Question: $f(x) = \begin{cases} (x-1)\cdot \sin\left(\frac{1}{x-1}\right) & , x \neq 1 \\ 0 & , x = 1 \end{cases}$ is differentiable at $x=1$

To be able show differentiability at given point function has to be continuous, then right-handed derivative and left-handed derivative should same.

① Continuity [for $(x=1)$]

$f(1) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x)$ a) $\lim_{x\to 1^+} \frac{\sin\left(\frac{1}{x-1}\right)}{\left(\frac{1}{x-1}\right)} = \lim_{u\to +\infty} \frac{\sin u}{u} = 0$

$f(1)=0$ let $\frac{1}{x-1} = u$ ↳ from sandwich theorem

$-1 \leq \sin u \leq 1$

b) $\lim_{u\to -\infty} \frac{\sin u}{u} = 0$ $-\frac{1}{u} \leq \frac{\sin u}{u} \leq \frac{1}{u}$

$\underbrace{\lim_{u\to \infty} -\frac{1}{u}}_{0} \leq \lim_{u\to \infty} \frac{\sin u}{u} \leq \underbrace{\lim_{u\to \infty} \frac{1}{u}}_{0}$ $\underbrace{\lim_{u\to -\infty} -\frac{1}{u}}_{0} \leq \lim_{u\to -\infty} \frac{\sin u}{u} \leq \underbrace{\lim_{u\to -\infty} \frac{1}{u}}_{0}$

$f(1) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = 0$ So, function is continuous at given $(x=1)$ number

② Derivatives

$f_+'(1)$
$\lim_{h\to 0} \left(\frac{f(1+h) - f(1)}{h}\right) = f_+'(1) = \lim_{h\to 0} \frac{(\cancel{(1+h)-1})\ \sin\left(\frac{1}{(1+h)-1}\right)}{\cancel{h}} = \lim_{h\to 0} \left(\sin\left(\frac{1}{h}\right)\right) = +\infty$

$f_-'(1)$
$\lim_{h\to 0} \frac{f(1-h) - f(1)}{-h} = f_-'(1) = \lim_{h\to 0} \frac{(\cancel{(1-h)-1})\ \sin\left(\frac{1}{(1-h)-1}\right)}{\cancel{-h}} = \lim_{h\to 0} \left(-\sin\left(\frac{1}{h}\right)\right) = -\infty$

$f_+'(1) \neq f_-'(1) \implies$ left and right handed derivatives are not same so, function is not differentiable at $x=1$.

Question: Let $f(x)$ be a function that has an inverse function $f^{-1}(x)$. If the normal line to the curve $y=f(x)$ at the point $P(x_0,-1)$ is $y+2x-1=0$, find $(f^{-1})'(-1)$

$y = -2x+1$ $\frac{-1}{f'(x_0)} = -2$ $x_0 = 1$, $f'(1) = \frac{1}{2}$

$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$

$(f^{-1})'(-1) = \frac{1}{1/2} = 2$

Question: For the function $f(x) = \frac{x^2-x+1}{x}$

i) domain ii) asymptotes iii) increasing - decreasing and local extrema iv) concavity and inflection p.

v) sketching

i) $f(x) = \frac{x^2-x+1}{x}$ → numerator, x → denominator

$(-1)^2-4\cdot1\cdot1=-3$ $\Delta<0$ numerator is always positive and defined for $\mathbb{R}$

denominator is not defined for $x=0$

$D_f: (-\infty,0)\cup(0,\infty)$

ii) $\lim_{x\to0^-}\frac{x^2-x+1}{x} = \lim_{x\to0^-}\frac{x-1+\frac{1}{x}}{1} = \underline{-\infty}$, $\lim_{x\to0^+}\frac{x-1+\frac{1}{x}}{1} = \underline{+\infty}$

So, $x=0$ namely y-axis is a vertical asymptote

$\lim_{x\to-\infty}\frac{x-1+\frac{1}{x}}{1} = \underline{-\infty}$, $\lim_{x\to+\infty}\frac{x-1+\frac{1}{x}}{1} = \underline{+\infty}$

So, there is no horizontal asymptote

$$\begin{array}{r|l} x^2-x+1 & x \\ \hline -x^2 & x-1 \\ \hline -x+1 & \\ -+x & \\ \hline 1 & \end{array}$$

$\frac{x^2-x+1}{x} = \frac{x(x-1)+1}{x} = \underbrace{(x-1)}_{\text{oblique asy}} + \frac{1}{x}$ $y=x-1$ is an oblique asymptote

$\underbrace{y=mx+n}_{\text{oblique asy}}$ $m=\lim_{x\to\mp\infty}\frac{f(x)}{x}$, $n=\lim_{x\to\mp\infty}(f(x)-mx)$

iii) $f'(x) = \frac{(2x-1)\cdot x-(x^2-x+1)\cdot1}{x^2} = \frac{x^2-1}{x^2}$

		-1		0		1	
$f'(x)$	+	|	−	||	−	|	+
$f(x)$	↗		↘		↘		↗
		(−1,−3) local max				(1,1) local min	

f is increasing on $(-\infty,-1)\cup(1,\infty)$

f is decreasing on $(-1,0)\cup(0,1)$

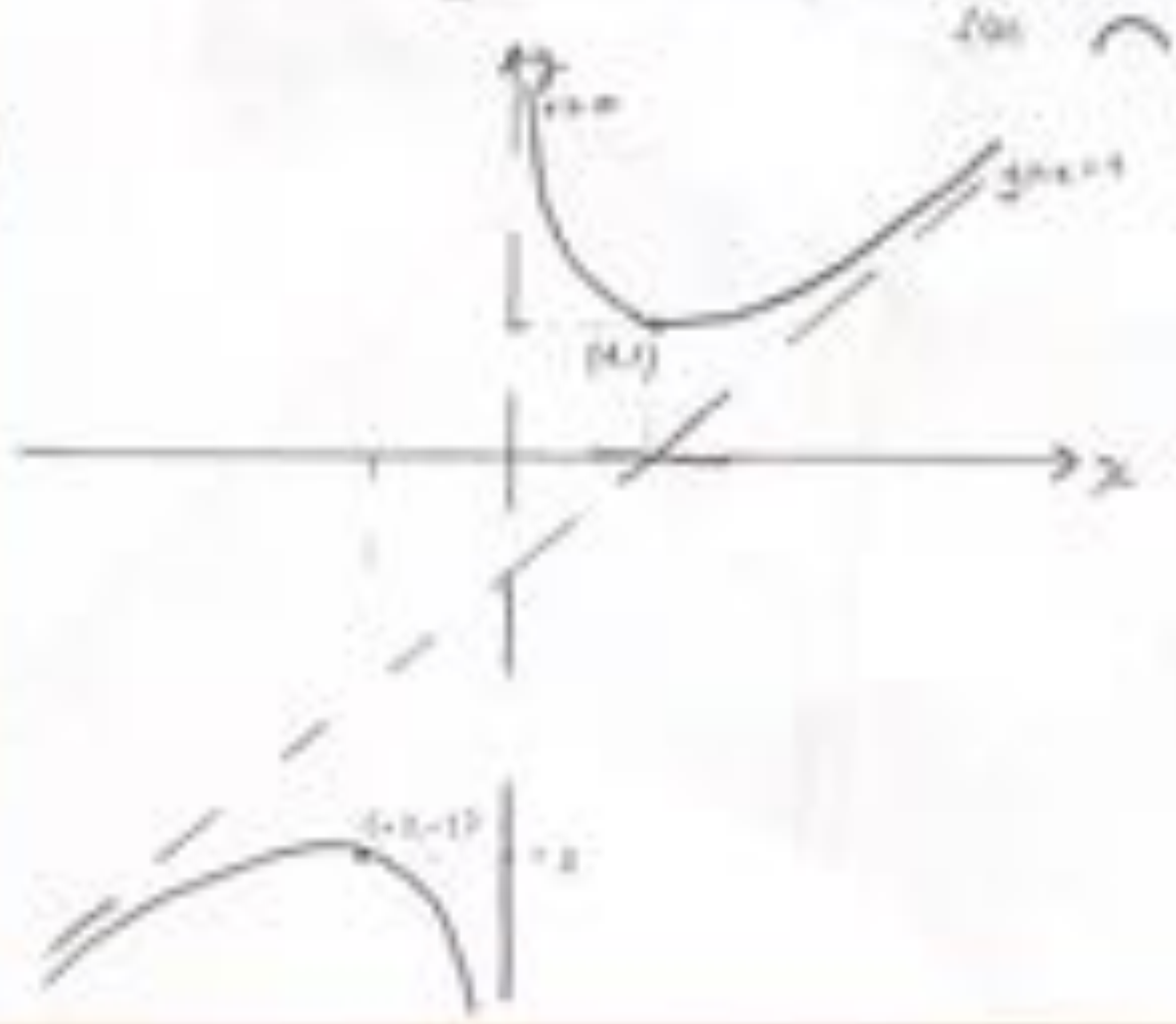
iv) $f''(x) = \frac{2x\cdot x^2-2x\cdot(x^2-1)}{x^4} = \frac{2}{x^3}$

		0	
$f''(x)$	−	||	+
$f(x)$	⌢		⌣

f is concave up on $(0,\infty)$

f is concave down on $(-\infty,0)$

v)



Question: $f(x) = \sinh(x)$, find the slope of the tangent $y = f^{-1}(x)$, at the point $P(0,0)$

$f(f^{-1}(x)) = x$ $y = \sinh(x)$ $y' = \cosh(x)$

$f'(f^{-1}(x)) \cdot \frac{d}{dx} f^{-1}(x) = 1$ $\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$ $(f^{-1}(0))' = \frac{1}{1} = 1$

Question: For the function $f(x) = \frac{x^2-1}{x}$ i) $D_f: (-\infty,0) \cup (0,\infty)$ x-intercepts $(-1,0)$ and $(1,0)$

i) domain   ii) asymptotes   iii) decreasing - increasing   iv) concavity   v) sketching

ii) $\lim_{x\to0^+} \frac{x^2-1}{x} = \lim_{x\to0^+} \frac{x-\frac{1}{x}}{1} = -\infty$, $\lim_{x\to0^-} \frac{x^2-1}{x} = \lim_{x\to0^-} \frac{x-\frac{1}{x}}{1} = +\infty$ x=0 (being y-axis) is a vertical asymptote

$\lim_{x\to+\infty} \frac{x-\frac{1}{x}}{1} = +\infty$, $\lim_{x\to-\infty} \frac{x-\frac{1}{x}}{1} = -\infty$ so there is no horizontal asymptote.

$$\begin{array}{r|l} x^2-1 & x \\ \hline -x^2 & x \\ \hline -1 & \end{array}$$

$\frac{(x\cdot x)-1}{x} = \boxed{x} - \frac{1}{x}$ y=x is an oblique asymptote.

or: $\lim_{x\to\mp\infty} \frac{f(x)}{x} = m$ and $\lim_{x\to\mp\infty} (f(x)-mx) = n$ and y=mx+n is an oblique asymptote

$\frac{\frac{x^2-1}{x}}{x} = 1 - \frac{1}{x^2}$ m=1 $\frac{x^2-1}{x} - x$ n=0

iii) $f'(x) = \frac{2x\cdot x - 1\cdot(x^2-1)}{x^2} = \frac{x^2+1}{x^2}$

		0	
$f'(x)$	+	‖	+
$f(x)$	↗		↗

f is increasing on $(-\infty,0) \cup (0,\infty)$

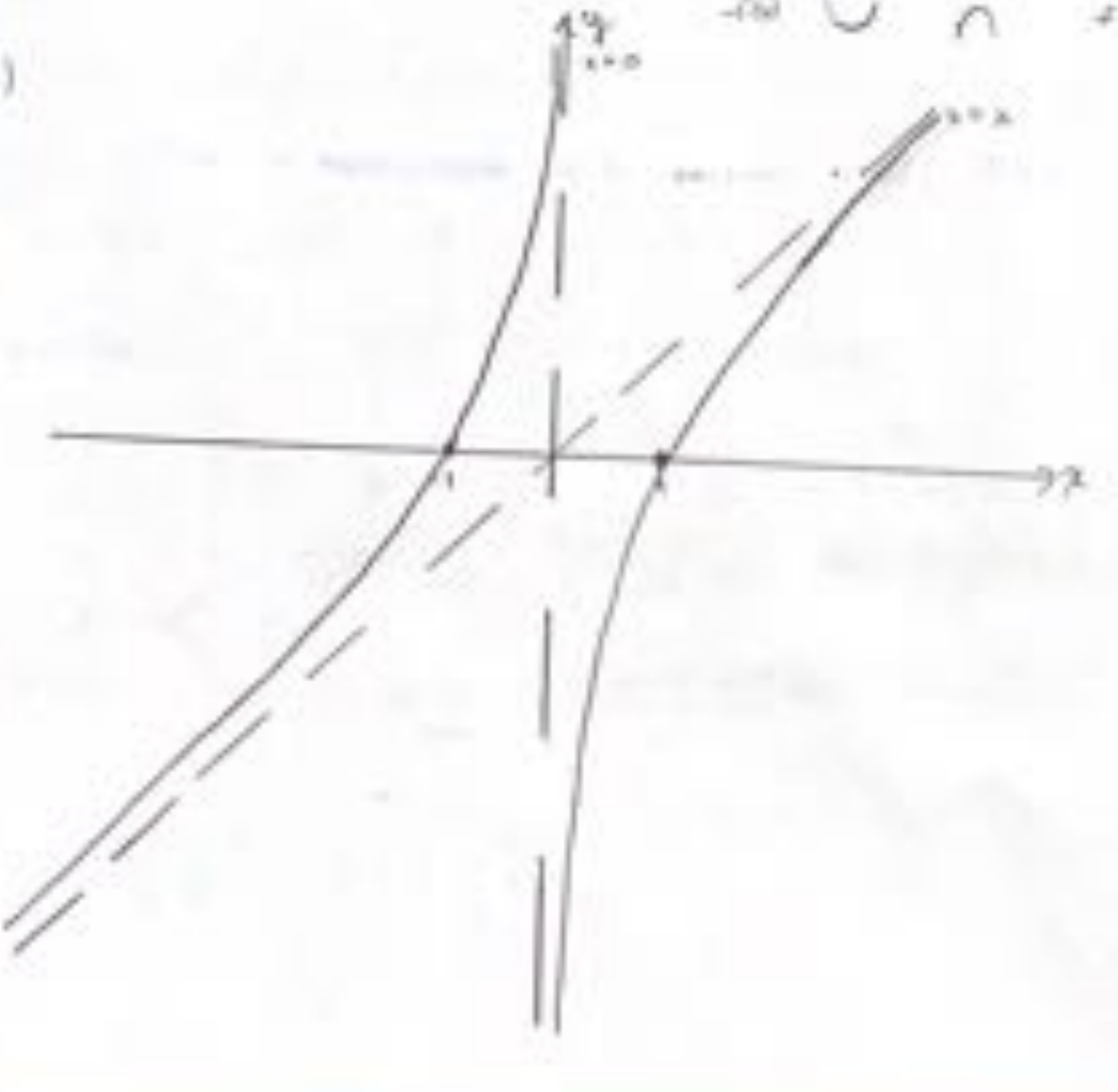
iv) $f''(x) = \frac{2x\cdot x^2 - 2x\cdot(x^2+1)}{x^4} = \frac{-2}{x^3}$

		0	
$f''(x)$	+	‖	−
$f(x)$	∪		∩

f is concave up on $(-\infty,0)$

f is concave down on $(0,\infty)$

v)



Question: Find the tangent and normal lines at $t = \frac{\pi}{3}$

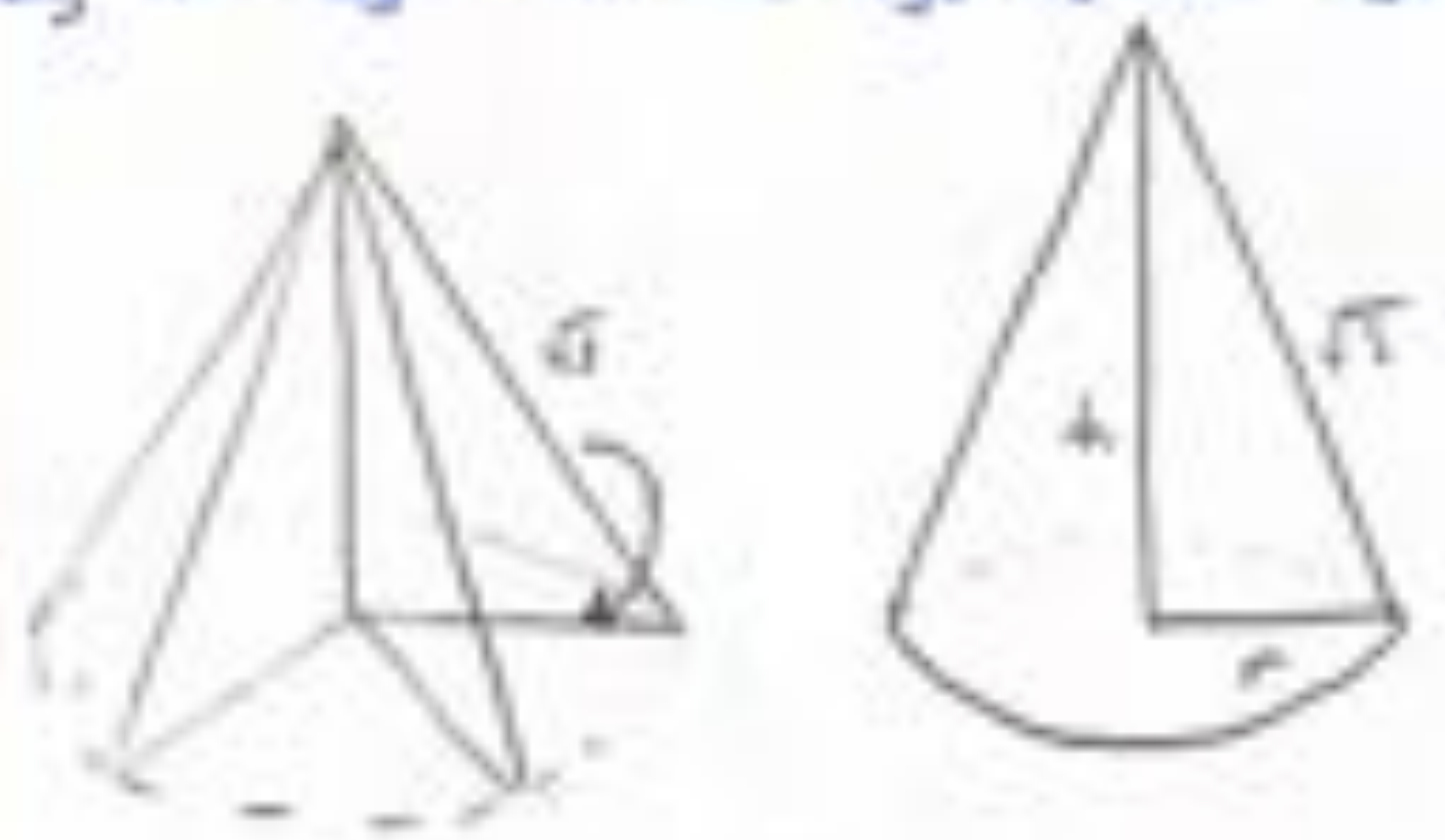
$\begin{cases} x(t) = 6t\cos(t) \\ y(t) = 6\sqrt{3}t\sin(t) \end{cases}$ $\frac{dy}{dx} = \frac{dy(t)}{dx(t)} = \frac{6\sqrt{3}\sin(t) + 6\sqrt{3}t\cos(t)}{6\cos(t) + 6t(-\sin(t))} = \frac{\sqrt{3}(\sin(t) + t\cos(t))}{\cos(t) - t\sin(t)}\Big|_{t=\frac{\pi}{3}}$

$\Rightarrow \frac{\sqrt{3}\left(\frac{\sqrt{3}}{2} + \frac{\pi}{3}\frac{1}{2}\right)}{\frac{1}{2} - \frac{\pi}{3}\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}\left(1 + \frac{\sqrt{3}\pi}{3}\right)}{1 - \frac{\pi}{3}} = \frac{6\sqrt{3} + 2\pi}{6\sqrt{3} - \pi} = m_T$ $\quad -\frac{6\sqrt{3} - \pi}{6\sqrt{3} + 2\pi} = m_N$

$x\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}\pi}{1}$ $\quad y\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}\pi}{2}$

$d_T: \left(y - \frac{\sqrt{3}\pi}{2}\right) = \frac{6\sqrt{3} + 2\pi}{6\sqrt{3} - \pi}\left(x - \frac{\sqrt{3}\pi}{2}\right)$ $\quad d_N: \left(y - \frac{\sqrt{3}\pi}{2}\right) = \left(-\frac{6\sqrt{3} - \pi}{6\sqrt{3} + 2\pi}\right)\left(x - \frac{\sqrt{3}\pi}{2}\right)$

Question: A right triangle with hypotenuse of $\sqrt{3}$ is rotated about one of its legs to generate a right circular cone. Find the greatest possible value of volume of a cone by determining the lengths of the legs of the right triangle $\left(V = \frac{1}{3}\pi r^2 h\right)$

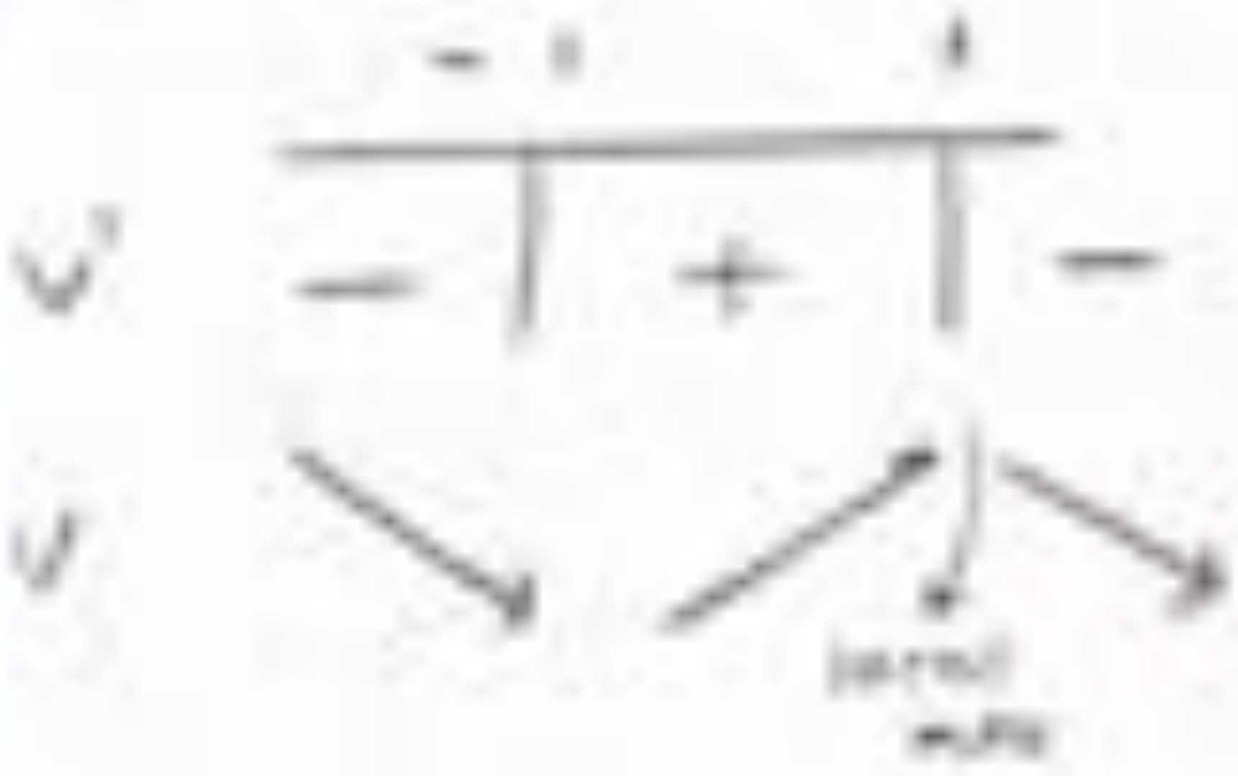


$V = \frac{1}{3}\pi r^2 h$ $\quad (\sqrt{3})^2 = h^2 + r^2$

$3 - h^2 = r^2$

$V = \frac{1}{3}\pi(3 - h^2)h$ $\quad V = \frac{\pi}{3}(3h - h^3)$

$V' = \frac{\pi}{3}(3 - 3h^2)$ $\quad h = \mp 1$



$h = 1 \quad r = \sqrt{2}$

Question: $f(x) = \begin{cases} x^2 - 1, & x \le 1 \\ k(x-1), & x > 1 \end{cases}$ $k \in \mathbb{R}$, Find k for f is differentiable at x=1

a) continuity (no need!)

$f(1) = 0 \quad f(1) = \lim_{x\to1^-}(x^2-1) = \lim_{x\to1^+}k(x-1) = 0$

b) derivative

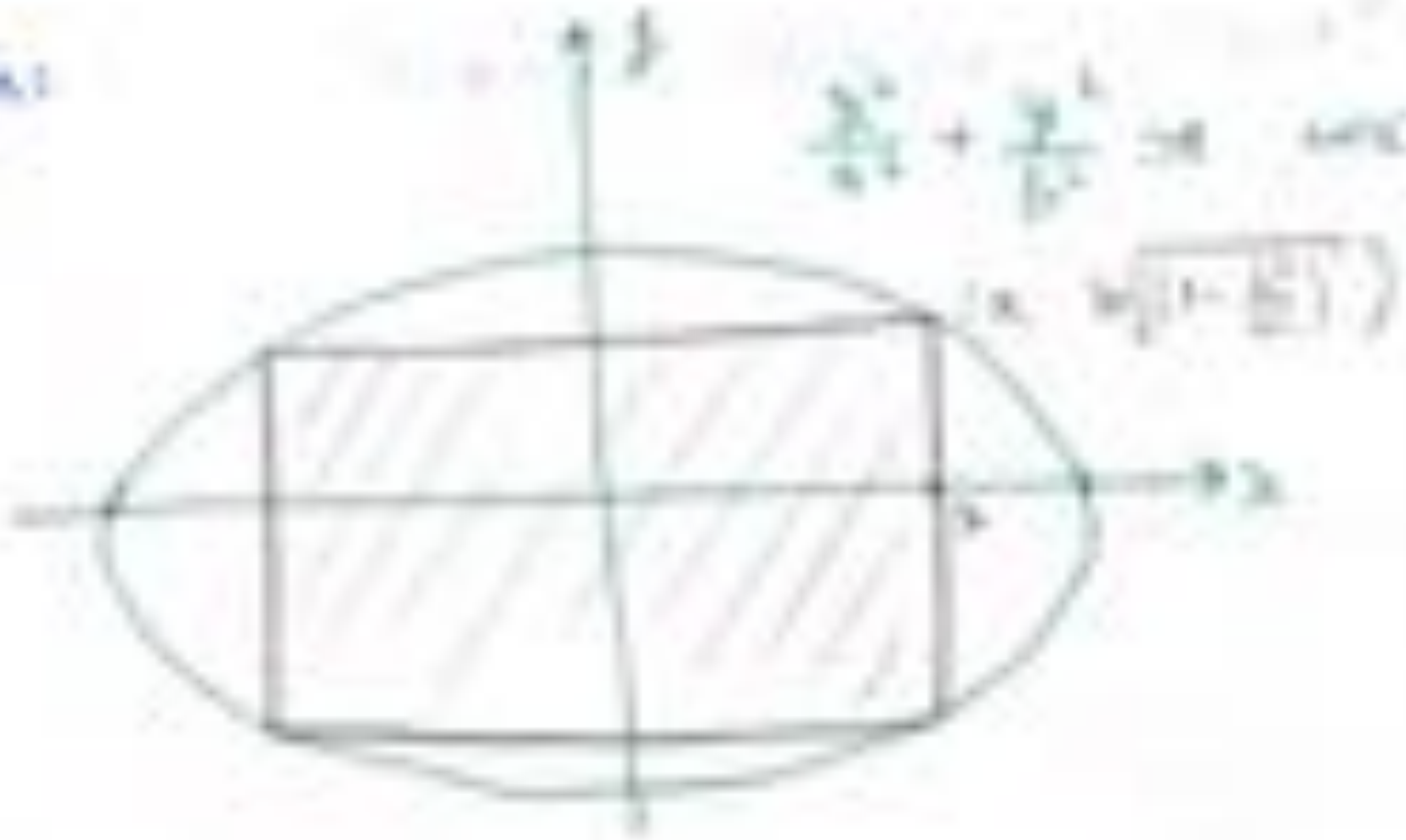
definition

$\lim_{\Delta x\to0}\frac{f(x+\Delta x) - f(x)}{\Delta x}$ $\quad \lim_{h\to0}\frac{f(1+h) - f(1)}{h} \Rightarrow \frac{k((1+h)-1) - 0}{h} = \lim_{h\to0} k = k$

$\lim_{h\to0}\frac{f(1-h) - f(1)}{-h} \Rightarrow \frac{(1-h)^2 - 1 - 0}{-h} = \lim_{h\to0}(2-h) = 2$

$\Big\rangle\, k = 2$

Question:



$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ max rectangle area , 4y(x)

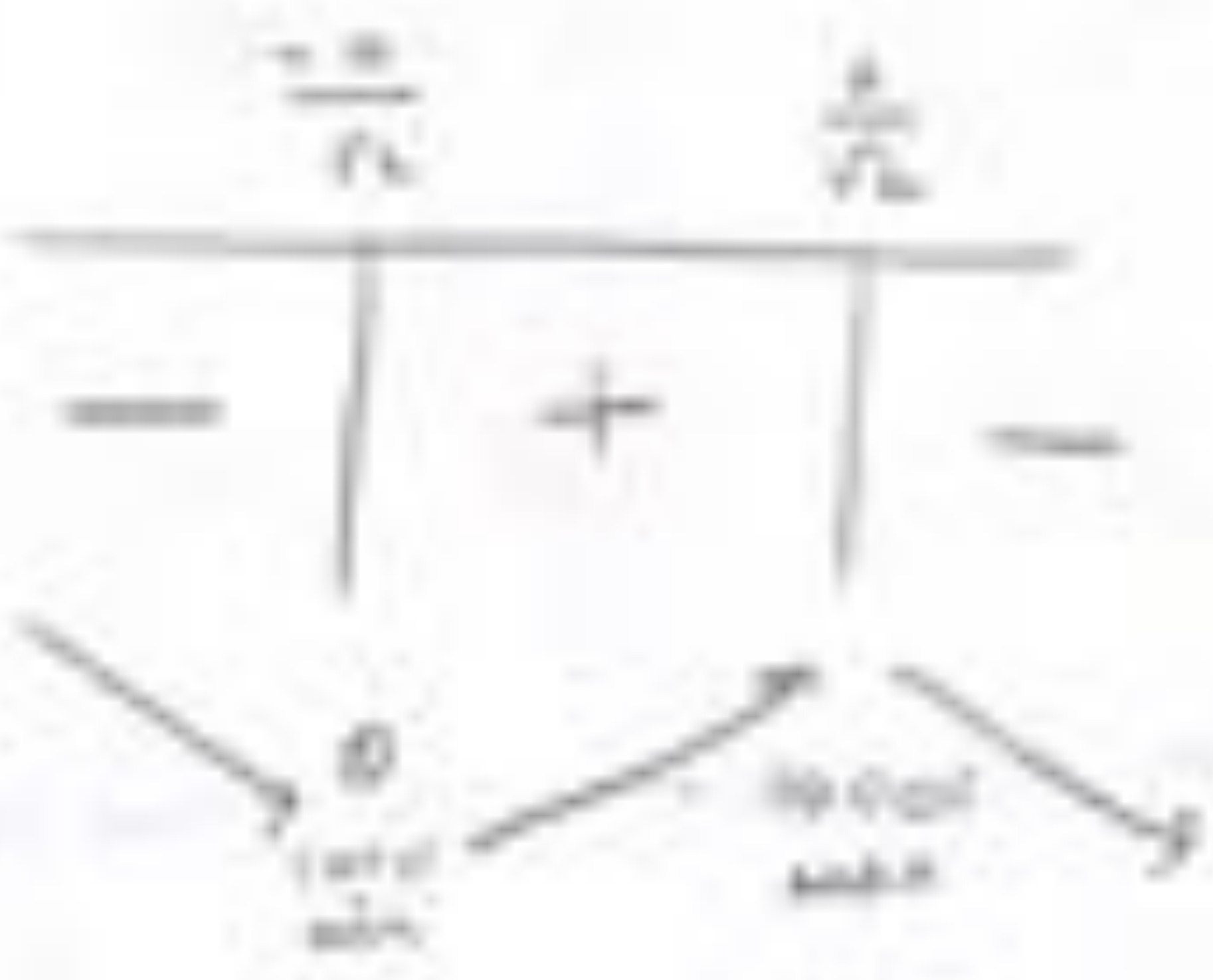
$A = 2x \cdot 2b\sqrt{1-\frac{x^2}{a^2}}$

$A = 4\frac{b}{a}\sqrt{a^2x^2-x^4}$

$A' = \frac{4\frac{b}{a}}{2\sqrt{a^2x^2-x^4}}(2a^2x-4x^3)$

$A' = \frac{4b}{a}\frac{x(a^2-2x^2)}{x\sqrt{a^2-x^2}}$ $\pm\frac{a}{\sqrt2}$ $\pm a$

$x = \mp\frac{a}{\sqrt2}$ $y = \frac{b}{\sqrt2}$



③ $x = -\frac{a}{\sqrt2}$

$2x \cdot 2y = A$ $\frac{2a}{\sqrt2}\cdot\frac{2b}{\sqrt2} = 2ab$ br²

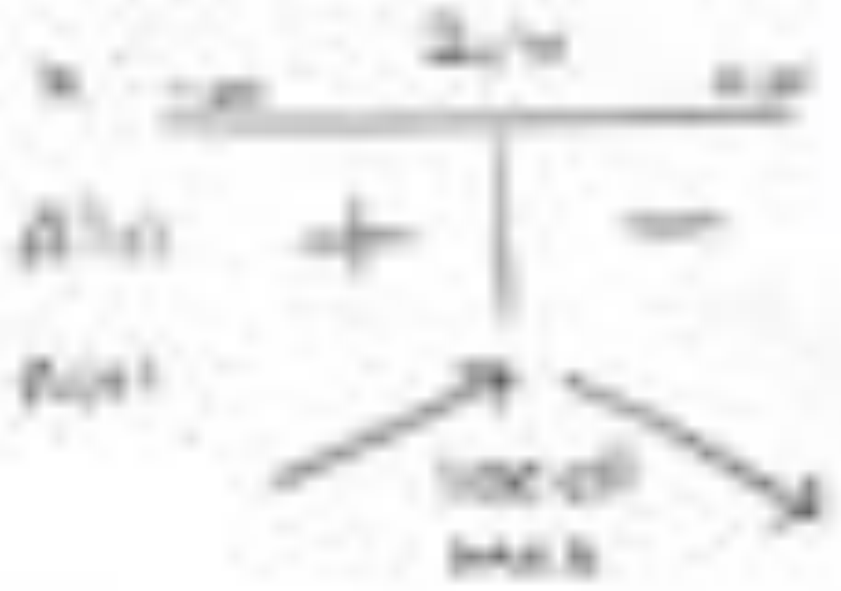
Question:

max area ; perimeter = 9

$\frac{\overset{9}{\cancel{18}}x}{\cancel{8}_4} + 2y = 9$ $y = \frac{36-9x}{8}$

$y = \frac{36-9x}{8}$ $A(x) = \underbrace{\frac{3x}{8}\cdot\frac{x}{2}}_{\text{triangle}} + \underbrace{x\cdot\frac{36-9x}{8}}_{\text{rectangle}}$

$A(x) = \frac{3x^2+72x-18x^2}{16} = \frac{72x-15x^2}{16}$ $A'(x) = \frac{72-30x}{16}$ $\frac{12}{5}$



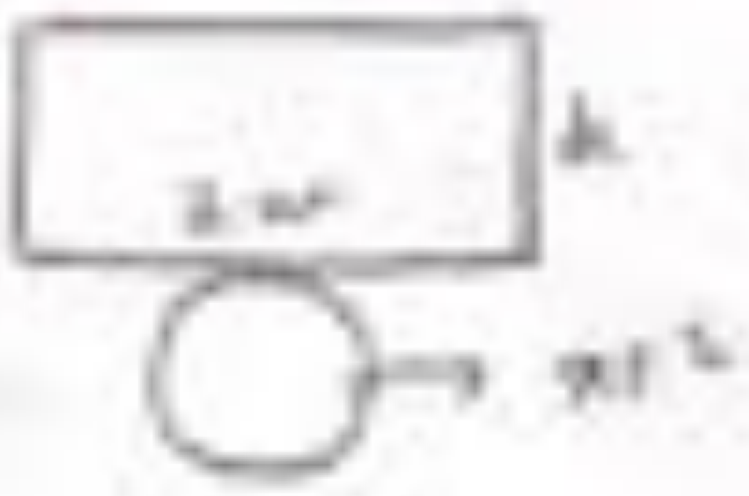
$x = 2.4$ $y = 1.8$ $z = 1.5$ $h = 0.9$

Question:

bottom is more expensive (k→2k)

volume is 300 m³

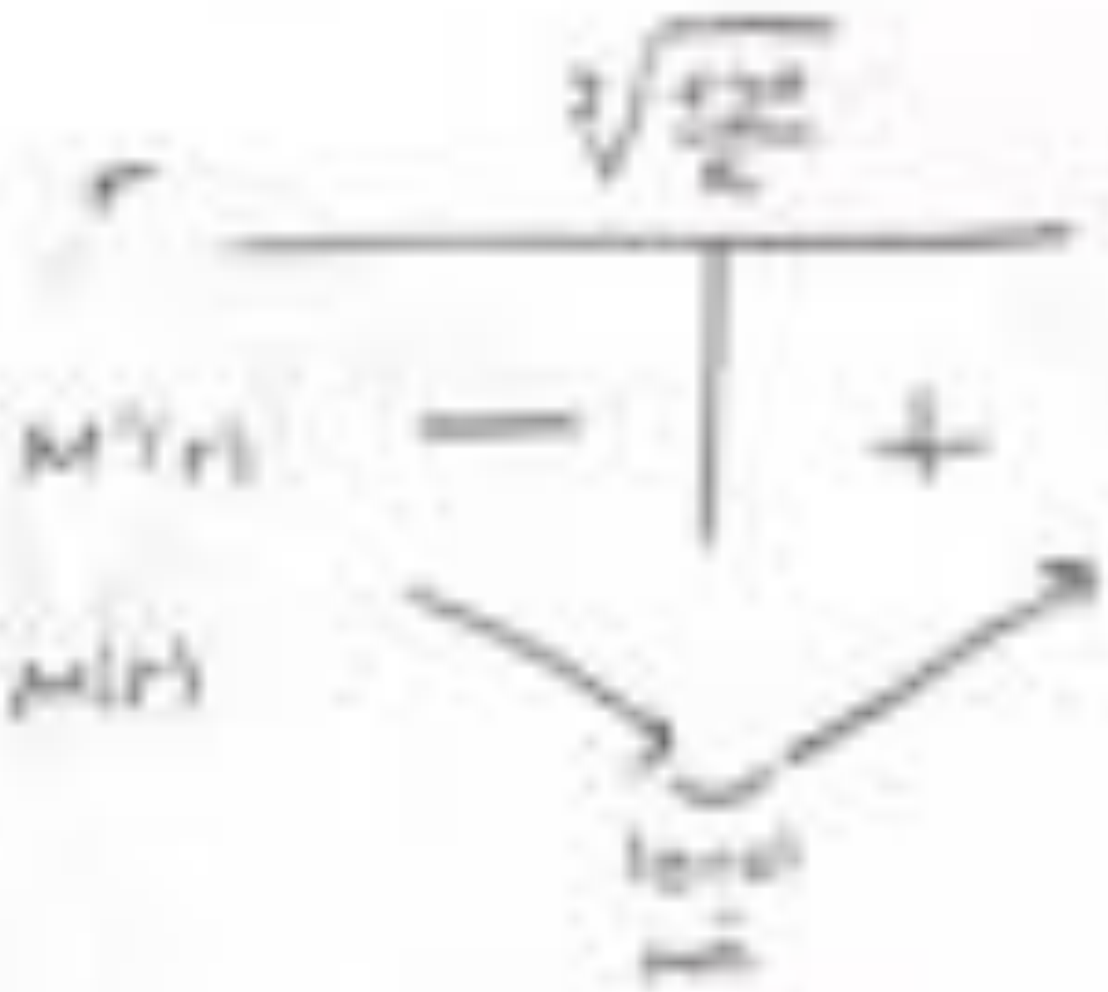
$\pi r^2 h = 300$ $h = \frac{300}{\pi r^2}$



$M = \overbrace{k \cdot 2\pi r h}^{\text{rectangle}} + \overbrace{\pi r^2 \cdot 2k}^{\text{circle}}$

$M = \frac{600k}{r} + 2\pi r^2 k$

$M'(r) = -\frac{600k}{r^2} + 4\pi r k \Rightarrow 4(\pi r^3)k = \overset{150}{\cancel{600}}k$ $r = \sqrt[3]{\frac{150}{\pi}}$



$r = \sqrt[3]{\frac{150}{\pi}}$ $h = 2\sqrt[3]{\frac{150}{\pi}}$

Question: For the curve given by $\begin{cases} t^2 \sin(x) + x^3 = e^t \\ \sin y = t\,x(t) - 2t \end{cases}$ $x = x(t)$, $y = y(t)$

Find the normal line at $t = 0$ $\qquad t=0 \quad x(0)=1 \quad y(0)=0$

$2t \cdot \sin(x) + t^2 \cos(x)\, x'(t) + 3x^2 \cdot x'(t) = e^t$ $\qquad \cos(y)\, y' = x(t) + t \cdot x'(t) - 2$

$3x^2\, x'(0) = 1 \qquad x'(0) = \frac{1}{3}$ $\qquad y'(0) = -1$

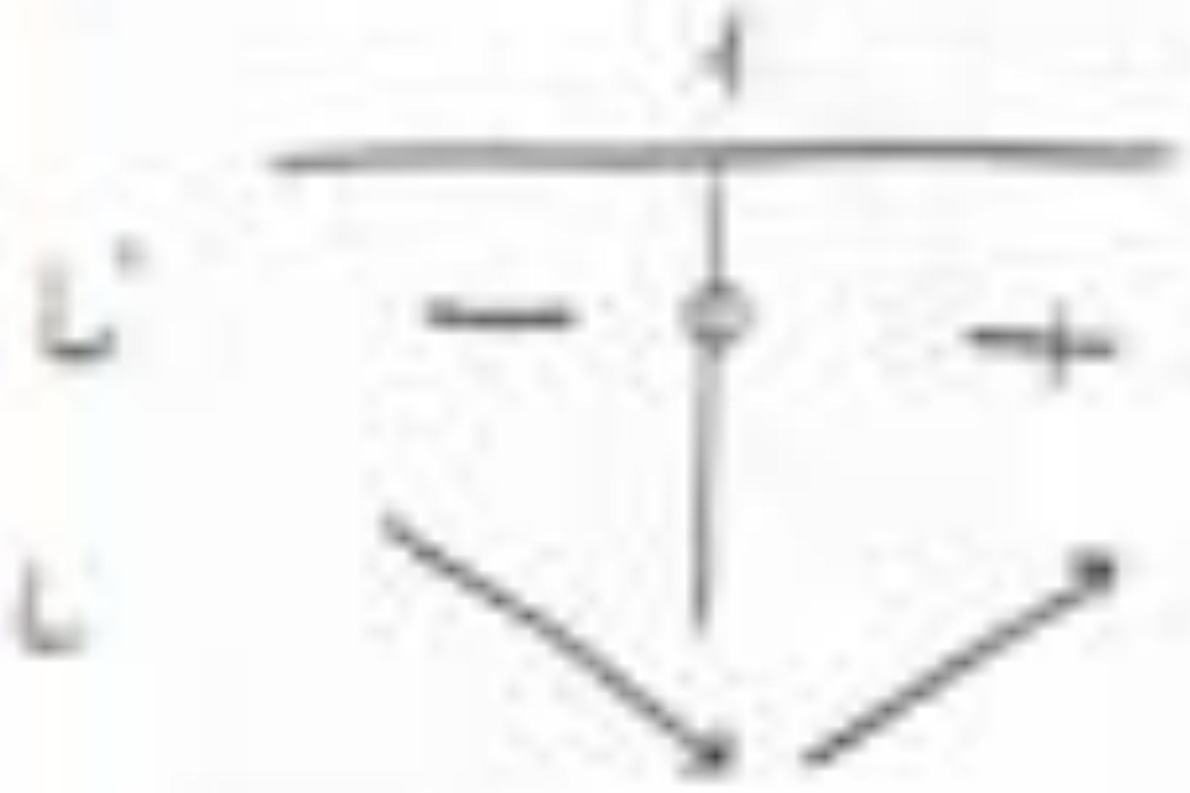
$\frac{dy}{dx} = \frac{dy(t)/dt}{dx(t)/dt} = \frac{-1}{1/3} = -3 = m_T \qquad m_N \cdot m_T = -1$
$\qquad \frac{1}{3} \quad -3$

$(y - y_0) = m_N (x - x_0) \Rightarrow (y - 0) = \frac{1}{3}(x - 1) \Rightarrow x - 3y - 1 = 0$

Question: Find a point $P(a,b)$ on the curve $y = x^2$ that is closest to point $B(3,0)$. Also find the distance $L = |PB|$. $\quad b = a^2$

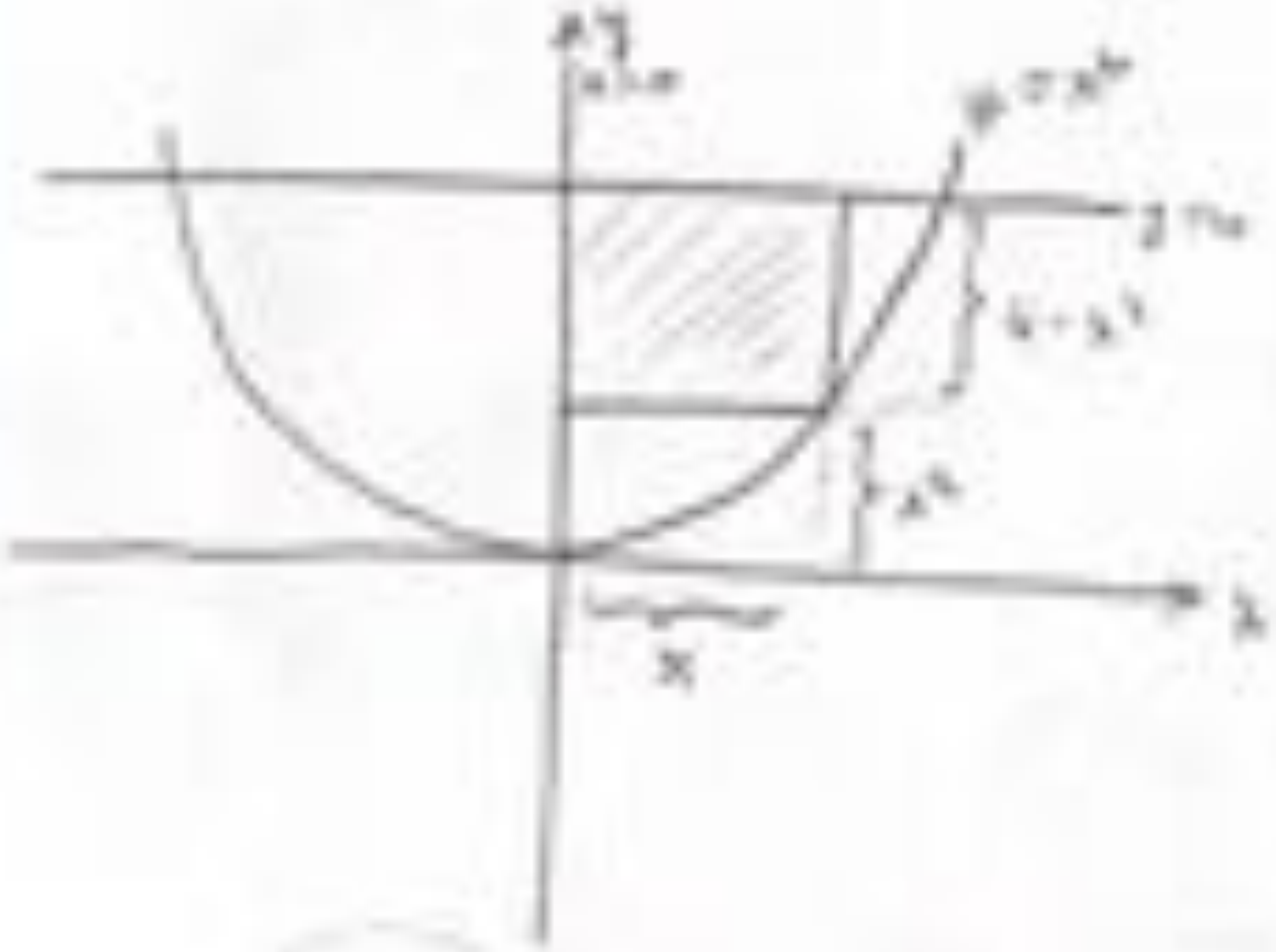
$L = \sqrt{(3-a)^2 + (0-b)^2} = \sqrt{(3-a)^2 + (0-a^2)^2} = \sqrt{9 - 6a + a^2 + a^4}$

$L' = \frac{4a^3 + 2a - 6}{2\sqrt{9 - 6a + a^2 + a^4}} \Rightarrow \frac{2a^3 + a - 3}{\sqrt{9 - 6a + a^2 + a^4}} = \frac{2(a^3 - 1) + a - 1}{\sqrt{9 - 6a + a^2 + a^4}} = \frac{\overset{a=1}{(a-1)}\left[2a^2 + 2a + 3\right]}{\sqrt{9 - 6a + a^2 + a^4}}$



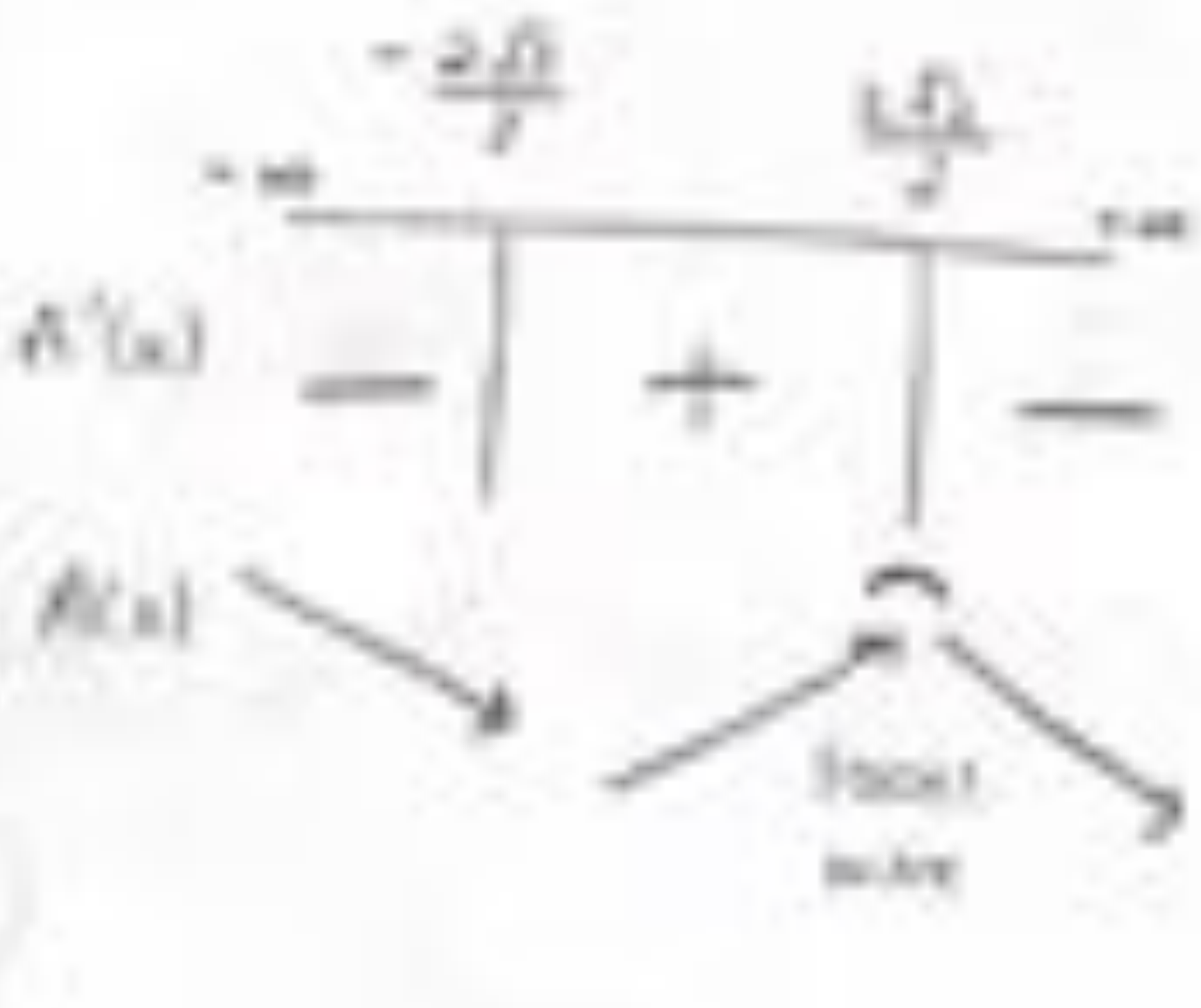
$(1, 1)$ local min point $\quad L = \sqrt{5}$

Question: $y = x^2$; $x = 0$, $y = 4$ max area rectangle



$A = x(4 - x^2) \qquad A = 4x - x^3 \qquad A' = 4 - 3x^2$

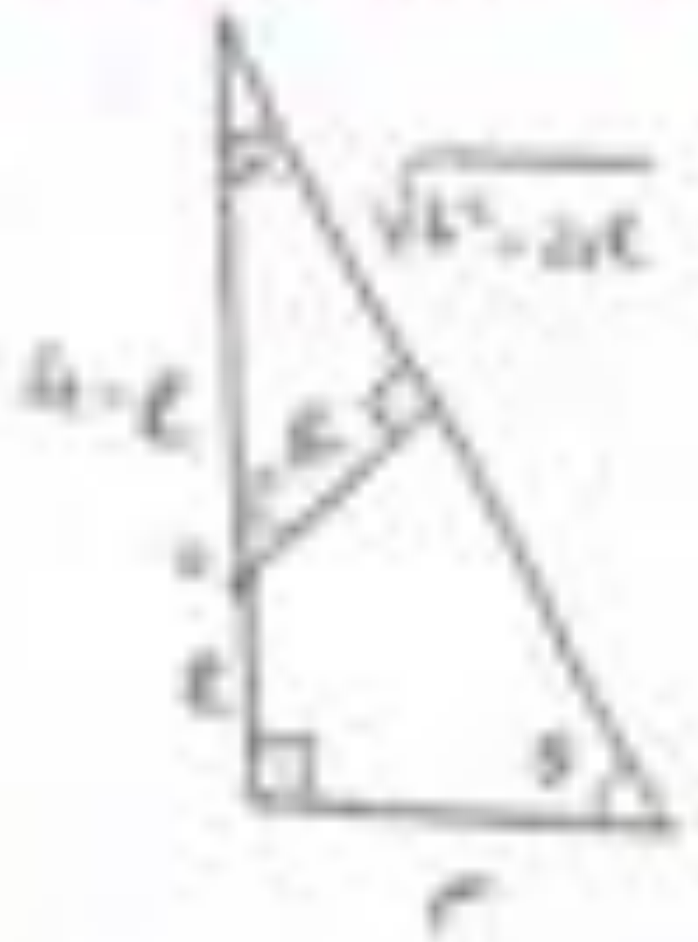
$A' = 0 \quad x = \mp \frac{2\sqrt{3}}{3}$



$x = \frac{2\sqrt{3}}{3}$ (length(x)) $\qquad 4 - x^2 = \frac{8}{3}$ (length(y)) $\qquad \frac{2\sqrt{3}}{3} \cdot \frac{8}{3} = $ Area $\frac{16\sqrt{3}}{9}$

Question:

Min volume for cone



$\frac{r}{h} = \frac{R}{\sqrt{h^2-2hR}}$ $\quad r = \frac{R\,h}{\sqrt{h^2-2hR}}$

$V(h) = \frac{1}{3}\pi \cdot \frac{R^2h^2}{(h^2-2hR)} \cdot h = \frac{1}{3}\pi\frac{R^2h^2}{h-2R}$ $\quad V'(h) = \frac{1}{3}\pi R^2\left(\frac{2h(h-2R)-h^2}{(h-2R)^2}\right)$

$V'(h) \Rightarrow \frac{\pi R^2(h^2-4hR)}{3(h-2R)^2} = 0$ $\quad h^2-4hR = 0$ $\quad h(h-4R) = 0$ $\quad h_1 = 0$ $\quad h_2 = 4R$

h		0		2R		4R	
V'(h)	+	|	−	||	−	|	+
V(h)	↗		↘		↘	local min	↗

$h = 4R$ $\quad r = \sqrt{2}R$ $\quad V(h) = \frac{1}{3}\pi \cdot 2R^2 \cdot 4R = \frac{8\pi R^3}{3}$

Question:

Max volume inscribed cone

$\frac{h-x}{r} = \frac{h}{R}$ $\quad h - x = \frac{hr}{R}$ $\quad h - \frac{hr}{R} = x$

$V(r) = \frac{1}{3}\pi r^2 \cdot \underbrace{h\left(1-\frac{r}{R}\right)}_{x}$ $\quad V'(r) \Rightarrow \frac{1}{3}\pi h\left(2r - \frac{3r^2}{R}\right) = 0$ $\quad \frac{2rR-3r^2}{R} = 0$

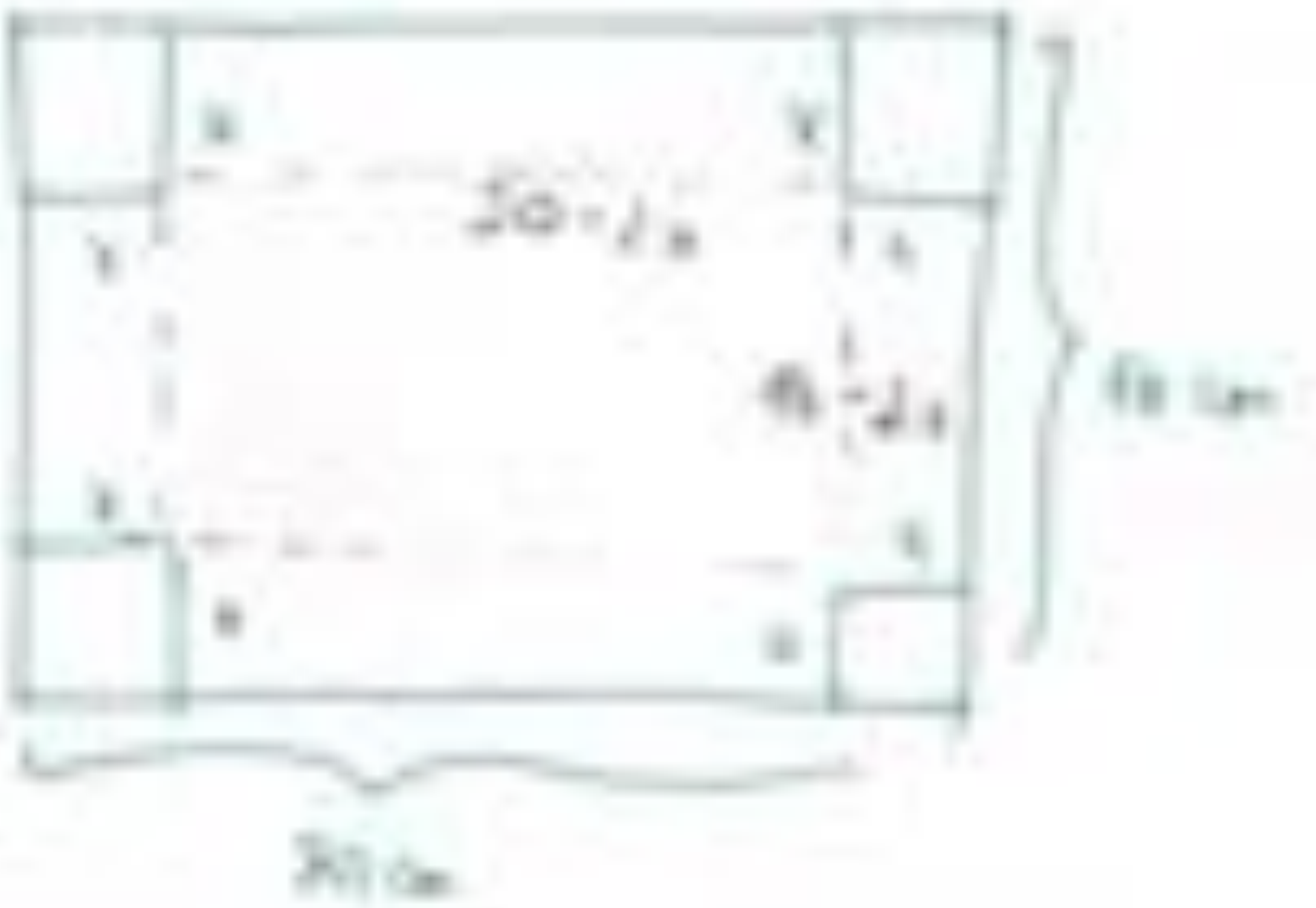
$r(2R-3r) = 0$ $\quad r_1 = 0$ $\quad r_2 = \frac{2R}{3}$

r		0		2R/3	
V'(r)	−	|	+	|	−
V(r)	↘		↗	local max	↘

$r = \frac{2R}{3}$ $\quad x = \frac{h}{3}$ $\quad V(2R/3) = \frac{1}{3}\pi\frac{4R^2}{9}\frac{h}{3}$

$V_{max} = \frac{4\pi R^2h}{81}$

Question:



. Max volume

$V(x) = (30-2x)(16-2x)\cdot x = 4\cdot(15-x)(8x-x^2) = 480x - 92x^2 + 4x^3$

$V'(x) = 480 - 184x + 12x^2 = 0$ $\quad 3x^2 - 46x + 120 = 0$

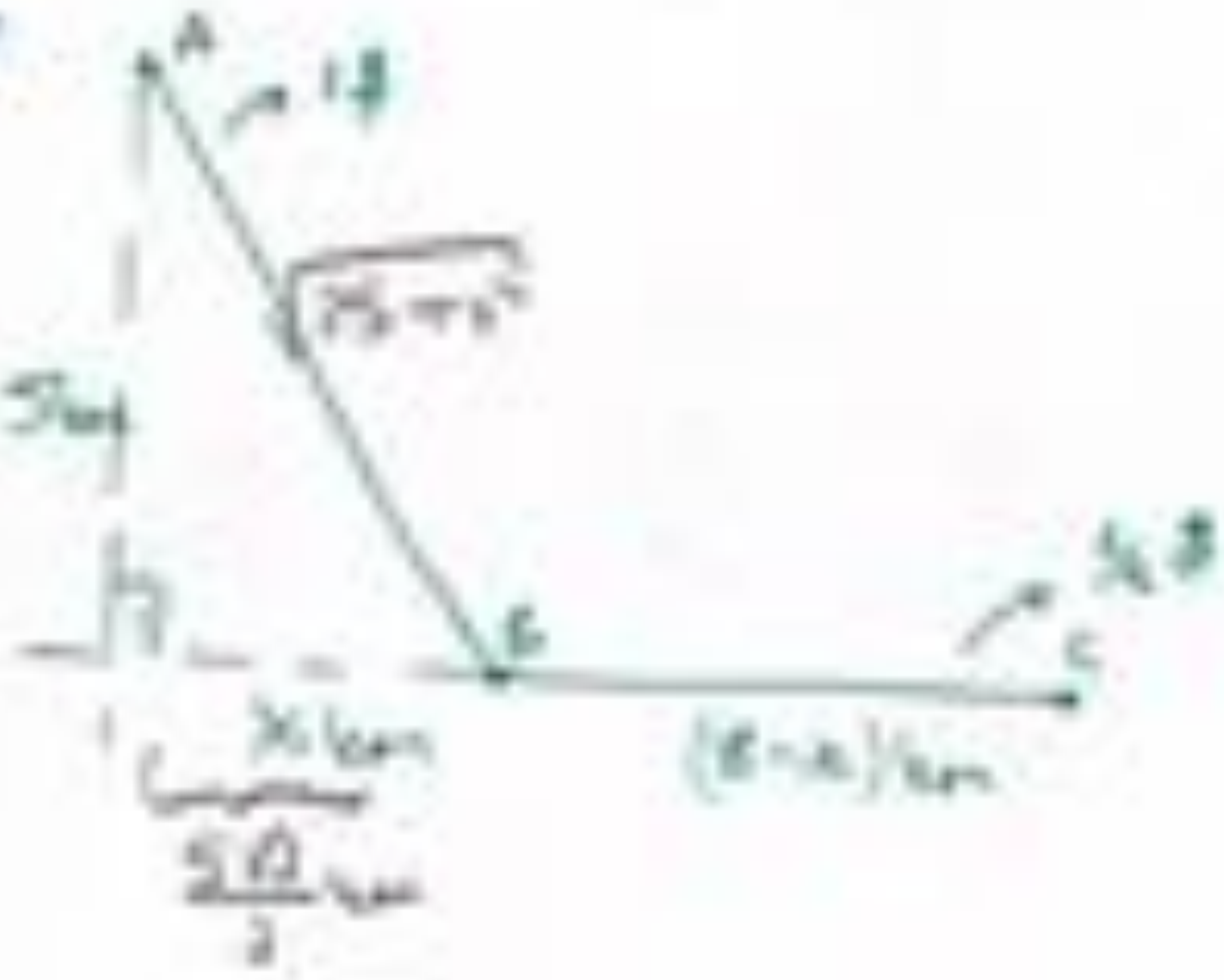
$\begin{matrix} x & -12 \\ 3x & -10 \end{matrix}$ $\quad (x-12)(3x-10)$

x	$-\infty$	$\frac{10}{3}$		12	∞
$V'(x)$	+	|	−	|	+
$V(x)$	↗	local max	↘		↗

$x = \frac{10}{3}$ $\quad V(10/3) = 4\left(15 - \frac{10}{3}\right)\left(\frac{80}{3} - \frac{100}{9}\right)$

$= 4\cdot\frac{35}{3}\cdot\frac{140}{9} = \frac{19600}{27}$ cm³

Question:



Min cost, find B point

$C(x) = 1\cdot\sqrt{25+x^2} + \frac{1}{2}(8-x)$

$C'(x) = \frac{2x}{2\sqrt{25+x^2}} - \frac{1}{2} \Rightarrow \frac{2x - \sqrt{25+x^2}}{2\sqrt{25+x^2}} = 0$

$4x^2 = 25 + x^2 \Rightarrow x = \pm\frac{5\sqrt{3}}{3}$

x		$-\frac{5\sqrt{3}}{3}$		$\frac{5\sqrt{3}}{3}$	
$C'(x)$	+	|	−	|	+
$C(x)$	↗		↘	local min	↗

Question:

Max volume cylinder inside given cone

$\frac{5}{3}\frac{\cancel{10}}{\cancel{6}} = \frac{10-h}{r}$ $\quad 5r = 30 - 3h$ $\quad h = \frac{30-5r}{3}$

$V(r) = \pi r^2\left(\frac{30-5r}{3}\right) = 10\pi r^2 - \frac{5\pi r^3}{3}$

$V'(r) \Rightarrow 20\pi r - 5\pi r^2 = 0$ $\quad 5\pi r(4-r) = 0$ $\quad r_1 = 0$ $\; r_2 = 4$

r		0		4	
$V'(r)$	−	|	+	|	−
$V(r)$	↘		↗	local max	↘

$r = 4$ $\quad h = \frac{10}{3}$ $\quad \pi\cdot 16\cdot\frac{10}{3} = \boxed{\frac{160\pi}{3}}$

Question: check the continuity of the given function [illegible]

$$f(x) = \frac{(x-3)\arcsin(x)}{\underset{\to 0}{x}\cdot\left(1-e^{\frac{1}{\underset{1}{x-1}}}\right)} \qquad D[\arcsin x] : [-1,1]$$

and possible discontinuous x values are 0 and 1

for x=0

$\lim_{x\to 0} f(x) = f(0)$ $\quad \lim_{x\to 0^-} \frac{(x-3)\arcsin(x)}{x\left(1-e^{\frac{1}{x-1}}\right)} = \frac{\arcsin(x) + (x-3)\cos(\arcsin(x))}{\left(1-e^{\frac{1}{x-1}}\right) + \frac{1}{(x-1)^2}\cdot x \cdot e^{\frac{1}{x-1}}} = \frac{-3e}{e-1}$

because of existence of the indeterminacy it is applicable L'Hospital

$f(0) = \frac{-3e}{e-1}$

$\lim_{x\to 0^+} = \frac{-3}{e-1}$ so, function is continuous at x=0

for x=1

$\lim_{x\to 1} f(x) = f(1)$ $\quad \lim_{x\to 1^-} \frac{(x-3)\arcsin(x)}{x\cdot\left(1-e^{\frac{1}{x-1}}\right)} = \frac{-2\ (\pi/2)}{1\cdot 1} = -\pi$

as the function is undefined at the right side of 1 so, limit value is $-\pi$ and then if we set f(1) as $-\pi$ we can say that function is continuous at x=1

Question: $f(x) = \sinh(x) + e^x + 1$, for $f^{-1}(x)$ find tangent line at P(0,2)

$(f^{-1})' = \frac{1}{f'(f^{-1}(x))}$ $\qquad f'(x) = \cosh(x) + e^x$ $\qquad f^{-1}(2) = 0$

$(f^{-1}(2))' = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(0)} = \frac{1}{2}$ $\qquad (y-0) = \frac{1}{2}(x-2)$ $\qquad y = \frac{x-2}{2}$

Question:

min area for the isosceles triangle

$\tan\alpha = \frac{x}{h} = \frac{r}{\sqrt{h^2-2hr}}$ $\qquad x = \frac{hr}{\sqrt{h^2-2hr}}$ $\qquad A(h) = \frac{h^2 r}{\sqrt{h^2-2hr}}$

$$A'(h) = \frac{2hr\sqrt{h^2-2hr} - \frac{h^2 r\,(\overset{(h-r)}{\cancel{2h-2r}})}{\cancel{2}\sqrt{h^2-2hr}}}{(h^2-2hr)} = \frac{2hr(h^2-2hr) - h^2 r(h-r)}{(h^2-2hr)\sqrt{h^2-2hr}}$$

$$\Rightarrow \frac{\overset{-3h^2r^2}{2h^3r - 4h^2r^2} - h^3r + h^2r^2}{h(h-2r)\sqrt{h^2-2hr}} = \frac{\cancel{h}\cdot hr(2h^2 - 3r - h)}{\cancel{h}(h-2r)\sqrt{h^2-2hr}}$$

$\neq 0 \quad 2r \qquad \neq 0 \quad 2r$

Question:

Linear approximation for r=3.01 cm (volume of cylinder)

$V = \pi r^2(2r) \quad V = 2\pi r^3 \quad V' = 6\pi r^2$

$V(r) \cong L(r) = V(r) + V'(r)(r-a)$ for $a=3$

$L(3) = V(3) + V'(3)(3{,}01-3)$

$L(3) = 54\pi + 54\pi(0{,}01) \quad L(3) = 54{,}54\pi\ cm^3$

Question: $y = \frac{e^x}{x^2-1}$ ⓐ $D: (-\infty,-1) \cup (-1,1) \cup (1,\infty)$ ⓑ $(0,-1)$ x intercept

ⓒ $\lim_{x\to -1^-} \frac{e^x}{x^2-1} = +\infty \quad \lim_{x\to -1^+} \frac{e^x}{x^2-1} = -\infty$ | $\lim_{x\to 1^-} \frac{e^x}{x^2-1} = -\infty \quad \lim_{x\to 1^+} \frac{e^x}{x^2-1} = +\infty$

x=−1 (vertical asymptote) | x=1 vertical asymptote

ⓓ $\lim_{x\to -\infty} \frac{e^x}{x^2-1} = \lim_{x\to -\infty} \frac{\frac{e^x}{x^2}}{1-\frac{1}{x^2}} = 0 \quad \lim_{x\to +\infty} \frac{\frac{e^x}{x^2}}{1-\frac{1}{x^2}} = 0$

There is no oblique asymptote

y=0 namely x-axis is a horizontal asymptote

ⓔ $y' = \frac{e^x(x^2-1) - 2x\,e^x}{(x^2-1)^2} \Rightarrow y' = \frac{e^x(x^2-2x-1)}{(x^2-1)^2}$ $\quad 1+\sqrt{2},\ 1-\sqrt{2}$; $\quad -1, 1$

x	−∞	−1		1−√2		1		1+√2	+∞
y'	+	‖	+		−	‖	−		+
y	↗		↗	local max	↘		↘	local min	↗

y is increasing on $(-\infty,-1) \cup (-1, 1-\sqrt{2}) \cup (1+\sqrt{2}, \infty)$

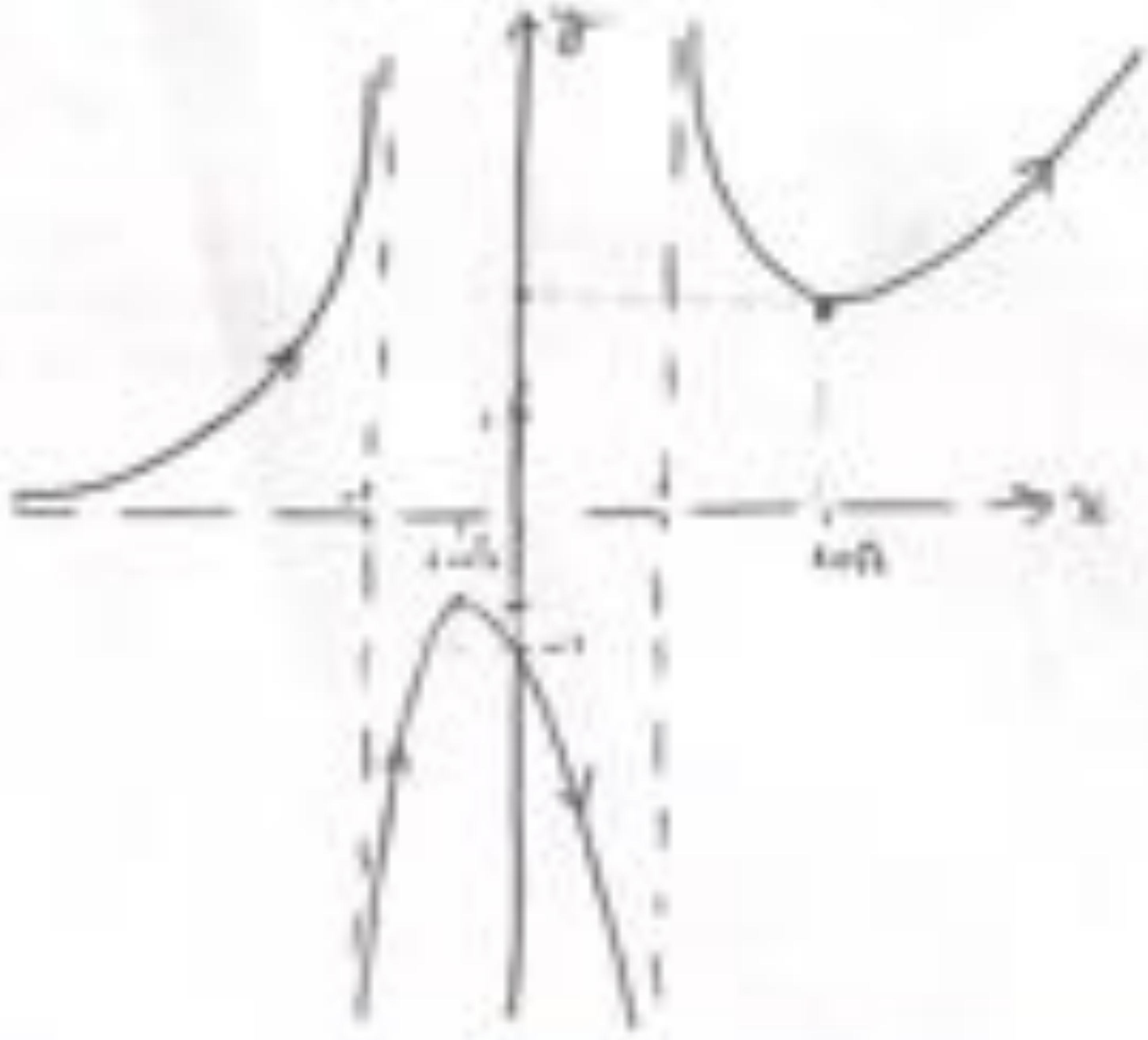
y is decreasing on $(1-\sqrt{2}, 1) \cup (1, 1+\sqrt{2})$

local max point $\left(1+\sqrt{2}, \frac{e^{1+\sqrt{2}}}{2+2\sqrt{2}}\right)$ local min point $\left(1-\sqrt{2}, \frac{e^{1-\sqrt{2}}}{2-2\sqrt{2}}\right)$

ⓕ $y'' = \frac{\left[e^x(x^2-2x-1) + e^x(2x-2)\right]\cdot(x^2-1)^2 - 2(x^2-1)\,2x\cdot e^x(x^2-2x-1)}{(x^2-1)^4}$

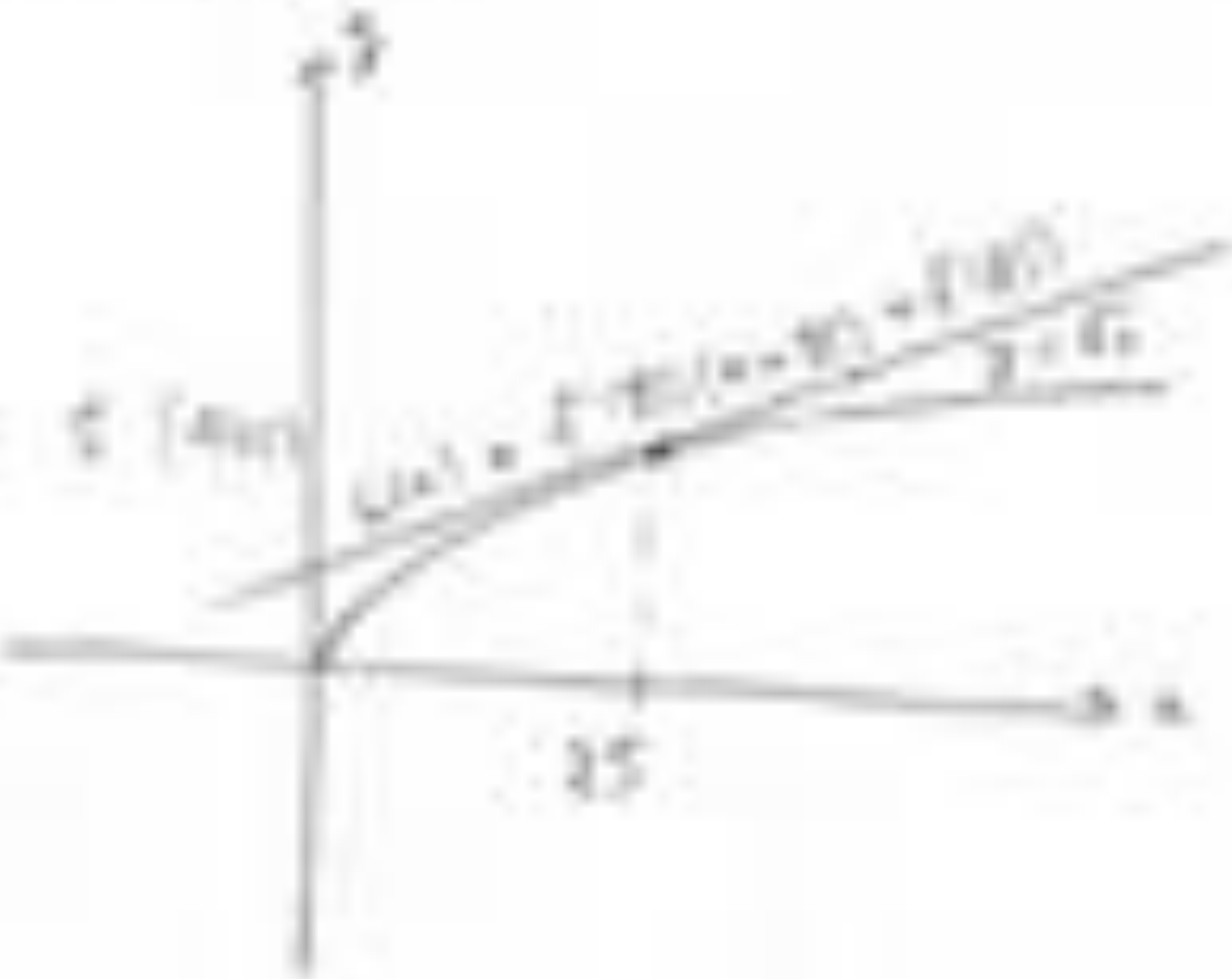
$\frac{e^x(x^2-3)\cdot(x^2-1)^2 - 4x(x^2-1)\cdot e^x(x^2-2x-1)}{(x^2-1)^4} = \frac{e^x(x^2-1)\left((x^2-1)(x^2-3) - 4x\cdot(x^2-2x-1)\right)}{(x^2-1)^4}$ $\quad x^4-4x^2+3-4x^3+8x^2+4x$

$y'' = \frac{e^x(x^2-1)\cdot(x^4-4x^3+4x^2+4x+3)}{(x^2-1)^4}$ ([illegible])



Question: $\sqrt{26}$, linear approximation

let $f(x) = \sqrt{x}$



$f(x) \cong L(x) = f'(25)(x-25) + f(25)$

$f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(25) = \frac{1}{10}$

$L(26) = \frac{1}{10}(26-25) + 5 \Rightarrow \sqrt{26} \cong 5.1$

Question: $\sqrt[4]{626} \cong 5{,}002$

$f(x) \cong L(x) = f'(a)(x-a) + f(a)$ let $f(x) = \sqrt[4]{x}$ central point (a) = 625

$f'(x) = \frac{1}{4\sqrt[4]{x^3}}$ $f(625) = 5$ $f'(625) = \frac{1}{500}$

$f(626) \cong L(626) = \frac{1}{500}(626-625) + 5$

Question: $(1{,}05)^{1{,}05}$

let $f(x) = x^x$ $f(x) \cong L(x) = f'(a)(x-a) + f(a)$ central point (a) = 1

$\ln[f(x)] = x \ln x \Rightarrow \frac{1}{\underset{x^x}{f(x)}} f'(x) = \ln x + x \frac{1}{x} \Rightarrow f'(x) = x^x(\ln x + 1)$

$f(1) = 1$ $f'(1) = 1$ $L(1{,}05) = 1(1{,}05 - 1) + 1 \Rightarrow (1{,}05)^{1{,}05} \cong 1{,}05$

Question: $\sin 25°$

let $f(x) = \sin x$ $f(x) \cong L(x) = f'(a)(x-a) + f(a)$ central point (a) = 30° $\boxed{1° = \frac{\pi}{180}}$

$f'(x) = \cos x$ $f(30) = \frac{1}{2}$ $f'(30) = \frac{\sqrt{3}}{2}$

$f(25) \cong L(25) = \frac{\sqrt{3}}{2}(25 - 30) + \frac{1}{2} \Rightarrow \sin 25° \cong \frac{1}{2} - \frac{\sqrt{3}\pi}{180}$

Question: $\arctan(0.98)$

let $f(x) = \arctan x$ $f(x) \cong L(x) = f'(a)(x-a) + f(a)$ central point (a) = 1

$\frac{d}{dx}(\arctan x) = \frac{1}{\sec^2(\arctan x)}$

$f'(x) = \frac{1}{1+x^2} \Rightarrow f'(1) = \frac{1}{2}$ $f(1) = \frac{\pi}{4}$

$f(0.98) \cong L(0.98) = \frac{1}{2}(0.98 - 1) + \frac{\pi}{4} \Rightarrow \arctan(0.98) \cong \frac{\pi}{4} - 0{,}01$

Question: $\sqrt{1.02}$ Let $f(x) = \sqrt{1+x}$ $f'(x) = \frac{1}{2\sqrt{1+x}}$ at 0 $f(0) = 1$ $f'(0) = \frac{1}{2}$

$f(1.02) \approx L(1.02) = 1 + \frac{1}{2}(1.02-1) \Rightarrow \sqrt{1.02} \approx 1.01$

or Let $g(x) = \sqrt{x}$ $g'(x) = \frac{1}{2\sqrt{x}}$ at 1 $g(1) = 1$ $g'(1) = \frac{1}{2}$

Question: $f(x) = \cos x$; at $\frac{\pi}{2}$ find linearization

$f'(x) = -\sin x$ $f(\frac{\pi}{2}) = 0$ $f'(\frac{\pi}{2}) = -1$ $\cos(1.45) \approx L(1.45) = 0 + (-1)(1.45 - \frac{\pi}{2})$

$\cos(1.45) \approx \frac{\pi}{2} - 1.45 \approx 0.12$

Question: $f(x) = x^3 + x$; $f^{-1}(10.1)$ $\underset{10}{f(x)} = \underset{x=2}{x(x^2+1)}$ $\frac{d(f^{-1}(x))}{dx} = \frac{1}{f'(f^{-1}(x))}$ ★

$f'(x) = 3x^2+1$ $f'(2) = 13$

$f^{-1}(10.1) \approx L(10.1) = \underbrace{f^{-1}(10)}_{2} + (f^{-1}(10))'(10.1-10)$

$2 + \frac{1}{13}(0.1)$ $1000 \div 13 = 0.00769$ $f^{-1}(10.1) \approx 2.00769$

Question: $(1.02)^{1/3}$ $f(x) = \sqrt[3]{x}$ at 1 $f'(x) = \frac{1}{3\sqrt[3]{x^2}}$ $f(1) = 1$ $f'(1) = \frac{1}{3}$

$f(1.02) \approx L(1.02) = 1 + \frac{1}{3}(1.02-1)$ $0.02 \div 3 = 0.006$ $(1.02)^{1/3} \approx 1.006$

Question: $t > 0$ and $F(t) = \int_{2}^{\sqrt{t}} \cos(x^2)\,dx$, $F'(\frac{\pi}{4}) = ?$

$F'(x) = \cos(\sqrt{t})^2 \cdot \frac{1}{2\sqrt{t}} - \cancel{\cos(2)^2 \cdot 0}^{0} \Rightarrow F'(x) = \frac{\cos t}{2\sqrt{t}} = \frac{\frac{\sqrt{2}}{2}}{2\sqrt{\frac{\pi}{4}}} = \frac{\sqrt{2\pi}}{2\pi}$

Question: $F(x) = \frac{1}{x}\underbrace{\int_1^x [2t - F'(t)]\,dt}_{K(x)} \Rightarrow F'(1) = ?$

$F'(x) = \left(-\frac{1}{x^2}\right)\cdot\left(\int_1^x (2t - F'(t))\,dt\right) + \frac{1}{x}\left((2x - F'(x))\cdot 1 - (2\cdot 1 - F'(1))\cdot 0\right)$

$F'(1) = (-1)\cancel{\left(\int_1^1 (2t - F'(t))\,dt\right)}^{0} + 1\,(2 - F'(1))\cdot 1 - 0$ $F'(1) = 2 - F'(1)$ $F'(1) = 1$

Question: $F(x) = \frac{1}{x}\left(\int_1^x \left[e^{1-\sqrt{t}} - F'(2-t)\right]dt\right)$, $F'(1) = ?$

$F'(x) = \left(-\frac{1}{x^2}\right)\left(\int_1^x (e^{1-\sqrt{t}} - F'(2-t))\right) + \frac{1}{x}\left((e^{1-\sqrt{x}} - F'(2-x))\cdot 1 - \cancel{(e^{1-\sqrt{1}} - F'(2-1))}\cdot 0\right)$

$F'(1) = 0 + 1\cdot(e^0 - F'(1) - 0)$ $F'(1) = 1/2$

Question: $f(x) = \frac{x^2+1}{x^2+x+1}$, find the absolute min and max at $[-2,2]$

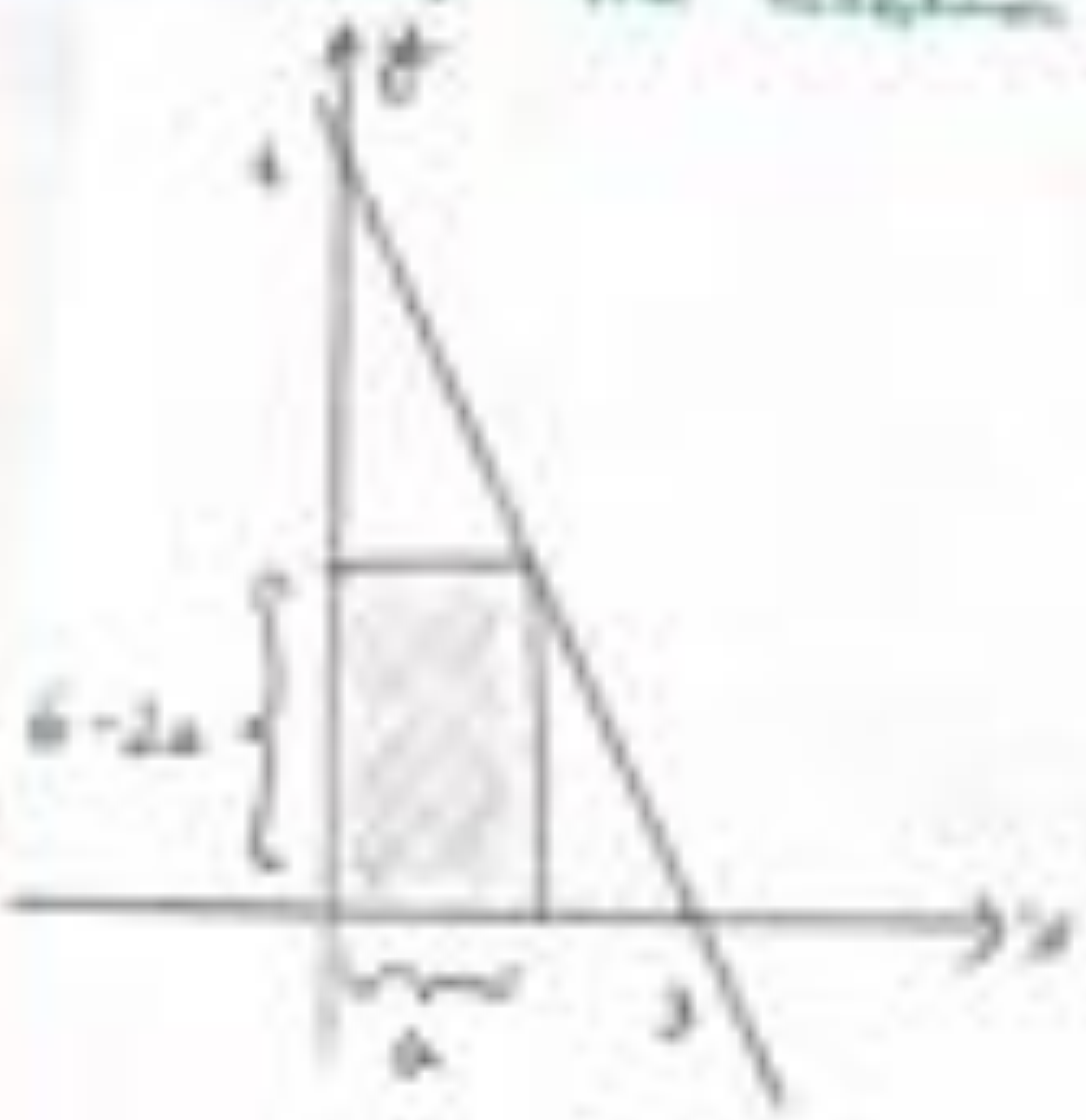
$f'(x) = \frac{2x\,(x^2+x+1) - (x^2+1)(2x+1)}{(x^2+x+1)^2} = \frac{(x^2-1)}{(x^2+x+1)^2}$ $x_1 = 1$, $x_2 = -1$

$f(1) = \frac{2}{3}$, $f(-1) = 2$ / $f(-2) = \frac{5}{3}$, $f(2) = \frac{5}{7}$

$f(1) = \frac{2}{3}$ $(1, \frac{2}{3}) \Rightarrow$ abs min $f(x)$ $x \in [-2,2]$

$f(-1) = 2$ $(-1, 2) \Rightarrow$ abs max $f(x)$ $x \in [-2,2]$

Question: Find the maximum rectangle area between x-axis, y-axis and $2x+y=6$



$A = a\cdot(6-2a) = 6a-2a^2$ $A' = 6-4a$ $a = \frac{3}{2}$

$A = \frac{3}{2}\cdot 3 = \frac{9}{2}$ br²

Question: Linearization $L(x) = f(a) + f'(a)(x-a)$

i) $\sqrt{37} = ?$ 6,083   ii) $\sqrt{4.2} = ?$ 2.05   iii) $\ln(1.02) = ?$

$f(x) = \sqrt{x}$ $f'(x) = \frac{1}{2\sqrt{x}}$

$a = 36$

$L(37) = 6 + \frac{1}{12}\cdot(37-36) = 6,08\bar{3}$

$100 \,|\, 12$, -96, 40, $0,08\bar{3}$ $\sqrt{37} \cong 6,08\bar{3}$

$g(x) = \ln x$ $g'(x) = \frac{1}{x}$ $a=1$

$g(1)=0$ $g'(1)=1$

$L(1.02) = 0 + 1\cdot(1.02-1)$

$\ln(1.02) \cong 0.02$

$L(4.2) = 2 + \frac{1}{4}(4.2-4) = 2 + \frac{1}{20}$ $100 \,|\, 20$, -100, 0, 0.05

$\Rightarrow \sqrt{4.2} \cong 2,05$

(v) $(2.001)^5$ $L(x) = f(a) + f'(a)(x-a)$

$f(x) = x^5$ $f'(x) = 5x^4$ $a = 2$

$L(2.001) = 32 + 80\,(2.001-2)$

$32 + 80 \cdot \frac{1}{1000} \Rightarrow 32.08$

Question: $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3ki+2n}{8in+n^2} = \lim_{n\to\infty} \sum_{i=1}^{n} \overset{f(x_i)}{\frac{3k_i+2n}{8i+n}} \cdot \overset{dx}{\frac{1}{n}}$ $\Delta x = \frac{b-a}{n}$ $x_i = a + \Delta x\, i$ $a=0$ $b=1$

$\frac{3k\frac{i}{n}+2}{8\frac{i}{n}+1}$ $\frac{3kx+2}{8x+1}$

HINT: interval $[0,1]$

$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{3i+2n}{8i+n} \Rightarrow \int_0^1 \left(\frac{3x+2}{8x+1}\right) dx$

$\frac{3\frac{i}{n}+2}{8\frac{i}{n}+1}$

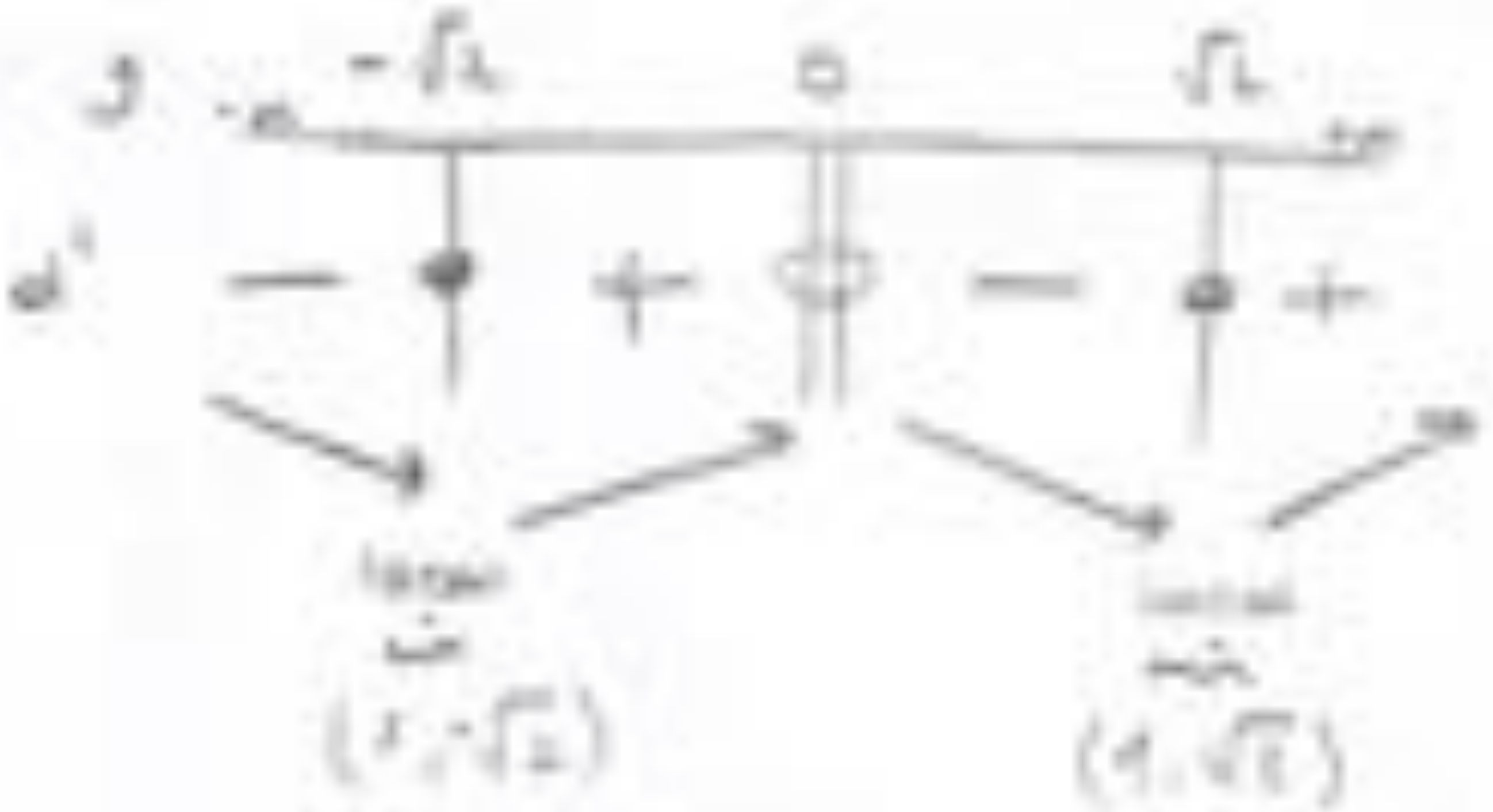
$\frac{3x+2}{8x+1}$

Question: $x \cdot y^2 = 2$, find point(s) closest to the origin

$\left(\frac{2}{y^2}, y\right)$ $\quad d = \sqrt{(y-0)^2 + \left(\frac{2}{y^2}-0\right)^2} = \sqrt{y^2 + \frac{4}{y^4}} = \sqrt{\frac{y^6+4}{y^4}}$

$$d' = \frac{1}{2\sqrt{\frac{y^6+4}{y^4}}} \cdot \frac{6 y^5 \cdot y^4 - 4 \cdot y^3 (y^6+4)}{y^8} = \frac{2y^9 - 16y^3}{2 \cdot y^8\sqrt{\frac{y^6+4}{y^4}}} = \frac{(y^6-8)}{y^3\sqrt{y^6+4}}$$

$$\Rightarrow \frac{\overset{\pm\sqrt{2}}{(y^2-2)}\,(y^4+2y^2+4)}{\underset{\to 0}{y^3}\sqrt{y^6+4}}$$



Question: $\int_0^2 (x^2+2x+3)\,dx$ $\quad \int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x$ $\quad b=2 \quad a=0 \quad \Delta x = \frac{2}{n}$

$x_i = a + \Delta x \cdot i$ $\quad \boxed{x_i = \frac{2i}{n}}$

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^2 + 2\left(\frac{2i}{n}\right) + 3\right]\cdot\frac{2}{n}\right) = \lim_{n\to\infty}\left[\sum_{i=1}^{n}\left(\frac{8i^2}{n^3} + \frac{8i}{n^2} + \frac{6}{n}\right)\right]$$

$$\Rightarrow \lim_{n\to\infty}\left(\underbrace{\frac{8}{n^3}\sum_{i=1}^{n} i^2}_{} + \underbrace{\frac{8}{n^2}\sum_{i=1}^{n} i}_{} + \frac{6}{n}\underbrace{\sum_{i=1}^{n} 1}_{n}\right) = \lim_{n\to\infty}\left[\frac{4}{3}\left(2+\frac{3}{n}+\frac{1}{n^2}\right) + 4\left(1+\frac{1}{n}\right) + 6\right]$$

$\frac{8}{n^3}\frac{n(n+1)(2n+1)}{6}$ $\quad \frac{8}{n^2}\frac{n(n+1)}{2}$

$\frac{4}{3}\frac{2n^2+3n+1}{n^2}$ $\quad = 4\left(1+\frac{1}{n}\right)$

$\frac{4}{3}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)$

$\frac{8}{3} + \cancel{\frac{1}{n}} + 4 + 6 = \frac{38}{3}$

Question: $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left[8\left(1+\frac{i}{n}\right)^3 + 2\left(1+\frac{i}{n}\right)^2\right]$ $\quad \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)\,dx$

$\frac{b-a}{n} = \Delta x$ $\quad a + \Delta x\, i = x_i$, $1 + \frac{1}{n} i$ $\quad a=1 \quad b=2$ $\quad f(x) = 8x^3 + 2x^2$ $\quad \int_1^2 (8x^3+2x^2)\,dx$

Question: $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi^2 i}{n^2}\cos^4\left(\frac{\pi i}{n}\right)$ $\quad \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)\,dx$

$\boxed{\int_0^\pi (x\cos^4 x)\,dx}$ $\quad \underbrace{\frac{\pi}{n}}_{\Delta x}\,\underbrace{\cos^4\left(\frac{\pi i}{n}\right)\cdot\frac{\pi i}{n}}_{f(x_i)}$ $\quad \frac{b-a}{n} = \Delta x = \frac{\pi}{n}$ $\quad x_i = \underset{0}{a} + \underset{\frac{\pi}{n}}{\Delta x}\, i$ $\quad \boxed{x_i = \frac{\pi i}{n}}$ $\quad \boxed{b=\pi \quad a=0}$

$f(x) = x \cdot \cos^4 x$

Question: $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\frac{i}{n^2}}{\left(\frac{2i}{n}+1\right)^3}$ $\quad \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)\,dx$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\underset{\Delta x}{\frac{1}{n}}\cdot\frac{\frac{i}{n}}{\left(\frac{2i}{n}+1\right)^3} \qquad \int_0^1\frac{x}{(2x+1)^3}\,dx$$

$x_i = \underset{0}{a} + \underset{\frac{1}{n}}{\Delta x}\, i$

$a=0 \quad b=1 \quad f(x) = \frac{x}{(2x+1)^3}$

Question: $x = \tan 2\theta \quad y = -4 + \sec 2\theta \quad \left.\frac{d^2y}{dx^2}\right|_{\theta=\frac{\pi}{8}} = ?$

$(\tan 2\theta)' = \left(\frac{\sin 2\theta}{\cos 2\theta}\right)' = \frac{2\cos 2\theta \cdot \cos 2\theta + 2\sin 2\theta \cdot \sin 2\theta}{\cos^2 2\theta}$ $= 2\sec^2 2\theta$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2 \sin 2\theta \cdot \sec^2 2\theta}{2 \sec^2 2\theta} = \sin 2\theta$

$(-4+\sec 2\theta)' = \left(-4 + \frac{1}{\cos 2\theta}\right)' = 0 - \frac{-2\sin 2\theta \cdot 1}{(\cos^2 2\theta)}$ $= 2\sin 2\theta \cdot \sec^2 2\theta$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{2\cos 2\theta}{2\sec^2 2\theta} = \cos^3 2\theta \Rightarrow \cos^3\left(2\cdot\frac{\pi}{8}\right) = \frac{1}{[illegible]}$

Question: $\lim_{x\to 0^+}(6^x - 2^x + 1)^{1/x} = L$ $\hookrightarrow 1^\infty$ $\quad \exp\left[\lim_{x\to 0^+}\left(\frac{\ln(6^x - 2^x + 1)}{x}\right)\right] = L$

Note: $(\ln y)' = (\ln a^x)' \Rightarrow \frac{y'}{y} = \ln a \Rightarrow y' = y \cdot \ln a \quad (6^x)' = 6^x \ln 6 \quad (2^x)' = 2^x \ln 2$

$\Rightarrow \exp\left[\lim_{x\to 0^+}\left(\frac{\frac{6^x \ln 6 - 2^x \ln 2 + 0}{6^x - 2^x + 1}}{1}\right)\right] = \exp\left(\lim_{x\to 0^+}(\ln 6 - \ln 2)\right) = \exp\left(\ln\left(\frac{6}{2}\right)\right) = e^{\ln 3} = 3$

Question: $f(x) = x^5 - 2x^2 + x$, find a root on $[0,1]$ by using Rolle Theorem

For the continuous $f: [a,b] \to \mathbb{R}$ function and f is differentiable for $\forall x \in (a,b)$. If $f(a) = f(b)$, there should exists at least one c point satisfies: $f'(c) = 0$.

$a = 0 \;,\; b = 1 \qquad f(0) = 0 \quad f(1) = 0$

$f'(x) = 5x^4 - 4x + 1 \qquad f''(x) = 20x^3 - 4 = 4(5x^3 - 1) = 4(\sqrt[3]{5}x - 1)\underbrace{(\sqrt[3]{25}x^2 + \sqrt[3]{5}x + 1)}_{\text{always positive}}$

$\hookrightarrow \frac{1}{\sqrt[3]{5}}$

	$-\infty$	$\frac{1}{\sqrt[3]{5}}$	∞
$f''(x)$	$-$		$+$

Because of the sign change there should exists at least one root that satisfies $f'(x) = 0$

Question: $\int_{-3}^{1}(1 - x^3)\,dx \Rightarrow \int_a^b f(x_i)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x \quad b = 1 \quad a = -3$

$f(x) = 1 - x^3 \quad \Delta x = \frac{1-(-3)}{n} = \frac{4}{n}$

$x_i = a + \Delta x\, i = -3 + \frac{4}{n} i$

$\lim_{n\to\infty}\sum_{i=1}^{n}\left[1 - \left(-3 + \frac{4i}{n}\right)^3\right]\frac{4}{n}$

$\binom{3}{0}\left(\frac{4i}{n}\right)^3(-3)^0 + \binom{3}{1}\left(\frac{4i}{n}\right)^2(-3)^1 + \binom{3}{2}\left(\frac{4i}{n}\right)^1(-3)^2 + \binom{3}{3}\left(\frac{4i}{n}\right)^0(-3)^3$

$= \frac{-16 \cdot i^3}{n^4} + \frac{32 i^2}{n^3} + \frac{-108 i}{n^2} + \frac{56}{n}$

$\lim_{n\to\infty}\left(\frac{-16}{n^4}\sum_{i=1}^{n} i^3 + \frac{32}{n^3}\sum_{i=1}^{n} i^2 + \frac{-108}{n^2}\sum_{i=1}^{n} i + \frac{56}{n}\sum_{i=1}^{n} 1\right)$

$\lim_{n\to\infty}\left(\frac{-16}{n^4}\cdot\left[\frac{n(n+1)}{2}\right]^2 + \frac{32}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{-108}{n^2}\left(\frac{n\cdot(n+1)}{2}\right) + \frac{56}{n}\, n\right)$

$\lim_{n\to\infty}\left(\frac{-4(n^2+2n+1)}{n^2} + \frac{32(n+1)(2n+1)}{n^2} + \frac{-54(n+1)}{n} + 56\right) = 22$

$\to (1 + \frac{2}{n} + \frac{1}{n^2})$ $\quad \frac{16\left(\frac{2n^2}{n^2} + \frac{3n}{n^2} + \frac{1}{n^2}\right)}{1}$ $\quad -54 \quad 56$

Question : Sketch $f(x) = \frac{x^2-5}{x^2-4}$ i) Domain : $(-\infty,-2) \cup (-2,2) \cup (2,\infty)$. ii) $(-\sqrt{5},0)$ [illegible] $(\sqrt{5},0)$ [illegible] $(0, f(0))$ [illegible]

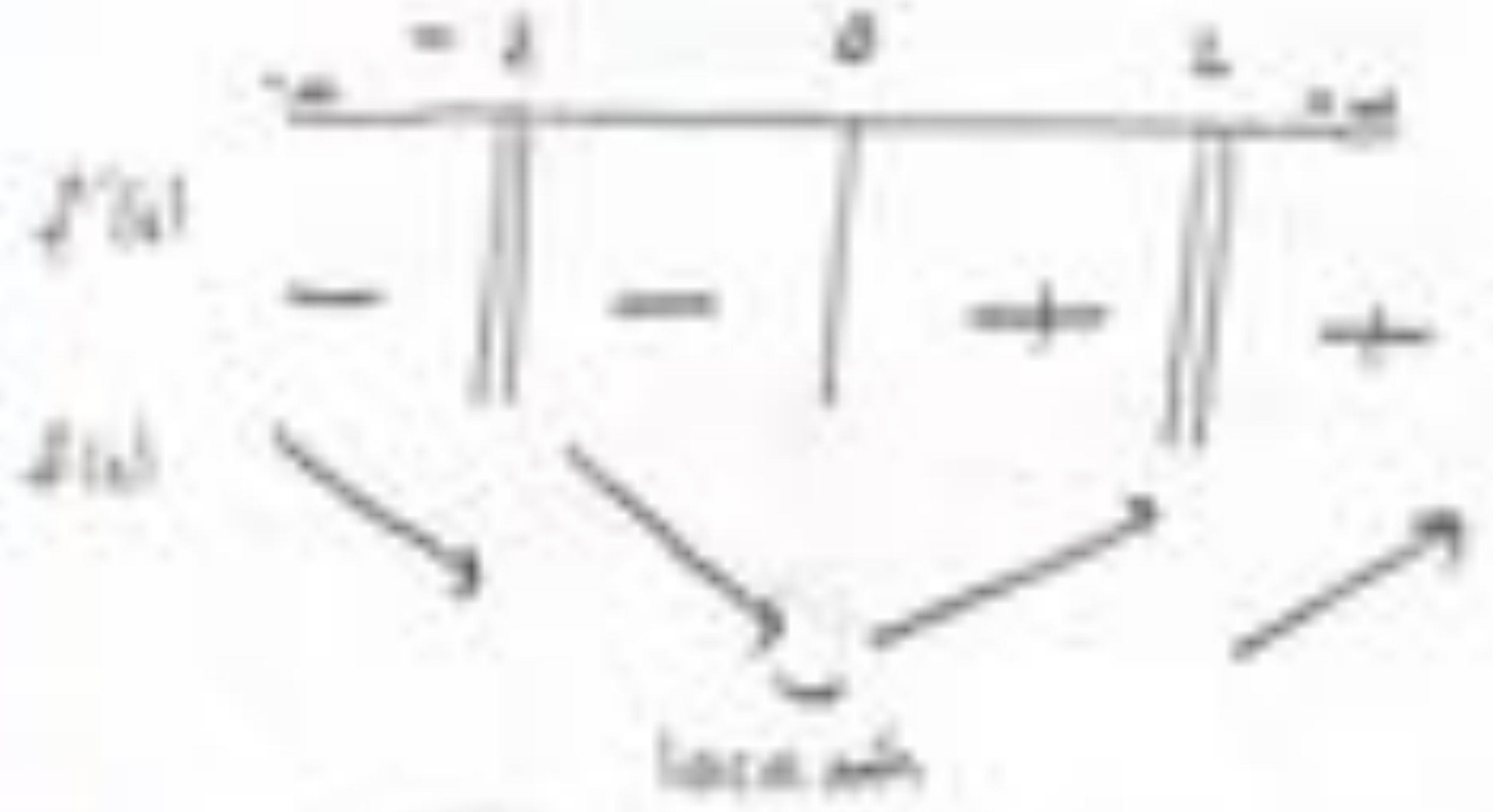
iii) $\lim_{x\to -2^-}\left(\frac{x^2-5}{x^2-4}\right) = -\infty$ $\lim_{x\to -2^+}\left(\frac{x^2-5}{x^2-4}\right) = +\infty$ $x=-2$ is a vertical asymptote

$\lim_{x\to 2^-}\left(\frac{x^2-5}{x^2-4}\right) = +\infty$ $\lim_{x\to 2^+}\left(\frac{x^2-5}{x^2-4}\right) = -\infty$ $x=2$ is a vertical asymptote

$\lim_{x\to -\infty}\left(\frac{x^2-5}{x^2-4}\right) = 1$ $\lim_{x\to +\infty}\left(\frac{x^2-5}{x^2-4}\right) = 1$ $y=1$ is a horizontal asymptote

$$\begin{array}{r|l} x^2-5 & x^2-4 \\ \hline -x^2+4 & 1 \\ \hline -1 & \end{array} \Rightarrow 1 + \frac{-1}{x^2-4} \qquad \lim_{x\to\pm\infty}\left(\left(1+\frac{-1}{x^2-4}\right)-1\right) = \lim_{x\to\pm\infty}\left(\frac{-1}{x^2-4}\right) = 0$$

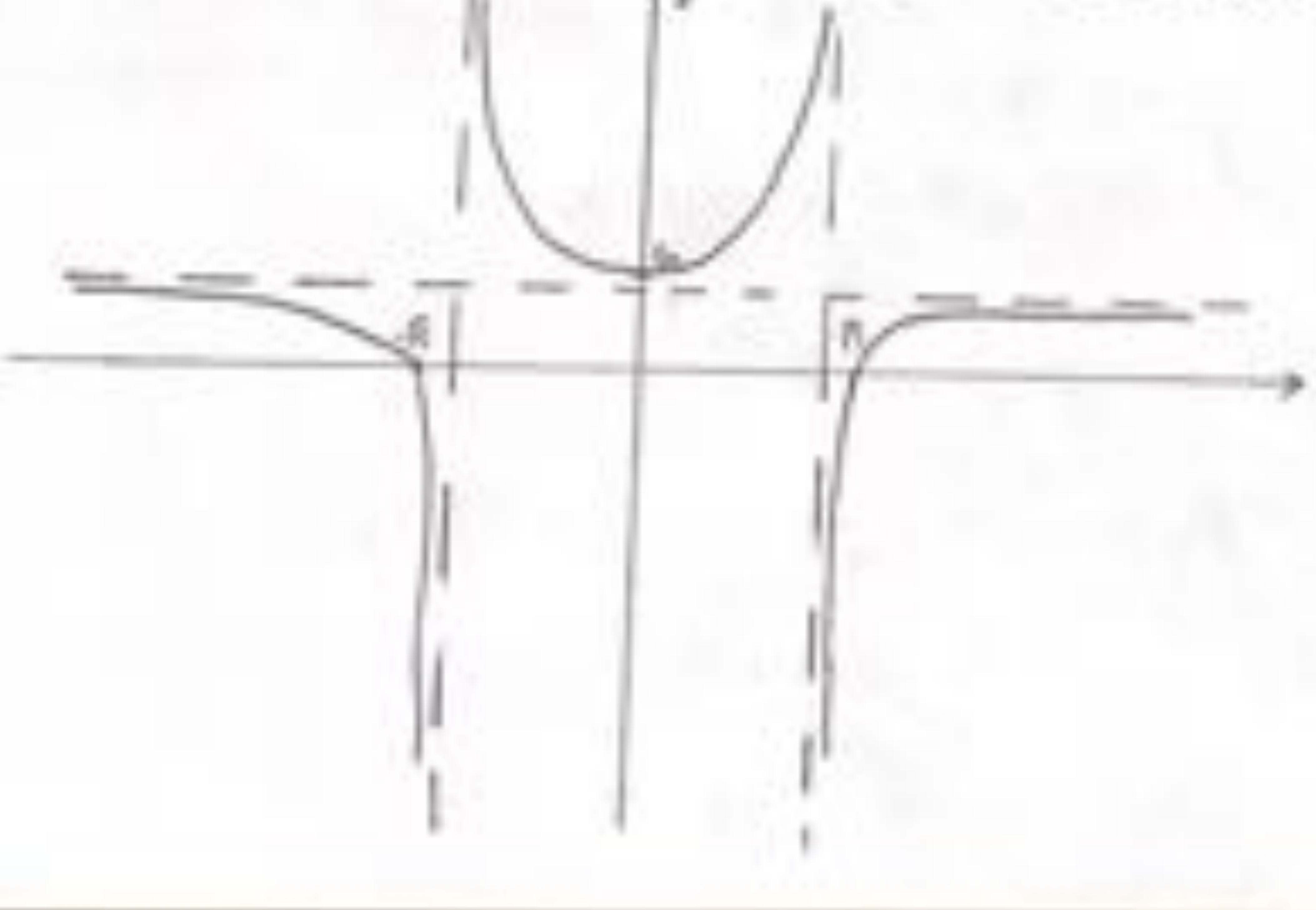
iv) $$f'(x) = \frac{2x(x^2-4) - 2x.(x^2-5)}{(x^2-4)^2} = \frac{2x(x^2-4-x^2+5)}{(x^2-4)^2} = \frac{2x^{\to x=0}}{\underbrace{(x^2-4)^2}_{x\neq\mp 2}}$$



- f is increasing on $(0,2) \cup (2,\infty)$
- f is decreasing on $(-\infty,-2) \cup (-2,0)$
- $(0, \frac{5}{4})$ is a local minima (extremum)

v) $$f''(x) = \frac{\overset{2(x^2-4)(x^2-4)}{2.(x^2-4)^2} - 2x.2(x^2-4).2x}{(x^2-4)^4} = \frac{2(x^2-4)\left[(x^2-4)-4x^2\right]}{(x^2-4)^4} = \frac{-2.(x^2-4)(3x^2+4)}{(x^2-4)^4}$$

- f is concave upward on $(-2,2)$
- f is concave downward on $(-\infty,-2) \cup (2,\infty)$
- There is no existing inflection point since the values change sign of second derivative are not in domain



Question: $\int_0^x f(t)\,dt = x + \int_x^1 \frac{f(t)}{1+t^2}\,dt \Rightarrow \int_0^x \frac{(2+t^2)\cdot f(t)}{1+t^2}\,dt = \underbrace{x}_{g(x)}$ $g(x) = x$

$\left(\frac{(2+x^2)\cdot f(x)}{(1+x^2)}\cdot 1 - 0\right) = g'(x) \Rightarrow f(x) = \frac{1+x^2}{2+x^2}$

$\left(\int_{a(t)}^{b(t)} f(x,t)\,dx\right)'_t = \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t}\,dx + f(b(t),t)\cdot b'(t) - f(a(t),t)\cdot a'(t)$

Question: $F(t) = \int_0^{t^2} \arcsin\left(\frac{x}{t^2}\right)dx$, $F'(t) = \pi t$ $(\arcsin(x))' = \frac{1}{\sqrt{1-x^2}}$

$F'(t) = \int_0^{t^2} \underbrace{\frac{\partial\left(\arcsin\left(\frac{x}{t^2}\right)\right)}{\partial t}}\,dx + \left(\underbrace{\arcsin\left(\frac{t^2}{t^2}\right)\cdot 2t}_{\pi t} - \cancel{\arcsin\left(\frac{0}{t^2}\right)\cdot 0}\right)$

$\int_0^{t^2}\left(\frac{\frac{-2x}{t^3}}{\sqrt{1-\left(\frac{x}{t^2}\right)^2}}\right)dx \Rightarrow \int_0^{t^2} \frac{-2x}{t^3\frac{\sqrt{t^4-x^2}}{t^2}}\,dx \Rightarrow \int_0^{t^2} \frac{-2x}{t\sqrt{t^4-x^2}}\,dx$ $t^4-x^2=u$ $-2x\,dx = du$

$\Rightarrow \frac{1}{t}\int_0^{t^4}\frac{1}{\sqrt{u}}\,du \Rightarrow \frac{1}{t}\left(\frac{\sqrt{u}}{2}\Big|_0^{t^4}\right) = 0$

Question: $F(u) = \int_u^{u^2}\frac{\sin(ut)}{t}\,dt \Rightarrow F'(u)$

$F'(u) = \int_u^{u^2}\frac{\partial \sin(t,u)}{\partial u}\,dt + \left(\underbrace{\frac{\sin(u\cdot u^2)}{u^2}\cdot 2u - \frac{\sin(u\cdot u)}{u}\cdot 1}_{\frac{2\sin(u^3) - \sin(u^2)}{u}}\right)$

$\int_u^{u^2}\frac{t\cdot\cos(ut)}{t}\,dt$ $ut = a$ $u\,dt = da$

$\int_{u^2}^{u^3}\cos(a)\,\frac{da}{u} \Rightarrow \frac{1}{u}\left(\sin(a)\Big|_{u^2}^{u^3}\right) = \frac{\sin(u^3)-\sin(u^2)}{u}$

$F'(u) = \frac{3\sin(u^3) - 2\sin(u^2)}{u}$

Question 1 : $r = 1 - \cos\theta$, sketch $r^2 = x^2 + y^2$

$x = (1 - \cos\theta)\cos\theta \Rightarrow x = \cos\theta - \cos^2\theta$ $y = (1 - \cos\theta)\sin\theta \Rightarrow y = \sin\theta - \sin\theta\cos\theta$

1) x-axis $(r,\theta) \longrightarrow (r,-\theta)$ or $(-r, \pi-\theta)$ ✓

2) y-axis $(r,\theta) \longrightarrow (r, \pi-\theta)$ or $(-r,-\theta)$ ✗

3) origin $(r,\theta) \longrightarrow (-r,\theta)$ or $(r, \pi+\theta)$ ✗

i) $r(\theta) = r(-\theta)$ is symmetric with respect to x-axis

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2

ii) $r = 1 - \frac{x}{r} \Rightarrow r^2 - r + x = 0$ $x^2 + y^2 - \sqrt{x^2+y^2} + x = 0$ (cardioid)

iii) $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta \sin\theta + \cos\theta(1-\cos\theta)}{-\sin\theta - 2\cos\theta(-\sin\theta)} = \frac{\cos\theta - \cos 2\theta}{-\sin\theta(1 - 2\cos\theta)}\Big|_{\theta=\frac{\pi}{2}} = -1$

iv) $\frac{dy}{d\theta} = 0$ $\frac{dx}{d\theta} = 0$

$\cos\theta - \cos 2\theta = 0$

$-(2\cos^2\theta - \cos\theta - 1) = 0$

$2\cos\theta \quad 1$

$\cos\theta \quad -1$

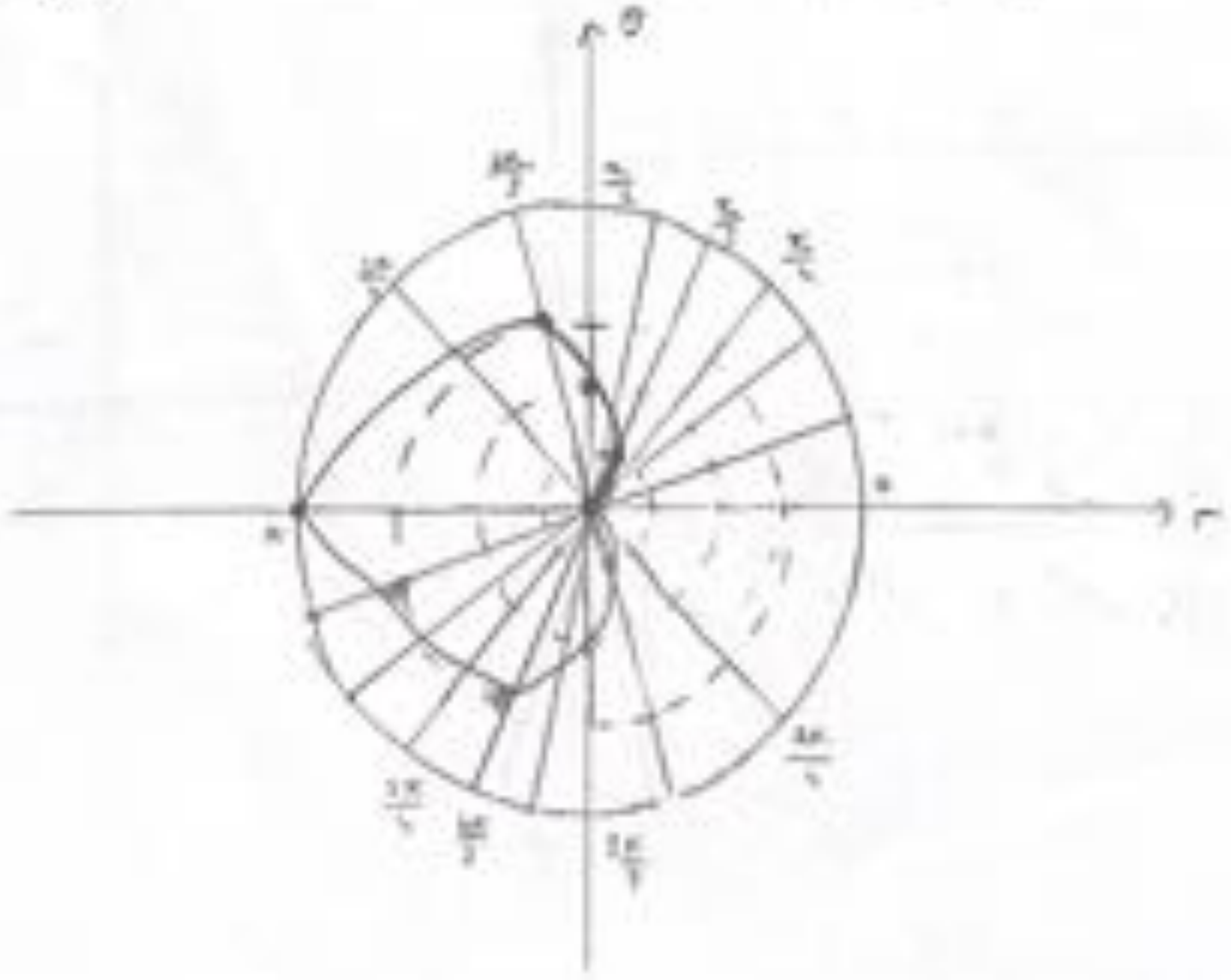
$-(2\cos\theta + 1)(\cos\theta - 1)$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ $\theta = 0, 2\pi$

$-\sin\theta(1 - 2\cos\theta) = 0$ $\theta = 0, \pi, 2\pi$ $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

vertical $\Rightarrow (r,\theta)$: $\left(\frac{1}{2}, \frac{\pi}{3}\right), \left(\frac{1}{2}, \frac{5\pi}{3}\right), (2, \pi)$

horizontal $\Rightarrow (r,\theta)$: $\left(\frac{3}{2}, \frac{2\pi}{3}\right), \left(\frac{3}{2}, \frac{4\pi}{3}\right)$



Question: $x=\frac{t}{t^2-1}$, $y=\frac{2t}{t+1}$; Sketch the graph

① x(t) is defined when $t^2-1\neq 0 \Rightarrow t\neq\pm1$ y(t) is undefined at $t=-1$

Domain : $\mathbb{R}-\{-1,1\}$ / (0,0) and (0,2) [illegible]

② $\dot{x}=\frac{dx}{dt}=\frac{1(t^2-1)-2t.t}{(t^2-1)^2}=\frac{-(t^2+1)}{(t^2-1)^2}<0$ x is decreasing

$\dot{y}=\frac{dy}{dt}=\frac{2(t+1)-1.2t}{(t+1)^2}=\frac{2}{(t+1)^2}>0$ y is increasing

③ $\lim_{t\to-1}\frac{t}{t^2-1}=-\infty$ $\lim_{t\to-1}\frac{2t}{t+1}=-\infty$ } may have oblique asymptote ④

$\lim_{t\to1}\frac{t}{t^2-1}=\infty$ $\lim_{t\to1}\frac{2t}{t+1}=1$ } y=1 is a horizontal asymptote

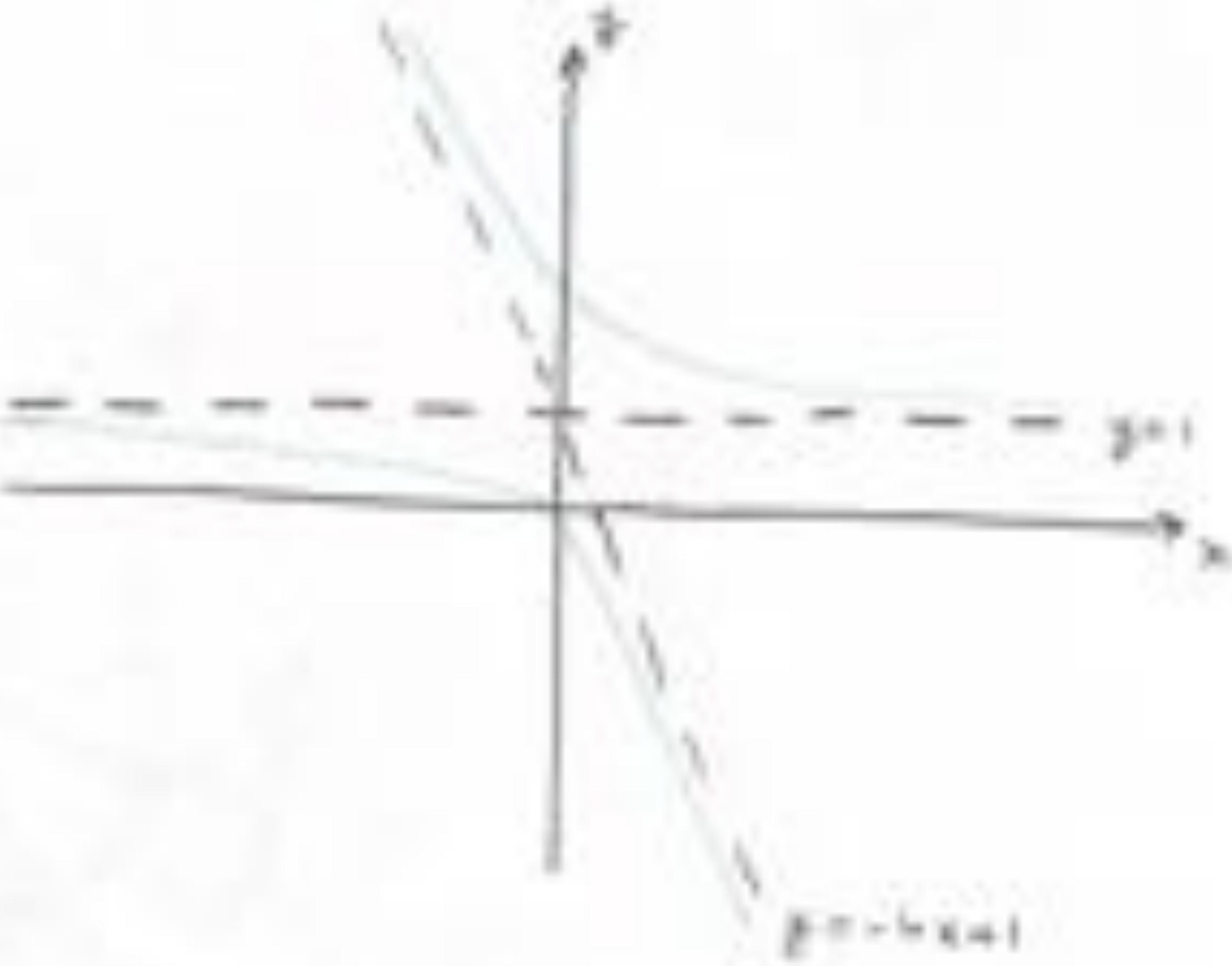
④ $m=\lim_{x\to-\infty}\frac{f(x)}{x}=\lim_{t\to-1}\frac{\frac{2t}{t+1}}{\frac{t}{t^2-1}}=\lim_{t\to-1}2(t-1)=-4$

$n=\lim_{x\to-\infty}(f(x)-mx)=\lim_{t\to-1}\left[\frac{2t}{t+1}+4\left(\frac{t}{t^2-1}\right)\right]=\lim_{t\to-1}\left[\frac{2t(t+1)}{(t-1)\cancel{(t+1)}}\right]=1$ } $y=-4x+1$ is an oblique asymptote

④ $\lim_{t\to-\infty}\frac{t}{t^2-1}=0$, $\lim_{t\to-\infty}\frac{2t}{t+1}=2$ $\lim_{t\to+\infty}\frac{t}{t^2-1}=0$, $\lim_{t\to+\infty}\frac{2t}{t+1}=2$

⑤

t	$-\infty$		-1		0		1		$+\infty$
$\dot{x}$		−		−		−		−	
$\dot{y}$		+		+		+		+	
x	0	↘	$-\infty$ ‖ $+\infty$	↘	0	↘	$-\infty$ ‖ $+\infty$	↘	0
y	2	↗	$+\infty$ ‖ $-\infty$	↗	0	↗	1	↗	2



Question: $r = 4\cos(2\theta)$ $x = r\cos\theta$ $y = r\sin\theta$ $r^2 = x^2+y^2$

1) x-axis $(r,\theta) \longrightarrow \underline{(r,-\theta)}$ or $(-r, \pi-\theta)$ ✓

2) y-axis $(r,\theta) \longrightarrow \underline{(r, \pi-\theta)}$ or $(-r, -\theta)$ ✓

3) origin $(r,\theta) \longrightarrow (-r, \theta)$ or $(r, \pi+\theta)$ ×

i) $r(\theta) = r(-\theta)$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	4	2	0	-2	-4

ii) $r = 4(\cos^2\theta - \sin^2\theta) \Rightarrow r = 4\left(\frac{x^2}{r^2} - \frac{y^2}{r^2}\right) \Rightarrow r^3 = 4(x^2-y^2)$

$(x^2+y^2)^2 - 4(x^2-y^2) = 0$ (cartesian)

iii) $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-8\sin2\theta \cdot \sin\theta + 4\cos2\theta \cdot \cos\theta}{-8\sin2\theta \cdot \cos\theta - 4\cos2\theta(-\sin\theta)} = \frac{2\sin2\theta\sin\theta - \cos2\theta\cos\theta}{2\sin2\theta\cos\theta - \cos2\theta\sin\theta}$

$\sin2\theta = 2\sin\theta\cos\theta \Rightarrow \frac{\cos\theta(2\sin^2\theta - (1-2\sin^2\theta))}{\sin\theta(2\cos^2\theta - (1-2\sin^2\theta))} = \frac{\cos\theta(2\sin\theta-1)(2\sin\theta+1)}{\sin\theta}$

↳ $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$ $\frac{dy}{dx} = -1$

iv) $\frac{dx}{d\theta} = \sin\theta$ $(\theta: 0, \pi, 2\pi)$ $(r,\theta): (4,0), (4,\pi), (4,2\pi)$ vertical

$\frac{dy}{d\theta} = \cos\theta(4\sin^2\theta - 1)$ $(\theta = \frac{\pi}{2}, \frac{3\pi}{2})$ $(r,\theta): (-4, \frac{\pi}{2}), (-4, \frac{3\pi}{2})$ horizontal

$(\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6})$ $(r,\theta): (0, \frac{\pi}{6}), (0, \frac{5\pi}{6}), (0, \frac{7\pi}{6}), (0, \frac{11\pi}{6})$