Linearization

In general, the tangent to $y = f(x)$ at a point $x = a$, where f is differentiable, passes through the point $(a, f(a))$, so its point-slope equation is

$$y = f(a) + f'(a)(x - a).$$

Thus, this tangent line is the graph of the linear function

$$L(x) = f(a) + f'(a)(x - a).$$

For as long as this line remains close to the graph of f as we move off the point of tangency, $L(x)$ gives a good approximation to $f(x)$.

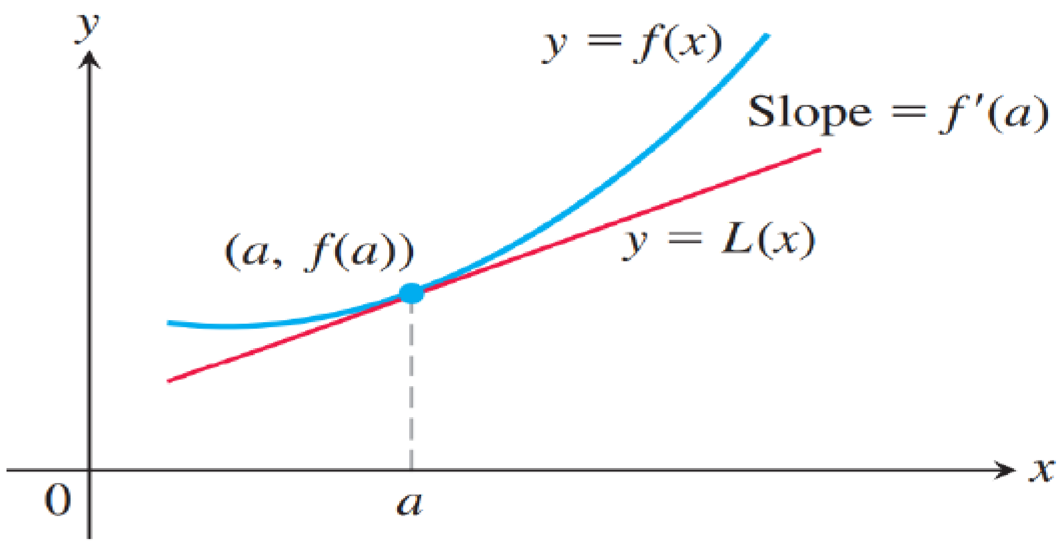
If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the **linearization** of f at a. The approximation

$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** (or tangent line approximation) of f at a. The point $x = a$ is the center of the approximation.



Find the linearization of $f(x) = \sqrt{1+x}$ at $x = 0$. Then find aprroximate values for $\sqrt{1.2}$, $\sqrt{1.05}$, and $\sqrt{1.005}$ using the linear approximation.

Solution

Since

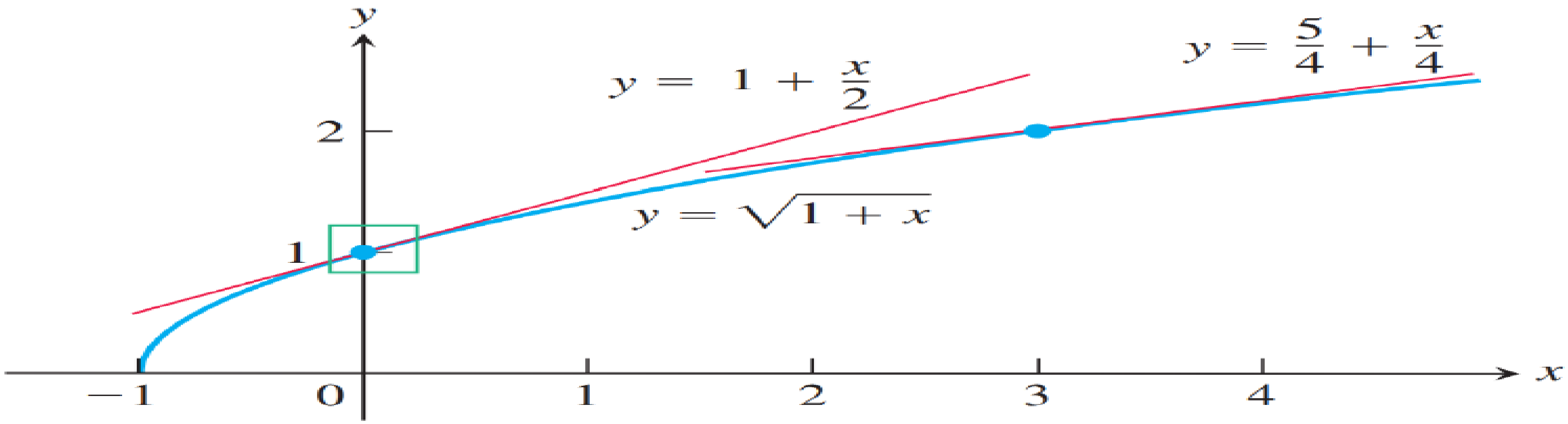
$$f'(x) = \frac{1}{2}(1+x)^{-1/2}$$

we have $f(0) = 1$ and $f'(0) = 1/2$, giving the linearization

$$L(x) = f(a) + f'(a)(x-a) = 1 + \frac{1}{2}(x-0) = 1 + \frac{x}{2}.$$

Thus, we have:

$$f(x) \approx L(x) \implies \sqrt{1+x} \approx 1 + (x/2)$$



The following table shows how accurate the approximation $\sqrt{1+x} \approx 1+(x/2)$ from Example is for some values of x near 0. As we move away from zero, we lose accuracy.

Approximation	True value	|True value - approximation|
$\sqrt{1.2} \approx 1+\frac{0.2}{2} = 1.10$	1.095445	$0.004555 < 10^{-2}$
$\sqrt{1.05} \approx 1+\frac{0.05}{2} = 1.025$	1.024695	$0.000305 < 10^{-3}$
$\sqrt{1.005} \approx 1+\frac{0.005}{2} = 1.00250$	1.002497	$0.000003 < 10^{-5}$

Determine the linear approximation for $f(x) = \sqrt[3]{x}$ at $x = 8$. Use the linear approximation to approximate the value of $\sqrt[3]{8.05}$ and $\sqrt[3]{25}$.

Solution.

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$$

The linear approximation is then,

$$L(x) = 2 + \frac{1}{12}(x-8) = \frac{1}{12}x + \frac{4}{3}$$

$$L(8.05) = 2.00416667, \quad \sqrt[3]{8.05} = 2.00415802$$

$$L(25) = 3.4166667, \quad \sqrt[3]{25} = 2.92401774$$

Differential

Let $y = f(x)$ be a differentiable function. The differential dx, called differential of x, is an independent variable. The differential dy, called differential of y, is

$$dy = f'(x)dx$$

(a) Find dy if $y = x^5 + 37x$.

(b) Find the value of dy when $x = 1$ and $dx = 0.2$.

Solution. (a) We calculate dy as:

$$dy = (5x^4 + 37)dx$$

(b) Substituting $x = 1$ and $dx = 0.2$ into the expression for dy, we have:

$$dy = (5 \cdot 1^4 + 37) \cdot 0.2 = 8.4.$$

Every differentiation formula, such as:

$$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}, \quad \textit{or} \quad \frac{d(\sin u)}{dx} = \cos u \frac{du}{dx},$$

has a corresponding differential form:

$$d(u+v) = du + dv, \quad \textit{or} \quad d(\sin u) = \cos u \, du.$$

We can use the Chain Rule and other differentiation rules to find differentials of functions.

(a) *For* $d(\tan 2x)$*:*

$$d(\tan 2x) = \sec^2(2x)\, d(2x) = 2\sec^2(2x)\, dx.$$

(b) *For* $d\left(\frac{x}{x+1}\right)$*:*

$$d\left(\frac{x}{x+1}\right) = \frac{(x+1)\,dx - x\,d(x+1)}{(x+1)^2} = \frac{x\,dx + dx - x\,dx}{(x+1)^2} = \frac{dx}{(x+1)^2}.$$

Geometrical Interpretation of Differential

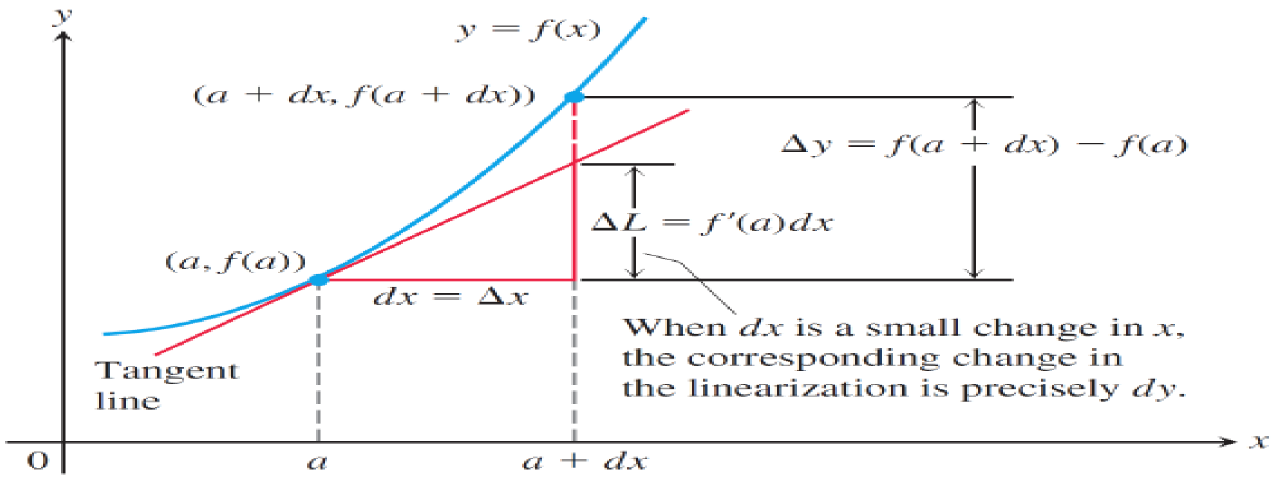
The geometric meaning of differentials is shown in Fig. . *Let $x = a$ and set $dx = \Delta x$. The corresponding change in $y = f(x)$ is:*

$$\Delta y = f(a + dx) - f(a).$$

The corresponding change in the tangent line L is:

$$\Delta L = L(a + dx) - L(a) = f(a) + f'(a)\big[(a + dx) - a\big] - f(a) = f'(a)dx.$$

Thus, $\Delta L = f'(a)dx$ represents the linear approximation to Δy.



Geometrically, the differential dy is the change ΔL in the linearization of f when $x = a$ changes by an amount $dx = \Delta x$.

Estimating with Differentials

Suppose we know the value of a differentiable function $f(x)$ at a point a and want to estimate how much this value will change if we move to a nearby point $a + dx$. If $dx = \Delta x$ is small, then we can see from Fig. that Δy is approximately equal to the differential dy. Since:

$$f(a + dx) = f(a) + \Delta y, \quad \Delta x = dx,$$

the differential approximation gives:

$$f(a + dx) \approx f(a) + dy.$$

Thus, the approximation $\Delta y \approx dy$ can be used to estimate $f(a + dx)$ when $f(a)$ is known, dx is small, and $dy = f'(a)dx$.

Let us find approximate values of $\sqrt{1.2}$, $\sqrt{1.05}$, $\sqrt{1.005}$ by using differential.

Solution. Let us select $f(x) = \sqrt{1+x}$.

The initial point is $x_0 = 0$. For $x_0 = 0$, we have $f(x_0) = f(0) = \sqrt{1+0} = 1$

We will calculate the approximate value of the function $f(x) = \sqrt{1+x}$ at the points $x = 0.2, 0.05, 0.005$.

$$f'(x) = \frac{1}{2\sqrt{x+1}}$$

First, let us consider $dx = 0.2$

$$dy = f'(x_0)dx$$

$$\Rightarrow f'(0) = \frac{1}{2\sqrt{0+1}} = \frac{1}{2}$$

$$\Rightarrow dy = f'(0)dx = \frac{1}{2}\frac{2}{10} = \frac{1}{10}$$

So, by differential, the approximate value is $f(x_0) + dy = 1 + 0.1 = 1.1$.

$$f(x_0 + dx) \approx f(x_0) + dy \implies f(0 + 0.2) \approx f(0) + 0.1 \implies \sqrt{1.2} \approx 1.1.$$

True value of $f(x_0 + dx) = f(0 + 0.2) = f(0.2) = \sqrt{1.2} \approx 1.09545$

Absolute error is:

$$|\text{True value- approximate value}| = 1.09545 - 1.1 = 0.00455488$$

Similarly, we can evaluate $\sqrt{1.05}, \sqrt{1.005}$ by using differential. The results are given in the following table.

dx	Approximate value $f(x_0) + dy$	True Value $f(x_0 + dx)$	Error
0.2	1.1	1.09545	0.00455488
0.05	1.025	1.0247	0.000304923
0.005	1.0025	1.0025	3.11721×10^{-6}

Where $x_0 = 0$

Find a reasonable approximation to the value of

$$(1.001)^7 - 2(1.001)^{4/3} + 3.$$

Solution. Let us assume that

$$F(x) = x^7 - 2x^{4/3} + 3$$

First, we will find the value of the function at $x_0 = 1$. Then, when the x value changes by 0.001 units (that is $dx = 0.001$), we will find out how much the change in the y axis will be. That is, we will find dy. So, $F(x_0) + dy$ is approximately equal to $F(x_0 + dx) = F(1.001)$.

At the beginning, for $x_0 = 1$, we have

$$y_0 = F(x_0) = F(1) = 2.$$

The tangent to this curve $y = F(x)$ at $(1, 2)$ will remain near the curve for values of x close to x_0. As x changes from $x_0 = 1$ to $x_0 + dx = 1.001$, the change in y along this tangent line will be

$$dy = F'(x_0)dx.$$

Since

$$F'(x) = 7x^6 - \frac{8}{3}x^{1/3}$$

has the value

$$F'(x_0) = \frac{13}{3}$$

at $x_0 = 1$, when we take $dx = 0.001$, we have

$$dy = \frac{13}{3}(0.001) = 0.0043.$$

When this change in y is added to y_0, we have

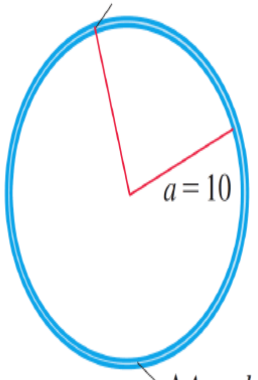
$$F(x_0) + dy = y_0 + dy = 2.0043$$

Thus,

$$(1.001)^7 - 2(1.001)^{4/3} + 3 \approx 2.0043$$

The radius r of a circle increases from $a = 10\,\text{m}$ to $10.1\,\text{m}$. Use dA to estimate the increase in the circle's area A. Estimate the area of the enlarged circle and compare your estimate to the true area found by direct calculation.

Solution. Since $A = \pi r^2$, the estimated increase is:



$$dA = A'(a)\,dr = 2\pi a\,dr = 2\pi(10)(0.1) = 2\pi\ \text{m}^2.$$

Thus, since $A(r + \Delta r) \approx A(r) + dA$, we have:

$$A(10 + 0.1) \approx A(10) + 2\pi = \pi(10)^2 + 2\pi = 100\pi + 2\pi = 102\pi.$$

The area of a circle of radius $10.1\,\text{m}$ is approximately $102\pi\ \text{m}^2$. The true area is:

$$A(10.1) = \pi(10.1)^2 = 102.01\pi\ \text{m}^2.$$

The error in our estimate is $0.01\pi\ \text{m}^2$, which is the difference $\Delta A - dA$.

Find the approximate value of $(1.05)^{1.05}$ *using linear approximation.*

Solution.

$$y = f(x) = x^x \quad ; \quad a = 1 \quad ; \quad x = 1.05$$

$$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$

$$f(x) = x^x$$

$$\ln y = \ln(x^x) = x \cdot \ln x$$

$$\frac{y'}{y} = [\ln x + 1] \implies y' = [\ln x + 1]x^x \implies y'(1) = 1$$

$$f(1.05) \approx f(1) + f'(1)(1.05 - 1)$$

$$\approx 1^1 + [\ln 1 + 1] \cdot 1^1(0.05)$$

$$\approx 1 + 0.05$$
$$\approx 1.05$$

Find the approximate value of $\sin 29^\circ$ *using approximate calculation.*

Solution.

$$f(x) \approx L(x) = f(a) + f'(a)(x-a)$$

$$f(x) = \sin x \quad ; \quad a = 30^\circ \quad ; \quad x = 29^\circ$$

$$f'(x) = \cos x$$

$$f(29^\circ) \approx f(30^\circ) + f'(30^\circ)(29^\circ - 30^\circ)$$

$$\sin 29^\circ \approx \sin 30^\circ + \cos 30^\circ(29^\circ - 30^\circ)$$

$$\approx \frac{1}{2} + \frac{\sqrt{3}}{2}(-1^\circ) \quad ; \quad 1^\circ = \frac{\pi}{180}$$

$$\approx \frac{1}{2} - \frac{\sqrt{3}\pi}{360}$$

Differential Approach:

The differential approach models how a function behaves for an infinitesimally small change. In mathematics, the differential of a function, denoted by df, is defined as the product of the function's derivative and the change dx:

$$df = f'(x) \cdot dx$$

This expression indicates how a small increase in x by dx would affect the value of the function. It provides an accurate estimate for very small changes and is typically used in differential calculations.

Linearization:

Linearization is used to obtain a linear approximation of a function at a point. This means finding the tangent line to the function at that point and using this line as the local approximate value of the function. The formula for linearization is as follows:

$$L(x) = f(a) + f'(a)(x - a)$$

Here, $L(x)$ represents the value of the tangent line at a, and $f(a)$ and $f'(a)$ are the value of the function and its derivative at a, respectively. x represents a deviation from the point a. Linearization provides a good estimate of the function's behavior around the point a even for larger values of dx.

Optimization Problems

In this section, we use derivatives to solve a variety of optimization problems in mathematics, physics, economics, and business.

What is Optimization?

In mathematics, **optimization** is the process of finding the ”best” result (maximum or minimum) achievable within given constraints. Calculus is one of the most powerful tools for solving these problems

Optimization in the Real World

- **Packaging and Design (Volume and Area):** A shipping company designing a box to achieve the **maximum volume** using a fixed amount of cardboard. Similarly, a factory finding the dimensions for a cylindrical can to use the **minimum amount of metal** for a fixed volume.
- **Business and Economics (Profit and Cost):** A company determining the specific selling price that yields **maximum profit** by analyzing the relationship between price and consumer demand.

- **Logistics and Engineering (Minimum Distance/Cost):** Calculating the **minimum construction cost** for laying a pipeline between two locations by balancing different terrain costs (e.g., land vs. water).

- **Agriculture (The Fencing Problem):** A farmer with a limited length of fencing calculating the optimal dimensions to enclose the **maximum possible area** for livestock.

Steps for Optimization Problems

1. ***Draw and label a picture.***

   Optimization problems often involve geometry. Draw a picture of the situation. Include any information you are given in the problem. Identify the quantities under your control and assign variables to them.

2. ***Find the objective function.***

   What is the quantity you want to maximize or minimize? Write a formula for it in terms of the variables in your picture.

3. ***Identify the constraints.***

   If your objective function has more than one variable, you will need to use one or more constraints in the problem to write equations that relate the variables together.

4. ***Reduce the objective function to one variable.***

   Solve each of the constraint equations for one of the variables and substitute this into the objective function. At the end, the objective function should contain just one variable.

5. ***Identify the domain of the objective function.***

   Now that your objective function has a single variable, what values of that variable make sense in the problem? Think about what the variable means in the problem—for example, lengths cannot be negative. There may be both a lower bound and an upper bound for meaningful values of the variable; if so, make sure you identify both of them. If the domain has both a lower and an upper bound, and the endpoints of this interval do not cause any mathematical problems in the objective function (such as division by zero), consider including the endpoints even if they don't quite make sense in the problem; this will allow you to use the Closed Interval Method.

6. ***Differentiate the objective function.***

7. ***Find the critical numbers.***

   Critical numbers are values of the variable that cause the derivative to equal zero or be undefined. These numbers are potential local maxima or minima. Ignore any critical numbers that are outside the domain of the objective function.

8. ***Test the critical numbers*** *(and possibly the endpoints of the domain). The critical numbers are not automatically the answer. You need to test each one to see if it's a local minimum, a local maximum, or neither. Usually you can use one of the following tests:*

   i. ***Closed Interval Method.***

      Requirements: The domain of the objective function must be a closed, bounded interval (that is, an interval that has endpoints on both ends, and includes those endpoints).

      Test: Evaluate the objective function at all critical numbers and at the endpoints of the domain, and choose the largest value (if you are maximizing) or the smallest value (if you are minimizing).

   ii. ***First Derivative Test.***

      Requirements: There must be only one critical number in the domain.

      Test: Evaluate the first derivative of the objective function at two numbers in the domain, one less than the critical number and one greater. If the first derivative is positive to the left of the critical number and negative to the right, then the critical number represents the absolute maximum. If the first derivative is negative to the left and positive to the right, the critical number represents the absolute minimum.

iii. ***Second Derivative Test.***

Requirements: There must be only one critical number in the domain.

Test: Find the second derivative of the objective function and evaluate it at the critical number. If this value is negative, then the critical number represents the absolute maximum. If the value is positive, then the critical number represents the absolute minimum.

9. ***Answer the question.*** *The critical number itself may not be the answer to the question. Reread the question to see exactly what it is asking for. If it is asking for the dimensions of a rectangle, for example, make sure you give the width and height of the rectangle, not just the area. And don't forget units!*

Find two positive numbers whose sum is 300 *and whose product is a maximum. What is the value of this maximum product?*

Solution. We will follow the systematic steps for optimization to solve this problem:

1. **Draw and label a picture:** In this case, since we are dealing with two numbers rather than a geometric shape, we assign variables:
   Let x be the first positive number and y be the second positive number.
2. **Find the objective function:** The quantity we want to maximize is the product, P. The formula is:
$$P = x \cdot y$$
3. **Identify the constraints:** The problem states that the sum of the two numbers must be 300. This gives us the constraint equation:
$$x + y = 300$$
4. **Reduce the objective function to one variable:** Solve the constraint for y:
$$y = 300 - x$$
   Substitute this into the objective function P:
$$P(x) = x(300 - x) = 300x - x^2$$
5. **Identify the domain of the objective function:** Since x and y must be positive numbers:
   $x > 0$
   $y > 0 \implies 300 - x > 0 \implies x < 300$

Thus, the domain is the open interval $(0, 300)$.

6. **Differentiate the objective function:**

$$P'(x) = \frac{d}{dx}(300x - x^2) = 300 - 2x$$

7. **Find the critical numbers:** Set the derivative equal to zero:

$$300 - 2x = 0 \implies 2x = 300 \implies x = 150$$

Since 150 is within the domain $(0, 300)$, it is a valid critical number.

8. **Test the critical numbers.**

   We choose the **Second Derivative Test** (Method iii) for efficiency:

   - *Requirement:* There is only one critical number (150) in the domain.
   - *Test:* Find the second derivative: $P''(x) = -2$.
   - *Evaluation:* Since $P''(150) = -2$ is negative, the critical number represents the absolute maximum.
   - *Note on alternatives:* The **First Derivative Test** would also work by checking signs around $x = 150$, but the second derivative is faster here because $P''(x)$ is a constant. The **Closed Interval Method** is not directly applicable because our domain is an open interval $(0, 300)$.

9. **Answer the question.** The two numbers are:

$$x = 150, \quad y = 300 - 150 = 150$$

   The maximum product is:

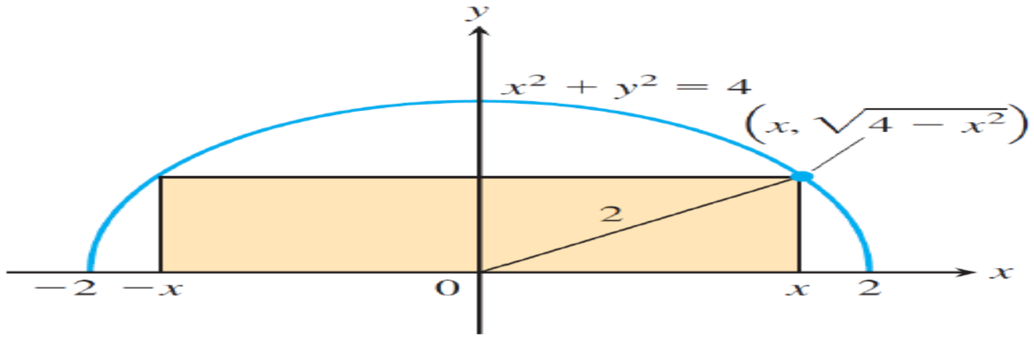
$$P_{max} = 150 \cdot 150 = 22,500$$

Conclusion: The two positive numbers are 150 and 150, yielding a maximum product of 22,500.

A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?

Solution. To find the maximum area, we apply the systematic steps for optimization:

1. **Draw and label a picture:** Imagine a semicircle centered at the origin $(0, 0)$ on a Cartesian plane with radius $r = 2$.



2. **Find the objective function:** The quantity to maximize is the area A of the rectangle. The base of the rectangle is $2x$ and the height is y.

$$A = 2x \cdot y$$

3. **Identify the constraints:** The upper corners of the rectangle must lie on the semicircle, providing the constraint:

$$y = \sqrt{4 - x^2}$$

4. **Reduce the objective function to one variable:** Substitute the constraint into the area formula:

$$A(x) = 2x\sqrt{4 - x^2}$$

5. **Identify the domain of the objective function:** Since x is a half-length and must stay within the radius of 2: The domain is $x \in (0, 2)$.

6. **Differentiate the objective function:** Using the product rule and chain rule:

$$A'(x) = 2\sqrt{4 - x^2} + 2x \cdot \frac{1}{2\sqrt{4 - x^2}} \cdot (-2x)$$

$$A'(x) = 2\sqrt{4 - x^2} - \frac{2x^2}{\sqrt{4 - x^2}}$$

Simplify by finding a common denominator:

$$A'(x) = \frac{2(4 - x^2) - 2x^2}{\sqrt{4 - x^2}} = \frac{8 - 4x^2}{\sqrt{4 - x^2}}$$

7. **Find the critical numbers:** Set $A'(x) = 0$:

$$8 - 4x^2 = 0 \implies 4x^2 = 8 \implies x^2 = 2 \implies x = \sqrt{2}$$

(We ignore $x = -\sqrt{2}$ as it is outside our domain).

8. **Test the critical numbers.**

   i. **Closed Interval Method:** For the Closed Interval Method, we can use the domain: $x \in [0, 2]$.

   Evaluate $A(x)$ at endpoints and critical numbers:

   - $A(0) = 2(0)\sqrt{4 - 0} = 0$
   - $A(\sqrt{2}) = 2(\sqrt{2})\sqrt{4 - 2} = 4$
   - $A(2) = 2(2)\sqrt{4 - 4} = 0$

   Result: Absolute maximum is 4 at $x = \sqrt{2}$.

   ii. **First Derivative Test:** Check the sign of $A'(x) = \frac{8-4x^2}{\sqrt{4-x^2}}$ around $x = \sqrt{2} \approx 1.41$:

   - Pick $x = 1$: $A'(1) = \frac{8-4(1)}{\sqrt{3}} > 0$ (Positive).
   - Pick $x = 1.5$: $A'(1.5) = \frac{8-4(2.25)}{\sqrt{4-2.25}} = \frac{-1}{\sqrt{1.75}} < 0$ (Negative).

   Result: Since $A'(x)$ changes from positive to negative, $x = \sqrt{2}$ is a maximum.

iii. **Second Derivative Test:** We find $A''(x)$ using the quotient rule:

$$A''(x) = \frac{4x\left(x^2 - 6\right)}{\left(4 - x^2\right)^{3/2}}$$

Evaluating at $x = \sqrt{2}$ (where $8 - 4x^2 = 0$):

$$A''(\sqrt{2}) = -8$$

Result: Since $A''(\sqrt{2}) < 0$, it is an absolute maximum.

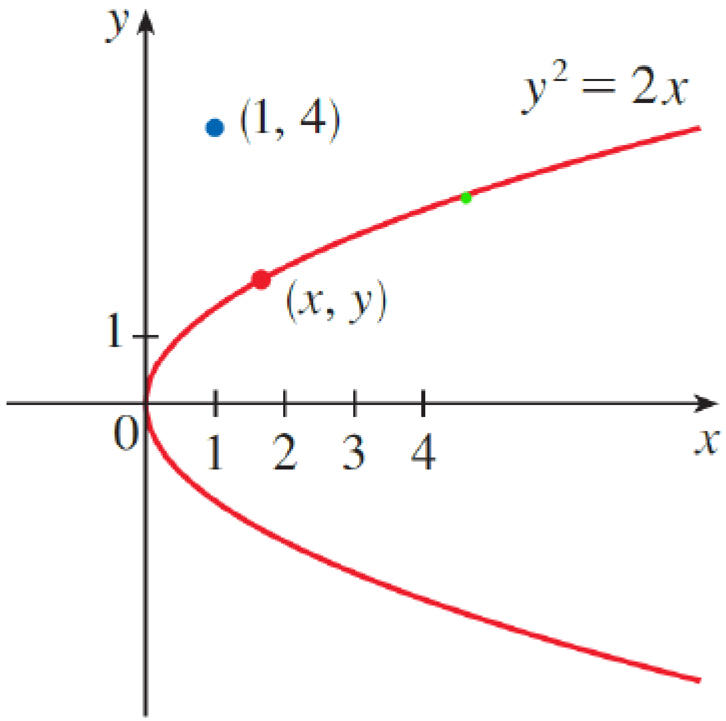
9. **Answer the question.** The question asks for the largest area and the dimensions:
   - **Height:** $y = \sqrt{4 - (\sqrt{2})^2} = \sqrt{2}$
   - **Base:** $2x = 2\sqrt{2}$
   - **Maximum Area:** $A_{max} = (2\sqrt{2})(\sqrt{2}) = 4$

Conclusion: The largest area is 4, achieved with a rectangle of base $2\sqrt{2}$ and height $\sqrt{2}$.

Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$. Find the distance.

Solution. To find the closest point, we minimize the distance between a generic point on the parabola and the fixed point $(1, 4)$.

1. **Draw and label a picture:** Consider the parabola $x = \frac{y^2}{2}$ which opens to the right. Let $P(x, y)$ be an arbitrary point on this parabola and $Q(1, 4)$ be the fixed point.



2. **Find the objective function:** The distance d between $P(x, y)$ and $Q(1, 4)$ is given by the distance formula:

$$d = \sqrt{(x-1)^2 + (y-4)^2}$$

To simplify, we minimize the square of the distance, $f = d^2$:

$$f = (x-1)^2 + (y-4)^2$$

(You should convince yourself that the minimum of d occurs at the same point as the minimum of d^2, but d^2 is easier to work with.)

3. **Identify the constraints:** The point P must lie on the parabola, so our constraint is:

$$x = \frac{y^2}{2}$$

(Alternatively, we could have substituted $y = \sqrt{2x}$ to get d in terms of x alone.)

4. **Reduce the objective function to one variable:** Substitute $x = \frac{y^2}{2}$ into the formula for f:

$$f(y) = \left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2$$

Expanding the terms:

$$f(y) = \left(\frac{y^4}{4} - y^2 + 1\right) + (y^2 - 8y + 16)$$

$$f(y) = \frac{1}{4}y^4 - 8y + 17$$

5. **Identify the domain of the objective function:** Since the parabola extends infinitely in both y directions, the domain for y is $(-\infty, \infty)$.

6. **Differentiate the objective function:**

$$f'(y) = \frac{d}{dy}\left(\frac{1}{4}y^4 - 8y + 17\right) = y^3 - 8$$

7. **Find the critical numbers:** Set $f'(y) = 0$:

$$y^3 - 8 = 0 \implies y^3 = 8 \implies y = 2$$

8. **Test the critical numbers.** We evaluate the three methods to confirm $y = 2$ is the absolute minimum:

   i. **Closed Interval Method:** This method is **not applicable** here because the domain $(-\infty, \infty)$ is not a closed, bounded interval.

   ii. **First Derivative Test:** There is only one critical number. We test points around $y = 2$:

   - For $y = 0$: $f'(0) = 0^3 - 8 = -8$ (Negative/Decreasing).
   - For $y = 3$: $f'(3) = 3^3 - 8 = 19$ (Positive/Increasing).

   Since the derivative changes from negative to positive, $y = 2$ is the absolute minimum.

   iii. **Second Derivative Test:** Find the second derivative: $f''(y) = 3y^2$. Evaluate at $y = 2$:

$$f''(2) = 3(2)^2 = 12$$

   Since $f''(2) > 0$ (positive), the critical number represents an absolute minimum.

9. **Answer the question.** Find the x-coordinate using the constraint $x = \frac{1}{2}y^2$:

$$x = \frac{1}{2}(2)^2 = 2$$

   The point is $(2, 2)$.

Conclusion: The point on the parabola closest to $(1, 4)$ is $(2, 2)$. Thus, the distance $d = \sqrt{(x-1)^2 + (y-4)^2} =$?