

Derivative Rules

Derivative of a Constant Function

If f has the constant value, that is $f(x) = c$, then

$$\frac{d}{dx} = \frac{d}{dx}(c) = 0.$$

Power Rule

If n is any real number, then

$$\frac{d}{dx}x^n = nx^{n-1},$$

for all x where the powers x^n and x^{n-1} are defined.

Constant Multiple Rule

If u is a differentiable function of x , and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

Sum Rule

If u and v are differentiable functions of x , then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

Product Rule

If u and v are differentiable at x , then their product uv is differentiable at x , and

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}.$$

Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x , and

$$\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}.$$

Derivatives of Hyperbolic Functions The six hyperbolic functions, being rational combinations of the differentiable functions e^x and e^{-x} , have derivatives at every point at which they are defined. Again, there are similarities to trigonometric functions.

The derivative formulas are obtained from the derivative of e^x :

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx} \left(\frac{e^x - e^{-x}}{2} \right) \quad \text{Definition of } \sinh x$$

$$= \frac{e^x + e^{-x}}{2} \quad \text{Derivative of } e^x$$

$$= \cosh x \quad \text{Definition of } \cosh x$$

$$\frac{d}{dx}(\operatorname{csch} x) = \frac{d}{dx} \left(\frac{1}{\sinh x} \right) \quad \text{Definition of } \operatorname{csch} x$$

$$= -\frac{\cosh x}{\sinh^2 x} \quad \text{Quotient Rule for derivatives}$$

$$= -\frac{1}{\sinh x} \cdot \frac{\cosh x}{\sinh x} \quad \text{Rearrange factors.}$$

$$= -\operatorname{csch} x \coth x \quad \text{Definitions of } \operatorname{csch} x \text{ and } \coth x$$

Derivatives of Hyperbolic Functions

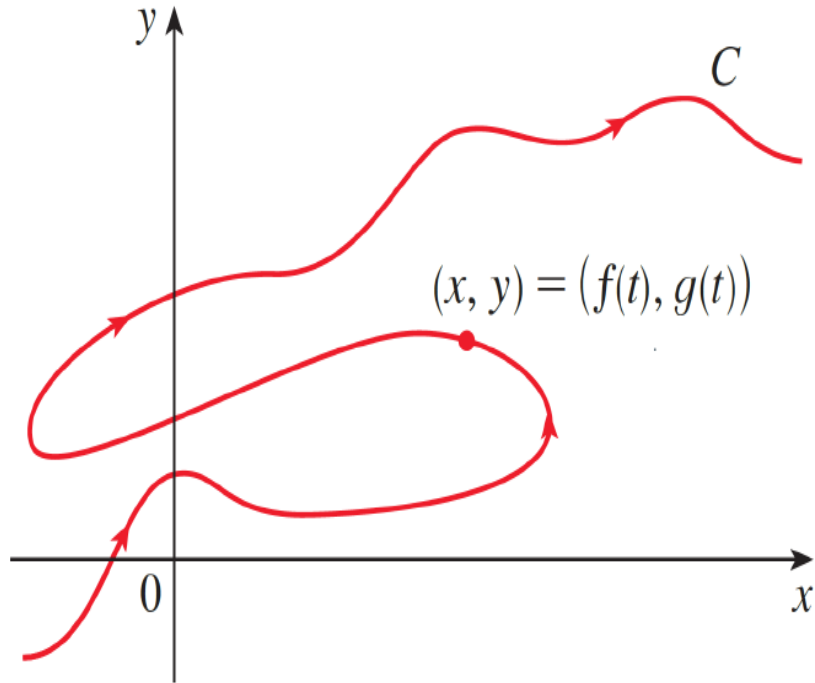
$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

Parametric Functions

Imagine that a particle moves along the curve C shown in fig. It is impossible to describe C by an equation of the form $y = f(x)$ because C fails the Vertical Line Test. But the x - and y -coordinates of the particle are functions of time t and so we can write $x = f(t)$ and $y = g(t)$. Such a pair of equations is often a convenient way of describing a curve.



If x and y are given as functions

$$x = f(t), \quad y = g(t)$$

over an interval I of t -values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a *parametric curve*. The equations are *parametric equations* for the curve.

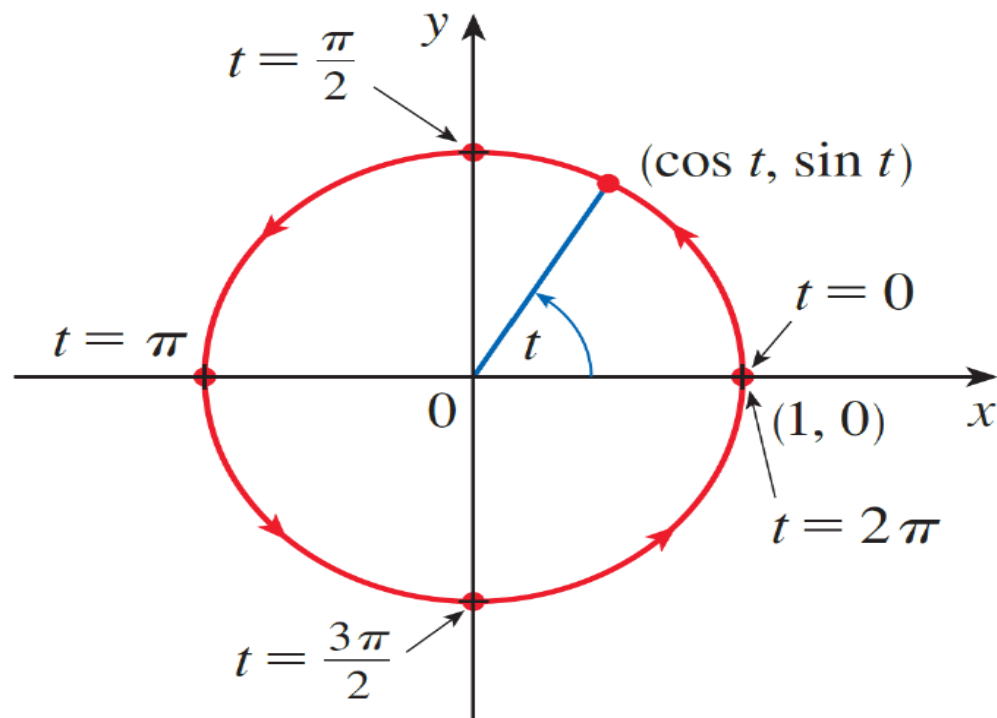
What curve is represented by the following parametric equations?

$$x = \cos t, \quad y = \sin t, \quad 0 \leq t \leq 2\pi$$

Solution. If we plot points, it appears that the curve is a circle. We can confirm this by eliminating the parameter t . Observe that

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1.$$

Because $x^2 + y^2 = 1$ is satisfied for all pairs of x - and y -values generated by the parametric equations, the point (x, y) moves along the unit circle $x^2 + y^2 = 1$.



Given $x(t) = t^2 + 1$ and $y(t) = 2 + t$, eliminate the parameter, and write the parametric equations as a Cartesian equation.

Solution. We will begin with the equation for y because the linear equation is easier to solve for t :

$$y = 2 + t \quad \Rightarrow \quad t = y - 2.$$

Next, substitute $t = y - 2$ into $x(t)$:

$$x = t^2 + 1 \quad \Rightarrow \quad x = (y - 2)^2 + 1.$$

Simplify:

$$x = (y - 2)^2 + 1 = y^2 - 4y + 4 + 1 = y^2 - 4y + 5.$$

Thus, the Cartesian form is:

$$x = y^2 - 4y + 5.$$

Derivative of Parametric Functions

Let C be a curve defined by the parametric equations $x = f(t)$, $y = g(t)$ for some interval I , and let f' , g' be continuous functions on I

(i) For every $t \in I$, if $f'(t) \neq 0$ (that is $f'(t) > 0$ or $f'(t) < 0$), then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}$$

holds true.

(ii) For every $t \in I$, if $f'(t) \neq 0$, then

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\frac{g'(t)}{f'(t)} \right) = \frac{d}{dt} \left(\frac{g'(t)}{f'(t)} \right) \frac{dt}{dx} = \frac{f'(t)g''(t) - g'(t)f''(t)}{(f'(t))^3}$$

Calculate the derivative $\frac{dy}{dx}$ where $x(t) = 5 \cos t$, $y(t) = 5 \sin t$,

Solution.

$$x'(t) = -5 \sin t$$

$$y'(t) = 5 \cos t.$$

Next substitute these into the equation:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dx} = \frac{5 \cos t}{-5 \sin t}$$

$$\frac{dy}{dx} = -\cot t$$

Find $\frac{d^2y}{dx^2}$ if $x = at^2$ and $y = 2at$.

Solution. From the given equations:

$$x = at^2 \quad \Rightarrow \quad \frac{dx}{dt} = 2at,$$

$$y = 2at \quad \Rightarrow \quad \frac{dy}{dt} = 2a.$$

Using the chain rule:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}.$$

Differentiating both sides of $\frac{dy}{dx} = \frac{1}{t}$ with respect to x :

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\frac{1}{t} \right).$$

Using the chain rule again:

$$\frac{d^2y}{dx^2} = \frac{d}{dt} \left(\frac{1}{t} \right) \cdot \frac{dt}{dx}.$$

Calculating $\frac{d}{dt} \left(\frac{1}{t} \right)$:

$$\frac{d}{dt} \left(\frac{1}{t} \right) = -\frac{1}{t^2}.$$

From (1), we know $\frac{dx}{dt} = 2at$, so:

$$\frac{dt}{dx} = \frac{1}{2at}.$$

Substitute these into the second derivative:

$$\frac{d^2y}{dx^2} = \left(-\frac{1}{t^2} \right) \cdot \frac{1}{2at} = -\frac{1}{2at^3}.$$

Thus,

$$\frac{d^2y}{dx^2} = -\frac{1}{2at^3}.$$

If $x = a \sec^3 \theta$ and $y = a \tan^3 \theta$, find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{4}$.

Solution. We are given:

$$x = a \sec^3 \theta \quad \Rightarrow \quad \frac{dx}{d\theta} = 3a \sec^2 \theta \cdot \frac{d}{d\theta}(\sec \theta) = 3a \sec^2 \theta \cdot \sec \theta \tan \theta = 3a \sec^3 \theta \tan \theta.$$

Similarly:

$$y = a \tan^3 \theta \quad \Rightarrow \quad \frac{dy}{d\theta} = 3a \tan^2 \theta \cdot \frac{d}{d\theta}(\tan \theta) = 3a \tan^2 \theta \cdot \sec^2 \theta = 3a \tan^2 \theta \sec^2 \theta.$$

The first derivative $\frac{dy}{dx}$ is:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a \tan^2 \theta \sec^2 \theta}{3a \sec^3 \theta \tan \theta}.$$

Simplify:

$$\frac{dy}{dx} = \frac{\tan^2 \theta \sec^2 \theta}{\sec^3 \theta \tan \theta} = \frac{\tan \theta}{\sec \theta} = \frac{\sin \theta}{\cos \theta} \cdot \cos \theta = \sin \theta.$$

Substitute $\theta = \frac{\pi}{4}$:

$$\left. \frac{dy}{dx} \right|_{\theta=\frac{\pi}{4}} = \sin \left(\frac{\pi}{4} \right) = \frac{\sqrt{2}}{2}.$$

Derivative of Inverse Functions

- Write $y = f(x)$
- Switch x and y
- Solve for y ,
- Replace y with $f^{-1}(x)$

If $f(x)$ is a continuous one-to-one function defined on an interval, then its inverse is also continuous. Moreover, if $f(x)$ is a differentiable function, then its inverse is also a differentiable function.

Let's find the inverse of the function $f(x) = 2x + 3$.

1. Write $y = 2x + 3$
2. Switch x and y :
 $x = 2y + 3$
3. Solve for y :

$$\begin{aligned}x &= 2y + 3 \\x - 3 &= 2y \\ \frac{x - 3}{2} &= y\end{aligned}$$

4. Replace y with $f^{-1}(x)$:

$$f^{-1}(x) = \frac{x - 3}{2}$$

Hence, the inverse of the function f is $f^{-1}(x) = \frac{x-3}{2}$.

The inverse of the function $f(x) = \frac{1}{2}x + 1$ as $f^{-1}(x) = 2x - 2$.

If we calculate their derivatives, we see that

$$\frac{d}{dx}f(x) = \frac{d}{dx} \left(\frac{1}{2}x + 1 \right) = \frac{1}{2},$$

and

$$\frac{d}{dx}f^{-1}(x) = \frac{d}{dx} (2x - 2) = 2.$$

The derivatives are reciprocals of one another, so the slope of one line is the reciprocal of the slope of its inverse line

The reciprocal relationship between the slopes of f and f^{-1} holds for other functions as well, but we must be careful to compare slopes at corresponding points. If the slope of $y = f(x)$ at the point $(a, f(a))$ is $f'(a)$ and $f'(a) \neq 0$, then the slope of $y = f^{-1}(x)$ at the point $(f(a), a)$ is the reciprocal $1/f'(a)$.

If we set $b = f(a)$, then

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}.$$

If $y = f(x)$ has a horizontal tangent line at $(a, f(a))$, then the inverse function f^{-1} has a vertical tangent line at $(f(a), a)$, and this infinite slope implies that f^{-1} is not differentiable at $f(a)$. Theorem gives the conditions under which f^{-1} is differentiable in its domain (which is the same as the range of f).

Proof.

$$f(f^{-1}(x)) = x$$

(Inverse function relationship)

$$\frac{d}{dx}f(f^{-1}(x)) = 1$$

(Differentiating both sides)

$$f'(f^{-1}(x)) \cdot \frac{d}{dx}f^{-1}(x) = 1$$

(Chain Rule)

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

(Solving for the derivative).

If we set $b = f(a)$, then

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}.$$

Find the derivative of the inverse function of $f(x) = x^3 + 1$ at the point -7 .

Given $f(x) = x^3 + 1$, the inverse function is $f^{-1}(x) = \sqrt[3]{x-1}$. We find that $f^{-1}(-7) = \sqrt[3]{-7-1}$, that is $f^{-1}(-7) = -2$. (or $f(-2) = -7$). The derivative of $f(x)$ is $f'(x) = 3x^2$. The derivative of the inverse function at -7 is then calculated as:

$$(f^{-1})'(-7) = \frac{1}{f'(-2)} = \frac{1}{3 \times (-2)^2} = \frac{1}{12}$$

Let $f(x) = x^3 - 2, x > 0$. Find the value of $\frac{df^{-1}}{dx}$ at $x = 6 = f(2)$ without finding a formula for $f^{-1}(x)$.

Solution

We are given $f(x) = x^3 - 2$, and we know that $f(2) = 6$. Therefore, $f^{-1}(6) = 2$. Let us compute the value of $\frac{df^{-1}}{dx}$ at $x = 6$ step by step.

Step 1: Find the inverse $f^{-1}(x)$

The function $f(x) = x^3 - 2$ is strictly increasing for $x > 0$, so it has an inverse. To find $f^{-1}(x)$, we solve:

$$f(x) = x^3 - 2 \Rightarrow x^3 = y + 2 \Rightarrow x = (y + 2)^{1/3}.$$

Thus, the inverse function is:

$$f^{-1}(x) = (x + 2)^{1/3}.$$

Step 2: Compute the derivative of $f^{-1}(x)$

Using the formula for the derivative of a power function, the derivative of $f^{-1}(x) = (x + 2)^{1/3}$ is:

$$\frac{df^{-1}}{dx} = \frac{1}{3}(x + 2)^{-2/3}.$$

Step 3: Evaluate at $x = 6$


Substituting $x = 6$ into the derivative:

$$\left. \frac{df^{-1}}{dx} \right|_{x=6} = \frac{1}{3}(6 + 2)^{-2/3} = \frac{1}{3}(8)^{-2/3} = \left. \frac{df^{-1}}{dx} \right|_{x=6} = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}.$$

Thus:

$$\boxed{\left. \frac{df^{-1}}{dx} \right|_{x=6} = \frac{1}{12}}$$

Find the derivative of $f^{-1}(x) = \sin^{-1} x$.

$$\begin{aligned}(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} && \text{Formula} \\&= \frac{1}{\cos(\sin^{-1} x)} && f'(u) = \cos u \\&= \frac{1}{\sqrt{1 - \sin^2(\sin^{-1} x)}} && \cos u = \sqrt{1 - \sin^2 u} \\&= \frac{1}{\sqrt{1 - x^2}}. && \sin(\sin^{-1} x) = x\end{aligned}$$


If u is a differentiable function of x with $|u| < 1$, we apply the Chain Rule to get the general formula

$$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx}, \quad |u| < 1.$$

Let us find the derivative of $y = \tan^{-1} x$ by applying

the derivative of $\tan x$ is positive for $-\frac{\pi}{2} < x < \frac{\pi}{2}$:

$$\begin{aligned}(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} \\&= \frac{1}{\sec^2(\tan^{-1} x)} \quad f'(u) = \sec^2 u \\&= \frac{1}{1 + \tan^2(\tan^{-1} x)} \quad \sec^2 u = 1 + \tan^2 u \\&= \frac{1}{1 + x^2} \quad \tan(\tan^{-1} x) = x.\end{aligned}$$

The derivative is defined for all real numbers. If u is a differentiable function of x , we get the Chain Rule form:

$$\boxed{\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1 + u^2} \frac{du}{dx}.}$$

The Derivative of $y = \sec^{-1} u$

$$y = \sec^{-1} x \quad \text{Inverse function relationship}$$

$$\sec y = x$$

$$\frac{d}{dx}(\sec y) = \frac{d}{dx}x \quad \text{Differentiate both sides.}$$

$$\sec y \tan y \frac{dy}{dx} = 1 \quad \text{Chain Rule}$$

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}.$$

To express the result in terms of x , we use the relationships:

$$\sec y = x \quad \text{and} \quad \tan y = \pm \sqrt{\sec^2 y - 1} = \pm \sqrt{x^2 - 1}.$$

Thus:

$$\frac{dy}{dx} = \pm \frac{1}{x\sqrt{x^2 - 1}}.$$

Can we do anything about the \pm sign? A glance at Figure below shows that the slope of the graph $y = \sec^{-1} x$ is always positive. Thus,

$$\frac{d}{dx} \sec^{-1} x = \begin{cases} \frac{1}{x\sqrt{x^2 - 1}}, & \text{if } x > 1, \\ -\frac{1}{x\sqrt{x^2 - 1}}, & \text{if } x < -1. \end{cases}$$

With the absolute value symbol, we can write a single expression that eliminates the \pm ambiguity:

$$\frac{d}{dx} \sec^{-1} x = \frac{1}{|x|\sqrt{x^2 - 1}}.$$

If u is a differentiable function of x with $|u| > 1$, we have the formula:

$$\frac{d}{dx}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2 - 1}} \frac{du}{dx}, \quad |u| > 1.$$

Derivatives of the inverse trigonometric functions

1.	$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}, \quad u < 1$
2.	$\frac{d}{dx}(\cos^{-1} u) = -\frac{1}{\sqrt{1-u^2}} \frac{du}{dx}, \quad u < 1$
3.	$\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1+u^2} \frac{du}{dx}$
4.	$\frac{d}{dx}(\cot^{-1} u) = -\frac{1}{1+u^2} \frac{du}{dx}$
5.	$\frac{d}{dx}(\sec^{-1} u) = \frac{1}{ u \sqrt{u^2-1}} \frac{du}{dx}, \quad u > 1$
6.	$\frac{d}{dx}(\csc^{-1} u) = -\frac{1}{ u \sqrt{u^2-1}} \frac{du}{dx}, \quad u > 1$

Derivatives of inverse hyperbolic functions

<i>Function</i>	<i>Derivative Formula</i>
$\frac{d}{dx}(\sinh^{-1} u)$	$= \frac{1}{\sqrt{1+u^2}} \frac{du}{dx}$
$\frac{d}{dx}(\cosh^{-1} u)$	$= \frac{1}{\sqrt{u^2-1}} \frac{du}{dx}, \quad u > 1$
$\frac{d}{dx}(\tanh^{-1} u)$	$= \frac{1}{1-u^2} \frac{du}{dx}, \quad u < 1$
$\frac{d}{dx}(\coth^{-1} u)$	$= \frac{1}{1-u^2} \frac{du}{dx}, \quad u > 1$
$\frac{d}{dx}(\operatorname{sech}^{-1} u)$	$= \frac{-1}{u\sqrt{1-u^2}} \frac{du}{dx}, \quad 0 < u < 1$
$\frac{d}{dx}(\operatorname{csch}^{-1} u)$	$= \frac{-1}{ u \sqrt{1+u^2}} \frac{du}{dx}, \quad u \neq 0$

Show that if u is a differentiable function of x whose values are greater than 1, then:

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2 - 1}} \frac{du}{dx}.$$

Solution. First, we find the derivative of $y = \cosh^{-1} x$ for $x > 1$. $f(x) = \cosh x$ and $f^{-1}(x) = \cosh^{-1} x$. Theorem can be applied because the derivative of $\cosh x$ is positive for $x > 0$.

$$\begin{aligned}(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} \\ &= \frac{1}{\sinh(\cosh^{-1} x)} \\ &= \frac{1}{\sqrt{\cosh^2(\cosh^{-1} x) - 1}} \quad \cosh^2 u - \sinh^2 u = 1 \\ &= \frac{1}{\sqrt{x^2 - 1}} \quad \cosh(\cosh^{-1} x) = x.\end{aligned}$$

The Chain Rule gives the final result:

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2 - 1}} \frac{du}{dx}.$$

Derivatives of Logarithmic Functions

$$\frac{d}{dx} (\log_b x) = \frac{1}{x \ln b}$$

Let $y = \log_b x$. Then

$$b^y = x.$$

Differentiating this equation implicitly with respect to x

$$(b^y \ln b) \frac{dy}{dx} = 1,$$

$$\frac{dy}{dx} = \frac{1}{b^y \ln b} = \frac{1}{x \ln b}.$$

If we put $b = e$ $\frac{d}{dx}(\ln x) = \frac{1}{x}.$

or

$$\frac{d}{dx} \ln |x| = \frac{1}{x}, \quad x \neq 0.$$

$$\frac{d}{dx} (\ln x) = \frac{1}{x}, \quad x > 0.$$

$f(x) = e^x$ is differentiable everywhere

find the derivative of its inverse $f^{-1}(x) = \ln x$:

$$y = \ln x \quad x > 0 \quad (\text{Inverse function relationship})$$

$$e^y = x \quad \text{Inverse function relationship}$$

$$\frac{d}{dx}(e^y) = \frac{d}{dx}(x) \quad \text{Differentiate implicitly.}$$

$$e^y \frac{dy}{dx} = 1 \quad \text{Chain Rule}$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}. \quad e^y = x$$

The Chain Rule extends this formula to positive functions $u(x)$:

$$\boxed{\frac{d}{dx} \ln u = \frac{1}{u} \frac{du}{dx}, \quad u > 0.}$$

Find $f'(x)$ if $f(x) = \ln |x|$.

Solution. Since

$$f(x) = \begin{cases} \ln x & \text{if } x > 0, \\ \ln(-x) & \text{if } x < 0, \end{cases}$$

it follows that

$$f'(x) = \begin{cases} \frac{1}{x} & \text{if } x > 0, \\ \frac{1}{-x}(-1) = \frac{1}{x} & \text{if } x < 0. \end{cases}$$

Thus, $f'(x) = \frac{1}{x}$ for all $x \neq 0$.

Find $\frac{d}{dx} \ln(\sin x)$.

$$\frac{d}{dx} \ln(\sin x) = \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x) = \frac{1}{\sin x} \cdot \cos x = \cot x.$$

Differentiate $y = \ln(x^3 + 1)$.

Solution. To use the Chain Rule, we let $u = x^3 + 1$. Then $y = \ln u$, so

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot \frac{du}{dx} = \frac{1}{x^3 + 1} \cdot (3x^2) = \frac{3x^2}{x^3 + 1}.$$

$$\frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}.$$

Differentiate $f(x) = \sqrt{\ln x}$.

Solution. This time the logarithm is the inner function, so the Chain Rule gives

$$f'(x) = \frac{1}{2}(\ln x)^{-1/2} \cdot \frac{d}{dx}(\ln x) = \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}.$$

Find $\frac{d}{dx} \ln \frac{x+1}{\sqrt{x-2}}$.

If we first expand the given function using the laws of logarithms, then the differentiation becomes easier:

$$\frac{d}{dx} \ln \frac{x+1}{\sqrt{x-2}} = \frac{d}{dx} [\ln(x+1) - \frac{1}{2} \ln(x-2)].$$

Differentiating term-by-term, we get

$$\frac{1}{x+1} - \frac{1}{2} \cdot \frac{1}{x-2}.$$

Combining into a single fraction gives the same result:

$$\frac{x-5}{2(x+1)(x-2)}.$$

Definition

Suppose we want to differentiate a function of the form

$$y = (f(x))^{g(x)} \quad (\text{for } f(x) > 0).$$

Since the variable appears in both the base and the exponent, neither the general power rule, $\frac{dx^a}{dx} = ax^{a-1}$, nor the exponential rule, $\frac{dx^a}{dx} = a^x \ln a$, can be directly applied.

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to expand the expression.
2. Differentiate implicitly with respect to x .
3. Solve the resulting equation for y' and replace y by $f(x)$.

Let us find the derivative of $y = x^x$

Solution. The derivative can be obtained by taking natural logarithms of both sides of the equation $y = x^x$ and differentiating implicitly:

$$\begin{aligned}\ln y &= x \ln x \\ \frac{1}{y} \frac{dy}{dx} &= \ln x + \frac{x}{x} = 1 + \ln x \\ \frac{dy}{dx} &= y(1 + \ln x) = x^x(1 + \ln x).\end{aligned}$$

Differentiate $y = x^{\sqrt{x}}$.

Since both the base and the exponent are variable, we use logarithmic differentiation:

$$\ln y = \ln x^{\sqrt{x}} = \sqrt{x} \ln x.$$

Differentiating implicitly with respect to x :

$$\frac{y'}{y} = \sqrt{x} \cdot \frac{1}{x} + (\ln x) \cdot \frac{1}{2\sqrt{x}}.$$

Simplify:

$$y' = y \left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}} \right) = x^{\sqrt{x}} \left(\frac{2 + \ln x}{2\sqrt{x}} \right).$$

Find dy/dt if $y = (\sin t)^{\ln t}$, where $0 < t < \pi$.

Solution. We have $\ln y = \ln t \ln \sin t$. Thus,

$$\begin{aligned}\frac{1}{y} \frac{dy}{dt} &= \frac{1}{t} \ln \sin t + \ln t \frac{\cos t}{\sin t} \\ \frac{dy}{dt} &= y \left(\frac{\ln \sin t}{t} + \ln t \cot t \right) = (\sin t)^{\ln t} \left(\frac{\ln \sin t}{t} + \ln t \cot t \right).\end{aligned}$$

Differentiate $y = [(x+1)(x+2)(x+3)]/(x+4)$.

Find $\left. \frac{du}{dx} \right|_{x=1}$ if $u = \sqrt{(x+1)(x^2+1)(x^3+1)}$.

Given the function

$$f(x) = (\cos^4 x)^{\arctan(x^2)}$$

we need to compute the derivative $f'(x)$ and evaluate $f'(0)$.

Find the derivative of the function:

$$y = 5 \sinh^3 \frac{x}{15} + 3 \cosh \frac{x}{15}.$$