① $w = x^2+y^2+z^2$

$z = x^2+y^2$ ise

a) $\left(\frac{\partial w}{\partial y}\right)_z$ b) $\left(\frac{\partial w}{\partial z}\right)_x$ c) $\left(\frac{\partial w}{\partial z}\right)_y$

a) $w = w(y,z)$

$w = \underbrace{x^2}_{z-y^2}+y^2+z^2 = z+z^2$

$\left(\frac{\partial w}{\partial y}\right)_z = 0$

b) $w = w(z,x)$

$w = x^2+\underbrace{y^2}_{z-x^2}+z^2 = z+z^2$

$\left(\frac{\partial w}{\partial z}\right)_x = 1+2z$

c) $w = w(z,y)$

$w = z+z^2$

$\left(\frac{\partial w}{\partial z}\right)_y = 1+2z$

② $U = f(P,V,T)$ ve $PV = nRT$ ise $(n, R\ sbt)$

a) $\left(\frac{\partial U}{\partial P}\right)_V$ b) $\left(\frac{\partial U}{\partial T}\right)_V = ?$

a) $U = U(P,V)$

$U = f(P,V,T) = f(P,V,\underbrace{\frac{PV}{nR}}_{T})$ $\left(T = \frac{PV}{nR} \Rightarrow T_P = \frac{V}{nR}\right)$

$\left(\frac{\partial U}{\partial P}\right)_V = f_P + \underset{y_f}{f_T} \cdot \frac{V}{nR}$

b) $\left(\frac{\partial U}{\partial T}\right)_V = ?$

$U = U(T,V)$

$U = f(\underbrace{\frac{nRT}{V}}_{P}, V, T)$

$\Downarrow$

$= f_T + \frac{nR}{V} f_P$ $\left(P = \frac{NRT}{V} \Rightarrow P_T = \frac{nR}{V}\right)$

③ $w = x^2 y^2 + yz - z^3$

$x^2 + y^2 + z^2 = 6$

$(w, x, y, z) = (4, 2, 1, -1)$ noktasında

a) $\left(\frac{\partial w}{\partial y}\right)_x$ b) $\left(\frac{\partial w}{\partial y}\right)_z = ?$

a) $w = w(y, x)$

$z = z(y, x)$

$$\left(\frac{\partial w}{\partial y}\right)_x = x^2 . 2y + z + y . z_y - 3z^2 z_y$$

$z_y = ?$

$$2\underset{1}{y} + 2\underset{-1}{z}\, z_y = 0 \qquad z_y = \frac{-y}{z} = \frac{-1}{-1} = 1$$

$$\left(\frac{\partial w}{\partial y}\right)_x \Bigg|_{(\underset{x}{4}, \underset{y}{2}, \underset{z}{1}, -1)} = 4 . 2 . 1 - 1 + 1 . 1 - 3 . (-1)^2 . 1$$

$$= +8 - 1 + 1 - 3 = 5$$

b) $w = w(y, z)$

$x = x(y, z)$

$$\left(\frac{\partial w}{\partial y}\right)_z = 2x . x_y . y^2 + 2x^2 y + z$$

$x_y = ?$

$$2\underset{2}{x} . x_y + 2\underset{1}{y} = 0 \qquad x_y = \frac{-2}{4} = -\frac{1}{2}$$

$$\left(\frac{\partial w}{\partial y}\right)_z \Bigg|_{(4, 2, 1, -1)} = 2 . \cancel{2} . -\frac{1}{\cancel{2}} . 1 + 2 . 4 . 1 - 1$$

$$= 5 \text{ //}$$

(4) $w = 2x^2 + 3y^2 + z^2$ ise

$z = x^2 + 2y^2$

$\left(\frac{\partial w}{\partial y}\right)_z = ?$

$w = w(y,z) \qquad x = x(y,z)$

$\left(\frac{\partial w}{\partial y}\right)_z = 4x\,x_y + 6y$

$x_y = ?$

$z = x^2 + 2y^2$

$0 = 2x\,x_y + 4y$

$x_y = -\frac{4y}{2x} = -\frac{2y}{x}$

$\left(\frac{\partial w}{\partial y}\right)_z = 4 \cdot x \cdot \left(-\frac{2y}{x}\right) + 6y$

$= -8y + 6y = -2y$

(5) $w = 2x^2 + y - 4z + \cos t$

$x - y = t \qquad \left(\frac{\partial w}{\partial t}\right)_{x,z} = ?$

$w = w(t,x,z)$

$y = y(t,x,z)$

$\left(\frac{\partial w}{\partial t}\right)_{x,z} = y_t - \sin t$

$y_t = ?$

$x - y = t \Rightarrow -y_t = 1$

$y_t = -1$

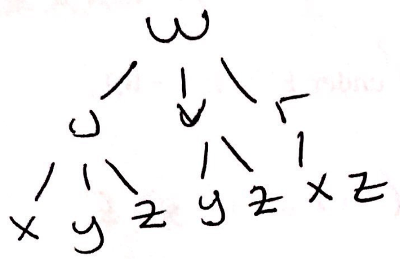
$\left(\frac{\partial w}{\partial t}\right)_{(x,z)} = -1 - \sin t$

$W=f\,(2xz-y-x^2,\ y-z^2,\ z-x)$ ise

$\dfrac{\partial w}{\partial x}+h(z)\dfrac{\partial w}{\partial y}+\dfrac{\partial w}{\partial z}=0$ eşitliğini

sağlayan h fonk nedir?

$$w=f(\underbrace{2xz-y-x^2}_{u},\ \underbrace{y-z^2}_{v},\ \underbrace{z-x}_{r})$$



$$\frac{\partial w}{\partial x}=f_u\overset{2z-2x}{u_x}+f_r\overset{-1}{r_x}$$

$u=2xz-y-x^2$

$r=z-x$

$$\frac{\partial w}{\partial y}=f_u u_y+f_v v_y$$
$$=f_u\cdot(-1)+f_v\cdot 1$$

$$\frac{\partial w}{\partial z}=f_u u_z+f_v v_z+f_r r_z$$
$$=2x f_u+f_v\cdot(-2z)+f_r\cdot 1$$

yerine yazalım denklemde:

$$f_u(2z-2x)-f_r+h(z)[-f_u+f_v]$$
$$-2z f_v+f_r=0$$
$$+2x f_u$$
$$f_u(2z-\cancel{2x}-h(z)+\cancel{2x})+(h(z)-2z)f_v$$
$$+f_r(-1+1)=0$$
$$(-f_u+f_v)(h(z)-2z)=0$$

$-f_u+f_v\neq 0 \Rightarrow h(z)=2z$

$-f_u+f_v=0 \Rightarrow h(z)=\varphi(z)$

$z = xy - \cos(z^2-1)$ denklemi ile kapalı olarak tanımlı $z=f(x,y)$ fonksiyonunun $z_{xx}|_{(2,1,1)}$ nok. değeri?

$z_x = y + 2z z_x \sin(z^2-1)$

$z_{xx} = (2z_x^2 \sin(z^2-1) + 2z z_{xx} \sin(z^2-1)$

$+ 2z z_x\, 2z z_x \cos(z^2-1)$

$z_{xx}|_{(2,1,1)} = ?$

$z_x|_{(2,1,1)} \Rightarrow z_x|_{(2,1,1)} = 1 + 2.1.z_x \sin 0 = 1$

$z_{xx} = 2.1.\cancel{\sin 0} + 2.1\, \cancel{z_{xx}.\sin 0}$

$+ 4.1^2.1^2.\cos 0 = 4$

$x^2-y^2+u^2-v^3+3=0$

$xy+y^2-u^3+2v^2-9=0$ ise $P(1,2,-1,1)$ de

a) $\left(\dfrac{\partial u}{\partial x}\right)_y \Big|_P$ b) $\left(\dfrac{\partial u}{\partial y}\right)_x \Big|_P = ?$

a) $v=v(x,y)$, $u=u(x,y)$

$x^2-y^2+u^2-v^3+3=0 \xrightarrow[\text{türev}]{x\text{e göre}} 2x+2uu_x-3v^2v_x=0$

$xy+y^2-u^3+2v^2-9=0 \xrightarrow{x\text{e göre}} y-3u^2u_x+4vv_x=0$

P nokt. de

$2u_x+3v_x=2$

$3u_x-4v_x=2$

$$u_x=\frac{\begin{vmatrix}2&3\\2&-4\end{vmatrix}}{\begin{vmatrix}2&3\\3&-4\end{vmatrix}}=\frac{-8-6}{-8-9}=\frac{14}{17}$$

b) $u=u(x,y)$

$x^2-y^2+u^2-v^3+3=0 \xrightarrow{y\text{e göre}} -2y+2uu_y-3v^2v_y=0$

$xy+y^2-u^3+2v^2-9=0 \xrightarrow{y\text{e göre}} x+2y-3u^2u_y+4vv_y=0$

P de

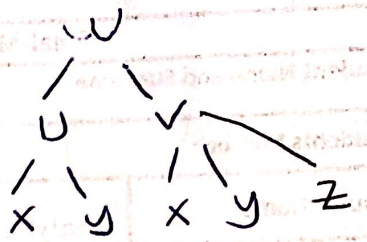
$-2u_y-3v_y=4$

$-3u_y+4v_y=-5$

$u_y=-1/17$

$w=f\left(\frac{y-x}{xy}, \frac{z-y}{yz}\right)$ ise

$x^2 w_x + y^2 w_y + z^2 w_z = 0$; bağıntısını gerçekleyin.



$$w = f\left(\underbrace{\frac{y-x}{xy}}_{u}, \underbrace{\frac{z-y}{yz}}_{v}\right)$$

$$u = \frac{1}{x} - \frac{1}{y} \quad , \quad v = \frac{1}{y} - \frac{1}{z}$$

$$w_x = w_u \cdot u_x + w_v \cdot v_x = -\frac{1}{x^2} w_u$$

$$w_y = w_u u_y + w_v v_y = +\frac{1}{y^2} w_u - \frac{1}{y^2} w_v$$

$$w_z = w_v v_z = \frac{1}{z^2} w_v$$

$$x^2 w_x + y^2 w_y + z^2 w_z$$

$$= -w_u + w_u - w_v + w_v = 0 \checkmark$$

$U = f(s) + g(r)$ fonk. s ve r ye bağlı 2. mertebeden sürekli kısmi türevlere sahip bir fonk olsun. $s = 5x+y$, $r = y-5x$ ise

$U_{xx} - 25\,U_{yy} = 0$ old. göst.

$$\begin{array}{ccc} & U & \\ s & & r \\ x \quad y & & x \quad y \end{array}$$

$$U_x = U_s s_x + U_r r_x = 5U_s - 5U_r$$

$$U_{xx} = 5(U_{ss}\cdot 5 + U_{sr}(-5)) - 5(U_{rs}\cdot 5 + U_{rr}(-5))$$

$$= 25U_{ss} - 50\underbrace{U_{sr}}_{0} + 25U_{rr}$$

$$= 25f''(s) + 25g''(r)$$

$$U_y = U_s s_y + U_r r_y = U_s + U_r$$

$$U_{yy} = U_{ss}\cdot 1 + \underbrace{\cancel{U_{sr}\cdot 1}}_{0} + \underbrace{\cancel{U_{sr}}\cdot 1}_{0} + U_{rr}\cdot 1$$

$$= f''(s) + g''(r)$$

$$U_{xx} - 25U_{yy} = 25f''(s) + 25g''(r) - 25f''(s) - 25g''(r) = 0$$

—o—

$z = x^2 e^{y/x} \Rightarrow x z_{xy} + (y-x) z_{yy} = 0$ old. göst.

$$z_x = 2x e^{y/x} + \left(\frac{-y}{x^2}\right) x^2 e^{y/x} = (2x-y)e^{y/x}$$

$$z_{xy} = -e^{y/x} + (2x-y)\cdot\frac{1}{x}\,e^{y/x}$$

$$z_y = x^2\cdot\frac{1}{x}\cdot e^{y/x} = x\cdot e^{y/x}$$

$$z_{yy} = x\cdot\frac{1}{x}\,e^{y/x} = e^{y/x}$$

$$x\cdot z_{xy} = -x e^{y/x} + 2x e^{y/x} - y e^{y/x} = (x-y)e^{y/x}$$

$$(y-x) z_{yy} = (y-x)e^{y/x}$$

$$+ \overline{\qquad\qquad\qquad}$$

$$x z_{xy} + (y-x) z_{yy} = 0$$

(12)

İki değişkenli fonksiyonlarda Taylor Formülü

Tek değişkenli fonksiyonlarda $[a, a+h]$ aralığında $(n+1)$. mertebeye kadar sürekli türevlere sahip bir $F(x)$ fonks. için Taylor Formülü:

$$F(a+h) = F(a) + F'(a)h + \frac{F''(a)}{2!}h^2 + \dots \frac{F^{(n)}(a)}{n!}h^n + \underbrace{\frac{F^{(n+1)}(x)}{(n+1)!}h^{n+1}}_{\substack{\text{kalan} \\ \text{(Lagrange kalanı)}}}$$

burada x a ile $a+h$ arasında bir sayıdır

özel olarak $a=0$ ve $h=1$ alırsak

$$F(1) = F(0) + F'(0) + \frac{F''(0)}{2!} + \dots + \frac{F^{(n)}(0)}{n!} + \frac{F^{(n+1)}(\theta)}{(n+1)!} \quad \text{dir.} \quad 0<\theta<1$$

Şimdi ise $f(x,y)$'nin kendi tanım kümesinde (a,b) ve $(a+h, b+k)$ noktalarını birleştiren, bir açık kümedeki tüm noktalarda $(n+1)$. mertebeye kadar sürekli kısmi türevlere sahip olduğunu kabul edelim.

$$f(x,y) = F(t) = f(\underbrace{a+th}_{x}, \underbrace{b+tk}_{y}) \quad 0 \le t \le 1$$

fonk. tek değişkenli Taylor formülünü bulalım formunu kullanarak

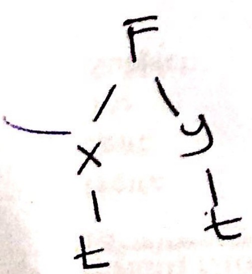
$F(0) = f(a,b) \qquad F(1) = f(a+h, b+k)$

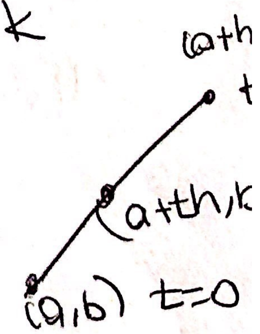
$F'(t) = hf_1 + kf_2 \quad \Rightarrow \quad F'(0) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a,b)$

$F''(t) = h^2 f_{11} + 2hk f_{12} + k^2 f_{22}$

$= (f_{11} \cdot h + f_{12} \cdot k)h + (f_{21}h + kf_{22})k$

$\Rightarrow F''(0) = \left(h \frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a,b)$





r'inci

$$F^{(r)}(0) = \left(h \cdot \frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^r f(a,b)$$

(13)

$$F(1)=F(0)+F'(0)+\frac{F''(0)}{2!}+\cdots$$

$$f(a+h,b+k)=f(a,b)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(a,b)$$

$$+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(a,b)$$

$$+\cdots+\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(a,b)$$

$$+\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1} f(a+\theta h,b+\theta k)$$

$f(x,y)$ fonksiyonunun (a,b) nok. civarındaki Taylor Formülü

$$f(x,y)=f(a,b)+\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]f(a,b)+\frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]^2 f(a,b)$$

$$=f(\underbrace{a+h}_{x},\underbrace{b+k}_{y})$$

$$\cdots+\frac{1}{(n)!}\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]^n f(a,b)$$

$$+\frac{1}{(n+1)!}\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]^{n+1}\left(f(a+\theta(x-a),\ b+\theta(y-b))\right)$$

Örn

$$F(x,y)=-x^2+2xy+3y^2-6x-2y-4$$ fonksiyonunun $(-2,1)$ nok. Taylor açılımı

$$F(x,y)=F(2,-1)+\left[\frac{\partial}{\partial x}(x+2)+\frac{\partial}{\partial y}(y-1)\right]f(-2,1)$$

$$+\left[\frac{\partial}{\partial x}(x+2)+\frac{\partial}{\partial y}(y-1)\right]^2 f(-2,1)+\cdots$$

$$F_x=-2x+2y-6\big|_{(2,-1)}=0$$

$$F_y=2x+6y-2\big|_{(2,-1)}=0$$

(14)

$F_{xx}|_{(-2,1)} = -2$

$F_{xy}|_{(-2,1)} = 2$

$F_{yy}|_{(-2,1)} = 6$

$$F(x,y) = F(2,-1) + (x+2)\overbrace{F_x(-2,1)}^{0} + (y-1)\overbrace{F_y(-2,1)}^{0}$$
$$+ \left[(x+2)^2 F_{xx}(-2,1) + 2(x+2)(y-1)F_{xy}(-2,1) + (y-1)^2 F_{yy}(-2,1)\right] \cdot \frac{1}{2!}$$

$$= 1 + \frac{1}{2}\left[-2(x+2)^2 + 4(x+2)(y-1) + (6y-1)^2\right]$$

— o —

Örn

$f(x,y) = \sqrt{x^2+y^3}$ fonksiyonunun $(1,2)$ noktasındaki 2. dereceden Taylor açılımını yapınız. Bu açılımı kullanarak $\sqrt{(1.02)^2 + (1.97)^3} = ?$

$$f(x,y) = f(1,2) + (x-1)f_x(1,2) + (y-2)f_y(1,2)$$
$$+ \frac{1}{2!}\left[(x-1)^2 f_{xx}(1,2) + 2(x-1)(y-2)f_{xy}(1,2) + (y-2)^2 f_{yy}(1,2)\right]$$

$f(1,2) = \sqrt{1+2^3} = 3$ $\quad f_x|_{(1,2)} = \frac{1}{\sqrt{9}} = \frac{1}{3}$

$f_x(x,y) = \frac{x}{\sqrt{x^2+y^3}}$

$$f_{xx}(x,y) = \frac{\sqrt{x^2+y^3} - x \cdot \frac{x}{\sqrt{x^2+y^3}}}{x^2+y^3} \Rightarrow \frac{3 - 1/3}{9} = \frac{8}{27} = f_{xx}|_{(1,2)}$$

$$f_y = \frac{3y^2}{2\sqrt{x^2+y^3}} \Rightarrow f_y|_{(1,2)} = \frac{3 \cdot 4}{2 \cdot 3} = 2$$

$$f_{yy} = \frac{3}{2}\left(\frac{2y\sqrt{x^2+y^3} - y^2 \frac{3y^2}{2\sqrt{x^2+y^3}}}{x^2+y^3}\right) \Rightarrow f_{yy}|_{(1,2)} = \frac{3}{2}\frac{(4.3-4.2}{9} = 2/3$$

$$f_{xy} = \frac{0.\sqrt{x^2+y^3} - \frac{3y^2 \cdot x}{2\sqrt{x^2+y^3}}}{x^2+y^3} \Rightarrow f_{xy}|_{(1,2)} = \frac{0 - \frac{3.4.1}{3}}{9} = -2/9$$

$$f(x,y) = 3 + \frac{1}{3}(x-1) + 2(y-2) + \frac{1}{2}\left[(x-1)^2 \cdot \frac{8}{27} + 2(x-1)(y-2) \cdot \left(-\frac{2}{9}\right)\right.$$

(15)

$$f(1.02, 1.97) = 3 + \frac{1}{3}(0.02) + 2(-0.03) + \frac{4}{27}(0.02)^2 + \left(\frac{-2}{9}\right)(0.02)\cdot(-0.03) + \frac{1}{3}\cdot(-0.03)^2$$

$$= 3 + 0.0066 - 0.06 + 0.00005 + 0.00013 + 0.0003$$

$$\approx 2.94714$$

$$\sqrt{(1.02)^2 + (1.97)^3} \approx 2.94714$$

Gerçek değer: 2.94716

Örn

(16) $x+2y+z+e^{2z}=1$ denkleminin $f(0,0)=0$ da $z=f(x,y)$ formunda bir çözüme sahip olduğunu gösterin. Ayrıca $f(x,y)$ fonksiyonunu x ve $y$nin kuvvetlerine göre 2. dereceden Taylor polinomunu açınız.

$$F(x,y,z)=x+2y+z+e^{2z}-1=0$$

$$F(P_0)=0+0+0+e^{0}-1=0$$

$$F_z(x,y,z)=1+2.e^{2z}$$

$$F_z(P_0)=1+2.e^{0}=3\neq 0$$

ve F_x, F_y de her yerde sürekli olduğundan $z=f(x,y)$ yazılabilir ve z fonk. x ve y cinsinden diferansiyellenebilir.

$$z=f(0,0)+[(x-0)f_x(0,0)+(y-0)f_y(0,0)]+\frac{1}{2}[(x-0)^2 f_{xx}(0,0)+2(x-0)(y-0)f_{xy}(0,0)+(y-0)^2 f_{yy}(0,0)]$$

$f(0,0)=0$

$$F(x,y,z)=x+2y+z+e^{2z}-1=0$$

x e göre türetelim:

$$1+0+z_x+2z_x e^{2z}=0 \Rightarrow z_x(1+2.e^{2z})=-1$$

$$z_x(0,0)=-1/3$$

y'ye göre türetelim:

$$0+2+z_y+2z_y e^{2z}=0 \quad z_y(1+2e^{2z})=-2$$

$$z_y|_{(0,0)}=-2/3$$

Tekrar x e göre:

$$z_{xx}+2z_{xx}e^{2z}+2z_x.2z_x e^{2z}=0$$

$$z_x(0,0)=4.1/9.3=4/27$$

Tekrar y e göre:

$$z_{yy}+2z_{yy}e^{2z}+2\underbrace{z_y}_{-2/3}\underbrace{2z_y}_{-2/3}e^{2z}=0$$

$$z_{yy}(0,0)=-16/27$$

(17) $z_{yx} \Rightarrow z_{yx} + 2z_{yx} e^{2z} + 4 z_y z_x e^{2z} \Rightarrow$

$z_{yx}(0,0) = -8/27$

$$f(x,y) = 0 + x \cdot \left(-1/3\right) + y\left(-2/3\right) + \frac{1}{2} x^2 \left(\frac{-4}{27}\right) + xy \cdot \left(\frac{-8}{27}\right) + \frac{y^2}{2}\left(-\frac{16}{27}\right)$$

$$f(x,y) = -\frac{x}{3} - \frac{2y}{3} - \frac{2}{27}x^2 - \frac{8}{27}xy - \frac{8}{27}y^2$$