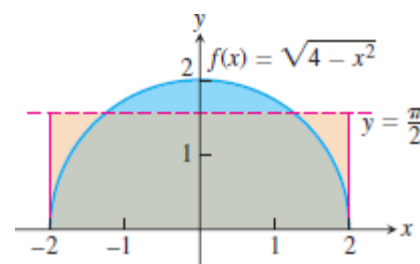
$$\text{Area} = \frac{1}{2} \cdot \pi r^2 = \frac{1}{2} \cdot \pi (2)^2 = 2\pi.$$

Because f is nonnegative, the area is also the value of the integral of f from -2 to 2,

$$\int_{-2}^{2} \sqrt{4 - x^2}\, dx = 2\pi.$$

Therefore, the average value of f is

$$\text{av}(f) = \frac{1}{2 - (-2)} \int_{-2}^{2} \sqrt{4 - x^2}\, dx = \frac{1}{4}(2\pi) = \frac{\pi}{2}.$$
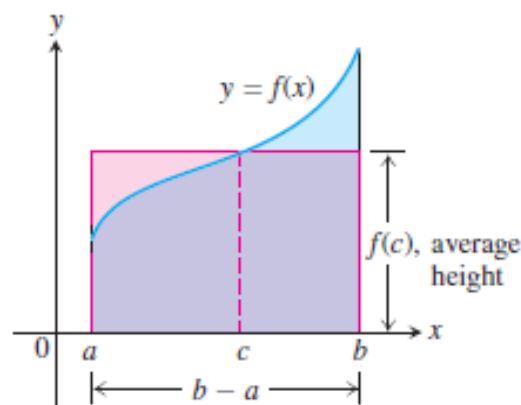
The Fundamental Theorem of Calculus

Mean Value Theorem for Definite Integrals

In the previous section we defined the average value of a continuous function over a closed interval $[a, b]$ as the definite integral $\int_a^b f(x)\, dx$ divided by the length or width $b - a$ of the interval. The Mean Value Theorem for Definite Integrals asserts that this average value is *always* taken on at least once by the function f in the interval.

The graph in Figure   shows a *positive* continuous function $y = f(x)$ defined over the interval $[a, b]$. Geometrically, the Mean Value Theorem says that there is a number c in $[a, b]$ such that the rectangle with height equal to the average value $f(c)$ of the function and base width $b - a$ has exactly the same area as the region beneath the graph of f from a to b.

The value $f(c)$ in the Mean Value Theorem is, in a sense, the average (or *mean*) height of f on $[a, b]$. When $f \geq 0$, the area of the rectangle is the area under the graph of f from a to b,

$$f(c)(b - a) = \int_a^b f(x)\, dx.$$

THEOREM —The Mean Value Theorem for Definite Integrals   If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b - a} \int_a^b f(x)\, dx.$$

EXAMPLE 1   Show that if f is continuous on $[a, b], a \neq b$, and if

$$\int_a^b f(x)\,dx = 0,$$

then $f(x) = 0$ at least once in $[a, b]$.

Solution   The average value of f on $[a, b]$ is

$$\mathrm{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a} \cdot 0 = 0.$$

By the Mean Value Theorem, f assumes this value at some point $c \in [a, b]$.

Fundamental Theorem, Part 1

If $f(t)$ is an integrable function over a finite interval I, then the integral from any fixed number $a \in I$ to another number $x \in I$ defines a new function F whose value at x is

$$F(x) = \int_a^x f(t)\, dt. \tag{1}$$

For example, if f is nonnegative and x lies to the right of a, then $F(x)$ is the area under the graph from a to x . The variable x is the upper limit of integration of an integral, but F is just like any other real-valued function of a real variable. For each value of the input x, there is a well-defined numerical output, in this case the definite integral of f from a to x.

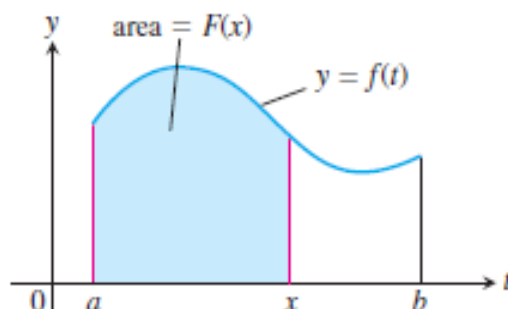Equation (1) gives a way to define new functions   , but its importance now is the connection it makes between integrals and derivatives. If f is any continuous function, then the Fundamental Theorem asserts that F is a differentiable function of x whose derivative is f itself. At every value of x, it asserts that
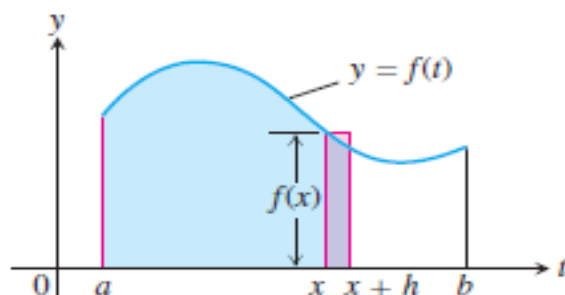
$$\frac{d}{dx} F(x) = f(x).$$

To gain some insight into why this result holds, we look at the geometry behind it.

If $f \geq 0$ on $[a, b]$, then the computation of $F'(x)$ from the definition of the derivative means taking the limit as $h \to 0$ of the difference quotient

$$\frac{F(x + h) - F(x)}{h}.$$

The function $F(x)$ defined by Equation (1) gives the area under the graph of f from a to x when f is nonnegative and $x > a$.

For $h > 0$, the numerator is obtained by subtracting two areas, so it is the area under the graph of f from x to $x + h$. If h is small, this area is approximately equal to the area of the rectangle of height $f(x)$ and width h, which can be seen from Figure ⎺⎺⎺. That is,

$$F(x + h) - F(x) \approx hf(x).$$

Dividing both sides of this approximation by h and letting $h \to 0$, it is reasonable to expect that

$$F'(x) = \lim_{h \to 0} \frac{F(x + h) - F(x)}{h} = f(x).$$

This result is true even if the function f is not positive, and it forms the first part of the Fundamental Theorem of Calculus.

THEOREM —The Fundamental Theorem of Calculus, Part 1   If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\, dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\, dt = f(x). \tag{2}$$

EXAMPLE 2   Use the Fundamental Theorem to find dy/dx if

(a) $y = \int_a^x (t^3 + 1)\, dt$ (b) $y = \int_x^5 3t \sin t\, dt$

(c) $y = \int_1^{x^2} \cos t\, dt$ (d) $y = \int_{1+3x^2}^4 \frac{1}{2 + e^t}\, dt$

Solution   We calculate the derivatives with respect to the independent variable x.

(a) $\dfrac{dy}{dx} = \dfrac{d}{dx}\displaystyle\int_a^x (t^3 + 1)\, dt = x^3 + 1$ Eq. (2) with $f(t) = t^3 + 1$

(b) $\dfrac{dy}{dx} = \dfrac{d}{dx}\displaystyle\int_x^5 3t \sin t\, dt = \dfrac{d}{dx}\left(-\int_5^x 3t \sin t\, dt\right)$

$$= -\frac{d}{dx}\int_5^x 3t \sin t\, dt$$

$$= -3x \sin x \qquad \text{Eq. (2) with } f(t) = 3t \sin t$$

(c) The upper limit of integration is not x but x^2. This makes y a composite of the two functions,

$$y = \int_1^u \cos t \, dt \qquad \text{and} \qquad u = x^2.$$

We must therefore apply the Chain Rule when finding dy/dx.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \left(\frac{d}{du} \int_1^u \cos t \, dt \right) \cdot \frac{du}{dx}$$

$$= \cos u \cdot \frac{du}{dx}$$

$$= \cos(x^2) \cdot 2x$$

$$= 2x \cos x^2$$

(d) $\dfrac{d}{dx} \displaystyle\int_{1+3x^2}^{4} \dfrac{1}{2 + e^t} \, dt = \dfrac{d}{dx} \left(- \displaystyle\int_{4}^{1+3x^2} \dfrac{1}{2 + e^t} \, dt \right)$ Rule 1

$$= -\frac{d}{dx} \int_{4}^{1+3x^2} \frac{1}{2 + e^t} \, dt$$

$$= -\frac{1}{2 + e^{(1+3x^2)}} \frac{d}{dx}(1 + 3x^2) \qquad \text{\color{blue}{Eq. (2) and the Chain Rule}}$$

$$= -\frac{6x}{2 + e^{(1+3x^2)}}$$

Find $\dfrac{d}{dx} \displaystyle\int_1^{x^4} \sec t \, dt$.

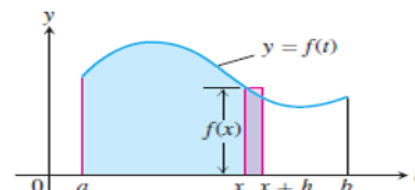**SOLUTION** Here we have to be careful to use the Chain Rule in conjunction with FTC1. Let $u = x^4$. Then

$$\dfrac{d}{dx} \int_1^{x^4} \sec t \, dt = \dfrac{d}{dx} \int_1^{u} \sec t \, dt$$

$$= \dfrac{d}{du} \left(\int_1^{u} \sec t \, dt \right) \dfrac{du}{dx} \qquad \text{(by the Chain Rule)}$$

$$= \sec u \, \dfrac{du}{dx} \qquad \text{(by FTC1)}$$

$$= \sec(x^4) \cdot 4x^3$$

Proof of Theorem   We prove the Fundamental Theorem, Part 1, by applying the definition of the derivative directly to the function $F(x)$, when x and $x + h$ are in (a, b). This means writing out the difference quotient

$$\frac{F(x + h) - F(x)}{h} \tag{3}$$

and showing that its limit as $h \to 0$ is the number $f(x)$ for each x in (a, b). Thus,

$$
\begin{aligned}
F'(x) &= \lim_{h \to 0} \frac{F(x + h) - F(x)}{h} \\
&= \lim_{h \to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)\, dt - \int_a^x f(t)\, dt \right] \\
&= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\, dt
\end{aligned}
$$

According to the Mean Value Theorem for Definite Integrals, the value before taking the limit in the last expression is one of the values taken on by f in the interval between x and $x + h$. That is, for some number c in this interval,

$$\frac{1}{h} \int_x^{x+h} f(t)\, dt = f(c). \tag{4}$$

As $h \to 0$, $x + h$ approaches x, forcing c to approach x also (because c is trapped between x and $x + h$). Since f is continuous at x, $f(c)$ approaches $f(x)$:

$$\lim_{h \to 0} f(c) = f(x).$$

In conclusion, we have

$$
\begin{aligned}
F'(x) &= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\, dt \\
&= \lim_{h \to 0} f(c) &&\text{Eq. (4)} \\
&= f(x). &&\text{Eq. (5)}
\end{aligned}
$$

THEOREM —The Mean Value Theorem for Definite Integrals   If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

If $x = a$ or b, then the limit of Equation (3) is interpreted as a one-sided limit with $h \to 0^+$ or $h \to 0^-$, respectively. F is continuous for every point in $[a, b]$. This concludes the proof.

We now come to the second part of the Fundamental Theorem of Calculus. This part describes how to evaluate definite integrals without having to calculate limits of Riemann sums. Instead we find and evaluate an antiderivative at the upper and lower limits of integration.

THEOREM **(Continued)—The Fundamental Theorem of Calculus, Part 2** If f is continuous at every point in $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Proof Part 1 of the Fundamental Theorem tells us that an antiderivative of f exists, namely

$$G(x) = \int_a^x f(t)\, dt.$$

Thus, if F is *any* antiderivative of f, then $F(x) = G(x) + C$ for some constant C for $a < x < b$.

Since both F and G are continuous on $[a, b]$, we see that $F(x) = G(x) + C$ also holds when $x = a$ and $x = b$ by taking one-sided limits (as $x \to a^+$ and $x \to b^-$). Evaluating $F(b) - F(a)$, we have

$$
\begin{aligned}
F(b) - F(a) &= [G(b) + C] - [G(a) + C] \\
&= G(b) - G(a) \\
&= \int_a^b f(t)\, dt - \int_a^a f(t)\, dt \\
&= \int_a^b f(t)\, dt - 0 \\
&= \int_a^b f(t)\, dt.
\end{aligned}
$$