(1)

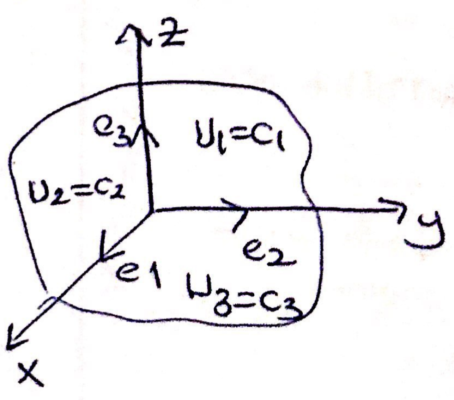
Eğrisel Koordinatlar:

Dik koordinat sisteminde bir P noktası, birbirine dik olan x, y, z sabit düzlemleri üzerinde hareket eden ve bu noktanın konumu (x, y, z) sayı ölçüsüyle tanımlanır.

Fizik ve matematikte kullanılan ve sistemin geometrisine uygun olarak tanımlanan küresel silindirik parabolik, elipsoidal... koordinat yüzeyleri kartezyen koord. sistemindeki gibi bir düzlem olarak zorunda değildir.

Bu nedenle, kartezyen koordinatlar dışında tanımlanan bu sistemlere "eğrisel koordinat sistemleri" denir.

Bir birlerine dik olmaları gerekmeyen ve ortak P noktaları olan bu yüzeylere, yani c_1, c_2, c_3 sabit olarak üzere $u_1 = c_1$, $u_2 = c_2$, $u_3 = c_3$ yüzeylerine "koordinat yüzeyleri" denir.



Koordinat düzlemleri birbirini dik kesiyorsa, eğrisel koordinat sistemine ortogonal eğrisel koordinatlar denir

Ölçek Çarpanı ve Diferansiyel eleman

$x = x(u_1, u_2, u_3)$
$y = y(u_1, u_2, u_3)$
$z = z(u_1, u_2, u_3)$

ve ters dönüşümler

$u_1 = u_1(x, y, z)$
$u_2 = u_2(x, y, z)$
$u_3 = u_3(x, y, z)$ ise

Her hangi bir P noktasının yer değiştirme vektörü

$$\vec{r} = \underbrace{x\vec{i} + y\vec{j} + z\vec{k}}_{\text{kartezyen koord. sist}} \Rightarrow \vec{r} = \underbrace{x(u_1,u_2,u_3)\vec{i} + y(u_1,u_2,u_3)\vec{j} + z(u_1,u_2,u_3)\vec{k}}_{\text{eğrisel koord. sist.}}$$

şeklinde ve $d\vec{r}$ diferansiyel ötelemesi de

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k} \quad (*)$$

$$\begin{cases} dx = \dfrac{\partial x}{\partial u_1} du_1 + \dfrac{\partial x}{\partial u_2} du_2 + \dfrac{\partial x}{\partial u_3} du_3 \\ dy = \dfrac{\partial y}{\partial u_1} du_1 + \dfrac{\partial y}{\partial u_2} du_2 + \dfrac{\partial y}{\partial u_3} du_3 \\ dz = \dfrac{\partial z}{\partial u_1} du_1 + \dfrac{\partial z}{\partial u_2} du_2 + \dfrac{\partial z}{\partial u_3} du_3 \end{cases}$$

eşitlikleri (*) da yazılırsa

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u_1} du_1 + \frac{\partial \vec{r}}{\partial u_2} du_2 + \frac{\partial \vec{r}}{\partial u_3} du_3$$

$$= \sum_{i=1}^{3} \frac{\partial \vec{r}}{\partial u_i} du_i \quad \text{olur.}$$

③

u_i koordinat eğrisine P noktasında teğet olan vektör $\gamma_i = \frac{\partial \vec{r}}{\partial u_i}$ olarak tanımlanırsa, bu doğrultudaki birim vektör

$$\boxed{\vec{e_i} = \frac{\vec{\gamma_i}}{|\vec{\gamma_i}|}} = \frac{\frac{\partial \vec{r}}{\partial u_i}}{\left|\frac{\partial r}{\partial u_i}\right|} = \frac{\frac{\partial \vec{r}}{\partial u_i}}{h_i} \text{ olur.}$$

ve

$$\frac{\partial \vec{r}}{\partial u_i} = h_i \vec{e_i} \text{ olur.}$$

O zaman $d\vec{r}$ diferansiyel elemanı

$$d\vec{r} = \sum_{i=1}^{3} h_i \vec{e_i}\, du_i = h_1 e_1 du_1 + h_2 du_2 \vec{e_2} + h_3 du_3 e_3$$

$$\boxed{dr = h_1 du_1 \vec{e_1} + h_2 du_2 \vec{e_2} + h_3 du_3 \vec{e_3}}$$

$h_1, h_2, h_3 \rightarrow$ ölçek çarpanı

Örn: Silindirik koordinat sisteminin ortogonal old. ispatlayın.

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = \rho\cos\theta\,\vec{i} + \rho\sin\theta\,\vec{j} + z\vec{k}$$

$$\left.\begin{aligned} x &= \rho\cos\theta \\ y &= \rho\sin\theta \\ z &= z \end{aligned}\right\}$$

$$\frac{\partial \vec{r}}{\partial \rho} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}, \quad \frac{\partial r}{\partial \theta} = -\rho\sin\theta\,\vec{i} + \rho\cos\theta\,\vec{j}$$

$$\frac{\partial r}{\partial z} = \vec{k}$$

$$\left.\begin{aligned} e_\rho &= \cos\theta\,\vec{i} + \sin\theta\,\vec{j} \\ e_\theta &= -\sin\theta\, i + \cos\theta\, J \\ e_z &= \vec{k} \end{aligned}\right\} \quad \left.\begin{aligned} e_\rho \cdot e_\theta &= 0 \\ e_\rho \cdot e_z &= 0 \\ e_\theta \cdot e_z &= 0 \end{aligned}\right\} \text{ ortogonal!}$$

(4)

ölçek çarpanları:

$$\frac{\partial \vec{r}}{\partial \rho} = h_\rho \vec{e}_\rho \Rightarrow h_\rho = \frac{\cos\theta i + \sin\theta j}{\cos\theta i + \sin\theta j} = 1$$

$$\frac{\partial \vec{r}}{\partial \theta} = h_\theta \vec{e}_\theta \Rightarrow h_\theta = \frac{-\rho\sin\theta \vec{i} + \rho\cos\theta}{-\sin\theta i + \cos\theta j} = \rho$$

$$\frac{\partial \vec{r}}{\partial z} = h_z \vec{e}_z \Rightarrow h_z = \frac{\vec{k}}{\vec{k}} = 1 \text{ dir.}$$

Örn

Küresel koord. sisteminin ortogonal old. göst.

$x = \rho\cos\theta\sin\phi$

$y = \rho\sin\theta\sin\phi$

$z = \rho\cos\phi$

$$r = \rho\cos\theta\sin\phi\vec{i} + \rho\sin\theta\sin\phi\vec{j} + \rho\cos\phi\vec{k}$$

$$\frac{\partial r}{\partial \rho} = \cos\theta\sin\phi\vec{i} + \sin\theta\sin\phi\vec{j} + \cos\phi\vec{k}$$

$$\frac{\partial r}{\partial \theta} = -\rho\sin\theta\sin\phi\vec{i} + \rho\cos\theta\sin\phi j$$

$$\frac{\partial r}{\partial \phi} = \rho\cos\theta\cos\phi\vec{i} + \rho\sin\theta\cos\phi j - \rho\sin\phi k$$

$$e_\rho = \frac{\cos\theta\sin\phi\vec{i} + \sin\theta\sin\phi j + \cos\phi k}{\sqrt{\cos^2\theta\sin^2\phi + \sin^2\theta\sin^2\phi + \cos^2\phi}}$$

(zaten birim vektör!)

$$e_\theta = \frac{-\rho\sin\theta\sin\phi i + \rho\cos\theta\sin\phi j}{\sqrt{\rho^2\sin^2\theta\sin^2\phi + \rho^2\cos^2\theta\sin^2\phi}}$$

$$= -\sin\theta\sin\phi i + \cos\theta\sin\phi j$$

(5)

$$e_\phi = \frac{\rho\cos\theta\cos\phi\,\vec{i} + \rho\sin\theta\cos\phi\,\vec{j} - \rho\sin\phi\,\vec{k}}{\sqrt{\rho^2\cos^2\theta\cos^2\phi + \rho^2\sin^2\theta\cos^2\phi + \rho^2\sin^2\phi}}$$

$y \qquad \rho$

$$e_\phi = \cos\theta\cos\phi\,\vec{i} + \sin\theta\cos\phi\,\vec{j} - \sin\phi\,\vec{k}$$

$$\begin{aligned} e_\rho \cdot e_\phi &= \cos^2\theta\sin\phi\cos\phi \\ &\quad + \sin^2\theta\sin\phi\cos\phi \\ &\quad - \sin\phi\cos\phi \\ &= \sin\phi\cos\phi(\cos^2\theta + \sin^2\theta) \\ &\quad - \sin\phi\cos\phi \\ &= 0 \checkmark \end{aligned}$$

$$\begin{aligned} e_\rho e_\theta &= -\sin\theta\cos\theta\sin\phi\cos\phi \\ &\quad + \sin\theta\cos\theta\sin\phi\cos\phi = 0 \checkmark \end{aligned}$$

$$\begin{aligned} e_\theta e_\phi &= -\cos\theta\cos\phi\sin\theta\sin\phi \\ &\quad + \sin\theta\cos\theta\sin\phi\cos\phi \\ &= 0 \checkmark \end{aligned}$$

oldışından ortogonaldir. (h_ρ, h_θ, h_ϕ ? bulun)

Özel Ortogonal koordinat sistemleri:

(1) silindirik koordinatlar (ρ, θ, z)

$x = \rho\cos\theta \qquad \rho \geq 0,\ 0 \leq \theta < 2\pi,\ -\infty < z < \infty$

$y = \rho\sin\theta \qquad h_\rho = 1,\ h_\theta = \rho,\ h_z = 1$

$z = z$

(2) Küresel koordinatlar (ρ, θ, ϕ)

$x = \rho\sin\phi\cos\theta = \rho\cos\theta\sin\phi$

$y = \rho\sin\theta\sin\phi \qquad \rho \geq 0 \quad 0 \leq \theta \leq 2\pi$

$z = \rho\cos\phi \qquad 0 \leq \theta \leq \pi$

$h_\rho = 1 \quad h_\theta = \rho$

$h_\phi = \rho\sin\phi$

3. Parabolik Silindirik koordinatlar:

$x = \frac{u^2 - v^2}{2}$ $\quad -\infty < u < \infty, \ v \geq 0, \ -\infty < z < \infty$

$y = uv$ $\quad h_u = h_v = \sqrt{u^2 + v^2}$

$z = z$ $\quad h_z = 1$

4. Paraboloidal Koordinatlar

$x = uv\cos\phi$ $\quad u \geq 0, v \geq 0, \ 0 \leq \phi \leq 2\pi$

$y = uv\sin\phi$

$z = \frac{u^2 - v^2}{2}$ $\quad h_u = h_v = \sqrt{u^2 + v^2}$

$h_\phi = u.v$

5. Eliptik silindirik koordinatlar:

$u \geq 0, \ 0 \leq v \leq 2\pi$

$x = a.\cosh u.\cos v$

$y = a\sinh u \sin v$ $\quad h_u = h_v = a\sqrt{\sin^2 hu + \cos^2 hu}$

$z = z$ $\quad h_z = 1$

6. Bipolar koordinatlar:

$x = \frac{a\sinh v}{\cosh v - \cos u}$, $\quad y = \frac{a\sin u}{\cosh v - \cos u}$, $z = z$

$0 \leq u \leq 2\pi$ $\quad -\infty < v < \infty, \ -\infty < z < \infty$

$h_u = h_v = \frac{a}{\cosh v - \cos u}$, $h_z = 1$

Not: $\vec{A} = A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$ vektörü eğrisel koordinatların birim baz vektörleri ile $\vec{A} = A_1'\vec{e}_1 + A_2'\vec{e}_2 + A_3'\vec{e}_3$ şeklinde gösterilebilir.

⑦

Örn

$\vec{A} = z\vec{i} - 2x\vec{j} + y\vec{k}$ vektörünü silindirik koordinatlar cinsinden ifade edip A_ρ, A_θ, A_z'i bulun.

$x = \rho\cos\theta$

$y = \rho\sin\theta$

$z = z$

$\vec{A} = A_\rho \vec{e}_\rho + A_\theta \vec{e}_\theta + A_z \vec{e}_z$

$\vec{r} = \rho\cos\theta\,\vec{i} + \rho\sin\theta\,\vec{j} + z\vec{k}$

$\frac{\partial r}{\partial \rho} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$

$\frac{\partial r}{\partial \theta} = -\rho\sin\theta\,\vec{i} + \rho\cos\theta\,\vec{j}$

$\frac{\partial r}{\partial z} = \vec{k}$

$$\left.\begin{aligned} \vec{e}_\rho &= \cos\theta\,\vec{i} + \sin\theta\,\vec{j} \\ \vec{e}_\theta &= -\sin\theta\,\vec{i} + \cos\theta\,\vec{j} \end{aligned}\right\} \text{çözelim}$$

$\vec{e}_z = \vec{k}$

$$\frac{\Delta_i}{\Delta} = \vec{i} = \frac{\begin{vmatrix} \vec{e}_\rho & \sin\theta \\ \vec{e}_\theta & \cos\theta \end{vmatrix}}{1} = \cos\theta\,\vec{e}_\rho - \sin\theta\,\vec{e}_\theta$$

$$\frac{\Delta_j}{\Delta} = \vec{j} = \frac{\begin{vmatrix} \cos\theta & e_\rho \\ -\sin\theta & e_\theta \end{vmatrix}}{1} = \cos\theta\,\vec{e}_\theta + \sin\theta\,\vec{e}_\rho$$

$\vec{k} = \vec{e}_z$

$$A = z\vec{i} - 2x\vec{j} + y\vec{k}$$

$$= z(\cos\theta\,\vec{e}_\rho - \sin\theta\,\vec{e}_\theta) - 2\rho\cos\theta(\cos\theta\,\vec{e}_\theta + \sin\theta\,\vec{e}_\rho)$$

$$+ \rho\sin\theta\,\vec{e}_z$$

$$= (z\cos\theta - 2\rho\cos\theta\sin\theta)\vec{e}_\rho$$

$$+ (-\sin\theta - 2\rho\cos^2\theta)\vec{e}_\theta$$

$$+ \rho\sin\theta\,\vec{e}_z$$

$$= A_\rho \vec{e}_\rho + A_\theta \vec{e}_\theta + A_z \vec{e}_z$$

(6)

(1) $g\left(\frac{x}{z}\right)=yz$ ise $x.z_x-yz_y=z$ old. gösterin.

$F: g\left(\frac{x}{z}\right)-yz=0$ (kapalı fonk)

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{\frac{1}{z}g'\left(\frac{x}{z}\right)}{-\frac{x}{z^2}g'\left(\frac{x}{z}\right)-y}=\frac{z.\,g'(x/z)}{xg'(x/z)+yz^2}$$

$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=\frac{-z}{-\frac{x}{z^2}g'(x/z)-y}=\frac{z^3}{xg'(x/z)+yz^2}$$

$$xz_x-yz_y=\frac{xz\,g'(x/z)}{xg'(x/z)+yz^2}+\frac{yz^3}{xg'(x/z)+yz^2}$$

$$=\frac{z\,\cancel{(xg'(x/z)+yz^2)}}{\cancel{xg'(x/z)+yz^2}}=z$$

(2) $f(x,y,z)=x.e^{yz}$ fonksiyonunun

$C:\begin{cases}x=t^2\\x=t+1\\z=2t\end{cases}$ eğrisinin $(4,3,4)$ noktasındaki teğeti boyunca $(4,3,4)$ noktasındaki türevi?

$\nabla f=e^{yz}\vec{i}+xze^{yz}\vec{j}+xye^{yz}\vec{k}$

$\nabla f|_{(4,3,4)}=(e^{12},16e^{12},12e^{12})$

C eğrisi: $\vec{r}(t)=t^2\vec{i}+(t+1)\vec{j}+2t\vec{k}$

$r'(t)=2t\vec{i}+\vec{j}+2\vec{k}$

$(4,3,4)$ için $t=2$ olur

$\vec{r}'(2)=(4,1,2)$

$|\vec{r}'(2)|=\sqrt{16+1+4}=\sqrt{21}$

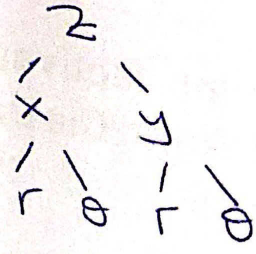
$\vec{u}=\frac{\vec{r}'(2)}{|r'(2)|}=\left(\frac{4}{\sqrt{21}},\frac{1}{\sqrt{21}},\frac{2}{\sqrt{21}}\right)$

$D_uf|_{(4,3,4)}=\nabla f.u=\frac{44}{\sqrt{21}}e^{12}$

(3) $y\dfrac{\partial z}{\partial x} - x\dfrac{\partial z}{\partial y} = 0$ veriliyor Kutupsal koordinatlarda $\dfrac{\partial z}{\partial \theta} = ?$

$x = r\cos\theta$

$y = r\sin\theta$



$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= \frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta) = -y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0$$

(4) $f(x,y) = \sqrt{1+x^2y^2}$ ise ekst. değerler?

$f_x = \dfrac{xy^2}{\sqrt{1+x^2y^2}}$

$f_y = -\dfrac{x^2y}{\sqrt{1+x^2y^2}}$

$\left.\begin{array}{l} f_x = 0 \Rightarrow \\ f_y = 0 \Rightarrow \end{array}\right\} \begin{array}{l}(a,0) \\ (0,b)\end{array}$

$\forall x,y \in \mathbb{R}^2$ için $\sqrt{1+x^2y^2} \geq 1$ old.dan

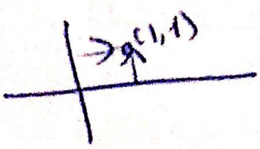
$f(a,0) = 1 = f(0,b)$

$f(x,y) \geq f(a,0)$

$\geq f(0,b)$

min değerdir.

(5) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{y\sin\pi x}{x+y-2}$ limitinin varlığı?



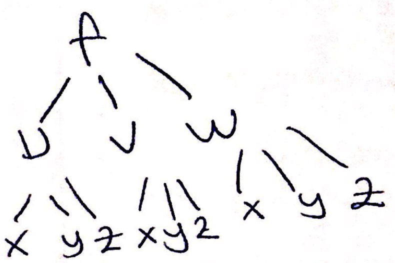
$f(x,1) = \dfrac{\sin\pi x}{x-1}$

$\displaystyle\lim_{x\to 1} \frac{\sin\pi x}{x-1} \left(\frac{0}{0}\right) = \lim_{x\to 1} \frac{\pi\cos\pi x}{1} = -\pi$

$f(1,y) = 0/y-1 = 0$

$\lim\limits_{y \to 1} 0 = 0 \neq -\pi$ limit yoktur.

6 $u = z - x$
$v = y - z$
$w = x - y$ ise $\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y} + \dfrac{\partial f}{\partial z} \overset{?}{=} 0$



$$f_x = f_u \underbrace{u_x}_{-1} + f_v \underbrace{v_x}_{0} + f_w \underbrace{w_x}_{1}$$

$$f_y = f_u \underbrace{u_y}_{0} + f_v \underbrace{v_y}_{1} + f_w \underbrace{w_y}_{-1}$$

$$f_z = f_u \underbrace{u_z}_{-1} + f_v \underbrace{v_z}_{-1} + f_w \underbrace{w_z}_{0}$$

$+$

(8) $w = x^2+y^2+z^2$

$y\sin z + z\sin x = 0$ ise $(x,y,z)=(0,1,\pi)$

$\left(\frac{\partial w}{\partial y}\right)_x = ?$ $\left(\frac{\partial w}{\partial y}\right)_z = ?$

$\rightarrow w = w(y,x)$

$z = z(y,x)$

$$\frac{\partial w}{\partial y} = 2y + 2z\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\sin z}{y\cos z + \sin x}$$

$$\frac{\partial w}{\partial y} = 2y + 2z \cdot \frac{-\sin z}{y\cos z + \sin x}$$

$$\left(\frac{\partial w}{\partial y}\right)\Bigg|_{(0,1,\pi)} = 2.1 + 2\pi \cdot \frac{-\sin\pi}{1.\cos\pi + \sin 0} = 2$$

$\rightarrow w = w(y,z)$

$x = x(y,z)$

$$\frac{\partial w}{\partial y} = 2x\frac{\partial x}{\partial y} + 2y = 2x \cdot \frac{-\sin z}{z.\cos x} + 2y$$

$$\frac{\partial x}{\partial y} = -\frac{F_y}{F_x} = -\frac{\sin z}{z.\cos x}$$

$$\frac{\partial w}{\partial y}\Bigg|_{(0,1,\pi)} = 0 + 2.1 = 2$$

(9) $f(x,y) = e^y \sin(y^2-x)$ ile tanımlı f fonksiyonun $(\pi,0)$ noktasındaki yönlü türevi, hangi yönlerde 0 olur

$$\nabla f = -e^y \cos(y^2-x)\vec{i} + [e^y \sin(y^2-x^2) + 2ye^y \cos(y^2-x)]\vec{J}$$

$$\nabla f(\pi,0) = (1,0)$$

$$\vec{U} = (a,b) \quad , \quad a^2+b^2=1$$

$$D_{\vec{U}}f(\pi,0) = \nabla f(\pi,0) \cdot \vec{U} = a = 0 \Rightarrow b = \mp 1$$

$$\vec{U} = (0, \mp 1)$$

veya

$$\vec{U} = \mp J$$

(10) $x^2+y^2=80$ çemberi için (çemberinde) $(1,2)$ noktasına en yakın ve en uzak noktasını bulun

$$d = \sqrt{(x-1)^2+(y-2)^2} = f(x,y)$$

$$g(x,y) = x^2+y^2-80$$

$$L(x,y,\lambda) = \sqrt{(x-1)^2+(y-2)^2} + \lambda(x^2+y^2-80)$$

$$L_x = \frac{x-1}{\sqrt{(x-1)^2+(y-2)^2}} + 2\lambda x = 0$$

$$L_y = \frac{y-2}{\sqrt{(x-1)^2+(y-2)^2}} + 2\lambda y = 0$$

$$L_\lambda = x^2+y^2-80 = 0$$

$$2\lambda x = \frac{-(x-1)}{\sqrt{}} \qquad 2\lambda y = \frac{-(y-2)}{\sqrt{}}$$

$$2\lambda = \frac{-(x-1)}{x\cancel{\sqrt{\;}}} = \frac{-(y-2)}{y\cancel{\sqrt{\;}}}$$

$$\frac{x-1}{x} = \frac{y-2}{y}$$

$$xy - y = xy - 2x$$

$$y = 2x$$

$$x^2 + y^2 - 80 = 0$$

$$x^2 + 4x^2 - 80 = 0$$

$$5x^2 = 80 \qquad x = \mp 4$$

$$(4,8) \qquad (-4,-8)$$

$$f(4,8) = 45 \qquad f(-4,-8) = 125$$

$$\Downarrow \qquad\qquad \Downarrow$$

en yakın   en uzak nokta