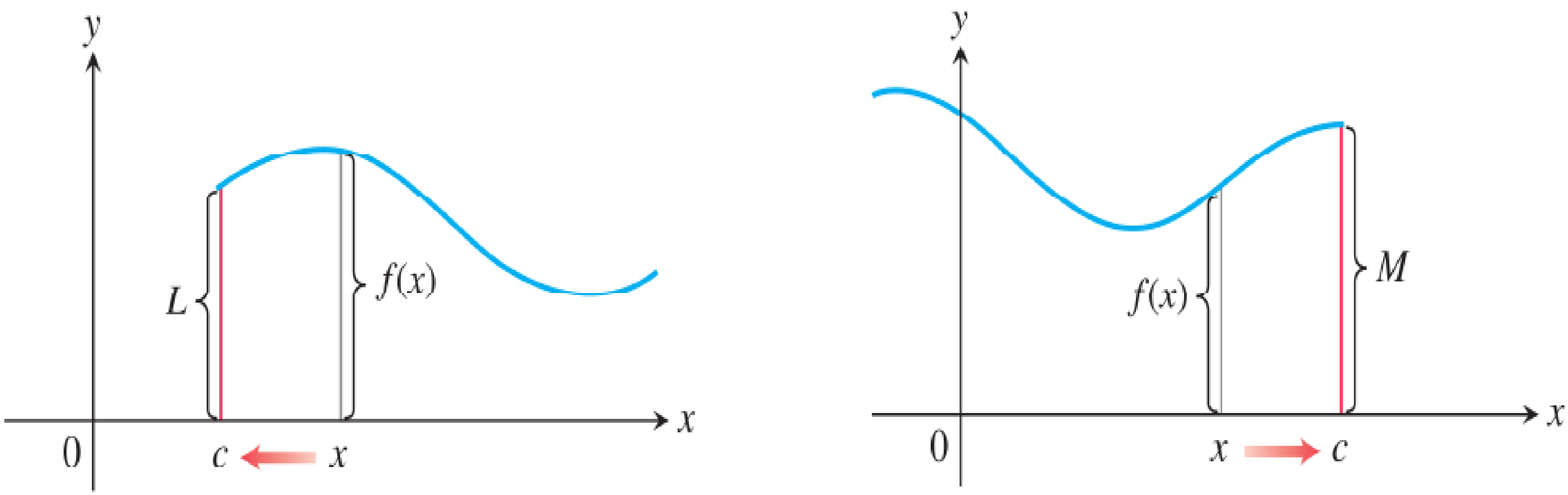
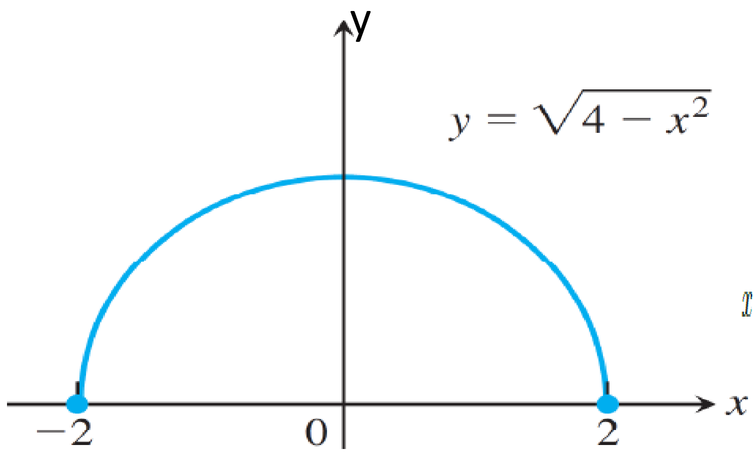
Left and Right Limit



(a) $\lim_{x\to c^+} f(x) = L$ (b) $\lim_{x\to c^-} f(x) = M$

(a) Right-hand limit as x approaches c. (b) Left-hand limit as x approaches c.



The function $f(x) = \sqrt{4 - x^2}$ has right-hand limit 0 at $x = -2$ and left-hand limit 0 at $x = 2$.

The domain of $f(x) = \sqrt{4 - x^2}$ *is* $[-2, 2]$; *its graph is the semicircle*

$$\lim_{x \to -2^+} \sqrt{4 - x^2} = 0 \quad \text{and} \quad \lim_{x \to 2^-} \sqrt{4 - x^2} = 0.$$

The function does not have a left-hand limit at $x = -2$ or a right-hand limit at $x = 2$. It does not have a two-sided limit at either −2 or 2 because each point does not belong to an open interval over which f is defined.

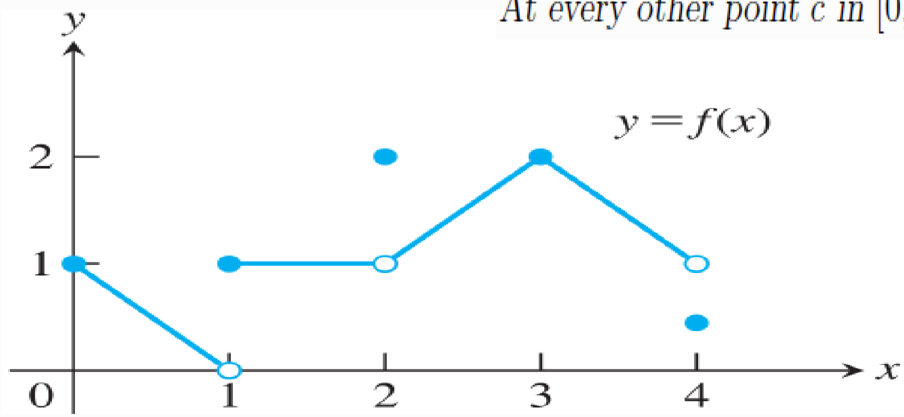
Theorem

A function $f(x)$ has a limit as x approaches c ***if and only if*** *it has left-handed and right-handed limits at c and these one-sided limits are equal:*

$$\lim_{x \to c} f(x) = L \iff \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

For the function graphed in the following figure,

At every other point c in $[0,4]$ except 1, $f(x)$ has limit $f(c)$.



- At $x = 0$:

$$\lim_{x \to 0^-} f(x) \text{ does not exist,}$$

$$\lim_{x \to 0^+} f(x) = 1, \quad \lim_{x \to 0} f(x) = 1.$$

  f is not defined to the left of $x = 0$.
  f has a right-hand limit at $x = 0$.
  f has a limit at domain endpoint $x = 0$.

- At $x = 1$:

$$\lim_{x \to 1^-} f(x) = 0, \quad \lim_{x \to 1^+} f(x) = 1, \quad \lim_{x \to 1} f(x) \text{ does not exist.}$$

  Even though $f(1) = 1$.
  Right- and left-hand limits are not equal.

- At $x = 2$:

$$\lim_{x \to 2^-} f(x) = 1, \quad \lim_{x \to 2^+} f(x) = 1, \quad \lim_{x \to 2} f(x) = 1.$$

  Even though $f(2) = 2$.

- At $x = 3$:

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = \lim_{x \to 3} f(x) = f(3) = 2.$$

- At $x = 4$:

$$\lim_{x \to 4^-} f(x) = 1, \quad \lim_{x \to 4^+} f(x) \text{ does not exist,} \quad \lim_{x \to 4} f(x) = 1.$$

  Even though $f(4) \neq 1$.
  f is not defined to the right of $x = 4$.
  f has a limit at domain endpoint $x = 4$.

If the following limit exists as $x \to 2$, find the value of a.

$$\lim_{x\to 2} \frac{3-\sqrt{a-x}}{x-2}$$

As $x \to 2$, we see that $x-2=0$, so the denominator $\frac{1}{x-2} \to \pm\infty$. Normally, the function is undefined at $x=2$, but since it is stated that the limit exists, we proceed by simplifying the numerator. First, observe the numerator as $x \to 2$:

$$\lim_{x\to 2}\left(3-\sqrt{a-x}\right)=0.$$

This implies:

$$3=\sqrt{a-x} \quad \text{at } x=2, \quad \text{so } \sqrt{a-2}=3 \implies a=11.$$

Now let us verify it by rewriting $a=11$ to the limit:

$$\lim_{x\to 2} \frac{3-\sqrt{11-x}}{x-2}.$$

Rationalizing the numerator:

$$=\lim_{x\to 2} \frac{\left(3-\sqrt{11-x}\right)\left(3+\sqrt{11-x}\right)}{(x-2)\left(3+\sqrt{11-x}\right)}.$$

Simplify:

$$=\lim_{x\to 2} \frac{9-(11-x)}{(x-2)\left(3+\sqrt{11-x}\right)}=\lim_{x\to 2} \frac{x-2}{(x-2)\left(3+\sqrt{11-x}\right)}.$$

Cancel the common factor $x - 2$:

$$= \lim_{x \to 2} \frac{1}{3 + \sqrt{11 - x}}.$$

Substitute $x = 2$:

$$= \frac{1}{3 + \sqrt{11 - 2}} = \frac{1}{3 + 3} = \frac{1}{6}.$$

Thus:

$$\lim_{x \to 2} \frac{3 - \sqrt{11 - x}}{x - 2} = \frac{1}{6}.$$

(a) *Assume the domain of f contains a* **open** *interval (c, d) to the right of c. We say that $f(x)$ has* **right-hand limit** *L at c, and write*

$$\lim_{x \to c^+} f(x) = L$$

if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad c < x < c + \delta.$$

(b) *Assume the domain of f contains an interval (b, c) to the left of c. We say that f has* **left-hand limit** *L at c, and write*

$$\lim_{x \to c^-} f(x) = L$$

if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad c - \delta < x < c.$$

Prove that

$$\lim_{x\to 0^+} \sqrt{x} = 0.$$

Solution Let $\varepsilon > 0$ be given. Here $c = 0$ and $L = 0$, so we want to find a $\delta > 0$ such that for all x,

$$0 < x < \delta \quad \Rightarrow \quad |\sqrt{x} - 0| < \varepsilon,$$

or

$$0 < x < \delta \quad \Rightarrow \quad \sqrt{x} < \varepsilon.$$

Squaring both sides of this last inequality gives

$$x < \varepsilon^2 \quad \text{if} \quad 0 < x < \delta.$$

If we choose $\delta = \varepsilon^2$, we have

$$0 < x < \delta = \varepsilon^2 \quad \Rightarrow \quad \sqrt{x} < \varepsilon,$$

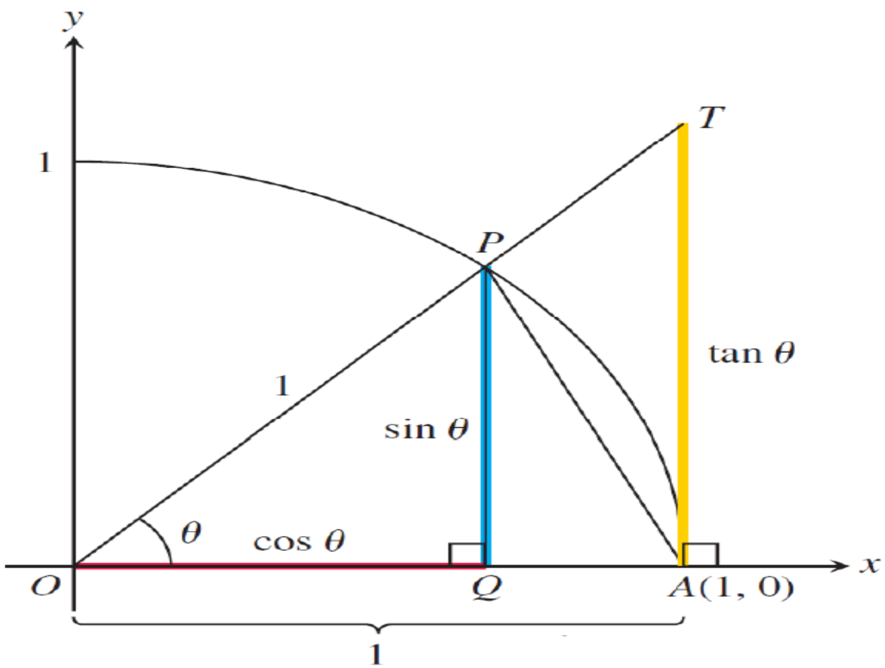
or

$$0 < x < \varepsilon^2 \quad \Rightarrow \quad |\sqrt{x} - 0| < \varepsilon.$$

According to the definition, this shows that $\lim_{x\to 0+} \sqrt{x} = 0$.

Theorem

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \quad (\theta \textit{ in radians})$$



$$\sin \theta < \theta < \tan \theta.$$

$$1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}.$$

Taking the reciprocals reverses the inequalities:

$$\cos \theta < \frac{\sin \theta}{\theta} < 1.$$

As $\theta \to 0$, $\cos \theta \to 1$. Therefore, by the Squeeze Theorem,

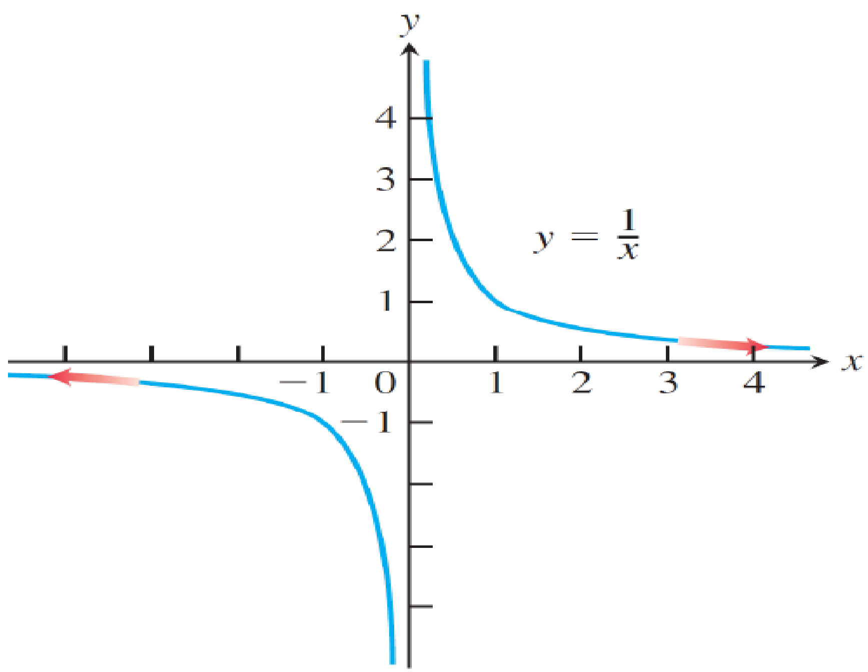
$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1.$$

Limits Involving Infinity

Finite Limits as $x \to \pm\infty$

The symbol for infinity (∞) <u>does not</u> represent **a real number.** We use ∞ to describe the behavior of a function when the values in its domain or range outgrow all finite bounds.

For example, the function $f(x) = \frac{1}{x}$ is defined for all $x \neq 0$.



The graph of $y = \frac{1}{x}$ approaches 0 as $x \to +\infty$ or $x \to -\infty$.

We say that $f(x)$ has the *limit* L *as* x *approaches infinity* and write

$$\lim_{x \to \infty} f(x) = L$$

if, for every number $\varepsilon > 0$, there exists a corresponding number M such that for all x

$$x > M \Rightarrow |f(x) - L| < \varepsilon.$$

We say that $f(x)$ has the *limit* L *as* x *approaches minus infinity* and write

$$\lim_{x \to -\infty} f(x) = L$$

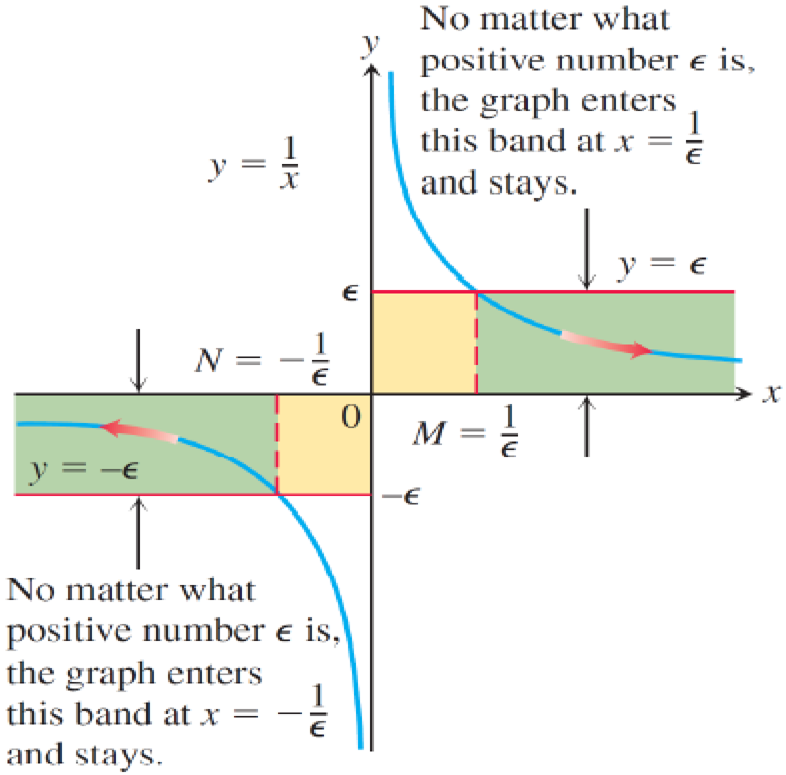
if, for every number $\varepsilon > 0$, there exists a corresponding number N such that for all x

$$x < N \Rightarrow |f(x) - L| < \varepsilon.$$

Show that

(a) $\lim_{x\to\infty} \frac{1}{x} = 0$

(b) $\lim_{x\to-\infty} \frac{1}{x} = 0$



The geometry behind the argument in Example above.

Show that

(a) $\lim_{x\to\infty} \frac{1}{x} = 0$

(b) $\lim_{x\to-\infty} \frac{1}{x} = 0$

Solution (a) Let $\varepsilon > 0$ be given. We must find a number M such that for all x

$$x > M \Rightarrow \left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \varepsilon.$$

The implication will hold if $M = 1/\varepsilon$ or any larger positive number. This proves $\lim_{x\to\infty} \frac{1}{x} = 0$.
(b) Let $\varepsilon > 0$ be given. We must find a number N such that for all x

$$x < N \Rightarrow \left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \varepsilon.$$

The implication will hold if $N = -1/\varepsilon$ or any number less than $-1/\varepsilon$. This proves $\lim_{x\to-\infty} \frac{1}{x} = 0$.
Limits at infinity have properties similar to those of finite limits.

All the Limit Laws *are* ***true*** *when we replace* $\lim_{x\to c}$ *by* $\lim_{x\to\infty}$ *or* $\lim_{x\to-\infty}$. *That is, the variable* x *may approach a finite number* c *or* $\pm\infty$.

Remark

In general, the Limit Laws can't be applied to infinite limits because ∞ *is not a number* ($\infty - \infty$ *can't be defined).*
However, we can write:

$$\lim_{x\to\infty}\left(x^2 - x\right) = \lim_{x\to\infty} x(x-1) = \infty$$

because both x *and* $x-1$ *become arbitrarily large and so their product does too.*

(a)

$$\lim_{x\to\infty}\left(5+\frac{1}{x}\right) = \lim_{x\to\infty} 5 + \lim_{x\to\infty}\frac{1}{x} = 5+0 = 5$$

(b)

$$\lim_{x\to\infty}\frac{\pi\sqrt{3}}{x^2} = \lim_{x\to\infty}\pi\sqrt{3}\cdot\lim_{x\to\infty}\frac{1}{x^2} = \pi\sqrt{3}\cdot 0 = 0$$

Limits at Infinity of Rational Functions

$$\lim_{x\to\infty} \frac{5x^2+8x-3}{3x^2+2} = \lim_{x\to\infty} \frac{5+(8/x)-(3/x^2)}{3+(2/x^2)} = \frac{5+0-0}{3+0} = \frac{5}{3}$$

$$\lim_{x\to\infty} \frac{11x+2}{2x^3-1} = \lim_{x\to\infty} \frac{(11/x^2)+(2/x^3)}{2-(1/x^3)} = \frac{0+0}{2-0} = 0$$

Find $\lim_{x\to 1^+} \frac{1}{x-1}$ *and* $\lim_{x\to 1^-} \frac{1}{x-1}$.

Analytic Solution Think about the number $x-1$ and its reciprocal. As $x \to 1^+$, we have $(x-1) \to 0^+$ and $1/(x-1) \to \infty$. As $x \to 1^-$, we have $(x-1) \to 0^-$ and $1/(x-1) \to -\infty$.

Discuss the behavior of

$$f(x) = \frac{1}{x^2} \quad \textit{as} \quad x \to 0.$$

$$\lim_{x\to 0^+} \frac{1}{x^2} = \lim_{x\to 0^-} \frac{1}{x^2} = \infty.$$

The function $y = 1/x$ shows no consistent behavior as $x \to 0$. We have $1/x \to \infty$ if $x \to 0^+$, but $1/x \to -\infty$ if $x \to 0^-$. All we can say about $\lim_{x\to 0}(1/x)$ is that it does not exist. The function $y = 1/x^2$ is different. Its values approach infinity as x approaches zero from either side, so we can say that $\lim_{x\to 0}(1/x^2) = \infty$.

(a) $\lim_{x\to 2^-} \frac{(x-2)^2}{x^2-4} = 0$

(b) $\lim_{x\to 2^+} \frac{x-2}{(x-2)(x+2)} = \frac{1}{4}$

Find $\lim_{x\to -\infty} \frac{2x^5-6x^4+1}{3x^2+x-7}$.

Solution. We are asked to find the limit of a rational function as $x \to -\infty$, so we divide the numerator and denominator by x^2, the highest power of x in the denominator:

$$\lim_{x\to-\infty} \frac{2x^5-6x^4+1}{3x^2+x-7} = \lim_{x\to-\infty} \frac{2x^3-6x^2+\frac{1}{x^2}}{3+\frac{1}{x}-\frac{7}{x^2}}.$$

As $x \to -\infty$, the numerator tends to $-\infty$ while the denominator approaches 3 as $x \to -\infty$. Therefore,

$$\lim_{x\to-\infty} \frac{2x^5-6x^4+1}{3x^2+x-7} = -\infty.$$

1. *We say that* $f(x)$ ***approaches infinity*** *as* x *approaches* c*, and write*

$$\lim_{x \to c} f(x) = \infty,$$

if for every positive real number B *there exists a corresponding* $\delta > 0$ *such that for all* x

$$0 < |x - c| < \delta \Rightarrow f(x) > B.$$

2. *We say that* $f(x)$ ***approaches minus infinity*** *as* x *approaches* c*, and write*

$$\lim_{x \to c} f(x) = -\infty,$$

if for every negative real number $-B$ *there exists a corresponding* $\delta > 0$ *such that for all* x

$$0 < |x - c| < \delta \Rightarrow f(x) < -B.$$

Prove that $\lim_{x\to 0}\frac{1}{x^2}=\infty$.

Solution. Given $B>0$, we want to find $\delta>0$ such that

$$0<|x-0|<\delta \Rightarrow \frac{1}{x^2}>B.$$

Now,

$$\frac{1}{x^2}>B \quad \text{if and only if} \quad x^2<\frac{1}{B}$$

or, equivalently,

$$|x|<\frac{1}{\sqrt{B}}.$$

Thus, choosing $\delta=1/\sqrt{B}$ (or any smaller positive number), we see that

$$|x|<\delta \Rightarrow \frac{1}{x^2}\geq B.$$

Therefore, by definition,

$$\lim_{x\to 0}\frac{1}{x^2}=\infty.$$

Find (a) $\lim_{x\to\infty} \sin\left(\frac{1}{x}\right)$ *and (b)* $\lim_{x\to\pm\infty} x\sin\left(\frac{1}{x}\right)$.

Solution. (a) We introduce the new variable $t = \frac{1}{x}$. We know that $t \to 0^+$ as $x \to \infty$ Therefore,

$$\lim_{x\to\infty} \sin\left(\frac{1}{x}\right) = \lim_{t\to 0^+} \sin t = 0.$$

(b) We calculate the limits as $x \to \infty$ and $x \to -\infty$:

$$\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = \lim_{t\to 0^+} \frac{\sin t}{t} = 1 \quad \text{and} \quad \lim_{x\to-\infty} x\sin\left(\frac{1}{x}\right) = \lim_{t\to 0^-} \frac{\sin t}{t} = 1.$$

Find $\lim_{x\to 0^-} e^{1/x}$.

Solution. We let $t = 1/x$. we can see that $t \to -\infty$ as $x \to 0^-$. Therefore,

$$\lim_{x\to 0^-} e^{1/x} = \lim_{t\to-\infty} e^t = 0$$

If $$y = 2 + \frac{\sin x}{x}.$$

Solution. We are interested in the behavior as $x \to \pm\infty$. Since

$$0 \leq \left| \frac{\sin x}{x} \right| \leq \frac{1}{|x|}$$

and $\lim_{x\to\pm\infty} \frac{1}{|x|} = 0$, we have $\lim_{x\to\pm\infty} \frac{\sin x}{x} = 0$ by the Sandwich Theorem. Hence,

$$\lim_{x\to\pm\infty} \left(2 + \frac{\sin x}{x} \right) = 2 + 0 = 2,$$

Find $\lim_{x\to\infty} \left(x - \sqrt{x^2 + 16}\right)$.

$$\lim_{x\to\infty} \left(x - \sqrt{x^2+16}\right) = \lim_{x\to\infty} \frac{\left(x - \sqrt{x^2+16}\right)\left(x + \sqrt{x^2+16}\right)}{x + \sqrt{x^2+16}} = \lim_{x\to\infty} \frac{x^2 - (x^2+16)}{x + \sqrt{x^2+16}}.$$

Simplifying, we get

$$\lim_{x\to\infty} \frac{-16}{x + \sqrt{x^2+16}} = 0.$$

The Indeterminate Form $\frac{0}{0}$ This form occurs when both the numerator and the denominator approach zero as x approaches a certain value. To resolve this type of indeterminate form, one can often factorize the numerator and the denominator, simplify the expression, or apply trigonometric identities if applicable.

If the expression does not have a common factor that can eliminate the indeterminate form, and either the numerator or the denominator contains a radical expression, we can try multiplying both the numerator and the denominator by the conjugate of the radical expression. This approach often simplifies the expression and removes the indeterminate form. Once the indeterminate form is eliminated, we can calculate the limit using the simplified expression.

Evaluate the limit:

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

Factor the numerator:

$$= \lim_{x \to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x \to 2}(x+2) = 4$$

Evaluate the following limit:

$$\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$$

Rewrite the expression by rationalizing the numerator:

$$\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} \cdot \frac{\sqrt{x} + 2}{\sqrt{x} + 2} = \lim_{x \to 4} \frac{x - 4}{x - 4} \cdot \frac{1}{\sqrt{x} + 2}$$

This simplifies to:

$$= \lim_{x \to 4} \frac{1}{\sqrt{x} + 2}$$

Now, substitute $x = 4$:

$$= \frac{1}{\sqrt{4} + 2} = \frac{1}{2 + 2} = \frac{1}{4}$$

Therefore:

$$\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = \frac{1}{4}$$

Evaluate the limit:

$$\lim_{x \to 0} \frac{\sin(2x)}{x\cos(x)}$$

Solution. To evaluate this limit, we first observe that as $x \to 0$:

$$\lim_{x \to 0} \sin(2x) = \sin(0) = 0$$

and

$$\lim_{x \to 0} x\cos(x) = 0 \cdot \cos(0) = 0$$

Thus, we have an indeterminate form of type $\frac{0}{0}$. To resolve this, we use trigonometric identities. Using the double-angle identity for sine, we have:

$$\sin(2x) = 2\sin(x)\cos(x)$$

Substituting this into the limit expression:

$$\lim_{x \to 0} \frac{\sin(2x)}{x\cos(x)} = \lim_{x \to 0} \frac{2\sin(x)\cos(x)}{x\cos(x)}$$

Now, we can simplify by canceling $\cos(x)$ from the numerator and the denominator:

$$= \lim_{x \to 0} \frac{2\sin(x)}{x}$$

We can further separate the constant factor:

$$= 2\lim_{x \to 0} \frac{\sin(x)}{x}$$

Using the standard trigonometric limit $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$, we find:

$$= 2 \cdot 1 = 2$$

Therefore, the solution is:

$$\lim_{x \to 0} \frac{\sin(2x)}{x\cos(x)} = 2$$

Evaluate the limit:

$$\lim_{x\to\infty} \frac{2x^2+3x}{x^2-x}$$

Divide both the numerator and denominator by x^2:

$$= \lim_{x\to\infty} \frac{2+\frac{3}{x}}{1-\frac{1}{x}} = \frac{2+0}{1-0} = 2$$

The Indeterminate Form $0 \times \infty$ This form arises when one part of a product approaches zero while the other approaches infinity.

To resolve the $0 \cdot \infty$ indeterminate form, we can transform the expression into either a $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form. After that, we can apply appropriate techniques to find the limit.

Transformation Steps:

To convert the $0 \cdot \infty$ form to a $\frac{0}{0}$ indeterminate form, take the term approaching ∞ and place it in the denominator as a reciprocal:

$$0 \cdot \infty \Rightarrow \frac{0}{\frac{1}{\infty}} \Rightarrow \frac{0}{0}$$

Alternatively, to convert the $0 \cdot \infty$ form to a $\frac{\infty}{\infty}$ indeterminate form, take the term approaching 0 and place it in the denominator as a reciprocal:

$$0 \cdot \infty \Rightarrow \frac{\infty}{\frac{1}{0}} \Rightarrow \frac{\infty}{\infty}$$

Evaluate the limit:

$$\lim_{x\to\infty} 3x \cdot \tan\left(\frac{1}{4x}\right)$$

Solution. To evaluate this limit, observe that as $x \to \infty$, the term $3x$ approaches $+\infty$ and $\tan\left(\frac{1}{4x}\right)$ approaches 0. Therefore, this limit is an indeterminate form of type $0 \times \infty$. To resolve this, we use a substitution to simplify the expression.
Let

$$t = \frac{1}{x} \Rightarrow x = \frac{1}{t}$$

As $x \to \infty$, $t \to 0^+$. Substituting into the limit, we get:

$$\lim_{x\to\infty} 3x \cdot \tan\left(\frac{1}{4x}\right) = \lim_{t\to 0^+} 3 \cdot \frac{1}{t} \cdot \tan\left(\frac{t}{4}\right)$$

This simplifies to:

$$= \lim_{t\to 0^+} \frac{3 \cdot \tan\left(\frac{t}{4}\right)}{t}$$

Using the trigonometric limit identity $\lim_{u\to 0} \frac{\tan(au)}{u} = a$, we can apply it here with $a = \frac{1}{4}$:

$$= 3 \cdot \frac{1}{4} = \frac{3}{4}$$

Therefore, the solution is:

$$\lim_{x\to\infty} 3x \cdot \tan\left(\frac{1}{4x}\right) = \frac{3}{4}$$

The Indeterminate Form $\infty - \infty$ The $\infty - \infty$ indeterminate form occurs when the limit involves the difference of two expressions, each with an infinite limit.

$$\lim_{x \to a}[f(x) - g(x)]$$

If

$$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty$$

then this limit represents an $\infty - \infty$ indeterminate form.

To resolve this indeterminate form, we should first convert the expression into a rational form of $\frac{f(x)}{g(x)}$. Once transformed, we can then apply techniques for resolving $\frac{0}{0}$ or $\frac{\infty}{\infty}$ indeterminate forms to find the limit.

Evaluate the limit:

$$\lim_{x\to\infty}\left(\sqrt{x^2-x}-x\right)$$

Solution. As $x \to \infty$, both $\sqrt{x^2-x}$ and x approach infinity, resulting in an indeterminate form of type $\infty - \infty$.
To resolve this indeterminate form, we rationalize the expression by multiplying the numerator and the denominator by the conjugate.

$$\lim_{x\to\infty}\left(\sqrt{x^2-x}-x\right) = \lim_{x\to\infty}\frac{\left(\sqrt{x^2-x}-x\right)\left(\sqrt{x^2-x}+x\right)}{\sqrt{x^2-x}+x}$$

This simplifies to:

$$= \lim_{x\to\infty}\frac{x^2-x-x^2}{\sqrt{x^2-x}+x}$$

$$= \lim_{x\to\infty}\frac{-x}{\sqrt{x^2-x}+x}$$

Now, factor x out of the terms in the denominator:

$$= \lim_{x\to\infty}\frac{-x}{x\sqrt{1-\frac{1}{x}}+x}$$

Simplify by canceling x from the numerator and the denominator:

$$= \lim_{x\to\infty}\frac{-1}{\sqrt{1-\frac{1}{x}}+1}$$

As $x \to \infty$, the term $\frac{1}{x}$ approaches zero:

$$= \frac{-1}{\sqrt{1-0}+1} = \frac{-1}{1+1} = \frac{-1}{2}$$

Evaluate the limit:

$$\lim_{x \to 2} \left(\frac{1}{x-2} - \frac{4}{x^2-4} \right)$$

Solution. First, note that as $x \to 2$ from both the left and the right, the limit results in an $\infty - \infty$ indeterminate form. Therefore, we should convert the expression into a single fraction of the form $\frac{f(x)}{g(x)}$ by finding a common denominator.

$$\lim_{x \to 2} \left(\frac{1}{x-2} - \frac{4}{(x-2)(x+2)} \right)$$

Rewrite with a common denominator:

$$= \lim_{x \to 2} \frac{x+2-4}{(x-2)(x+2)}$$

Simplify the numerator:

$$= \lim_{x \to 2} \frac{x-2}{(x-2)(x+2)}$$

Cancel out the $(x-2)$ terms:

$$= \lim_{x \to 2} \frac{1}{x+2}$$

With the indeterminate form removed, we can now evaluate the limit directly:

$$= \frac{1}{2+2} = \frac{1}{4}$$

Thus, the solution is:

$$\lim_{x \to 2} \left(\frac{1}{x-2} - \frac{4}{x^2-4} \right) = \frac{1}{4}$$

Exponential Indeterminates In cases of indeterminate forms, three $(0^0, 1^\infty, \infty^0)$ of the seven types appear in the limits of expressions of the form $f(x)^{g(x)}$.
We can resolve three of these exponential indeterminate forms by following the steps below:

- Define the expression being limited as a function.
- Take the natural logarithm of both sides of the function, then take the limit.
- Convert the resulting indeterminate exponential form to either $\frac{0}{0}$ or $\frac{\infty}{\infty}$.
- Using techniques up to now, find the limit of the expression (L).
- Since we took the natural logarithm initially, the limit of the original function becomes e^L.

The Indeterminate Form 0^0 This form is encountered in limits where a base approaching zero is raised to a power that also approaches zero. Rewriting the expression in exponential form can often resolve this form.

The Indeterminate Form 1^∞ This form arises when a base that approaches 1 is raised to a power that approaches infinity.

The Indeterminate Form ∞^0 This form occurs when a base approaching infinity is raised to a power that approaches zero. Rewriting the expression in exponential form can help determine the behavior of the limit.

Examples

$$\lim_{x\to 0}\frac{5-\sqrt{x+25}}{x}=?$$

$$\lim_{x\to 0}\frac{5-\sqrt{x+25}}{x}=\lim_{x\to 0}\frac{(5-\sqrt{x+25})(5+\sqrt{x+25})}{x(5+\sqrt{x+25})}=\lim_{x\to 0}\frac{25-(x+25)}{x(5+\sqrt{x+25})}=-\frac{1}{10}$$

Evaluate the following limit:

$$\lim_{x\to -1}\frac{x+1}{1-\sqrt{x+2}}$$

Notice that we have an indeterminate limit case of $\frac{0}{0}$. We will multiply the numerator and the denominator by the conjugate of the denominator. The conjugate of the denominator is

$$1+\sqrt{x+2}$$

Let us multiply the numerator and the denominator by this conjugate

$$\lim_{x\to -1}\frac{x+1}{1-\sqrt{x+2}}\cdot\frac{1+\sqrt{x+2}}{1+\sqrt{x+2}}$$

The remainder of the problem requires a number of algebraic steps, shown below.

$$\begin{aligned}&\lim_{x\to -1}\frac{(x+1)(1+\sqrt{x+2})}{1+\sqrt{x+2}-\sqrt{x+2}-(x+2)}\\ &=\lim_{x\to -1}\frac{(x+1)(1+\sqrt{x+2})}{1-x-2}=\lim_{x\to -1}\frac{(x+1)(1+\sqrt{x+2})}{-x-1}\\ &=\lim_{x\to -1}-(1+\sqrt{x+2})=-(1+\sqrt{-1+2})\end{aligned}$$

$$\implies \lim_{x\to -1}\frac{x+1}{1-\sqrt{x+2}}=-2$$

Evaluate the following limit:

$$\lim_{x\to\frac{1}{2}} \frac{2x^2+x-1}{2x-1}$$

Notice that we have an indeterminate limit case of $\frac{0}{0}$.
We compute as follows:

$$\begin{aligned}\lim_{x\to\frac{1}{2}} \frac{2x^2+x-1}{2x-1} &= \lim_{x\to\frac{1}{2}} \frac{(2x-1)(x+1)}{2x-1} \\ &= \lim_{x\to\frac{1}{2}} (x+1) \\ &3\end{aligned}$$

Compute $\lim_{x\to 0^+} e^{1/x}$, $\lim_{x\to 0^-} e^{1/x}$ *and* $\lim_{x\to 0} e^{1/x}$.

We have:

$$\lim_{x\to 0+} e^{\frac{1}{x}} = e^{\frac{1}{0+}} = e^{+\infty} = \infty.$$

$$\lim_{x\to 0-} e^{\frac{1}{x}} = e^{\frac{1}{0-}} = e^{-\infty} = 0.$$

Thus, as left-hand limit $\neq$ right-hand limit,

$$\lim_{x\to 0} e^{\frac{1}{x}} = \text{ DNE.}$$

DNE is an abbreviation for "does not exist".

Show that

$$\lim_{x\to\infty} \frac{\sin(x)}{x} = 0.$$

Solution. Notice that $\sin(x)$ oscillates between -1 and 1 for all x. Therefore, we have:

$$-1 \leq \sin(x) \leq 1.$$

Dividing each part of this inequality by x (for $x > 0$. Because $x \to +\infty$), we get:

$$-\frac{1}{x} \leq \frac{\sin(x)}{x} \leq \frac{1}{x}.$$

As $x \to \infty$, both $\frac{1}{x}$ and $-\frac{1}{x}$ approach 0. By the Squeeze Theorem, it follows that:

$$\lim_{x\to\infty} \frac{\sin(x)}{x} = 0.$$

Thus,

$$\lim_{x\to\infty} \frac{\sin(x)}{x} = 0.$$

Note that

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$

Show that

$$\lim_{x \to -\infty} \frac{\cos(x)}{x} = 0.$$

Solution. We know that $\cos(x)$ oscillates between -1 and 1 for all x. Thus, we have:

$$-1 \leq \cos(x) \leq 1.$$

Dividing each part of this inequality by x, and noting that x is negative as $x \to -\infty$, we need to reverse the inequalities:

$$\frac{1}{x} \leq \frac{\cos(x)}{x} \leq -\frac{1}{x}.$$

Now, as $x \to -\infty$, both $\frac{1}{x}$ and $-\frac{1}{x}$ approach 0. That is

$$\lim_{x \to -\infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x \to -\infty} -\frac{1}{x} = 0.$$

Therefore, by the Squeeze Theorem:

$$\lim_{x \to -\infty} \frac{\cos(x)}{x} = 0.$$

To evaluate the limit

$$\lim_{x \to 0} \frac{\sin(ax)}{bx},$$

where a and b are real constants and $b \neq 0$.

Solution. Let $u = ax$. As $x \to 0$, we also have $u \to 0$ since a is a constant. We can now rewrite the expression in terms of u as follows:

$$\lim_{x \to 0} \frac{\sin(ax)}{bx} = \lim_{u \to 0} \frac{\sin(u)}{b \cdot \frac{u}{a}}.$$

Simplifying inside the limit, we get:

$$= \lim_{u \to 0} \frac{a \cdot \sin(u)}{b \cdot u}.$$

Now, we can separate the constant $\frac{a}{b}$ from the limit using the Constant Multiple Rule, which states that $\lim_{x \to c} (k \cdot f(x)) = k \cdot \lim_{x \to c} f(x)$ for any constant k:

$$= \frac{a}{b} \cdot \lim_{u \to 0} \frac{\sin(u)}{u}.$$

We now use a well-known Trigonometric Limit, which states that $\lim_{u \to 0} \frac{\sin(u)}{u} = 1$:

$$= \frac{a}{b} \cdot 1 = \frac{a}{b}.$$

Therefore,

$$\lim_{x \to 0} \frac{\sin(ax)}{bx} = \frac{a}{b}.$$

This completes the evaluation of the limit.

Evaluate

$$\lim_{x\to 0}\frac{\sin(ax)}{\sin(bx)}$$

Solution. We know from trigonometric limits that:

$$\lim_{x\to 0}\frac{\sin(kx)}{kx}=1,\quad \text{for any constant } k.$$

The fraction can be split into parts as follows:

$$\frac{\sin(ax)}{\sin(bx)}=\frac{ax\cdot\frac{\sin(ax)}{ax}}{bx\cdot\frac{\sin(bx)}{bx}}.$$

Now, we evaluate each component as $x\to 0$:

- $\lim_{x\to 0}\frac{\sin(ax)}{ax}=1$, using the property of the sine function.
- $\lim_{x\to 0}\frac{bx}{\sin(bx)}=1$, again from the same property.

Substituting these values into the expression:

$$\lim_{x\to 0}\frac{\sin(ax)}{\sin(bx)}=\lim_{x\to 0}\frac{ax\cdot\frac{\sin(ax)}{ax}}{bx\cdot\frac{\sin(bx)}{bx}}=\lim_{x\to 0}\frac{a\cdot\frac{\sin(ax)}{ax}}{b\cdot\frac{\sin(bx)}{bx}}=\frac{a\lim_{x\to 0}\cdot\frac{\sin(ax)}{ax}}{b\lim_{x\to 0}\cdot\frac{\sin(bx)}{bx}}=\frac{a}{b}.$$

Thus, we have shown that:

$$\lim_{x\to 0}\frac{\sin(ax)}{\sin(bx)}=\frac{a}{b}.$$

Consider the limit:

$$\lim_{x\to 0} \frac{\tan(x)}{x}$$

We can rewrite this limit using the identity for $\tan(x)$ *as:*

$$\lim_{x\to 0} \frac{\sin(x)}{x \cdot \cos(x)}$$

This expression can then be separated as:

$$= \lim_{x\to 0} \frac{\sin(x)}{x} \cdot \frac{1}{\cos(x)} = \lim_{x\to 0} \frac{\sin(x)}{x} \cdot \lim_{x\to 0} \frac{1}{\cos(x)}$$

Since $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ *and* $\lim_{x\to 0} \cos(x) = 1$*, we get:*

$$= 1 \cdot 1 = 1$$

Therefore:

$$\lim_{x\to 0} \frac{\tan(x)}{x} = 1$$

Using a similar approach, consider the limit:

$$\lim_{x \to 0} \frac{\tan(kx)}{kx} \quad \textit{for some constant } k$$

Solution. Let $u = kx$. As $x \to 0$, it follows that $u \to 0$. Then we can rewrite the limit as:

$$\lim_{x \to 0} \frac{\tan(kx)}{kx} = \lim_{u \to 0} \frac{\tan(u)}{u}$$

Since we know that $\lim_{u \to 0} \frac{\tan(u)}{u} = 1$, it follows that:

$$\lim_{x \to 0} \frac{\tan(kx)}{kx} = 1$$

Evaluate the limit:

$$\lim_{u \to 0} \frac{\sin(3u)}{\tan(5u)}$$

Solution. To solve this limit, we start by rewriting the expression in a form that allows us to use standard trigonometric limits.

$$\lim_{u \to 0} \frac{\sin(3u)}{\tan(5u)} = \lim_{u \to 0} \frac{\sin(3u) \cdot \frac{3u}{3u}}{\tan(5u) \cdot \frac{5u}{5u}}$$

Here, we have multiplied the numerator by $\frac{3u}{3u}$ and the denominator by $\frac{5u}{5u}$ to create terms that can utilize the limits $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$ and $\lim_{x \to 0} \frac{\tan(x)}{x} = 1$.
Next, we rewrite the limit by grouping terms:

$$= \lim_{u \to 0} \frac{\left(\frac{\sin(3u)}{3u}\right) \cdot 3}{\left(\frac{\tan(5u)}{5u}\right) \cdot 5}$$

Now, we can apply the standard trigonometric limits:

$$= \frac{1 \cdot 3}{1 \cdot 5} = \frac{3}{5}$$

Therefore, the solution is:

$$\lim_{u \to 0} \frac{\sin(3u)}{\tan(5u)} = \frac{3}{5}$$

Evaluate the following limit:

$$\lim_{x \to 3} (x-3) \sin\left(\frac{1}{x-3}\right).$$

Solution. Let $t = x - 3$. As $x \to 3$, we have $t \to 0$. Substituting into the limit, we rewrite:

$$\lim_{x \to 3} (x-3) \sin\left(\frac{1}{x-3}\right) = \lim_{t \to 0} t \sin\left(\frac{1}{t}\right).$$

Now, observe that $\sin\left(\frac{1}{t}\right)$ oscillates between -1 and 1 as $t \to 0$. Therefore:

$$-1 \leq \sin\left(\frac{1}{t}\right) \leq 1.$$

Multiply through by t (note that $t \to 0$):

$$-t \leq t \sin\left(\frac{1}{t}\right) \leq t.$$

As $t \to 0$, both $-t \to 0$ and $t \to 0$. By the Squeeze Theorem:

$$\lim_{t \to 0} t \sin\left(\frac{1}{t}\right) = 0.$$

Thus:

$$\lim_{x \to 3} (x-3) \sin\left(\frac{1}{x-3}\right) = 0.$$

Use the Squeeze Theorem to determine the value of $\lim_{x\to 0} x^4 \sin\left(\frac{\pi}{x}\right)$.

We first need to determine lower/upper functions. We'll start off by acknowledging that provided $x \neq 0$ (which we know it won't be because we are looking at the limit as $x \to 0$) we will have,

$$-1 \leq \sin\left(\frac{\pi}{x}\right) \leq 1$$

Now, simply multiply through this by x^4 to get,

$$-x^4 \leq x^4 \sin\left(\frac{\pi}{x}\right) \leq x^4$$

Before proceeding note that we can only do this because we know that $x^4 > 0$ for $x \neq 0$. Recall that if we multiply through an inequality by a negative number we would have had to switch the signs. So, for instance, had we multiplied through by x^3 we would have had issues because this is positive if $x > 0$ and negative if $x < 0$.
Now, let's get back to the problem. We have a set of lower/upper functions and clearly,

$$\lim_{x\to 0} x^4 = \lim_{x\to 0} \left(-x^4\right) = 0$$

Therefore, by the Squeeze Theorem, we must have,

$$\lim_{x\to 0} x^4 \sin\left(\frac{\pi}{x}\right) = 0$$

1. $\lim_{x\to 0} \frac{\sin 5x}{x} = \lim_{x\to 0} 5 \cdot \frac{\sin 5x}{5x} = 5 \cdot \lim_{x\to 0} \frac{\sin 5x}{5x} = 5 \lim_{y\to 0} \frac{\sin y}{y} = 5 \quad (y = 5x)$

2. $\lim_{x\to 0} \frac{\sin 3x}{\sin 5x} = \lim_{x\to 0} \frac{\frac{\sin 3x}{3x}}{\frac{\sin 5x}{5x}} = \frac{3\cdot \lim_{x\to 0} \frac{\sin 3x}{3x}}{5\cdot \lim_{x\to 0} \frac{\sin 5x}{5x}} = \frac{3}{5}$

3. $\lim_{x\to 0} \frac{\sin x}{2x^2-x} = \lim_{x\to 0} \frac{\sin x}{x\cdot(2x-1)} = \frac{\lim_{x\to 0} \frac{\sin x}{x}}{\lim_{x\to 0}(2x-1)} = -1$

4. $\lim_{x\to 0} \frac{\tan 2x}{7x} = \lim_{x\to 0} \frac{\sin 2x}{2x} \cdot \frac{2x}{\cos 2x} \cdot \frac{1}{7x} = \frac{2}{7} \lim_{x\to 0} \frac{\sin 2x}{2x} \cdot \lim_{x\to 0} \frac{1}{\cos 2x} = \frac{2}{7}$

5.
$$\lim_{x\to 0} \frac{\sin x}{|x|} = \begin{cases} \lim_{x\to 0^-} \frac{-\sin x}{x} = -1 & x < 0, \\ \lim_{x\to 0^+} \frac{\sin x}{x} = 1 & x \geq 0. \end{cases}$$

Thus, the limit does not exist.

6. $\lim_{x\to 0} \frac{x \sin x}{1-\cos x} = \lim_{x\to 0} \frac{x \sin x(1+\cos x)}{\sin^2 x} = \frac{\lim_{x\to 0}(1+\cos x)}{\lim_{x\to 0} \frac{\sin^2 x}{x^2}} = 2$

7. $\lim_{x\to \frac{\pi}{4}} \left(x - \frac{\pi}{4}\right) \cdot \tan 2x = \lim_{t\to 0} t \cdot \tan\left(2t + \frac{\pi}{2}\right)$, *where* $x = t + \frac{\pi}{4}$. *Then,*
$\lim_{t\to 0} t \cdot \tan\left(2t + \frac{\pi}{2}\right) = \lim_{t\to 0} \frac{t\cdot \sin\left(2t+\frac{\pi}{2}\right)}{\cos\left(2t+\frac{\pi}{2}\right)} = \lim_{t\to 0} \frac{\cos 2t}{-2\left(\frac{\sin 2t}{2t}\right)} = -\frac{1}{2}$

8. $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{(1-\cos x)(1+\cos x)}{(1+\cos x)\cdot x^2} = \lim_{x\to 0} \frac{\sin^2 x}{x^2(1+\cos x)} = \frac{\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2}{\lim_{x\to 0}(1+\cos x)} = \frac{1}{2}$

Evaluate the limit

$$\lim_{x\to 0} \frac{1-\cos x}{\sin^2 x}$$

Solution. Notice that we have an indeterminate limit case of $\frac{0}{0}$, since $\cos 0 = 1$ and $\sin 0 = 0$. So, we may use the identity

$$\sin^2 x + \cos^2 x = 1$$

to obtain

$$\sin^2 x = 1 - \cos^2 x$$

and write the limit as

$$\lim_{x\to 0} \frac{1-\cos x}{\sin^2 x} = \lim_{x\to 0} \frac{1-\cos x}{1-\cos^2 x} = \lim_{x\to 0} \frac{1-\cos x}{(1-\cos x)(1+\cos x)} = \lim_{x\to 0} \frac{1}{1+\cos x} = \frac{1}{2}$$

Evaluate the limit:

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = 0.$$

Solution. Using the half-angle identity for cosine:

$$\cos h = 1 - 2\sin^2\left(\frac{h}{2}\right),$$

we can rewrite the limit as:

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = \lim_{h \to 0} \frac{-2\sin^2\left(\frac{h}{2}\right)}{h}.$$

Let $\theta = \frac{h}{2}$. Then $h = 2\theta$, and as $h \to 0$, we also have $\theta \to 0$. Substituting:

$$\lim_{h \to 0} \frac{-2\sin^2\left(\frac{h}{2}\right)}{h} = \lim_{\theta \to 0} \frac{-2\sin^2(\theta)}{2\theta}.$$

Simplify:

$$\lim_{\theta \to 0} \frac{-2\sin^2(\theta)}{2\theta} = \lim_{\theta \to 0} -\sin(\theta) \cdot \frac{\sin(\theta)}{\theta}.$$

Using the standard limit $\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$ and $\sin(\theta) \to 0$ as $\theta \to 0$:

$$\lim_{\theta \to 0} -\sin(\theta) \cdot \frac{\sin(\theta)}{\theta} = -(0)(1) = 0.$$

Thus:

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = 0.$$

Evaluate the following limit:

$$\lim_{x\to\pi} \frac{\sin(x) - \tan(x)}{\sin(x)}$$

Solution. Notice that we have an indeterminate limit case of $\frac{0}{0}$. To solve this limit, we start by rewriting the expression inside the limit to simplify it.

$$\lim_{x\to\pi} \frac{\sin(x) - \tan(x)}{\sin(x)} = \lim_{x\to\pi} \left[1 - \frac{\tan(x)}{\sin(x)}\right]$$

Next, we rewrite $\frac{\tan(x)}{\sin(x)}$ in terms of sine and cosine functions:

$$= \lim_{x\to\pi} \left[1 - \frac{\sin(x)}{\cos(x)\cdot\sin(x)}\right]$$

Simplifying this expression, we get:

$$= \lim_{x\to\pi} \left[1 - \frac{1}{\cos(x)}\right]$$

Now, we need to evaluate $\lim_{x\to\pi}\cos(x)$. As $x \to \pi$, we have $\cos(x) \to -1$.
Thus, the limit becomes:

$$= \lim_{x\to\pi} \left[1 - \frac{1}{-1}\right] = 1 - (-1) = 2$$

Evaluate the limit

$$\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta}$$

Notice that we have an indeterminate limit case of $\frac{0}{0}$.

$$\begin{aligned}
\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} &= \lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} \cdot \frac{1 + \cos\theta}{1 + \cos\theta} \\
&= \lim_{\theta \to 0} \frac{1 - \cos^2\theta}{\theta(1 + \cos\theta)} \\
&= \lim_{\theta \to 0} \frac{\sin^2\theta}{\theta(1 + \cos\theta)} \\
&= \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \frac{\sin\theta}{1 + \cos\theta} \\
&= 1 \cdot \frac{0}{2} = 0
\end{aligned}$$

$$\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = 0.$$

Evaluate the following limit:

$$\lim_{n\to\infty} n \cdot \sin\frac{2}{n}.$$

Solution. Notice that we have an indeterminate limit case of $0 \times \infty$. Let us consider:

$$\lim_{n\to\infty} n \cdot \sin\frac{2}{n}.$$

Using the substitution $x = \frac{2}{n}$, as $n \to \infty$, $x \to 0$. Thus, we can rewrite the expression in terms of x:

$$\lim_{n\to\infty} n \cdot \sin\frac{2}{n} = \lim_{x\to 0} \frac{2}{x} \cdot \sin x = \lim_{x\to 0} \frac{\sin x}{x}.2$$

Since $\lim_{x\to 0} \frac{\sin x}{x} = 1$, it follows that:

$$\lim_{n\to\infty} n \cdot \sin\frac{2}{n} = 2 \cdot 1 = 2.$$

Thus, the limit of the sequence a_n is 2, and we conclude that the sequence is convergent.

Evaluate the following limit:

$$\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$$

Notice that we have an indeterminate limit case of $\frac{0}{0}$.

We compute as follows:

$$\begin{aligned}
\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} &= \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} \cdot \frac{\sqrt{x} + 2}{\sqrt{x} + 2} \\
&= \lim_{x \to 4} \frac{x - 4}{(x - 4)(\sqrt{x} + 2)} \\
&= \lim_{x \to 4} \frac{1}{\sqrt{x} + 2} \\
&= \frac{1}{\sqrt{4} + 2} \\
&= \frac{1}{4}
\end{aligned}$$

Let $f(x) : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x+1$. Show that $\lim_{x\to 2} f(x) = 5$ using the ε-δ technique.

Solution. For every $\varepsilon \in \mathbb{R}^+$, there should exist a $\delta \in \mathbb{R}^+$ such that when $|x-2| < \delta$, we have $|f(x) - 5| < \varepsilon$.

$$|x-2| < \delta \Rightarrow 2|x-2| < 2\delta$$

$$|2x-4| < 2\delta$$

$$|2x+1-5| < 2\delta$$

$$|f(x)-5| < 2\delta$$

If we choose $\delta = \delta(\varepsilon) = \frac{\varepsilon}{2}$, then for any $\varepsilon \in \mathbb{R}^+$, we can find at least one $\delta(\varepsilon)$ such that:

$$|f(x)-5| < \varepsilon$$

Thus:

$$\lim_{x\to 2}(2x+1) = 5$$

Prove that $\lim_{x \to 0}(x^3 + 2) = 2$ *using the* ε-δ *technique.*

Solution. For every $\varepsilon \in \mathbb{R}^+$, there should exist a $\delta(\varepsilon) \in \mathbb{R}^+$ such that when $|x - 0| < \delta$, we hav
$|f(x) - 2| < \varepsilon$.

$$|x - 0| < \delta \Rightarrow |f(x) - 2| < \varepsilon$$

Let $f(x) = x^3 + 2$. We want to show that $|(x^3 + 2) - 2| < \varepsilon$.

$$|(x^3 + 2) - 2| = |x^3| < \varepsilon$$

This implies:

$$|x|^3 < \varepsilon$$

Taking the cube root of both sides, we get:

$$|x| < \sqrt[3]{\varepsilon}$$

Therefore, we can choose $\delta = \delta(\varepsilon) = \sqrt[3]{\varepsilon}$.
Thus, for every $\varepsilon > 0$, if we choose $\delta = \sqrt[3]{\varepsilon}$, then $|x - 0| < \delta$ implies $|f(x) - 2| < \varepsilon$.
Hence, the limit is proven:

$$\lim_{x \to 0}(x^3 + 2) = 2$$

Prove that $\lim_{x\to 2} x^2 = 4$ *using* $\varepsilon - \delta$ *definition.*

$$\lim_{x\to x_0} f(x) = L \Leftrightarrow \forall\varepsilon > 0, \exists\delta(\varepsilon) > 0 \ni \forall x \in X, 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

In our case:

- $f(x) = x^2$
- $a = 2$
- $L = 4$

Assume that $0 < |x-2| < \delta$. For all ε, we want to find δ and our goal is to show that if $0 < |x-2| < \delta$, then $|x^2 - 4| < \varepsilon$.
Start by expressing $|x^2 - 4|$ in terms of $|x - 2|$:

$$|x^2 - 4| = |x - 2||x + 2| < \delta|x + 2|$$

To control $|x^2 - 4| < \varepsilon$, we need to establish a bound for the term $|x + 2|$. Let's ensure that x is close to 2 by choosing $\delta \leq 1$. (we can use many other reasonable choices replacing "1".)
First, let's take the case $\delta < 1$. It implies that $|x - 2| < \delta < 1$.

$$|x - 2| < 1$$

$$\implies -1 < x - 2 < 1$$

$$\implies -1 + 2 < x < 1 + 2$$

$$\implies 1 < x < 3$$

$$\implies 1 + 2 < x + 2 < 3 + 2$$

$$\implies 3 < x + 2 < 5$$

$$\implies -5 < 3 < x + 2 < 5$$

$$\implies -5 < x + 2 < 5$$

$$\implies |x + 2| < 5$$

, we have the following expression:

$$|x^2 - 4| = |x - 2||x + 2| < \delta|x + 2| < 5\delta,$$

so we obtain $|x^2 - 4| < 5\delta$.
Now, to ensure that $|x^2 - 4| < \varepsilon$, we need $5\delta = \varepsilon$. Thus, we can choose δ to satisfy the condition $\delta = \frac{\varepsilon}{5}$.
Besides, the choice $\delta = 1$ works, still by the previous estimate.
In the beginning, we assumed that $\delta \leq 1$. Therefore, we need to satisfy both conditions $\delta \leq 1$ and $\delta = \frac{\varepsilon}{5}$. By taking the minimum of these two values, we choose

$$\delta := \min\left(1, \frac{\varepsilon}{5}\right).$$

We have shown that for every $\varepsilon > 0$, there exists a $\delta > 0$ (specifically, $\delta = \min\{1,\ \varepsilon/5\}$) such that if $0 < |x - 2| < \delta$, then $|x^2 - 4| < \varepsilon$.
Therefore, by the ε-δ definition of a limit:

$$\lim_{x \to 2} x^2 = 4$$