When sketching the graph $y = f(x)$ of a function f, we have three sources of useful information:

(i) **the function f itself**, from which we determine the coordinates of some points on the graph, the symmetry of the graph, and any asymptotes;

(ii) **the first derivative, f'**, from which we determine the intervals of increase and decrease and the location of any local extreme values; and

(iii) **the second derivative, f''**, from which we determine the concavity and inflection points, and sometimes extreme values.

Items (ii) and (iii) were explored in the previous sections. In this section we consider what we can learn from the function itself about the shape of its graph, and then we illustrate the entire sketching procedure with several examples using all three sources of information.

Procedure for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.
2. Find the derivatives y' and y''.
3. Find the critical points of f, if any, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes that may exist
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve together with any asymptotes that exist.

The graph of $y = f(x)$ has a **vertical asymptote** at $x = a$ if

$$\text{either} \quad \lim_{x \to a-} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x \to a+} f(x) = \pm\infty, \qquad \text{or both.}$$

The graph of $y = f(x)$ has a **horizontal asymptote** $y = L$ if

$$\text{either} \quad \lim_{x \to \infty} f(x) = L \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L, \qquad \text{or both.}$$

The straight line $y = ax + b$ (where $a \neq 0$) is an **oblique asymptote** of the graph of $y = f(x)$ if

$$\text{either} \quad \lim_{x \to -\infty} \big(f(x) - (ax + b)\big) = 0 \quad \text{or} \quad \lim_{x \to \infty} \big(f(x) - (ax + b)\big) = 0,$$

or both.

Here is a checklist of things to consider when you are asked to make a careful sketch of the graph of $y = f(x)$. It will, of course, not always be possible to obtain every item of information mentioned in the list.

Checklist for curve sketching

1. Calculate $f'(x)$ and $f''(x)$, and express the results in factored form.
2. Examine $f(x)$ to determine its domain and the following items:
   (a) Any vertical asymptotes. (Look for zeros of denominators.)
   (b) Any horizontal or oblique asymptotes. (Consider $\lim_{x\to\pm\infty} f(x)$.)
   (c) Any obvious symmetry. (Is f even or odd?)
   (d) Any easily calculated intercepts (points with coordinates $(x, 0)$ or $(0, y)$) or endpoints or other "obvious" points. You will add to this list when you know any critical points, singular points, and inflection points. Eventually you should make sure you know the coordinates of at least one point on every component of the graph.
3. Examine $f'(x)$ for the following:
   (a) Any critical points.
   (b) Any points where f' is not defined. (These will include singular points, endpoints of the domain of f, and vertical asymptotes.)
   (c) Intervals on which f' is positive or negative. It's a good idea to convey this information in the form of a chart such as those used in the examples. Conclusions about where f is increasing and decreasing and classification of some critical and singular points as local maxima and minima can also be indicated on the chart.

4. Examine $f''(x)$ for the following:
   (a) Points where $f''(x) = 0$.
   (b) Points where $f''(x)$ is undefined. (These will include singular points, endpoints, vertical asymptotes, and possibly other points as well, where f' is defined but f'' isn't.)
   (c) Intervals where f'' is positive or negative and where f is therefore concave up or down. Use a chart.
   (d) Any inflection points.

When you have obtained as much of this information as possible, make a careful sketch that reflects *everything* you have learned about the function.

sketch the curve $y = \dfrac{2x^2}{x^2 - 1}$.

A. The domain is

$$\{x \mid x^2 - 1 \neq 0\} = \{x \mid x \neq \pm 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$$

B. The x- and y-intercepts are both 0.

C. Since $f(-x) = f(x)$, the function f is even. The curve is symmetric about the y-axis.

D.

$$\lim_{x \to \pm\infty} \frac{2x^2}{x^2 - 1} = \lim_{x \to \pm\infty} \frac{2}{1 - 1/x^2} = 2$$

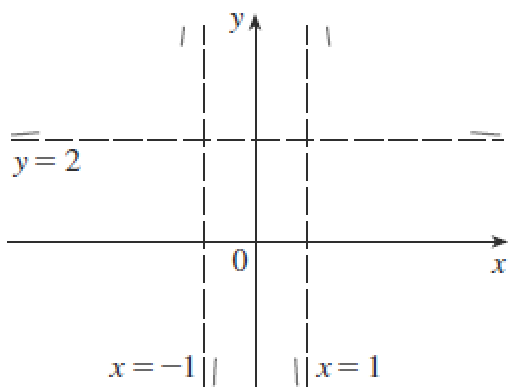
Therefore, the line $y = 2$ is a horizontal asymptote.

Since the denominator is 0 when $x = \pm 1$, we compute the following limits:

$$\lim_{x \to 1^+} \frac{2x^2}{x^2 - 1} = \infty \qquad \lim_{x \to 1^-} \frac{2x^2}{x^2 - 1} = -\infty$$

$$\lim_{x \to -1^+} \frac{2x^2}{x^2 - 1} = -\infty \qquad \lim_{x \to -1^-} \frac{2x^2}{x^2 - 1} = \infty$$

Therefore, the lines $x = 1$ and $x = -1$ are vertical asymptotes. This information about limits and asymptotes enables us to draw the preliminary sketch in Figure showing the parts of the curve near the asymptotes.



E.
$$f'(x) = \frac{4x(x^2 - 1) - 2x^2 \cdot 2x}{(x^2 - 1)^2} = \frac{-4x}{(x^2 - 1)^2}$$

Since $f'(x) > 0$ when $x < 0$ ($x \neq -1$) and $f'(x) < 0$ when $x > 0$ ($x \neq 1$), f is increasing on $(-\infty, -1)$ and $(-1, 0)$ and decreasing on $(0, 1)$ and $(1, \infty)$.

F. The only critical number is $x = 0$. Since f' changes from positive to negative at 0, $f(0) = 0$ is a local maximum by the First Derivative Test.

G.
$$f''(x) = \frac{-4(x^2 - 1)^2 + 4x \cdot 2(x^2 - 1)2x}{(x^2 - 1)^4} = \frac{12x^2 + 4}{(x^2 - 1)^3}$$

Since $12x^2 + 4 > 0$ for all x, we have

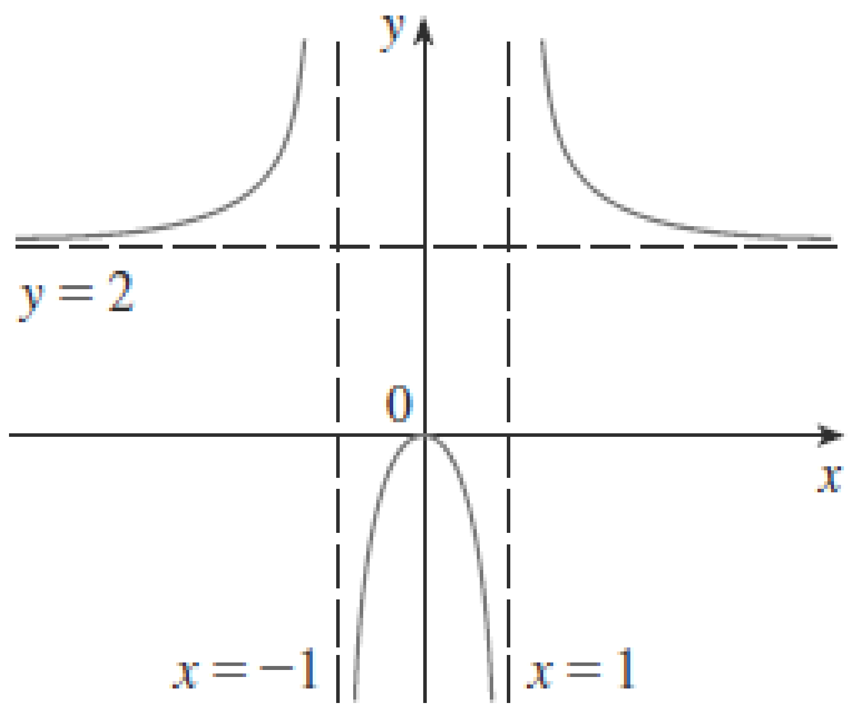
$$f''(x) > 0 \iff x^2 - 1 > 0 \iff |x| > 1$$

and $f''(x) < 0 \iff |x| < 1$. Thus, the curve is concave upward on the intervals $(-\infty, -1)$ and $(1, \infty)$ and concave downward on $(-1, 1)$. It has no point of inflection since 1 and -1 are not in the domain of f.

Concavity Test

a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I

b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.



Sketch the graph of $f(x) = \dfrac{x^2}{\sqrt{x+1}}$.

A. Domain $= \{x \mid x + 1 > 0\} = \{x \mid x > -1\} = (-1, \infty)$

B. The x- and y-intercepts are both 0.

C. Symmetry: None

D. Since

$$\lim_{x \to \infty} \frac{x^2}{\sqrt{x+1}} = \infty$$

there is no horizontal asymptote. Since $\sqrt{x+1} \to 0$ as $x \to -1^+$ and $f(x)$ is always positive, we have

$$\lim_{x \to -1^+} \frac{x^2}{\sqrt{x+1}} = \infty$$

and so the line $x = -1$ is a vertical asymptote.

E.

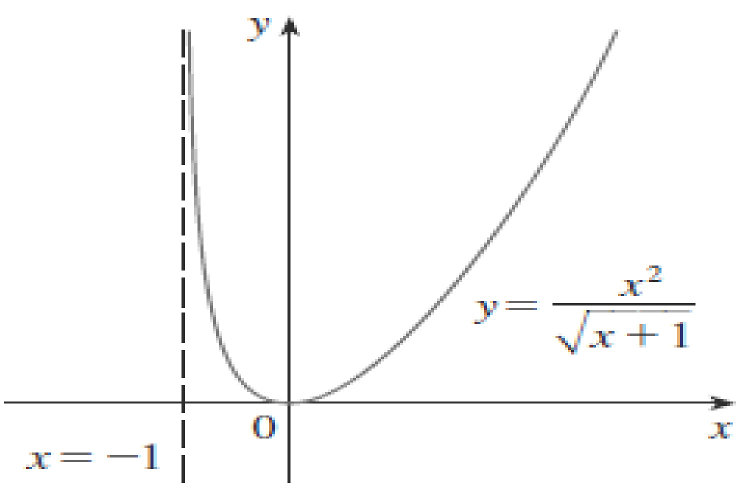
$$f'(x) = \frac{2x\sqrt{x+1} - x^2 \cdot 1/\left(2\sqrt{x+1}\right)}{x+1} = \frac{x(3x+4)}{2(x+1)^{3/2}}$$

We see that $f'(x) = 0$ when $x = 0$ (notice that $-\frac{4}{3}$ is not in the domain of f), so the only critical number is 0. Since $f'(x) < 0$ when $-1 < x < 0$ and $f'(x) > 0$ when $x > 0$, f is decreasing on $(-1, 0)$ and increasing on $(0, \infty)$.

F. Since $f'(0) = 0$ and f' changes from negative to positive at 0, $f(0) = 0$ is a local (and absolute) minimum by the First Derivative Test.

G. $$f''(x) = \frac{2(x+1)^{3/2}(6x+4) - (3x^2+4x)3(x+1)^{1/2}}{4(x+1)^3} = \frac{3x^2+8x+8}{4(x+1)^{5/2}}$$

Note that the denominator is always positive. The numerator is the quadratic $3x^2 + 8x + 8$, which is always positive because its discriminant is $b^2 - 4ac = -32$, which is negative, and the coefficient of x^2 is positive. Thus, $f''(x) > 0$ for all x in the domain of f, which means that f is concave upward on $(-1, \infty)$ and there is no point of inflection.



Sketch the graph of $f(x) = xe^x$.

A. The domain is $\mathbb{R}$.

B. The x- and y-intercepts are both 0.

C. Symmetry: None

D. Because both x and e^x become large as $x \to \infty$, we have $\lim_{x\to\infty} xe^x = \infty$. As $x \to -\infty$, however, $e^x \to 0$ and so we have an indeterminate product that requires the use of l'Hospital's Rule:

$$\lim_{x\to-\infty} xe^x = \lim_{x\to-\infty} \frac{x}{e^{-x}} = \lim_{x\to-\infty} \frac{1}{-e^{-x}} = \lim_{x\to-\infty} (-e^x) = 0$$

Thus, the x-axis is a horizontal asymptote.

E.

$$f'(x) = xe^x + e^x = (x+1)e^x$$

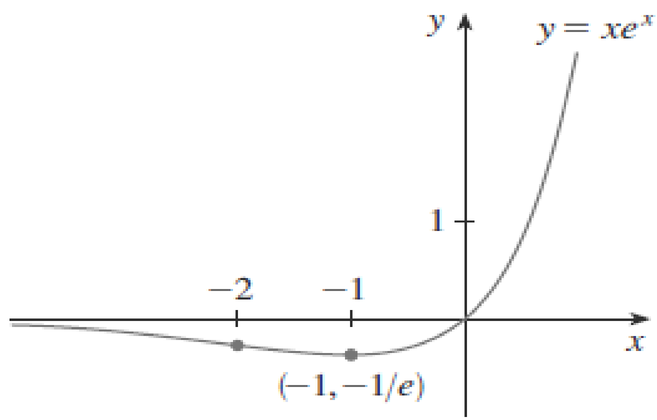
Since e^x is always positive, we see that $f'(x) > 0$ when $x + 1 > 0$, and $f'(x) < 0$ when $x + 1 < 0$. So f is increasing on $(-1, \infty)$ and decreasing on $(-\infty, -1)$.

F. Because $f'(-1) = 0$ and f changes from negative to positive at $x = -1$, $f(-1) = -e^{-1}$ is a local (and absolute) minimum.

G.

$$f''(x) = (x+1)e^x + e^x = (x+2)e^x$$

Since $f''(x) > 0$ if $x > -2$ and $f''(x) < 0$ if $x < -2$, f is concave upward on $(-2, \infty)$ and concave downward on $(-\infty, -2)$. The inflection point is $(-2, -2e^{-2})$.



y
$y = xe^x$
1
-2
-1
x
$(-1, -1/e)$

Sketch the graph of $f(x) = 2\cos x + \sin 2x$.

A. The domain is $\mathbb{R}$.

B. The y-intercept is $f(0) = 2$. The x-intercepts occur when

$$2\cos x + \sin 2x = 2\cos x + 2\sin x \cos x = 2\cos x\,(1 + \sin x) = 0$$

that is, when $\cos x = 0$ or $\sin x = -1$. Thus, in the interval $[0, 2\pi]$, the x-intercepts are $\pi/2$ and $3\pi/2$.

C. f is neither even nor odd, but $f(x + 2\pi) = f(x)$ for all x and so f is periodic and has period 2π. Thus, in what follows we need to consider only $0 \leqslant x \leqslant 2\pi$ and then extend the curve by translation in H.

D. Asymptotes: None

E.

$$\begin{aligned} f'(x) &= -2\sin x + 2\cos 2x = -2\sin x + 2(1 - 2\sin^2 x) \\ &= -2(2\sin^2 x + \sin x - 1) = -2(2\sin x - 1)(\sin x + 1) \end{aligned}$$

Thus, $f'(x) = 0$ when $\sin x = \frac{1}{2}$ or $\sin x = -1$, so in $[0, 2\pi]$ we have $x = \pi/6$, $5\pi/6$, and $3\pi/2$. In determining the sign of $f'(x)$ in the following chart, we use the

fact that $\sin x + 1 \geqslant 0$ for all x.

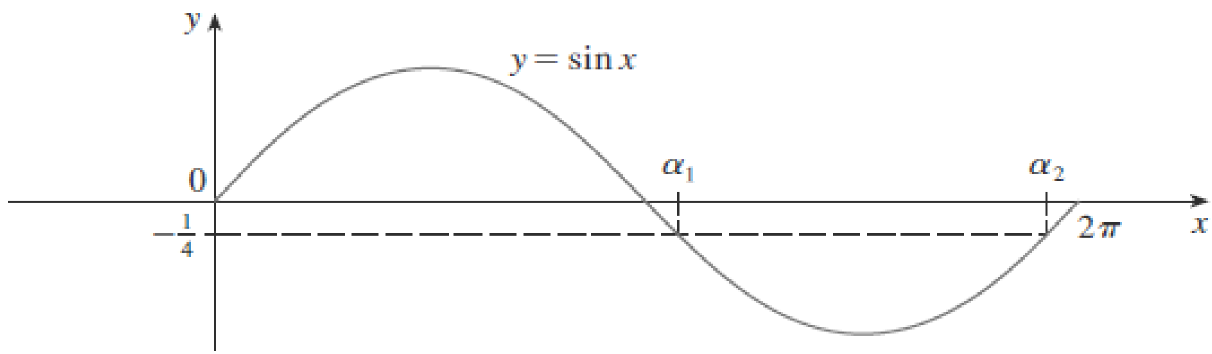
Interval	$f'(x)$	f
$0 < x < \pi/6$	$+$	increasing on $(0, \pi/6)$
$\pi/6 < x < 5\pi/6$	$-$	decreasing on $(\pi/6, 5\pi/6)$
$5\pi/6 < x < 3\pi/2$	$+$	increasing on $(5\pi/6, 3\pi/2)$
$3\pi/2 < x < 2\pi$	$+$	increasing on $(3\pi/2, 2\pi)$

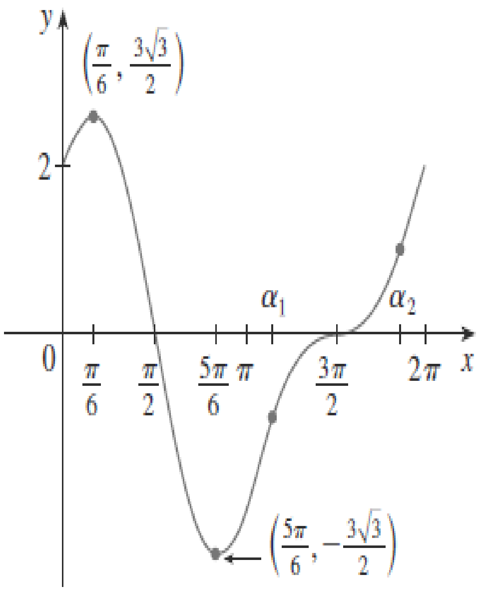
F. From the chart in E the First Derivative Test says that $f(\pi/6) = 3\sqrt{3}/2$ is a local maximum and $f(5\pi/6) = -3\sqrt{3}/2$ is a local minimum, but f has no maximum or minimum at $3\pi/2$, only a horizontal tangent.

G.

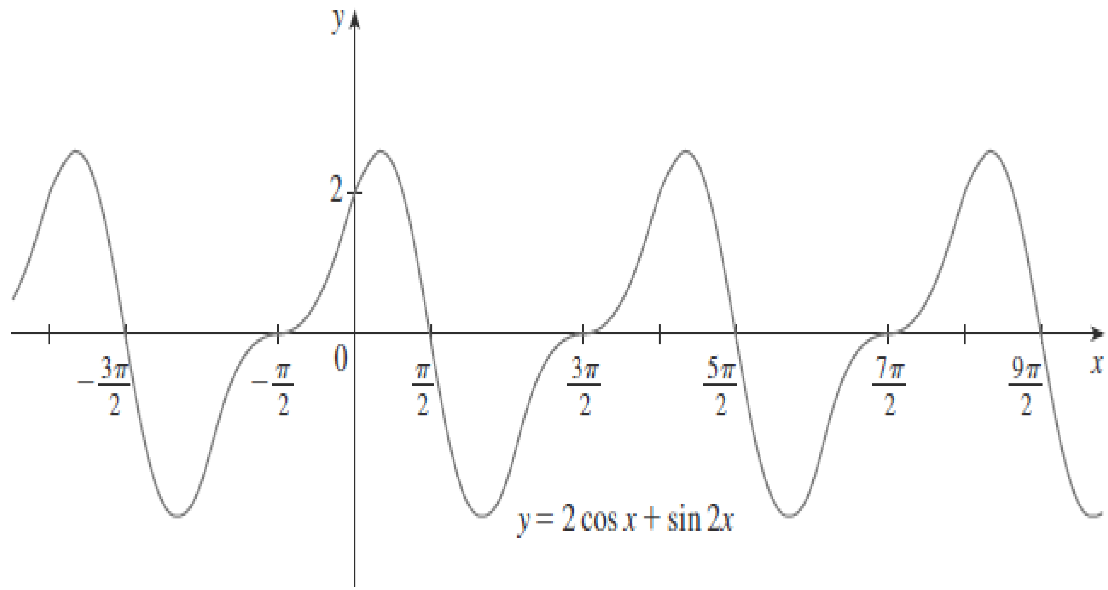
$$f''(x) = -2\cos x - 4\sin 2x = -2\cos x\,(1 + 4\sin x)$$

Thus, $f''(x) = 0$ when $\cos x = 0$ (so $x = \pi/2$ or $3\pi/2$) and when $\sin x = -\frac{1}{4}$. we see that there are two values of x between 0 and 2π for which $\sin x = -\frac{1}{4}$. Let's call them α_1 and α_2. Then $f''(x) > 0$ on $(\pi/2, \alpha_1)$ and $(3\pi/2, \alpha_2)$, so f is concave upward there. Also $f''(x) < 0$ on $(0, \pi/2)$, $(\alpha_1, 3\pi/2)$, and $(\alpha_2, 2\pi)$, so f is concave downward there. Inflection points occur when $x = \pi/2, \alpha_1, 3\pi/2$, and α_2.





y
x
0
2
$\left(\frac{\pi}{6}, \frac{3\sqrt{3}}{2}\right)$
$\left(\frac{5\pi}{6}, -\frac{3\sqrt{3}}{2}\right)$
α_1
α_2
$\frac{\pi}{6}$
$\frac{\pi}{2}$
$\frac{5\pi}{6}$
π
$\frac{3\pi}{2}$
2π



y
x
0
2
$-\frac{3\pi}{2}$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
$\frac{3\pi}{2}$
$\frac{5\pi}{2}$
$\frac{7\pi}{2}$
$\frac{9\pi}{2}$
$y = 2\cos x + \sin 2x$

Sketch the graph of $y = \ln(4 - x^2)$.

A. The domain is

$$\{x \mid 4 - x^2 > 0\} = \{x \mid x^2 < 4\} = \{x \mid |x| < 2\} = (-2, 2)$$

B. The y-intercept is $f(0) = \ln 4$. To find the x-intercept we set

$$y = \ln(4 - x^2) = 0$$

We know that $\ln 1 = \log_e 1 = 0$ (since $e^0 = 1$), so we have $4 - x^2 = 1 \Rightarrow x^2 = 3$ and therefore the x-intercepts are $\pm\sqrt{3}$.

C. Since $f(-x) = f(x)$, f is even and the curve is symmetric about the y-axis.

D. We look for vertical asymptotes at the endpoints of the domain. Since $4 - x^2 \to 0^+$ as $x \to 2^-$ and also as $x \to -2^+$, we have

$$\lim_{x \to 2^-} \ln(4 - x^2) = -\infty \qquad \lim_{x \to -2^+} \ln(4 - x^2) = -\infty$$

Thus, the lines $x = 2$ and $x = -2$ are vertical asymptotes.

E.

$$f'(x) = \frac{-2x}{4 - x^2}$$

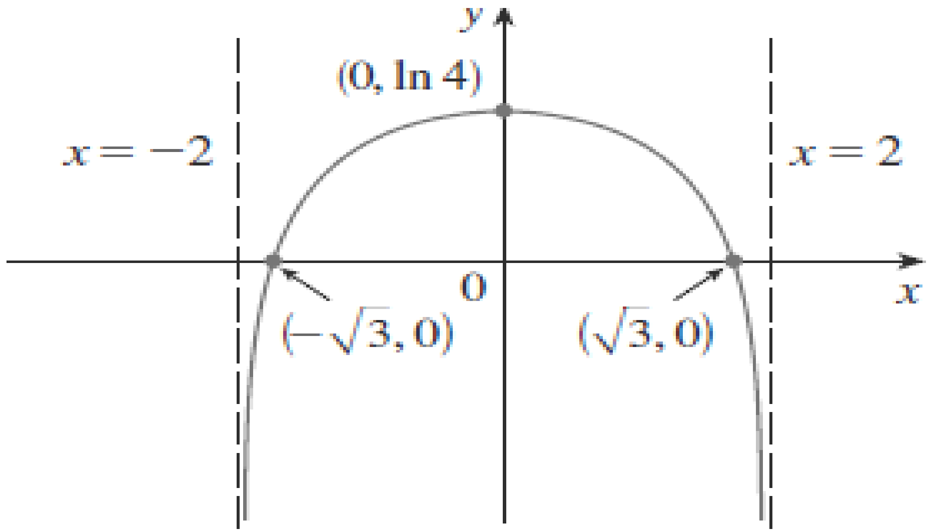
Since $f'(x) > 0$ when $-2 < x < 0$ and $f'(x) < 0$ when $0 < x < 2$, f is increasing on $(-2, 0)$ and decreasing on $(0, 2)$.

F. The only critical number is $x = 0$. Since f' changes from positive to negative at 0, $f(0) = \ln 4$ is a local maximum by the First Derivative Test.

G.

$$f''(x) = \frac{(4 - x^2)(-2) + 2x(-2x)}{(4 - x^2)^2} = \frac{-8 - 2x^2}{(4 - x^2)^2}$$

Since $f''(x) < 0$ for all x, the curve is concave downward on $(-2, 2)$ and has no inflection point.



$y = \ln(4 - x^2)$

Sketch the graph of $f(x) = \dfrac{x^3}{x^2 + 1}$.

A. The domain is $\mathbb{R} = (-\infty, \infty)$.

B. The x- and y-intercepts are both 0.

C. Since $f(-x) = -f(x)$, f is odd and its graph is symmetric about the origin.

D. Since $x^2 + 1$ is never 0, there is no vertical asymptote. Since $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$, there is no horizontal asymptote. But long division gives

$$f(x) = \frac{x^3}{x^2 + 1} = x - \frac{x}{x^2 + 1}$$

$$f(x) - x = -\frac{x}{x^2 + 1} = -\frac{\dfrac{1}{x}}{1 + \dfrac{1}{x^2}} \to 0 \quad \text{as} \quad x \to \pm\infty$$

So the line $y = x$ is a slant asymptote.

E.

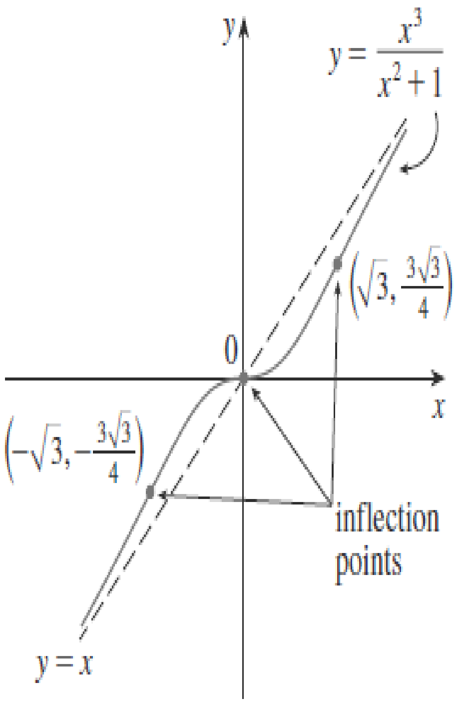
$$f'(x) = \frac{3x^2(x^2 + 1) - x^3 \cdot 2x}{(x^2 + 1)^2} = \frac{x^2(x^2 + 3)}{(x^2 + 1)^2}$$

Since $f'(x) > 0$ for all x (except 0), f is increasing on $(-\infty, \infty)$.

F. Although $f'(0) = 0$, f' does not change sign at 0, so there is no local maximum or minimum.

G. $$f''(x) = \frac{(4x^3 + 6x)(x^2 + 1)^2 - (x^4 + 3x^2) \cdot 2(x^2 + 1)2x}{(x^2 + 1)^4} = \frac{2x(3 - x^2)}{(x^2 + 1)^3}$$

Since $f''(x) = 0$ when $x = 0$ or $x = \pm\sqrt{3}$, we set up the following chart:



Interval	x	$3 - x^2$	$(x^2 + 1)^3$	$f''(x)$	f
$x < -\sqrt{3}$	$-$	$-$	$+$	$+$	CU on $(-\infty, -\sqrt{3})$
$-\sqrt{3} < x < 0$	$-$	$+$	$+$	$-$	CD on $(-\sqrt{3}, 0)$
$0 < x < \sqrt{3}$	$+$	$+$	$+$	$+$	CU on $(0, \sqrt{3})$
$x > \sqrt{3}$	$+$	$-$	$+$	$-$	CD on $(\sqrt{3}, \infty)$

The points of inflection are $(-\sqrt{3}, -3\sqrt{3}/4)$, $(0, 0)$, and $(\sqrt{3}, 3\sqrt{3}/4)$.

Sketch the graph of $y = \dfrac{x^2 + 2x + 4}{2x}$.

EXAMPLE Sketch the graph of $f(x) = \dfrac{(x + 1)^2}{1 + x^2}$.

EXAMPLE Sketch the graph of $f(x) = \dfrac{x^2 + 4}{2x}$.

1. The domain of f is all nonzero real numbers. There are no intercepts because neither x nor $f(x)$ can be zero. Since $f(-x) = -f(x)$, we note that f is an odd function, so the graph of f is symmetric about the origin.
2. We calculate the derivatives of the function, but first rewrite it in order to simplify our computations:

$$f(x) = \frac{x^2 + 4}{2x} = \frac{x}{2} + \frac{2}{x} \qquad \text{Function simplified for differentiation}$$

$$f'(x) = \frac{1}{2} - \frac{2}{x^2} = \frac{x^2 - 4}{2x^2} \qquad \text{Combine fractions to solve easily } f'(x) = 0.$$

$$f''(x) = \frac{4}{x^3} \qquad \text{Exists throughout the entire domain of } f$$

3. The critical points occur at $x = \pm 2$ where $f'(x) = 0$. Since $f''(-2) < 0$ and $f''(2) > 0$, we see from the Second Derivative Test that a relative maximum occurs at $x = -2$ with $f(-2) = -2$, and a relative minimum occurs at $x = 2$ with $f(2) = 2$.

4. On the interval $(-\infty, -2)$ the derivative f' is positive because $x^2 - 4 > 0$ so the graph is increasing; on the interval $(-2, 0)$ the derivative is negative and the graph is decreasing. Similarly, the graph is decreasing on the interval $(0, 2)$ and increasing on $(2, \infty)$.
5. There are no points of inflection because $f''(x) < 0$ whenever $x < 0$, $f''(x) > 0$ whenever $x > 0$, and f'' exists everywhere and is never zero throughout the domain of f. The graph is concave down on the interval $(-\infty, 0)$ and concave up on the interval $(0, \infty)$.

6. From the rewritten formula for $f(x)$, we see that

$$\lim_{x\to 0^+}\left(\frac{x}{2}+\frac{2}{x}\right)=+\infty \quad \text{and} \quad \lim_{x\to 0^-}\left(\frac{x}{2}+\frac{2}{x}\right)=-\infty,$$

so the y-axis is a vertical asymptote. Also, as $x\longrightarrow\infty$ or as $x\longrightarrow-\infty$, the graph of $f(x)$ approaches the line $y=x/2$. Thus $y=x/2$ is an oblique asymptote.

7. The graph of f is sketched in Figure

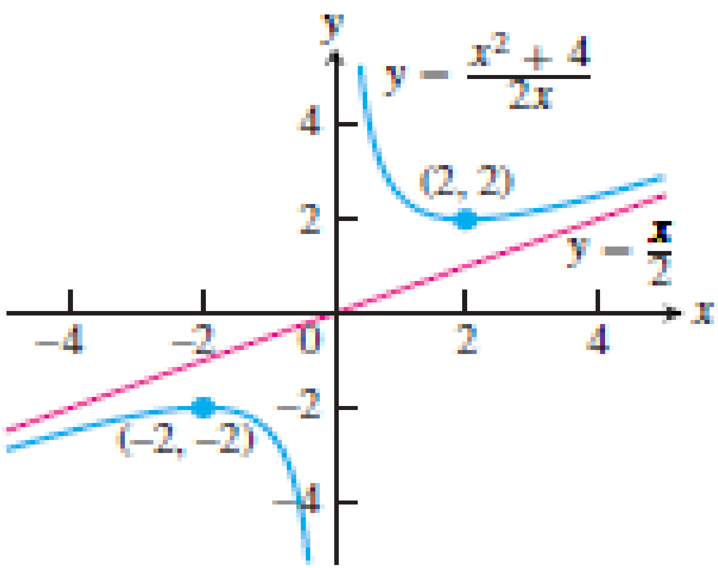


FIGURE The graph of $y=\dfrac{x^2+4}{2x}$

Sketch the graph of $y = \dfrac{1}{4 - x^2}$.

Sketch the graph of $y = \frac{4x}{x^2+1}$.

Solution:

Intercepts When $x = 0$, $y = 0$; when $y = 0$, $x = 0$. Thus, $(0, 0)$ is the only intercept. Since the denominator of y is always positive, we see that the sign of y is that of x. Here we dispense with a sign chart for y. From the observations so far it follows that the graph proceeds from the third quadrant (negative x and negative y), through $(0, 0)$ to the positive quadrant (positive x and positive y).

Symmetry There is symmetry about the origin:

$$y(-x) = \frac{4(-x)}{(-x)^2+1} = \frac{-4x}{x^2+1} = -y(x)$$

No other symmetry exists.

Asymptotes The denominator of this rational function is never 0, so there are no vertical asymptotes. Testing for horizontal asymptotes, we have

$$\lim_{x\to\pm\infty} \frac{4x}{x^2+1} = \lim_{x\to\pm\infty} \frac{4x}{x^2} = \lim_{x\to\pm\infty} \frac{4}{x} = 0$$

Thus, $y = 0$ (the x-axis) is a horizontal asymptote and the only nonvertical asymptote.

Maxima and Minima We have

$$y' = \frac{(x^2+1)(4) - 4x(2x)}{(x^2+1)^2} = \frac{4-4x^2}{(x^2+1)^2} = \frac{4(1+x)(1-x)}{(x^2+1)^2}$$

The critical values are $x = \pm 1$, so there are three intervals to consider in the sign chart.

We see that y is decreasing on $(-\infty, -1)$ and on $(1, \infty)$, increasing on $(-1, 1)$, with relative minimum at -1 and relative maximum at 1. The relative minimum is $(-1, y(-1)) = (-1, -2)$; the relative maximum is $(1, y(1)) = (1, 2)$.

	$-\infty$		-1		1		∞
$1+x$		$-$	0	$+$		$+$	
$1-x$		$+$		$+$	0	$-$	
$\frac{1}{(x^2+1)^2}$		$+$		$+$		$+$	
y'		$-$	0	$+$	0	$-$	
y							

Sign chart for y'.

Concavity Since $y' = \dfrac{4 - 4x^2}{(x^2+1)^2}$,

$$y'' = \frac{(x^2+1)^2(-8x) - (4-4x^2)(2)(x^2+1)(2x)}{(x^2+1)^4}$$

$$= \frac{8x(x^2+1)(x^2-3)}{(x^2+1)^4} = \frac{8x(x+\sqrt{3})(x-\sqrt{3})}{(x^2+1)^3}$$

Setting $y'' = 0$, we conclude that the possible points of inflection are when $x = \pm\sqrt{3}, 0$. There are four intervals to consider in the sign chart.

	$-\infty$		$-\sqrt{3}$		0		$\sqrt{3}$		∞
$x+\sqrt{3}$		$-$	0	$+$		$+$		$+$	
x		$-$		$-$	0	$+$		$+$	
$x-\sqrt{3}$		$-$		$-$		$-$	0	$+$	
$\dfrac{1}{(x^2+1)^3}$		$+$		$+$		$+$		$+$	
y''		$-$	0	$+$	0	$-$	0	$+$	
y		$\cap$		$\cup$		$\cap$		$\cup$	

Concavity analysis for $y = \dfrac{4x}{x^2+1}$.

Inflection points occur at $x = 0$ and $\pm\sqrt{3}$. The inflection points are

$$(-\sqrt{3}, y(\sqrt{3})) = (-\sqrt{3}, -\sqrt{3}) \quad (0, y(0)) = (0,0) \quad (\sqrt{3}, y(\sqrt{3})) = (\sqrt{3}, \sqrt{3})$$

x	0	1	$\sqrt{3}$	-1	$-\sqrt{3}$
y	0	2	$\sqrt{3}$	-2	$-\sqrt{3}$

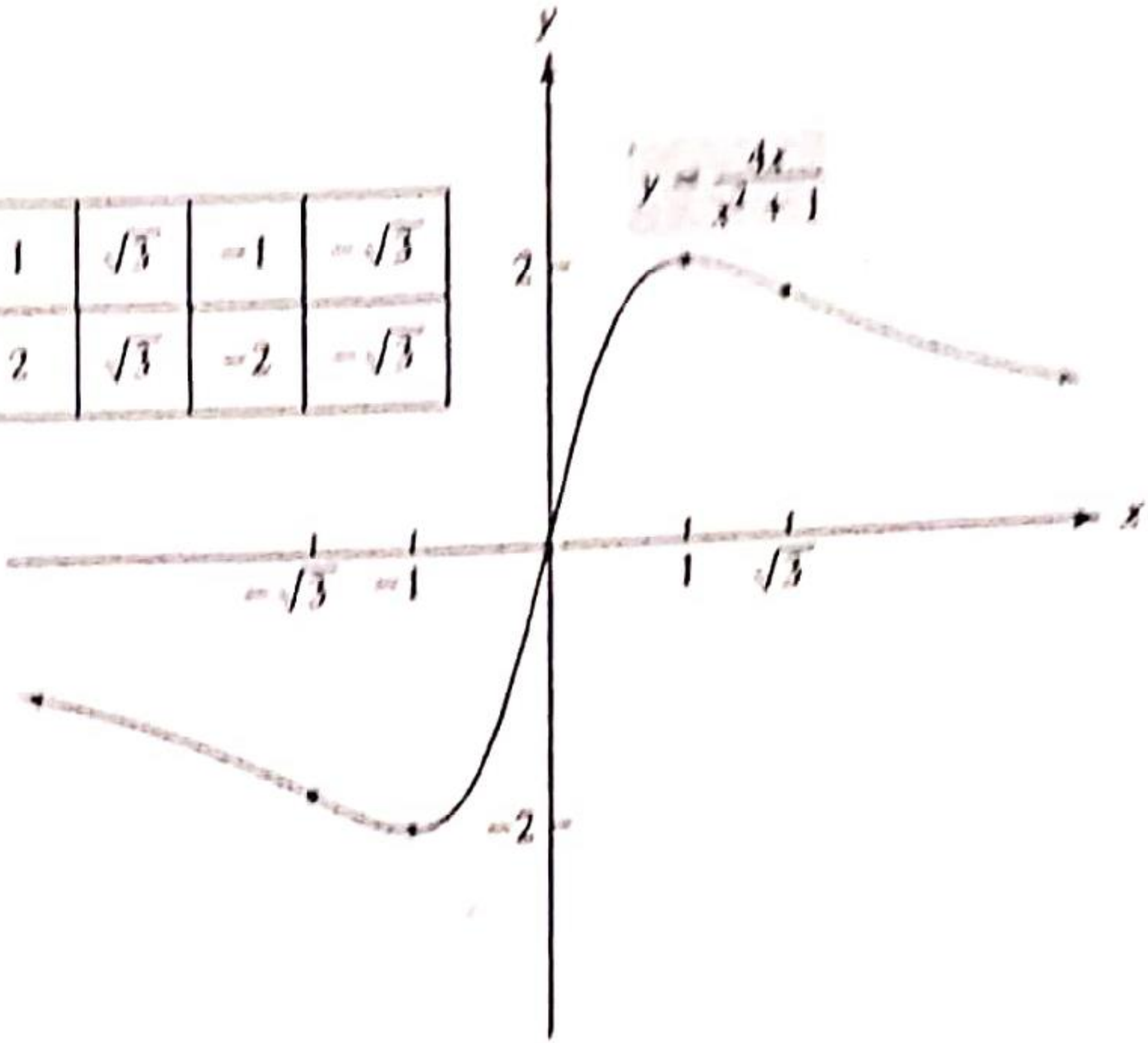


FIGURE Graph of $y = \frac{4x}{x^2+1}$.

$f(x) = y = x - \dfrac{1}{x}$

1) The domain is $\mathbb{R} - \{0\}$.

2) $y' = 1 + \dfrac{1}{x^2} = \dfrac{x^2+1}{x^2} > 0$ $y'' = -\dfrac{2}{x^3}$

x		0	
y''	$+$	$\Vert$	$-$

There is no local max and min. y' is always increasing.

3) $f(x)$ is not continuous at $x = 0$

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0^+} \frac{(h)^2 - 1}{h} = -\infty$$

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0^-} \frac{(-h)^2 - 1}{-h} = +\infty$$

while
$x \to 0 \Rightarrow y = \mp\infty$ so ; $\underline{\underline{x=0}}$ is <u>vertical</u> Asymptote

$\lim\limits_{x \to \mp\infty} \frac{x^2-1}{x} = \mp\infty$ It may have oblique asymptote

$\lim\limits_{x \to \infty} \frac{y}{x} = \frac{x^2-1}{x^2} = 1$; there is a oblique asymptote

$x^2 - 1 \,|\, \frac{x}{x}$
$-x^2$
$0 - 1$

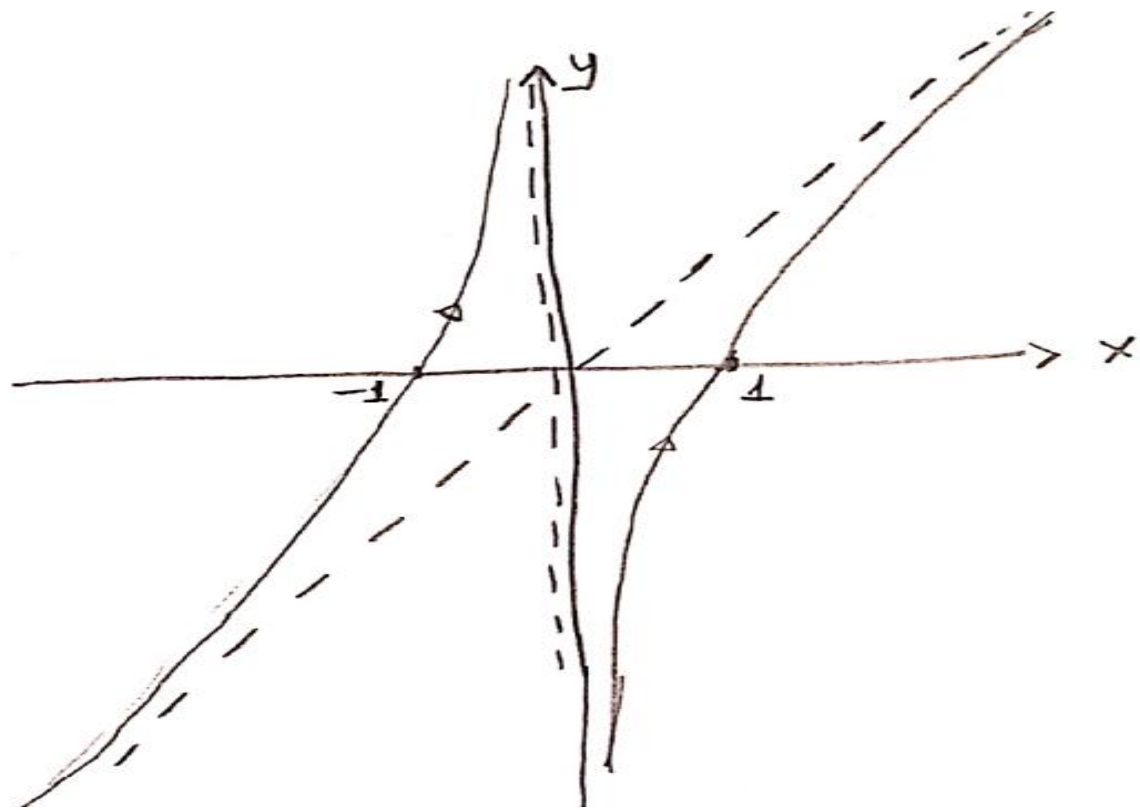
$\frac{x^2-1}{x} = x - \frac{1}{x}$

$\lim\limits_{x \to \infty} \left(\frac{x^2-1}{x} - x \right) = \lim\limits_{x \to \infty} \frac{x^2-1-x^2}{x} = \lim\limits_{x \to \infty} -\frac{1}{x} = 0$

$\boxed{y=x}$ <u>oblique</u> asymptote

4) if $x=0 \Rightarrow y \Rightarrow +\infty$
if $y=0 \Rightarrow x = \mp 1$

x	$-\infty$		-1		(V.A) 0		1		∞
y'		$+$		$+$	$\|$	$+$		$+$	
y	$-\infty$	$\nearrow$	0	$\nearrow$ $+\infty$	$\|$	$-\infty$ $\nearrow$	0	$\nearrow$	$+\infty$



$$y = f(x) = \frac{x}{\ln x - 1}$$ sketch the graph of $f(x)$.

① for $x > 0$ $\ln x - 1 > 0 \rightarrow x \neq e$

it is defined on $(0, e) \cup (e, \infty)$

② $$y' = \frac{\ln x - 1 - x \cdot \frac{1}{x}}{(\ln x - 1)^2} = \frac{\ln x - 2}{(\ln x - 1)^2}$$

$$y'' = \frac{\frac{1}{x}(\ln x - 1)^2 - 2(\ln x - 1) \cdot \frac{\ln x - 2}{x}}{(\ln x - 1)^4} = \frac{3 - \ln x}{x(\ln x - 1)^3}$$

$$y' = 0 \Rightarrow \ln x - 2 = 0 \Rightarrow e^2 = x \Rightarrow y = e^2$$

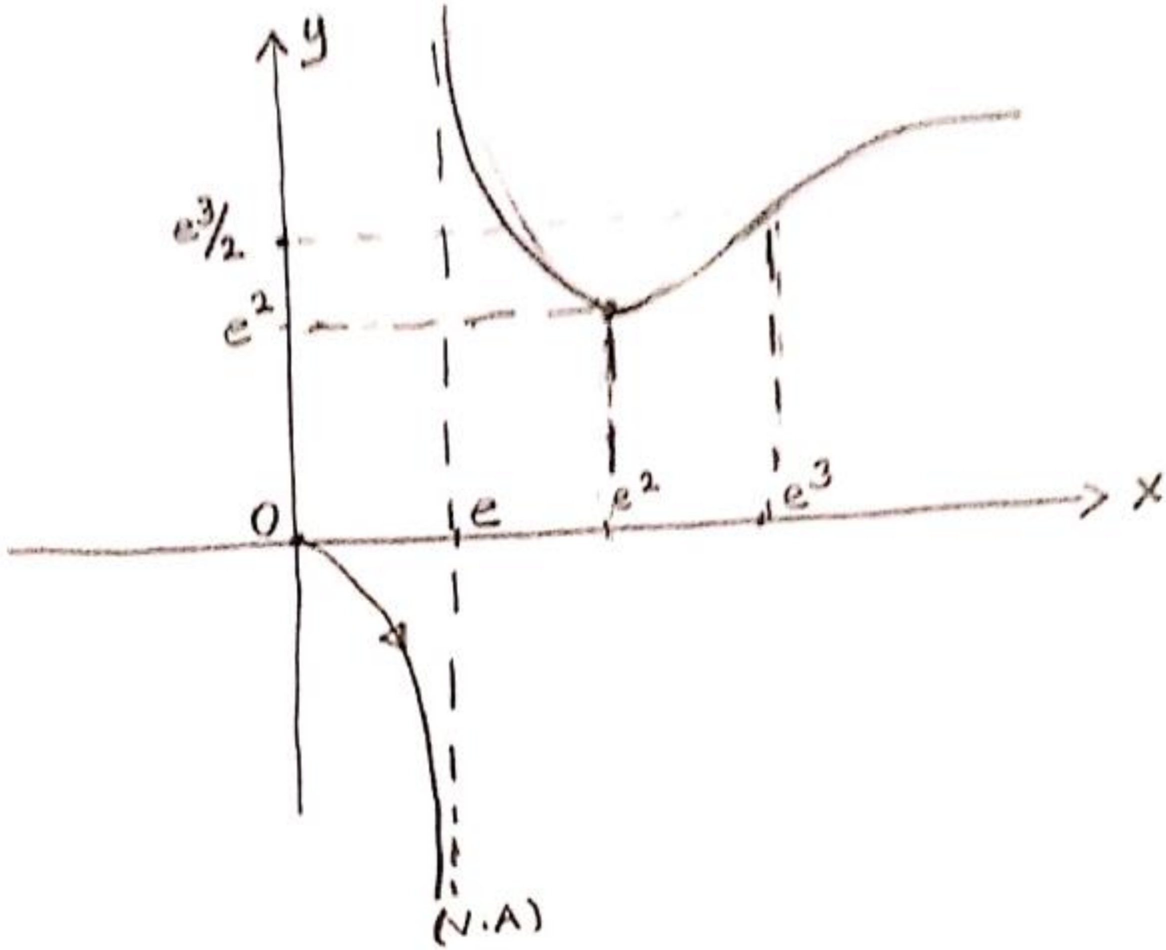
$$y''(e^2) = \frac{3 - \ln e^2}{e^2(\ln e^2 - 1)^3} = \frac{1}{e^2} > 0$$ local minimum at (e^2, e^2)

$y'' = 0 \Rightarrow x = e^3 \ , \ y = \frac{e^3}{2}$ $\quad (e^3, \frac{e^3}{2})$ inflection point

x	0		e		e^2		e^3		∞
y'		$-$		$-$	0	$+$		$+$	
y''		$+$		$+$		$+$		$-$	
y	0		$-\infty$ $\|$ $+\infty$		e^2		$\frac{e^3}{2}$		∞
			(V.A)						

$$\lim_{x \to 0^+} y = \lim_{x \to 0^+} \frac{x}{\ln x - 1} = 0$$

$$\lim_{x \to e^+} y = \infty \qquad \lim_{x \to e^-} y = -\infty$$



y
x
0
e
e^2
e^3
$e^{3}/2$
e^2
(V.A)

$$y = f(x) = x \cdot e^{-x}$$