Curve Sketching

Sketch the graphs of the following questions:

23. $y = x + \sin x, \quad 0 \leq x \leq 2\pi$

24. $y = x - \sin x, \quad 0 \leq x \leq 2\pi$

25. $y = \sqrt{3}x - 2\cos x, \quad 0 \leq x \leq 2\pi$

26. $y = \frac{4}{3}x - \tan x, \quad \frac{-\pi}{2} < x < \frac{\pi}{2}$

27. $y = \sin x \cos x, \quad 0 \leq x \leq \pi$

28. $y = \cos x + \sqrt{3}\sin x, \quad 0 \leq x \leq 2\pi$

29. $y = x^{1/5}$

30. $y = x^{2/5}$

45. $y = |x^2 - 1|$

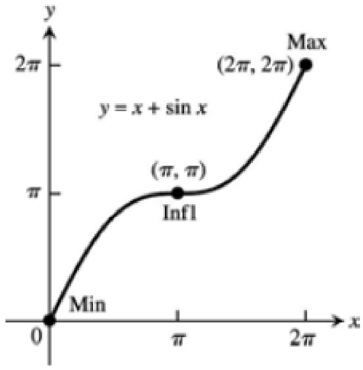
46. $y = |x^2 - 2x|$

51. $y = \ln(3 - x^2)$

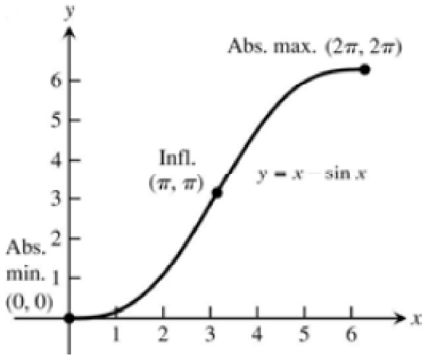
Short Solutions for the questions above

The solutions offered here are in a summarized form. In your exams, you should provide detailed explanations and illustrations.

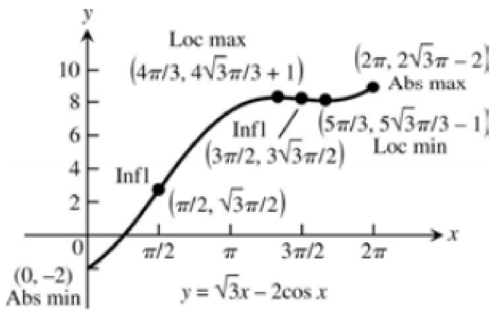
23. When $y = x + \sin x$, then $y' = 1 + \cos x$ and $y'' = -\sin x$. The curve rises on $(0, 2\pi)$. At $x = 0$ there is a local and absolute minimum and at $x = 2\pi$ there is a local and absolute maximum. The curve is concave down on $(0, \pi)$ and concave up on $(\pi, 2\pi)$. At $x = \pi$ there is a point of inflection.



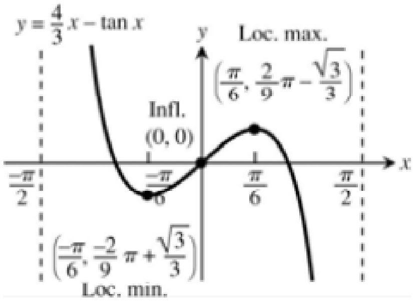
24. When $y = x - \sin x$, then $y' = 1 - \cos x$ and $y'' = \sin x$. The curve rises on $(0, 2\pi)$. At $x = 0$ there is a local and absolute minimum and at $x = 2\pi$ there is a local and absolute maximum. The curve is concave up on $(0, \pi)$ and concave down on $(\pi, 2\pi)$. At $x = \pi$ there is a point of inflection.



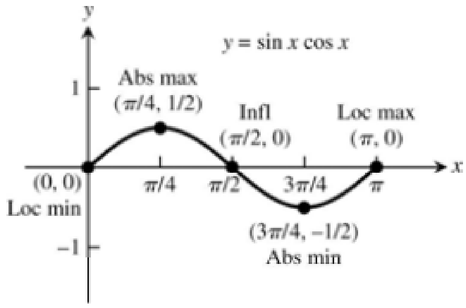
25. When $y = \sqrt{3}x - 2\cos x$, then $y' = \sqrt{3} + 2\sin x$ and $y'' = 2\cos x$. The curve is increasing on $\left(0, \frac{4\pi}{3}\right)$ and $\left(\frac{5\pi}{3}, 2\pi\right)$, and decreasing on $\left(\frac{4\pi}{3}, \frac{5\pi}{3}\right)$. At $x = 0$ there is a local and absolute minimum, at $x = \frac{4\pi}{3}$ there is a local maximum, at $x = \frac{5\pi}{3}$ there is a local minimum, and at $x = 2\pi$ there is a local and absolute maximum. The curve is concave up on $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right)$, and is concave down on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. At $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$ there are points of inflection.



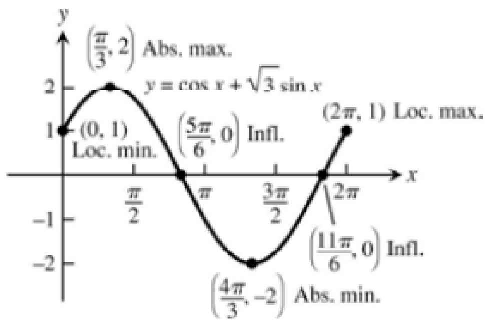
26. When $y=\frac{4}{3}x-\tan x$, then $y'=\frac{4}{3}-\sec^2 x$ and $y''=-2\sec^2 x\tan x$. The curve is increasing on $\left(-\frac{\pi}{6},\frac{\pi}{6}\right)$, and decreasing on $\left(-\frac{\pi}{2},-\frac{\pi}{6}\right)$ and $\left(\frac{\pi}{6},\frac{\pi}{2}\right)$. At $x=-\frac{\pi}{6}$ there is a local minimum, at $x=\frac{\pi}{6}$ there is a local maximum, there are no absolute maxima or absolute minima. The curve is concave up on $\left(-\frac{\pi}{2},0\right)$, and is concave down on $\left(0,\frac{\pi}{2}\right)$. At $x=0$ there is a point of inflection.



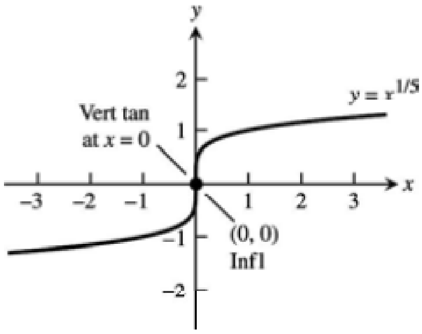
27. When $y=\sin x\cos x$, then $y'=-\sin^2 x+\cos^2 x$ $=\cos 2x$ and $y''=-2\sin 2x$. The curve is increasing on $\left(0,\frac{\pi}{4}\right)$ and $\left(\frac{3\pi}{4},\pi\right)$, and decreasing on $\left(\frac{\pi}{4},\frac{3\pi}{4}\right)$. At $x=0$ there is a local minimum, at $x=\frac{\pi}{4}$ there is a local and absolute maximum, at $x=\frac{3\pi}{4}$ there is a local and absolute minimum, and at $x=\pi$ there is a local maximum. The curve is concave down on $\left(0,\frac{\pi}{2}\right)$, and is concave up on $\left(\frac{\pi}{2},\pi\right)$. At $x=\frac{\pi}{2}$ there is a point of inflection.



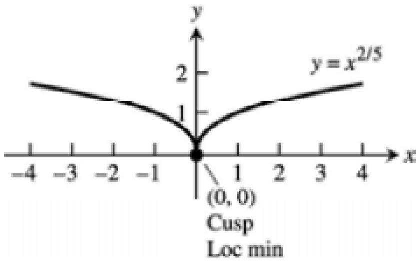
28. When $y=\cos x+\sqrt{3}\sin x$, then $y'=-\sin x+\sqrt{3}\cos x$ and $y''=-\cos x-\sqrt{3}\sin x$. The curve is increasing on $\left(0,\frac{\pi}{3}\right)$ and $\left(\frac{4\pi}{3},2\pi\right)$, and decreasing on $\left(\frac{\pi}{3},\frac{4\pi}{3}\right)$. At $x=0$ there is a local minimum, at $x=\frac{\pi}{3}$ there is a local and absolute maximum, at $x=\frac{4\pi}{3}$ there is a local and absolute minimum, and at $x=2\pi$ there is a local maximum. The curve is concave down on $\left(0,\frac{5\pi}{6}\right)$ and $\left(\frac{11\pi}{6},2\pi\right)$, and is concave up on $\left(\frac{5\pi}{6},\frac{11\pi}{6}\right)$. At $x=\frac{5\pi}{6}$ and $x=\frac{11\pi}{6}$ there are points of inflection.



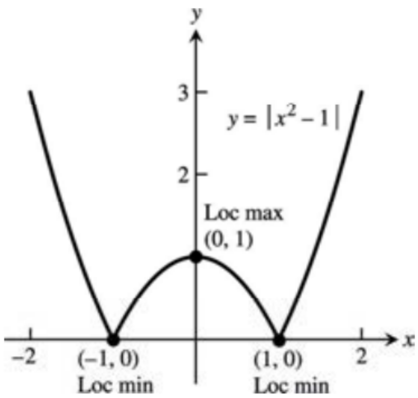
29. When $y = x^{1/5}$, then $y' = \frac{1}{5}x^{-4/5}$ and $y'' = -\frac{4}{25}x^{-9/5}$. The curve rises on $(-\infty, \infty)$ and there are no extrema. The curve is concave up on $(-\infty, 0)$ and concave down on $(0, \infty)$. At $x = 0$ there is a point of inflection.



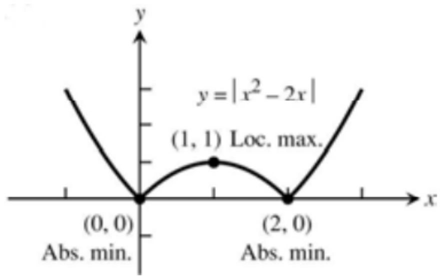
30. When $y = x^{2/5}$, then $y' = \frac{2}{5}x^{-3/5}$ and $y'' = -\frac{6}{25}x^{-8/5}$. The curve is rising on $(0, \infty)$ and falling on $(-\infty, 0)$. At $x = 0$ there is a local and absolute minimum. There is no local or absolute maximum. The curve is concave down on $(-\infty, 0)$ and $(0, \infty)$. There are no points of inflection, but a cusp exists at $x = 0$.



45. When $y = |x^2 - 1| = \begin{cases} x^2 - 1, & |x| \geq 1 \\ 1 - x^2, & |x| < 1 \end{cases}$, then $y' = \begin{cases} 2x, & |x| > 1 \\ -2x, & |x| < 1 \end{cases}$ and $y'' = \begin{cases} 2, & |x| > 1 \\ -2, & |x| < 1 \end{cases}$. The curve rises on $(-1, 0)$ and $(1, \infty)$ and falls on $(-\infty, -1)$ and $(0, 1)$. There is a local maximum at $x = 0$ and local minima at $x = \pm 1$. The curve is concave up on $(-\infty, -1)$ and $(1, \infty)$, and concave down on $(-1, 1)$. There are no points of inflection because y is not differentiable at $x = \pm 1$ (so there is no tangent line at those points).



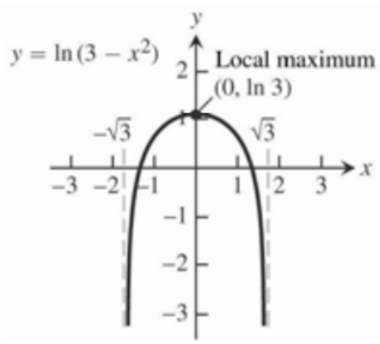
46. When $y = |x^2 - 2x| = \begin{cases} x^2 - 2x, & x < 0 \\ 2x - x^2, & 0 \le x \le 2, \\ x^2 - 2x, & x > 2 \end{cases}$



then $y' = \begin{cases} 2x - 2, & x < 0 \\ 2 - 2x, & 0 < x < 2, \\ 2x - 2, & x > 2 \end{cases}$ and $y'' = \begin{cases} 2, & x < 0 \\ -2, & 0 < x < 2 \\ 2, & x > 2 \end{cases}$.

The curve is rising on $(0, 1)$ and $(2, \infty)$, and falling on $(-\infty, 0)$ and $(1, 2)$. There is a local maximum at $x = 1$ and local minima at $x = 0$ and $x = 2$. The curve is concave up on $(-\infty, 0)$ and $(2, \infty)$, and concave down on $(0, 2)$. There are no points of inflection because y is not differentiable at $x = 0$ and $x = 2$ (so there is no tangent at those points).

51. $y = \ln(3 - x^2) \Rightarrow y' = \frac{-2x}{3-x^2} = \frac{2x}{x^2-3}$



$\Rightarrow y' = \underset{-\sqrt{3}}{(} \; + + + \underset{0}{|} - - - \; \underset{\sqrt{3}}{)} \Rightarrow$ the graph is rising on $\left(-\sqrt{3}, 0\right)$, falling on $\left(0, \sqrt{3}\right)$; a local minimum is ln 3 at $x = 0$; $y'' = \frac{(x^2-3)(2)-(2x)(2x)}{(x^2-3)^2} = \frac{-2(x^2+3)}{(x^2-3)^2}$

$\Rightarrow y'' = \underset{-\sqrt{3}}{(} \; - - - \; \underset{\sqrt{3}}{)} \Rightarrow$ the graph is concave down on $\left(-\sqrt{3}, \sqrt{3}\right)$.

Optimization problems

Question:

Physical Applications

37. Vertical motion The height above ground of an object moving vertically is given by

$$s = -16t^2 + 96t + 112,$$

with s in feet and t in seconds. Find

a. the object's velocity when $t = 0$;

b. its maximum height and when it occurs;

c. its velocity when $s = 0$.

Solution:

37. (a) $s(t) = -16t^2 + 96t + 112 \Rightarrow v(t) = s'(t) = -32t + 96$. At $t = 0$, the velocity is $v(0) = 96$ ft/sec.

(b) The maximum height occurs when $v(t) = 0$, when $t = 3$. The maximum height is $s(3) = 256$ ft and it occurs at $t = 3$ sec.

(c) Note that $s(t) = -16t^2 + 96t + 112 = -16(t+1)(t-7)$, so $s = 0$ at $t = -1$ or $t = 7$. Choosing the positive value of t, the velocity when $s = 0$ is $v(7) = -128$ ft/sec.

Question:

51. It costs you c dollars each to manufacture and distribute backpacks. If the backpacks sell at x dollars each, the number sold is given by

$$n = \frac{a}{x - c} + b(100 - x),$$

where a and b are positive constants. What selling price will bring a maximum profit?

Solution:

51. The profit is $p = nx - nc = n(x-c) = [a(x-c)^{-1} + b(100-x)](x-c) = a + b(100-x)(x-c)$ $= a + (bc + 100b)x - 100bc - bx^2$. Then $p'(x) = bc + 100b - 2bx$ and $p''(x) = -2b$. Solving $p'(x) = 0 \Rightarrow x = \frac{c}{2} + 50$. At $x = \frac{c}{2} + 50$ there is a maximum profit since $p''(x) = -2b < 0$ for all x.