NOW !!!

Let

$$F(t) = \int_a^b f(x, t)\, dx \quad .$$

$f(x,t)$ and $\frac{\partial f(x,t)}{\partial t} = f_t(x,t)$ are continuous

at $a \leq x \leq b$

$c \leq t \leq d$

$\Delta t$ is an increment on $t$.

remember mean value theorem for derivative
$f(b)-f(a) = f'(c)(b-a)$
$c \in (a,b)$

$$F(t+\Delta t) = \int_a^b f(x, t+\Delta t)\, dx$$

$$F(t+\Delta t) - F(t) = \int_a^b [f(x, t+\Delta t) - f(x,t)]\, dx$$

$(t+\Delta t - t) = \Delta t$

$$= \int_a^b f_t(x, t+\theta\Delta t)\Delta t \, dx \quad \boxed{0<\theta<1}$$

$$\frac{F(t+\Delta t)-F(t)}{\Delta t} = \int_a^b f_t(x, t+\theta\Delta t)\, dx$$

$$= \int_a^b f_t(x,t) + \int_a^b \left[f_t(x, t+\theta\Delta t)\,dx - f_t(x,t)\right]dx$$

for $\varepsilon>0$; since $f_t(x,t)$ is continuous, there is a $\delta>0$

$$|f_t(x, t+\theta\Delta t) - f_t(x,t)| < \varepsilon$$

since $|\Delta t| \leq \delta$

$$\left| \int_0^b [f_t(x, t+\theta \Delta t) - f_t(x,t)] dx \right| < \varepsilon (b-a)$$

$$\left| \frac{F(t+\Delta t) - F(t)}{\Delta t} - \int_a^b f_t(x,t) dx \right| \leq$$

$$\leq \int_a^b | f_t(x, t+\theta\Delta t) - f_t(x,t) dx| \leq \varepsilon (b-a)$$

Thus

$$\lim_{\Delta t \to 0} \frac{F(t+\Delta t) - F(t)}{\Delta t} = \int_a^b f_t(x,t) dx$$

$$\frac{dF(t)}{dt} = \frac{d}{dt}\int_a^b f(x,t)\,dx = \int_a^b \frac{\partial f(x,t)}{\partial t}\,dx$$

$$= \int_a^b f_t(x,t)\,dx$$

int. bounds (limits) are constant.

NOW !!!

$$F(t) = \int_{a(t)}^{b(t)} f(x,t)\, dx$$

$a(t), b(t)$ are continuous relative to $t$.

You know

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial b} \cdot \frac{db}{dt} + \frac{\partial F}{\partial a} \cdot \frac{da}{dt}$$

$$F(t) = \int_{a}^{b(t)} f(x,t)\,dx \Rightarrow \frac{\partial F}{\partial b} = f(b(t),t)$$

$$F(t) = \int_{a(t)}^{b} f(x,t)\,dx = -\int_{b}^{a(t)} f(x,t)\,dx \Longrightarrow$$

$$\frac{\partial F}{\partial a} = -f(a(t),t)$$

$$\Rightarrow F'(t) = \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t}\,dx + f(b(t),t) \cdot b'(t) - f((a(t),t) \cdot a'(t)$$

Leibniz Rule is obtained.

Namely;

$$\frac{d}{dx} F(x) = \frac{d}{dx} \int_a^x f(t)\, dt = f(x)$$

$$\int_{u(x)}^{v(x)} f(t)\, dt = \int_{u(x)}^{c} f(t)\, dt + \int_{c}^{v(x)} f(t)\, dt$$

$$\frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(t)\, dt\right] = f(v(x)) \cdot v'(x) - f(u(x)) \cdot u'(x)$$

Ex. $$F(t) = \int_{t}^{t^2} \sin x^2 \, dx$$

$$F'(t) = \frac{d}{dt} \int_{t}^{t^2} \sin x^2 \, dx$$

$$= \int_{t}^{t^2} 0 \cdot dx + 2t \cdot \sin(t^2)^2 - 1 \cdot \sin t^2$$

$$\to F'(t) = \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t} dx + f(b(t),t) \cdot b'(t) - f((a(t),t) \cdot a'(t)$$

Ex. $y = \int_0^{\sqrt{x}} e^{t^2}\, dt \Rightarrow \dfrac{dy}{dx} = ?$

$$y' = \frac{1}{2\sqrt{x}} \cdot e^{(\sqrt{x})^2} - 0 \cdot e^0$$

Ex: $\displaystyle \lim_{x \to \pi} \frac{\int_\pi^x \frac{dt}{2+\cos t}}{x-\pi} = \lim_{x \to \pi} \frac{\frac{1}{2+\cos x} - 0}{1} = 1$

$$F(x) = \int_{u(x)}^{v(x)} f(x,y)\, dy$$

$$F'(x) = \int_{u(x)}^{v(x)} \frac{\partial f}{\partial x}\, dy + v'(x) \cdot f(x, v(x)) - u'(x) \cdot f(x, u(x))$$

Ex:

$$f(x) = \int_{1-x}^{2-x} e^{t^2}\, dt \qquad f'(1) = ?$$

$$f'(x) = e^{(2-x)^2} \cdot (2-x)' - e^{(1-x)^2} \cdot (1-x)'$$

$$= e^{(2-x)^2} \cdot (-1) - e^{(1-x)^2} \cdot (-1)$$

$$f'(x) = e^{(1-x)^2} - e^{(2-x)^2}$$

$$f'(1) = e^0 - e^1 = 1 - e$$

Ex:

$$\lim_{x \to 0} \frac{\arctan x}{\int_0^{2x} e^{t^2} dt} = ? \qquad \frac{0}{0}$$

$$\lim_{x \to 0} \frac{\frac{1}{1+x^2}}{e^{(2x)^2} \cdot 2 - e^0 \cdot 0} = \frac{1}{2} \quad //$$

$$F(t) = \int_{x=1}^{x=t} \frac{\sin(x t)}{x}\, dx = ?$$

$$F'(t) = ?$$

$$F'(x) = \int_{u(x)}^{v(x)} \frac{\partial f}{\partial x}\, dy + v'(x)\cdot f(x, v(x)) - u'(x)\cdot f(x, u(x))$$

$$F'(t) = \int_1^t \frac{\cancel{x}\cdot\cos(x\cdot t)}{\cancel{x}}\, dx + \frac{\sin t^2}{t}\cdot 1 - \frac{\sin t}{1}\cdot 0$$

$$F'(t) = \frac{1}{t}\cdot \sin(x\cdot t)\Big|_1^t + \frac{\sin t^2}{t}$$

$$F'(t) = \frac{1}{t}(\sin t^2) - \frac{1}{t}\sin t + \frac{\sin t^2}{t}$$

Ex:

$$F(x) = \frac{1}{x}\int_1^x [2t - F'(t)]\,dt \implies F'(1) = ?$$

$$F'(x) = -\frac{1}{x^2}\cdot\int_1^x [2t - F'(t)]\,dt + \frac{1}{x}\cdot\left[1\cdot((2x - F'(x)) - 0\cdot\right]$$

$$F'(1) = \frac{1}{1}(2 - F'(1))$$

$$2F'(1) = 2$$

$$\boxed{F'(1) = 1}$$

Find $g'(x)$ if $g(x) = \displaystyle\int_{x^2}^{\ln x} \arctan t \, dt$

Let $F$ be any antiderivative of $\arctan t$. By the Evaluation Theorem, the integral in the question equals

$$F(\ln x) - F(x^2).$$

Taking the derivative using the chain rule (and remembering that, *by definition of antiderivative*, $F'(t) = \arctan t$), we get

$$\begin{aligned} g'(x) &= \left(\frac{1}{x}\right) \cdot F'(\ln x) - 2xF'(x^2) \\ &= \frac{\arctan(\ln x)}{x} - 2x \arctan(x^2). \end{aligned}$$