Exponential Indeterminates In cases of indeterminate forms, three $(0^0, 1^\infty, \infty^0)$ of the seven types appear in the limits of expressions of the form $f(x)^{g(x)}$.
We can resolve three of these exponential indeterminate forms by following the steps below:

- Define the expression being limited as a function.
- Take the natural logarithm of both sides of the function, then take the limit.
- Convert the resulting indeterminate exponential form to either $\frac{0}{0}$ or $\frac{\infty}{\infty}$.
- Using techniques up to now, find the limit of the expression (L).
- Since we took the natural logarithm initially, the limit of the original function becomes e^L.

The Indeterminate Form 0^0 This form is encountered in limits where a base approaching zero is raised to a power that also approaches zero. Rewriting the expression in exponential form can often resolve this form.

The Indeterminate Form 1^∞ This form arises when a base that approaches 1 is raised to a power that approaches infinity.

The Indeterminate Form ∞^0 This form occurs when a base approaching infinity is raised to a power that approaches zero. Rewriting the expression in exponential form can help determine the behavior of the limit.

Examples

$$\lim_{x\to 0} \frac{5-\sqrt{x+25}}{x} = ?$$

$$\lim_{x\to 0} \frac{5-\sqrt{x+25}}{x} = \lim_{x\to 0} \frac{(5-\sqrt{x+25})(5+\sqrt{x+25})}{x(5+\sqrt{x+25})} = \lim_{x\to 0} \frac{25-(x+25)}{x(5+\sqrt{x+25})} = -\frac{1}{10}$$

Evaluate the following limit:

$$\lim_{x\to -1} \frac{x+1}{1-\sqrt{x+2}}$$

Notice that we have an indeterminate limit case of $\frac{0}{0}$. We will multiply the numerator and the denominator by the conjugate of the denominator. The conjugate of the denominator is

$$1+\sqrt{x+2}$$

Let us multiply the numerator and the denominator by this conjugate

$$\lim_{x\to -1} \frac{x+1}{1-\sqrt{x+2}} \cdot \frac{1+\sqrt{x+2}}{1+\sqrt{x+2}}$$

The remainder of the problem requires a number of algebraic steps, shown below.

$$\lim_{x\to -1} \frac{(x+1)(1+\sqrt{x+2})}{1+\sqrt{x+2}-\sqrt{x+2}-(x+2)}$$

$$= \lim_{x\to -1} \frac{(x+1)(1+\sqrt{x+2})}{1-x-2} = \lim_{x\to -1} \frac{(x+1)(1+\sqrt{x+2})}{-x-1}$$

$$= \lim_{x\to -1} -(1+\sqrt{x+2}) = -(1+\sqrt{-1+2})$$

$$\Longrightarrow \lim_{x\to -1} \frac{x+1}{1-\sqrt{x+2}} = -2$$

Evaluate the following limit:

$$\lim_{x\to\frac{1}{2}} \frac{2x^2+x-1}{2x-1}$$

Notice that we have an indeterminate limit case of $\frac{0}{0}$.
We compute as follows:

$$\begin{aligned}\lim_{x\to\frac{1}{2}} \frac{2x^2+x-1}{2x-1} &= \lim_{x\to\frac{1}{2}} \frac{(2x-1)(x+1)}{2x-1}\\ &= \lim_{x\to\frac{1}{2}} (x+1)\\ &3\end{aligned}$$

Compute $\lim_{x\to 0^+} e^{1/x}$, $\lim_{x\to 0^-} e^{1/x}$ *and* $\lim_{x\to 0} e^{1/x}$.

We have:

$$\lim_{x\to 0+} e^{\frac{1}{x}} = e^{\frac{1}{0+}} = e^{+\infty} = \infty.$$

$$\lim_{x\to 0-} e^{\frac{1}{x}} = e^{\frac{1}{0-}} = e^{-\infty} = 0.$$

Thus, as left-hand limit $\neq$ right-hand limit,

$$\lim_{x\to 0} e^{\frac{1}{x}} = \text{ DNE.}$$

DNE is an abbreviation for "does not exist".

Show that

$$\lim_{x \to \infty} \frac{\sin(x)}{x} = 0.$$

Solution. Notice that $\sin(x)$ oscillates between -1 and 1 for all x. Therefore, we have:

$$-1 \leq \sin(x) \leq 1.$$

Dividing each part of this inequality by x (for $x > 0$. Because $x \to +\infty$), we get:

$$-\frac{1}{x} \leq \frac{\sin(x)}{x} \leq \frac{1}{x}.$$

As $x \to \infty$, both $\frac{1}{x}$ and $-\frac{1}{x}$ approach 0. By the Squeeze Theorem, it follows that:

$$\lim_{x \to \infty} \frac{\sin(x)}{x} = 0.$$

Thus,

$$\lim_{x \to \infty} \frac{\sin(x)}{x} = 0.$$

Note that

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$

Show that

$$\lim_{x \to -\infty} \frac{\cos(x)}{x} = 0.$$

Solution. We know that $\cos(x)$ oscillates between -1 and 1 for all x. Thus, we have:

$$-1 \leq \cos(x) \leq 1.$$

Dividing each part of this inequality by x, and noting that x is negative as $x \to -\infty$, we need to reverse the inequalities:

$$\frac{1}{x} \leq \frac{\cos(x)}{x} \leq -\frac{1}{x}.$$

Now, as $x \to -\infty$, both $\frac{1}{x}$ and $-\frac{1}{x}$ approach 0. That is

$$\lim_{x \to -\infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x \to -\infty} -\frac{1}{x} = 0.$$

Therefore, by the Squeeze Theorem:

$$\lim_{x \to -\infty} \frac{\cos(x)}{x} = 0.$$

To evaluate the limit

$$\lim_{x \to 0} \frac{\sin(ax)}{bx},$$

where a and b are real constants and $b \neq 0$.

Solution. Let $u = ax$. As $x \to 0$, we also have $u \to 0$ since a is a constant. We can now rewrite the expression in terms of u as follows:

$$\lim_{x \to 0} \frac{\sin(ax)}{bx} = \lim_{u \to 0} \frac{\sin(u)}{b \cdot \frac{u}{a}}.$$

Simplifying inside the limit, we get:

$$= \lim_{u \to 0} \frac{a \cdot \sin(u)}{b \cdot u}.$$

Now, we can separate the constant $\frac{a}{b}$ from the limit using the Constant Multiple Rule, which states that $\lim_{x \to c} (k \cdot f(x)) = k \cdot \lim_{x \to c} f(x)$ for any constant k:

$$= \frac{a}{b} \cdot \lim_{u \to 0} \frac{\sin(u)}{u}.$$

We now use a well-known Trigonometric Limit, which states that $\lim_{u \to 0} \frac{\sin(u)}{u} = 1$:

$$= \frac{a}{b} \cdot 1 = \frac{a}{b}.$$

Therefore,

$$\lim_{x \to 0} \frac{\sin(ax)}{bx} = \frac{a}{b}.$$

This completes the evaluation of the limit.

Evaluate

$$\lim_{x\to 0} \frac{\sin(ax)}{\sin(bx)}$$

Solution. We know from trigonometric limits that:

$$\lim_{x\to 0} \frac{\sin(kx)}{kx} = 1, \quad \text{for any constant } k.$$

The fraction can be split into parts as follows:

$$\frac{\sin(ax)}{\sin(bx)} = \frac{ax \cdot \frac{\sin(ax)}{ax}}{bx \cdot \frac{\sin(bx)}{bx}}.$$

Now, we evaluate each component as $x \to 0$:

- $\lim_{x\to 0} \frac{\sin(ax)}{ax} = 1$, using the property of the sine function.
- $\lim_{x\to 0} \frac{bx}{\sin(bx)} = 1$, again from the same property.

Substituting these values into the expression:

$$\lim_{x\to 0} \frac{\sin(ax)}{\sin(bx)} = \lim_{x\to 0} \frac{ax \cdot \frac{\sin(ax)}{ax}}{bx \cdot \frac{\sin(bx)}{bx}} = \lim_{x\to 0} \frac{a \cdot \frac{\sin(ax)}{ax}}{b \cdot \frac{\sin(bx)}{bx}} = \frac{a \lim_{x\to 0} \cdot \frac{\sin(ax)}{ax}}{b \lim_{x\to 0} \cdot \frac{\sin(bx)}{bx}} = \frac{a}{b}.$$

Thus, we have shown that:

$$\lim_{x\to 0} \frac{\sin(ax)}{\sin(bx)} = \frac{a}{b}.$$

Consider the limit:

$$\lim_{x \to 0} \frac{\tan(x)}{x}$$

We can rewrite this limit using the identity for $\tan(x)$ *as:*

$$\lim_{x \to 0} \frac{\sin(x)}{x \cdot \cos(x)}$$

This expression can then be separated as:

$$= \lim_{x \to 0} \frac{\sin(x)}{x} \cdot \frac{1}{\cos(x)} = \lim_{x \to 0} \frac{\sin(x)}{x} \cdot \lim_{x \to 0} \frac{1}{\cos(x)}$$

Since $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$ *and* $\lim_{x \to 0} \cos(x) = 1$*, we get:*

$$= 1 \cdot 1 = 1$$

Therefore:

$$\lim_{x \to 0} \frac{\tan(x)}{x} = 1$$

Using a similar approach, consider the limit:

$$\lim_{x \to 0} \frac{\tan(kx)}{kx} \quad \textit{for some constant } k$$

Solution. Let $u = kx$. As $x \to 0$, it follows that $u \to 0$. Then we can rewrite the limit as:

$$\lim_{x \to 0} \frac{\tan(kx)}{kx} = \lim_{u \to 0} \frac{\tan(u)}{u}$$

Since we know that $\lim_{u \to 0} \frac{\tan(u)}{u} = 1$, it follows that:

$$\lim_{x \to 0} \frac{\tan(kx)}{kx} = 1$$

Evaluate the limit:

$$\lim_{u\to 0}\frac{\sin(3u)}{\tan(5u)}$$

Solution. To solve this limit, we start by rewriting the expression in a form that allows us to use standard trigonometric limits.

$$\lim_{u\to 0}\frac{\sin(3u)}{\tan(5u)}=\lim_{u\to 0}\frac{\sin(3u)\cdot\frac{3u}{3u}}{\tan(5u)\cdot\frac{5u}{5u}}$$

Here, we have multiplied the numerator by $\frac{3u}{3u}$ and the denominator by $\frac{5u}{5u}$ to create terms that can utilize the limits $\lim_{x\to 0}\frac{\sin(x)}{x}=1$ and $\lim_{x\to 0}\frac{\tan(x)}{x}=1$.
Next, we rewrite the limit by grouping terms:

$$=\lim_{u\to 0}\frac{\left(\frac{\sin(3u)}{3u}\right)\cdot 3}{\left(\frac{\tan(5u)}{5u}\right)\cdot 5}$$

Now, we can apply the standard trigonometric limits:

$$=\frac{1\cdot 3}{1\cdot 5}=\frac{3}{5}$$

Therefore, the solution is:

$$\lim_{u\to 0}\frac{\sin(3u)}{\tan(5u)}=\frac{3}{5}$$

Evaluate the following limit:

$$\lim_{x\to 3}(x-3)\sin\left(\frac{1}{x-3}\right).$$

Solution. Let $t = x - 3$. As $x \to 3$, we have $t \to 0$. Substituting into the limit, we rewrite:

$$\lim_{x\to 3}(x-3)\sin\left(\frac{1}{x-3}\right) = \lim_{t\to 0} t\sin\left(\frac{1}{t}\right).$$

Now, observe that $\sin\left(\frac{1}{t}\right)$ oscillates between -1 and 1 as $t \to 0$. Therefore:

$$-1 \le \sin\left(\frac{1}{t}\right) \le 1.$$

Multiply through by t (note that $t \to 0$):

$$-t \le t\sin\left(\frac{1}{t}\right) \le t.$$

As $t \to 0$, both $-t \to 0$ and $t \to 0$. By the Squeeze Theorem:

$$\lim_{t\to 0} t\sin\left(\frac{1}{t}\right) = 0.$$

Thus:

$$\lim_{x\to 3}(x-3)\sin\left(\frac{1}{x-3}\right) = 0.$$

Use the Squeeze Theorem to determine the value of $\lim_{x\to 0} x^4 \sin\left(\frac{\pi}{x}\right)$.

We first need to determine lower/upper functions. We'll start off by acknowledging that provided $x \neq 0$ (which we know it won't be because we are looking at the limit as $x \to 0$) we will have,

$$-1 \leq \sin\left(\frac{\pi}{x}\right) \leq 1$$

Now, simply multiply through this by x^4 to get,

$$-x^4 \leq x^4 \sin\left(\frac{\pi}{x}\right) \leq x^4$$

Before proceeding note that we can only do this because we know that $x^4 > 0$ for $x \neq 0$. Recall that if we multiply through an inequality by a negative number we would have had to switch the signs. So, for instance, had we multiplied through by x^3 we would have had issues because this is positive if $x > 0$ and negative if $x < 0$.
Now, let's get back to the problem. We have a set of lower/upper functions and clearly,

$$\lim_{x\to 0} x^4 = \lim_{x\to 0} \left(-x^4\right) = 0$$

Therefore, by the Squeeze Theorem, we must have,

$$\lim_{x\to 0} x^4 \sin\left(\frac{\pi}{x}\right) = 0$$

1. $\lim_{x\to 0} \frac{\sin 5x}{x} = \lim_{x\to 0} 5\cdot \frac{\sin 5x}{5x} = 5\cdot \lim_{x\to 0} \frac{\sin 5x}{5x} = 5\lim_{y\to 0} \frac{\sin y}{y} = 5 \quad (y=5x)$

2. $\lim_{x\to 0} \frac{\sin 3x}{\sin 5x} = \lim_{x\to 0} \frac{\frac{\sin 3x}{3x}}{\frac{\sin 5x}{5x}} = \frac{3\cdot \lim_{x\to 0} \frac{\sin 3x}{3x}}{5\cdot \lim_{x\to 0}\frac{\sin 5x}{5x}} = \frac{3}{5}$

3. $\lim_{x\to 0} \frac{\sin x}{2x^2-x} = \lim_{x\to 0} \frac{\sin x}{x\cdot(2x-1)} = \frac{\lim_{x\to 0}\frac{\sin x}{x}}{\lim_{x\to 0}(2x-1)} = -1$

4. $\lim_{x\to 0} \frac{\tan 2x}{7x} = \lim_{x\to 0} \frac{\sin 2x}{2x}\cdot\frac{2x}{\cos 2x}\cdot\frac{1}{7x} = \frac{2}{7}\lim_{x\to 0}\frac{\sin 2x}{2x}\cdot \lim_{x\to 0}\frac{1}{\cos 2x} = \frac{2}{7}$

5.
$$\lim_{x\to 0}\frac{\sin x}{|x|} = \begin{cases} \lim_{x\to 0^-} \frac{-\sin x}{x} = -1 & x<0,\\ \lim_{x\to 0^+}\frac{\sin x}{x} = 1 & x\geq 0. \end{cases}$$

Thus, the limit does not exist.

6. $\lim_{x\to 0} \frac{x\sin x}{1-\cos x} = \lim_{x\to 0} \frac{x\sin x(1+\cos x)}{\sin^2 x} = \frac{\lim_{x\to 0}(1+\cos x)}{\lim_{x\to 0}\frac{\sin^2 x}{x^2}} = 2$

7. $\lim_{x\to\frac{\pi}{4}} \left(x-\frac{\pi}{4}\right)\cdot \tan 2x = \lim_{t\to 0} t\cdot \tan\left(2t+\frac{\pi}{2}\right)$, *where* $x = t+\frac{\pi}{4}$. *Then,*
$\lim_{t\to 0} t\cdot\tan\left(2t+\frac{\pi}{2}\right) = \lim_{t\to 0}\frac{t\cdot\sin\left(2t+\frac{\pi}{2}\right)}{\cos\left(2t+\frac{\pi}{2}\right)} = \lim_{t\to 0}\frac{\cos 2t}{-2\left(\frac{\sin 2t}{2t}\right)} = -\frac{1}{2}$

8. $\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{(1+\cos x)\cdot x^2} = \lim_{x\to 0}\frac{\sin^2 x}{x^2(1+\cos x)} = \frac{\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2}{\lim_{x\to 0}(1+\cos x)} = \frac{1}{2}$

Evaluate the limit

$$\lim_{x \to 0} \frac{1 - \cos x}{\sin^2 x}$$

Solution. Notice that we have an indeterminate limit case of $\frac{0}{0}$, since $\cos 0 = 1$ and $\sin 0 = 0$. So, we may use the identity

$$\sin^2 x + \cos^2 x = 1$$

to obtain

$$\sin^2 x = 1 - \cos^2 x$$

and write the limit as

$$\lim_{x \to 0} \frac{1 - \cos x}{\sin^2 x} = \lim_{x \to 0} \frac{1 - \cos x}{1 - \cos^2 x} = \lim_{x \to 0} \frac{1 - \cos x}{(1 - \cos x)(1 + \cos x)} = \lim_{x \to 0} \frac{1}{1 + \cos x} = \frac{1}{2}$$

Evaluate the limit:

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = 0.$$

Solution. Using the half-angle identity for cosine:

$$\cos h = 1 - 2\sin^2\left(\frac{h}{2}\right),$$

we can rewrite the limit as:

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = \lim_{h\to 0} \frac{-2\sin^2\left(\frac{h}{2}\right)}{h}.$$

Let $\theta = \frac{h}{2}$. Then $h = 2\theta$, and as $h \to 0$, we also have $\theta \to 0$. Substituting:

$$\lim_{h\to 0} \frac{-2\sin^2\left(\frac{h}{2}\right)}{h} = \lim_{\theta\to 0} \frac{-2\sin^2(\theta)}{2\theta}.$$

Simplify:

$$\lim_{\theta\to 0} \frac{-2\sin^2(\theta)}{2\theta} = \lim_{\theta\to 0} -\sin(\theta) \cdot \frac{\sin(\theta)}{\theta}.$$

Using the standard limit $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1$ and $\sin(\theta) \to 0$ as $\theta \to 0$:

$$\lim_{\theta\to 0} -\sin(\theta) \cdot \frac{\sin(\theta)}{\theta} = -(0)(1) = 0.$$

Thus:

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = 0.$$

Evaluate the following limit:

$$\lim_{x \to \pi} \frac{\sin(x) - \tan(x)}{\sin(x)}$$

Solution. Notice that we have an indeterminate limit case of $\frac{0}{0}$. To solve this limit, we start by rewriting the expression inside the limit to simplify it.

$$\lim_{x \to \pi} \frac{\sin(x) - \tan(x)}{\sin(x)} = \lim_{x \to \pi} \left[1 - \frac{\tan(x)}{\sin(x)}\right]$$

Next, we rewrite $\frac{\tan(x)}{\sin(x)}$ in terms of sine and cosine functions:

$$= \lim_{x \to \pi} \left[1 - \frac{\sin(x)}{\cos(x) \cdot \sin(x)}\right]$$

Simplifying this expression, we get:

$$= \lim_{x \to \pi} \left[1 - \frac{1}{\cos(x)}\right]$$

Now, we need to evaluate $\lim_{x \to \pi} \cos(x)$. As $x \to \pi$, we have $\cos(x) \to -1$.
Thus, the limit becomes:

$$= \lim_{x \to \pi} \left[1 - \frac{1}{-1}\right] = 1 - (-1) = 2$$

Evaluate the limit

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}$$

Notice that we have an indeterminate limit case of $\frac{0}{0}$.

$$\begin{aligned}
\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} &= \lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta}\\
&= \lim_{\theta\to 0}\frac{1-\cos^2\theta}{\theta(1+\cos\theta)}\\
&= \lim_{\theta\to 0}\frac{\sin^2\theta}{\theta(1+\cos\theta)}\\
&= \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\frac{\sin\theta}{1+\cos\theta}\\
&= 1\cdot\frac{0}{2} = 0
\end{aligned}$$

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = 0.$$

Evaluate the following limit:

$$\lim_{n\to\infty} n \cdot \sin\frac{2}{n}.$$

Solution. Notice that we have an indeterminate limit case of $0 \times \infty$. Let us consider:

$$\lim_{n\to\infty} n \cdot \sin\frac{2}{n}.$$

Using the substitution $x = \frac{2}{n}$, as $n \to \infty$, $x \to 0$. Thus, we can rewrite the expression in terms of x:

$$\lim_{n\to\infty} n \cdot \sin\frac{2}{n} = \lim_{x\to 0} \frac{2}{x} \cdot \sin x = \lim_{x\to 0} \frac{\sin x}{x}.2$$

Since $\lim_{x\to 0} \frac{\sin x}{x} = 1$, it follows that:

$$\lim_{n\to\infty} n \cdot \sin\frac{2}{n} = 2 \cdot 1 = 2.$$

Evaluate the following limit:

$$\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$$

Notice that we have an indeterminate limit case of $\frac{0}{0}$.

We compute as follows:

$$\begin{aligned}
\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} &= \lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} \\
&= \lim_{x\to 4} \frac{x-4}{(x-4)(\sqrt{x}+2)} \\
&= \lim_{x\to 4} \frac{1}{\sqrt{x}+2} \\
&= \frac{1}{\sqrt{4}+2} \\
&= \frac{1}{4}
\end{aligned}$$

Let $f(x) : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x+1$. Show that $\lim_{x\to 2} f(x) = 5$ using the ε-δ technique.

Solution. For every $\varepsilon \in \mathbb{R}^+$, there should exist a $\delta \in \mathbb{R}^+$ such that when $|x-2| < \delta$, we have $|f(x) - 5| < \varepsilon$.

$$|x-2| < \delta \Rightarrow 2|x-2| < 2\delta$$

$$|2x-4| < 2\delta$$

$$|2x+1-5| < 2\delta$$

$$|f(x)-5| < 2\delta$$

If we choose $\delta = \delta(\varepsilon) = \frac{\varepsilon}{2}$, then for any $\varepsilon \in \mathbb{R}^+$, we can find at least one $\delta(\varepsilon)$ such that:

$$|f(x)-5| < \varepsilon$$

Thus:

$$\lim_{x\to 2}(2x+1) = 5$$

Prove that $\lim_{x\to 0}(x^3+2)=2$ *using the* ε*-*δ *technique.*

Solution. For every $\varepsilon \in \mathbb{R}^+$, there should exist a $\delta(\varepsilon) \in \mathbb{R}^+$ such that when $|x-0| < \delta$, we hav
$|f(x)-2| < \varepsilon$.

$$|x-0| < \delta \Rightarrow |f(x)-2| < \varepsilon$$

Let $f(x) = x^3+2$. We want to show that $|(x^3+2)-2| < \varepsilon$.

$$|(x^3+2)-2| = |x^3| < \varepsilon$$

This implies:

$$|x|^3 < \varepsilon$$

Taking the cube root of both sides, we get:

$$|x| < \sqrt[3]{\varepsilon}$$

Therefore, we can choose $\delta = \delta(\varepsilon) = \sqrt[3]{\varepsilon}$.
Thus, for every $\varepsilon > 0$, if we choose $\delta = \sqrt[3]{\varepsilon}$, then $|x-0| < \delta$ implies $|f(x)-2| < \varepsilon$.
Hence, the limit is proven:

$$\lim_{x\to 0}(x^3+2) = 2$$

Prove that $\lim_{x\to 2} x^2 = 4$ *using* $\varepsilon - \delta$ *definition.*

$$\lim_{x\to x_0} f(x) = L \Leftrightarrow \forall\varepsilon > 0, \exists\delta(\varepsilon) > 0 \ni \forall x \in X, 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

In our case:

- $f(x) = x^2$
- $a = 2$
- $L = 4$

Assume that $0 < |x-2| < \delta$. For all ε, we want to find δ and our goal is to show that if $0 < |x-2| < \delta$, then $|x^2 - 4| < \varepsilon$.
Start by expressing $|x^2 - 4|$ in terms of $|x - 2|$:

$$|x^2 - 4| = |x - 2||x + 2| < \delta|x + 2|$$

To control $|x^2 - 4| < \varepsilon$, we need to establish a bound for the term $|x + 2|$. Let's ensure that x is close to 2 by choosing $\delta \leq 1$. (we can use many other reasonable choices replacing "1".)
First, let's take the case $\delta < 1$. It implies that $|x - 2| < \delta < 1$.

$$|x - 2| < 1$$

$$\implies -1 < x - 2 < 1$$

$$\implies -1 + 2 < x < 1 + 2$$

$$\implies 1 < x < 3$$

$$\implies 1 + 2 < x + 2 < 3 + 2$$

$$\implies 3 < x + 2 < 5$$

$$\implies -5 < 3 < x + 2 < 5$$

$$\implies -5 < x + 2 < 5$$

$$\implies |x + 2| < 5$$

we have the following expression:

$$|x^2 - 4| = |x - 2||x + 2| < \delta|x + 2| < 5\delta,$$

so we obtain $|x^2 - 4| < 5\delta$.
Now, to ensure that $|x^2 - 4| < \varepsilon$, we need $5\delta = \varepsilon$. Thus, we can choose δ to satisfy the condition $\delta = \frac{\varepsilon}{5}$.
Besides, the choice $\delta = 1$ works, still by the previous estimate.
In the beginning, we assumed that $\delta \leq 1$. Therefore, we need to satisfy both conditions $\delta \leq 1$ and $\delta = \frac{\varepsilon}{5}$. By taking the minimum of these two values, we choose

$$\delta := \min\left(1, \frac{\varepsilon}{5}\right).$$

We have shown that for every $\varepsilon > 0$, there exists a $\delta > 0$ (specifically, $\delta = \min\{1,\ \varepsilon/5\}$) such that if $0 < |x - 2| < \delta$, then $|x^2 - 4| < \varepsilon$.
Therefore, by the ε-δ definition of a limit:

$$\lim_{x \to 2} x^2 = 4$$

Prove that $\lim_{x\to a} x^2 = a^2$ *using* $\varepsilon - \delta$ *definition.*

Let $a \in \mathbb{R}$. If $x \in \mathbb{R}$, then

$$|x^2 - a^2| = |x-a||x+a|;$$

if in addition $|x-a| < 1$ (we can use many other reasonable choices replacing "1", say $|a|+1$; but these other choices may result in unnecessarily messy bounds), then

$$|x| - |a| \le |x-a| < 1$$

$$\implies |x| < 1 + |a|$$

$$\implies |x+a| \le |x| + |a| < 1 + |a| + |a| = 2|a| + 1$$

$$\implies |x-a||x+a| < |x-a|(2|a|+1)$$

For any $\varepsilon > 0$, if we have

$$|x-a|(2|a|+1) < \varepsilon$$

it implies

$$|x-a| < \varepsilon/(2|a|+1)$$

All in all, for every $a \in \mathbb{R}$ and every $\varepsilon > 0$ it holds that $|x-a| < \min\{1, \varepsilon/(2|a|+1)\}$ implies $|x^2-a^2| < \varepsilon$; this completes the proof.

Prove that $\lim_{x\to 1}(2x^2+x-1)=2$ *using the* ε-δ *technique.*

Solution. To prove that:

$$\lim_{x\to 1}(2x^2+x-1)=2$$

using the ε-δ technique, we need to show that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |x-1| < \delta$, it follows that $|f(x)-2| < \varepsilon$, where $f(x)=2x^2+x-1$.

- **Step 1: Express $|f(x)-2|$ in terms of $|x-1|$**
  - Compute the difference:

    $$|f(x)-2| = |(2x^2+x-1)-2| = |2x^2+x-3|$$

  - Factor the quadratic expression:

    $$2x^2+x-3=(2x+3)(x-1)$$

  - Therefore:

    $$|f(x)-2| = |(2x+3)(x-1)| = |2x+3|\cdot|x-1|$$

- **Step 2: Bound $|2x+3|$ when x is near 1**
  - To find an upper bound for $|2x+3|$, restrict x to be within a certain distance from 1. Let $\delta_0 = 1$, so that $0 < |x-1| < \delta_0$ implies x is in the interval $(0,2)$.

    $$|x-1|<1 \implies -1<x-1<1 \implies 0<x<2$$

  - Now, within this interval, we can find the maximum value of $|2x+3|$:

- For all x in $(0, 2)$:
$$3 < 2x + 3 < 7$$
- Thus,
$$-7 < 3 < 2x + 3 < 7 \implies -7 < 2x + 3 < 7 \implies |2x + 3| < 7$$

- **Step 3: Establish the relationship between $|f(x) - 2|$ and $|x - 1|$**
  - Using the bound on $|2x + 3|$:
$$|f(x) - 2| = |2x + 3| \cdot |x - 1| < 7|x - 1|$$

- **Step 4: Determine δ in terms of ε**
  - To ensure $|f(x) - 2| < \varepsilon$, we need:
$$7|x - 1| < \varepsilon$$
  - Solving for $|x - 1|$:
$$|x - 1| < \frac{\varepsilon}{7}$$
  - Recall that we initially set $|x - 1| < \delta_0 = 1$ to bound $|2x + 3|$. Therefore, our final δ must satisfy both conditions:
    * $|x - 1| < 1$
    * $|x - 1| < \frac{\varepsilon}{7}$
  - Thus, we choose:
$$\delta = \min\left\{1, \frac{\varepsilon}{7}\right\}$$

- **Step 5: Conclude the proof**
  - With this choice of δ, whenever $0 < |x - 1| < \delta$, the following holds:

- **Step 5: Conclude the proof**
  - With this choice of δ, whenever $0 < |x - 1| < \delta$, the following holds:

$$|f(x) - 2| \leq 7|x - 1| < 7 \cdot \frac{\varepsilon}{7} = \varepsilon$$

  - This satisfies the ε-δ definition of the limit.

For every $\varepsilon > 0$, by choosing $\delta = \min\left\{1, \frac{\varepsilon}{7}\right\}$, we ensure that $0 < |x - 1| < \delta$ implies $|f(x) - 2| < \varepsilon$. Therefore, we have proven that:

$$\lim_{x \to 1}(2x^2 + x - 1) = 2$$

Show that $\lim_{x\to c} \sin x = \sin c$.

Solution. For each given $\varepsilon > 0$, we need to find a $\delta(\varepsilon) > 0$ such that $|\sin x - \sin c| < \varepsilon$, ensuring $|x - c| < \delta(\varepsilon)$.
Using the trigonometric identity:

$$\sin(a+b) - \sin(a-b) = 2 \sin b \cos a,$$

we let $x = a + b$ and $c = a - b$. Then, we can define:

$$a = \frac{x+c}{2} \quad \text{and} \quad b = \frac{x-c}{2}.$$

This implies that

$$\sin x - \sin c = 2 \sin \frac{x-c}{2} \cos \frac{x+c}{2}.$$

Thus,

$$|\sin x - \sin c| = \left| 2 \sin \frac{x-c}{2} \cos \frac{x+c}{2} \right|.$$

We can further simplify this by observing that $|\cos \frac{x+c}{2}| \leq 1$, which gives us:

$$|\sin x - \sin c| \leq 2 \left| \sin \frac{x-c}{2} \right|.$$

To proceed, we use the fact that for small values, $|\sin u| \leq |u|$
Thus,

$$\left| \sin \frac{x-c}{2} \right| \leq \left| \frac{x-c}{2} \right|,$$

and we get:

$$|\sin x - \sin c| \leq 2 \left| \frac{x-c}{2} \right| = |x-c|.$$

Therefore, if we choose $\delta(\epsilon) = \epsilon$, then for $|x-c| < \epsilon$, we have $|\sin x - \sin c| < \epsilon$ as required. Thus, we have shown that $\lim_{x \to c} \sin x = \sin c$.

To see why the identity $\sin(a+b)-\sin(a-b)=2\sin b\cos a$ *holds, we can expand each term separately using the sum and difference formulas for sine:*

$$\sin(a+b)=\sin a\cos b+\cos a\sin b$$

and

$$\sin(a-b)=\sin a\cos b-\cos a\sin b.$$

Now, taking the difference $\sin(a+b)-\sin(a-b)$, *we get:*

$$\sin(a+b)-\sin(a-b)=(\sin a\cos b+\cos a\sin b)-(\sin a\cos b-\cos a\sin b).$$

Simplifying, the terms $\sin a\cos b$ *cancel out, leaving:*

$$\implies \sin(a+b)-\sin(a-b)=2\cos a\sin b,$$

which verifies the identity.

Show that

$$\lim_{x \to 3} \left(x^4 + 7x - 17 \right) = 43$$

using the formal definition of the limit.

Evaluate the limit

$$\lim_{x \to \infty} (\sqrt{x+1} - \sqrt{x+2})$$

Evaluate the limit and justify each step by indicating the appropriate properties of limits.

$$\lim_{x \to -\infty} \frac{(1-x)(2+x)}{(1+2x)(2-3x)}$$

Evaluate the following limit:

$$\lim_{x \to \infty} \frac{x+1}{\sqrt{x^2+1}}$$

For $f(x)=\frac{x^6-x^4+x^2-1}{7x^6+4x^3+10}$, answer each of the following questions:

(a) Evaluate $\lim_{x\to\infty} f(x)$.

(b) Evaluate $\lim_{x\to-\infty} f(x)$.

For $f(x)=\frac{\sqrt{7+9x^2}}{1-2x}$, answer each of the following questions:

(a) Evaluate $\lim_{x\to-\infty} f(x)$.

(b) Evaluate $\lim_{x\to\infty} f(x)$.

$\lim_{x\to 2^-}(x-\lfloor x\rfloor) = ?$

$$f : [-1, 5] \to \mathbb{R}$$

$$f(x) = \begin{cases} x & \text{if } -1 < x \leq 2, \\ 4 - x & \text{if } 2 < x \leq 5. \end{cases}$$

Evaluate $\lim_{x\to 2} f(x)$.

$\lim_{x\to\infty}(\sqrt{x^2 + x + 5} - \sqrt{x^2 + 7}) = ?$

$\lim_{x\to-\infty} \frac{\sqrt{x^2+|x|}}{|x|} = ?$

Solution. Notice that we have an indeterminate limit case of $\frac{\infty}{\infty}$. For $x < 0$, $|x| = -x$

$$\begin{aligned} \lim_{x\to-\infty} \frac{\sqrt{x^2 + |x|}}{|x|} &= \lim_{x\to-\infty} \frac{\sqrt{x^2 - x}}{-x} \\ &= \lim_{x\to-\infty} \frac{|x|\sqrt{1 - \frac{1}{x}}}{-x} \\ &= +1 \end{aligned}$$

Show that

$a \in \mathbb{R}^+ \quad \lim_{x\to\infty} \sqrt{ax^2+bx+c} = \sqrt{a}\lim_{x\to\infty}\left(x+\frac{b}{2a}\right)$ *is true.*

Solution. $ax^2+bx+c = a\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a}$

$\sqrt{ax^2+bx+c} = \sqrt{a\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a}}$

$\lim_{x\to\infty} ax^2 + k = \lim_{x\to\infty} ax^2$

$\sqrt{\lim ax^2 + k} = \sqrt{\lim ax^2}$

So, $\lim_{x\to\infty} \sqrt{ax^2+bx+c} = \lim_{x\to\infty} \sqrt{a\left(x+\frac{b}{2a}\right)^2}$

$= \sqrt{a}\lim_{x\to\infty} \sqrt{\left(x+\frac{b}{2a}\right)^2}$

$= \sqrt{a}\lim_{x\to\infty} \left|x+\frac{b}{2a}\right|$

$\left|x+\frac{b}{2a}\right| = x+\frac{b}{2a}$

So,

As $x\to\infty$,

$\left|x+\frac{b}{2a}\right| = x+\frac{b}{2a}$

Thus,

$\lim_{x\to\infty} \sqrt{ax^2+bx+c} = \sqrt{a}\lim_{x\to\infty}\left(x+\frac{b}{2a}\right)$

The number e is a mathematical constant approximately equal to 2.71828 that is the base of the natural logarithm and exponential function. It is sometimes called Euler's number.

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

Show that

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = e.$$

Let $x=-y$ so that $x\to-\infty \Rightarrow y\to+\infty$.
We have

$$\left(1+\frac{1}{x}\right)^x = \left(1-\frac{1}{y}\right)^{-y} = \left(\frac{y}{y-1}\right)^y$$
$$= \left(1+\frac{1}{y-1}\right)^y = \left(1+\frac{1}{y-1}\right)^{(y-1)}\left(1+\frac{1}{y-1}\right).$$

Thus,

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = \lim_{y\to+\infty}\left(1+\frac{1}{y-1}\right)^{y-1}\left(1+\frac{1}{y-1}\right).$$
$$= \lim_{y\to+\infty}\left(1+\frac{1}{y-1}\right)^{y-1}\cdot\lim_{y\to+\infty}\left(1+\frac{1}{y-1}\right).$$
$$= e\cdot 1 = e.$$

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = e.$$

Evaluate the limit

$$\lim_{x\to 1^+} x^{\frac{1}{x-1}}.$$

Solution. Notice that we have an indeterminate limit case of 1^∞.
To evaluate this limit, let's rewrite the expression by substituting $x = 1 + (x-1)$:

$$\lim_{x\to 1^+} x^{\frac{1}{x-1}} = \lim_{x\to 1^+} (1 + (x-1))^{\frac{1}{x-1}}.$$

This simplifies to:

$$\lim_{x\to 1^+} (1 + (x-1))^{\frac{1}{x-1}} = \lim_{x\to 1^+} \left(1 + \frac{1}{\frac{1}{x-1}}\right)^{\frac{1}{x-1}}.$$

To make this expression easier to evaluate, let's introduce a substitution. Let

$$u = \frac{1}{x-1}.$$

As $x \to 1^+$, $u \to \infty$. Then the limit becomes:

$$\lim_{x\to 1^+} \left(1 + \frac{1}{\frac{1}{x-1}}\right)^{\frac{1}{x-1}} = \lim_{u\to\infty} \left(1 + \frac{1}{u}\right)^u.$$

According to the exponential definition, we know:

$$\lim_{u\to\infty} \left(1 + \frac{1}{u}\right)^u = e.$$

Thus,

$$\lim_{x\to 1^+} x^{\frac{1}{x-1}} = e.$$

Evaluate the limit

$$\lim_{x\to 2^+} (x-1)^{\frac{1}{x-2}},$$

Solution. Notice that we have an indeterminate limit case of 1^∞.
To evaluate this limit, let's consider the definition of the exponential constant e.

$$\lim_{x\to 2^+} (x-1)^{\frac{1}{x-2}} = \lim_{x\to 2^+} (1+x-2)^{\frac{1}{x-2}} = \lim_{x\to 2^+} \left[1+\frac{1}{\frac{1}{x-2}}\right]^{\frac{1}{x-2}}$$

Here, we can rewrite the expression using substitution for easier evaluation. Let

$$u = \frac{1}{x-2}$$

As $x \to 2^+$, $u \to \infty$.
Then the limit becomes:

$$\lim_{u\to\infty} \left(1+\frac{1}{u}\right)^u$$

According to the exponential definition, we know:

$$\lim_{u\to\infty} \left(1+\frac{1}{u}\right)^u = e$$

Thus,

$$\lim_{x\to 2^+} (x-1)^{\frac{1}{x-2}} = e$$

Evaluate the limit

$$\lim_{x\to 2^-}(x-1)^{\frac{1}{x-2}}.$$

Solution. Notice that we have an indeterminate limit case of $1^{-\infty}$.
To evaluate this limit, let's consider the definition of the exponential constant e.

$$\lim_{x\to 2^-}(x-1)^{\frac{1}{x-2}} = \lim_{x\to 2^-}(1+(x-2))^{\frac{1}{x-2}} = \lim_{x\to 2^-}\left[1+\frac{1}{\frac{1}{x-2}}\right]^{\frac{1}{x-2}}.$$

We can rewrite the expression using substitution for easier evaluation. Let

$$u = \frac{1}{x-2}.$$

As $x \to 2^-$, $u \to -\infty$.
Then the limit becomes:

$$\lim_{u\to -\infty}\left(1+\frac{1}{u}\right)^u.$$

According to the exponential definition, we know:

$$\lim_{u\to -\infty}\left(1+\frac{1}{u}\right)^u = e.$$

Thus,

$$\lim_{x\to 2^-}(x-1)^{\frac{1}{x-2}} = e.$$

Since $\lim_{x\to 2^-}(x-1)^{\frac{1}{x-2}} = \lim_{x\to 2^-}(x-1)^{\frac{1}{x-2}} = e$, *note that*

$$\lim_{x\to 2}(x-1)^{\frac{1}{x-2}} = e.$$

Evaluate the limit:

$$\lim_{x\to 1^+}(2-x)^{1/(x-1)}.$$

Solution. Let us make the substitution $u = 1 - x$. Then, as $x \to 1^+$, $u \to 0^-$. Rewriting the function in terms of u:

$$(2-x)^{1/(x-1)} = [1+(1-x)]^{-1/(x-1)} = [(1+u)]^{-1/u}.$$

Thus, the limit becomes:

$$\lim_{x\to 1^+}(2-x)^{1/(x-1)} = \lim_{u\to 0^-}[(1+u)^{1/u}]^{-1}.$$

We know from the exponential limit property that:

$$\lim_{u\to 0}(1+u)^{1/u} = e.$$

Therefore:

$$\lim_{u\to 0^-}[(1+u)^{1/u}]^{-1} = e^{-1}.$$

Evaluate the limit:

$$\lim_{x \to 1^-} \frac{\ln(2-x)}{1-x}.$$

Solution. By reorganizing the expression inside the limit, we proceed as follows:

$$\lim_{x \to 1^-} \frac{\ln(2-x)}{1-x} = \lim_{x \to 1^-} \ln\left([1+(1-x)]^{1/(1-x)}\right) = \ln\left(\lim_{x \to 1^-} [1+(1-x)]^{1/(1-x)}\right).$$

From the exponential limit:

$$\lim_{x \to 1^-} [1+(1-x)]^{1/(1-x)} = e,$$

we find:

$$\lim_{x \to 1^-} \frac{\ln(2-x)}{1-x} = \ln(e) = 1.$$

Evaluate the limit:

$$\lim_{x\to 1^-} [1+\ln(2-x)]^{1/(1-x)}.$$

Solution. To evaluate the given limit:

$$\lim_{x\to 1^-} [1+\ln(2-x)]^{1/(1-x)},$$

we begin by rewriting the expression.
Using the property of exponents:

$$a^b = \exp\left(b\ln(a)\right),$$

we write:

$$[1+\ln(2-x)]^{1/(1-x)} = \exp\left(\frac{\ln\left(1+\ln(2-x)\right)}{1-x}\right).$$

To simplify the exponent, consider:

$$\frac{\ln\left(1+\ln(2-x)\right)}{1-x}.$$

This term appears complex, so we analyze it further.
Inside the curly brackets:

$$[1+\ln(2-x)]^{1/\ln(2-x)},$$

we focus on evaluating the limit:

$$\lim_{x\to 1^-} [1+\ln(2-x)]^{1/\ln(2-x)} = \lim_{x\to 1^-}\left[1+\frac{1}{\frac{1}{\ln(2-x)}}\right]^{1/\ln(2-x)}.$$

we focus on evaluating the limit:

$$\lim_{x\to 1^-} [1+\ln(2-x)]^{1/\ln(2-x)} = \lim_{x\to 1^-} \left[1+\frac{1}{\frac{1}{\ln(2-x)}}\right]^{1/\ln(2-x)}.$$

Let $u = \frac{1}{\ln(2-x)}$. As $x \to 1^-$, $2-x \to 0^+$, and thus $u \to -\infty$. Substituting, the term becomes:

$$\lim_{u\to -\infty} \left[1+\frac{1}{u}\right]^{u}.$$

For large negative u, the expression $1+u$ approaches u, and the general exponential property applies:

$$\lim_{u\to -\infty} \left(1+\frac{1}{u}\right)^{u} = e.$$

Thus, the limit simplifies to:

$$\lim_{x\to 1^-} [1+\ln(2-x)]^{1/\ln(2-x)} = e.$$

Returning to the original expression:

$$\lim_{x\to 1^-} [1+\ln(2-x)]^{1/(1-x)} = \exp\left(\ln(e)\right) = e.$$

Evaluate the limit:

$$\lim_{n\to\infty}\left(1-\frac{4}{n^2}\right)^n$$

Solution. Notice that we have an indeterminate limit case of 1^∞.
Consider the limit:

$$\lim_{n\to\infty}\left(1-\frac{4}{n^2}\right)^n.$$

We can rewrite this expression as:

$$\lim_{n\to\infty}\left(\left(1-\frac{2}{n}\right)\cdot\left(1+\frac{2}{n}\right)\right)^n.$$

This separates into two limits:

$$=\lim_{n\to\infty}\left(1-\frac{2}{n}\right)^n\cdot\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n.$$

Using the known limit $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=e^x$, we find:

$$=e^{-2}\cdot e^2=1.$$

Thus,

$$\lim_{n\to\infty}\left(1-\frac{4}{n^2}\right)^n=1.$$

$$\lim_{x\to\infty}\left(\frac{x+4}{x-1}\right)^{x+4} =?$$

We know that

$$\lim_{x\to\infty}\left(1 \pm \frac{k}{x}\right)^{x} = e^{\pm k}$$

We use a change of variables to make the limit resemble the definition of e:
We can rewrite $\frac{x+4}{x-1}$ as follows:

$$\frac{x+4}{x-1} = 1 + \frac{5}{x-1}$$

So, we get

$$\lim_{x\to\infty}\left(\frac{x+4}{x-1}\right)^{x+4} = \lim_{x\to\infty}\left(1+\frac{5}{x-1}\right)^{x+4}$$

We also know that

$$\lim_{x\to\infty}\frac{x-1}{x-1} = 1$$

So, we can multiply the power $x+4$ by $\frac{x-1}{x-1} = 1$.

$$\lim_{x\to\infty}\left(\frac{x+4}{x-1}\right)^{x+4} = \lim_{x\to\infty}\left(1+\frac{5}{x-1}\right)^{\frac{x-1}{x-1}(x+4)}$$

$$= \lim_{x\to\infty}\left(\left(1+\frac{5}{x-1}\right)^{x-1}\right)^{\frac{x+4}{x-1}} = \left(\lim_{x\to\infty}\left(1+\frac{5}{x-1}\right)^{x-1}\right)^{\frac{x+4}{x-1}}$$

Defining a new variable u such that:

$$x-1=u$$

So,

$$x=u+1$$

Then, as $x \Rightarrow \infty$, $u \Rightarrow \infty$. Thus, substituting into the original limit we obtain:

$$= \left(\lim_{u\to\infty}\left(1+\frac{5}{u}\right)^{u}\right)^{\frac{u+1+4}{u}} = \left(\lim_{u\to\infty}\left(1+\frac{5}{u}\right)^{u}\right)^{\frac{u+5}{u}} = \left(e^{5}\right)^{\lim_{u\to\infty}\frac{u+5}{u}} = \left(e^{5}\right)^{1} = e^{5}$$

(Here $\lim_{u\to\infty}\frac{u+5}{u} = 1$)
Consequently;

$$\lim_{x\to\infty}\left(\frac{x+4}{x-1}\right)^{x+4} = e^{5}$$

Evaluate the limit:

$$\lim_{x\to 2}\frac{e^{x-2}-1}{x-2}$$

Solution. Notice that we have an indeterminate limit case of $\frac{0}{0}$.
To resolve this, we make the substitution $x-2=u$, so $x=u+2$. As $x\to 2$, we have $u\to 0$.

$$\lim_{u\to 0}\frac{e^u-1}{u}=\frac{0}{0}$$

This indeterminate form can be further analyzed by defining $e^u-1=t$. Thus, as $u\to 0$, $t\to 0$ as well, and we rewrite $e^u=t+1$.
Taking the natural logarithm on both sides:

$$\ln(e^u)=\ln(t+1)\Rightarrow u=\ln(t+1)$$

Then, we have:

$$\lim_{t\to 0}\frac{t}{\ln(t+1)}$$

This can be rewritten as:

$$=\lim_{t\to 0}\frac{1}{\frac{\ln(t+1)}{t}}=\lim_{t\to 0}\frac{1}{\left(\frac{1}{t}\right)\ln(t+1)}=\lim_{t\to 0}\frac{1}{\ln(t+1)^{\frac{1}{t}}}$$

$$=\frac{1}{\ln\left(\lim_{t\to 0}(t+1)^{\frac{1}{t}}\right)}$$

$$=\frac{1}{\ln e}=\frac{1}{1}$$

Here we use a well-known limit property:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$$

we conclude:

$$\lim_{t\to 0}(1+t)^{\frac{1}{t}}=e$$

Evaluate the following limit:

$$\lim_{x\to 0} \tan\left(\frac{\sin 4x}{\pi x}\right).$$

Solution. We start by rewriting the limit:

$$\lim_{x\to 0} \tan\left(\frac{\sin 4x}{\pi x}\right) = \tan\left(\lim_{x\to 0} \frac{\sin 4x}{\pi x}\right).$$

Using the standard limit property $\lim_{x\to 0} \frac{\sin kx}{kx} = 1$, we simplify:

$$\lim_{x\to 0} \frac{\sin 4x}{\pi x} = \frac{4}{\pi}.$$

Thus:

$$\lim_{x\to 0} \tan\left(\frac{\sin 4x}{\pi x}\right) = \tan\left(\frac{4}{\pi}\right).$$

Evaluate the following limit:

$$\lim_{x\to 1} \frac{x}{\sqrt{2x+1}-\sqrt{3}}.$$

Solution. We start by rewriting the limit:

$$\lim_{x\to 1} \frac{x}{\sqrt{2x+1}-\sqrt{3}}.$$

Step 1: Analyze the numerator and denominator. As $x \to 1$:

$$\text{Numerator: } \lim_{x\to 1} x = 1.$$

$$\text{Denominator: } \lim_{x\to 1} \sqrt{2x+1}-\sqrt{3} = \sqrt{3}-\sqrt{3} = 0.$$

Step 2: Check left-hand and right-hand limits.
For $x \to 1^+$, the denominator approaches:

$$\sqrt{2x+1}-\sqrt{3} \to 0^+ \quad \text{(positive side of 0)}.$$

Thus:

$$\lim_{x\to 1^+} \frac{x}{\sqrt{2x+1}-\sqrt{3}} \to \frac{1}{0^+} = +\infty.$$

For $x \to 1^-$, the denominator approaches:

$$\sqrt{2x+1}-\sqrt{3} \to 0^- \quad \text{(negative side of 0)}.$$

Thus:

$$\lim_{x\to 1^-} \frac{x}{\sqrt{2x+1}-\sqrt{3}} \to \frac{1}{0^-} = -\infty.$$

Step 3: Combine the results. The left-hand and right-hand limits do not match:

$$\lim_{x\to 1^+} f(x) = +\infty, \quad \lim_{x\to 1^-} f(x) = -\infty.$$

Therefore, the limit does not exist (D.N.E.).

Evaluate the following limit:

$$\lim_{x\to\frac{\pi}{4}} \arctan\left(\frac{\sin(x-\frac{\pi}{4})}{x-\frac{\pi}{4}}\right).$$

Solution. We start by applying the limit to the argument of the arctan function:

$$\lim_{x\to\frac{\pi}{4}} \arctan\left(\frac{\sin(x-\frac{\pi}{4})}{x-\frac{\pi}{4}}\right) = \arctan\left(\lim_{x\to\frac{\pi}{4}} \frac{\sin(x-\frac{\pi}{4})}{x-\frac{\pi}{4}}\right).$$

Let $t = x - \frac{\pi}{4}$. As $x \to \frac{\pi}{4}$, $t \to 0$. Substituting into the limit:

$$\lim_{x\to\frac{\pi}{4}} \frac{\sin(x-\frac{\pi}{4})}{x-\frac{\pi}{4}} = \lim_{t\to 0} \frac{\sin t}{t}.$$

Using the standard limit property:

$$\lim_{t\to 0} \frac{\sin t}{t} = 1.$$

Thus:

$$\lim_{x\to\frac{\pi}{4}} \arctan\left(\frac{\sin(x-\frac{\pi}{4})}{x-\frac{\pi}{4}}\right) = \arctan(1).$$

Finally, we know that:

$$\arctan(1) = \frac{\pi}{4}.$$

Therefore:

$$\lim_{x\to\frac{\pi}{4}} \arctan\left(\frac{\sin(x-\frac{\pi}{4})}{x-\frac{\pi}{4}}\right) = \frac{\pi}{4}$$

Evaluate the limit:

$$\lim_{x\to 0} \cos\left(\frac{\pi - \pi\cos^2 x}{x^2}\right).$$

Solution. Rewrite the expression inside the cosine:

$$\frac{\pi - \pi\cos^2 x}{x^2} = \pi \cdot \frac{1-\cos^2 x}{x^2}.$$

Using the identity $1-\cos^2 x = \sin^2 x$, this becomes:

$$\pi \cdot \frac{\sin^2 x}{x^2}.$$

Now substitute back into the limit:

$$\lim_{x\to 0} \cos\left(\pi \cdot \frac{\sin^2 x}{x^2}\right).$$

Simplify $\frac{\sin^2 x}{x^2} \to 1$ as $x \to 0$, so:

$$\cos(\pi \cdot 1) = \cos\pi = -1.$$

Thus:

$$\lim_{x\to 0} \cos\left(\frac{\pi - \pi\cos^2 x}{x^2}\right) = -1.$$

Evaluate the limit:

$$\lim_{x\to 0}\frac{\tan(2x)}{x}.$$

Solution. Rewrite using the definition of tangent:

$$\frac{\tan(2x)}{x}=\frac{\sin(2x)}{x\cos(2x)}.$$

Split the limit:

$$\lim_{x\to 0}\frac{\sin(2x)}{2x}\cdot\lim_{x\to 0}\frac{2}{\cos(2x)}.$$

Using the standard limit $\lim_{x\to 0}\frac{\sin(2x)}{2x}=1$ and $\cos(2x)\to 1$ as $x\to 0$:

$$\frac{1\cdot 2}{1}=2.$$

Thus:

$$\lim_{x\to 0}\frac{\tan(2x)}{x}=2.$$

Evaluate the limit:

$$\lim_{x \to \pi} \sec(1 + \cos x).$$

Solution. To evaluate the given limit, we analyze the behavior of the function $\sec(1+\cos x)$ as $x \to \pi$. We know that the cosine function is continuous and periodic. At $x = \pi$:

$$\cos(\pi) = -1.$$

Thus, as $x \to \pi$, we have:

$$\cos x \to -1.$$

The argument of the secant function becomes:

$$1 + \cos x \to 1 + (-1) = 0.$$

Therefore, as $x \to \pi$:

$$\sec(1 + \cos x) \to \sec(0).$$

The secant function is defined as:

$$\sec y = \frac{1}{\cos y}.$$

At $y = 0$, we know:

$$\cos(0) = 1 \quad \Longrightarrow \quad \sec(0) = \frac{1}{\cos(0)} = \frac{1}{1} = 1.$$

Combining the above steps, the value of the limit is:

$$\lim_{x \to \pi} \sec(1 + \cos x) = 1.$$

Find the limit

$$\lim_{x \to 1} \frac{x^{20} - 1}{x^{10} - 1}.$$

Solution. Direct substitution of $x = 1$ yields the indeterminate form $\frac{0}{0}$ at the point $x = 1$. Therefore, we factor the numerator to get

$$\lim_{x \to 1} \frac{x^{20} - 1}{x^{10} - 1} = \lim_{x \to 1} \frac{\left(x^{10}\right)^2 - 1}{x^{10} - 1} = \lim_{x \to 1} \frac{\left(x^{10} - 1\right)\left(x^{10} + 1\right)}{x^{10} - 1} = \lim_{x \to 1} \left(x^{10} + 1\right) = 1^{10} + 1 = 2.$$

Calculate

$$\lim_{y\to-2}\frac{y^3+3y^2+2y}{y^2-y-6}$$

Solution. This is of the form $\frac{0}{0}$ at $y=-2$. We factor the numerator and the denominator:

$$y^3+3y^2+2y=y\left(y^2+3y+2\right)=y(y+1)(y+2).$$

Here we used the formula

$$ax^2+bx+c=a\left(x-x_1\right)\left(x-x_2\right)$$

where x_1, x_2 are the solutions of the quadratic equation.
Similarly,

$$y^2-y-6=(y-3)(y+2)$$

Thus, the limit is

$$\lim_{y\to-2}\frac{y^3+3y^2+2y}{y^2-y-6}=\lim_{y\to-2}\frac{y(y+1)(y+2)}{(y-3)(y+2)}=\lim_{y\to-2}\frac{y(y+1)}{y-3}=\frac{\lim_{y\to-2}y\cdot\lim_{y\to-2}(y+1)}{\lim_{y\to-2}(y-3)}=\frac{-2(-1)}{-5}=-\frac{2}{5}$$

(by the quotient and product rules for limits).

Calculate

$$\lim_{x\to\infty} \frac{x^3+3x+5}{2x^3-6x+1}.$$

Solution. Substituting $x \to \infty$ shows that this is of the form $\frac{\infty}{\infty}$. Divide the numerator and denominator by x^3 (the highest degree in this expression). Thus, we obtain

$$\lim_{x\to\infty} \frac{x^3+3x+5}{2x^3-6x+1} = \lim_{x\to\infty} \frac{\frac{x^3+3x+5}{x^3}}{\frac{2x^3-6x+1}{x^3}} = \lim_{x\to\infty} \frac{\frac{x^3}{x^3}+\frac{3x}{x^3}+\frac{5}{x^3}}{\frac{2x^3}{x^3}-\frac{6x}{x^3}+\frac{1}{x^3}}$$

$$= \lim_{x\to\infty} \frac{1+\frac{3}{x^2}+\frac{5}{x^3}}{2-\frac{6}{x^2}+\frac{1}{x^3}} = \frac{\lim_{x\to\infty}\left(1+\frac{3}{x^2}+\frac{5}{x^3}\right)}{\lim_{x\to\infty}\left(2-\frac{6}{x^2}+\frac{1}{x^3}\right)}$$

$$= \frac{\lim_{x\to\infty} 1+\lim_{x\to\infty}\frac{3}{x^2}+\lim_{x\to\infty}\frac{5}{x^3}}{\lim_{x\to\infty} 2-\lim_{x\to\infty}\frac{6}{x^2}+\lim_{x\to\infty}\frac{1}{x^3}} = \frac{1+0+0}{2-0-0} = \frac{1}{2}.$$

Calculate the limit

$$\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{x-1}$$

Solution. We write the denominator in the form

$$x-1=\left(\sqrt[3]{x}\right)^3-1^3$$

and factor it as difference of cubes:

$$x-1=\left(\sqrt[3]{x}\right)^3-1^3=\left(\sqrt[3]{x}-1\right)\left(\sqrt[3]{x^2}+\sqrt[3]{x}+1\right).$$

As a result we have $\left[\frac{0}{0}\right]$ indeterminate case.

$$\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{x-1}=\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{\left(\sqrt[3]{x}-1\right)\left(\sqrt[3]{x^2}+\sqrt[3]{x}+1\right)}=\lim_{x\to 1}\frac{1}{\sqrt[3]{x^2}+\sqrt[3]{x}+1}=\frac{1}{\sqrt[3]{1^2}+\sqrt[3]{1}+1}=\frac{1}{3}$$

Calculate

$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-\sqrt{x^2-1}\right)$$

Solution. If $x \to \infty$, then

$$\lim_{x\to\infty}\sqrt{x^2+1}=\infty \text{ and } \lim_{x\to\infty}\sqrt{x^2-1}=\infty$$

Thus, we deal here with an indeterminate form of type $\infty - \infty$. Multiply this expression (both the numerator and the denominator) by the corresponding conjugate expression.

$$L=\lim_{x\to\infty}\left(\sqrt{x^2+1}-\sqrt{x^2-1}\right)=\lim_{x\to\infty}\frac{\left(\sqrt{x^2+1}\right)^2-\left(\sqrt{x^2-1}\right)^2}{\left(\sqrt{x^2+1}+\sqrt{x^2-1}\right)}=$$

$$\lim_{x\to\infty}\frac{x^2+1-(x^2-1)}{\left(\sqrt{x^2+1}+\sqrt{x^2-1}\right)}=\lim_{x\to\infty}=\lim_{x\to\infty}\frac{\cancel{x^2}+1-\cancel{x^2}+1}{\left(\sqrt{x^2+1}+\sqrt{x^2-1}\right)}=\lim_{x\to\infty}\frac{2}{\left(\sqrt{x^2+1}+\sqrt{x^2-1}\right)}.$$

By using the product and the sum rules for limits, we obtain

$$L=\lim_{x\to\infty}\frac{2}{\left(\sqrt{x^2+1}+\sqrt{x^2-1}\right)}=\frac{\lim_{x\to\infty}2}{\lim_{x\to\infty}\sqrt{x^2+1}+\lim_{x\to\infty}\sqrt{x^2-1}}\sim\frac{2}{\infty+\infty}\sim\frac{2}{\infty}=0$$

Show that

$$\lim_{x \to -3} \left(x^4 + 7x - 17\right) = 43$$

using the formal definition of the limit.

Solution. For any given $\varepsilon > 0$, we have to find a $\delta > 0$ such that for all x,

$$0 < |x - (-3)| < \delta \implies \left|x^4 + 7x - 17 - 43\right| < \varepsilon.$$

We have

$$\left(x^4 + 7x - 17\right) - 43 = x^4 + 7x - 60 = (x+3)\left(x^3 - 3x^2 + 9x - 20\right).$$

Suppose that $0 < |x - (-3)| < \delta$ and $\delta \le 1$. Then $-4 \le x - \delta < x < -3 + \delta \le -2$. In particular, $|x| < 4$.

Therefore, using the triangle inequality, we obtain

$$\left|x^3 - 3x^2 + 9x - 20\right| \le |x|^3 + 3|x|^2 + 9|x| + 20 < 4^3 + 3 \cdot 4^2 + 9 \cdot 4 + 20 = 168.$$

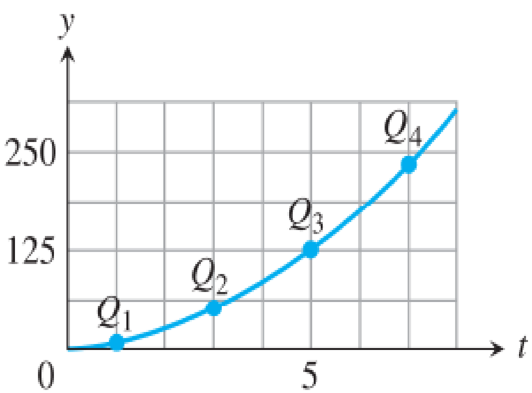
Now if we choose δ to satisfy $0 < \delta \le \min\left\{\frac{\varepsilon}{168}, 1\right\}$, then we have

$$\left|x^4 + 7x - 17 - 43\right| = \left|x^4 + 7x - 60\right| = |x+3| \cdot \left|x^3 - 3x^2 + 9x - 20\right| < \delta \cdot 168 \le \frac{\varepsilon}{168} \cdot 168 = \varepsilon.$$

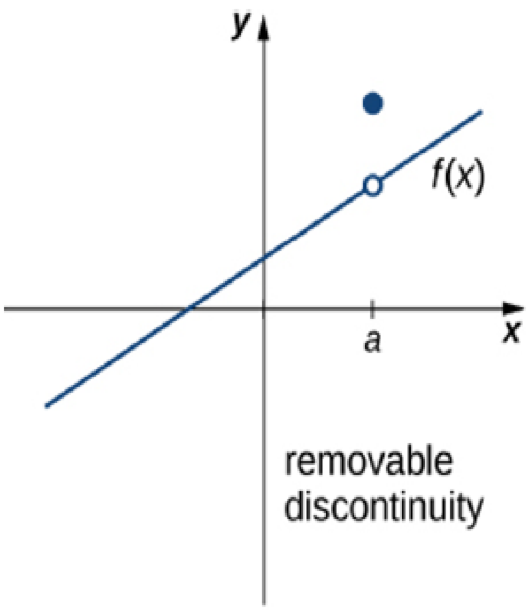
This holds whenever $0 < |x - (-3)| < \delta$. We are done.

Continuity

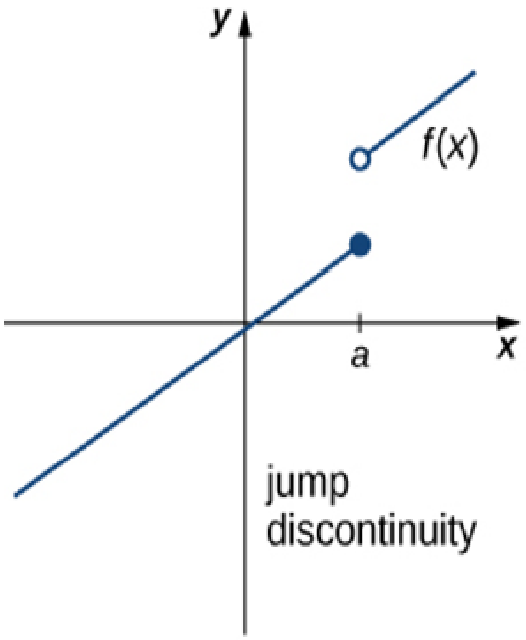
the graph of a continuous function has no breaks or jumps.



(a) Continuous



(b) Break



(c) Jump

A ***boundary point*** *(or an* ***endpoint****) is the beginning or ending point of a range or interval. It can be either inclusive or exclusive.*

An ***interior point*** *of an interval I is an element of I which is not an endpoint of I.*

Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ be a function. For $a \in A$, if

$$\lim_{x \to a} f(x) = f(a),$$

*then the function f is said to be **continuous** at $x = a$.*

When examining continuity, it is important that the point $x = a$ under consideration belongs to the domain of the function.

***

Continuity Test at a Interior Point

Notice that definition implicitly requires three conditions if f is continuous at an interior point a:

1. *The function $f(x)$ is defined at $x = a$. That is, $f(a)$ exists (must be defined).*
2. *The limit of the function exists as x approaches a, i.e., $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$.*
3. *The value of the limit equals the function's value at that point: $\lim_{x\to a} f(x) = f(a)$.*

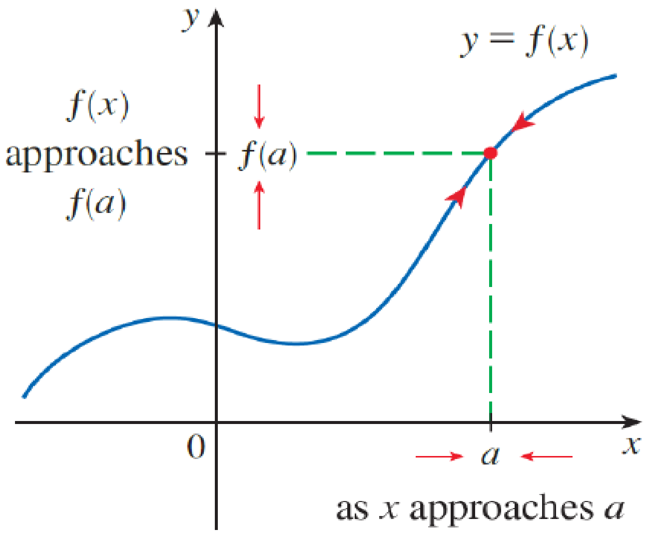
*If these three conditions are satisfied, the function is said to be **continuous** at $x = a$.*

*If any of these conditions are not satisfied, the function is said to be **discontinuous** at a (or f has a **discontinuity at** a).*

Relationships Between Limit and Continuity

- *If a function is* **continuous** *at* $x = x_0$, *then the* **limit** *of the function exists as* $x \to x_0$.
- *If the* **limit** *of a function does* <u>*not*</u> *exist as* $x \to x_0$, *then the function is* <u>*not continuous*</u> *at* $x = x_0$.

the continuity is a stronger condition than the existence of a limit.



Continuity

Geometrically, you can think of a function that is continuous at every number in an interval as a function whose graph has no break in it: the graph can be drawn without removing your pen from the paper.

Definition **Precise (Formal or $\varepsilon - \delta$) definition of Continuous Functions**

Let f be a real-valued function whose domain is a subset of $\mathbb{R}$.
Then f is **continuous** *at x_0 in $\text{dom}(f)$* ***if and only if***

for each $\varepsilon > 0$ there exists $\delta > 0$ such that $x \in \text{dom}(f)$ and $|x - x_0| < \delta$ imply $|f(x) - f(x_0)| < \varepsilon$.

Shortly,

f is **continuous** at x_0 $\Leftrightarrow \forall \varepsilon > 0, \quad \exists \delta(\varepsilon) > 0 \; (\delta(\varepsilon) \in \mathbb{R}^+) \; \ni \forall x \in X, \; |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$

If f fails to be continuous at x_0, then we say that f is **discontinuous at** x_0.

Key Differences Between Continuity and Limit

- *For continuity, the point x_0 must belong to the domain of f, i.e., $x_0 \in \text{dom}(f)$. This ensures $f(x_0)$ is well-defined.*
- *For limits, x_0 does not necessarily need to be in the domain of f. The behavior of $f(x)$ near x_0 is sufficient to define the limit.*
- *Continuity at x_0 requires $\lim_{x \to x_0} f(x) = f(x_0)$, meaning the limit of $f(x)$ as x approaches x_0 must equal the actual value of the function at x_0.*
- *The definition of a limit does not require $f(x_0)$ to exist, nor does it require $\lim_{x \to x_0} f(x)$ to equal $f(x_0)$ if $f(x_0)$ exists.*
- *Continuity focuses on the function's behavior in the immediate vicinity of x_0 and includes x_0 itself.*
- *Limits exclude the point x_0 itself by requiring $0 < |x - x_0| < \delta$, ensuring the limit depends solely on the behavior of $f(x)$ near x_0, not at x_0.*
- *Continuity ensures no "jumps" or "breaks" at x_0 since $f(x)$ transitions smoothly through x_0.*
- *The limit addresses how $f(x)$ behaves as x approaches x_0, independent of whether x_0 is in the domain or $f(x_0)$ exists.*
- *For continuity, think of $f(x)$ as a graph that you can draw without lifting your pencil at x_0.*
- *For a limit, the graph near x_0 (but possibly excluding x_0) must approach a specific height L, regardless of the value at x_0.*

Limit Definition:

$$\lim_{x \to x_0} f(x) = L \Leftrightarrow \forall \varepsilon > 0, \quad \exists \delta(\varepsilon) > 0 \; (\delta(\varepsilon) \in \mathbb{R}^+) \quad \ni \forall x \in X, \quad 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon. \tag{1}$$

Continuity Definition:

$$f \text{ is continuous at } x_0 \Leftrightarrow \forall \varepsilon > 0, \quad \exists \delta(\varepsilon) > 0 \; (\delta(\varepsilon) \in \mathbb{R}^+) \; \ni \forall x \in X, \cancel{0 <} |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon. \tag{2}$$

The key difference between the limit and continuity definitions is the inclusion of $x = x_0$ (so, we have $0 \le |x - x_0|$) in the continuity condition. In the continuity formula, L in the limit definition is replaced with $f(x_0)$, ensuring $\lim_{x \to x_0} f(x) = f(x_0)$.

Show that the function $f(x) = 2x + 6$ is continuous at $x = 4$ using the $\varepsilon - \delta$ (epsilon-delta) definition.

Solution. In fact, the process of showing that the function $f(x)$ is continuous at $x = 4$ using the epsilon-delta (ε-δ) definition is fundamentally equivalent to proving that the limit of the function as $x \to 4$ is equal to 14 (the value of the function at that point).

To prove the continuity of a function using the epsilon-delta definition, we need to find a δ in terms of $\varepsilon > 0$, such that for every x in the domain of the function:

$$|x - a| < \delta \implies |f(x) - f(a)| < \varepsilon.$$

Let us work to find δ in terms of ε that satisfies the above condition.

Using the given function $f(x) = 2x + 6$, and at $x = 4$, we compute:

$$f(4) = 2(4) + 6 = 14.$$

Thus, the condition becomes:

$$|x - 4| < \delta \implies |(2x + 6) - 14| < \varepsilon.$$

We simplify the inequality:

$$|(2x + 6) - 14| < \varepsilon.$$

This becomes:

$$|2x - 8| < \varepsilon.$$

Factoring out the constant 2:

$$2|x - 4| < \varepsilon.$$

Dividing both sides by 2:

$$|x - 4| < \frac{\varepsilon}{2}.$$

Our goal is to express δ in terms of ε. From the above inequality, we can choose:

$$\delta = \frac{\varepsilon}{2}.$$

Now, for $|x-4| < \delta$, substituting $\delta = \frac{\varepsilon}{2}$:

$$|x-4| < \frac{\varepsilon}{2} \implies 2|x-4| < \varepsilon \implies |(2x+6) - 14| < \varepsilon.$$

Thus, the condition is satisfied for all x in the domain of the function.

Since we found a δ in terms of ε such that the epsilon-delta condition is always satisfied, we conclude that $f(x) = 2x + 6$ is continuous at $x = 4$.

Let $f(x) = 2x^2 + 1$ for $x \in \mathbb{R}$. Prove f is continuous on $\mathbb{R}$ by

(a) *Using definition*

(b) *Using the $\varepsilon - \delta$ definition.*

(a) Suppose $\lim x_n = x_0$. Then we have

$$\lim f(x_n) = \lim(2x_n^2 + 1) = 2(\lim x_n)^2 + 1 = 2x_0^2 + 1 = f(x_0)$$

where the second equality is an application of the limit Theorems (multiplication and summation). Hence f is continuous at each x_0 in $\mathbb{R}$.

(b) Let x_0 be in $\mathbb{R}$ and let $\varepsilon > 0$. We want to show $|f(x) - f(x_0)| < \varepsilon$ provided $|x - x_0|$ is sufficiently small, i.e., less than some δ. We observe

$$|f(x) - f(x_0)| = |2x^2 + 1 - (2x_0^2 + 1)| = |2x^2 - 2x_0^2| = 2|x - x_0| \cdot |x + x_0|.$$

We need to get a bound for $|x + x_0|$ that does not depend on x. We notice that if $|x - x_0| < 1$, say, then $|x| < |x_0| + 1$ and hence

$$|x + x_0| \le |x| + |x_0| < 2|x_0| + 1.$$

Thus we have

$$|f(x) - f(x_0)| \le 2|x - x_0|(2|x_0| + 1)$$

provided $|x - x_0| < 1$. To arrange for $2|x - x_0|(2|x_0| + 1) < \varepsilon$, it suffices to have

$$|x - x_0| < \frac{\varepsilon}{2(2|x_0| + 1)}$$

and also $|x - x_0| < 1$. So we put

$$\delta = \min\left\{1, \frac{\varepsilon}{2(2|x_0| + 1)}\right\}.$$

The work above shows $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$, as desired.

Left and Right Continuity

The function f is **right-continuous** *at c (or continuous from the right) if*

$$\lim_{x \to c^+} f(x) = f(c).$$

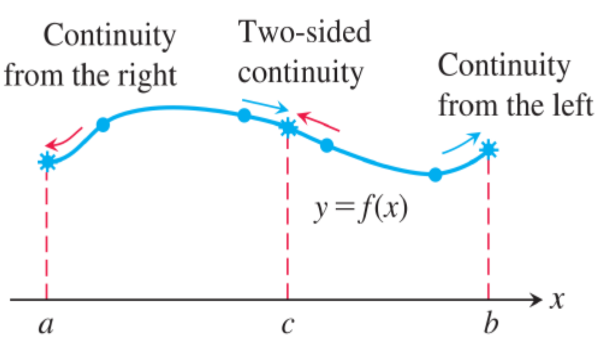
The function f is **left-continuous** *at c (or continuous from the left) if*

$$\lim_{x \to c^-} f(x) = f(c).$$

Reminder

A function $f(x)$ has a limit as x approaches an interior point c ***if and only if*** *it has left-hand and right-hand limits there, and these one-sided limits are equal:*

$$\lim_{x \to c} f(x) = L \iff \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$



For one-sided continuity and continuity at an endpoint of an interval, the limits in parts 2 and 3 of the test in *** should be replaced by the appropriate one-sided limits.

Theorem : Properties of Continuous Functions

If the functions f and g are continuous at $x = c$, then the following algebraic combinations are continuous at $x = c$:

1. **Sums:** $f + g$
2. **Differences:** $f - g$
3. **Constant multiples:** $k \cdot f$, *for any number* k
4. **Products:** $f \cdot g$
5. **Quotients:** $\frac{f}{g}$, *provided* $g(c) \neq 0$
6. **Powers:** f^n, n *a positive integer*
7. **Roots:** $\sqrt[n]{f}$, *provided it is defined on an open interval containing* c, *where* n *is a positive integer*

(a) Every polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ is continuous because

$$\lim_{x \to c} P(x) = P(c)$$

(b) If $P(x)$ and $Q(x)$ are polynomials, then the rational function $\frac{P(x)}{Q(x)}$ is continuous wherever it is defined $(Q(c) \neq 0)$.

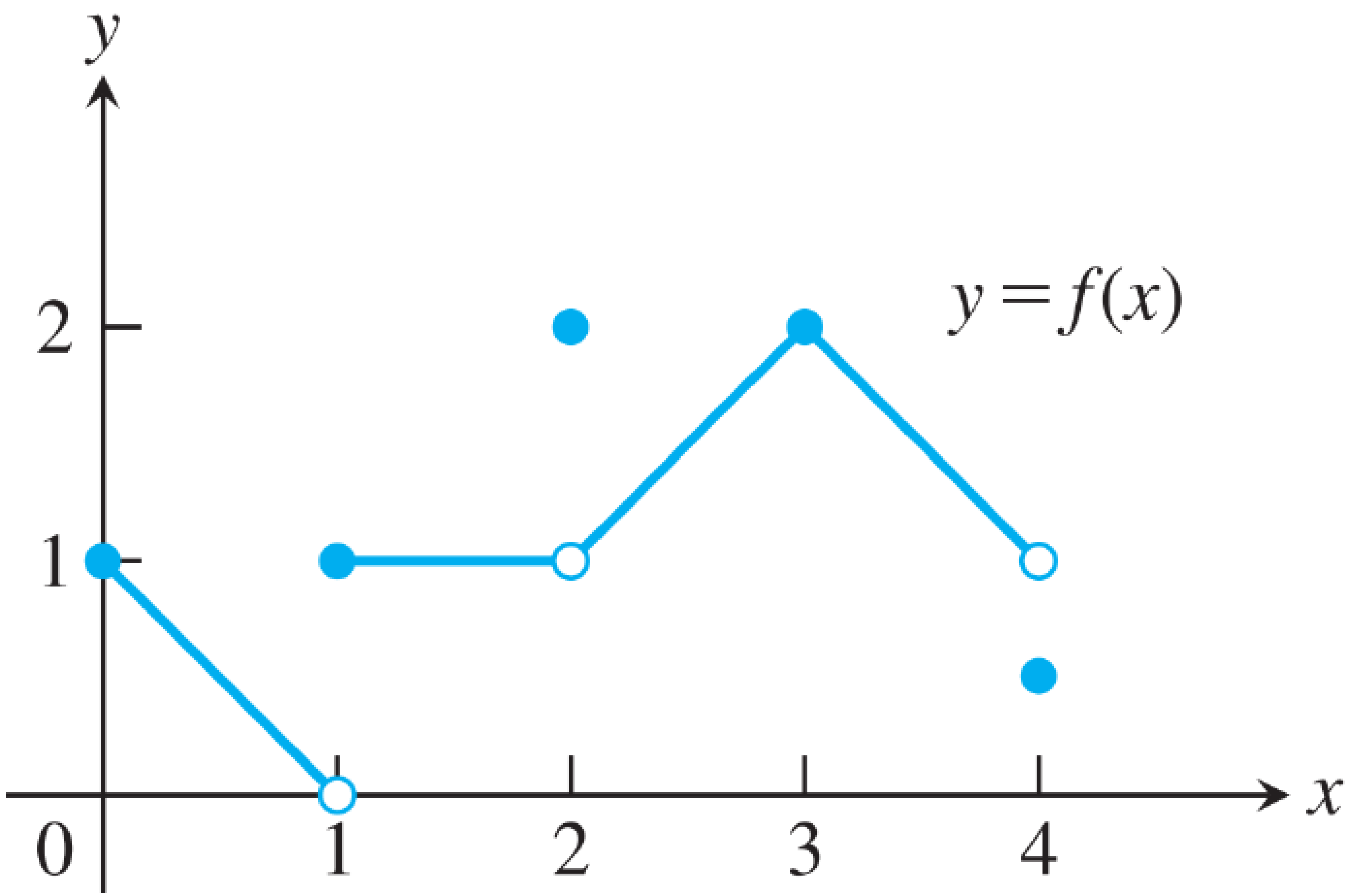
The function $f(x) = |x|$ is continuous. If $x > 0$, we have $f(x) = x$, a polynomial. If $x < 0$, we have $f(x) = -x$, another polynomial. Finally, at the origin,

$$\lim_{x \to 0} |x| = 0$$

Theorem

The following functions are **continuous** *within* <u>*their domains*</u>*:*

- *Polynomial functions*
- *Rational functions*
- *Trigonometric functions*
- *Inverse trigonometric functions*
- *Exponential functions*
- *Logarithmic functions*
- *Root functions*



The function is not continuous at $x = 1, 2,$ and 4

Theorem

If g is continuous at the point b and $\lim_{x\to c} f(x) = b$, then

$$\lim_{x\to c} g(f(x)) = g(b) = g\left(\lim_{x\to c} f(x)\right).$$

$$\lim_{x\to\pi/2} \cos\left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right) = \cos\left(\lim_{x\to\pi/2} 2x + \lim_{x\to\pi/2} \sin\left(\frac{3\pi}{2} + x\right)\right).$$

Evaluating the limits inside:

$$= \cos(\pi + \sin 2\pi) = \cos\pi = -1.$$

$$\lim_{x\to 1} \sin^{-1}\left(\frac{1-x}{1-x^2}\right) = \sin^{-1}\left(\lim_{x\to 1} \frac{1-x}{1-x^2}\right).$$

Simplifying the fraction:

$$\frac{1-x}{1-x^2} = \frac{1}{1+x}, \quad \text{so} \quad \lim_{x\to 1} \frac{1-x}{1-x^2} = \frac{1}{2}.$$

Hence:

$$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}.$$

$$
\begin{aligned}
\lim_{x\to 0} \sqrt{x+1}e^{\tan x} &= \lim_{x\to 0}\sqrt{x+1}\cdot \exp\left(\lim_{x\to 0}\tan x\right). \\
&= \sqrt{1}\cdot e^0 = 1\cdot 1 = 1.
\end{aligned}
$$

Theorem

If f is continuous at c, and g is continuous at $f(c)$, then the composition $g \circ f$ is continuous at c.

Consider the following function:

$$
f(x) = \begin{cases} x^2, & \text{if } x \neq 2 \\ 5, & \text{if } x = 2 \end{cases}
$$

Let's examine the limit of the function as x approaches 2:

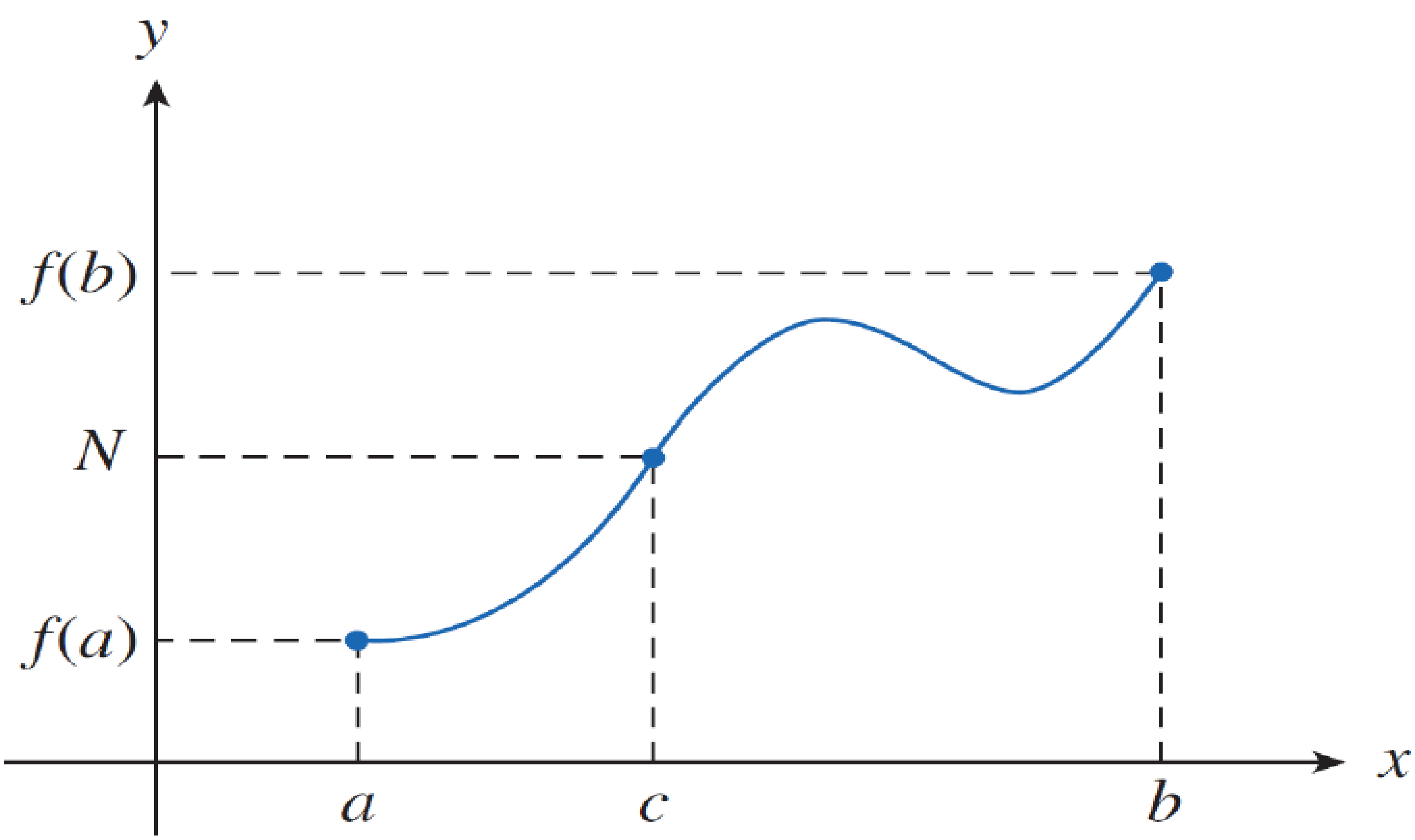
Solution.

$$
\lim_{x\to 2} f(x) = \lim_{x\to 2} x^2 = 4
$$

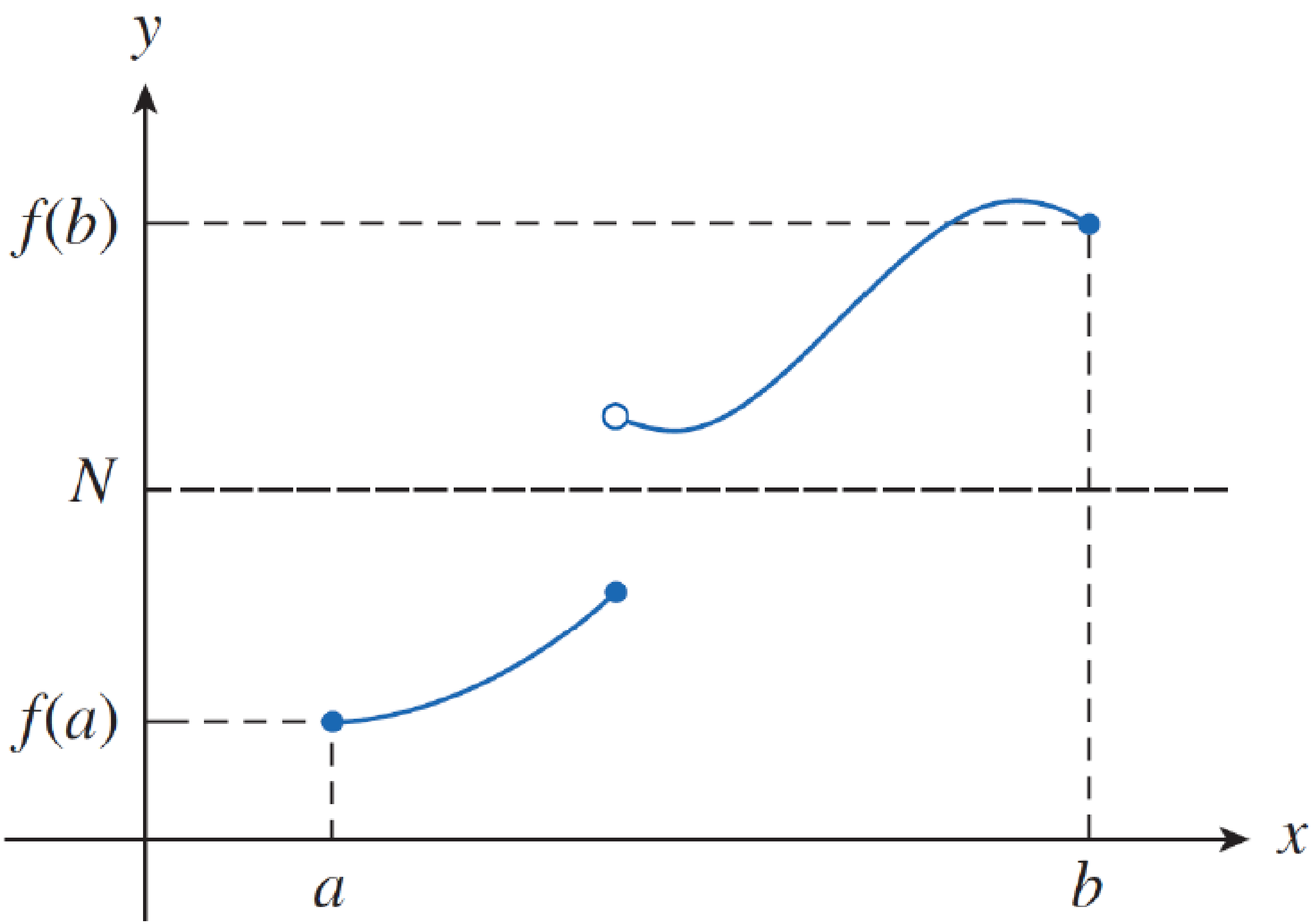
However, $f(2) = 5$. So, the limit as x approaches 2 is 4, but the function is defined to take the value 5 at $x = 2$. Thus, the function is not continuous at $x = 2$ because the limit does not equal the function's value at that point.

Intermediate Value Theorem

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.



When the function f is discontinuous in the interval $[a, b]$, there is no number c in (a, b) such that $f(c) = N$.



Bolzano's Theorem

If $f(x)$ is a continuous function on a closed interval $[a, b]$, and $f(a) \cdot f(b) < 0$, then there exists at least one $c \in (a, b)$ such that:

$$f(c) = 0.$$

Show that there is a root of the equation $x^3 - x - 1 = 0$ between 1 and 2.

Solution. Let $f(x) = x^3 - x - 1$. Since

$$f(1) = 1^3 - 1 - 1 = -1 < 0$$

and

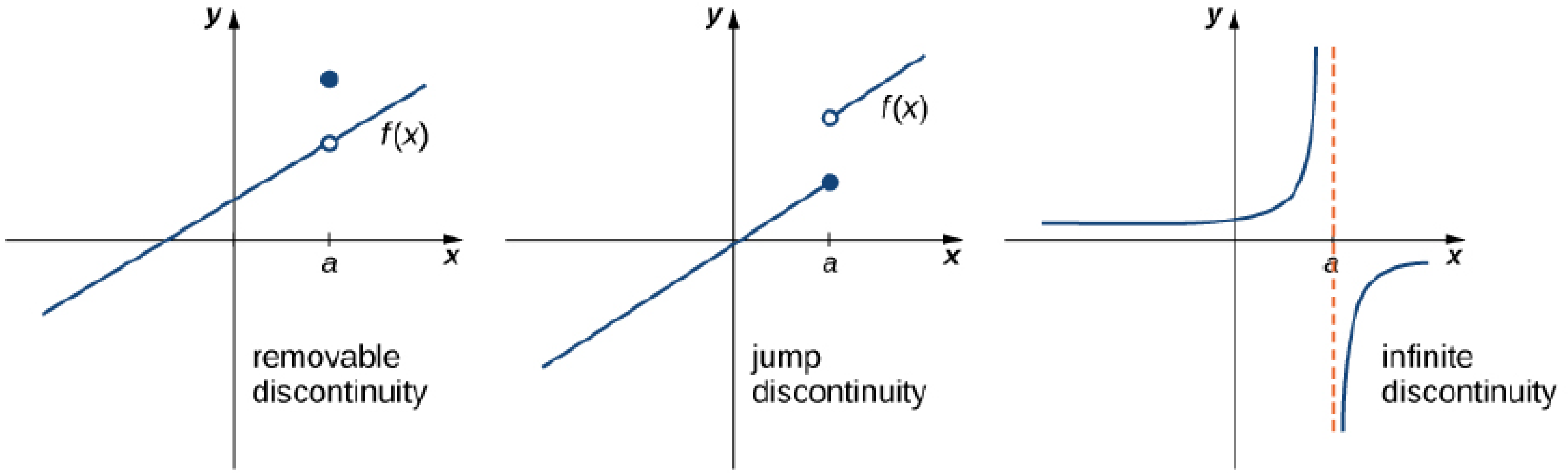
$$f(2) = 2^3 - 2 - 1 = 5 > 0,$$

we see that $y_0 = 0$ is a value between $f(1)$ and $f(2)$. Since f is continuous, the Intermediate Value Theorem says there is a zero of f between 1 and 2.

Classification of discontinuities

If $f(x)$ is discontinuous at a, then

- *f has a* **removable discontinuity** *at a if $\lim_{x\to a} f(x)$ exists but $f(a) \neq \lim_{x\to a} f(x)$ or $f(a)$ is undefined. (Note: When we state that $\lim_{x\to a} f(x)$ exists, we mean that $\lim_{x\to a} f(x) = L$, where L is a real number.)*
- *f has a* **jump discontinuity** *at a if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist, but $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$. (Note: When we state that $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist, we mean that both are real-valued and that neither take on the values $\pm\infty$.)*
- *f has an* **essential (infinite) discontinuity** *at a if $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$.*



Let us consider the following function:

$$f(x) = \frac{x^2 - 4}{x - 2}$$

It is discontinuous at $x = 2$. Classify this discontinuity as removable, jump, or infinite.

Solution. To classify the discontinuity at $x = 2$, we must evaluate $\lim_{x \to 2} f(x)$:

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 2)}{x - 2}.$$

Canceling the common factor $x - 2$, we have:

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} (x + 2) = 4.$$

Since f is discontinuous at $x = 2$ and $\lim_{x \to 2} f(x)$ exists, f has a **removable discontinuity** at $x = 2$.

Let us consider the following function:

$$f(x) = \begin{cases} -x^2 + 4 & \text{if } x \leq 3, \\ 4x - 8 & \text{if } x > 3 \end{cases}$$

It is discontinuous at $x = 3$. Classify this discontinuity as removable, jump, or infinite.

Solution. Earlier, we showed that f is discontinuous at $x = 3$ because $\lim_{x \to 3} f(x)$ does not exist. However, since

$$\lim_{x \to 3^-} f(x) = -5 \quad \text{and} \quad \lim_{x \to 3^+} f(x) = 4$$

both exist, we conclude that the function has a **jump discontinuity** at $x = 3$.

Determine whether

$$f(x) = \frac{x+2}{x+1}$$

is continuous at $x = -1$. *If the function is discontinuous at* $x = -1$, *classify the discontinuity as removable, jump, or infinite.*

Solution. The function value $f(-1)$ is undefined. Therefore, the function is not continuous at $x = -1$. To determine the type of discontinuity, we must determine the limit at $x = -1$. We see that:

$$\lim_{x \to -1^-} \frac{x+2}{x+1} = -\infty \quad \text{and} \quad \lim_{x \to -1^+} \frac{x+2}{x+1} = +\infty.$$

Therefore, the function has an **infinite discontinuity** at $x = -1$.

Bounded Function

A function $f : A \to \mathbb{R}$ *is said to be bounded on* A *if there exists a constant* $M > 0$ *such that* $|f(x)| \leq M$ *for all* $x \in A$.

Boundedness Theorem

Let $I := [a, b]$ *be a closed bounded interval and let* $f : I \to \mathbb{R}$ *be continuous on* I. *Then,* f *is* **bounded** *on* I.

Definition

Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. We say that f has an **absolute maximum** on A if there is a point $x^* \in A$ such that

$$f(x^*) \geq f(x) \quad \text{for all } x \in A.$$

We say that f has an **absolute minimum** on A if there is a point $x_* \in A$ such that

$$f(x_*) \leq f(x) \quad \text{for all } x \in A.$$

We say that x^* is an **absolute maximum point** for f on A, and that x_* is an **absolute minimum point** for f on A, if they exist.

Extreme Value Theorem

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f attains both its **minimum** and **maximum** values on $[a, b]$. That is, there exist $\alpha, \beta \in [a, b]$ such that:

$$m := \inf\{f(x) : x \in [a, b]\} = f(\alpha),$$

and

$$M := \sup\{f(x) : x \in [a, b]\} = f(\beta).$$

Show that the function

$$f(x) = x^2 - 4x + 3$$

has maximum and minimum values on the interval $[0, 5]$.

Solution. The function $f(x) = x^2 - 4x + 3$ is a **polynomial**. Polynomials are continuous everywhere, and in particular, $f(x)$ is continuous on the closed and bounded interval $[0, 5]$.

By the **Extreme Value Theorem** (**Weierstrass Theorem**), if a function is continuous on a closed interval $[a, b]$, it must attain both a global maximum and minimum in that interval.

Thus, $f(x)$ has an **absolute maximum** and **absolute minimum** on $[0, 5]$.

Uniform Continuity

Let f be a real-valued function defined on a set $S \subseteq \mathbb{R}$. Then f is **uniformly continuous** on S if

for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\forall x, y \in S$ and $|x - y| < \delta$ imply $|f(x) - f(y)| < \varepsilon$.

We will say that f is **uniformly continuous** if f is uniformly continuous on $\mathrm{dom}(f)$.

Every **uniformly continuous function** is **continuous**, but not every continuous function is **uniformly continuous.** (That is, the converse is not always true. But if the function is defined on a closed and bounded interval, the converse also holds.

Theorem

Let $f : [a, b] \to \mathbb{R}$ be **continuous**. Then f is **uniformly continuous**.

Uniform continuity is a **stronger condition** *than ordinary continuity.*

- *In* **uniform continuity**, *the δ corresponding to a given ε is globally applicable across the entire domain, depending* **solely** *on ε.*
- *In contrast,* **ordinary continuity** *allows δ to depend on* **both** *ε and the specific point x in the domain, making it locally defined.*
- *The absence of breaks or jumps in the graph is not sufficient for* **uniform continuity**. *In* **uniform continuity**, *the function must also avoid sudden increases or decreases, ensuring consistent and predictable changes within the given or selected interval. This sudden increase (decrease) disrupts the uniformity of continuous curves.*
- *However, if there are no gaps in this curve with the possibility of sudden increase (decrease), then the curve is (ordinary)* **continuous**.

Prove that $f(x) = \frac{1}{x}$ is uniformly continuous on $[1, 2]$ using ε-δ definition of uniform continuity.

Solution. We aim to prove that $f(x) = \frac{1}{x}$ is uniformly continuous on the closed interval $[1, 2]$ using the ε-δ definition of uniform continuity.

A function $f(x)$ is uniformly continuous on a set S if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in S$,

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

Step 1: Analyze $|f(x) - f(y)|$

Given $f(x) = \frac{1}{x}$, calculate $|f(x) - f(y)|$:

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{y - x}{xy}\right|.$$

On the interval $[1, 2]$, we know that $x, y \in [1, 2]$. Hence:

$$1 \leq x, y \leq 2 \implies 1 \leq xy \leq 4.$$

Thus:

$$\frac{1}{xy} \leq \frac{1}{1} = 1.$$

This implies:

$$|f(x) - f(y)| = \left|\frac{y - x}{xy}\right| \leq |y - x| \cdot \frac{1}{xy} \leq |y - x|.$$

Step 2: Choose δ in terms of ε

To ensure $|f(x) - f(y)| < \varepsilon$, it suffices to choose $\delta = \varepsilon$. Then, if $|x - y| < \delta$, we have:

$$|f(x) - f(y)| \leq |y - x| < \delta = \varepsilon.$$

Step 3: Verify the condition

For any $\varepsilon > 0$, choosing $\delta = \varepsilon$ ensures that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$, satisfying the uniform continuity condition.

Conclusion: Since $\delta = \varepsilon$ works for any $\varepsilon > 0$ and is independent of the choice of x and y in $[1, 2]$, the function $f(x) = \frac{1}{x}$ is uniformly continuous on $[1, 2]$.

Remark

The result above can also be deduced from the Heine–Cantor Theorem, which states: Every continuous function on a closed and bounded interval is uniformly continuous.
Since $f(x) = \frac{1}{x}$ is continuous on $[1, 2]$, and $[1, 2]$ is closed and bounded, it follows directly from the theorem that $f(x)$ is uniformly continuous on $[1, 2]$.

Prove that $f(x) = 3x + 1$ is uniformly continuous on $\mathbb{R}$ using ε-δ definition of uniform continuity.

Solution. To prove that $f(x) = 3x + 1$ is uniformly continuous on $\mathbb{R}$, we use the ε-δ definition of uniform continuity.

Definition: A function $f(x)$ is uniformly continuous on a set S if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in S$,

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

Step 1: Analyze $|f(x) - f(y)|$

Given $f(x) = 3x + 1$, calculate $|f(x) - f(y)|$:

$$|f(x) - f(y)| = |(3x + 1) - (3y + 1)| = |3x - 3y| = 3|x - y|.$$

Step 2: Choose δ in terms of ε

To ensure $|f(x) - f(y)| < \varepsilon$, we require:

$$3|x - y| < \varepsilon.$$

Dividing both sides by 3:

$$|x - y| < \frac{\varepsilon}{3}.$$

Thus, choose $\delta = \frac{\varepsilon}{3}$.

Step 3: Verify the condition

For $|x - y| < \delta$, where $\delta = \frac{\varepsilon}{3}$, we have:

$$|f(x) - f(y)| = 3|x - y| < 3 \cdot \frac{\varepsilon}{3} = \varepsilon.$$

This satisfies the uniform continuity condition.

Conclusion: Since $\delta = \frac{\varepsilon}{3}$ works for any $\varepsilon > 0$ and is independent of the choice of x and y, the function $f(x) = 3x + 1$ is uniformly continuous on $\mathbb{R}$.

Find the constant c that makes g continuous on $(-\infty, \infty)$.

$$g(x) = \begin{cases} x^2 - c^2 & \text{if } x < 4 \\ cx + 20 & \text{if } x \geq 4 \end{cases}$$

Solution. If a function g is continuous at $x = a$, then

$$\lim_{x \to a^+} g(x) = \lim_{x \to a^-} g(x) = g(a).$$

Our function $g(x)$ is piecewise defined. For $x < 4$, it is the polynomial $x^2 - c^2$, so it is continuous (polynomials are continuous). For $x > 4$ it is also a polynomial, so it will also be continuous in this region. The only point we don't know if the function $g(x)$ is continuous is at $x = 4$, not surprisingly the point where the definition changes.

We must choose c to make the function continuous at $x = 4$. We do this by by imposing that the following limits be equal:

$$\lim_{x \to 4^+} g(x) = \lim_{x \to 4^-} g(x).$$

Insert the proper definitions for $g(x)$:

$$\lim_{x \to 4^+} (cx + 20) = \lim_{x \to 4^-} \left(x^2 - c^2\right)$$

Evaluate by direct substitution:

$$c(4) + 20 = (4)^2 - c^2$$

A little algebraic rearranging gives us the following quadratic in c :

$$c^2 + 4c + 4 = 0$$

So if c satisfies this quadratic, then the left and right hand limits will be equal. The equality with $g(a)$ that is required for continuity follows automatically in this case. All that is left to do is solve the quadratic for c:

$$c(4) + 20 = (4)^2 + c^2 \longrightarrow (c+2)^2 = 0 \longrightarrow c = -2$$

So if $c = -2$, the function $g(x)$ will be continuous for $x \in \mathbb{R}$.

Consider the function defined by

$$f(x) = \begin{cases} ax^2 + x - b, & x < 2 \\ ax + b, & 2 \leq x \leq 5 \\ 2ax - 7, & x > 5. \end{cases}$$

for a and b of this piecewise function such that the function f(x) is continuous.

Solution. First, remember the definition of continuity:

A function f is continuous at a ($a \in \operatorname{Dom} f$) if:

$$\lim_{x \to a} f(x) = f(a)$$

Also, remember that a limit exists if and only if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

Now, we want f to be continuous at $x = 2$ and at $x = 5$, using the preceding definitions at $x = 2$:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = \lim_{x \to 2} f(x) = f(2)$$

So, we have the following equality:

$$4a + 2 - b = 2a + b \implies a = b - 1$$

Doing the same for $x = 5$:

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) = \lim_{x \to 5} f(x) = f(5)$$

Then, evaluating the lateral limits, we have:

$$5a + b = 10a - 7 \implies 5a - 7 = b$$

Remember we obtained $a = b - 1$, plugging in the last equation:

$$5a - 7 = b \implies 5(b - 1) - 7 = b \implies b = 3$$

Then $a = b - 1 \implies a = 3 - 1 = 2$.
Therefore, $a = 2$ and $b = 3$.