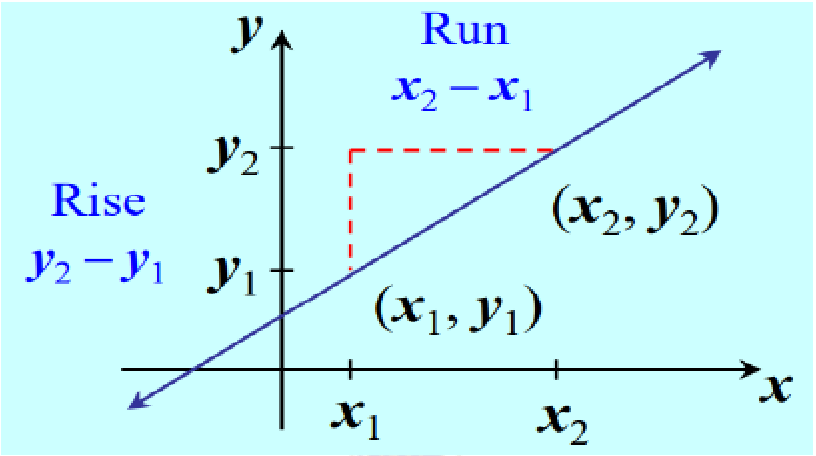
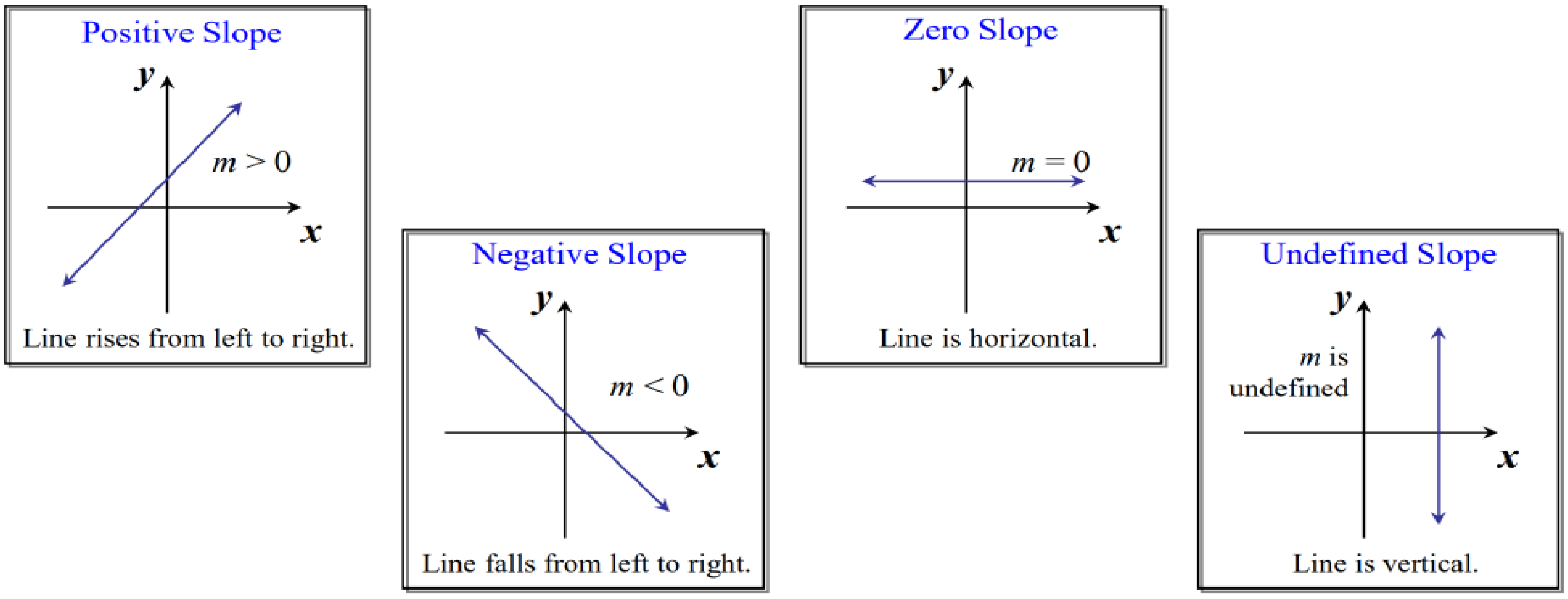
## Definition of Slope of A line

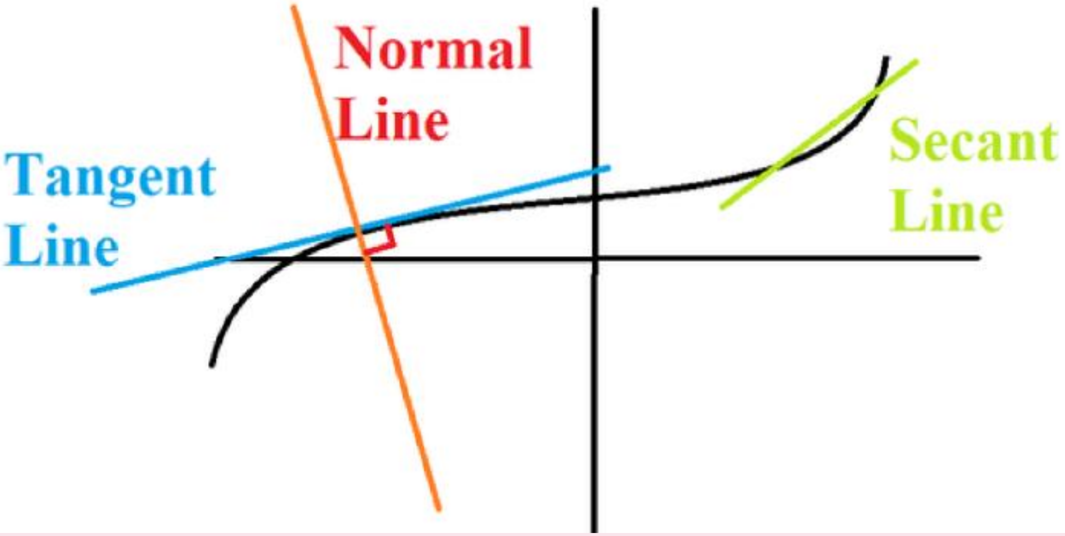
The slope of the line, denoted by $m$, through the distinct points $(x_1, y_1)$ and $(x_2, y_2)$ is

$$m = \frac{\text{Change in } y}{\text{Change in } x} = \frac{\text{Rise}}{\text{Run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$



*A* ***secant*** *line (or simply called a secant) is a line passing through two points of a curve.*



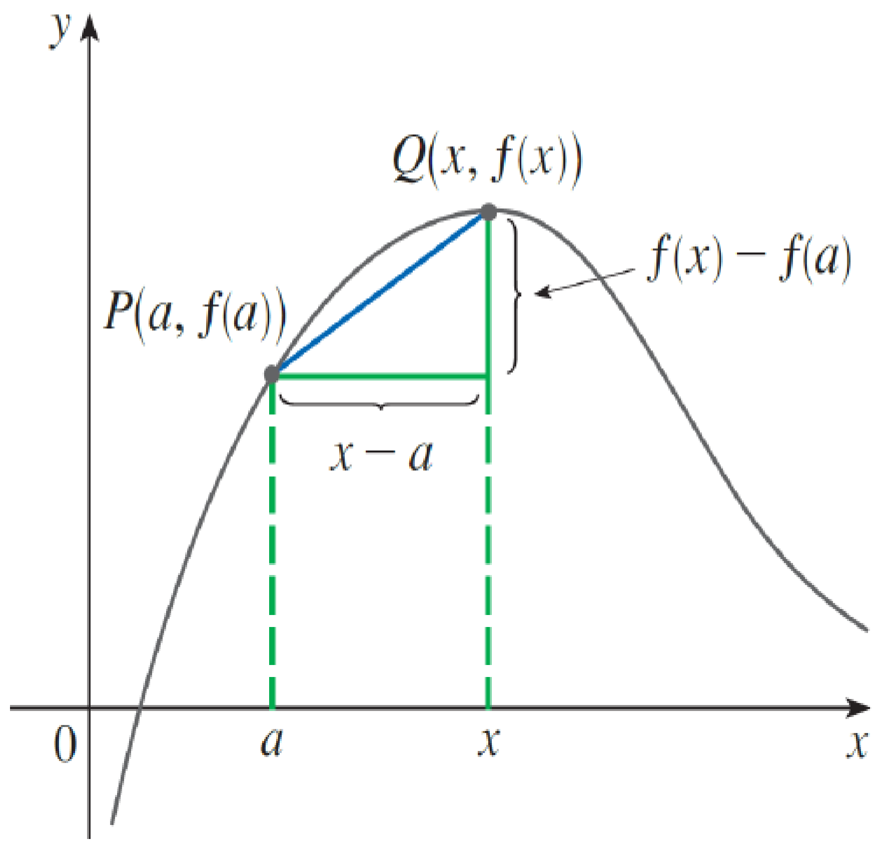

*The slope of a secant line represents the* ***average rate of change*** *between two points on the curve. It shows how much the function's output changes relative to its input between these points.*

We compute average rates of change:

$$\text{average rate of change} = \frac{\text{change in the function (output)}}{\text{change in the input}} = \frac{\Delta y}{\Delta x}.$$

More precisely, we compute the slope of the secant line $PQ$, which connects two points on a curve:

$$m_{\text{secant}} = m_{PQ} = \frac{\Delta y}{\Delta x} = \frac{f(x) - f(a)}{x - a}.$$



The slope of the secant line To refine this idea, let the horizontal change $\Delta x$ be represented by a small increment $h$. Thus, we set:

$$\Delta x = h,$$

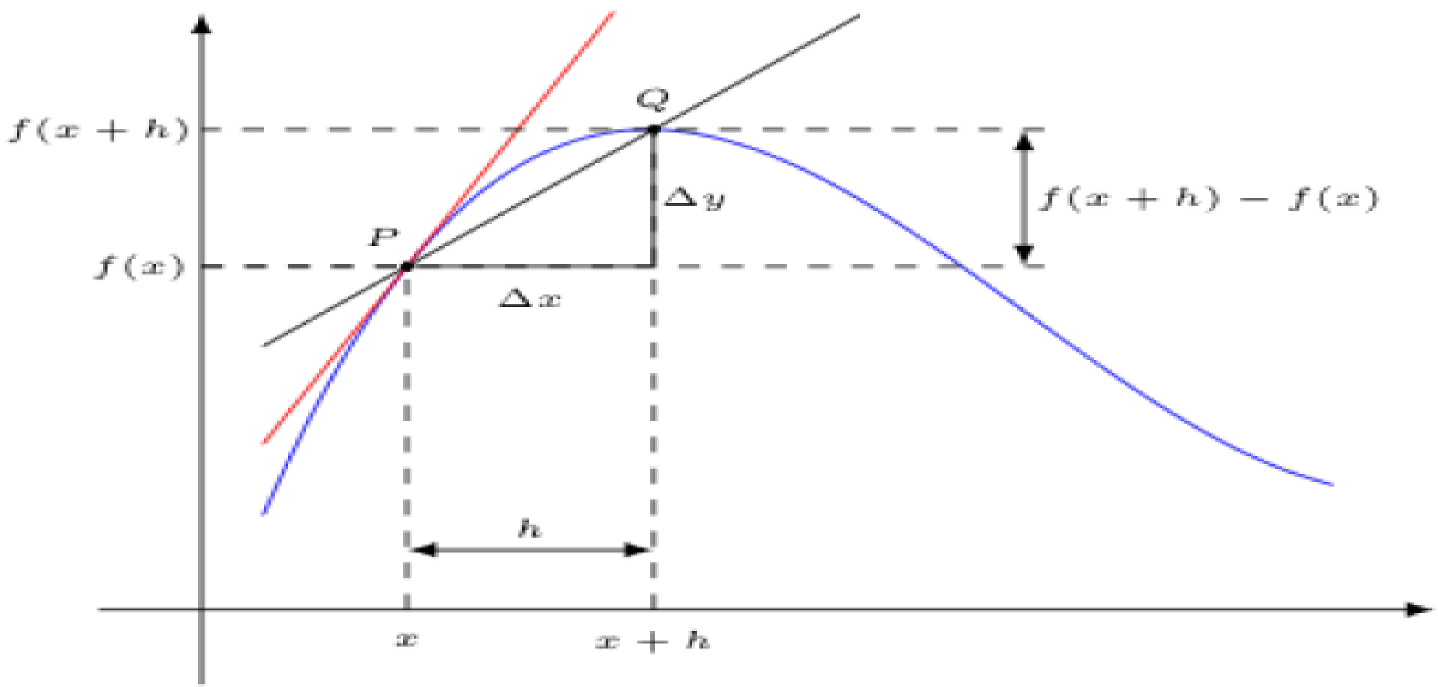
where $h$ represents the horizontal distance between the two points $x_0$ and $x_0 + h$.

Substituting $\Delta x = h$ into the formula for the change in $y$, we obtain:

$$\Delta y = f(x_0 + h) - f(x_0).$$

Hence, the slope of the secant line becomes:

$$m_{\text{secant}} = \frac{\Delta y}{\Delta x} = \frac{f(x_0 + h) - f(x_0)}{h}.$$



## The derivative of a function at a point

*The* **derivative** *of a function $f$ at a point $x_0$, denoted $f'(x_0)$, is*

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

*provided this limit exists.*
*We say that $f$ is* **differentiable** *at $x_0$. Otherwise, we say that $f$ is* **non-differentiable** *at $x_0$.*

## Alternative Formula for the Derivative at one point

*The alternative formula for the* **derivative** *of a function $f$ at a point $x_0$, denoted $f'(x_0)$, is*

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

*provided this limit exists.*

## Derivative as a function (The Derivative Function)

*The derivative of a function $f$ is another function $f'$ (read "$f$ prime") defined by*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

*at all points $x$ for which the limit exists (i.e., is a finite real number). If $f'(x)$ exists, we say that $f$ is differentiable at $x$.*

*Let $f : \mathbb{R} \to \mathbb{R}$ be a real function.*
*The* **right-hand derivative** *of $f$ is defined as the right-hand limit:*

$$f'_{+}(x_0) = \lim_{x \to x_0^{+}} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0^{+}} \frac{f(x_0 + h) - f(x_0)}{h}.$$

*If the* **right-hand derivative** *exists, then $f$ is said to be* **right-hand differentiable** *at $x_0$.*

*Let $f : \mathbb{R} \to \mathbb{R}$ be a real function.*
*The* **left-hand derivative** *of $f$ is defined as the left-hand limit:*

$$f'_{-}(x_0) = \lim_{x \to x_0^{-}} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0^{-}} \frac{f(x_0 + h) - f(x_0)}{h}.$$

*If the* **left-hand derivative** *exists, then $f$ is said to be* **left-hand differentiable** *at $x_0$.*

# The Derivative

**Definition** Let $I \subseteq \mathbb{R}$ be an interval, let $f : I \to \mathbb{R}$, and let $c \in I$. We say that a real number $L$ is the **derivative of $f$ at $c$** if given any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $x \in I$ satisfies $0 < |x - c| < \delta(\varepsilon)$, then

$$\left| \frac{f(x) - f(c)}{x - c} - L \right| < \varepsilon. \tag{1}$$

In this case we say that $f$ is **differentiable** at $c$, and we write $f'(c)$ for $L$.

In other words, the derivative of $f$ at $c$ is given by the limit

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \tag{2}$$

provided this limit exists. (We allow the possibility that $c$ may be the endpoint of the interval.)

Whenever the derivative of $f : I \to \mathbb{R}$ exists at a point $c \in I$, its value is denoted by $f'(c)$. In this way we obtain a function $f'$ whose domain is a subset of the domain of $f$. In working with the function $f'$, it is convenient to regard it also as a function of $x$. For example, if $f(x) := x^2$ for $x \in \mathbb{R}$, then at any $c$ in $\mathbb{R}$ we have

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{x^2 - c^2}{x - c} = \lim_{x \to c} (x + c) = 2c.$$

Thus, in this case, the function $f'$ is defined on all of $\mathbb{R}$ and $f'(x) = 2x$ for $x \in \mathbb{R}$.

Ex: show that if $f(x) = ax + b$ then $f'(x) = a$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{a(x+h) + b - (ax+b)}{h}$$

$$= \lim_{h \to 0} \frac{a \cdot h}{h} = a.$$

Ex: $y = x^n = f(x)$ is given $\quad y' = f'(x) = ?$

$$f'(x) = D_x f(x) = D_x x^n = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\overbrace{\binom{n}{0}}^{\to 1} x^n + \underbrace{\binom{n}{1}}_{n} x^{n-1} \cdot \Delta x + \binom{n}{2} x^{n-2} \cdot \Delta x^2 + \dots + \binom{n}{n} \Delta x^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left[ n \cdot x^{n-1} + \binom{n}{2} \overbrace{\cancel{x^{n-2} \cdot \Delta x}}^{\to 0} + \dots + \overbrace{\cancel{\binom{n}{n} \Delta x^{n-1}}}^{\to 0} \right] \Delta x}{\Delta x}$$

$$= n \cdot x^{n-1}$$

Ex 1 $y=\sqrt{x}$ , $y'=?$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x} \quad \left[\frac{0}{0}\right]$$

$$\lim_{\Delta x \to 0} \frac{x+\Delta x - x}{\Delta x(\sqrt{x+\Delta x}+\sqrt{x})} = \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}}{\cancel{\Delta x}(\sqrt{x+\Delta x}-\sqrt{x})} = \frac{1}{2\sqrt{x}}$$

Ex : $y=x^2$ $y'=?$

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cancel{x^2}+2\Delta x.x+\Delta x^2-\cancel{x^2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}(2x+\Delta x)}{\cancel{\Delta x}} = 2x.$$

Ex: $\lim\limits_{h\to 0} \dfrac{\psi(x^2+ah)-\psi(x^2+bh)}{h} = ?$

$$\lim_{h\to 0} \frac{\psi(x^2+ah)}{h} - \lim_{h\to 0} \frac{\psi(x^2+bh)}{h} =$$

$$= \lim_{h\to 0} \frac{\psi(x^2+ah)-\psi(x^2)+\psi(x^2)}{h} - \lim_{h\to 0} \frac{\psi(x^2+bh)-\psi(x^2)+\psi(x^2)}{h}$$

$$= \lim_{\substack{h\to 0\\(ah)\to 0}} \frac{a[\psi(x^2+ah)-\psi(x^2)]}{ah} + \cancel{\lim_{h\to 0} \frac{\psi(x^2)}{h}} - \lim_{h\to 0} \frac{b[\psi(x^2+bh)-\psi(x^2)]}{bh} - \cancel{\lim_{h\to 0} \frac{\psi(x^2)}{h}}$$

$$= a\psi'(x^2) - b\psi'(x^2) = (a-b)\psi'(x^2)$$

Ex: $f: R \to R$ $f(x) = |x|$ . Examine the derivative of the given function at $x_0 = 0$.

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x - 0}{x - 0} = 1$$

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{-x - 0}{x - 0} = -1$$

The derivative does not exist at "0" for the given function $f(x) = |x|$

**Theorem** *If $f : I \to \mathbb{R}$ has a derivative at $c \in I$, then $f$ is continuous at $c$.*

***Proof.*** For all $x \in I$, $x \neq c$, we have

$$f(x) - f(c) = \left(\frac{f(x) - f(c)}{x - c}\right)(x - c).$$

Since $f'(c)$ exists, we may apply Theorem concerning the limit of a product to conclude that

$$\begin{aligned}\lim_{x \to c}(f(x) - f(c)) &= \lim_{x \to c}\left(\frac{f(x) - f(c)}{x - c}\right)\left(\lim_{x \to c}(x - c)\right)\\ &= f'(c) \cdot 0 = 0.\end{aligned}$$

Therefore, $\lim_{x \to c} f(x) = f(c)$ so that $f$ is continuous at $c$.

The continuity of $f : I \to \mathbb{R}$ at a point does not assure the existence of the derivative at that point. For example, if $f(x) := |x|$ for $x \in \mathbb{R}$, then for $x \neq 0$ we have $(f(x) - f(0))/(x - 0) = |x|/x$, which is equal to 1 if $x > 0$, and equal to $-1$ if $x < 0$. Thus the limit at 0 does not exist , and therefore the function is not differentiable at 0. Hence, continuity at a point $c$ is *not* a sufficient condition for the derivative to exist at $c$.

Theorem: Let $I \subseteq \mathbb{R}$ be an interval, let $x_0 \in I$, and let $f: I \to \mathbb{R}$ and $g: I \to \mathbb{R}$ be functions that are differentiable at $x_0$. Then

a) if $\alpha \in \mathbb{R}$, then the function $\alpha f$ is differentiable at $x_0$ and

$$(\alpha f)'(x_0) = \alpha f'(x_0)$$

b) The function $f+g$ is differentiable at $x_0$ and

$$(f+g)'(x_0) = f'(x_0) + g'(x_0)$$

c) The function $f \cdot g$ is differentiable at $x_0$ and

$$(f \cdot g)'(x_0) = f'(x_0) \cdot g(x_0) + f(x_0) \cdot g'(x_0)$$

d) if $g(x_0) \neq 0$, then the function $f/g$ is differentiable at $x_0$ and

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)\,g(x_0) - f(x_0)\,g'(x_0)}{[g(x_0)]^2}$$

Proof: $(f+g)'(x) = (f'+g')(x)$

$(f+g)'(x_0) =$

b) $\lim\limits_{\Delta x \to 0} \frac{(f+g)(x_0+\Delta x) - (f+g)(x_0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{f(x_0+\Delta x) + g(x_0+\Delta x) - f(x_0) - g(x_0)}{\Delta x}$

$= \lim\limits_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} + \lim\limits_{\Delta x \to 0} \frac{g(x_0+\Delta x) - g(x_0)}{\Delta x}$

$= f'(x_0) + g'(x_0) = (f'+g')(x_0)$

c) $$\lim_{\Delta x \to 0} \frac{(f\cdot g)(x_0+\Delta x) - (f\cdot g)(x_0)}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)\cdot g(x_0+\Delta x) + f(x_0+\Delta x)\cdot g(x_0) - f(x_0+\Delta x)\cdot g(x_0) - f(x_0)\, g(x_0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[ f(x_0+\Delta x)\cdot\left(\frac{g(x_0+\Delta x) - g(x_0)}{\Delta x}\right) + g(x_0)\cdot \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} f(x_0+\Delta x)\cdot \lim_{\Delta x \to 0} \frac{g(x_0+\Delta x) - g(x_0)}{\Delta x} + \lim_{\Delta x \to 0} g(x_0)\cdot \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$$

$$= f(x_0)\cdot Dg(x_0) + g(x_0)\cdot Df(x_0) = f(x_0)\cdot g'(x_0) + g(x_0)\cdot f'(x_0)$$

d) $\left(\frac{f}{g}\right)'(x_0) = \lim\limits_{\Delta x \to 0} \dfrac{\frac{f}{g}(x_0+\Delta x) - \frac{f}{g}(x_0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{\frac{f(x_0+\Delta x)}{g(x_0+\Delta x)} - \frac{f(x_0)}{g(x_0)}}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{\dfrac{f(x_0+\Delta x) \cdot g(x_0) - g(x_0+\Delta x) \cdot f(x_0)}{g(x_0+\Delta x) \cdot g(x_0)}}{\Delta x}$$

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$$= \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) \cdot g(x_0) - f(x_0) g(x_0) + f(x_0) g(x_0) - g(x_0+\Delta x) \cdot f(x_0)}{\Delta x \cdot g(x_0+\Delta x)\, g(x_0)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{g(x_0+\Delta x) \cdot g(x_0)} \cdot \left[ \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} \cdot g(x_0) - f(x_0) \frac{g(x_0+\Delta x) - g(x_0)}{\Delta x} \right]$$

$$= \frac{1}{g^2(x_0)} \cdot \left( f'(x_0) \cdot g(x_0) - f(x_0) \cdot g'(x_0) \right)$$

* Derivative of the constant function is zero.

$a \in \mathbb{R}$ , let $f(x) = a$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{a - a}{x - x_0} = \lim_{x \to x_0} 0 = 0$$

for $\forall a \in \mathbb{R}$; $(a)' = 0$

$$D f(x) = D \sin x = (\sin x)' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x}$$

$$\sin a - \sin b = 2 \sin\left(\frac{a-b}{2}\right) \cos\left(\frac{a+b}{2}\right)$$

$$= \lim_{\Delta x \to 0} \frac{2 \sin\left(\frac{x+\Delta x - x}{2}\right) \cos\left(\frac{x+\Delta x + x}{2}\right)}{\Delta x}$$

$$\cos a - \cos b = 2 \sin\left(\frac{a+b}{2}\right) . \sin\left(\frac{a-b}{2}\right)$$

$$= \lim_{\Delta x \to 0} \frac{2 \sin \frac{\Delta x}{2} \cdot \cos\left(\frac{2x+\Delta x}{2}\right)}{\Delta x}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \frac{\Delta x}{2} \to 0}} \frac{\cancel{2} \sin \frac{\Delta x}{2}}{\cancel{2} \cdot \frac{\Delta x}{2}} \to 1 \circ \lim_{\Delta x \to 0} \cos\left(\frac{2x+\Delta x}{2}\right)$$

$$= \cos x$$

$$D(\sec x) = (\sec x)' = \lim_{\Delta x \to 0} \frac{\frac{1}{\cos(x+\Delta x)} - \frac{1}{\cos x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cos x - \cos(x+\Delta x)}{\Delta x . \cos x . \cos(x+\Delta x)}$$

$$-2\sin\left(\frac{2x+\Delta x}{2}\right)\sin\left(-\frac{\Delta x}{2}\right)$$

$$= \lim_{\Delta x \to 0} \frac{1}{\cos x . \cos(x+\Delta x)} \cdot \frac{\cancel{2} \sin \frac{\Delta x}{2}}{\cancel{2} \frac{\Delta x}{2}} \cdot \sin\left(x+\frac{\Delta x}{2}\right) \quad \to 1$$

$$\frac{\cancel{2}\ \sin \frac{\Delta x}{2}}{\cancel{2} . \frac{\Delta x}{2}} \cdot \sin\left(x+\frac{\Delta x}{2}\right) = \frac{\sin x}{\cos^2 x}$$

$y = \tan x \quad , \quad y' = ?$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\dfrac{\sin(x+\Delta x)}{\cos(x+\Delta x)} - \dfrac{\sin x}{\cos x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x).\cos x - \sin x.\cos(x+\Delta x)}{\Delta x.\ \cos(x+\Delta x).\cos x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} \cdot \lim_{\Delta x \to 0} \frac{1}{\cos(x+\Delta x).\cos x}$$

$$= \frac{1}{\cos^2 x}$$

$$\log_a a = 1$$

$$\log_a 1 = 0$$

$$\log_a (x \cdot t) = \log_a x + \log_a t$$

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

$$\log_a x^n = n \cdot \log_a x$$

$$\log_a x = \frac{1}{\log_x a}$$

$y = \ln x$ , $y' = ?$ $\qquad \lim\limits_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$

$$y' = \lim_{\Delta x \to 0} \frac{\ln(x+\Delta x) - \ln x}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \cdot \ln\left(\frac{x+\Delta x}{x}\right)$$

$$= \lim_{\Delta x \to 0} \ln\left(1 + \frac{\Delta x}{x}\right)^{\frac{1}{\Delta x}} = \lim_{\Delta x \to 0} \frac{1}{x} \cdot \ln\left(1 + \frac{1}{\frac{x}{\Delta x}}\right)^{\frac{x}{\Delta x}}$$

$$= \lim_{\Delta x \to 0} \frac{1}{x} \cdot \ln e = \frac{1}{x}$$

Given the function

$$f(x) = \begin{cases} x+2, & \text{for } \; x < 1 \\ ax^2+b, & \text{for } \; x \geq 1 \end{cases}$$

find the constants $a$ and $b$ such that the function is continuous *and* differentiable at $x = 1$.

**Solution** Notice that

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1-} (x+2) = 3, \quad f(1) = \lim_{x\to 1+} f(x) = \lim_{x\to 1+} (ax^2+b) = a+b$$

Therefore, the function is *continuous* at $x = 1$ if we require that

$$a + b = 3$$

Notice that the derivative at $x = 1$ is defined as

$$f'(1) = \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1}$$

We have

$$f'(1-) = \lim_{x\to 1-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1-} \frac{x+2-(a+b)}{x-1} \overset{use\ a+b=3}{=} \lim_{x\to 1-} \frac{x-1}{x-1} = 1$$

and

$$\begin{aligned} f'(1+) &= \lim_{x\to 1+} \frac{f(x)-f(1)}{x-1} \\ &= \lim_{x\to 1+} \frac{ax^2+b-(a+b)}{x-1} = \lim_{x\to 1+} \frac{a(x^2-1)}{x-1} = \lim_{x\to 1+} \frac{a(x-1)(x+1)}{x-1} = 2a \end{aligned}$$

For the function to be *differentiable* at $x = 1$ we require that $f'(1-) = f'(1+)$,

$$2a = 1 \quad thus \quad a = \frac{1}{2}$$

Then, since $a + b = 3$ we get the value of $b$ as

$$b = 3 - \frac{1}{2} = \frac{5}{2}$$

Find the values of $a$ that makes the following function *differentiable* for all $x$-values.

$$f(x)=\begin{cases} ax & x<0 \\ x^2-3x & x\geq 0 \end{cases}$$

Assuming that the given function $f(x)$ is differentiable for all $x$-values, we only need to worry about the differentiability at $x=0$. We then must have right-hand and left-hand derivatives must exist and equal each other at $x=0$. The right-hand derivative is

$$\begin{aligned} f'_+(0) &= \lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h} \\ &= \lim_{h\to 0^+}\frac{(0+h)^2-3(0+h)-0}{h} \\ &= \lim_{h\to 0^+}\frac{h^2-3h}{h} \\ &= \lim_{h\to 0^+}\frac{\cancel{h}(h-3)}{\cancel{h}} \\ &= \lim_{h\to 0^+}(h-3)=-3 \end{aligned}$$

The left-hand derivative is

$$\begin{aligned} f'_-(0) &= \lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h} \\ &= \lim_{h\to 0^-}\frac{a(0+h)-0}{h} \\ &= \lim_{h\to 0^-}\frac{ah}{h} \\ &= \lim_{h\to 0^-}\frac{\cancel{h}(a)}{\cancel{h}} \\ &= \lim_{h\to 0^-}(a)=a \end{aligned}$$

Hence equality forces $\boxed{a=-3}$. Conversely, when $a=-3$, the uniquely determined function

$$\boxed{f(x)=\begin{cases} -3x & x<0 \\ x^2-3x & x\geq 0 \end{cases}}$$

**Notation** If $I \subseteq \mathbb{R}$ is an interval and $f : I \to \mathbb{R}$, we have introduced the notation $f'$ to denote the function whose domain is a subset of $I$ and whose value at a point $c$ is the derivative $f'(c)$ of $f$ at $c$. There are other notations that are sometimes used for $f'$; for example, one sometimes writes $Df$ for $f'$.

$$D(f+g) = Df + Dg, \qquad D(fg) = (Df)\cdot g + f\cdot(Dg).$$

When $x$ is the "independent variable," it is common practice in elementary courses to write $df/dx$ for $f'$. Thus formula is sometimes written in the form

$$\frac{d}{dx}(f(x)g(x)) = \left(\frac{df}{dx}(x)\right)g(x) + f(x)\left(\frac{dg}{dx}(x)\right).$$

This last notation, due to Leibniz, has certain advantages. However, it also has certain disadvantages and must be used with some care.

**Chain Rule** *Let $I, J$ be intervals in $\mathbb{R}$, let $g : I \to \mathbb{R}$ and $f : J \to \mathbb{R}$ be functions such that $f(J) \subseteq I$, and let $c \in J$. If $f$ is differentiable at $c$ and if $g$ is differentiable at $f(c)$, then the composite function $g \circ f$ is differentiable at $c$ and*

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c).$$

Theorem: (Chain Rule)

if $f(x), g(x) \in D[a,b]$,

$$D(f \circ g)(x) = Df(g(x)) = f'(g(x)) \cdot \underbrace{g'(x)}_{\text{derivative of } x} \cdot 1$$

Proof:

$$D(f \circ g)(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x}$$

$$g(x+\Delta x) - g(x) = \Delta(g(x))$$
$$g(x+\Delta x) = \Delta(g(x)) + g(x)$$

$$= \lim_{\Delta x \to 0} \frac{f[g(x) + \Delta g(x)] - f(g(x))}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left\{ \frac{f[g(x) + \Delta g(x)] - f(g(x))}{\Delta g(x)} \cdot \frac{\Delta g(x)}{\Delta x} \right\}$$

$\Delta x \to 0$, $\Delta g(x) \to 0$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta g(x) \to 0}} \frac{f[g(x) + \Delta g(x)] - f(g(x))}{\Delta g(x)} \cdot \lim_{\Delta x \to 0} \frac{\Delta g(x)}{\Delta x}$$

$$= f'(g(x)) \cdot g'(x) \cdot 1$$

**Examples** (a) If $f : I \to \mathbb{R}$ is differentiable on $I$ and $g(y) := y^n$ for $y \in \mathbb{R}$ and $n \in \mathbb{N}$, then since $g'(y) = ny^{n-1}$, it follows from the Chain Rule

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x) \quad \text{for} \quad x \in I.$$

Therefore we have $(f^n)'(x) = n(f(x))^{n-1} f'(x)$ for all $x \in I$ as was seen

(b) Suppose that $f : I \to \mathbb{R}$ is differentiable on $I$ and that $f(x) \neq 0$ and $f'(x) \neq 0$ for $x \in I$. If $h(y) := 1/y$ for $y \neq 0$, then it is an exercise to show that $h'(y) = -1/y^2$ for $y \in \mathbb{R}$, $y \neq 0$. Therefore we have

$$\left(\frac{1}{f}\right)'(x) = (h \circ f)'(x) = h'(f(x))f'(x) = -\frac{f'(x)}{(f(x))^2} \quad \text{for} \quad x \in I.$$

Ex: $f(x) = \sin(x^2)$ , $f'(x) = ?$

Let $g(x) = x^2$ ; $h(x) = \sin x$

$(h \circ g)(x) = h(g(x)) = h(x^2) = \sin x^2$

$f'(x) = (h \circ g)'(x) = h'(g(x)) \cdot g'(x)$

$= \cos(g(x)) \cdot 2x$

$= \cos x^2 \cdot 2x = \underbrace{2x \cdot \cos x^2}$

or ;

Let $x^2 = u$ ; $f(u) = \sin u$

$f'(x) = \frac{df}{dx} = \frac{df}{du} \frac{du}{dx} = \cos u \cdot 2x = \underbrace{2x \cdot \cos x^2}$

Ex: $y = \tan^5(2x-1)$ $\quad y' = ?$

let $2x-1 = v$ , $\tan v = u$

$y = u^5$ . so

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx} = 5u^4 \cdot \frac{1}{\cos^2 v} \cdot 2 = 10(\tan v)^4 \cdot \frac{1}{\cos^2 v}$$

$$= 10 \tan^4(2x-1) \cdot \frac{1}{\cos^2(2x-1)}$$

Suppose that a function $f(x)$ has derivative $f'(x) = e^{x^2}$ and

$$g(x) = f(\sqrt{\sin x}).$$

Compute $g'(x)$

**Solution.**

- By the chain rule,

$$g'(x) = f'(\sqrt{\sin x}) \cdot \frac{1}{2\sqrt{\sin x}} \cdot \cos x.$$

- Using the expression for $f'(x)$ we then get that

$$g'(x) = e^{\sin x} \cdot \frac{1}{2\sqrt{\sin x}} \cdot \cos x.$$

Suppose $y = \tan^2(\sec(3t))$. Find the derivative $\dfrac{dy}{dt}$

$$= \boxed{6\tan(\sec(3t))\sec^2(\sec(3t))\sec(3t)\tan(3t)}$$

Suppose that $f$ is defined by

$$f(x) := \begin{cases} x^2 \sin(1/x) & \text{for} \quad x \neq 0, \\ 0 & \text{for} \quad x = 0. \end{cases}$$

If we use the fact that $D \sin x = \cos x$ for all $x \in \mathbb{R}$ and apply the Product Rule and the Chain Rule we obtain (why?)

$$f'(x) = 2x \sin(1/x) - \cos(1/x) \quad \text{for} \quad x \neq 0.$$

If $x = 0$, none of the calculational rules may be applied. (Why?) Consequently, the derivative of $f$ at $x = 0$ must be found by applying the definition of derivative. We find that

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin(1/x)}{x} = \lim_{x \to 0} x \sin(1/x) = 0.$$

Hence, the derivative $f'$ of $f$ exists at all $x \in \mathbb{R}$. However, the function $f'$ does not have a limit at $x = 0$ (why?), and consequently $f'$ is discontinuous at $x = 0$. Thus, a function $f$ that is differentiable at every point of $\mathbb{R}$ need not have a continuous derivative $f'$.

- Derivative operator is linear.

$$D\left(k_1 f(x) + k_2 g(x)\right) = k_1 D f(x) + k_2 D g(x)$$

Ex: $f(x) = \sqrt{x} - x^2$, $\quad (2D^2 - D + 3) f(x) = ?$

$$2D^2 f(x) - D f(x) + 3 f(x) = 2D^2(\sqrt{x} - x^2) - D(\sqrt{x} - x^2) + 3(\sqrt{x} - x^2)$$

$$= 2D^2\sqrt{x} - 2D^2x^2 - D\sqrt{x} + Dx^2 + 3\sqrt{x} - 3x^2$$

$$= 2\left(-\frac{1}{4x\sqrt{x}}\right) - 2(2) - \frac{1}{2\sqrt{x}} + 2 + 3\sqrt{x} - 3x^2$$

$$= -\frac{1}{2x\sqrt{x}} - 4 - \frac{1}{2\sqrt{x}} + 3\sqrt{x} - 3x^2 + 2$$

$F(x,y)=0$ is given. $y'=?$

$$dF(x,y)=\frac{dF(x,y)}{dx}\cdot\frac{dx}{dx}+\frac{dF(x,y)}{dy}\cdot\frac{dy}{dx}=0$$

$$y'=-\frac{\frac{dF(x,y)}{dx}}{\frac{dF(x,y)}{dy}}=-\frac{F_x}{F_y}$$

Assuming that $y$ is defined implicitly by the equation $x^3 \sin y + y = 4x + 3$, find $\frac{dy}{dx}$.

$\frac{d}{dx}(x^3 \sin y + y) = \frac{d}{dx}(4x+3)$

$\frac{d}{dx}(x^3 \sin y) + \frac{d}{dx}(y) = 4$

$(\frac{d}{dx}(x^3) \cdot \sin y + \frac{d}{dx}(\sin y) \cdot x^3) + \frac{dy}{dx} = 4$

$3x^2 \sin y + (\cos y \frac{dy}{dx}) \cdot x^3 + \frac{dy}{dx} = 4$

$x^3 \cos y \frac{dy}{dx} + \frac{dy}{dx} = 4 - 3x^2 \sin y$

$\frac{dy}{dx}(x^3 \cos y + 1) = 4 - 3x^2 \sin y$

$\frac{dy}{dx} = \frac{4-3x^2 \sin y}{x^3 \cos y + 1}$

Step 1: Differentiate both sides of the equation.

Step 1.1: Apply the sum rule on the left. On the right, $\frac{d}{dx}(4x+3) = 4$.

Step 1.2: Use the product rule to find $\frac{d}{dx}(x^3 \sin y)$. Observe that $\frac{d}{dx}(y) = \frac{dy}{dx}$.

Step 1.3: We know $\frac{d}{dx}(x^3) = 3x^2$. Use the chain rule to obtain $\frac{d}{dx}(\sin y) = \cos y \frac{dy}{dx}$.

Step 2: Keep all terms containing $\frac{dy}{dx}$ on the left. Move all other terms to the right.

Step 3: Factor out $\frac{dy}{dx}$ on the left.

Step 4: Solve for $\frac{dy}{dx}$ by dividing both sides of the equation by $x^3 \cos y + 1$.

Ex: $x^3 + y^3 + 2xy = 0 \qquad y'_x = ?$

$$3x^2 + 3y^2 \cdot y'_x + 2y + 2xy'_x = 0$$

$$y'_x(3y^2 + 2x) = -3x^2 - 2y$$

$$y'_x = -\frac{3x^2 + 2y}{3y^2 + 2x}$$

Ex: $f(x) = x^{2/3} \qquad y' = ?$

$$y = x^{2/3} \Rightarrow y^3 = x^2$$

$$3y^2 \cdot y'_x = 2x$$

$$y'_x = \frac{2x}{3y^2} = \frac{2x}{3 \cdot (x^{2/3})^2} = \frac{2x}{3 \cdot x^{4/3}}$$

$$y'_x = \frac{2}{3} x^{-1/3}$$

Ex: if $f(x)=\sqrt{2x+3}$ $\quad f(x).f'(x)=?$

$$f(x)=\sqrt{2x+3} \Rightarrow f^2(x)=2x+3$$

$$2f(x).f'(x)=2$$

$$f(x).f'(x)=1$$

Higher Order Derivatives.

$f'(x) \quad y' \quad D_x(f) \quad \frac{dy}{dx}$

$f''(x) \quad y'' \quad D_x^2(f) \quad \frac{d^2y}{dx^2} \quad \frac{d^2}{dx^2}(y)$

$\vdots \quad \vdots$

$f^{(n)}(x) \quad y^{(n)} \quad D_x^n(f) \quad \frac{d^ny}{dx^n}$

Ex: Find $\frac{dy}{dx}$ for $y = \sin\left(\frac{\pi}{x}\right)$

$$y' = \left(\frac{\pi}{x}\right)' \cdot \cos\left(\frac{\pi}{x}\right) = \left(-\pi x^{-2}\right) \cdot \cos\left(\frac{\pi}{x}\right)$$

Ex: $D_x^5(\underbrace{3x^4 - 2x^3 + x^2 - 4}_{y}) = ?$ $\quad y^5 = 0$

Use implicit differentiation to find $d^2y/dx^2$:

$$x^2 = 2 - \frac{3}{y}$$

$$2x = 0 + \frac{3}{y^2}\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2}{3}xy^2$$

$$\frac{d^2y}{dx^2} = \frac{2}{3}x\left(2y\frac{dy}{dx}\right) + \frac{2}{3}y^2$$

$$= \frac{4}{3}xy\left(\frac{2}{3}xy^2\right) + \frac{2}{3}y^2$$

$$= \frac{8}{9}x^2y^3 = \frac{2}{3}y^2$$

$$x^2 - x = 3\sin y$$

$$2x - 1 = 3\cos y\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2x-1}{3\cos y}$$

$$\frac{d^2y}{dx^2} = \frac{(3\cos y)(2) - (2x-1)(-3\sin y)\left(\frac{dy}{dx}\right)}{9\cos^2 y}$$

$$= \frac{(3\cos y)(2) - (2x-1)(-3\sin y)\left(\frac{2x-1}{3\cos y}\right)}{9\cos^2 y}$$

$$= \frac{6\cos y + (2x-1)^2\tan y}{9\cos^2 y}$$

Ex: Find a formula for $D_x^n\left(\frac{1}{x}\right)$.

$y = \frac{1}{x}$

$y' = -x^{-2}$

$y'' = -(-2)x^{-3} = 2x^{-3}$

$y''' = -3!\,x^{-4}$

$y^{IV} = 4!\,x^{-5}$

$y^{5} = -5!\,x^{-6}$

$\vdots$

$y^{20} = (-1)^{20} \cdot 20!\,x^{-21}$

$y^{(n)} = (-1)^{n}\,n!\,x^{-n-1}$

At what point does the tangent line to the curve

$$2x^2 + y^3 = x^3 + y^2$$

at $(x, y) = (2, 1)$ intersect the $x$-axis?

**Solution.**

- Differentiating the equation of the curve with respect to $x$, we get that

$$4x + 3y^2 \frac{dy}{dx} = 3x^2 + 2y \frac{dy}{dx}.$$

- Evaluation of this equation at $(x, y) = (2, 1)$ gives

$$8 + 3\frac{dy}{dx} = 12 + 2\frac{dy}{dx},$$

  so the slope of the tangent line is

$$\frac{dy}{dx} = 4.$$

- The equation of the tangent line is

$$y - 1 = 4(x - 2).$$

- The line intersect the $x$-axis when $y = 0$, so $x = 7/4$ and the point of intersection is

$$(x, y) = (7/4, 0).$$

Given the function

$$f(x) = x^2 - \sqrt{x} + 2$$

- find the value of the derivative $f'(1)$
- write the equation of the tangent line at $a = 1$.

**Solution** Notice that $\sqrt{x} = x^{\frac{1}{2}}$. Use the power rule to obtain the derivative function

$$f'(x) = 2x - \frac{1}{2\sqrt{x}}$$

Thus, at $x = 1$,

$$f'(1) = 2 - \frac{1}{2} = \frac{3}{2}$$

The equation of the tangent line at $a = 1$ is

$$y = f(1) + f'(1) \cdot (x - 1)$$

Notice that $f(1) = 1 - 1 + 2 = 2$.

Therefore, the equation of the tangent line at $a = 1$ is

$$y = 2 + \frac{3}{2} \cdot (x - 1)$$

which may be also written as

$$y = \frac{3}{2}x + \frac{1}{2}$$

Calculate $y'$ in the following expressions using the rules for differentiation i.e. derivatives.

SOLUTION. The easiest is by logarithmic differentiation.

(a) $y = \left[\dfrac{(1+x)(x^2+1)}{(x^3+1)(x^4+1)}\right]^{\frac{1}{3}}$ at $x = 1$ $\rightarrow$ $\log y = \dfrac{1}{3}\left[\log(x+1) + \log(x^2+1) - \log(x^3+1) - \log(x^4+1)\right]$

Taking the derivatives of both sides with respect to x:

$$\frac{y'}{y}\bigg|_{x=1} = \frac{1}{3}\left[\frac{1}{(1+x)} + \frac{2x}{(x^2+1)} - \frac{3x^2}{(x^3+1)} - \frac{4x^3}{(x^4+1)}\right]\bigg|_{x=1} = \frac{1}{3}\left[\frac{1}{2} + \frac{2}{2} - \frac{3}{2} - \frac{4}{2}\right] \rightarrow$$

$$\frac{y'}{y}\bigg|_{x=1} = \frac{-2}{3} \qquad y\bigg|_{x=1} = \left[\frac{(1+x)(x^2+1)}{(x^3+1)(x^4+1)}\right]^{\frac{1}{3}}\bigg|_{x=1} = \left[\frac{(2)(2)}{(2)(2)}\right]^{\frac{1}{3}} = 1$$

$$y'\bigg|_{x=1} = \frac{-2}{3}$$

(b) $xy^2 + x^3 \sin y + xy \log(y^2+1) + e^{-y} \cos x = 1 + x \quad \text{at } x = 0$

(Hint: You must calculate the value of $y$ for $x = 0$ from the equation.)

SOLUTION. The only way is by implicit function differentiation.

First, set x = 0 in order to find the corresponding $y$:

$$0 + 0 + 0 + e^{-y}\, 1 = 1 \quad \rightarrow \quad e^{-y} = 1 \quad \rightarrow \quad y = 0$$

Now differentiate implicitly :

$$\Big(y^2 + x(2yy')\Big) + \Big(3x^2 \sin y + x^3 \cos y\; y'\Big) + \Big(y \log(y^2+1) + xy' \log(y^2+1) + xy \frac{2y}{(y^2+1)}\Big)$$

$$+ \Big(-y'\, e^{-y} \cos x \;-\; e^{-y} \sin x\Big)\Big|_{x=0, y=0} = 0 + 1 \quad \rightarrow$$

$$\Big(0+0\Big) + \Big(0 + 0\, y'\Big) + \Big(0+0+0\Big) + \Big(-y'\Big|_{x=0,y=0} - 0\Big) = 1 \quad \rightarrow \quad -y'\Big|_{x=0,y=0} = 1 \quad \rightarrow$$

$$y'\Big|_{x=0,y=0} = -1$$

Use the Intermediate Value Theorem to find an interval of length 1 containing a root of the equation

$$2^x - x^3 = 0$$

**Solution** Since

$$f(x) = 2^x - x^3$$

is a continuous function, to apply the IVT all we need is to find an interval $(a, b)$ such that $b - a = 1$ (that is of length 1) and

$$f(a) \cdot f(b) < 0$$

Notice that

$$f(1) = 2 - 1 = 1 > 0, \qquad f(2) = 2^2 - 2^3 = 4 - 8 = -4 < 0$$

Thus (1, 2) is an interval of length 1 containing a root (solution) of the equation