Simplify the expression $\cos(\tan^{-1} x)$.

Solution. Let $y = \tan^{-1} x$. In this case, $\tan y = x$ and $-\pi/2 < y < \pi/2$. To find $\cos y$, we use the trigonometric identity involving $\tan y$:

$$\sec^2 y = 1 + \tan^2 y = 1 + x^2.$$

Thus:

$$\sec y = \sqrt{1 + x^2} \quad (\text{since } -\pi/2 < y < \pi/2 \text{ ensures } \sec y > 0).$$

Now, since $\cos y = \frac{1}{\sec y}$, we have:

$$\cos(\tan^{-1} x) = \cos y = \frac{1}{\sec y} = \frac{1}{\sqrt{1 + x^2}}.$$

Find $\sec\left(\tan^{-1}\frac{x}{3}\right)$.

Solution. Let $\theta = \tan^{-1}\frac{x}{3}$ (just to assign a name to the angle), and note that:

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{x}{3}.$$

We represent θ in a right triangle. The hypotenuse of the triangle is:

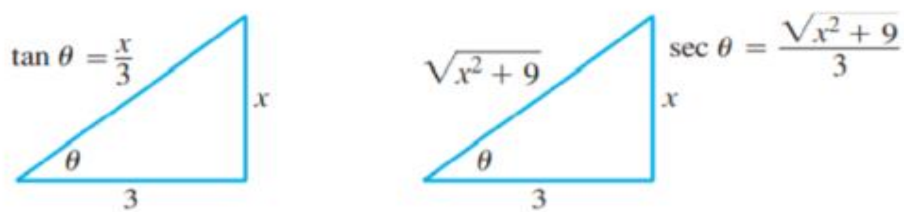
$$\sqrt{x^2+3^2} = \sqrt{x^2+9}.$$

Using the triangle:

$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{\sqrt{x^2+9}}{3}.$$

Thus:

$$\sec\left(\tan^{-1}\frac{x}{3}\right) = \sec\theta = \frac{\sqrt{x^2+9}}{3}.$$



Summary:

$$\sec\left(\tan^{-1}\frac{x}{3}\right) = \frac{\sqrt{x^2+9}}{3}.$$

Find the domain of the function:

$$f(x) = 5 + \log_{\frac{1}{2}}\left((x^2 - 4)^3\right).$$

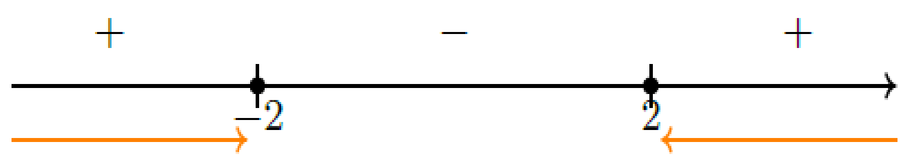
Solution. For the logarithmic function to be defined, the argument must be positive:

$$x^2 - 4 > 0.$$

Solving the inequality:

$$x^2 > 4 \implies x > 2 \quad \text{or} \quad x < -2.$$

Using a sign chart for $x^2 - 4$:



Thus, the domain of $f(x)$ is:

$$\text{Domain: } (-\infty, -2) \cup (2, \infty).$$

Alternatively:

$$\text{Domain: } \mathbb{R} \setminus [-2, 2].$$

Find the domain and range of each of the following functions:

Solution. (a) $f(x) = \sqrt{x-2} + 3, \quad 2 \leq x \leq 4$

To satisfy the square root's domain, $x - 2 \geq 0$, so:

$$x \geq 2.$$

Since $x \leq 4$ is also given:

$$\text{Domain: } D_f = [2, 4].$$

For the range:

$$f(x) = \sqrt{x-2} + 3, \quad \text{where } x - 2 \text{ varies from 0 to 2.}$$

Thus:

$$\text{Range: } R_f = [3, 3 + \sqrt{2}].$$

(b) $f(x) = \frac{x}{1-x}$

The function is undefined when $1 - x = 0$, i.e., $x = 1$. Thus:

$$\text{Domain: } D_f = (-\infty, 1) \cup (1, \infty).$$

To find the range, solve $f(x) = y$:

$$\frac{x}{1-x} = y \implies x = y - xy \implies x(1+y) = y \implies x = \frac{y}{1+y}.$$

The function $x = \frac{y}{1+y}$ is undefined when $y = -1$. Thus:

$$\text{Range: } R_f = (-\infty, -1) \cup (-1, \infty).$$

(c) $f(x) = \frac{1}{\sqrt{x-2}}$

For the square root to be defined, $x - 2 > 0$, so:

$$x > 2.$$

Thus:

$$\text{Domain: } D_f = (2, \infty).$$

For the range:

$$f(x) = \frac{1}{\sqrt{x-2}}.$$

As $x \to \infty$, $f(x) \to 0^+$, and as $x \to 2^+$, $f(x) \to \infty$. Thus:

$$\text{Range: } R_f = (0, \infty).$$

Let $f(x) = \frac{1}{x-2}$ and $g(x) = \frac{x}{1-x}$.

1. *Compute $(f \circ g)(x)$ and its domain.*
2. *Compute $(g \circ f)(x)$ and its domain.*

Solution. (a) Compute $f(g(x))$:

$$f(g(x)) = \frac{1}{g(x) - 2} = \frac{1}{\frac{x}{1-x} - 2}.$$

Simplify the denominator:

$$\frac{x}{1-x} - 2 = \frac{x - 2(1-x)}{1-x} = \frac{x - 2 + 2x}{1-x} = \frac{3x-2}{1-x}.$$

Thus:

$$f(g(x)) = \frac{1}{\frac{3x-2}{1-x}} = \frac{1-x}{3x-2}.$$

The domain of $f(g(x))$ is determined by ensuring that $g(x) \neq 2$ and $f(g(x))$ remains defined:

$$g(x) = \frac{x}{1-x} \neq 2$$

Additionally, $g(x)$ is undefined at $x = 1$. Therefore:

$$D_{f \circ g} = (-\infty, \frac{2}{3}) \cup (\frac{2}{3}, 1) \cup (1, \infty).$$

Let $f(x) = \frac{1}{x-2}$ and $g(x) = \frac{x}{1-x}$.

(b) Compute $g(f(x))$:

$$g(f(x)) = \frac{f(x)}{1-f(x)} = \frac{\frac{1}{x-2}}{1-\frac{1}{x-2}}.$$

Simplify the denominator:

$$1 - \frac{1}{x-2} = \frac{x-2-1}{x-2} = \frac{x-3}{x-2}.$$

Thus:

$$g(f(x)) = \frac{\frac{1}{x-2}}{\frac{x-3}{x-2}} = \frac{1}{x-3}.$$

The domain of $g(f(x))$ is determined by ensuring $f(x) \neq 1$ and $g(f(x))$ remains defined:

$$f(x) = \frac{1}{x-2} \neq 1 \implies \frac{1}{x-2} \neq 1 \implies x \neq 3.$$

Additionally, $f(x)$ is undefined at $x = 2$. Therefore:

$$D_{g \circ f} = (-\infty, 2) \cup (2, 3) \cup (3, \infty).$$

Given the functions $f(x) = e^{3x-1}$ and $g(x) = \ln(x-1)$*:*

(a) *Find the domain of* $g \circ f$*.*

(b) *Find the domain of* $\frac{f}{g}$*.*

Solution. (a) To find the domain of $g \circ f$, we first note that:

$$g \circ f(x) = g(f(x)) = \ln\left(e^{3x-1} - 1\right).$$

For $\ln\left(e^{3x-1} - 1\right)$ to be defined, we require:

$$e^{3x-1} - 1 > 0 \implies e^{3x-1} > 1.$$

Taking the natural logarithm of both sides:

$$3x - 1 > 0 \implies x > \frac{1}{3}.$$

Therefore, the domain of $g \circ f$ is:

$$D_{g \circ f} = \left(\frac{1}{3}, \infty\right).$$

(b) To find the domain of $\frac{f}{g}$, we start with the definition:

$$\frac{f}{g}(x) = \frac{e^{3x-1}}{\ln(x-1)}.$$

For $\frac{f}{g}(x)$ to be defined, both e^{3x-1} and $\ln(x-1)$ must exist, and $\ln(x-1) \neq 0$.

- The function $\ln(x-1)$ is defined for $x-1>0$, i.e., $x>1$.
- $\ln(x-1) \neq 0$ implies:

$$\ln(x-1) \neq 0 \implies x-1 \neq e^0 = 1 \implies x \neq 2.$$

Combining these restrictions:

$$x > 1 \quad \text{and} \quad x \neq 2.$$

Thus, the domain of $\frac{f}{g}$ is:

$$D_{f/g} = (1,2) \cup (2,\infty).$$

Given the functions $f(x) = \arcsin(x)$ and $g(x) = \sin(x)$:

(a) *Find the domain of f and g.*

(b) *Find the domain of $f + g$.*

(c) *Find the domain of $g \circ f$.*

(d) *Find the domain of $\frac{f}{g}$.*

Solution. (a) **Finding the domain of f and g:**

- For $f(x) = \arcsin(x)$: The function $\arcsin(x)$ is defined only when $-1 \le x \le 1$. Therefore, the domain of f is:
$$D_f = [-1, 1].$$
- For $g(x) = \sin(x)$: The sine function is defined for all real numbers. Therefore, the domain of g is:
$$D_g = (-\infty, \infty).$$

(b) **Finding the domain of $f+g$:** The sum $f+g$ is defined only when both $f(x)$ and $g(x)$ are defined. Thus, the domain of $f+g$ is the intersection of the domains of f and g:

$$D_{f+g} = D_f \cap D_g.$$

Substituting the domains:

$$D_f = [-1,1], \quad D_g = (-\infty, \infty).$$

The intersection is:

$$D_{f+g} = [-1,1].$$

Final Result:

- Domain of f: $[-1,1]$.
- Domain of g: $(-\infty,\infty)$.
- Domain of $f+g$: $[-1,1]$.

(c) **Finding the domain of $g \circ f$:** The composition $g \circ f(x)$ is defined as:

$$(g \circ f)(x) = g(f(x)) = \sin(\arcsin(x)).$$

Since $\arcsin(x)$ is the inverse of $\sin(x)$, we have:

$$\sin(\arcsin(x)) = x.$$

Therefore, $g \circ f(x)$ is defined wherever $\arcsin(x)$ is defined. From part (a), we know that $\arcsin(x)$ is defined for $x \in [-1, 1]$. Thus:

$$D_{g \circ f} = [-1, 1].$$

(d) **Finding the domain of $\frac{f}{g}$:** The function $\frac{f}{g}(x)$ is defined as:

$$\frac{f}{g}(x) = \frac{\arcsin(x)}{\sin(x)}.$$

For $\frac{f}{g}(x)$ to be defined, both $\arcsin(x)$ and $\sin(x)$ must be defined, and $\sin(x) \neq 0$. From part (a), we know:

$$D_f = [-1, 1], \quad D_g = (-\infty, \infty).$$

The intersection of D_f and D_g is:

$$D_f \cap D_g = [-1, 1].$$

Next, exclude the points where $\sin(x) = 0$. The sine function is zero at $x = 0$. Therefore:

$$D_{f/g} = [-1, 1] \setminus \{0\} = [-1, 0) \cup (0, 1].$$

Final Results:

- Domain of $g \circ f$: $[-1, 1]$.
- Domain of $\frac{f}{g}$: $[-1, 0) \cup (0, 1]$.

Absolute value function:

Let $A \subset R$ and $f: A \to R$

$$|f(x)| = \begin{cases} f(x) & , f(x) \geqslant 0 \\ -f(x) & , f(x) < 0 \end{cases}$$

$$f(x) = |x| = \begin{cases} x & x \geqslant 0 \\ -x & x < 0 \end{cases}$$

$$f(x) = \frac{|x|}{x} = \begin{cases} \frac{x}{x} & x \geqslant 0 \\ \frac{-x}{x} & x < 0 \end{cases}$$

$$= \begin{cases} 1 & x \geqslant 0 \\ -1 & x < 0 \end{cases}$$

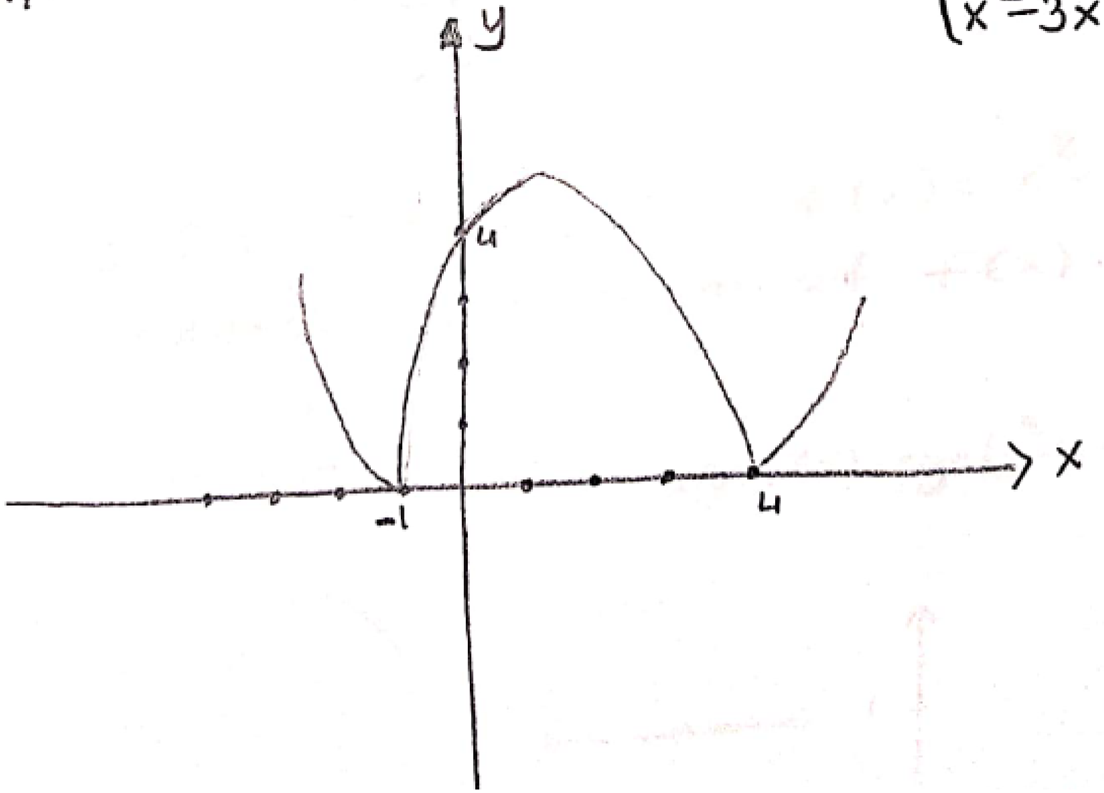
Let

$f(x) = |-x^2+3x+4|$; $f: R \to R$

$-x^2+3x+4=0$

$x_1=-1 \quad x_2=4$

$$f(x) = \begin{cases} x^2-3x-4 & x<-1 \\ -x^2+3x+4 & -1 \le x \le 4 \\ x^2-3x-4 & x>4 \end{cases}$$



$y=|-x^2+3x+4|$

Solve the inequality

$|x-3| + |x+2| < 11$

$$|x-3| = \begin{cases} x-3 & \text{if } x-3 \geq 0 \\ -(x-3) & \text{if } x-3 < 0 \end{cases} = \begin{cases} x-3 & \text{if } x-3 \geq 0 \\ -x+3 & \text{if } x-3 < 0 \end{cases}$$

$$|x+2| = \begin{cases} x+2 & \text{if } x \geq -2 \\ -x-2 & \text{if } x < -2 \end{cases}$$

$x < -2 \qquad -2 \leq x < 3 \qquad x \geq 3$

Case 1: if $x < -2$

$|x-3| + |x+2| < 11$

$-x+3 - x-2 < 11$

$x > -5$

Case 2: if $-2 \leq x < 3$

$|x-3| + |x+2| < 11$

$-x+3 + x+2 < 11$

$5 < 11$ always true

Case 3:

if $x \geq 3$

$|x-3| + |x+2| < 11$

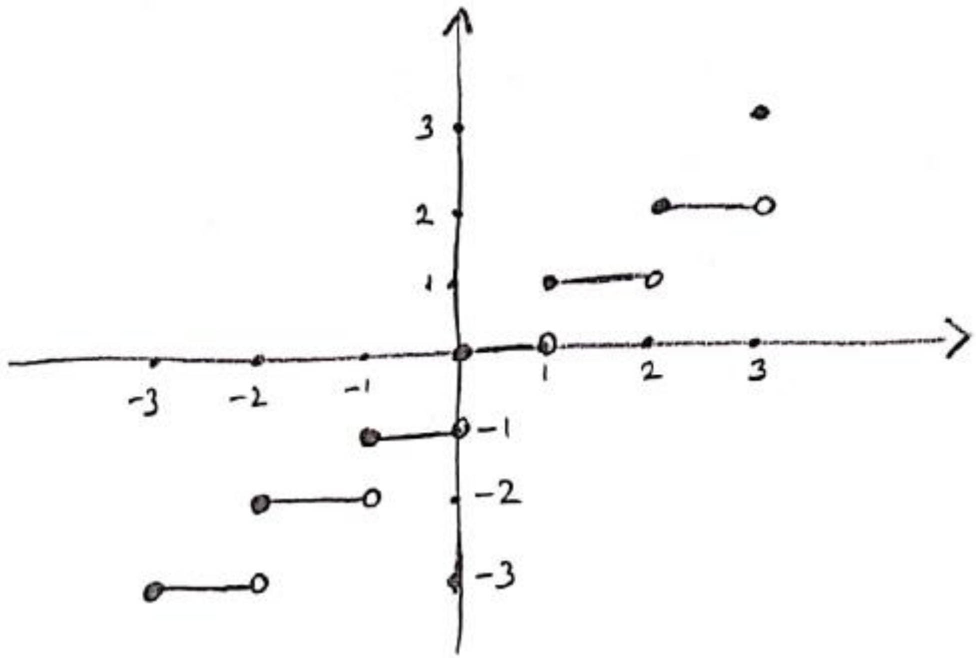
$x-3 + x+2 < 11$

$x < 6$

Combining cases I, II and III,
the inequality is satisfed when $-5 < x < 6$

So the solution is in the interval $(-5, 6)$

Ex: $y = [\![x]\!]$, $f : [-3,3] \longrightarrow R$



$0 \le x < 1 \Rightarrow [\![x]\!] = 0$

$1 \le x < 2 \Rightarrow [\![x]\!] = 1$

$2 \le x < 3 \Rightarrow [\![x]\!] = 2$

$-3 \le x < -2 \Rightarrow [\![x]\!] = -3$

$-2 \le x \le -1 \Rightarrow [\![x]\!] = -2$

$-1 \le x < 0 \Rightarrow [\![x]\!] = -1$

(Sgn) function:

Let $f: A \to R$ $\quad (A \subset R)$

$$g(x) = \begin{cases} \dfrac{|f(x)|}{f(x)} & f(x) \neq 0 \\ 0 & f(x) = 0 \end{cases}$$

$$\operatorname{sgn} f(x) = \begin{cases} 1 & f(x) > 0 \\ 0 & f(x) = 0 \\ -1 & f(x) < 0 \end{cases}$$

Ex: Let $f: [-3,3] \longrightarrow$, $\quad f(x) = x^2 - x - 2$

sketch the graph of $f(x)$.

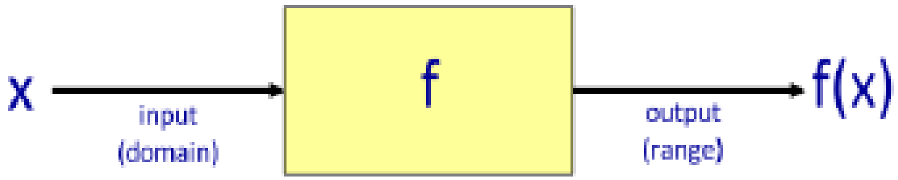
x		-3		-1		2		3	
x^2-x-2	$+$		$+$	0	$-$	0	$+$		$+$

$x^2 - x - 2 = 0$

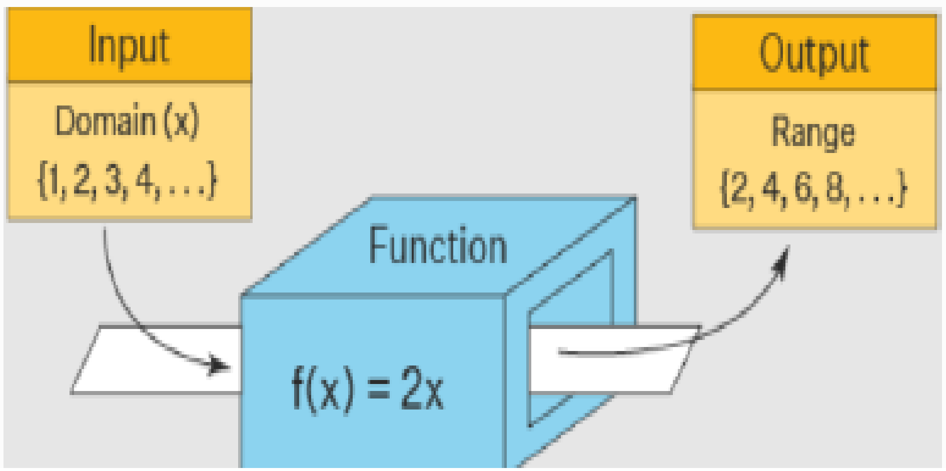
$x_1 = -1 \qquad x_2 = 2$

$$\operatorname{sgn} f(x) = \operatorname{sgn}(x^2 - x - 2) = \begin{cases} 1 & , \ x \in [-3,-1) \cup (2,3] \\ 0 & x \in \{-1, 2\} \\ -1 & x \in (-1, 2) \end{cases}$$

*A **function** f from a set D to a set Y is a rule that assigns a unique (single) element $f(x) \in Y$ to each element $x \in D$.*



For example; let us take the function $f(x) = 2x$.



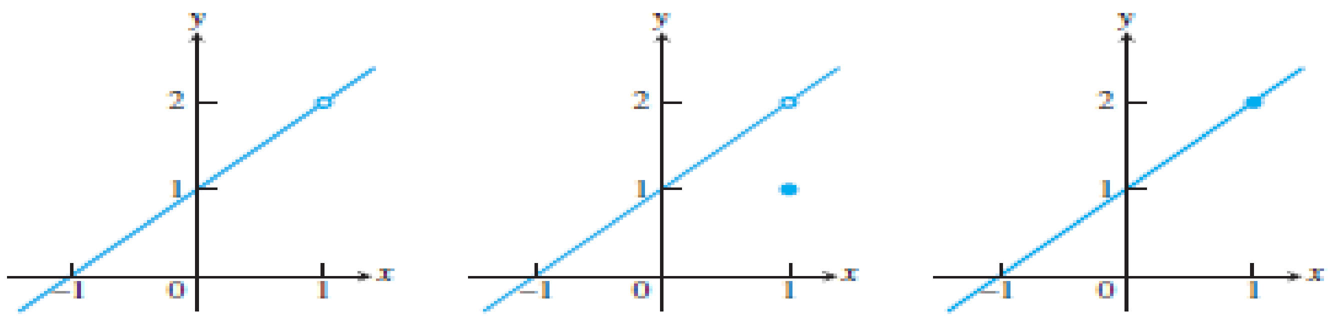
Limit of a Function and Limit Laws

Suppose $f(x)$ is defined on an open interval about x_0, *except possibly at x_0 itself.* If $f(x)$ is arbitrarily close to L (as close to L as we like) for all x sufficiently close to x_0, we say that f approaches the **limit** L as x approaches x_0, and write

$$\lim_{x \to x_0} f(x) = L,$$

which is read "the limit of $f(x)$ as x approaches x_0 is L."

This example illustrates that the limit value of a function does not depend on how the function is defined at the point being approached. Consider the three functions in Figure . The function f has limit 2 as $x \to 1$ even though f is not defined at $x = 1$.



(a) $f(x) = \frac{x^2 - 1}{x - 1}$ (b) $g(x) = \begin{cases} \frac{x^2 - 1}{x - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$ (c) $h(x) = x + 1$

The limits of $f(x)$, $g(x)$, and $h(x)$ all equal 2 as x approaches 1. However, only $h(x)$ has the same function value as its limit at $x = 1$ (

(a) If f is the **identity function** $f(x) = x$, then for any value of x_0 ,

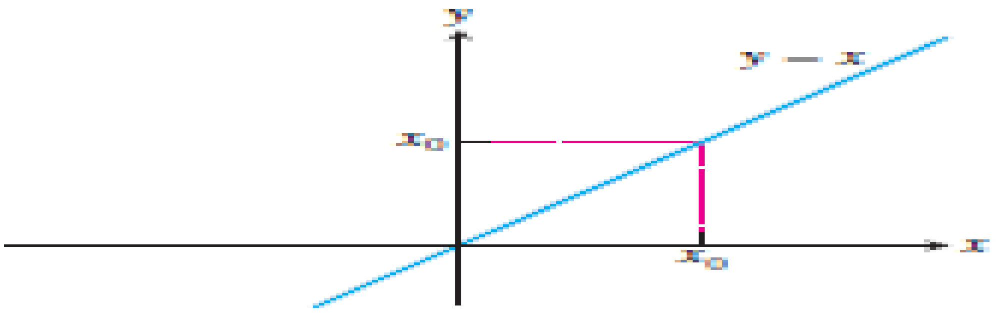
$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x = x_0 .$$

(b) If f is the **constant function** $f(x) = k$ (function with the constant value k), then for any value of x_0 (Figure 2.9b),

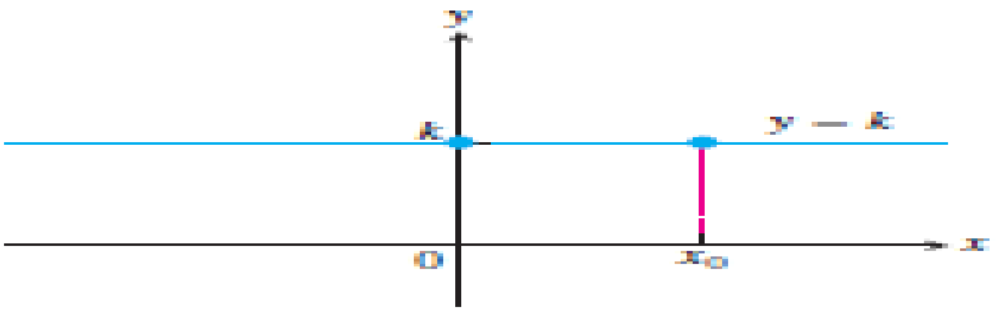
$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} k = k .$$

For instances of each of these rules we have

$$\lim_{x \to 3} x = 3 \qquad \text{and} \qquad \lim_{x \to -7} (4) = \lim_{x \to 2} (4) = 4 .$$



(a) Identity function



(b) Constant function

Discuss the behavior of the following functions as $x \to 0$.

(a) $U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$

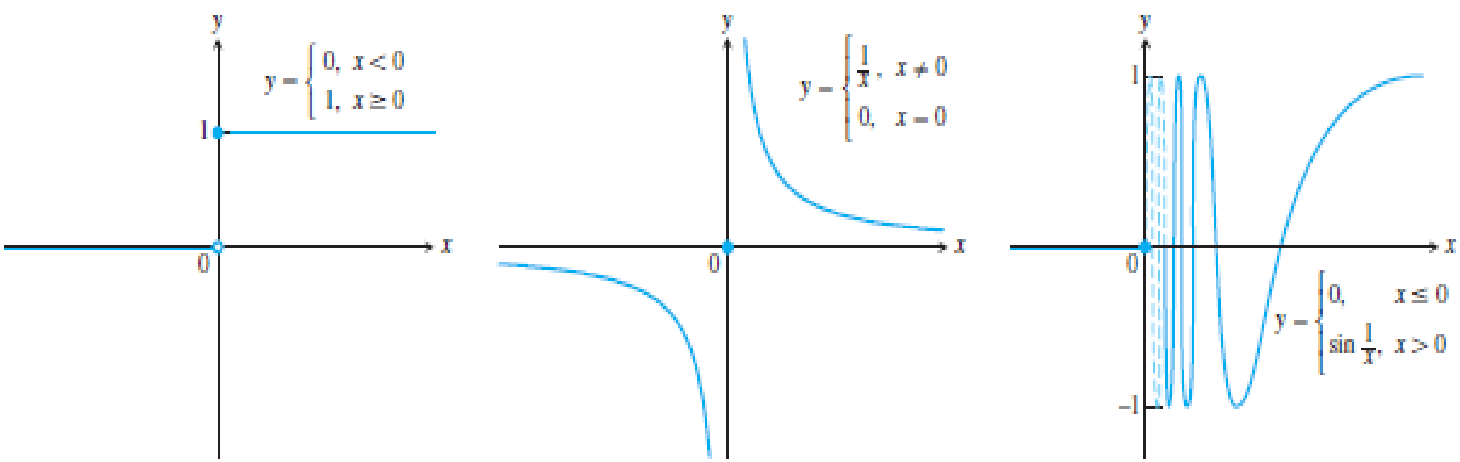
(b) $g(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(c) $f(x) = \begin{cases} 0, & x \leq 0 \\ \sin\frac{1}{x}, & x > 0 \end{cases}$

(a) It *jumps*: The **unit step function** $U(x)$ has no limit as $x \to 0$ because its values jump at $x = 0$. For negative values of x arbitrarily close to zero, $U(x) = 0$. For positive values of x arbitrarily close to zero, $U(x) = 1$. There is no *single* value L approached by $U(x)$ as $x \to 0$

(b) It *grows too "large" to have a limit*: $g(x)$ has no limit as $x \to 0$ because the values of g grow arbitrarily large in absolute value as $x \to 0$ and do not stay close to *any* fixed real number

(c) It *oscillates too much to have a limit*: $f(x)$ has no limit as $x \to 0$ because the function's values oscillate between $+1$ and -1 in every open interval containing 0. The values do not stay close to any one number as $x \to 0$



(a) Unit step function $U(x)$ (b) $g(x)$ (c) $f(x)$

THEOREM —Limit Laws If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x \to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x \to c}[f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $\lim_{x \to c} f(x) = L > 0$.)

Let $f(x)$ be defined on an open interval about c, except possibly at c itself. We say that the ***limit of $f(x)$ as x approaches c is the number L****, and write*

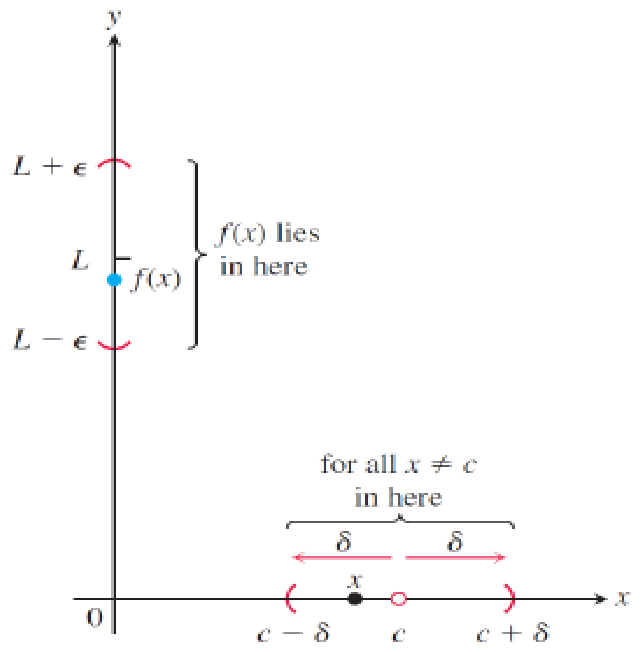
$$\lim_{x \to c} f(x) = L,$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - c| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

Epsilon-Delta Definition of Limit of Functions

$$\lim_{x \to x_0} f(x) = L \Leftrightarrow \forall \varepsilon > 0, \quad \exists \delta(\varepsilon) > 0 \ (\delta(\varepsilon) \in \mathbb{R}^+) \quad \forall x \in X, \quad 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$



Show that

$$\lim_{x \to 2} 2x - 3 = 1.$$

$$\lim_{x \to x_0} f(x) = L \Leftrightarrow \forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \ni \forall x \in X, 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

Assume that $0 < |x-2| < \delta$. For all ε, we want to find δ and our goal is to show that if $0 < |x-2| < \delta$, then $|(2x-3)-1| < \varepsilon$.
(To prove any statement of the form "If this, then that," we begin by assuming "this" and trying to get "that.")

$$\begin{aligned}
|(2x-3)-1| &= |2x-4| \\
&= |2(x-2)| \\
&= |2||x-2| && \text{property of absolute values: } |ab| = |a||b| \\
&= 2|x-2| \\
&< 2 \cdot \delta && \text{here's where we use the assumption that } 0 < |x-2| < \delta \\
&= 2 \cdot \frac{\varepsilon}{2} = \varepsilon && \text{here's where we use our choice of } \delta = \varepsilon/2
\end{aligned}$$

Show that

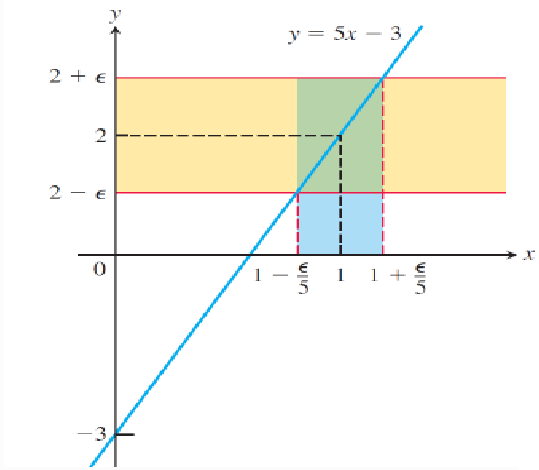
$$\lim_{x \to 1}(5x - 3) = 2.$$

Solution *Set $c = 1$, $f(x) = 5x - 3$, and $L = 2$ in the definition of limit. For any given $\varepsilon > 0$, we have to find a suitable $\delta > 0$ so that if $x \neq 1$ and x is within distance δ of $c = 1$, that is, whenever*

$$0 < |x - 1| < \delta,$$

it is true that $f(x)$ is within distance ε of $L = 2$, so

$$|f(x) - 2| < \varepsilon.$$



We find δ by working backward from the ε-inequality:

$$|(5x - 3) - 2| = |5x - 5| < \varepsilon$$

$$5|x - 1| < \varepsilon$$

$$|x - 1| < \frac{\varepsilon}{5}.$$

Thus, we can take $\delta = \frac{\varepsilon}{5}$. If $0 < |x - 1| < \delta = \varepsilon/5$, then

$$|(5x - 3) - 2| = |5x - 5| = 5|x - 1| < 5\left(\frac{\varepsilon}{5}\right) = \varepsilon,$$

which proves that $\lim_{x \to 1}(5x - 3) = 2$.
The value of $\delta = \varepsilon/5$ is not the only value that will make $0 < |x - 1| < \delta$ imply $|f(x) - 2| < \varepsilon$. Any smaller positive δ will do as well. The definition does not ask for a "best" positive δ, just one that will work.

(a) $\lim_{x\to c}(x^3+4x^2-3)$

(b) $\lim_{x\to c}\frac{x^4+x^2-1}{x^2+5}$

(c) $\lim_{x\to 2}\sqrt{4x^2-3}$

(a)

$$\begin{aligned}\lim_{x\to c}(x^3+4x^2-3) &= \lim_{x\to c}x^3+\lim_{x\to c}4x^2-\lim_{x\to c}3 \quad \text{Sum and Difference Rules}\\ &= c^3+4c^2-3 \quad \text{Power and Multiple Rules}\end{aligned}$$

(b)

$$\begin{aligned}\lim_{x\to c}\frac{x^4+x^2-1}{x^2+5} &= \frac{\lim_{x\to c}(x^4+x^2-1)}{\lim_{x\to c}(x^2+5)} \quad \text{Quotient Rule}\\ &= \frac{\lim_{x\to c}x^4+\lim_{x\to c}x^2-\lim_{x\to c}1}{\lim_{x\to c}x^2+\lim_{x\to c}5} \quad \text{Sum and Difference Rules}\\ &= \frac{c^4+c^2-1}{c^2+5} \quad \text{Power or Product Rule}\end{aligned}$$

(c)

$$\begin{aligned}\lim_{x\to 2}\sqrt{4x^2-3} &= \sqrt{\lim_{x\to 2}(4x^2-3)} \quad \text{Root Rule with } n=2\\ &= \sqrt{\lim_{x\to 2}4x^2-\lim_{x\to 2}3} \quad \text{Difference Rule}\\ &= \sqrt{4(2)^2-3}=\sqrt{16-3}=\sqrt{13} \quad \text{Product and Multiple Rules}\end{aligned}$$

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, *then*

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0.$$

If $P(x)$ *and* $Q(x)$ *are polynomials and* $Q(c) \neq 0$, *then*

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

$$\lim_{x \to -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{-1 + 4 - 3}{1 + 5} = \frac{0}{6} = 0.$$

Eliminating Common Factors from Zero Denominators

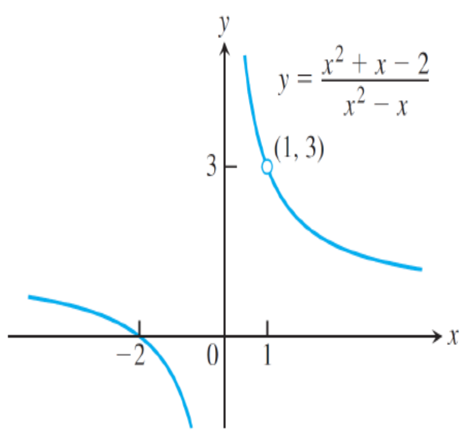
Theorem applies only if the denominator of the rational function is not zero at the limit point c. If the denominator is zero, canceling common factors in the numerator and denominator may reduce the fraction to one whose denominator is no longer zero at c. If this happens, we can find the limit by substitution in the simplified fraction.

Evaluate

$$\lim_{x\to 1} \frac{x^2+x-2}{x^2-x}.$$

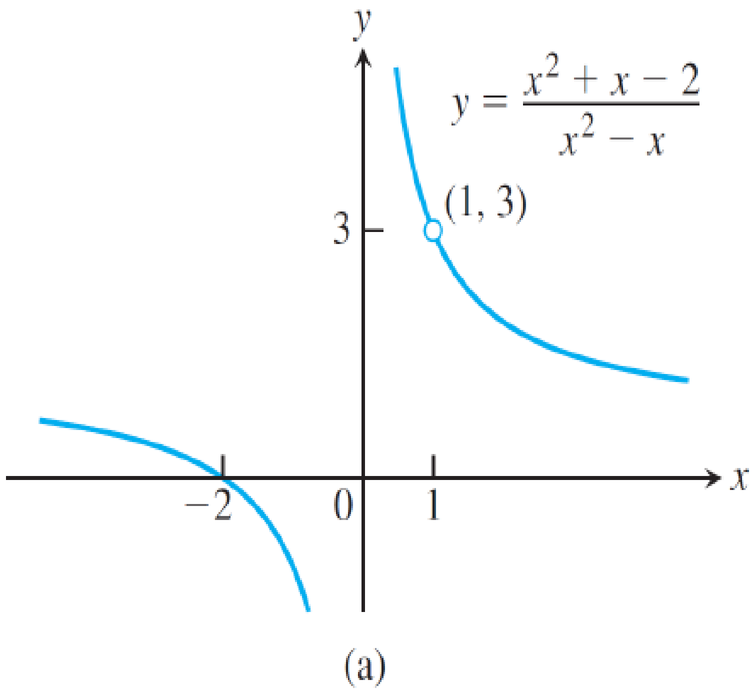
Solution We cannot substitute $x = 1$ because it makes the denominator zero. We test the numerator to see if it, too, is zero at $x = 1$. It is, so it has a factor of $(x-1)$ in common with the denominator. Canceling this common factor gives a simpler fraction with the same values as the original for $x \neq 1$:

$$\frac{x^2+x-2}{x^2-x} = \frac{(x-1)(x+2)}{x(x-1)} = \frac{x+2}{x}, \quad \text{if } x \neq 1.$$

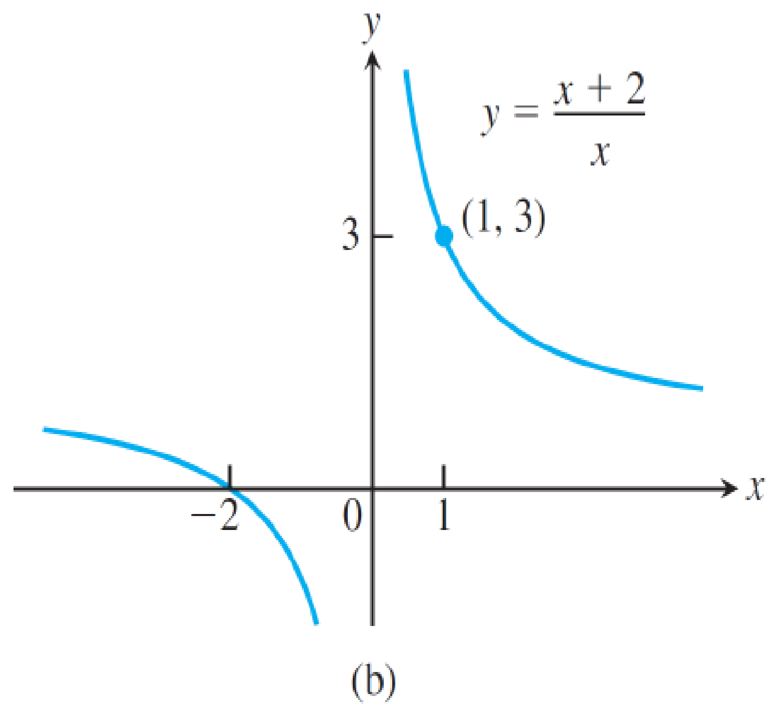


Using the simpler fraction, we find the limit of these values as $x \to 1$

$$\lim_{x\to 1} \frac{x^2+x-2}{x^2-x} = \lim_{x\to 1} \frac{x+2}{x} = \frac{1+2}{1} = 3.$$



(a) Graph of $f(x) = \frac{x^2+x-2}{x^2-x}$



(b) Graph of $g(x) = \frac{x+2}{x}$

The graph of $f(x) = \frac{x^2+x-2}{x^2-x}$ in part (a) is the same as the graph of $g(x) = \frac{x+2}{x}$ in part (b) except at $x = 1$, where f is undefined. The functions have the same limit as $x \to 1$

Evaluate

$$\lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2}.$$

Solution We can create a common factor by multiplying both the numerator and the denominator by the conjugate radical expression $\sqrt{x^2+100}+10$ (obtained by changing the sign after the square root). The preliminary algebra rationalizes the numerator:

$$\frac{\sqrt{x^2+100}-10}{x^2}=\frac{\left(\sqrt{x^2+100}-10\right)\cdot\left(\sqrt{x^2+100}+10\right)}{x^2\cdot\left(\sqrt{x^2+100}+10\right)}$$

$$=\frac{x^2+100-100}{x^2\left(\sqrt{x^2+100}+10\right)}=\frac{x^2}{x^2\left(\sqrt{x^2+100}+10\right)}=\frac{1}{\sqrt{x^2+100}+10}.$$

Therefore,

$$\lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2}=\lim_{x\to 0}\frac{1}{\sqrt{x^2+100}+10}.$$

Since the denominator is not zero at $x=0$, we substitute:

$$=\frac{1}{\sqrt{0^2+100}+10}=\frac{1}{20}=0.05.$$

Squeeze Theorem

Let $A \subseteq \mathbb{R}$, *let* $f, g, h : A \to \mathbb{R}$.
If

$$f(x) \leq g(x) \leq h(x) \quad \text{for all} \quad x \in A, x \neq c,$$

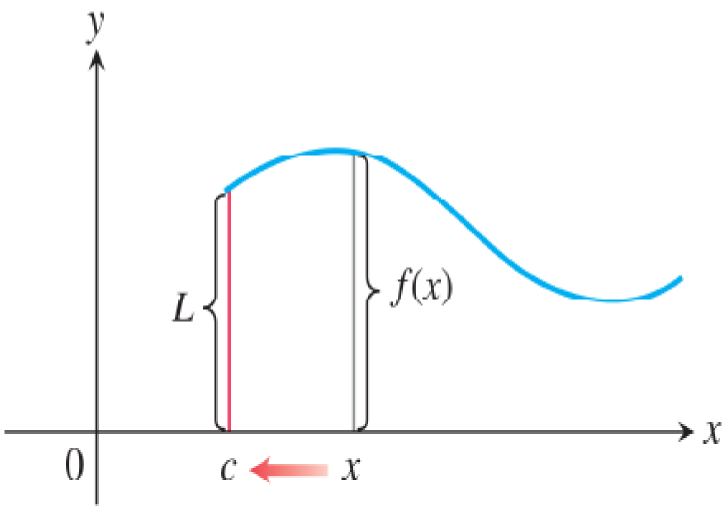
and
if $\lim_{x \to c} f = L = \lim_{x \to c} h$,
then $\lim_{x \to c} g = L$.

Theorem

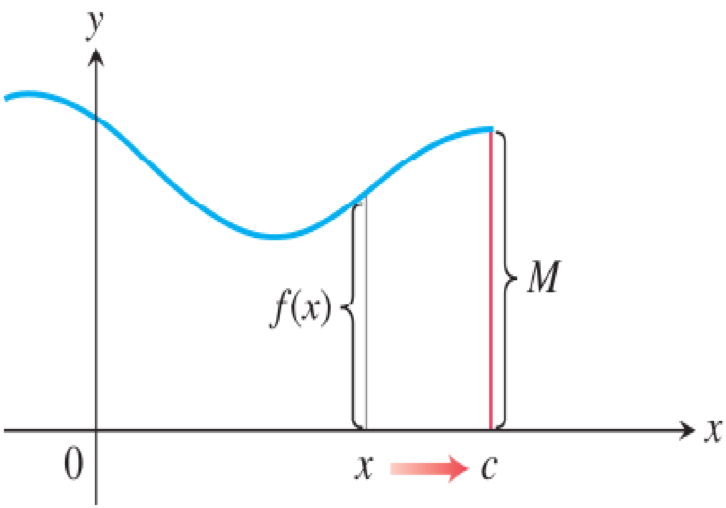
If $f(x) \leq g(x)$ *for all* x *in some open interval containing* c, *except possibly at* $x = c$ *itself, and the limits of* f *and* g *both exist as* x *approaches* c, *then*

$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x).$$

Left and Right Limit

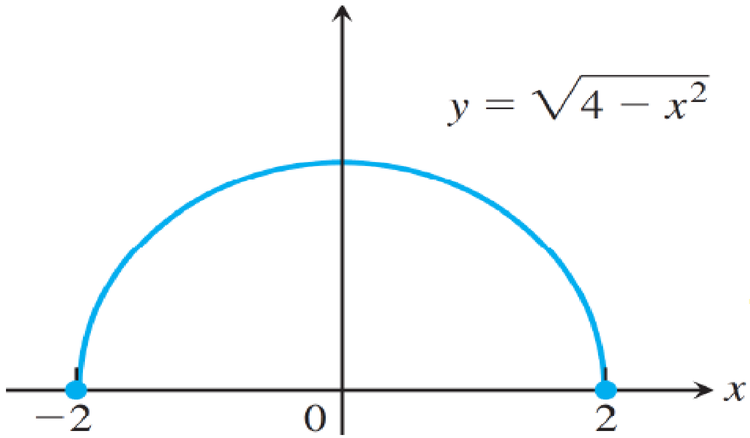


(a) $\lim_{x\to c^+} f(x) = L$



(b) $\lim_{x\to c^-} f(x) = M$

(a) Right-hand limit as x approaches c. (b) Left-hand limit as x approaches c.



The function $f(x) = \sqrt{4 - x^2}$ has right-hand limit 0 at $x = -2$ and left-hand limit 0 at $x = 2$.

The domain of $f(x) = \sqrt{4 - x^2}$ *is* $[-2, 2]$; *its graph is the semicircle*

$$\lim_{x \to -2^+} \sqrt{4 - x^2} = 0 \quad \text{and} \quad \lim_{x \to 2^-} \sqrt{4 - x^2} = 0.$$

The function does not have a left-hand limit at $x = -2$ or a right-hand limit at $x = 2$. It does not have a two-sided limit at either -2 or 2 because each point does not belong to an open interval over which f is defined.

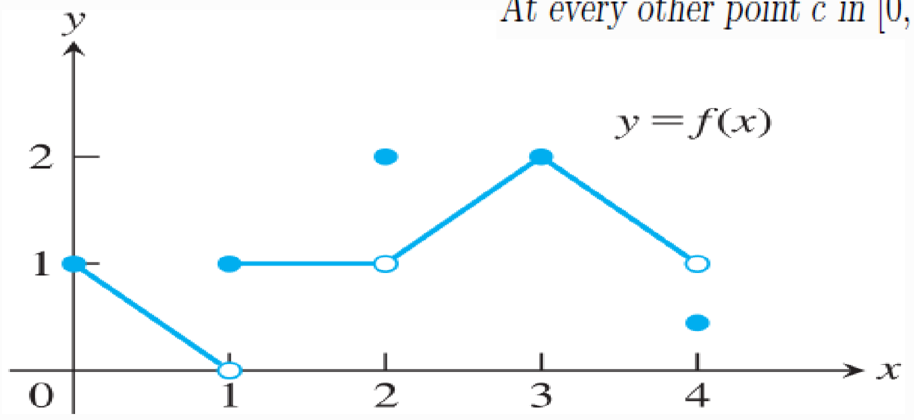
Theorem

A function $f(x)$ has a limit as x approaches c ***if and only if*** *it has left-handed and right-handed limits at c and these one-sided limits are equal:*

$$\lim_{x \to c} f(x) = L \quad \Longleftrightarrow \quad \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

For the function graphed in the following figure,

At every other point c in $[0,4]$ except 1, $f(x)$ has limit $f(c)$.



- *At $x = 0$:*

$$\lim_{x \to 0^-} f(x) \text{ does not exist,}$$

$$\lim_{x \to 0^+} f(x) = 1, \quad \lim_{x \to 0} f(x) = 1.$$

f is not defined to the left of $x = 0$.
f has a right-hand limit at $x = 0$.
f has a limit at domain endpoint $x = 0$.

- *At $x = 1$:*

$$\lim_{x \to 1^-} f(x) = 0, \quad \lim_{x \to 1^+} f(x) = 1, \quad \lim_{x \to 1} f(x) \text{ does not exist.}$$

Even though $f(1) = 1$.
Right- and left-hand limits are not equal.

- *At $x = 2$:*

$$\lim_{x \to 2^-} f(x) = 1, \quad \lim_{x \to 2^+} f(x) = 1, \quad \lim_{x \to 2} f(x) = 1.$$

Even though $f(2) = 2$.

- *At $x = 3$:*

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = \lim_{x \to 3} f(x) = f(3) = 2.$$

- *At $x = 4$:*

$$\lim_{x \to 4^-} f(x) = 1, \quad \lim_{x \to 4^+} f(x) \text{ does not exist,} \quad \lim_{x \to 4} f(x) = 1.$$

Even though $f(4) \neq 1$.
f is not defined to the right of $x = 4$.
f has a limit at domain endpoint $x = 4$.

If the following limit exists as $x \to 2$, find the value of a.

$$\lim_{x \to 2} \frac{3 - \sqrt{a - x}}{x - 2}$$

As $x \to 2$, we see that $x - 2 = 0$, so the denominator $\frac{1}{x-2} \to \pm\infty$. Normally, the function is undefined at $x = 2$, but since it is stated that the limit exists, we proceed by simplifying the numerator. First, observe the numerator as $x \to 2$:

$$\lim_{x \to 2} \left(3 - \sqrt{a - x}\right) = 0.$$

This implies:

$$3 = \sqrt{a - x} \quad \text{at } x = 2, \quad \text{so } \sqrt{a - 2} = 3 \implies a = 11.$$

Now let us verify it by rewriting $a = 11$ to the limit:

$$\lim_{x \to 2} \frac{3 - \sqrt{11 - x}}{x - 2}.$$

Rationalizing the numerator:

$$= \lim_{x \to 2} \frac{\left(3 - \sqrt{11 - x}\right)\left(3 + \sqrt{11 - x}\right)}{(x - 2)\left(3 + \sqrt{11 - x}\right)}.$$

Simplify:

$$= \lim_{x \to 2} \frac{9 - (11 - x)}{(x - 2)\left(3 + \sqrt{11 - x}\right)} = \lim_{x \to 2} \frac{x - 2}{(x - 2)\left(3 + \sqrt{11 - x}\right)}.$$

Cancel the common factor $x-2$:

$$= \lim_{x \to 2} \frac{1}{3+\sqrt{11-x}}.$$

Substitute $x=2$:

$$= \frac{1}{3+\sqrt{11-2}} = \frac{1}{3+3} = \frac{1}{6}.$$

Thus:

$$\lim_{x \to 2} \frac{3-\sqrt{11-x}}{x-2} = \frac{1}{6}.$$

(a) Assume the domain of f contains a **open** interval (c, d) to the right of c. We say that $f(x)$ has **right-hand limit** L at c, and write

$$\lim_{x \to c^+} f(x) = L$$

if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad c < x < c + \delta.$$

(b) Assume the domain of f contains an interval (b, c) to the left of c. We say that f has **left-hand limit** L at c, and write

$$\lim_{x \to c^-} f(x) = L$$

if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad c - \delta < x < c.$$

Prove that

$$\lim_{x \to 0^+} \sqrt{x} = 0.$$

Solution Let $\varepsilon > 0$ be given. Here $c = 0$ and $L = 0$, so we want to find a $\delta > 0$ such that for all x,

$$0 < x < \delta \quad \Rightarrow \quad |\sqrt{x} - 0| < \varepsilon,$$

or

$$0 < x < \delta \quad \Rightarrow \quad \sqrt{x} < \varepsilon.$$

Squaring both sides of this last inequality gives

$$x < \varepsilon^2 \quad \text{if} \quad 0 < x < \delta.$$

If we choose $\delta = \varepsilon^2$, we have

$$0 < x < \delta = \varepsilon^2 \quad \Rightarrow \quad \sqrt{x} < \varepsilon,$$

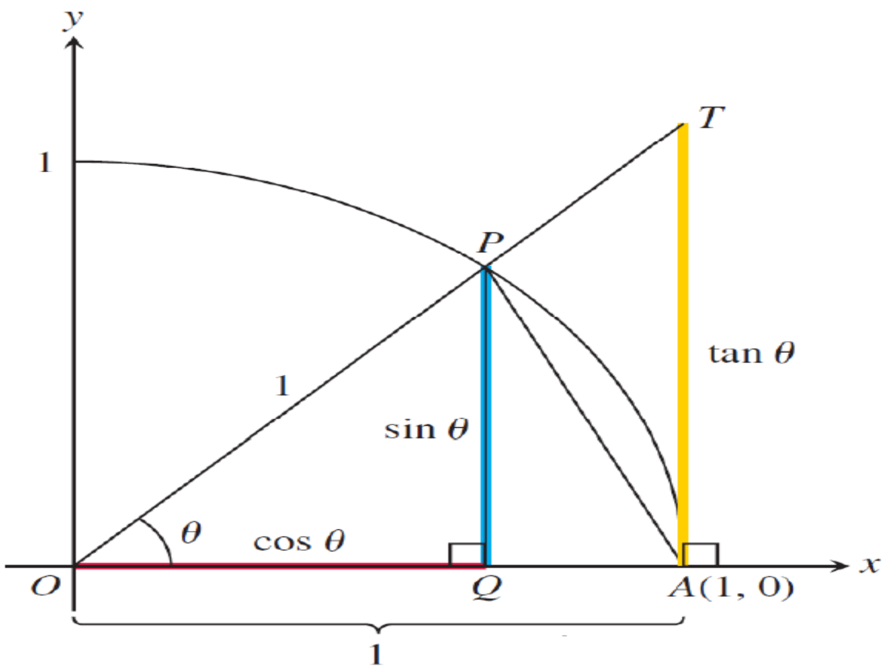
or

$$0 < x < \varepsilon^2 \quad \Rightarrow \quad |\sqrt{x} - 0| < \varepsilon.$$

According to the definition, this shows that $\lim_{x \to 0^+} \sqrt{x} = 0$.

Theorem

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \quad (\theta \textit{ in radians})$$



$$\sin\theta < \theta < \tan\theta.$$

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Taking the reciprocals reverses the inequalities:

$$\cos\theta < \frac{\sin\theta}{\theta} < 1.$$

As $\theta \to 0$, $\cos\theta \to 1$. Therefore, by the Squeeze Theorem,

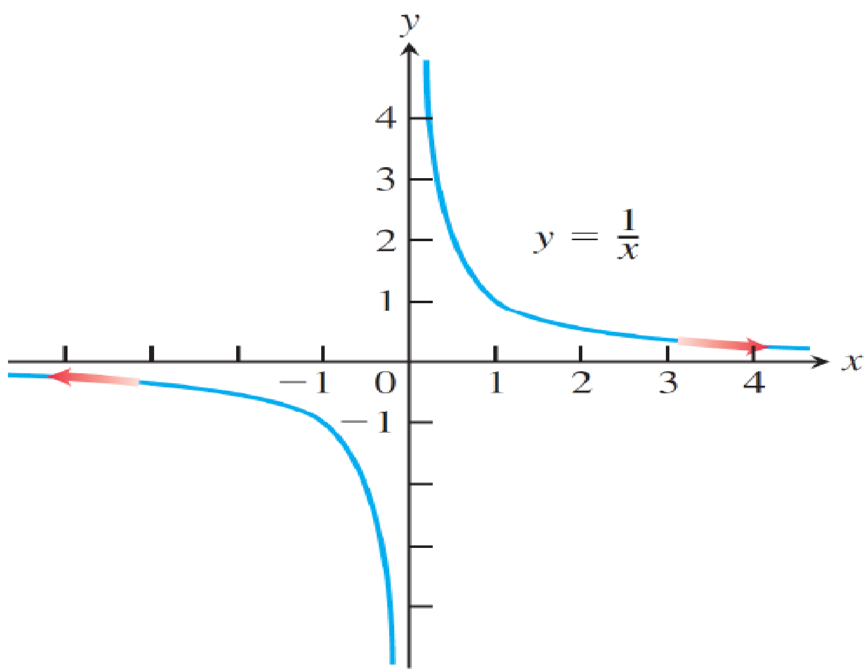
$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.$$

Limits Involving Infinity

Finite Limits as $x \to \pm\infty$

The symbol for infinity (∞) does not represent **a real number.** We use ∞ to describe the behavior of a function when the values in its domain or range outgrow all finite bounds.

For example, the function $f(x) = \frac{1}{x}$ is defined for all $x \neq 0$.



The graph of $y = \frac{1}{x}$ approaches 0 as $x \to +\infty$ or $x \to -\infty$.

We say that $f(x)$ has the *limit* L *as* x *approaches infinity* and write

$$\lim_{x \to \infty} f(x) = L$$

if, for every number $\varepsilon > 0$, there exists a corresponding number M such that for all x

$$x > M \Rightarrow |f(x) - L| < \varepsilon.$$

We say that $f(x)$ has the *limit* L *as* x *approaches minus infinity* and write

$$\lim_{x \to -\infty} f(x) = L$$

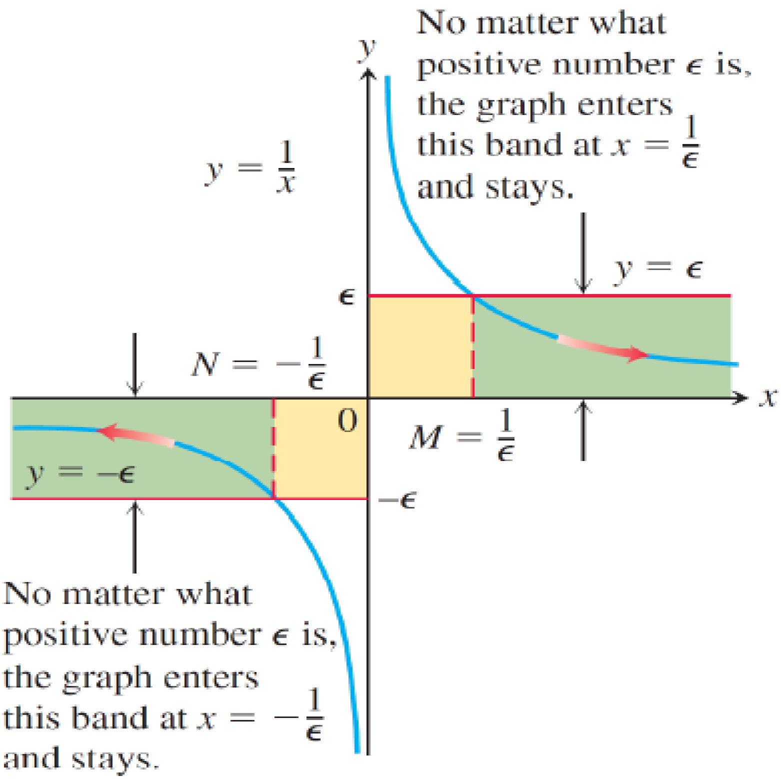
if, for every number $\varepsilon > 0$, there exists a corresponding number N such that for all x

$$x < N \Rightarrow |f(x) - L| < \varepsilon.$$

Show that

(a) $\lim_{x\to\infty} \frac{1}{x} = 0$

(b) $\lim_{x\to-\infty} \frac{1}{x} = 0$



The geometry behind the argument in Example above.

Show that

(a) $\lim_{x\to\infty} \frac{1}{x} = 0$

(b) $\lim_{x\to-\infty} \frac{1}{x} = 0$

Solution (a) Let $\varepsilon > 0$ be given. We must find a number M such that for all x

$$x > M \Rightarrow \left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \varepsilon.$$

The implication will hold if $M = 1/\varepsilon$ or any larger positive number. This proves $\lim_{x\to\infty} \frac{1}{x} = 0$.
(b) Let $\varepsilon > 0$ be given. We must find a number N such that for all x

$$x < N \Rightarrow \left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \varepsilon.$$

The implication will hold if $N = -1/\varepsilon$ or any number less than $-1/\varepsilon$. This proves $\lim_{x\to-\infty} \frac{1}{x} = 0$.
Limits at infinity have properties similar to those of finite limits.

All the Limit Laws *are **true** when we replace* $\lim_{x\to c}$ *by* $\lim_{x\to\infty}$ *or* $\lim_{x\to-\infty}$*.*
That is, the variable x may approach a finite number c or $\pm\infty$.

Remark

In general, the Limit Laws can't be applied to infinite limits because ∞ is not a number ($\infty - \infty$ can't be defined).
However, we can write:

$$\lim_{x\to\infty} (x^2 - x) = \lim_{x\to\infty} x(x-1) = \infty$$

because both x and $x-1$ become arbitrarily large and so their product does too.

(a)

$$\lim_{x\to\infty} \left(5 + \frac{1}{x}\right) = \lim_{x\to\infty} 5 + \lim_{x\to\infty} \frac{1}{x} = 5 + 0 = 5$$

(b)

$$\lim_{x\to\infty} \frac{\pi\sqrt{3}}{x^2} = \lim_{x\to\infty} \pi\sqrt{3} \cdot \lim_{x\to\infty} \frac{1}{x^2} = \pi\sqrt{3} \cdot 0 = 0$$

Limits at Infinity of Rational Functions

$$\lim_{x\to\infty} \frac{5x^2+8x-3}{3x^2+2} = \lim_{x\to\infty} \frac{5+(8/x)-(3/x^2)}{3+(2/x^2)} = \frac{5+0-0}{3+0} = \frac{5}{3}$$

$$\lim_{x\to\infty} \frac{11x+2}{2x^3-1} = \lim_{x\to\infty} \frac{(11/x^2)+(2/x^3)}{2-(1/x^3)} = \frac{0+0}{2-0} = 0$$

Find $\lim_{x\to 1^+} \frac{1}{x-1}$ *and* $\lim_{x\to 1^-} \frac{1}{x-1}$.

Analytic Solution Think about the number $x-1$ and its reciprocal. As $x \to 1^+$, we have $(x-1) \to 0^+$ and $1/(x-1) \to \infty$. As $x \to 1^-$, we have $(x-1) \to 0^-$ and $1/(x-1) \to -\infty$.

Discuss the behavior of

$$f(x) = \frac{1}{x^2} \quad \text{as} \quad x \to 0.$$

$$\lim_{x\to 0^+} \frac{1}{x^2} = \lim_{x\to 0^-} \frac{1}{x^2} = \infty.$$

The function $y = 1/x$ shows no consistent behavior as $x \to 0$. We have $1/x \to \infty$ if $x \to 0^+$, but $1/x \to -\infty$ if $x \to 0^-$. All we can say about $\lim_{x\to 0}(1/x)$ is that it does not exist. The function $y = 1/x^2$ is different. Its values approach infinity as x approaches zero from either side, so we can say that $\lim_{x\to 0}(1/x^2) = \infty$.

(a) $\lim_{x\to 2^-} \frac{(x-2)^2}{x^2-4} = 0$

(b) $\lim_{x\to 2^+} \frac{x-2}{(x-2)(x+2)} = \frac{1}{4}$

Find $\lim_{x\to -\infty} \frac{2x^5-6x^4+1}{3x^2+x-7}$.

Solution. We are asked to find the limit of a rational function as $x \to -\infty$, so we divide the numerator and denominator by x^2, the highest power of x in the denominator:

$$\lim_{x\to -\infty} \frac{2x^5-6x^4+1}{3x^2+x-7} = \lim_{x\to -\infty} \frac{2x^3-6x^2+\frac{1}{x^2}}{3+\frac{1}{x}-\frac{7}{x^2}}.$$

As $x \to -\infty$, the numerator tends to $-\infty$ while the denominator approaches 3 as $x \to -\infty$. Therefore,

$$\lim_{x\to -\infty} \frac{2x^5-6x^4+1}{3x^2+x-7} = -\infty.$$

1. *We say that* $f(x)$ ***approaches infinity*** *as* x *approaches* c*, and write*

$$\lim_{x \to c} f(x) = \infty,$$

if for every positive real number B *there exists a corresponding* $\delta > 0$ *such that for all* x

$$0 < |x - c| < \delta \Rightarrow f(x) > B.$$

2. *We say that* $f(x)$ ***approaches minus infinity*** *as* x *approaches* c*, and write*

$$\lim_{x \to c} f(x) = -\infty,$$

if for every negative real number $-B$ *there exists a corresponding* $\delta > 0$ *such that for all* x

$$0 < |x - c| < \delta \Rightarrow f(x) < -B.$$

Prove that $\lim_{x\to 0} \frac{1}{x^2} = \infty$.

Solution. Given $B > 0$, we want to find $\delta > 0$ such that

$$0 < |x - 0| < \delta \Rightarrow \frac{1}{x^2} > B.$$

Now,

$$\frac{1}{x^2} > B \quad \text{if and only if} \quad x^2 < \frac{1}{B}$$

or, equivalently,

$$|x| < \frac{1}{\sqrt{B}}.$$

Thus, choosing $\delta = 1/\sqrt{B}$ (or any smaller positive number), we see that

$$|x| < \delta \Rightarrow \frac{1}{x^2} \geq B.$$

Therefore, by definition,

$$\lim_{x\to 0} \frac{1}{x^2} = \infty.$$

Find (a) $\lim_{x\to\infty} \sin\left(\frac{1}{x}\right)$ *and (b)* $\lim_{x\to\pm\infty} x\sin\left(\frac{1}{x}\right)$.

Solution. (a) We introduce the new variable $t = \frac{1}{x}$. We know that $t \to 0^+$ as $x \to \infty$ Therefore,

$$\lim_{x\to\infty} \sin\left(\frac{1}{x}\right) = \lim_{t\to 0^+} \sin t = 0.$$

(b) We calculate the limits as $x \to \infty$ and $x \to -\infty$:

$$\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = \lim_{t\to 0^+} \frac{\sin t}{t} = 1 \quad \text{and} \quad \lim_{x\to-\infty} x\sin\left(\frac{1}{x}\right) = \lim_{t\to 0^-} \frac{\sin t}{t} = 1.$$

Find $\lim_{x\to 0^-} e^{1/x}$.

Solution. We let $t = 1/x$. we can see that $t \to -\infty$ as $x \to 0^-$. Therefore,

$$\lim_{x\to 0^-} e^{1/x} = \lim_{t\to-\infty} e^t = 0$$

If $$y = 2 + \frac{\sin x}{x}.$$

Solution. We are interested in the behavior as $x \to \pm\infty$. Since

$$0 \le \left|\frac{\sin x}{x}\right| \le \frac{1}{|x|}$$

and $\lim_{x\to\pm\infty} \frac{1}{|x|} = 0$, we have $\lim_{x\to\pm\infty} \frac{\sin x}{x} = 0$ by the Sandwich Theorem. Hence,

$$\lim_{x\to\pm\infty}\left(2 + \frac{\sin x}{x}\right) = 2 + 0 = 2,$$

Find $\lim_{x\to\infty}\left(x - \sqrt{x^2+16}\right)$.

$$\lim_{x\to\infty}\left(x - \sqrt{x^2+16}\right) = \lim_{x\to\infty}\frac{\left(x - \sqrt{x^2+16}\right)\left(x + \sqrt{x^2+16}\right)}{x + \sqrt{x^2+16}} = \lim_{x\to\infty}\frac{x^2 - (x^2+16)}{x + \sqrt{x^2+16}}.$$

Simplifying, we get

$$\lim_{x\to\infty}\frac{-16}{x + \sqrt{x^2+16}} = 0.$$