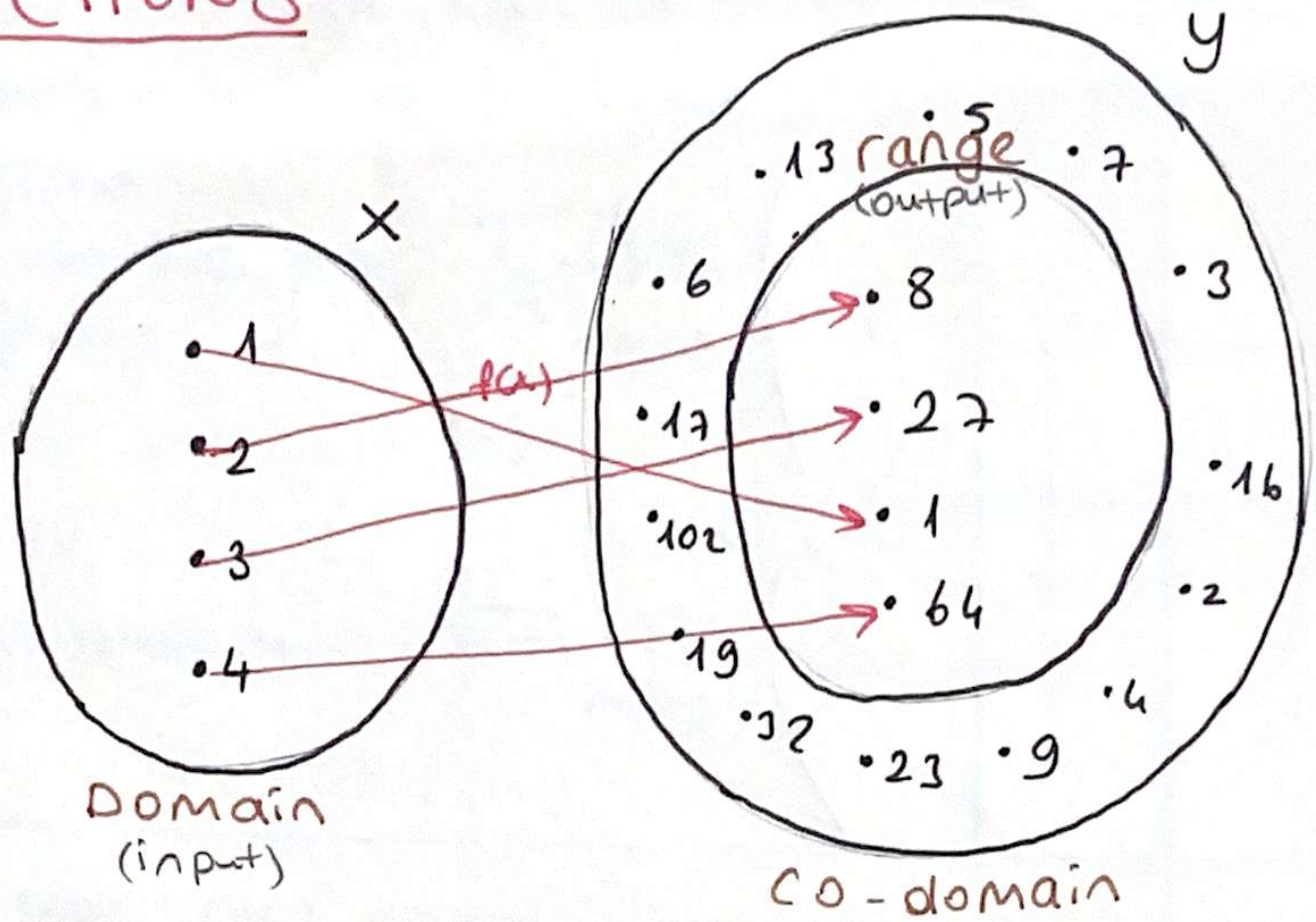
FUNCTIONS



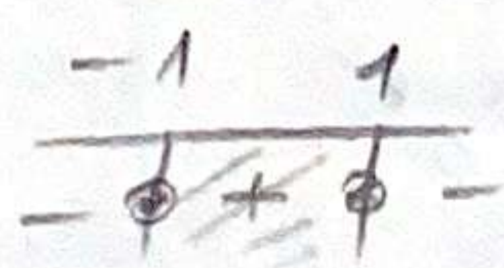
$X = \{1, 2, 3, 4\}$ $\quad f(x) = \{(1,1), (2,8), (3,27), (4,64)\}$

- The set of D of all possible input values is called the **domain** of the function.
- The theoretical set that contains all the outputs of the function is called co-domain.
- The set of values of $f(x)$ as x varies D is called the **range** of the function.

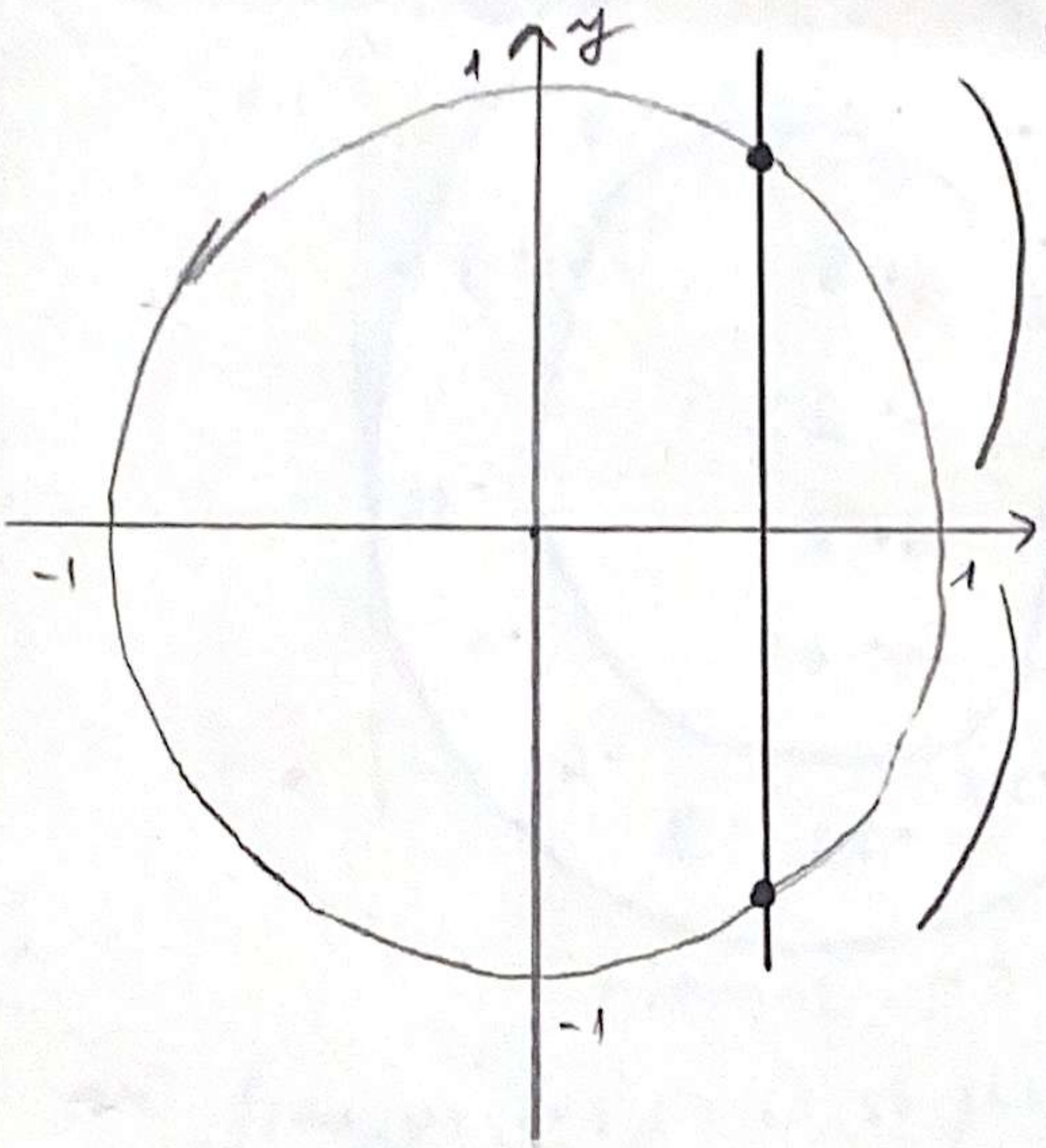
example:

function	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4-x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1-x^2}$	$[-1, 1]$	$[0, 1]$

$\sqrt{(1-x)\cdot(1+x)}$



The vertical test



1) ($x^2+y^2=1$) is not a graph of a function

2) ($\sqrt{1-x^2}$) upper semicircle is a graph

3) ($-\sqrt{1-x^2}$) lower semicircle is a graph

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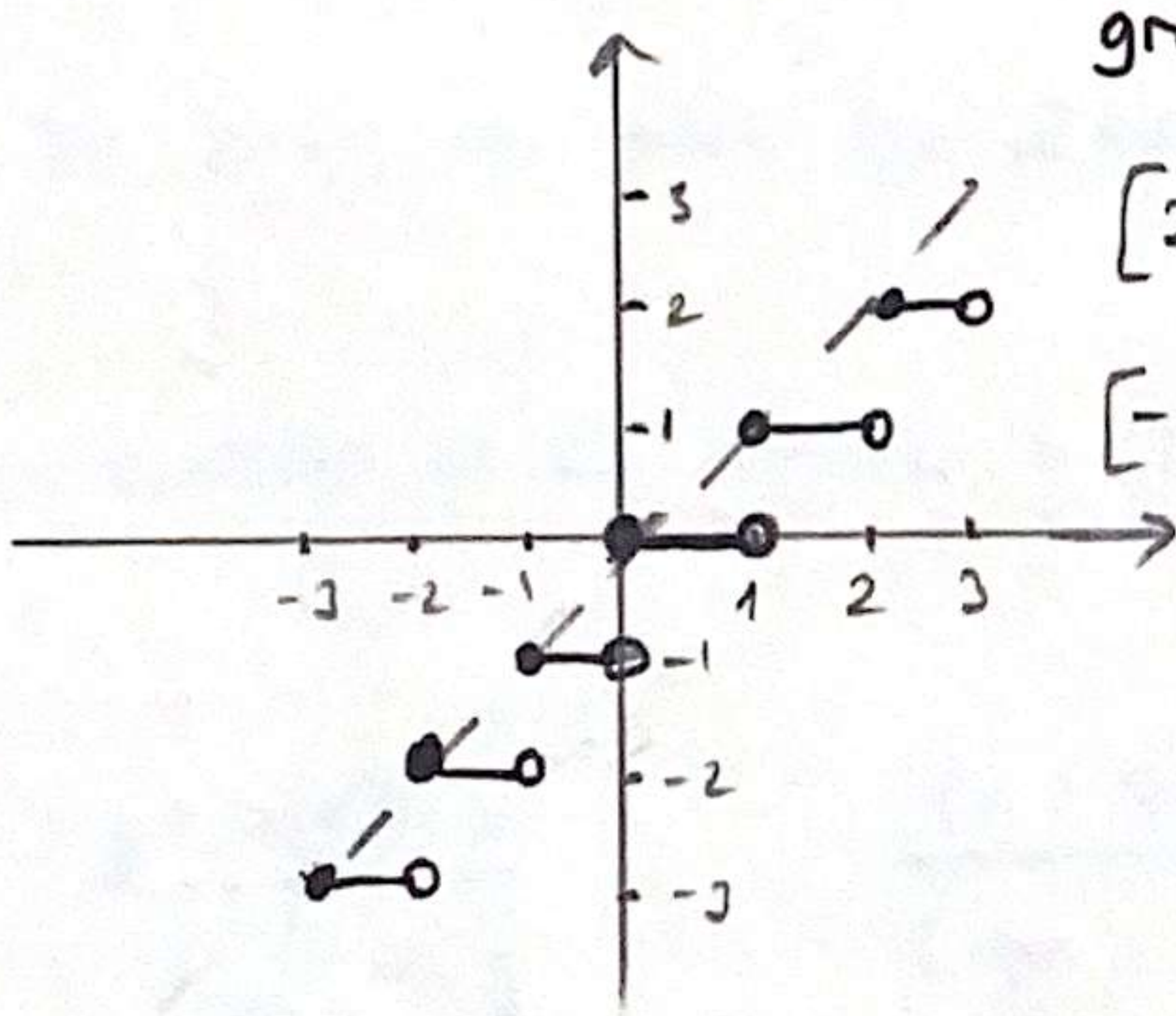
greatest integer function

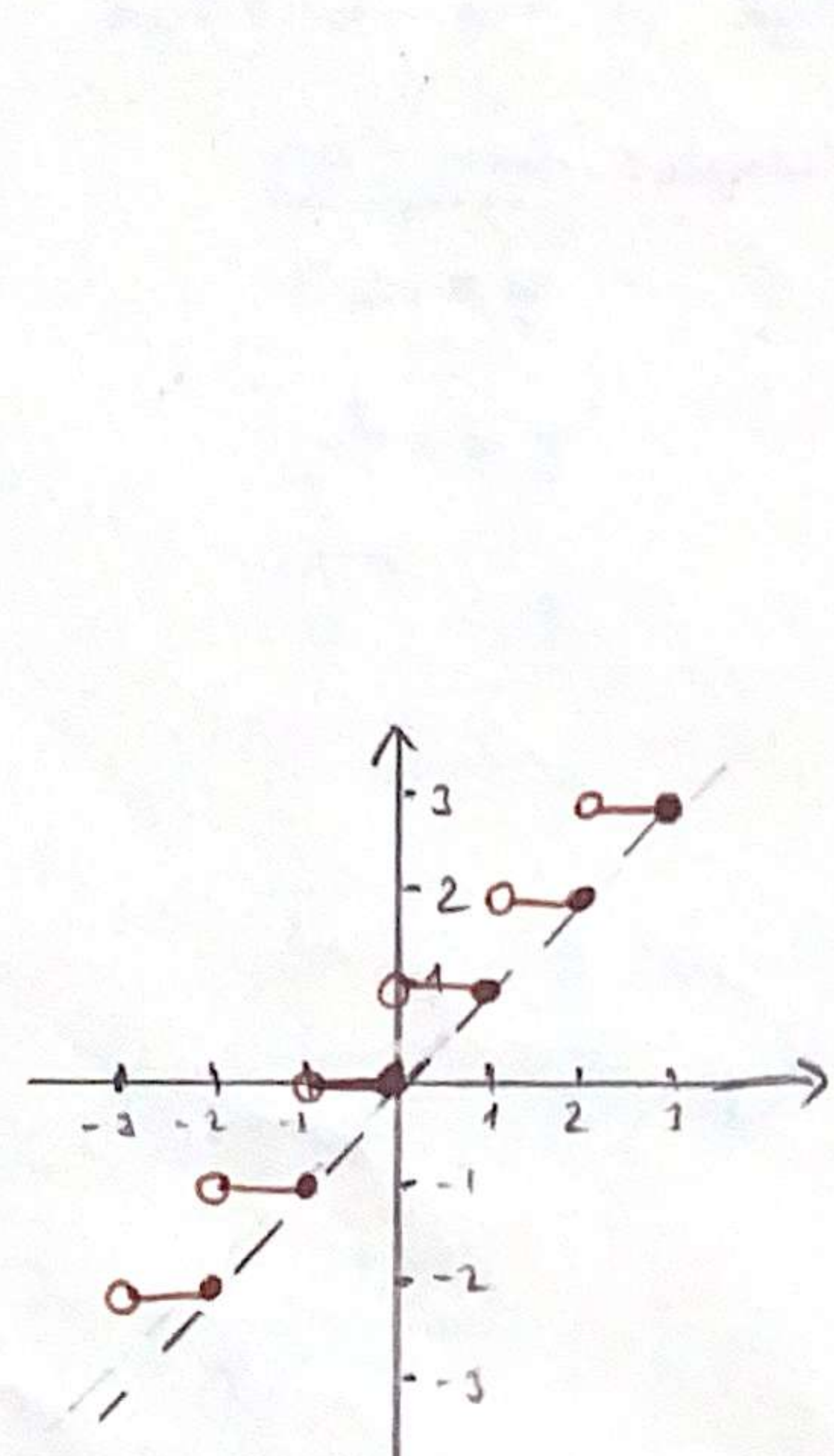
[2.4] = 2   [-1,2] = -2

[-0,3] = -1   [2] = 2



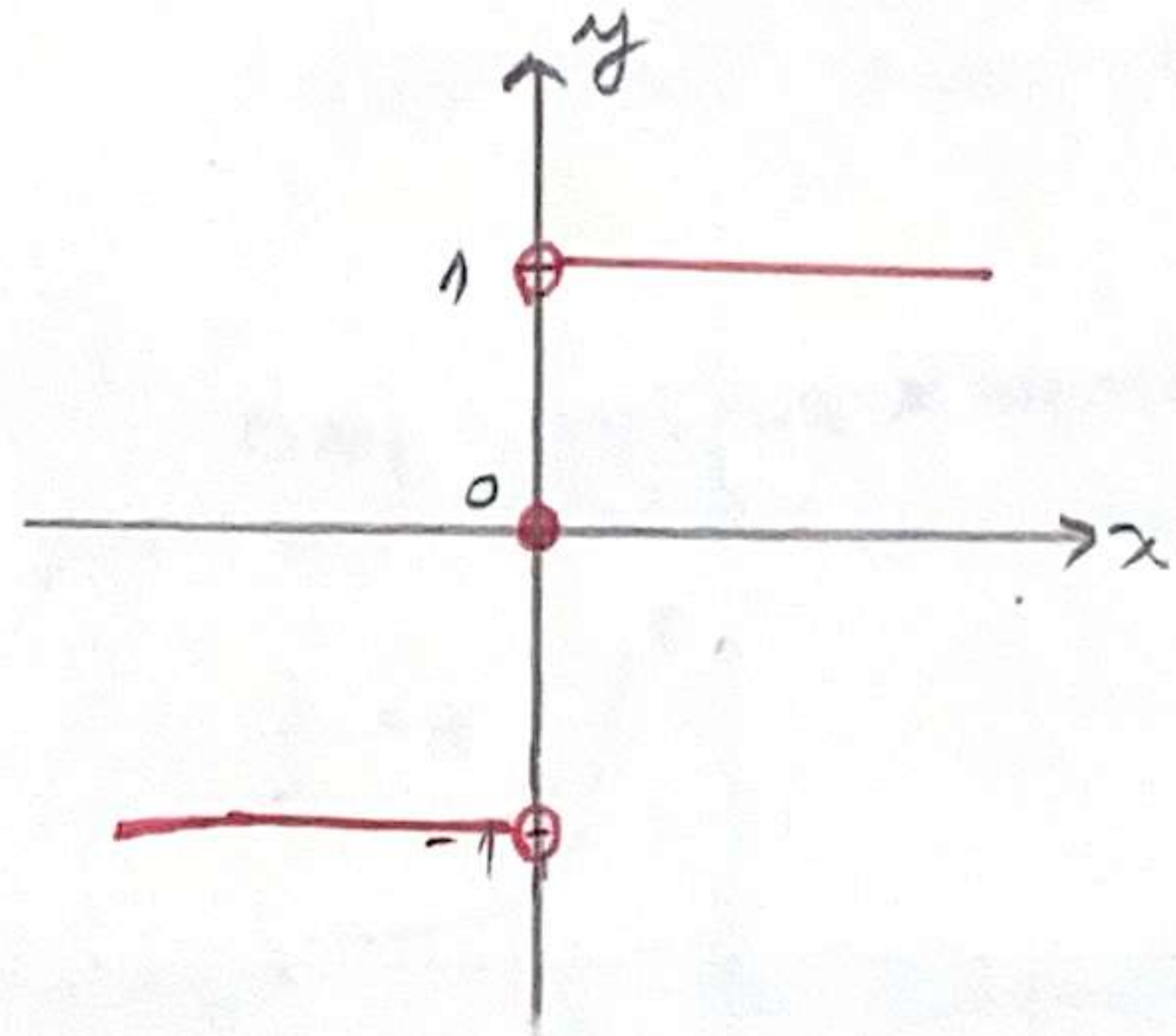
least integer function

[3.2] = 4   [-1.7] = -1



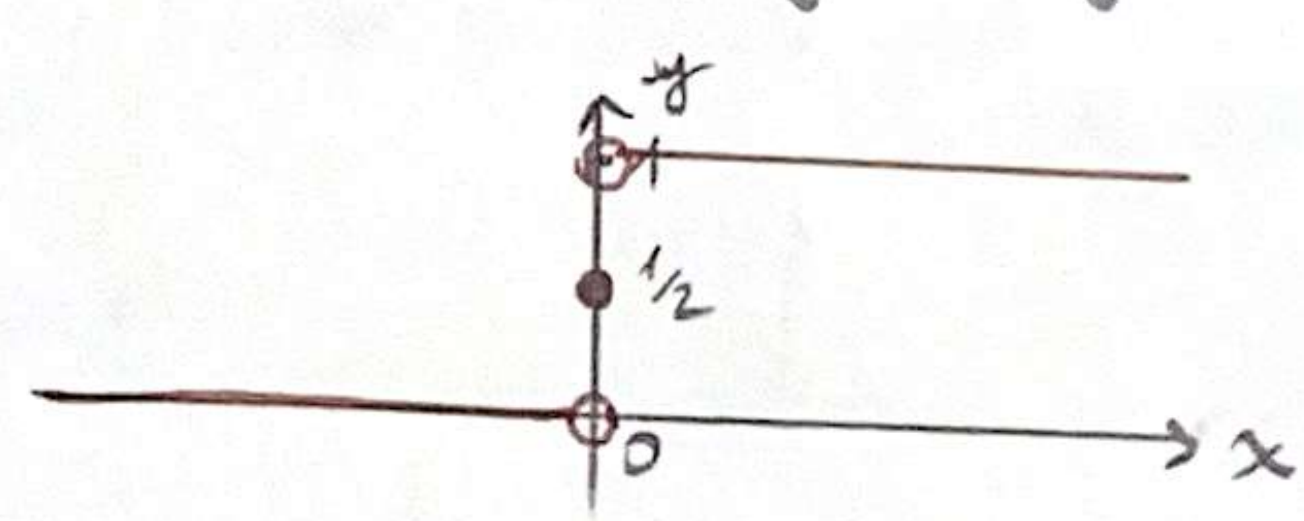
Signum (sign) function: The signum function of a real number x is a piecewise (parçalı) function which is defined as follows:

$$\operatorname{sgn} x := \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 1. \end{cases}$$



Heaviside step function: When defined as a piecewise constant (sabit) function, the heaviside step function is given by:

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2} & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$



Increasing & Decreasing functions

Let f be a function defined on an interval I and let x_1 and x_2 any two points in I.

1. If $f(x_2) > f(x_1)$ whenever $x_1 < x_2$, then f is said to be increasing on I.

2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be decreasing on I.

Even functions and odd functions: symmetry

A function $y = f(x)$ is an

even function of x if $f(-x) = f(x)$,

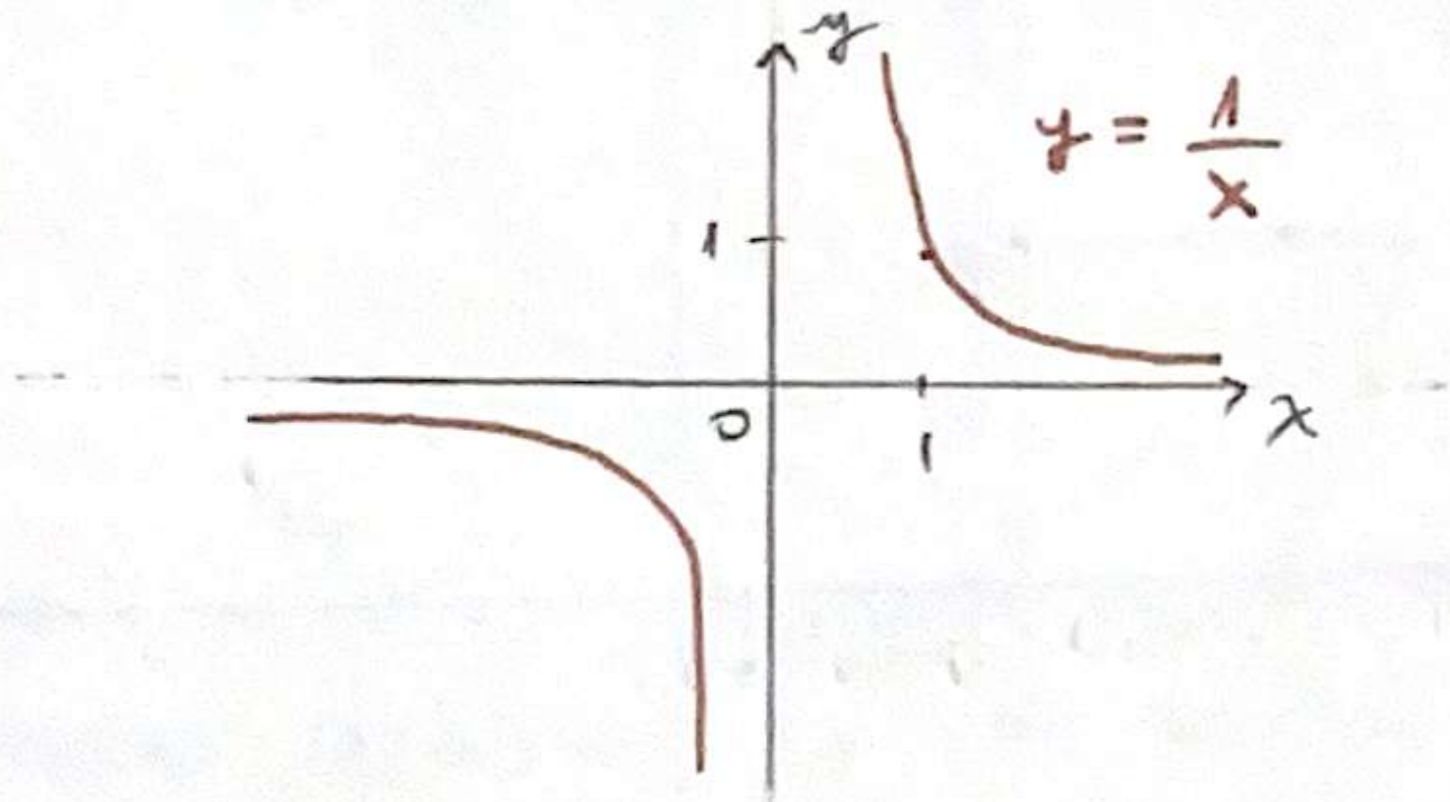
odd function of x if $f(-x) = -f(x)$,

for every x in the function's domain.

proportional: orantılı inversely proportional: ters orantılı

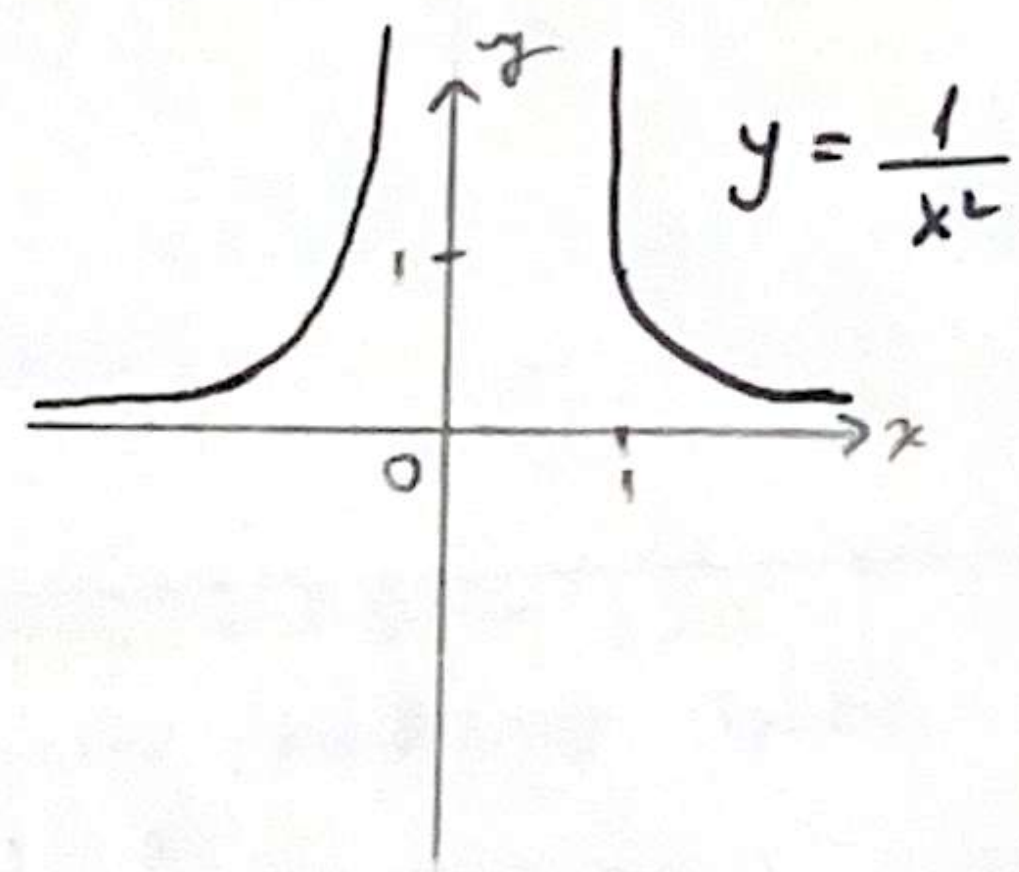
Power functions: A function $f(x) = x^a$, where "a" is a constant, is called power function.

Domain: $x \neq 0$ | Range: $y \neq 0$



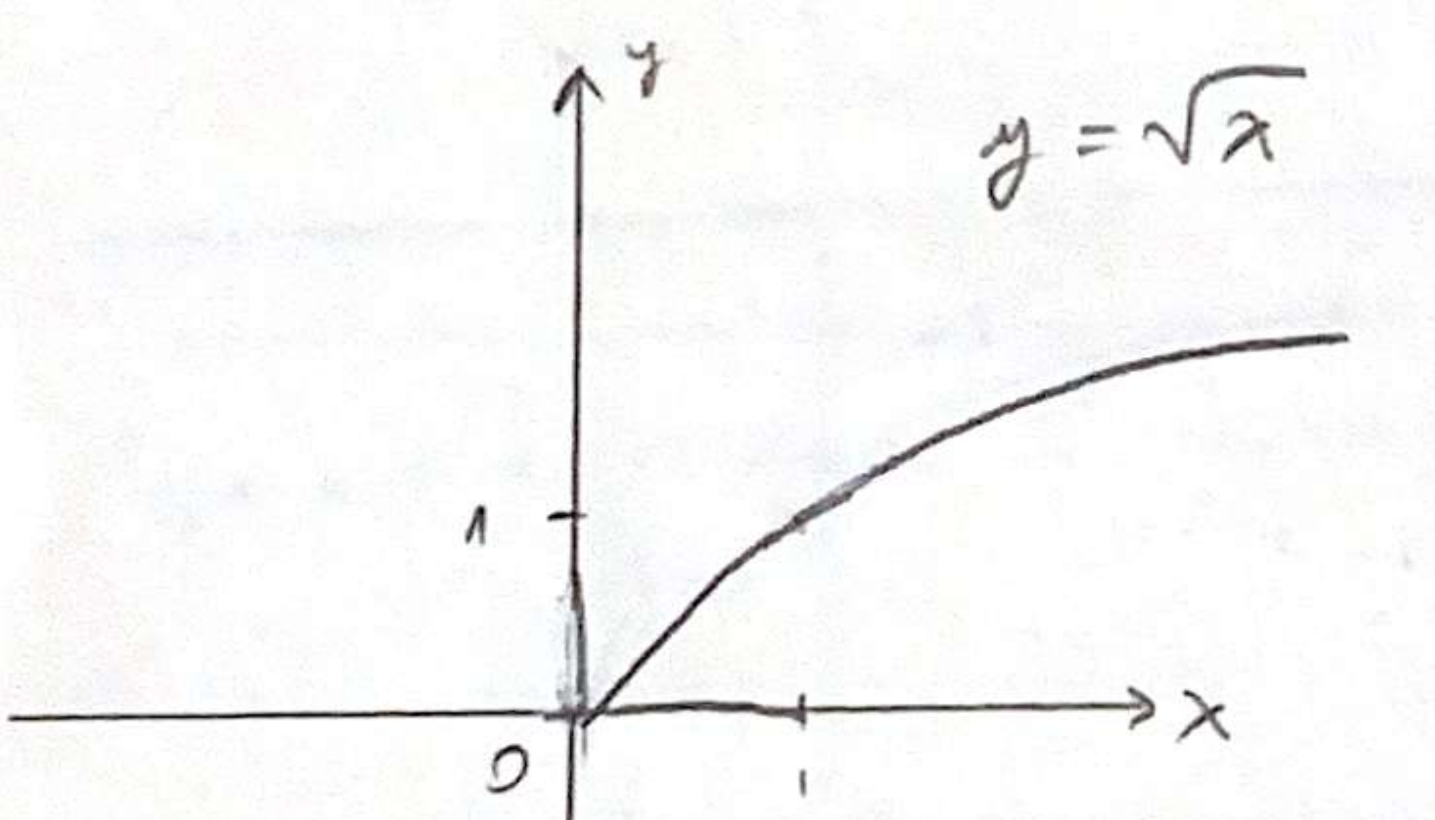
(a)

Domain: $x \neq 0$ | Range: $y > 0$



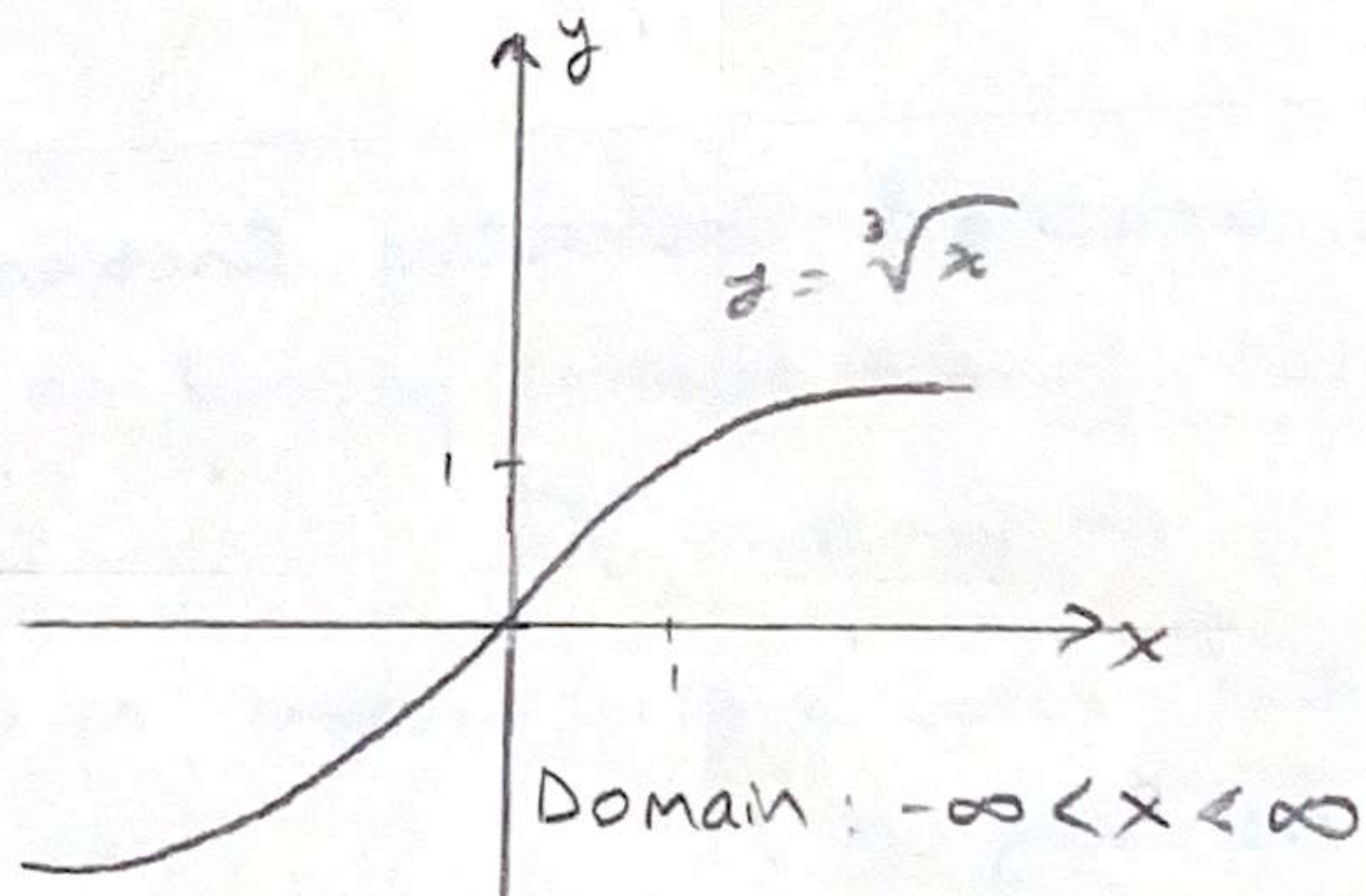
(b)

(a) $a = -1$ (b) $a = -2$



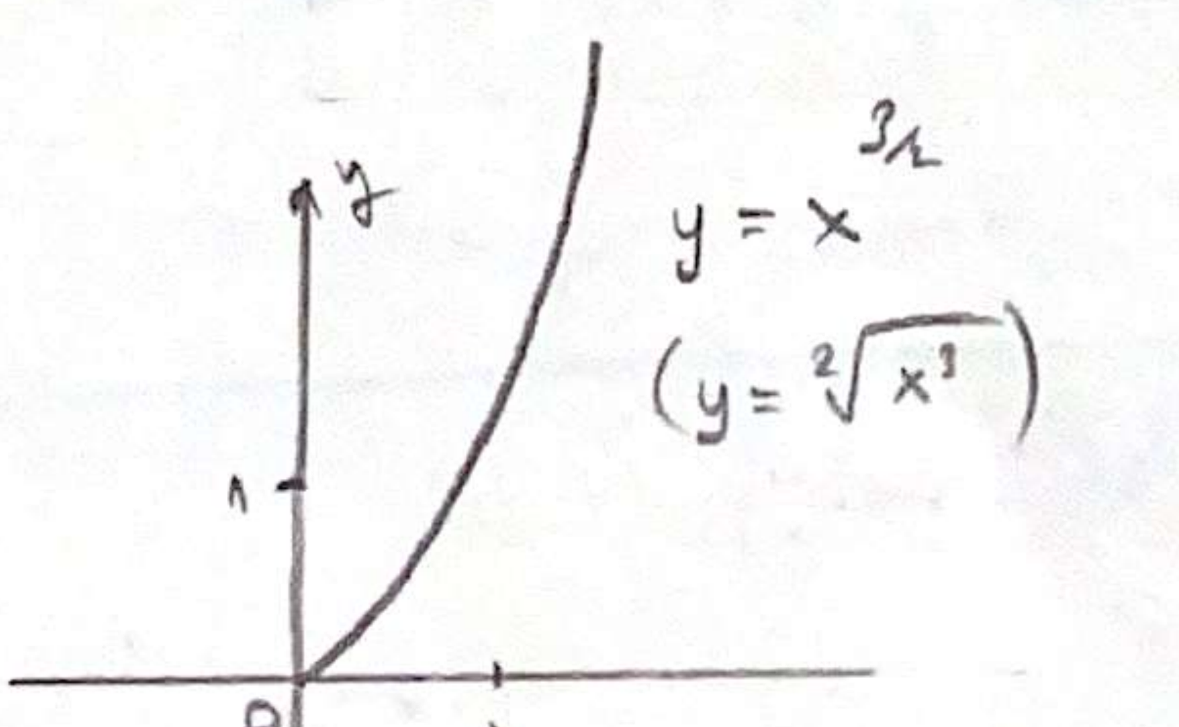
Domain: $0 \leq x < \infty$

Range: $0 \leq y < \infty$



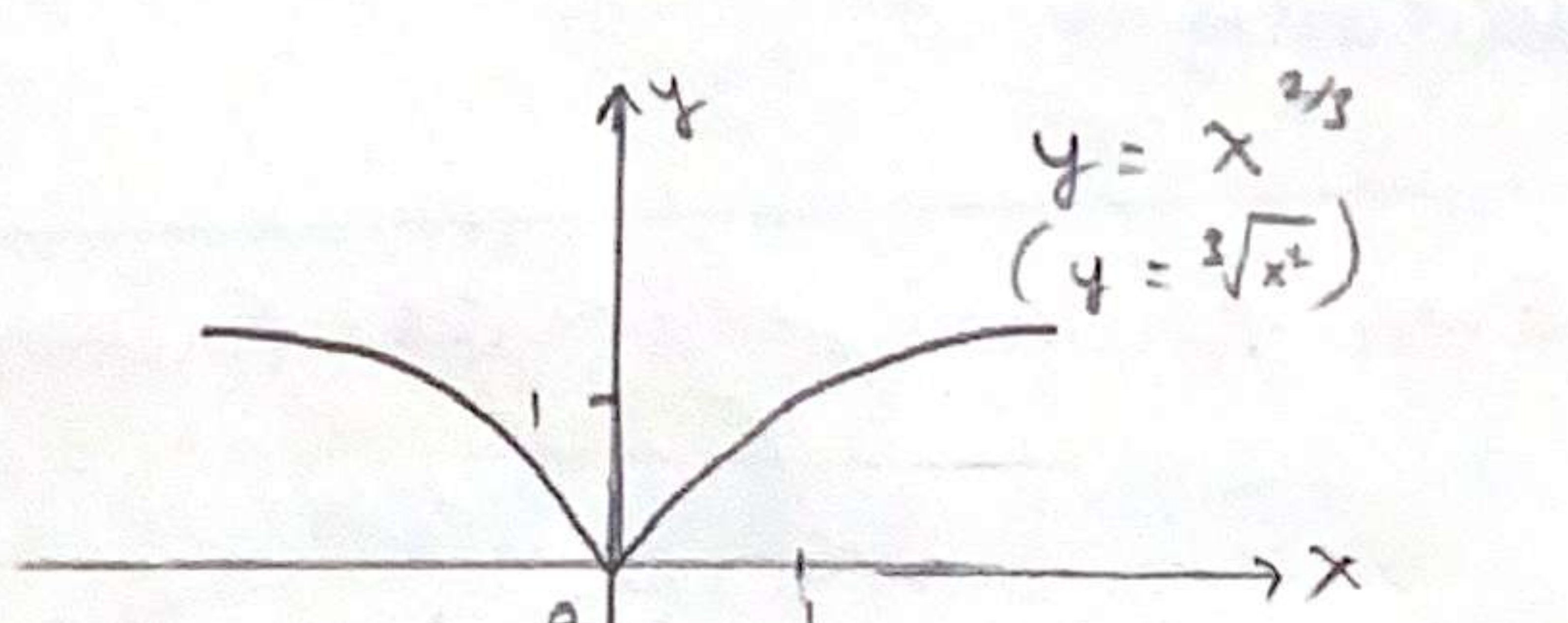
Domain: $-\infty < x < \infty$

Range: $-\infty < y < \infty$



Domain: $0 \leq x < \infty$

Range: $0 \leq y < \infty$



Domain: $-\infty < x < \infty$

Range: $0 \leq y < \infty$

Polynomials: $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$

n : nonnegative integer

$a_0, a_1, a_2 \ldots, a_n$: real constants

domain : $(-\infty, \infty)$

$a_n \neq 0$, $n > 0 \Rightarrow n$: degree

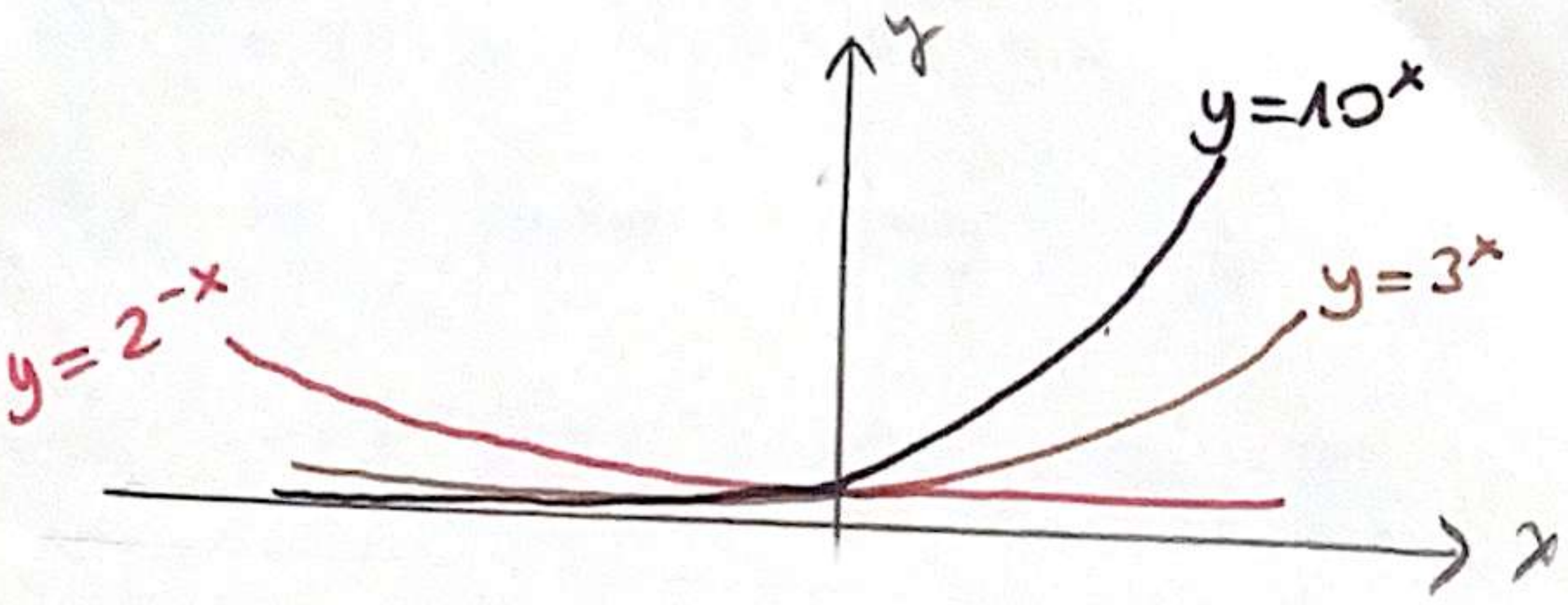
Rational: $f(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials

and $q(x) \neq 0$

Algebric: Constructed from polynomials using algebric operations like "addition, subtraction, multiplication, division and taking roots"

Exponential: $f(x) = a^x$ $a > 0$ $a \neq 1$ domain $(-\infty, \infty)$

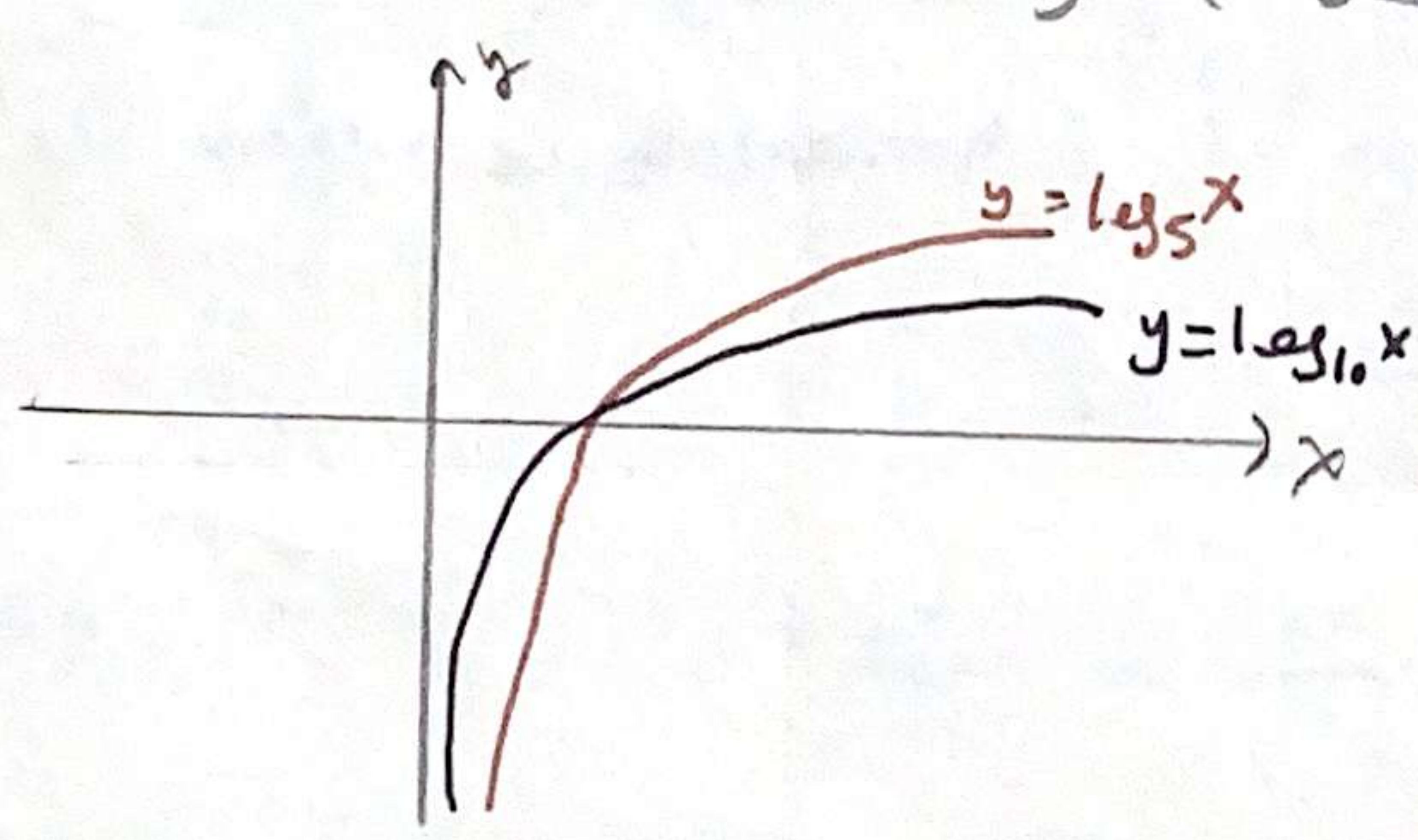
range $(0, \infty)$



inverse func.

Logarithmic: $f(x) = \log_a x$ $a \neq 1$ a positive constant

domain $(0, \infty)$ range $(-\infty, \infty)$



$f(x) = \sqrt{x}$ $g(x) = x+1$

a) $(f \circ g)(x) = \sqrt{x+1}$ Domain $[-1, \infty)$

c) $(f \circ f)(x) = \sqrt[4]{x}$ Domain $[0, \infty)$

b) $(g \circ f)(x) = \sqrt{x} + 1$ Domain $[0, \infty)$

d) $(g \circ g)(x) = x + 2$ Domain $(-\infty, \infty)$

$f(x) = x^2 \quad g(x) = \sqrt{x} \quad (f \circ g)(x) = (\sqrt{x})^2 = x$ Domain $[0, \infty)$

~~$(-\infty, \infty)$~~

$\sqrt{x} \geq 0$

Shift formulas

vertical

$y = f(x) + k$ $k > 0$ shift up $k < 0$ shift down

horizontal

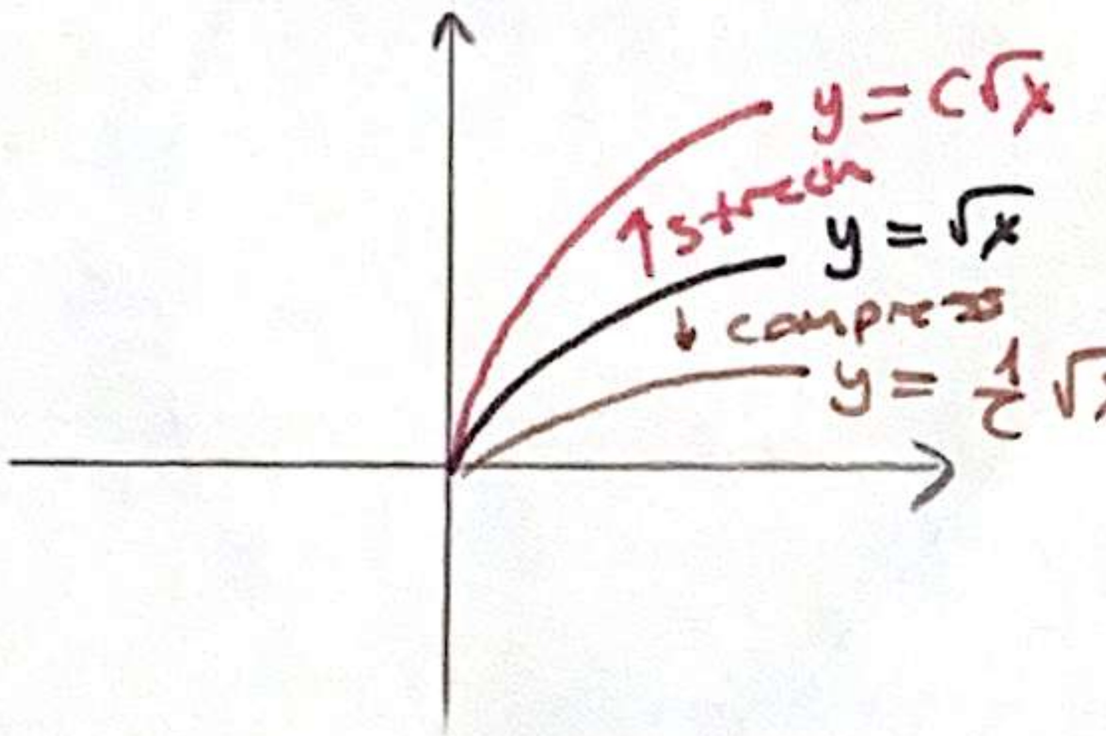
$y = f(x + h)$ $h > 0$ shift left $h < 0$ shift right

Scaling and reflecting

for $c > 1$

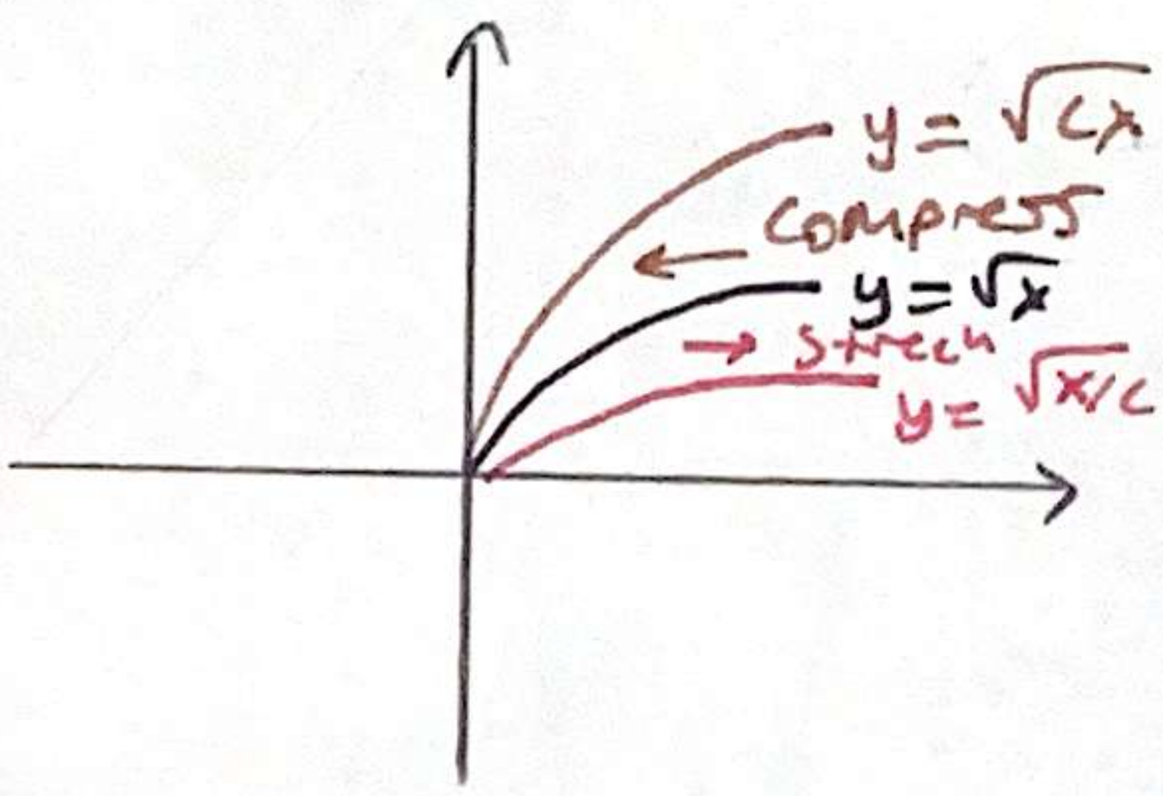
$y = cf(x)$ Streches the graph of f vertically by a factor of c

$y = \frac{1}{c} f(x)$ Compresses the graph of f vertically by a factor of c



$y = f(cx)$ Compresses the graph of f horizontally by a factor of c

$y = f(x/c)$ Streches the graph of f horizontally by a factor of c



for $c = -1$

$y = c f(x)$ Reflects the graph of f across the x-axis

$y = f(cx)$ Reflects the graph of f across the y-axis.

Periodic functions: If there is a positive number p such that $f(x+p) = f(x)$ for every value of x. The smallest such value of p is the period of f

Period π: $\tan(x+\pi) = \tan x$
$\cot(x+\pi) = \cot x$

Period 2π: $\sin(x+2\pi) = \sin x$
$\cos(x+2\pi) = \cos x$
cosec, sec

- Period of $\sin(ax+b)$ and $\cos(ax+b)$: $\frac{2\pi}{|a|}$
  cosec, sec

- Period of $\tan(ax+b)$ and $\cot(ax+b)$: $\frac{\pi}{|a|}$

- Period of $a\sin x + b$ and $a\cos x + b$: 2π
  cosec, sec

- Period of $a\tan x + b$ and $a\cot x + b$: π

$f(x) = a.\sin^n(cx+d) + b$ $\quad$ $g(x) = a.\cos^n(cx+d) + b$

If n is odd $\quad T_f = T_g = \frac{2\pi}{|c|}$

If n is even $\quad T_f = T_g = \frac{\pi}{|c|}$

Period of the Sum/Difference of periodic functions

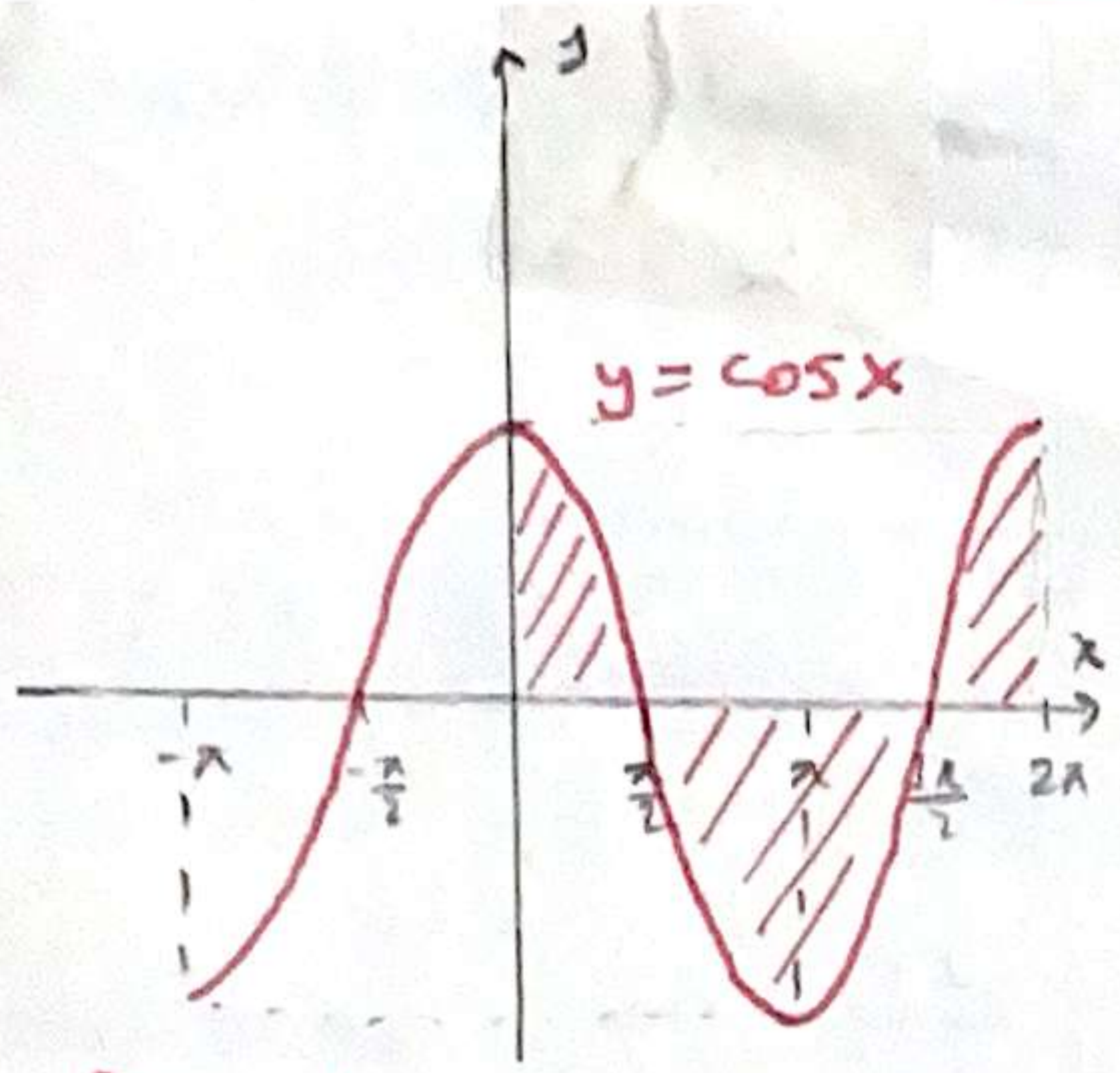
$f(x)$, $g(x)$ periodic $\quad$ LCM = least common multiple

$T_f = \frac{a}{b}$ $\quad T_g = \frac{c}{d}$ $\quad$ GCD = greatest common divider

$$T_{f \pm g} = LCM\left(\frac{a}{b}, \frac{c}{d}\right) = \frac{LCM(a,c)}{GCD(b,d)}$$

e.g. $T_f = \frac{3\pi}{4}$, $T_g = \frac{2\pi}{5}$ $\Rightarrow \frac{6\pi}{1} = 6\pi$

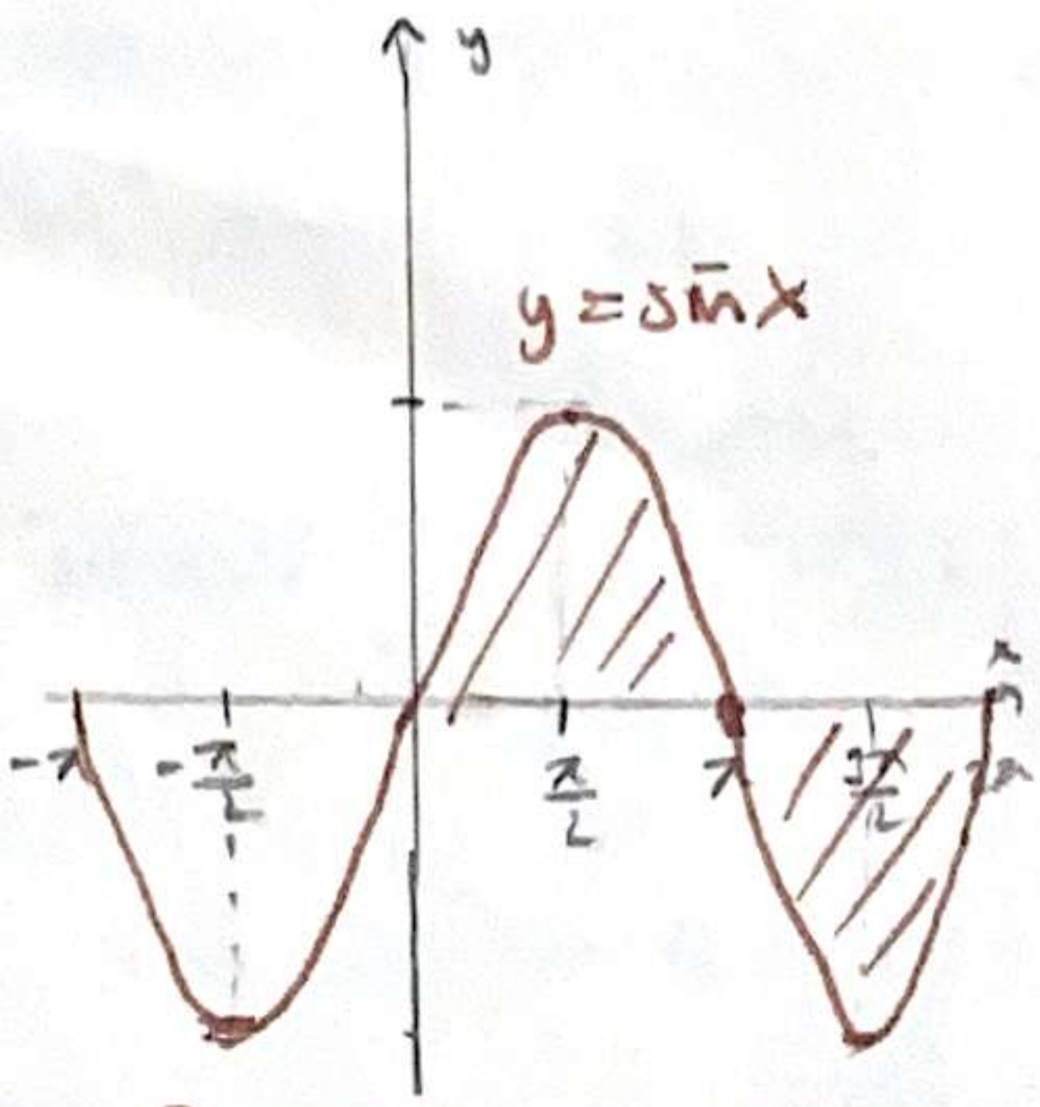
e.g.(1) $f(x) = \underbrace{\tan 3x}_{T_1 = \frac{\pi}{3}} + \underbrace{\cos 5x}_{T_2 = \frac{2\pi}{5}}$ $\quad \left(\frac{\pi}{3}, \frac{2\pi}{5}\right)$ $\quad \frac{2\pi}{1} = 2\pi$



Domain : $-\infty < x < \infty$

Range : $-1 \leq y \leq 1$

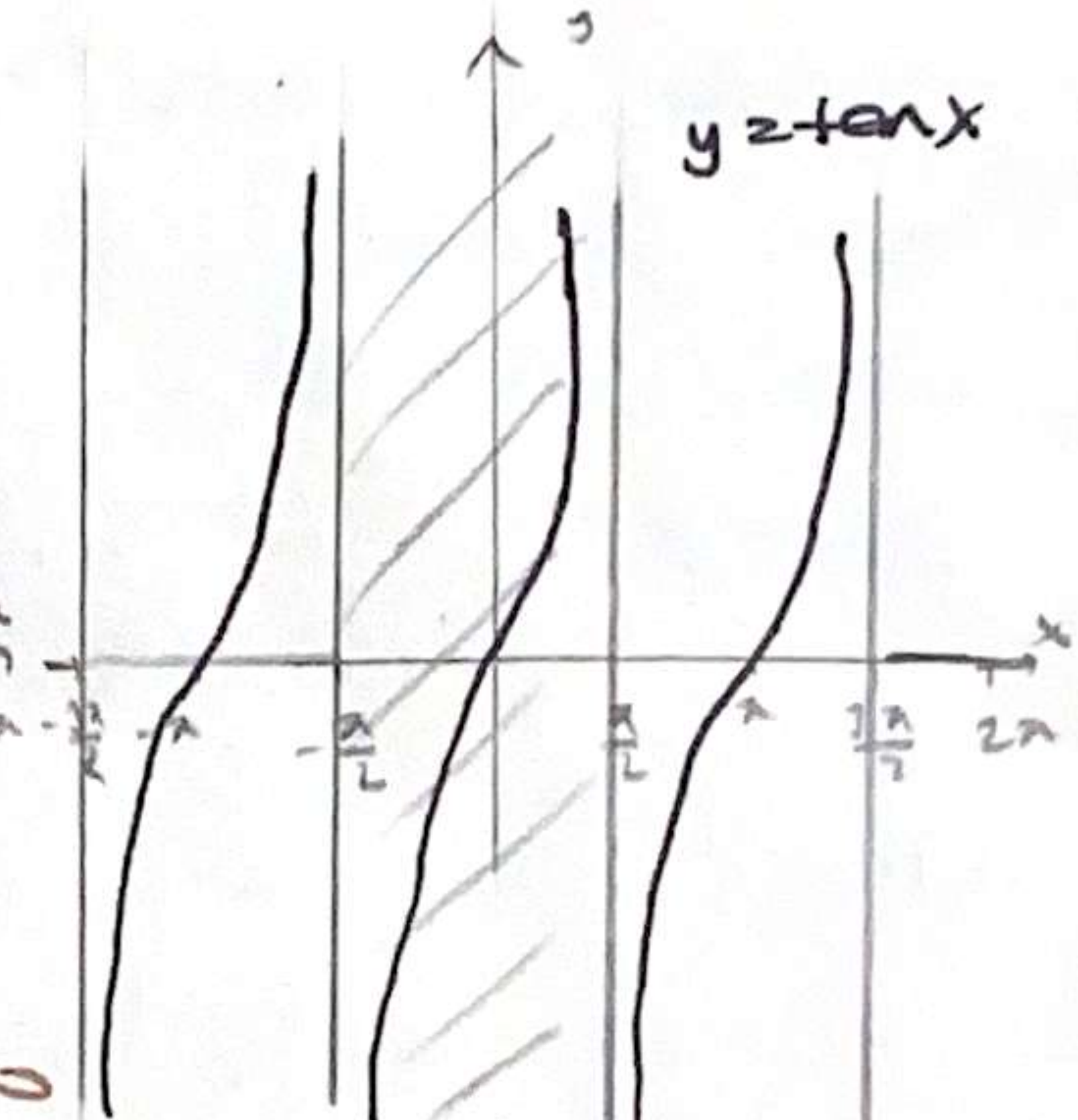
Period : 2π



Domain : $-\infty < x < \infty$

Range : $-1 \leq y \leq 1$

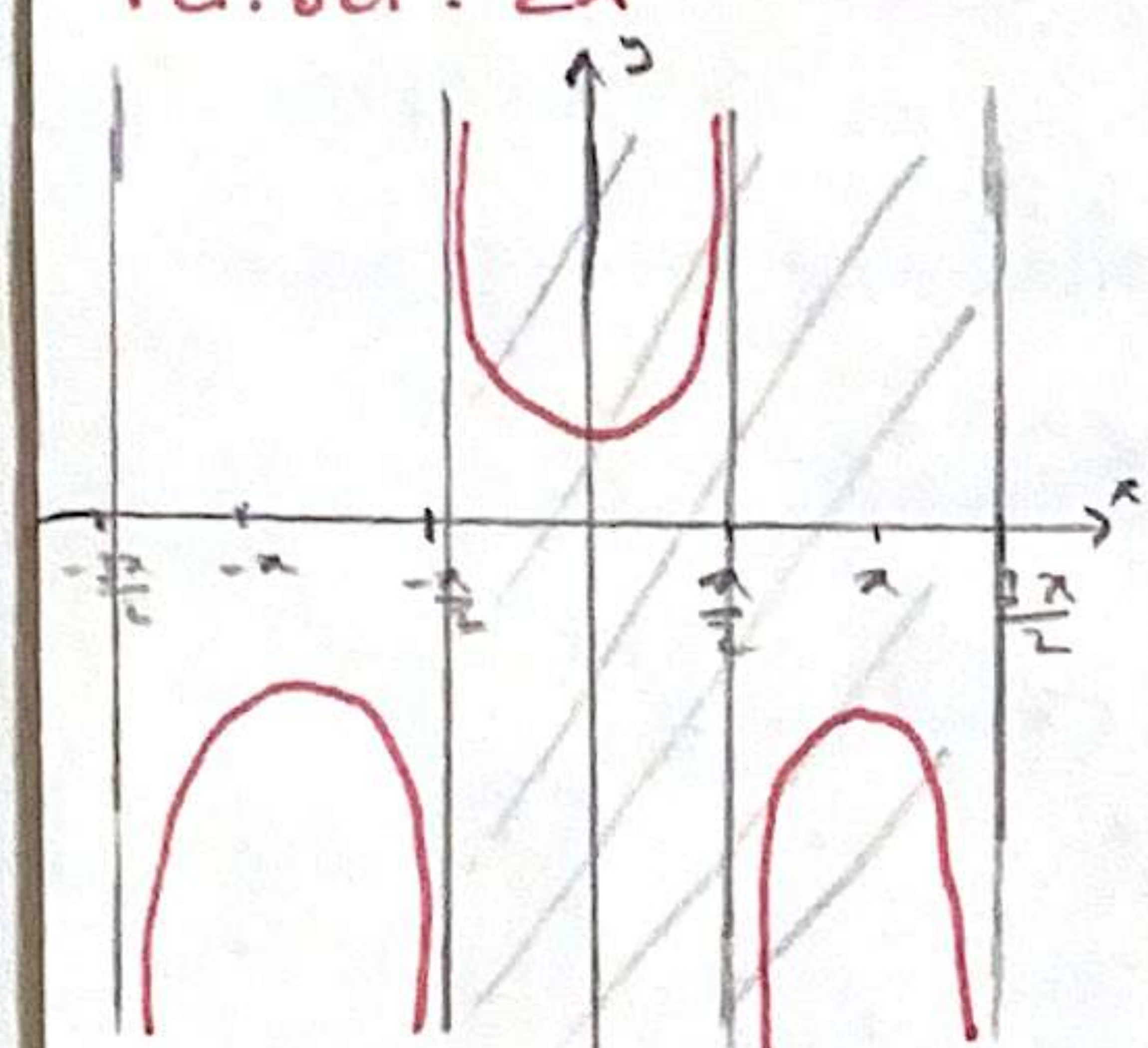
Period : 2π



Domain $\neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \dots$

Range : $-\infty < y < \infty$

Period : π

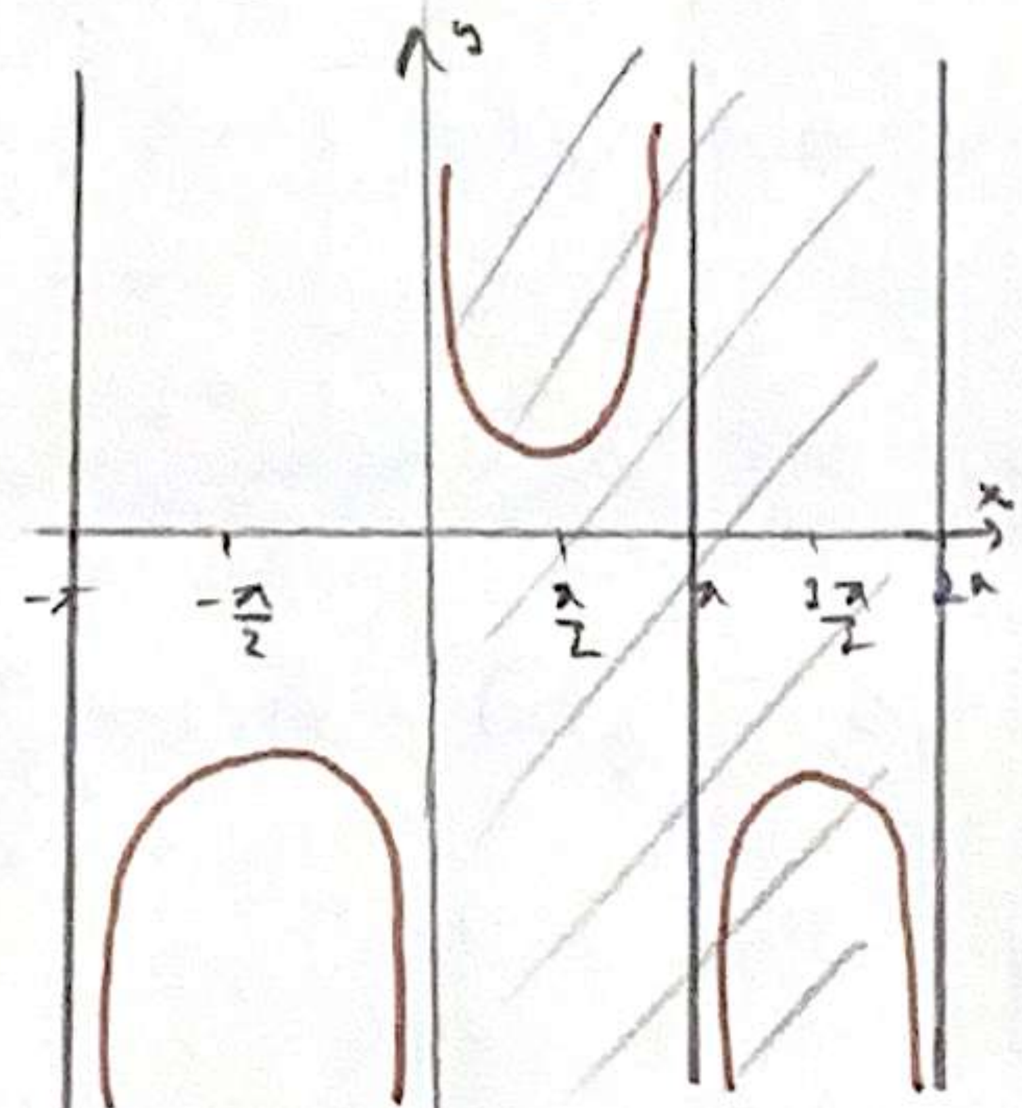


$y = \sec x$

Domain : $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \dots$

Range : $y \leq -1$ or $y \geq 1$

Period : 2π

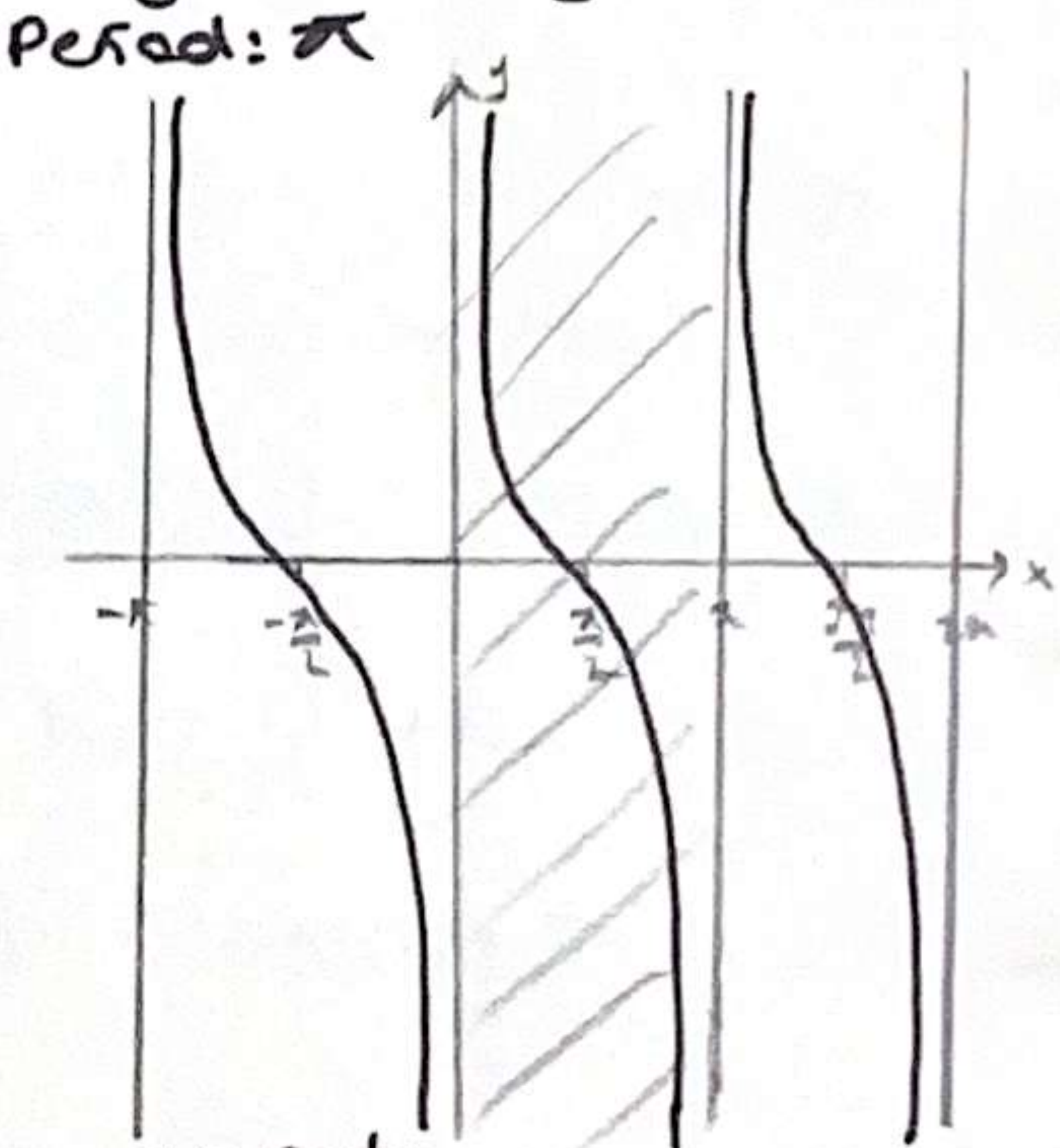


$y = \csc x$

Domain: $x \neq 0, \pm\pi, \pm 2\pi, \dots$

Range: $y \leq -1$ or $y \geq 1$

Period: 2π



$y = \cot x$

Domain : $x \neq 0, \pm\pi, \pm 2\pi$

Range: $-\infty < y < \infty$

Period : π

Trigonometry

$\boxed{\cos^2\theta + \sin^2\theta = 1}$ $\quad$ $\boxed{1 + \tan^2\theta = \sec^2\theta}$ $\quad$ $\boxed{1 + \cot^2\theta = \csc^2\theta}$

$\tan^2\theta = \sec^2\theta - 1$

$$\frac{1}{\cos^2\theta} - \frac{\cos^2\theta}{\cos^2\theta} = \frac{\sin^2\theta}{\cos^2\theta}$$

$\cos(\alpha + \theta) = \cos\alpha \cdot \cos\theta - \sin\alpha \cdot \sin\theta$

$\cos(\alpha - \theta) = \cos\alpha \cdot \cos\theta + \sin\alpha \cdot \sin\theta$

$\sin(\alpha + \theta) = \sin\alpha \cdot \cos\theta + \cos\alpha \cdot \sin\theta$

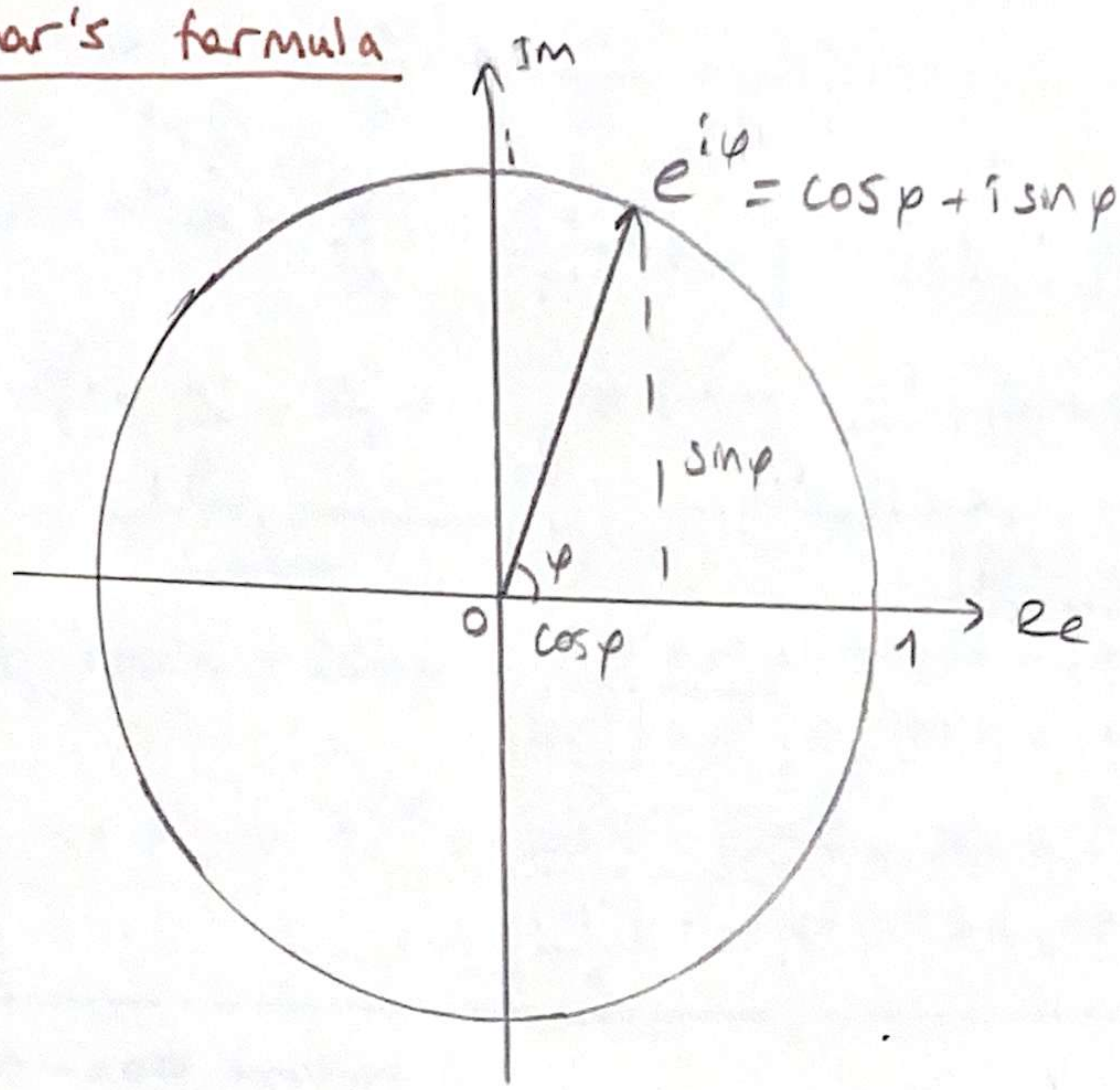
$\sin(\alpha - \theta) = \sin\alpha \cdot \cos\theta - \cos\alpha \cdot \sin\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$ $\qquad$ $\cos^2\theta = \frac{1 + \cos 2\theta}{2}$ $\quad$ $(\cos 2\theta = 2\cos^2\theta - 1$

$\sin 2\theta = 2\sin\theta \cdot \cos\theta$ $\qquad$ $\sin^2\theta = \frac{1 - \cos 2\theta}{2}$ $\quad$ $(\cos 2\theta = 1 - 2\sin^2\theta$

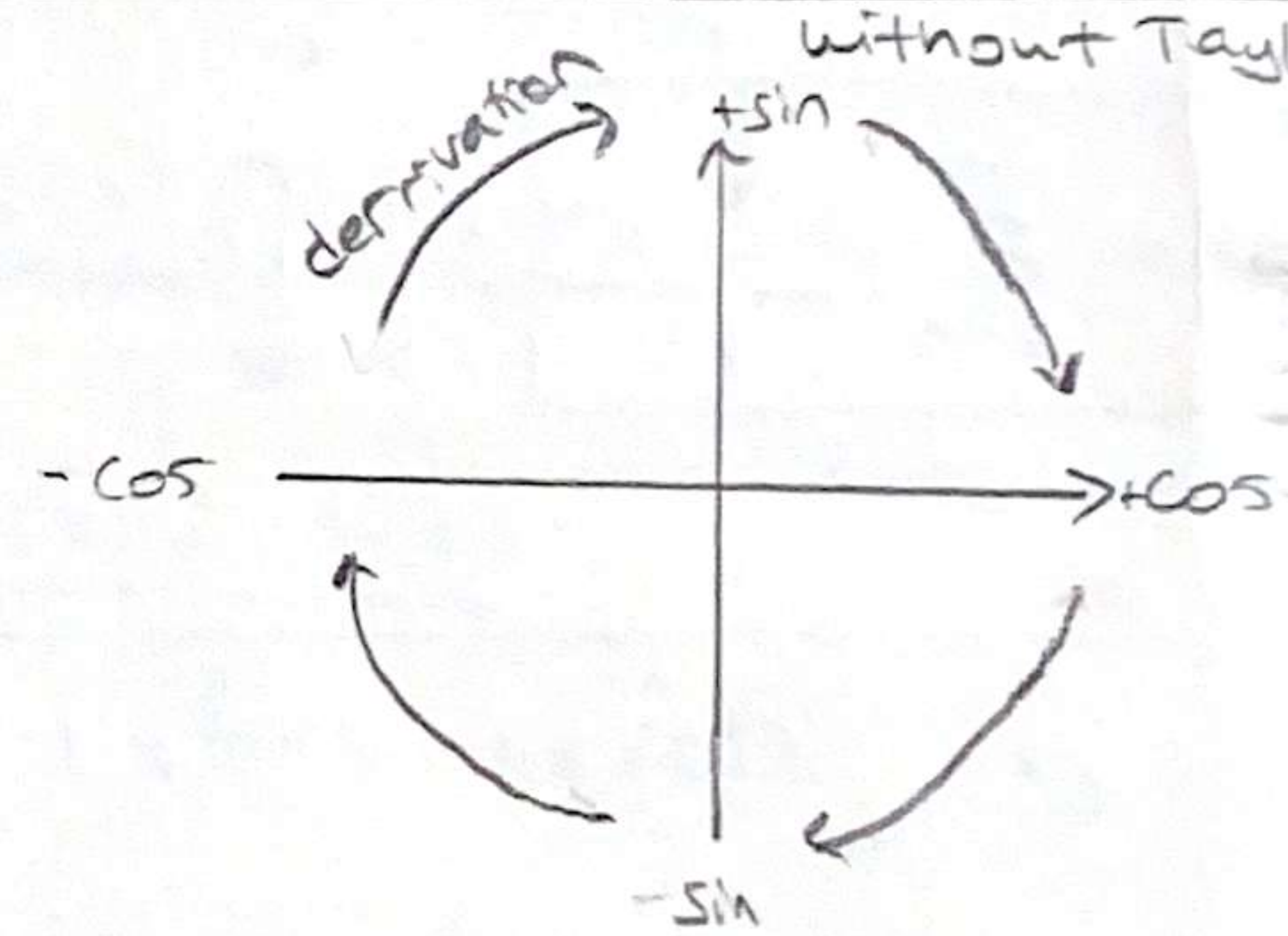
Eular's formula

$i = \sqrt{-1}$



without Taylor

Proof: $e^{i\theta} = \cos\theta + i\sin\theta$



$z = \cos\theta + i\sin\theta$

$$\frac{dz}{d\theta} = -\sin\theta + i\cos\theta$$

$$= i\cos\theta + i^2\sin\theta$$

$$= i\underbrace{(\cos\theta + i\sin\theta)}_{z}$$

$$\frac{dz}{d\theta} = iz \Rightarrow \frac{dz}{z} = i\,d\theta \quad \int\frac{dz}{z} = \int i\,d\theta \Rightarrow \ln z = i\theta + c$$

$$e^{\ln z} = e^{i\theta + c}$$

$$z = e^{i\theta}\cdot c \qquad z = \overbrace{\cos 0}^{1} + \cancel{i\sin 0} \qquad 1 = \overset{1}{\cancel{e^{i0}}}\cdot c \qquad c = 1$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

Taylor series

$e^{ix} = \cos x + i\sin x$

$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n$

$$\left.\begin{array}{l} f(0) = P(0) \\ f'(0) = P'(0) \\ f''(0) = P''(0) \\ \vdots \end{array}\right\}$$

$P(0) = \boxed{a_0 = f(0)}$

$P'(0) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots$

$P'(0) = \boxed{a_1 = f'(0)}$

$P''(0) = 2a_2 + 6a_3 x + 12 a_4 x^2 + \ldots$

$P''(0) = 2a_2 = f''(0) \Rightarrow \boxed{a_2 = \frac{f''(0)}{2}}$

$P'''(0) = 6a_3 + 24a_4 x + \ldots$

$P'''(0) = 6a_3 = f'''(0) \Rightarrow \boxed{a_3 = \frac{f'''(0)}{6}}$

$$P(x) = f(0) + \frac{f'(0)}{1!} \cdot x^1 + \frac{f''(0)}{2!} \cdot x^2 + \frac{f'''(0)}{3!} \cdot x^3 + \ldots$$

$$\boxed{P(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0) \cdot x^k}{k!}} \qquad \left[P(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a) \cdot (x-a)^k}{k!}\right]$$

↳ Maclaurin → Taylor

$f(x) = e^x \quad f(0) = 1 \quad f'(0) = 1 \quad f''(0) = 1 \quad \ldots$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$f(x) = \sin x \quad f(0) = 0 \quad f'(0) = 1 \quad f''(0) = 0 \quad f'''(0) = -1 \quad \ldots$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$f(x) = \cos x \quad f(0) = 1 \quad f'(0) = 0 \quad f''(0) = -1 \quad f'''(0) = 0 \quad \ldots$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \dots$$

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} + \dots$$

$$e^{ix} = \underbrace{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right)}_{\cos x} + i\underbrace{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right)}_{\sin x}$$

$$e^{ix} = \cos x + i\sin x$$

for $x = \pi \Rightarrow e^{i\pi} = \cos\pi + \cancel{i\sin\pi}^{0} \Rightarrow e^{i\pi} = -1 \Rightarrow \boxed{e^{i\pi} + 1 = 0}$

Even - odd function

Even: $g(x) = g(-x)$

$$g(x) = \frac{f(x) + f(-x)}{2} \qquad g(-x) = \frac{f(-x) + f(-(-x))}{2}$$

$$= \frac{f(x) + f(-x)}{2}$$

$$g(x) = g(-x)$$

$$\frac{f(x) + f(-x)}{2} = \frac{f(x) + f(-x)}{2}$$

Odd: $h(x) = -h(-x)$

$$h(x) = \frac{M(x) - M(-x)}{2} \qquad h(-x) = \frac{M(-x) - M(-(-x))}{2}$$

$$= \frac{M(-x) - M(x)}{2}$$

$$= \frac{-M(-x) + M(x)}{2} \Rightarrow \frac{M(x) - M(-x)}{2}$$

$$h(x) = -h(-x)$$

$$\frac{M(x) - M(-x)}{2} = -\left(\frac{M(-x) - M(x)}{2}\right)$$

a function can be considered as sum of an even and odd functions

$$g(x) : \frac{f(x)+f(-x)}{2} \qquad h(x) = \frac{f(x)-f(-x)}{2}$$

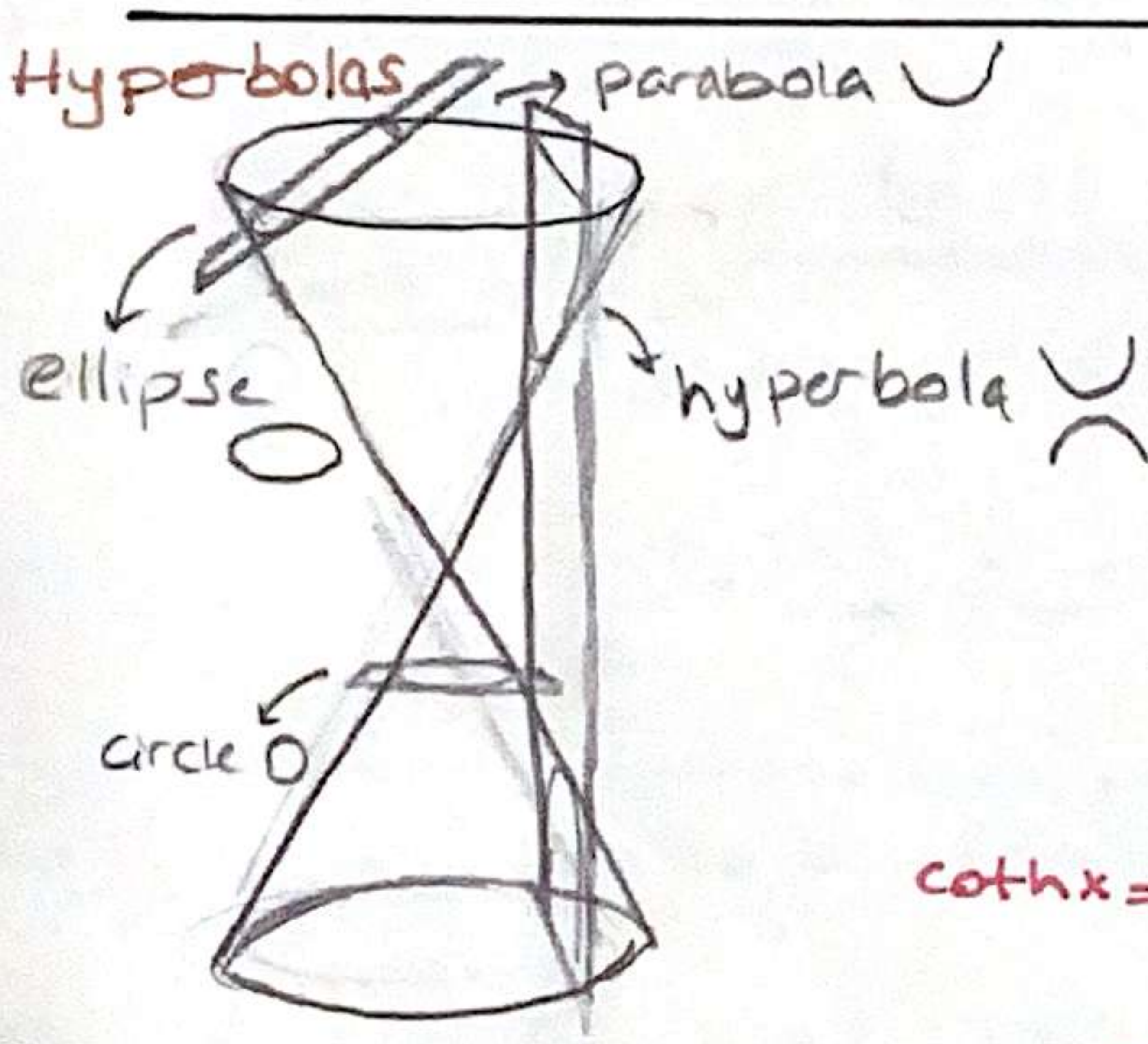
$$f(x) = g(x) + h(x) \Rightarrow \frac{f(x)+\cancel{f(-x)}}{2} + \frac{f(x)-\cancel{f(-x)}}{2} = f(x)$$

$f(x) = e^x$

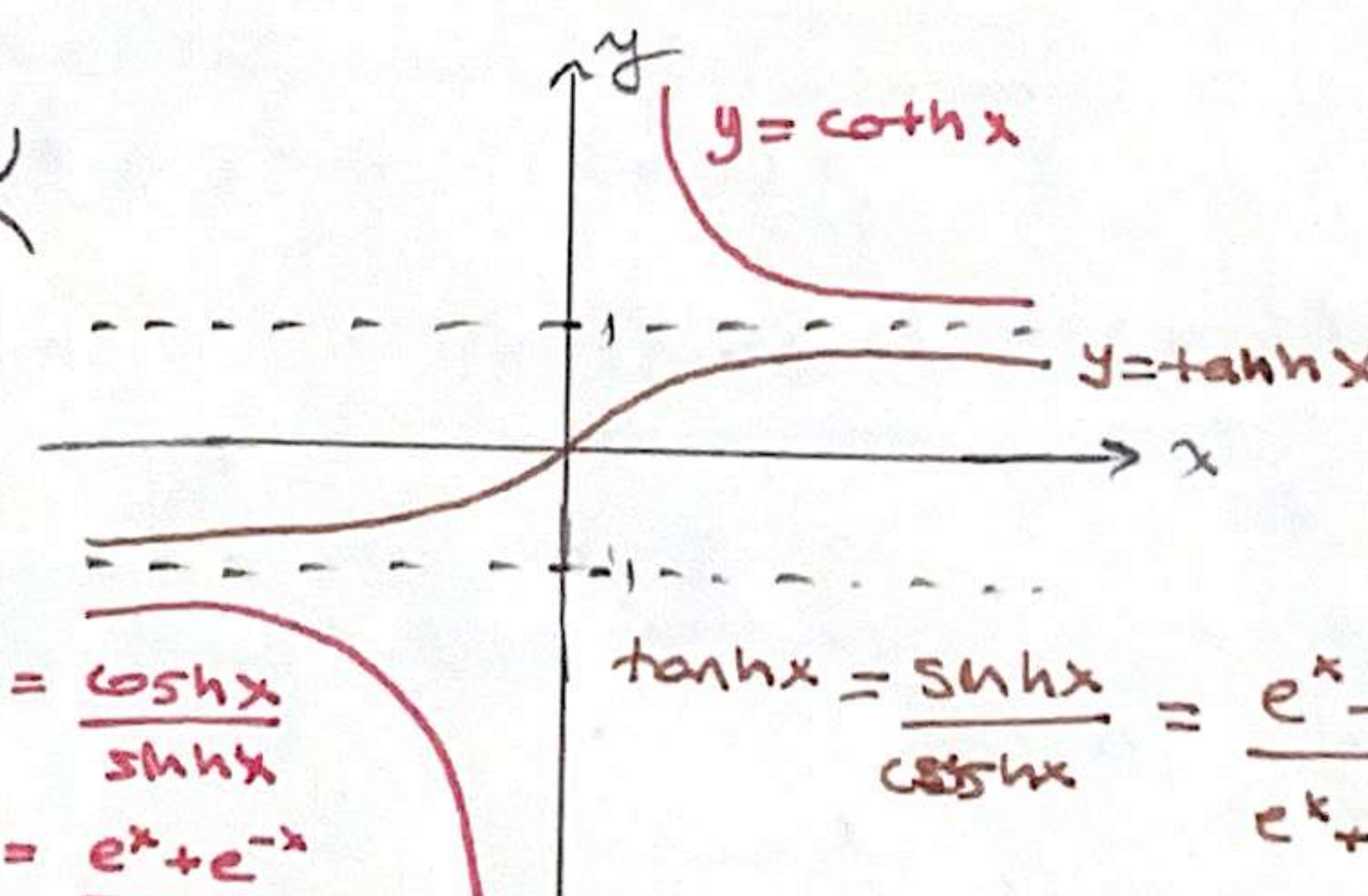
$$f_e(x) = \frac{f(x)+f(-x)}{2} = \frac{e^x+e^{-x}}{2} \qquad f_e(x) = \cosh(x)$$

$$f_o(x) = \frac{f(x)-f(-x)}{2} = \frac{e^x-e^{-x}}{2} \qquad f_o(x) = \sinh(x)$$

$e^x = \cosh(x) + \sinh(x)$

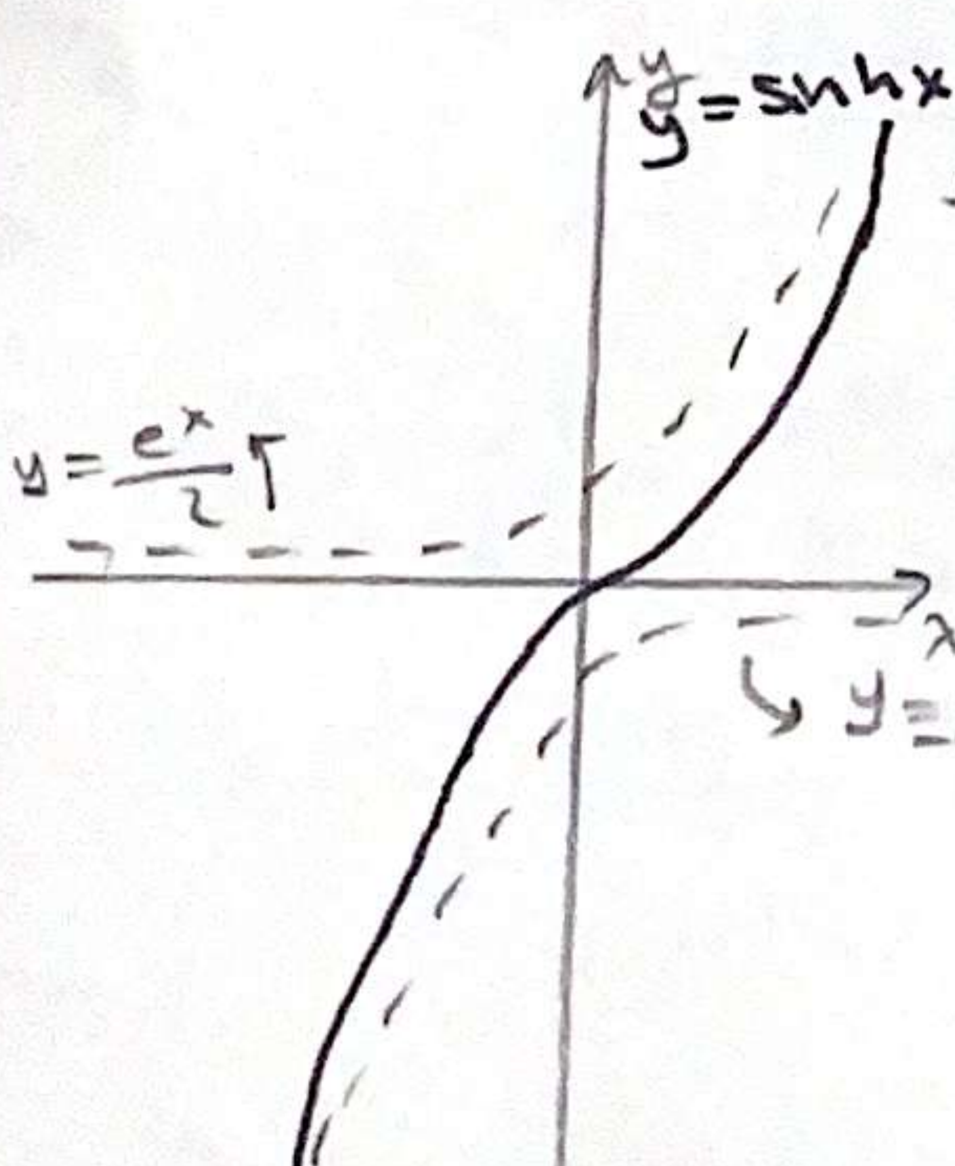


$\cosh^2 x - \sinh^2 x = 1$

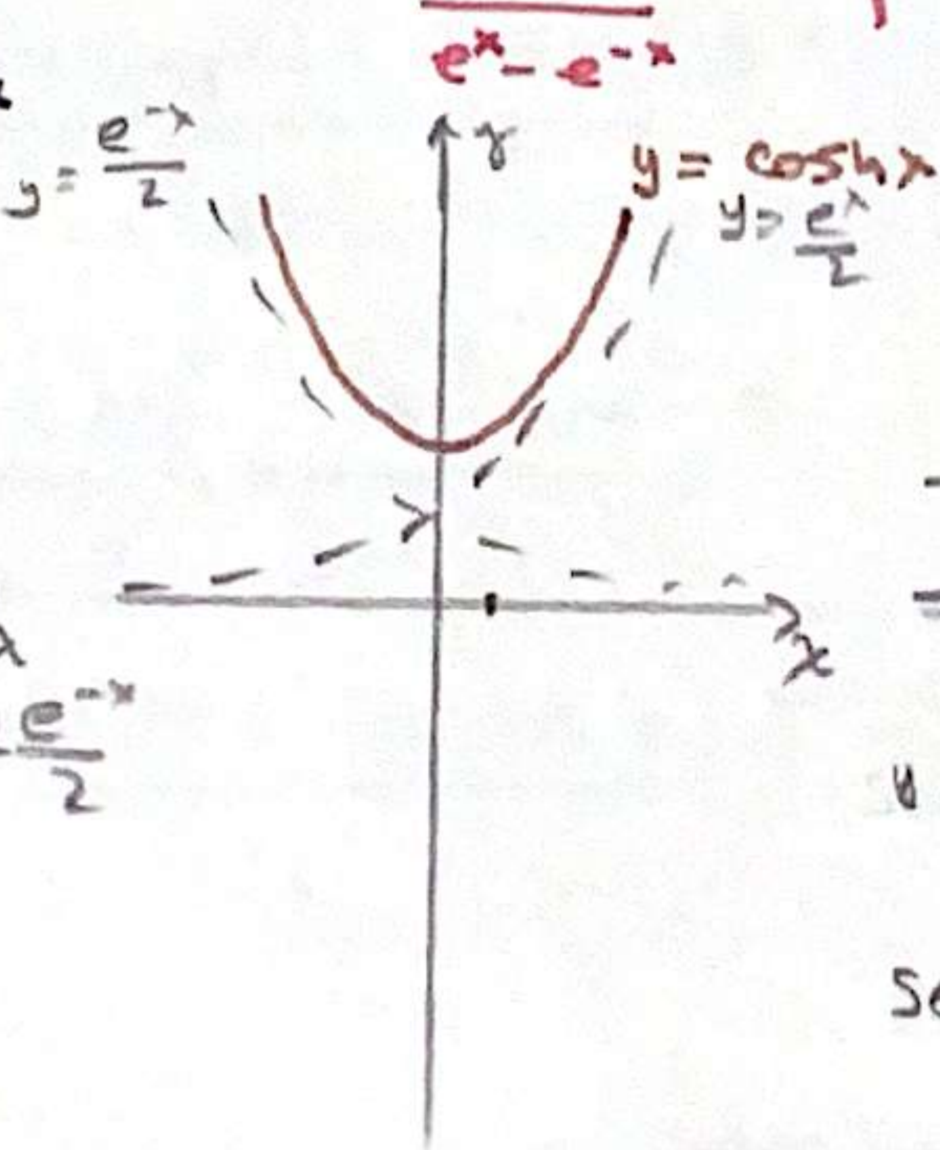


$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x+e^{-x}}{e^x-e^{-x}}$$

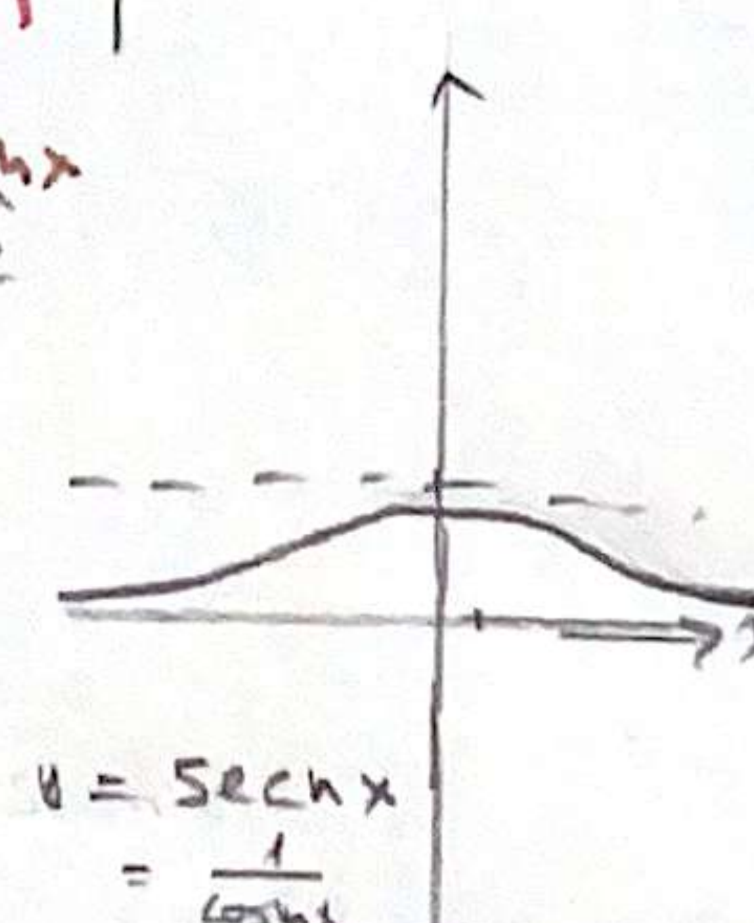
$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x-e^{-x}}{e^x+e^{-x}}$$



$$\sinh x = \frac{e^x-e^{-x}}{2}$$

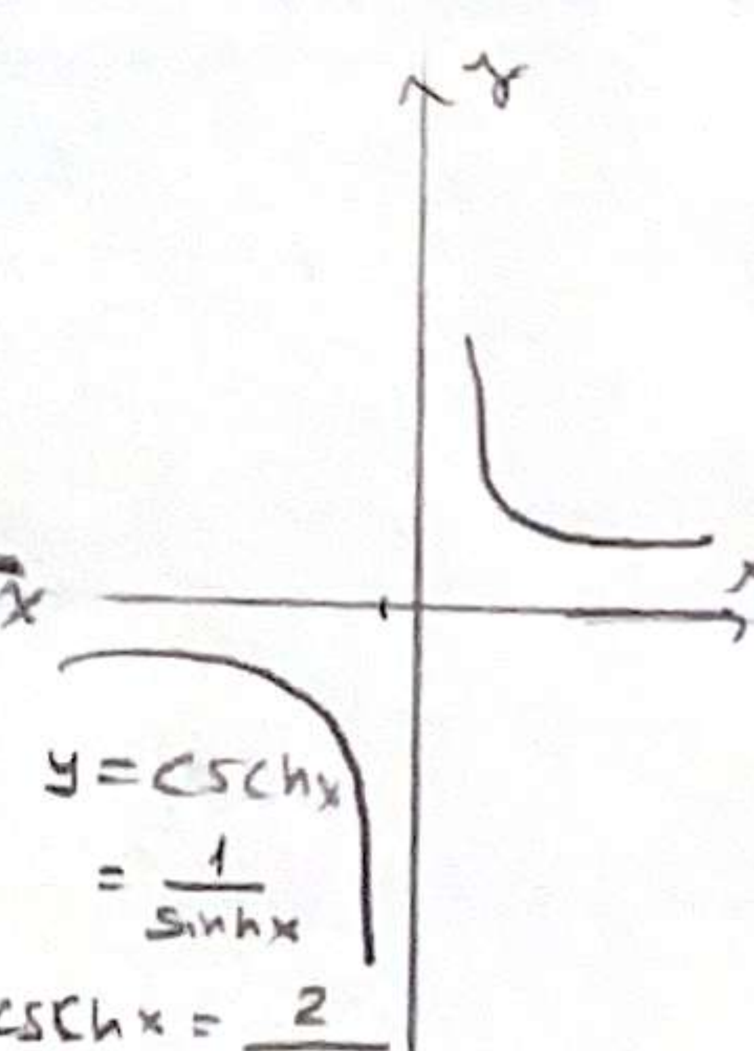


$$\cosh x = \frac{e^x+e^{-x}}{2}$$



$$y = \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\operatorname{sech} x = \frac{2}{e^x+e^{-x}}$$



$$y = \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\operatorname{csch} x = \frac{2}{e^x-e^{-x}}$$

$\cosh^2 x - \sinh^2 x = 1$ | $\sinh(-x) = -\sinh x$ | $\cosh(-x) = \cosh x$

$\sinh 2x = 2\sinh x . \cosh x$

$\cosh 2x = \cosh^2 x + \sinh^2 x$

$\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$ $\qquad$ $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

$\tanh^2 x = 1 - \operatorname{sech}^2 x$ $\qquad$ $\coth^2 x = 1 + \operatorname{csch}^2 x$

$\sinh(x+y) = \sinh x . \cosh y + \cosh x \sinh y$ | $\cosh(x+y) = \cosh x . \cosh y + \sinh x \sinh y$

Hyperbolic sine: $\sinh x = -i \sin(ix)$ $\quad (\operatorname{csch} x = i \csc(ix))$

Hyperbolic cosine: $\cosh x = \cos ix$ $\quad (\operatorname{sech} x = \sec(ix))$

Hyperbolic tangent: $\tanh x = i \tan(ix)$

Hyperbolic cotangent: $\coth x = i \cot(ix)$

$\boxed{i^2 = -1}$

One-to-one /Onto
(injective) (surjective)

- f is one to one if f maps every element of A to a unique element in B. In other words no element of B are mapped to by two or more elements of A

$(\forall a, b \in A)\ f(a) = f(b) \Rightarrow a = b$

- f is onto if every element of B is mapped to by some element of A

$(\forall b \in B)(\exists a \in A)\ f(a) = b$

- f is one-to-one onto (bijective) if it both one-to-one and onto

e.g: $f : \mathbb{N} \to \mathbb{N}$

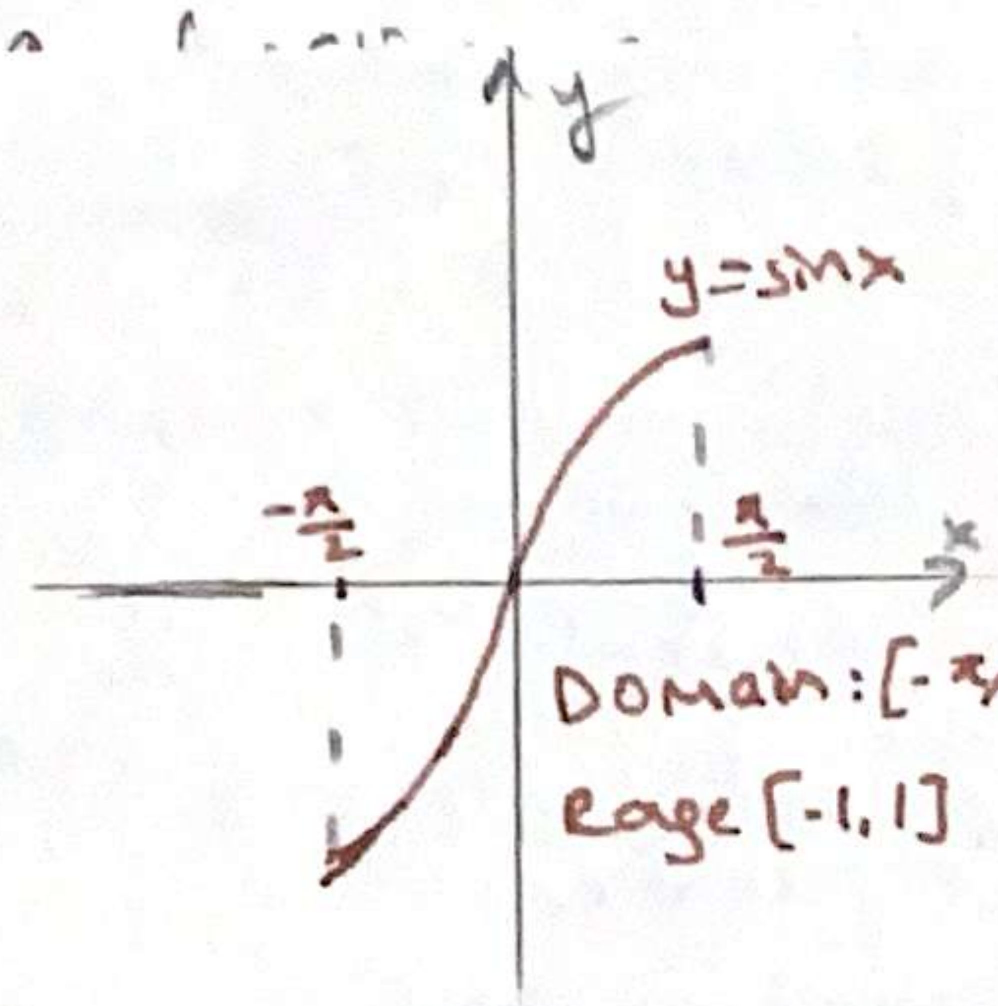
$f(x) = x^2$; $f(m) = f(n)$ then $m = n$ $(m, n \in \mathbb{N})$ $m^2 = n^2$

$m = \pm n$ for $n \neq 0$, $-n \notin \mathbb{N}$, therefore $m = n$

for $n = 0$, $m = n = 0$

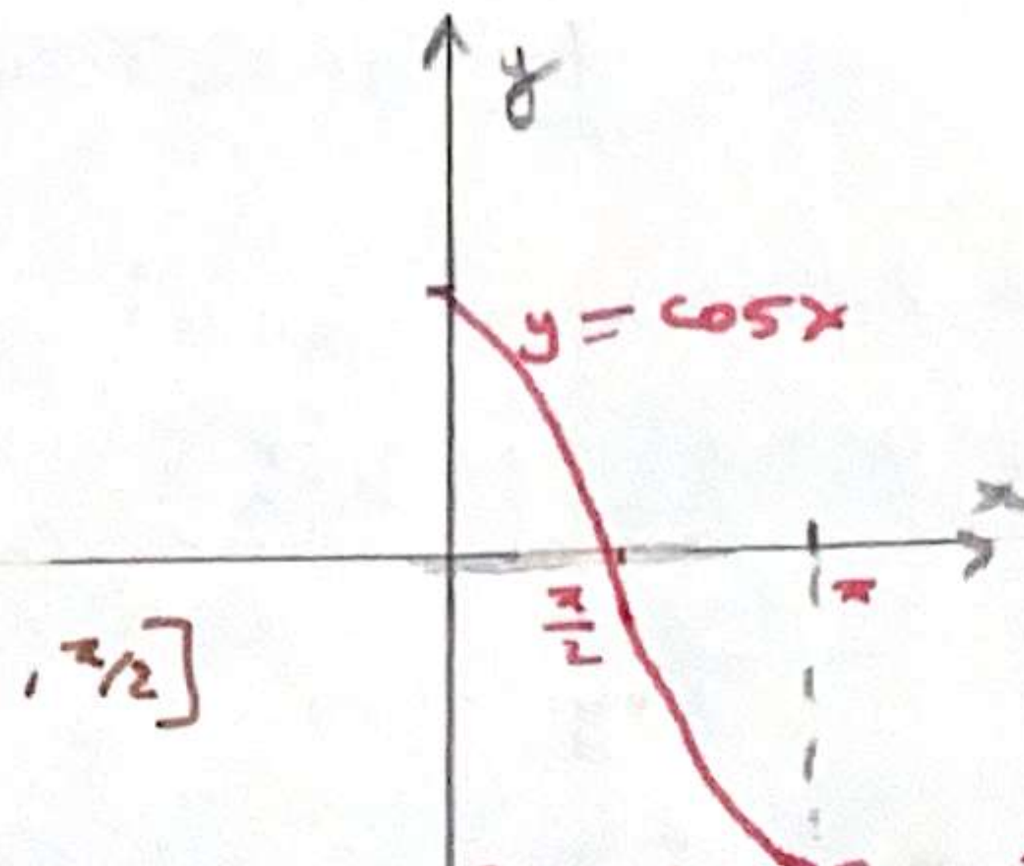
$a^x = e^{\ln(a^x)} \Rightarrow a^x = e^{x \ln a}$ $\qquad$ $a^x = e^{kx} \Rightarrow k = \ln a$

$\ln x = \ln(a^{\log_a x}) \Rightarrow \ln x = (\log_a x)(\ln a)$



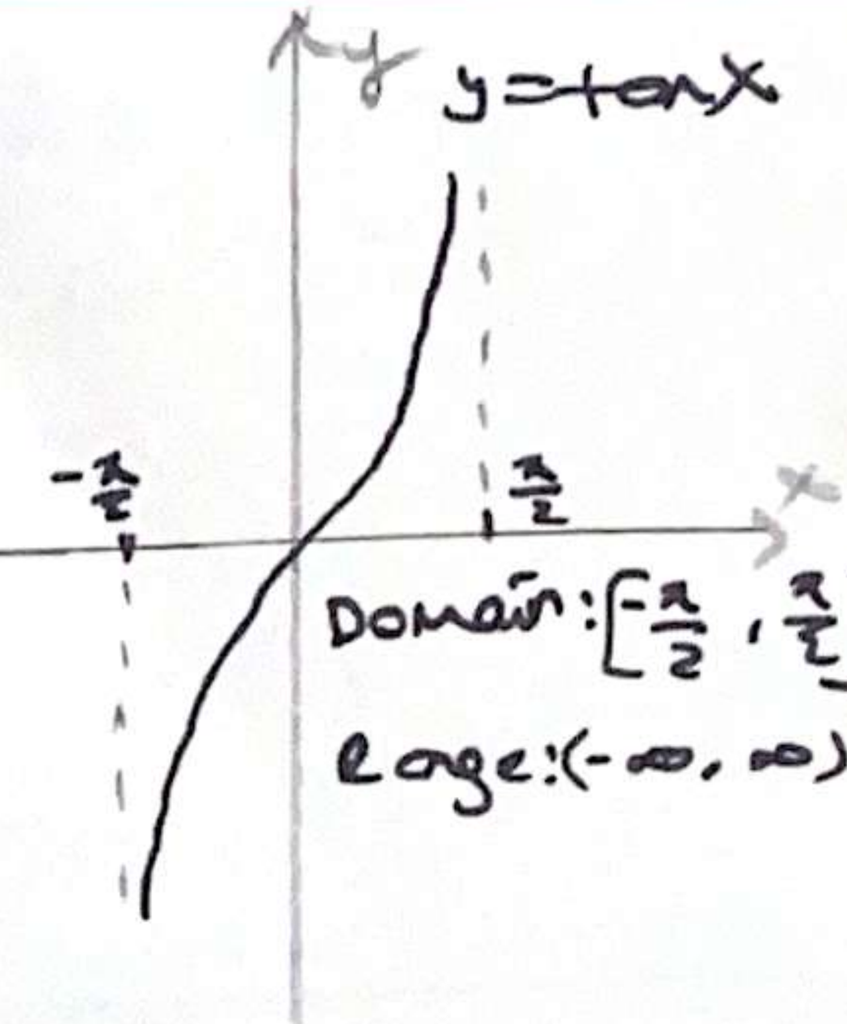
Domain: $[-\pi/2, \pi/2]$

Range $[-1, 1]$



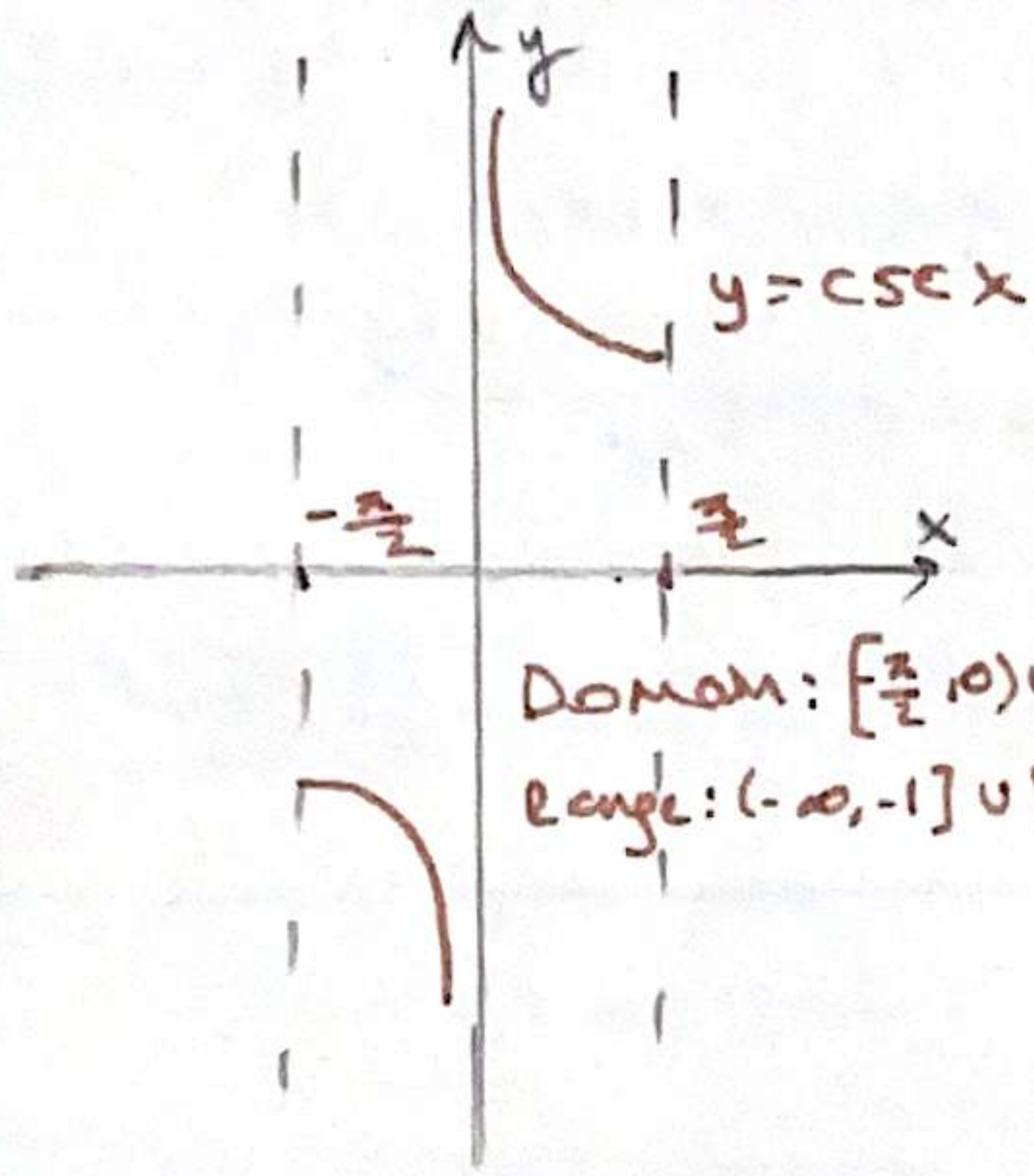
Domain: $[0, \pi]$

Range: $[-1, 1]$



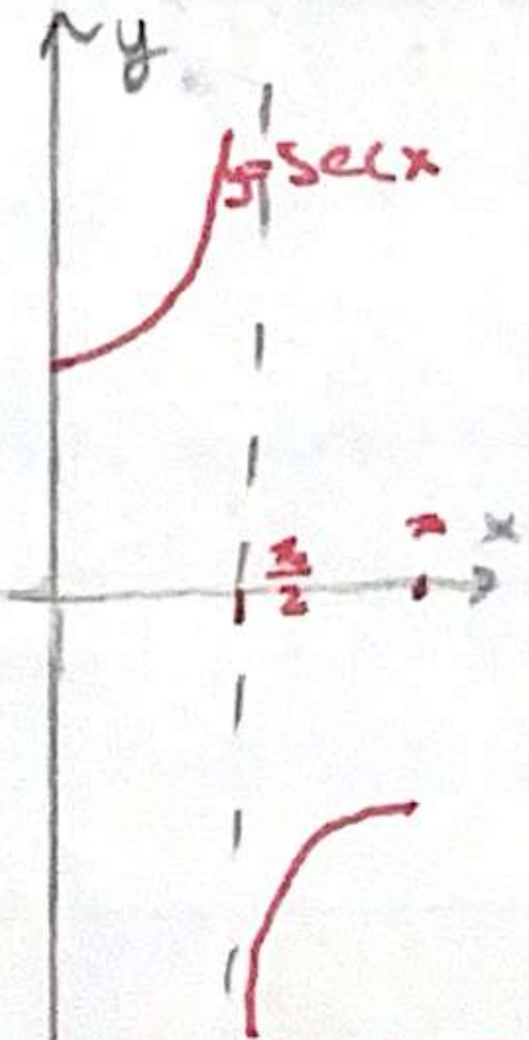
Domain: $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Range: $(-\infty, \infty)$



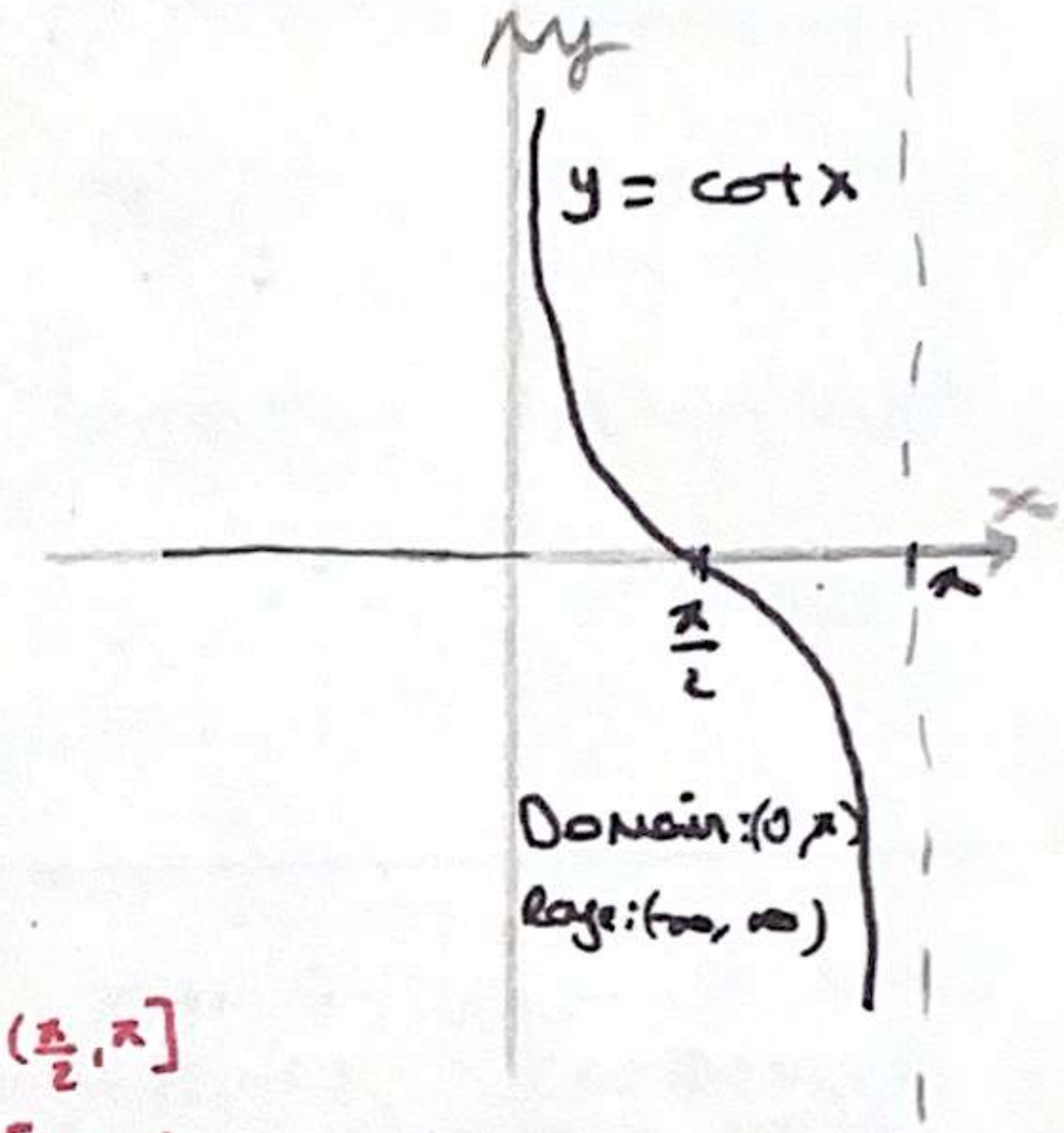
Domain: $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$

Range: $(-\infty, -1] \cup [1, \infty)$



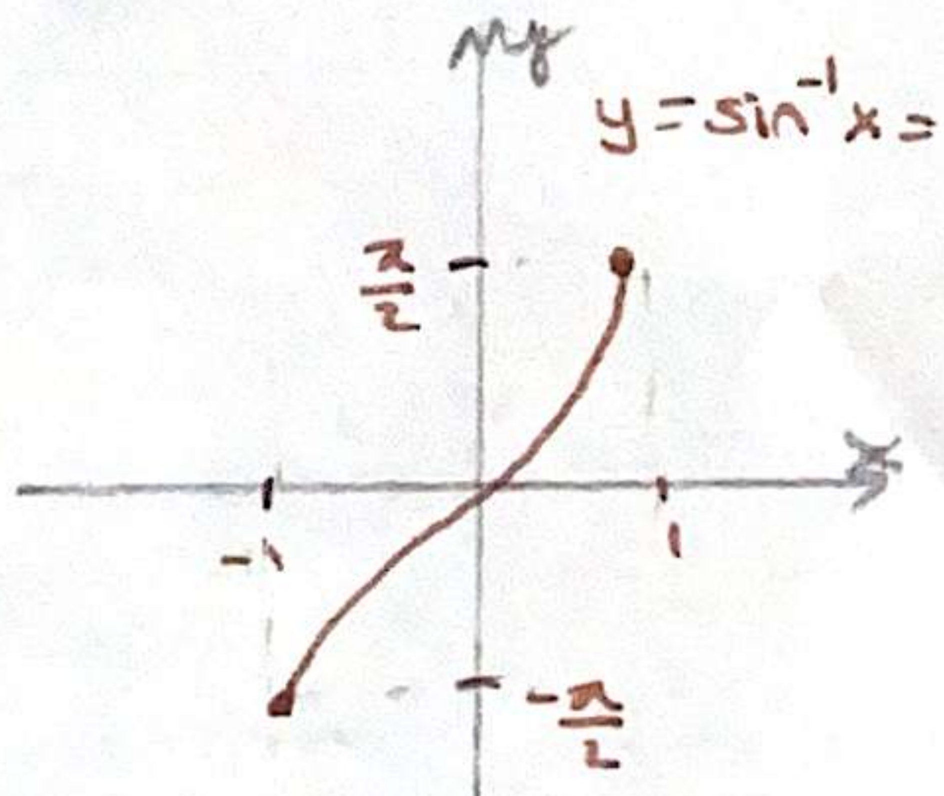
Domain: $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$

Range: $(-\infty, -1] \cup [1, \infty)$



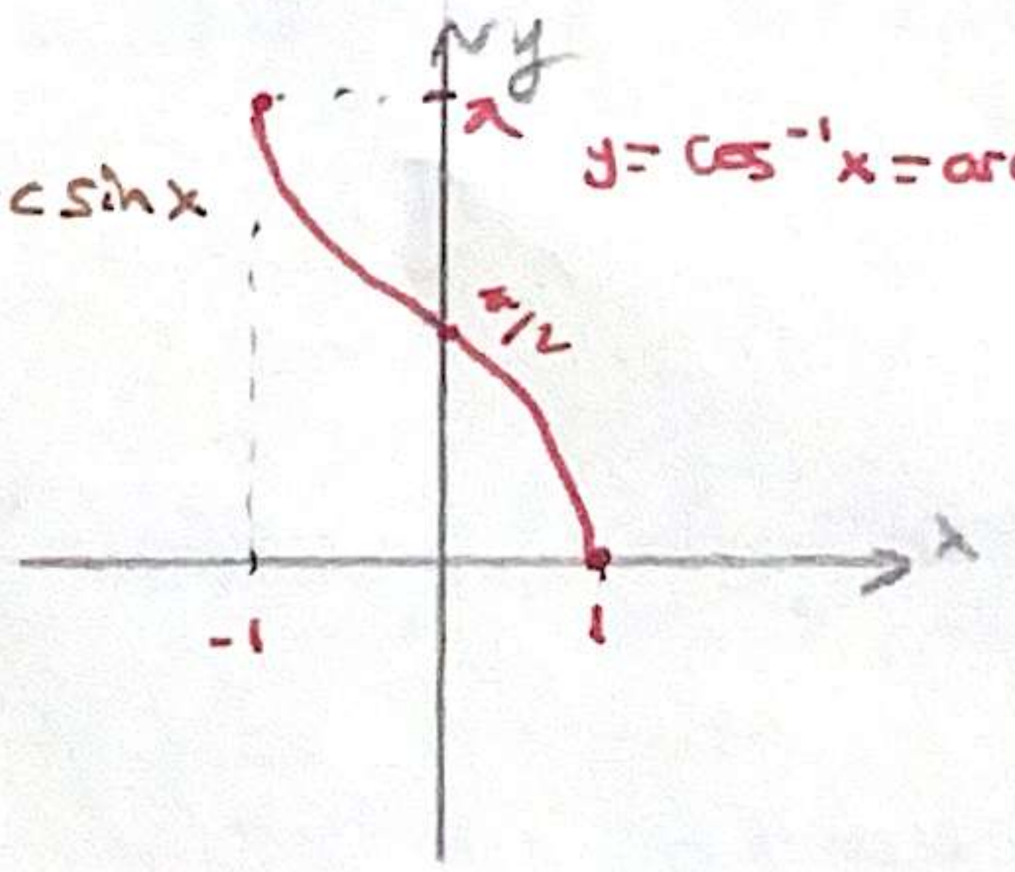
Domain: $(0, \pi)$

Range: $(-\infty, \infty)$



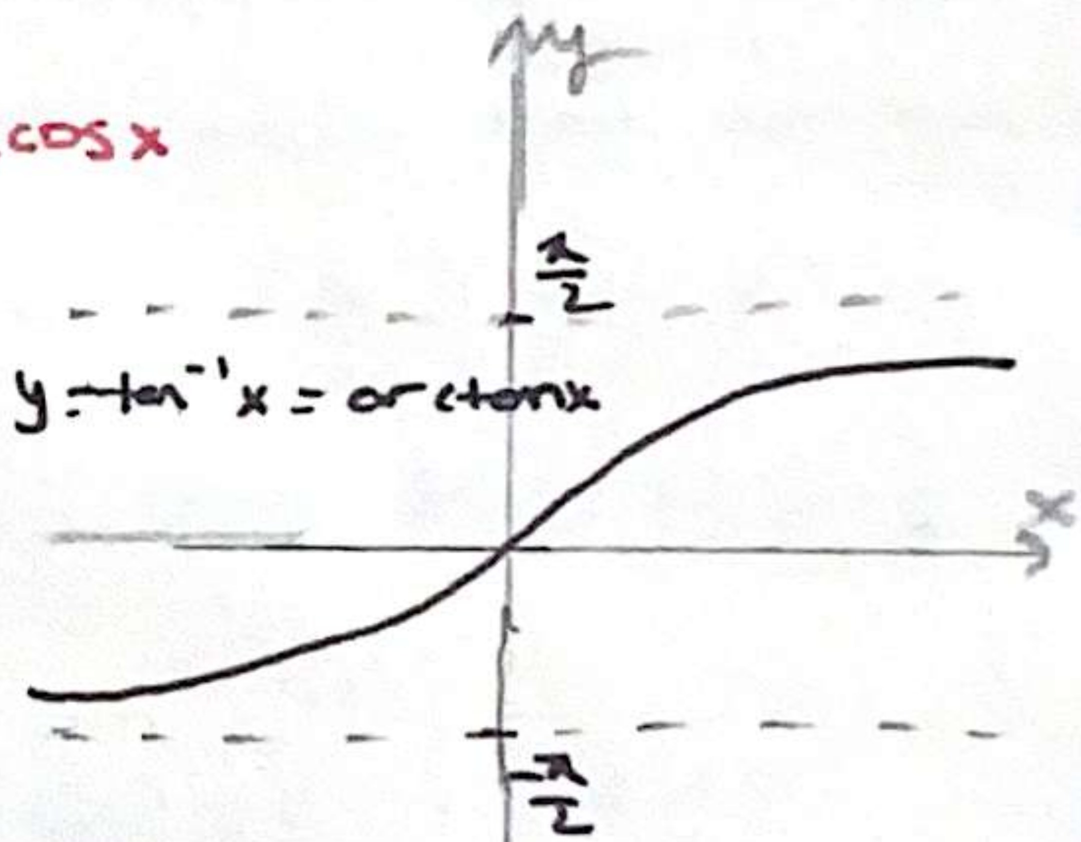
Domain: $-1 \leq x \leq 1$

Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$



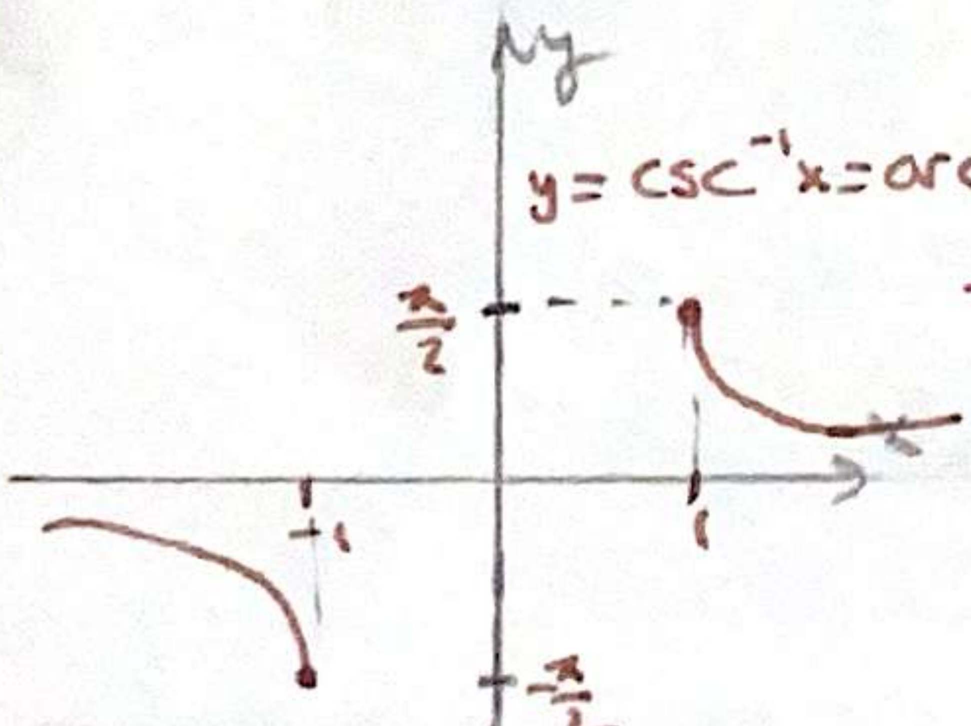
Domain: $-1 \leq x \leq 1$

Range: $0 \leq y \leq \pi$



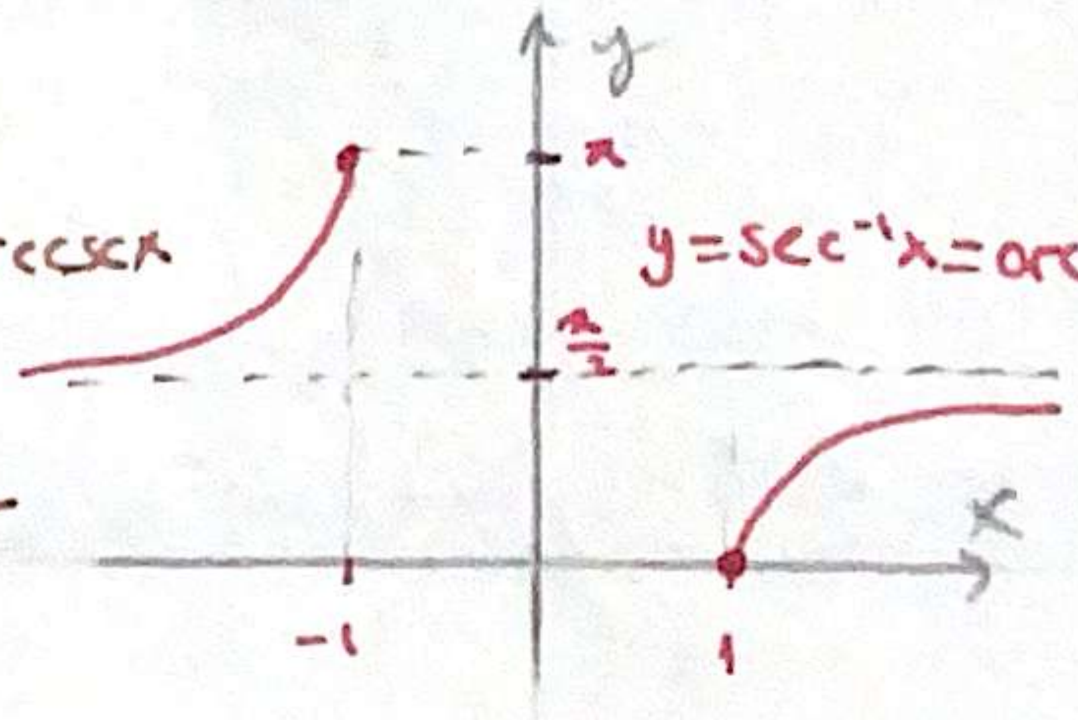
Domain: $-\infty < x < \infty$

Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$



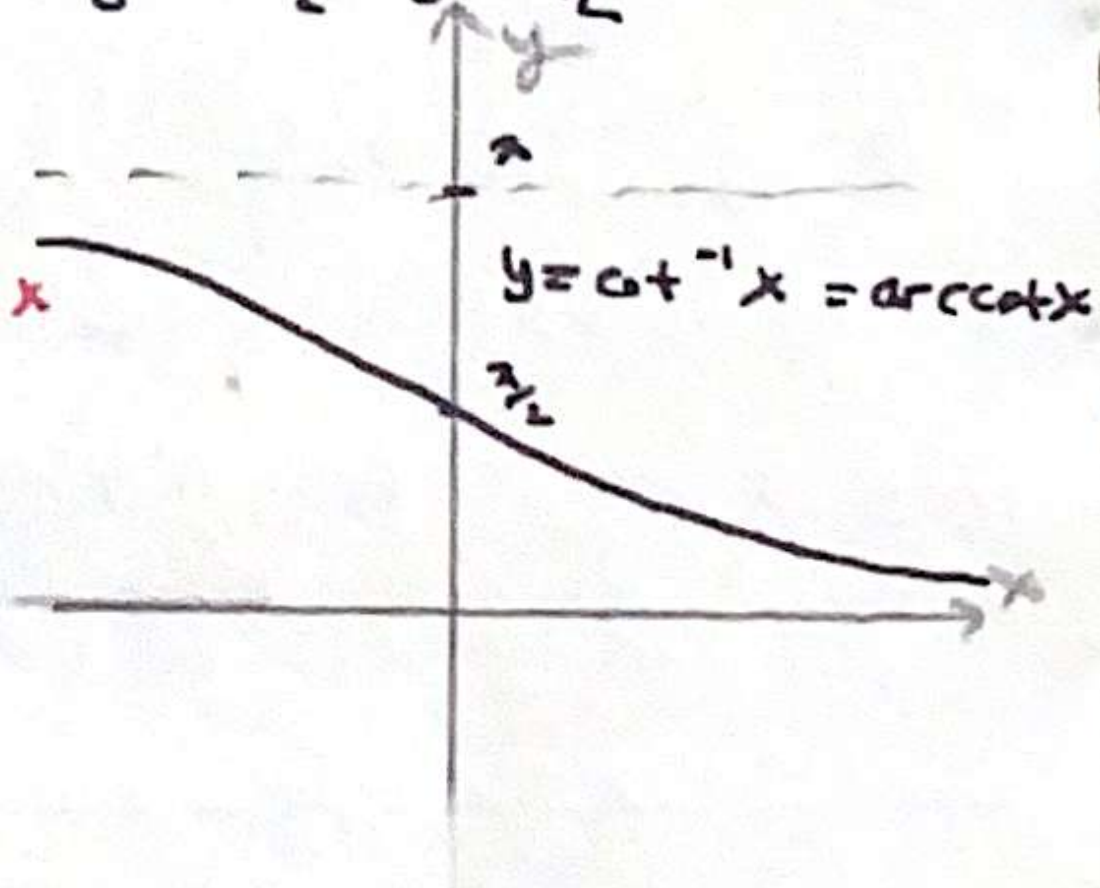
Domain: $x \leq -1$ or $x \geq 1$

Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, $y \neq 0$



Domain: $x \leq -1$ or $x \geq 1$

Range: $0 \leq y \leq \pi$, $y \neq \frac{\pi}{2}$



Domain: $-\infty < x < \infty$

Range: $0 < y < \infty$

Domain restrictions that make the trigonometric functions one to one

Identities for inverse hyperbolic functions

$\operatorname{sech}^{-1}(x) = \cosh^{-1}\left(\frac{1}{x}\right)$ if $0 < x \le 1$

$\operatorname{csch}^{-1}(x) = \sinh^{-1}\left(\frac{1}{x}\right)$

$\coth^{-1}(x) = \tanh^{-1}\left(\frac{1}{x}\right)$

$\operatorname{sech}\left(\cosh^{-1}\left(\frac{1}{x}\right)\right) = \frac{1}{\cosh\left(\cosh^{-1}\left(\frac{1}{x}\right)\right)} = \frac{1}{\frac{1}{x}} = x$

hyperbolic secant is one-to-one on $(0,1]$

---

e.g: Simplify the expression $\cos(\tan^{-1}(x))$

Let $y = \tan^{-1} x$ $\quad x = \tan y$

$$\frac{\sin^2 y + \cos^2 y}{\cos^2 y} = \frac{1}{\cos^2 y}$$

$$\tan^2 y + 1 = \sec^2 y$$

$$x^2 + 1 = \sec^2 y \qquad \sec y = \sqrt{1+x^2}$$

(since $-\frac{\pi}{2} < y < \frac{\pi}{2}$ ensures $\sec y > 0$)

Now since $\cos y = \frac{1}{\sec y}$, we have

$$\cos(\tan^{-1} x) = \cos y = \frac{1}{\sec y} = \frac{1}{\sqrt{1+x^2}}$$

e.g: Find $\sec\left(\tan^{-1}\left(\frac{x}{3}\right)\right)$

$\tan^{-1}\left(\frac{x}{3}\right) = \theta$ $\qquad \sec\theta = ?$

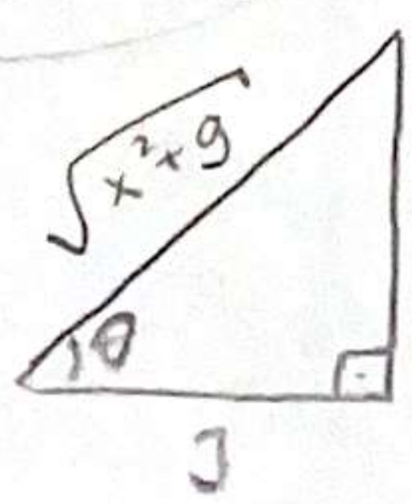
1st way

$\tan = \frac{\text{opposite}}{\text{adjacent}} = \frac{x}{3}$

$\sqrt{x^2+3^2} = \sqrt{x^2+9}$

$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{\sqrt{x^2+9}}{3}$

2nd way



$\frac{\sqrt{x^2+9}}{3} = \sec\theta$

---

e.g: $f(x) = \frac{1}{\sqrt{x-2}}$ $\qquad x - 2 > 0$

Domain $(2, \infty)$ $\qquad$ Range $(0, \infty)$

as $x \to \infty$, $f(x) \to 0^+$, and as $x \to 2^+$, $f(x) \to \infty$

e.g: $g(x) = \frac{x}{1-x}$ $\qquad 1 - x \neq 0$, $1 \neq x$

Domain $(-\infty, 1) \cup (1, \infty)$ $\qquad$ Range $(-\infty, -1) \cup (-1, \infty)$

$y = \frac{x}{1-x} \Rightarrow y - yx = x \mid y = x(y+1) \Rightarrow \frac{y}{y+1} = x$ } undefined when $y = -1$

e.g: $f(x) = \arcsin x \quad g(x) = \sin x$

$D_{g \circ f} = ?$ $\quad \sin(\arcsin(x)) = x \to [-1, 1]$ $\quad D_{g \circ f} = [-1, 1]$

$D_{\frac{f}{g}} = ?$ $\quad \dfrac{\arcsin x}{\sin x}$ $\quad D_f = [-1, 1] \quad D_g = (-\infty, \infty)$

$D_f \cap D_g = [-1, 1]$ $\quad \sin x \neq 0$

$\sin 0 = 0$

$$D_{\frac{f}{g}} = [-1, 1] \setminus \{0\}$$

e.g: solve the inequality $|x-3| + |x+2| < 11$

$$|x-3| = \begin{cases} x-3, & x \geq 3 \\ -x+3, & x < 3 \end{cases} \qquad |x+2| = \begin{cases} x+2, & x \geq -2 \\ -x-2, & x < -2 \end{cases}$$

$\underline{x < -2}$

$-x+3-x-2 < 11$

$-2x+1 < 11$

$-10 < 2x$

$-5 < x$

$\boxed{x > -5}$

$\underline{-2 \leq x < 3}$

$\cancel{-x}+3+\cancel{x}+2 < 11$

$\boxed{5 < 11}$

$\boxed{\text{always true}}$

$\underline{x \geq 3}$

$x-3+x+2 < 11$

$2x-1 < 11$

$\boxed{x < 6}$

Solution is in the interval $(-5, 6)$

e.g: Let $f: [-3, 3] \to \quad , f(x) = x^2 - x - 2$

$(x-2)(x+1)$

x	-3		-1		2		3	
$f(x)$	+	+		−		+		+

$f(-1) = 0 \quad f(2) = 0$

$$\operatorname{sgn} f(x) = \operatorname{sgn}(x^2 - x - 2) = \begin{cases} 1, & x \in [-3, 1) \cup (2, 3] \\ 0, & x \in \{-1, 2\} \\ -1, & x \in (-1, 2) \end{cases}$$

Limit of a function and limit laws

$\lim_{x\to c} f(x)=L$ $\lim_{x\to c} g(x)=M$

$\lim_{x\to x_0} f(x)=L$

- constant multiple rule: $\lim_{x\to c} (k \cdot f(x)) = k \cdot L$
- product rule: $\lim_{x\to c} (g(x) \cdot f(x)) = M \cdot L$
- quotient rule: $\lim \frac{f(x)}{g(x)} = \frac{L}{M}$ $(M \neq 0)$
- Power rule: $\lim_{x\to c} [f(x)]^n = L^n$ (n: a positive integer)
- root rule: $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$ (n: a positive integer / If n is even, we assume that $\lim_{x\to c} f(x) = L > 0$)

Epsilon-delta definition of limit of functions

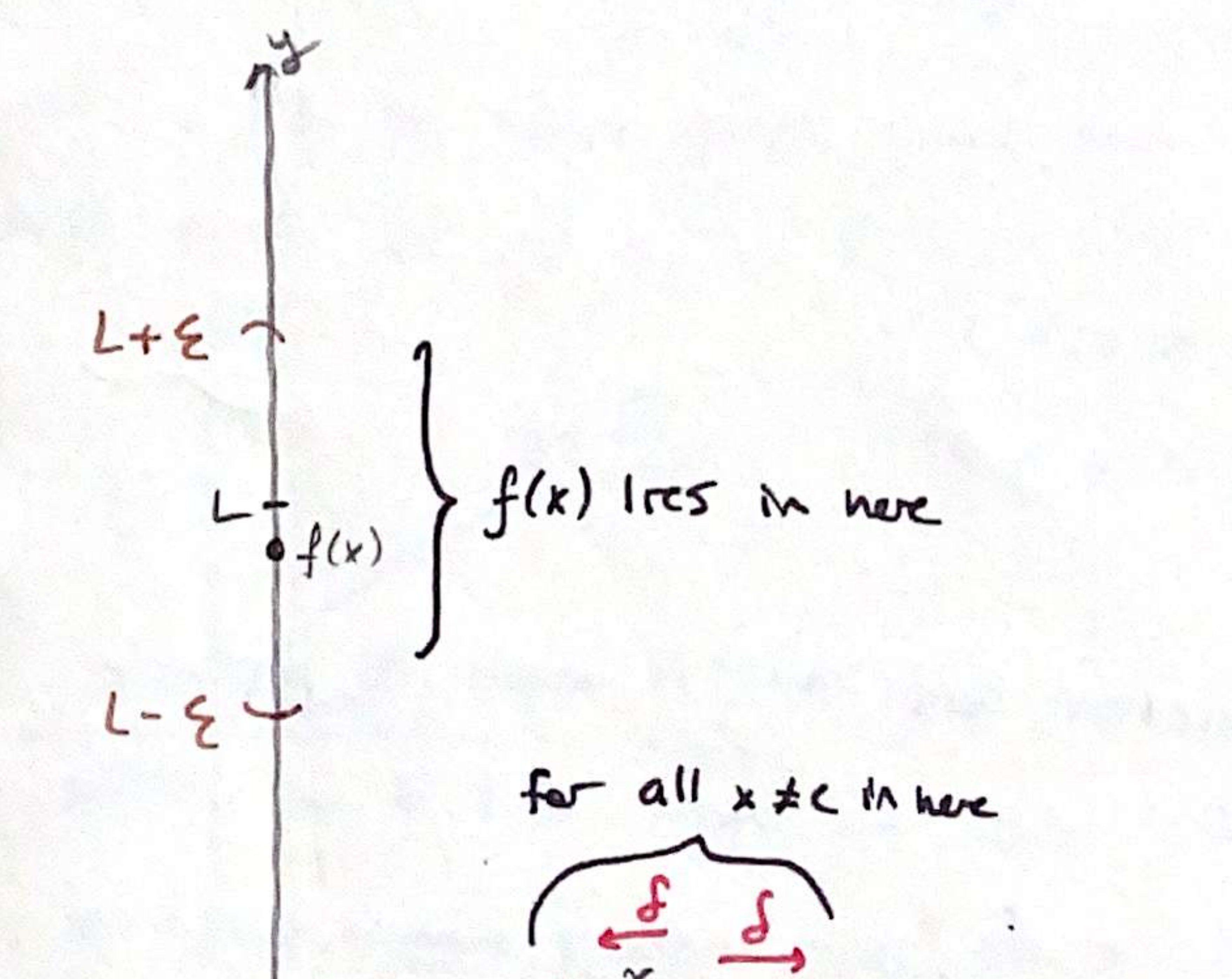
Let $f(x)$ be defined on an open interval about c, except possibly at c itself. We say that the limit of $f(x)$ as x approaches c is the number L, and write

$\lim_{x\to c} f(x) = L$ If, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x

$$0 < |x-c| < \delta \Rightarrow |f(x)-L| < \varepsilon$$

$$\lim_{x\to x_0} f(x) = L \Leftrightarrow \forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \ (\delta(\varepsilon) \in \mathbb{R}^+)$$

$$\forall x \in X, \ 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$$



e.g: Show that $\lim_{x \to 2} 2x-3 = 1$

$\lim_{x \to x_0} f(x) = L \Leftrightarrow \forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \ni \forall x \in X, 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$

Assume that $0 < |x-2| < \delta$, For all ε, we want to find δ and our goal is to show that if $0 < |x-2| < \delta$, then $|(2x-3) - 1| < \varepsilon$

$0 < |x - x_0| < \delta$ $\qquad$ $|f(x) - L| < \varepsilon$

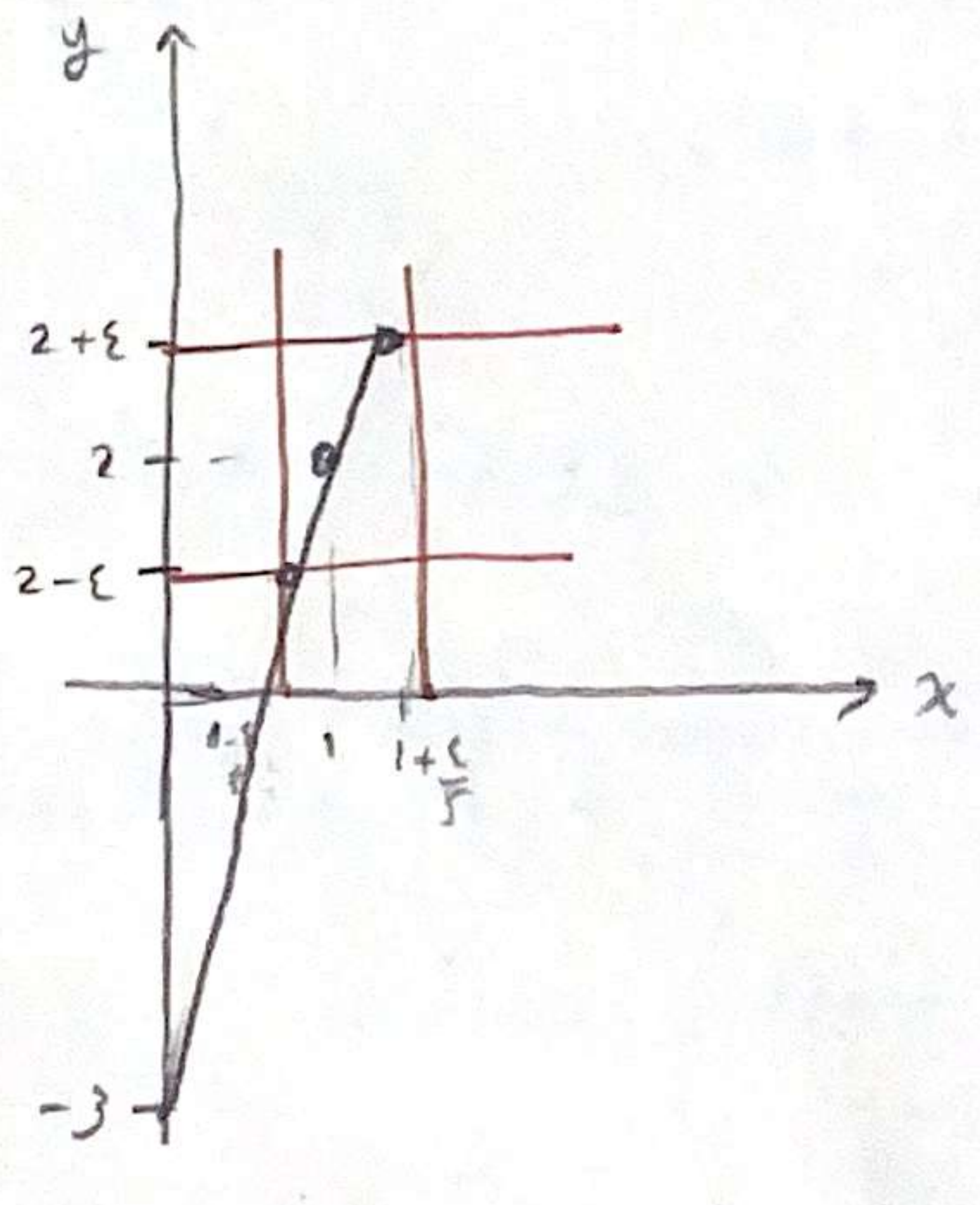
$|(2x-3) - 1| = |2x-4| = |2(x-2)| = 2|x-2|$

$2|x-2| < \varepsilon \qquad |x-2| < \frac{\varepsilon}{2} \qquad \frac{\varepsilon}{2} < \delta \qquad \varepsilon < 2\delta$

e.g: Show that $\lim_{x \to 1} (5x-3) = 2$

$0 < |x-1| < \delta \qquad |(5x-3) - 2| < \varepsilon \qquad |5(x-1)| < \varepsilon \qquad |(x-1)| < \frac{\varepsilon}{5}$

$\frac{\varepsilon}{5} = \delta$



Note: Normally we can say if a function has limit, it needs to provide that its right-handed and left-handed limit are equal, but if we tackle a function in a closed interval and we can just discuss about one side limit and it couldn't be checked the other side because of the interval) we can conclude by expressing that function's limit is equal to one side limit at that point.

Assume the domain of f contains an open interval (c,d) to the right of c. We say that $f(x)$ has right-hand limit L at c, and write

$$\lim_{x \to c^+} f(x) = L$$

If for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad c < x < c + \delta$$

Assume the domain of f contains an interval (b,c) to the left of c. We say f has left-hand limit L at c, and write

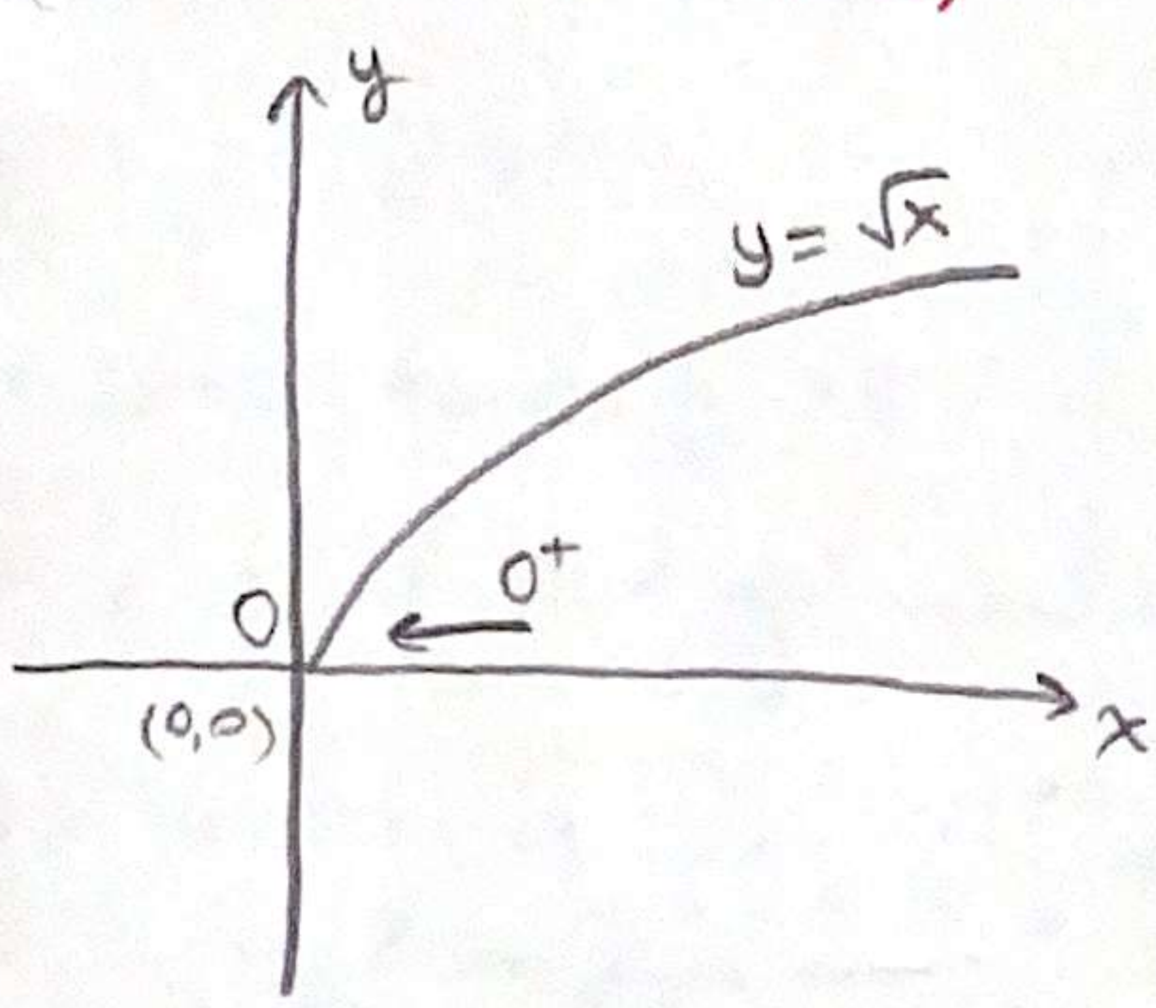
$$\lim_{x \to c^-} f(x) = L$$

If for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \qquad c - \delta < x < c$$

e.g. Prove that $\lim_{x \to 0^+} \sqrt{x} = 0$

Let $\varepsilon > 0$ be given. Here $c = 0$ and $L = 0$, so we want to find a $\delta > 0$ such that for all x,



$$c < x < c + \delta$$

$$0 < x < \delta \Rightarrow |\sqrt{x} - 0| < \varepsilon$$

$$\sqrt{x} < \varepsilon$$

$$x < \varepsilon^2$$

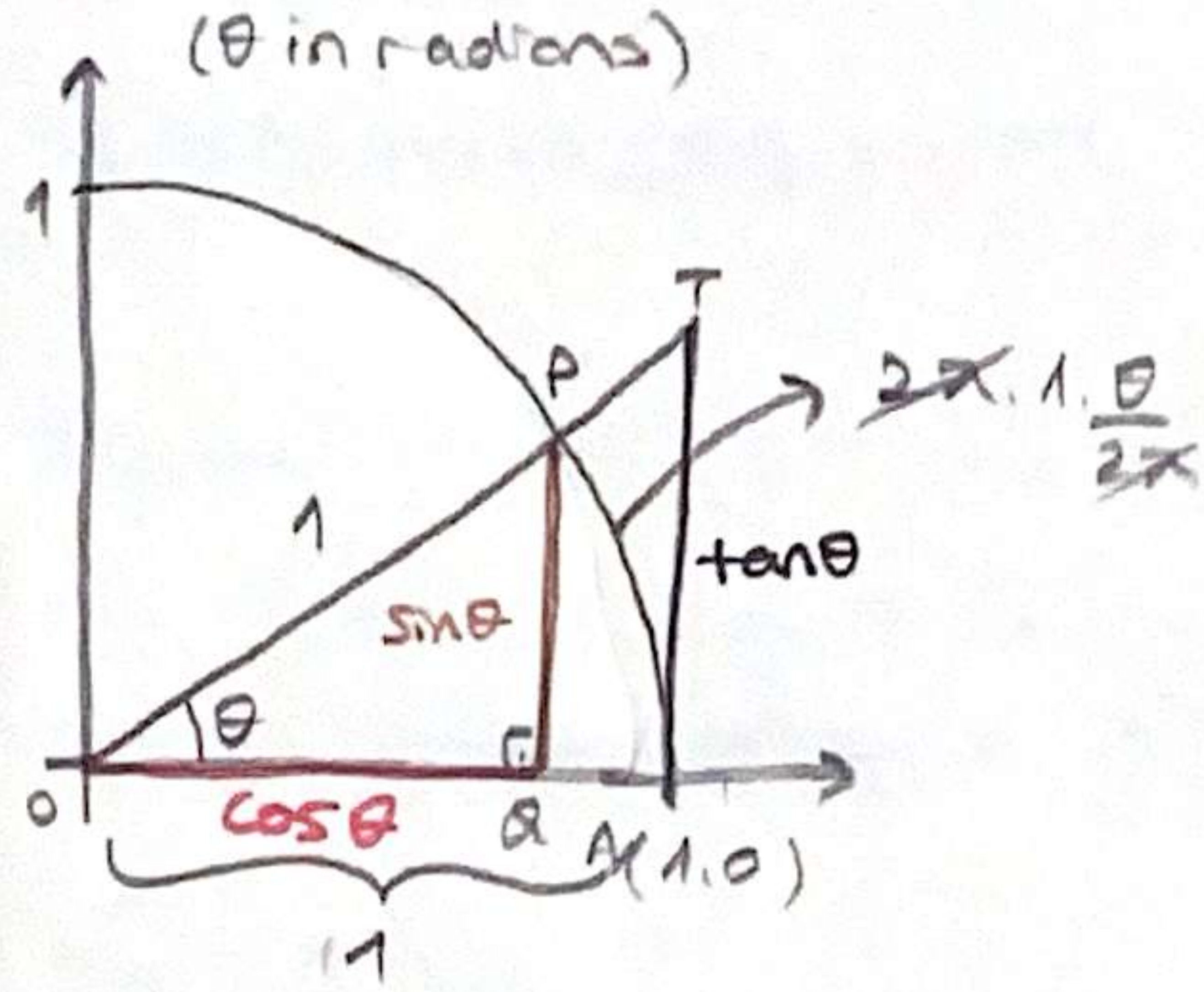
If $0 < x < \delta$

$$\delta = \varepsilon^2$$

$$0 < x < \varepsilon^2 \Rightarrow |\sqrt{x} - 0| < \varepsilon$$

Theorem

(θ in radians)



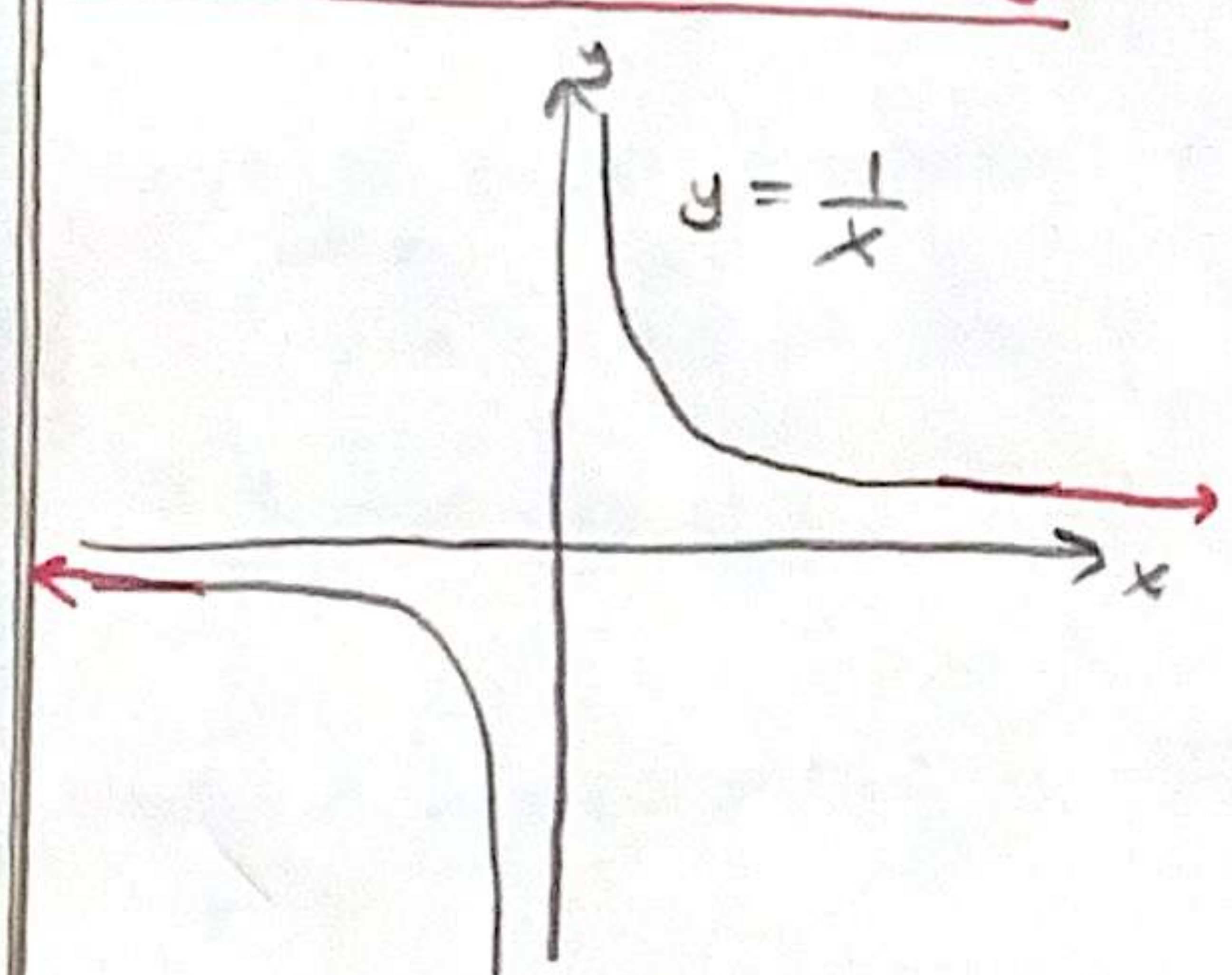
① $\frac{\sin\theta}{\sin\theta} < \frac{\theta}{\sin\theta} < \frac{\tan\theta}{\sin\theta}$ ② $1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$

③ $\cos\theta < \frac{\sin\theta}{\theta} < 1$

As $\theta \to 0$, $\cos\theta \to 1$; therefore, by squeeze theorem,

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

Limits involving infinity



The graph of $y = \frac{1}{x}$ approaches 0 as $x \to +\infty$ or $x \to -\infty$

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

$\varepsilon > 0$, $x > M \Rightarrow \left|\frac{1}{x} - 0\right| < \varepsilon$ $\quad M = \frac{1}{\varepsilon}$

$$\lim_{x \to -\infty}$$

$\varepsilon > 0$, $x < N \Rightarrow \left|\frac{1}{x} - 0\right| < \varepsilon$ $\quad N = -\frac{1}{\varepsilon}$

e.g. $\lim_{x \to \infty} \frac{5x^2 + 8x - 3}{3x^2 + 2} \Rightarrow \lim_{x \to \infty} \frac{5 + 8/x - 3/x^2}{3 + 2/x^2} = \frac{5 + 0 - 0}{3 + 0} = 5/3$

e.g. $\lim_{x \to 1^+} \frac{1}{x-1} = +\infty$ | $\lim_{x \to 1^-} \frac{1}{x-1} = -\infty$

$x \to 1^+ \Rightarrow (x-1) \to 0^+ \Rightarrow \frac{1}{x-1} \to +\infty$ | $x \to 1^- \Rightarrow (x-1) \to 0^- \Rightarrow \frac{1}{x-1} \to -\infty$

e.g. $\lim\limits_{x \to -\infty} \dfrac{2x^5 - 6x^4 + 1}{3x^2 + x - 7}$ (divide the numerator and denominator by x^2; the highest power of denominator)

$$\lim_{x \to -\infty} \frac{2x^3 - 6x^2 + \frac{1}{x^2}}{3 + \frac{1}{x} - \frac{7}{x^2}} = \frac{\lim\limits_{x \to -\infty} \left(2x^3 - 6x^2 + \frac{1}{x^2}\right)}{\lim\limits_{x \to -\infty} \left(3 + \frac{1}{x} - \frac{7}{x^2}\right)} = \frac{-\infty}{3} \Rightarrow -\infty$$

---

1. We say that $f(x)$ approches **infinity** as x approches c, and write

$$\lim_{x \to c} f(x) = \infty,$$

If every **positive** real number B there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - c| < \delta \Rightarrow f(x) > B$$

2. We say that $f(x)$ approches **minus infinity** as x approches c, and write

$$\lim_{x \to c} f(x) = -\infty,$$

If every **negative** real number $-B$ there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - c| < \delta \Rightarrow f(x) < -B$$

e.g. Prove that $\lim\limits_{x \to 0} \dfrac{1}{x^2} = \infty$

$B > 0 \rightarrow \delta > 0$

$0 < |x - 0| < \delta \Rightarrow \underset{f(x)}{\dfrac{1}{x^2}} > B$

$x^2 < \dfrac{1}{B} \rightarrow |x| < \dfrac{1}{\sqrt{B}}$ $\rightarrow$ $\delta = \dfrac{1}{\sqrt{B}}$ $\quad 0 < |x| < \delta \quad \dfrac{1}{x^2} > B$

other solution: $x_n \neq 0 \quad x_n \to 0 \quad \dfrac{1}{x_n^2} \to +\infty \quad \lim\limits_{n \to \infty} x_n = 0$

$y_n = \dfrac{1}{x_n^2} \quad x_n \to 0; \quad M > 0 \quad |x_n| < \dfrac{1}{\sqrt{M}} \Rightarrow x_n^2 < \dfrac{1}{M} \quad \dfrac{1}{x_n^2} > M$

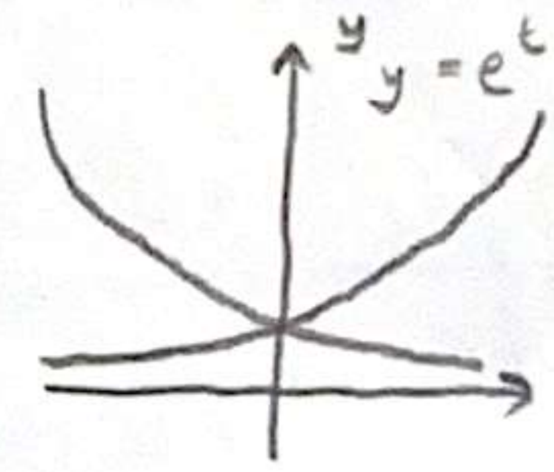
$\lim\limits_{n \to \infty} y_n = +\infty$

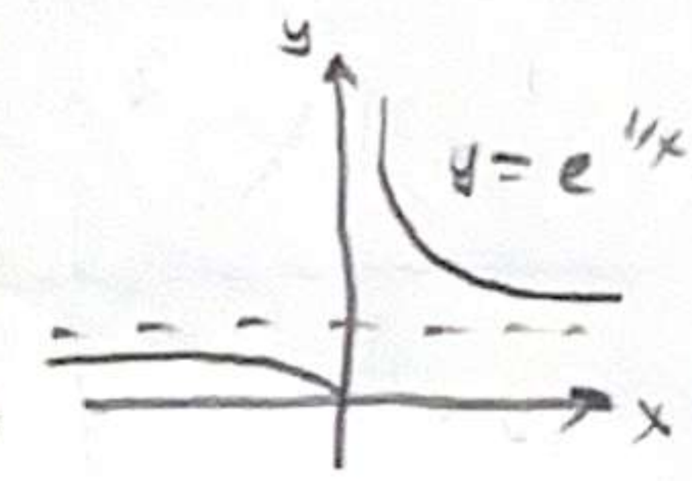
e.g. a) $\lim\limits_{x\to\infty} \sin\left(\frac{1}{x}\right)$ and b) $\lim\limits_{x\to\pm\infty} x\sin\left(\frac{1}{x}\right)$

a) $\frac{1}{x} = t$ $\quad \lim\limits_{t\to 0^+} \sin t = 0$

b) 1) $\lim\limits_{x\to\infty} x \cdot \sin\left(\frac{1}{x}\right)$ $\quad \frac{1}{x} = t$ $\quad \lim\limits_{t\to 0^+} \frac{\sin t}{t} = 1$

2) $\lim\limits_{x\to -\infty} x \sin\left(\frac{1}{x}\right)$ $\quad \frac{1}{x} = t$ $\quad \lim\limits_{t\to 0^-} \frac{\sin t}{t} = 1$





e.g. $\lim\limits_{x\to 0^-} e^{1/x}$ $\quad \frac{1}{x} = t$ $\quad \lim\limits_{t\to -\infty} e^t = 0$

e.g. $y = 2 + \frac{\sin x}{x}$ $\quad \lim\limits_{x\to\pm\infty} \left(2 + \frac{\sin x}{x}\right)$ $\quad -1 \le \sin x \le 1$

$x \to \pm\infty$

$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$

$0 \le \left|\frac{\sin x}{x}\right| \le \frac{1}{x}$

$\lim\limits_{x\to\pm\infty} 2 + \lim\limits_{x\to\pm\infty} \frac{\sin x}{x} = 2 + 0 = 2$ $\qquad \lim\limits_{x\to\pm\infty} \frac{1}{x} = 0$

e.g. $\left(\frac{0}{0}\right)$ $\lim\limits_{x\to 0} \frac{\sin 2x}{x\cos x} \Rightarrow \lim\limits_{x\to 0} \frac{2\sin x \cdot \cancel{\cos x}}{x\cancel{\cos x}} \Rightarrow 2 \cdot \lim\limits_{x\to 0} \frac{\sin x}{x} = 2$

note: $0 \cdot \infty \Rightarrow \frac{0}{\frac{1}{\infty}} \Rightarrow \frac{0}{0}$ $\qquad 0 \cdot \infty \Rightarrow \frac{\infty}{\frac{1}{0}} \Rightarrow \frac{\infty}{\infty}$

e.g. $\lim\limits_{x\to\infty} 3x \cdot \tan\left(\frac{1}{4x}\right) \Rightarrow \frac{1}{x} = t \Rightarrow \lim\limits_{t\to 0^+} 3 \cdot \frac{1}{t} \tan\left(\frac{t}{4}\right) \Rightarrow \lim\limits_{t\to 0^+} \frac{3 \cdot \tan\left(\frac{t}{4}\right)}{t}$

★ $\lim\limits_{u\to 0} \frac{\tan(au)}{u} = a$ $\quad a = \frac{1}{4}$ $\quad \lim\limits_{t\to 0^+} \frac{3 \cdot \tan\left(\frac{t}{4}\right)}{t} \frac{1}{4} = \frac{3}{4}$

e.g. $\lim\limits_{x\to\infty} \left(\sqrt{x^2 - x} - x\right)$ $\quad \lim\limits_{x\to\infty} \left(\sqrt{x^2-x} - x\right) \frac{\left(\sqrt{x^2-x} + x\right)}{\left(\sqrt{x^2-x} + x\right)} \Rightarrow \lim\limits_{x\to\infty} \frac{\cancel{x^2} - x - \cancel{x^2}}{\sqrt{x^2-x} + x}$

$\lim\limits_{x\to\infty} \frac{-x}{\sqrt{x^2\left(1-\frac{1}{x}\right)} + x} = \lim\limits_{x\to\infty} \frac{-\cancel{x}}{\cancel{x}\left(\sqrt{1-\frac{1}{x}} + 1\right)} \Rightarrow \lim\limits_{x\to\infty} \frac{-1}{\sqrt{1-\frac{1}{x}} + 1} = -\frac{1}{2}$

e.g. $\lim\limits_{x\to 0^+} e^{\frac{1}{x}} = e^{\frac{1}{0^+}} = e^{+\infty} = \infty$ $\qquad \lim\limits_{x\to 0^-} e^{\frac{1}{0^-}} = e^{-\infty} = 0$

$\lim\limits_{x\to 0} e^{\frac{1}{x}} = \text{DNE}$

e.g. $\lim_{x \to -\infty} \frac{\cos x}{x} = 0$ $\quad -1 \leq \cos x \leq 1 \quad -\frac{1}{x} \leq \frac{\cos x}{x} \leq \frac{1}{x}$

$x \to -\infty$

$0 \leq \frac{\cos x}{x} \leq \frac{1}{x} \Rightarrow \frac{1}{x} = 0 \Rightarrow 0 \leq \frac{\cos x}{x} \leq 0 \quad \lim_{x \to -\infty} \frac{\cos x}{x} = 0$

e.g. a and b real constants and $b \neq 0$

$\lim_{x \to 0} \frac{\sin(ax)}{bx} \quad ax = u \quad \left(x = \frac{u}{a}\right) \quad \lim_{u \to 0} \frac{\sin u}{b \cdot \frac{u}{a}} \Rightarrow \frac{a}{b} \underbrace{\lim_{u \to 0} \frac{\sin u}{u}}_{1} = \frac{a}{b}$

$\lim_{x \to 0} \frac{\sin ax}{bx} = \frac{a}{b}$

e.g. $\lim_{x \to 0} \frac{\sin(ax)}{\sin(bx)} = \frac{ax \cdot \frac{\sin(ax)}{ax}}{bx \cdot \frac{\sin bx}{bx}} \Rightarrow \frac{ax}{bx} = \frac{a}{b}$

e.g. $\lim_{x \to 0} \frac{\tan(x)}{x} = \lim_{x \to 0} \frac{\sin x}{x} \frac{1}{\cos x} = 1 \quad \lim_{x \to 0} \frac{\tan(x)}{x} = 1$

e.g. $\lim_{x \to 0} \frac{\tan(kx)}{kx} \Rightarrow kx = u \Rightarrow \lim_{u \to 0} \frac{\tan(u)}{u} = 1 \quad \lim_{x \to 0} \frac{\tan(kx)}{kx} = 1$

(for some constant k)

e.g. $\lim_{u \to 0} \frac{\sin(3u)}{\tan(5u)} = \lim_{u \to 0} \frac{\sin(3u) \cdot \frac{3u}{3u}}{\tan(5u) \cdot \frac{5u}{5u}} = \frac{\overbrace{\lim_{u \to 0} \frac{\sin(3u)}{3u}}^{1} \cdot 3u}{\underbrace{\lim_{u \to 0} \frac{\tan(5u)}{5u}}_{1} \cdot 5u} = \frac{3u}{5u} = \frac{3}{5}$

e.g. $\lim_{x \to 3} (x-3) \sin\left(\frac{1}{x-3}\right)$

$x - 3 = u$

$\lim_{u \to 0} u \cdot \sin\left(\frac{1}{u}\right) \Rightarrow -u \leq u \cdot \sin\left(\frac{1}{u}\right) \leq u \quad u \to 0 \quad -u \to 0 \quad +u \to 0$

$0 \leq u \cdot \sin\left(\frac{1}{u}\right) \leq 0$

$\lim_{x \to 3} (x-3) \cdot \sin \frac{1}{x-3}$

e.g. $\lim_{x \to 0} x^4 \cdot \sin\left(\frac{\pi}{x}\right) \quad -x^4 \leq x^4 \cdot \sin\left(\frac{\pi}{x}\right) \leq x^4$

$\lim_{x \to 0} -x^4 \leq \lim_{x \to 0} x^4 \cdot \sin\left(\frac{\pi}{x}\right) \leq \lim_{x \to 0} x^4$

$0 \leq \lim_{x \to 0} x^4 \cdot \sin\left(\frac{\pi}{x}\right) \leq 0 \Rightarrow \lim_{x \to 0} x^4 \cdot \sin\left(\frac{\pi}{x}\right) = 0$

1. $\lim\limits_{x \to 0} \dfrac{\sin 5x}{x} = \lim\limits_{x \to 0} \dfrac{5}{5} \dfrac{\sin 5x}{x} = 5 \lim\limits_{x \to 0} \dfrac{\sin 5x}{5x} = 5$

2. $\lim\limits_{x \to 0} \dfrac{\sin 3x}{\sin 5x} = \lim\limits_{x \to 0} \dfrac{\sin(3x) \cdot \frac{3x}{3x}}{\sin(5x) \cdot \frac{5x}{5x}} = \dfrac{3}{5} \cdot \dfrac{\lim\limits_{x \to 0} \frac{\sin(3x)}{3x}}{\lim\limits_{x \to 0} \frac{\sin(5x)}{5x}} = \dfrac{3}{5}$

3. $\lim\limits_{x \to 0} \dfrac{\sin x}{2x^2 - x} = \lim\limits_{x \to 0} \dfrac{\sin x}{x} \cdot \lim\limits_{x \to 0} \dfrac{1}{(2x-1)} = 1 \cdot \dfrac{1}{-1} = -1$

$2x^2 - x = x(2x-1)$

4. $\lim\limits_{x \to 0} \dfrac{\tan 2x}{7x} = \lim\limits_{x \to 0} \dfrac{\sin 2x}{2x} \cdot \dfrac{2x}{7x \cdot \cos 2x} = \dfrac{2}{7}$

5. $\lim\limits_{x \to 0} \dfrac{\sin x}{|x|} = \begin{cases} \lim\limits_{x \to 0^-} \dfrac{-\sin x}{x} & x < 0 \Rightarrow -1 \\ \lim\limits_{x \to 0^+} \dfrac{\sin x}{x} & x \geq 0 \Rightarrow 1 \end{cases}$ Limit DNE

6. $\lim\limits_{x \to 0} \dfrac{x \cdot \sin x}{1 - \cos x} = \lim\limits_{x \to 0} \dfrac{x \cdot \sin x}{1 - \cos x} \cdot \dfrac{(1+\cos x)}{(1+\cos x)} = \lim\limits_{x \to 0} \dfrac{x \cdot \cancel{\sin x} \cdot (1+\cos x)}{\sin^{\cancel{2}} x} = \lim\limits_{x \to 0} \dfrac{1+\cos x}{\left(\frac{\sin x}{x}\right)}$

$= \dfrac{\lim\limits_{x \to 0} (1+\cos x)}{\lim\limits_{x \to 0} \left(\frac{\sin x}{x}\right)} = \dfrac{2}{1} = 2$

7.* $\lim\limits_{x \to \frac{\pi}{4}} \left(x - \dfrac{\pi}{4}\right) \cdot \tan 2x \Rightarrow x - \dfrac{\pi}{4} = a \quad x = a + \dfrac{\pi}{4}$

$\lim\limits_{a \to 0} a \cdot \tan\left(2a + \dfrac{\pi}{2}\right) = \lim\limits_{a \to 0} a \cdot \dfrac{\sin\left(2a + \frac{\pi}{2}\right)}{\cos\left(2a + \frac{\pi}{2}\right)} = \lim\limits_{a \to 0} \dfrac{a \cdot \cos 2a}{-\sin 2a}$

$= \dfrac{\lim\limits_{a \to 0} \cos 2a}{-2 \lim\limits_{a \to 0} \frac{\sin 2a}{2a}} = \dfrac{1}{-2}$

8. $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x^2} = \lim\limits_{x \to 0} \dfrac{(1-\cos x)}{x^2} \dfrac{(1+\cos x)}{(1+\cos x)} = \lim\limits_{x \to 0} \dfrac{\sin^2 x}{x^2 \cdot (1+\cos x)} = \lim\limits_{x \to 0} \left(\dfrac{\sin x}{x}\right)^2 \cdot \lim\limits_{x \to 0} \dfrac{1}{1+\cos x}$

$= 1 \cdot \dfrac{1}{2} = \dfrac{1}{2}$

e.g. $\lim_{x\to 0} \frac{1-\cos x}{\sin^2 x} \Rightarrow \lim_{x\to 0} \frac{(1-\cos x).(1+\cos x)}{\sin^2 x\,(1+\cos x)} = \underbrace{\lim_{x\to 0} \frac{\sin^2 x}{\sin^2 x}}_{1} . \lim_{x\to 0} \frac{1}{1+\cos x}$

$= 1 . \frac{1}{2} = \frac{1}{2}$

e.g. $\lim_{h\to 0} \frac{\cos h - 1}{h}$ $\quad [\cos h = 1 - 2\sin^2(\frac{h}{2})] \Rightarrow \lim_{h\to 0} \frac{1-2\sin^2(\frac{h}{2}) - 1}{h}$

$\lim_{h\to 0} \frac{-2\sin^2(\frac{h}{2})}{h} = - \underbrace{\lim_{h\to 0} \frac{\sin(\frac{h}{2})}{\frac{h}{2}}}_{-1} . \underbrace{\lim_{h\to 0} \sin(\frac{h}{2})}_{0} = 1$

e.g. $\lim_{x\to \pi} \frac{\sin x - \tan x}{\sin x} = \lim_{x\to \pi} 1 - \lim_{x\to \pi} \frac{1}{\cos x} = 1 - (-1) = 2$

e.g. $\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} . \frac{(1+\cos\theta)}{(1+\cos\theta)} = \lim_{\theta\to 0} \frac{\sin^2\theta}{\theta.(1+\cos\theta)}$

$= \lim_{\theta\to 0} \frac{\sin\theta}{\theta} . \frac{\sin\theta}{(1+\cos\theta)} = 1 . \frac{0}{2} = 0$

e.g. $\lim_{n\to \infty} n . \sin(\frac{2}{n}) \Rightarrow \frac{2}{n} = x$ $\quad n\to\infty \Rightarrow x \to 0$ $\quad n = \frac{2}{x}$

$= \lim_{x\to 0} \frac{2}{x} \sin x = 2 . \lim_{x\to 0} \frac{\sin x}{x} = 2.1 = 2$

e.g. $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4} \frac{\cancel{\sqrt{x}-2}}{(\cancel{\sqrt{x}-2})(\sqrt{x}+2)} = \lim_{x\to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{4}$

ε-δ technique

e.g. $f(x) : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x+1$. Show that $\lim_{x\to 2} f(x) = 5$ using the ε-δ technique.

For every $\varepsilon \in \mathbb{R}^+$, there should exist a $\delta \in \mathbb{R}^+$ such that when $|x-2| < \delta$, we have $|f(x) - 5| < \varepsilon$.

$|x-2| < \delta \Rightarrow 2|x-2| < 2\delta$ $\quad |2x-4| < 2\delta$

$|(2x+1) - 5| < \varepsilon$ $\quad |2x-4| < \varepsilon$

if we choose $\delta = \delta(\varepsilon) = \frac{\varepsilon}{2}$, then for any $\varepsilon \in \mathbb{R}^+$, we can find at least one $\delta(\varepsilon)$ such that

$|f(x) - 5| < \varepsilon$

thus: $\lim_{x\to 2} (2x+1) = 5$

e.g. Prove that $\lim_{x \to 0} (x^3+2) = 2$ $\quad \varepsilon \in \mathbb{R}^+, \delta(\varepsilon) \in \mathbb{R}^+$

$|x-0| < \delta \Rightarrow |(x^3+2)-2| < \varepsilon$

$|x| < \delta \qquad |x| < \sqrt[3]{\varepsilon} \qquad \delta = \sqrt[3]{\varepsilon}$

e.g. Prove that $\lim_{x \to 2} x^2 = 4$ $\quad \varepsilon \in \mathbb{R}^+, \delta(\varepsilon) \in \mathbb{R}^+$

$\lim_{x \to x_0} f(x) = L \Leftrightarrow \forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \ni \forall x \in X, 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$

$f(x) = x^2 \quad a = 2 \quad L = 4$

$0 < |x-2| < \delta \Rightarrow |x^2-4| < \varepsilon$

$|x-2| . |x+2| < \varepsilon$

$x \to 2$

$\delta < 1 \qquad |x-2| < \delta < 1$

$|x-2| < 1$

$-1 < x-2 < 1 \qquad 1 < x < 3 \qquad 3 < x+2 < 5$

$-5 < 3 < x+2 < 5 \qquad -5 < x+2 < 5 \qquad |x+2| < 5$

$|x^2-4| = |x-2| . |x+2| < \delta |x+2| < 5\delta$

To ensure $|x^2-4| < \varepsilon$, we need $5\delta = \varepsilon \quad \delta = \frac{\varepsilon}{5} \quad \delta \leq 1$

$\delta : \min\left(1, \frac{\varepsilon}{5}\right)$

e.g. Prove that $\lim_{x \to a} x^2 = a^2$

Let $a \in \mathbb{R}$, if $x \in \mathbb{R}$

$|x-a| < \delta \Rightarrow |x^2 - a^2| < \varepsilon$

$|x-a| < \delta \Rightarrow |x-a| . |x+a| < \varepsilon$

$|x-a| < \delta \leq 1$

$|x| - |a| \leq |x-a| < 1$

$|x| < 1 + |a|$

$|x+a| \leq |x| + |a| < 1 + |a| + |a| = 2|a| + 1$

$|x-a| . |x+a| < |x-a| (2|a|+1)$

For any $\varepsilon > 0$

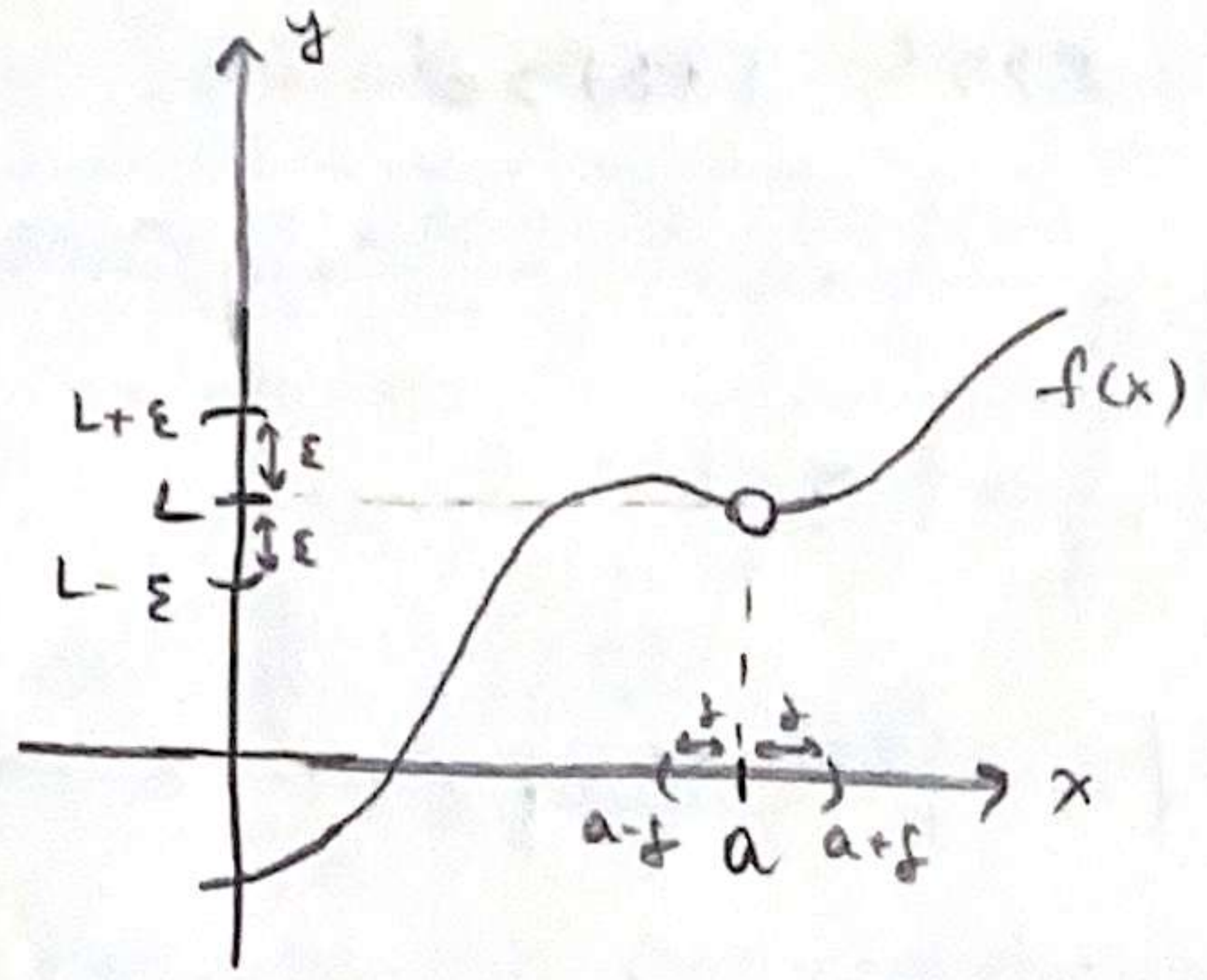
$|x-a| (2|a|+1) < \varepsilon$

$|x-a| < \varepsilon / (2|a|+1)$

for every $a \in \mathbb{R}$ and every $\varepsilon > 0$ it holds that

$|x-a| < \min\{1, \varepsilon/(2|a|+1)\}$ implies

$|x^2 - a^2| < \varepsilon$



$$\lim_{x \to a} f(x) = L$$

if $\forall \varepsilon > 0$, there exists a $\delta > 0$ such that

$0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

e.g. Prove that $\lim_{x \to 3} 4x - 1 = 11$

if $\forall \varepsilon > 0$, there exists a $\delta > 0$ such that

$$\left.\begin{array}{l} 0 < |x-3| < \delta \Rightarrow |(4x-1) - 11| < \varepsilon \\ \qquad |x-3| < \frac{\varepsilon}{4} \end{array}\right\} \text{choose } \delta = \frac{\varepsilon}{4}$$

to show that $\delta = \frac{\varepsilon}{4}$ works!

$|(4x-1) - 11| = |4x - 12| = 4|x-3| < 4\delta$

$|(4x-1) - 11| < 4 \cdot \frac{\varepsilon}{4} \Rightarrow |f(x) - L| < \varepsilon$

e.g. Prove that $\lim_{x \to 1} (2x^2 + x - 1) = 2$

$0 < |x-1| < \delta \Rightarrow |(2x^2 + x - 1) - 2| < \varepsilon$

$|(2x+3)(x-1)| < \varepsilon$

$|2x+3| \cdot |x-1| < \varepsilon$

$0 < |x-1| < \delta < 1$

$|x-1| < 1$

$-1 < x-1 < 1$

$0 < x < 2$, $3 < 2x+3 < 7 \Rightarrow -7 < 3 < 2x+3 < 7$ $|2x+3| < 7$

$|x-1| < \frac{\varepsilon}{7}$

$|x-1| < 1$ $|x-1| < \frac{\varepsilon}{7}$ $\delta = \min\left\{1, \frac{\varepsilon}{7}\right\}$

$|f(x) - 2| \leq 7|x-1| < 7 \cdot \frac{\varepsilon}{7} = \varepsilon$

e.g. Show that $\lim\limits_{x \to c} \sin x = \sin c$ $\qquad \varepsilon > 0 \, , \, \delta(\varepsilon) > 0$

$$\boxed{\sin(a+b) - \sin(a-b) = 2\sin b \sin a}$$

$0 < |x-c| < \delta \Rightarrow |\sin x - \sin c| < \varepsilon$

Let $x = a+b$, $c = a-b$ $\qquad a = \dfrac{x+c}{2} \qquad b = \dfrac{x-c}{2}$

$|\sin x - \sin c| = \left| 2 . \sin \dfrac{x-c}{2} . \cos \dfrac{x+c}{2} \right| \qquad \left| \cos \dfrac{x+c}{2} \right| \leq 1$

$|\sin x - \sin c| \leq 2 . \left| \sin \dfrac{x-c}{2} \right| \qquad \boxed{|\sin u| \leq |u|}$

$\left| \sin \dfrac{x-c}{2} \right| \leq \left| \dfrac{x-c}{2} \right| \qquad |\sin x - \sin c| \leq 2 \left| \dfrac{x-c}{2} \right| = |x-c|$

e.g. Show that $\lim\limits_{x \to 3} (x^4 - 7x - 17) = 43$

$0 < |x-3| < \delta \Rightarrow |f(x) - 43| < \varepsilon$

$$\begin{array}{l|l}
\cancel{x^4} - 7x - 60 & x-3 \\ \hline
-\cancel{x^4} + 3x^3 & x^3 + 3x^2 + 9x + 20 \\ \hline
3\cancel{x^3} - 7x - 60 & \\
-\cancel{3x^3} + 9x^2 & \\ \hline
9\cancel{x^2} - 7x - 60 & \\
-\cancel{9x^2} + 27x & \\ \hline
20\cancel{x} - \cancel{60} & \\
-\cancel{20x} + \cancel{60} & \\ \hline
0 &
\end{array}$$

$|x-3| \, . \, |x^3 + 3x^2 + 9x + 20| < \varepsilon$

$0 < |x-3| < \delta < 1 \qquad |x-3| < 1$

$-1 < x-3 < 1 \; | \; -4 < 2 < x < 4 \qquad |x| < 4$

$|x^3 + 3x^2 + 9x + 20| \leq |x|^3 + 3|x|^2 + 9|x| + 20 \ldots$

$\ldots < |4|^3 + 3|4|^2 + 9 . |4| + 20 = 168$

$168 \, |x-3|$

$168 \, |x-3| < \varepsilon \qquad |x-3| < \dfrac{\varepsilon}{168}$

$\delta = \min \left\{ 1, \dfrac{\varepsilon}{168} \right\} \qquad 0 < |x-3| < \delta \Rightarrow \begin{cases} |x-3| < 1, \\ |x-3| < \dfrac{\varepsilon}{168}, \end{cases}$

$|f(x) - 43| < 168 \, |x-3| < 168 . \dfrac{\varepsilon}{168} = \varepsilon$

e.g. Evaluate the limit $\lim_{x \to -\infty} \frac{(1-x).(2+x)}{(1+2x).(2-3x)} \Rightarrow \frac{(x-1).(x+2)}{(2x+1)(3x-2)}$

$$\frac{x^2+x-2}{6x^2-x-2} \qquad \frac{1+\frac{1}{x}-\frac{2}{x^2}}{6-\frac{1}{x}-\frac{2}{x^2}} \Rightarrow \frac{1+0-0}{6-0-0} = \frac{1}{6}$$

$$\lim_{x \to -\infty} \frac{(1-x).(2+x)}{(1+2x).(2-3x)} = \frac{1}{6}$$

e.g. Evaluate the limit $\lim_{x \to \infty} \frac{x+1}{\sqrt{x^2+1}}$

$$\lim_{x \to \infty} \frac{\cancel{x}\left(1+\frac{1}{x}\right)}{\cancel{x}\sqrt{1+\frac{1}{x^2}}} = \lim_{x \to \infty} \frac{\left(1+\frac{1}{x}\right)}{\sqrt{1+\frac{1}{x^2}}} = \frac{(1+0)}{\sqrt{1+0}} = \frac{1}{1} = 1$$

e.g. $f(x) = \frac{\sqrt{7+9x^2}}{1-2x}$

a) Evaluate $\lim_{x \to -\infty} f(x)$

$x \to -\infty \Rightarrow |x| \to -x$

$$= \frac{|x|\sqrt{\frac{7}{x^2}+9}}{x\left(\frac{1}{x}-2\right)} = \frac{\overset{-1}{\cancel{-x}}}{\cancel{x}} \cdot \frac{\sqrt{\frac{7}{x^2}+9}}{\left(\frac{1}{x}-2\right)}$$

$$\Rightarrow \lim_{x \to -\infty} \frac{-\sqrt{\frac{7}{x^2}+9}}{\left(\frac{1}{x}\right)-2} = \frac{-\sqrt{0+9}}{0-2}$$

$$= \lim_{x \to -\infty} f(x) = \frac{3}{2}$$

b) Evaluate $\lim_{x \to \infty} f(x)$

$x \to \infty \Rightarrow |x| \to x$

$$= \frac{\cancel{x}.\sqrt{\frac{7}{x^2}+9}}{\cancel{x}.\left(\frac{1}{x}-2\right)}$$

$$\Rightarrow \lim_{x \to \infty} \frac{\sqrt{\frac{7}{x^2}+9}}{\left(\frac{1}{x}-2\right)} = \frac{\sqrt{0+9}}{0-2} = -\frac{3}{2}$$

$$= \lim_{x \to \infty} f(x) = -\frac{3}{2}$$

e.g. Evaluate $\lim_{x \to 2^-} (x - \lfloor x \rfloor) = 2-1=1$

e.g. $f : [-1, 5] \to \mathbb{R}$

$$f(x) = \begin{cases} x & \text{if } -1 < x \leq 2 \\ 4 - x & \text{if } 2 < x \leq 5 \end{cases}$$

Evaluate $\lim_{x \to 2} f(x) = 2$

$f(2) = 2$

$\lim_{x \to 2^-} x = 2$

$\lim_{x \to 2^+} 4 - x = 2$

e.g. $\lim_{x \to \infty} \left(\sqrt{x^2 + x + 5} - \sqrt{x^2 + 7}\right) =$

$$\left(\sqrt{x^2+x+5} - \sqrt{x^2+7}\right) = x\left(\sqrt{1 + \frac{1}{x} + \frac{5}{x^2}} - \sqrt{1 + \frac{7}{x^2}}\right)$$

$$x \cdot \frac{\left(\sqrt{x + \frac{1}{x} + \frac{5}{x^2}} - \sqrt{1 + \frac{7}{x^2}}\right) \cdot \left(\sqrt{1 + \frac{1}{x} + \frac{5}{x^2}} + \sqrt{1 + \frac{7}{x^2}}\right)}{\left(\sqrt{1 + \frac{1}{x} + \frac{5}{x^2}} + \sqrt{1 + \frac{7}{x^2}}\right)}$$

$$= \frac{x\left(1 + \frac{1}{x} + \frac{5}{x^2} - 1 - \frac{7}{x^2}\right)}{\sqrt{1 + \frac{1}{x} + \frac{5}{x^2}} + \sqrt{1 + \frac{7}{x^2}}} = \frac{x \cdot \left(\frac{1}{x} - \frac{2}{x^2}\right)}{\sqrt{1 + \frac{1}{x} + \frac{5}{x^2}} + \sqrt{1 + \frac{7}{x^2}}}$$

$$= \lim_{x \to \infty} \frac{\left(1 - \frac{2}{x}\right)}{\sqrt{1 + \frac{1}{x} + \frac{5}{x^2}} + \sqrt{1 + \frac{7}{x^2}}} = \frac{(1 - 0)}{\sqrt{1 + 0 + 0} + \sqrt{1 + 0}} = \frac{1}{1 + 1} = \frac{1}{2}$$

e.g. $\lim_{x \to -\infty} \frac{\sqrt{x^2 + |x|}}{|x|} = ?$

$x \to -\infty \Rightarrow |x| \to -x$

$$\lim_{x \to -\infty} \frac{\sqrt{x^2 - x}}{-x} = \frac{\cancel{-x}\sqrt{1 - \frac{1}{x}}}{\cancel{-x}} = \sqrt{1 - 0} = 1$$

$|x|\sqrt{1 - \frac{1}{x}}$

eg. Show that $(a \in \mathbb{R}^+)$

$$\lim_{x\to\infty} \sqrt{ax^2+bx+c} = \sqrt{a} \lim_{x\to\infty} \left(x+\frac{b}{2a}\right) \text{ is true.}$$

$$\sqrt{a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)} = \sqrt{a \cdot \left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}}$$

$$\sqrt{\lim_{x\to\infty} ax^2+k} = \sqrt{\lim_{x\to\infty} ax^2}$$

so, $\lim_{x\to\infty} \sqrt{ax^2+bx+c} = \lim_{x\to\infty} \sqrt{a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}} \Rightarrow \lim_{x\to\infty} \sqrt{a\left(x+\frac{b}{2a}\right)^2}$

$$= \sqrt{a} \cdot \lim_{x\to\infty} \sqrt{\left(x+\frac{b}{2a}\right)^2} = \sqrt{a} \lim_{x\to\infty} \left|x+\frac{b}{2a}\right| \overset{x\to\infty}{=\!=} \sqrt{a} \lim_{x\to\infty} \left(x+\frac{b}{2a}\right)$$

$$e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \quad \Big| \quad e = \lim_{x\to 0} (1+x)^{\frac{1}{x}}$$

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = L \qquad n \lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right) = \ln L$$

$$\frac{1}{n} = u \qquad \frac{1}{u} \cdot \lim_{u\to 0} \ln(1+u) = \ln L$$

$n\to\infty \Rightarrow u\to 0$

$$\lim_{u\to 0} \underbrace{\left(\frac{\ln(1+u)}{u}\right)}_{1} = \ln L \qquad 1 = \ln L \qquad e = L$$

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eg. $\lim_{x\to 1^+} x^{\frac{1}{x-1}}$ $\qquad x = 1+(x-1) \qquad \lim_{x\to 1^+} (1+(x-1))^{\frac{1}{x-1}}$

$$\Rightarrow \lim_{x\to 1^+} \left(1+\frac{1}{\left(\frac{1}{x-1}\right)}\right)^{\frac{1}{x-1}} \qquad \text{Let } \frac{1}{x-1} = u$$

$$= \lim_{u\to\infty} \left(1+\frac{1}{u}\right)^u = e \qquad \lim_{x\to 1^+} x^{\frac{1}{x-1}} = e$$

e.g. $\lim_{x\to 2^+} (x-1)^{\frac{1}{x-2}}$ $\qquad x-1 = 1+(x-2)$

$$\Rightarrow \lim_{x\to 2^+} (1+(x-2))^{\frac{1}{x-2}} = \lim_{x\to 2^+} \left(1+\frac{1}{\frac{1}{x-2}}\right)^{\frac{1}{x-2}} \qquad \text{let } \frac{1}{x-2} = u$$

$x\to 2^+ \Rightarrow u\to\infty$

$$= \underbrace{\lim_{u\to\infty} \left(1+\frac{1}{u}\right)^{u}}_{e} \Rightarrow \lim_{x\to 2^+} (x-1)^{\frac{1}{x-2}} = e = \lim_{x\to 2^-} (x-1)^{\frac{1}{x-2}} = e$$

$x\to 2^- \Rightarrow u\to -\infty$

---

e.g. $\lim_{x\to 1^+} (2-x)^{\frac{1}{x-1}}$ $\qquad$ Let $(2-x) = 1+(1-x)$

$$\Rightarrow \lim_{x\to 1^+} \left(1+(1-x)\right)^{\frac{1}{x-1}} \qquad \boxed{\text{Let } 1-x=u}$$

$x\to 1^+ \Rightarrow u\to 0^-$

$$= \lim_{u\to 0^-} (1+u)^{\frac{-1}{u}} \Rightarrow \frac{1}{\lim_{u\to 0} (1+u)^{\frac{1}{u}}} = e^{-1}$$

$$\Rightarrow \lim_{u\to 0^-} \left((1+u)^{\frac{1}{u}}\right)^{-1} = e^{-1}$$

---

e.g. Evaluate the limit $\lim_{x\to 1^-} \frac{\ln(2-x)}{1-x}$ $\qquad$ let $(2-x) = 1+(1-x)$

$$\Rightarrow \lim_{x\to 1^-} \frac{\ln(1+(1-x))}{1-x} = \lim_{x\to 1^-} \frac{1}{1-x} \ln(1+(1-x))$$

$$\Rightarrow \ln\left(\lim_{x\to 1^-} (1+(1-x))^{\frac{1}{1-x}}\right) \qquad \text{Let } 1-x=u$$

$x\to 1^- \Rightarrow u\to 0^+$

$$= \ln\Big(\underbrace{\lim_{u\to 0^+} (1+u)^{\frac{1}{u}}}_{e}\Big) = \ln e = 1$$

eg. $\lim_{x\to\infty}\left(\frac{x+4}{x-1}\right)^{x+4}$ $\qquad \frac{x+4}{x-1} = 1+\frac{x+5}{x-1}$

$\Rightarrow \lim_{x\to\infty}\left(1+\frac{5}{x-1}\right)^{x+4\cdot\left(\frac{(x-1)}{(x-1)}\right)} \Rightarrow \left[\lim_{x\to\infty}\left(1+\frac{5}{x-1}\right)^{(x-1)}\right]^{\left(\frac{(x+4)}{(x-1)}\right)}$

Let $x-1=u \quad x=u+1$

$\Rightarrow \underbrace{\left[\lim_{u\to\infty}\left(1+\frac{5}{u}\right)^{u}\right]}_{e^5}^{\frac{u+5}{u}} = \left(e^5\right)^{\lim_{u\to\infty}\frac{u+5}{u}} = \left(e^5\right)^1 = e^5$

eg. $\lim_{x\to 2}\frac{e^{x-2}-1}{x-2}$ $\quad$ Let $x-2=t$ $\quad x\to 2 \Rightarrow t\to 0$ $\quad \lim_{t\to 0}\frac{e^t-1}{t}$

$\Rightarrow$ Let $e^t-1=u \quad e^t=u+1 \quad t=\ln(u+1)$

$t\to 0 \Rightarrow u\to 0$

$\Rightarrow \lim_{u\to 0}\frac{u}{\ln(u+1)} = \lim_{u\to 0}\frac{1}{\frac{\ln(u+1)}{u}} = \lim_{u\to 0}\frac{1}{\ln(u+1)\cdot\frac{1}{u}}$

$= \frac{1}{\ln\left(\left(\lim_{u\to 0}(u+1)\right)^{\frac{1}{u}}\right)} = \frac{1}{\ln(e)} = \frac{1}{1} = 1$

eg. Evaluate $\lim_{x\to 0}\tan\left(\frac{\sin 4x}{\pi x}\right)$ $\qquad \tan\left(\underbrace{\lim_{x\to 0}\left(\frac{\sin 4x}{\pi x}\right)}_{\frac{4}{\pi}}\right)$

$\Rightarrow \tan\left(\frac{4}{\pi}\right)$

eg. Evaluate $\lim_{x\to 1}\frac{x}{\sqrt{2x+1}-\sqrt{3}}$ $\qquad$ Step 1: $\frac{\lim_{x\to 1} x}{\lim_{x\to 1}\sqrt{2x+1}-\sqrt{3}} = \frac{1}{0}$

Step 2: $x\to 1^+$

$\lim_{x\to 1^+}\frac{x}{\sqrt{2x+1}-\sqrt{3}} \Rightarrow \frac{1}{0^+} \to +\infty$

Step 3: $x\to 1^-$

$\lim_{x\to 1^-}\frac{x}{\sqrt{2x+1}-\sqrt{3}} \Rightarrow \frac{1}{0^-} \to -\infty$

$\lim_{x\to 1}\frac{x}{\sqrt{2x+1}-\sqrt{3}} = \text{D.N.E.}$

eg. Evaluate $\lim_{x \to 0} \cos\left(\frac{\pi - \pi \cos^2 x}{x^2}\right)$

$$\Rightarrow \lim_{x \to 0} \cos\left(\frac{\pi(1-\cos^2 x)}{x^2}\right) = \lim_{x \to 0} \cos\left(\frac{\pi \cdot \sin^2 x}{x^2}\right)$$

$$= \underbrace{\cos \pi}_{-1} \cdot \underbrace{\lim_{x \to 0}\left(\frac{\sin^2 x}{x}\right)}_{1} = -1$$

eg. Evaluate $\lim_{x \to 0} \frac{\tan(2x)}{x}$ $\qquad \lim_{x \to 0} \frac{\sin 2x}{x \cdot \cos 2x}$

$$\Rightarrow \underbrace{\lim_{x \to 0} \frac{\sin 2x}{2x}}_{1} \cdot \underbrace{\lim_{x \to 0} \frac{2}{\cos 2x}}_{2} = 2$$

eg. $\lim_{x \to \pi} \sec(1 + \cos x) = \lim_{x \to \pi} \sec(0) \Rightarrow \frac{1}{\cos(0)} = 1$

eg. Calculate $\lim_{x \to 1} \frac{\sqrt[3]{x} - 1}{x - 1}$ $\qquad x - 1 = (\sqrt[3]{x})^3 - 1^3 = (\sqrt[3]{x} - 1)(\sqrt[3]{x^2} + \sqrt[3]{x} + 1)$

$$\Rightarrow \lim_{x \to 1} \frac{\cancel{(\sqrt[3]{x} - 1)}}{\cancel{(\sqrt[3]{x} - 1)}(\sqrt[3]{x^2} + \sqrt[3]{x} + 1)} = \lim_{x \to 1} \frac{1}{\sqrt[3]{x^2} + \sqrt[3]{x} + 1} = \frac{1}{1 + 1 + 1} = \frac{1}{3}$$

eg. $\lim_{x \to \infty} \left(\sqrt{x^2 + 1} - \sqrt{x^2 - 1}\right)$

$$\Rightarrow \lim_{x \to \infty} \left(\sqrt{x^2+1} - \sqrt{x^2-1}\right) \cdot \frac{\left(\sqrt{x^2+1} + \sqrt{x^2-1}\right)}{\left(\sqrt{x^2+1} + \sqrt{x^2-1}\right)}$$

we just want to get rid of $\infty - \infty$

$$= \lim_{x \to \infty} \frac{\cancel{x^2} + 1 - \cancel{x^2} + 1}{\left(\sqrt{x^2+1} + \sqrt{x^2-1}\right)} \Rightarrow \frac{\lim_{x \to \infty} 2}{\lim_{x \to \infty}\left(\sqrt{x^2+1} + \sqrt{x^2-1}\right)} \sim \frac{2}{\infty + \infty} = \frac{2}{\infty} = 0$$

$$\lim_{x \to \infty} \left(\sqrt{x^2+1} - \sqrt{x^2-1}\right) = 0$$

Notes:

- For limits, x_0 does not necessarily need to be in the domain of f. The behavior of $f(x)$ near x_0 is sufficient to define the limit.

Limit definition:

$$\lim_{x \to x_0} f(x) = L \Leftrightarrow \forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \; (\delta(\varepsilon) \in \mathbb{R}^+) \ni \forall x \in X,$$

$$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Continuity definition:

f is continuous at $x_0 \Leftrightarrow \forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \; (\delta(\varepsilon) \in \mathbb{R}^+) \ni \forall x \in X,$

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

The key difference between the limit and continuity definitions is the inclusion of $x = x_0$ (so, we have $0 \leq |x - x_0|$) in the continuity condition. In the continuity formula, L in the limit definition is replaced with $f(x_0)$, ensuring $\lim_{x \to x_0} f(x) = f(x_0)$.

e.g. Show that $f(x) = 2x + 6$ is continuous at $x = 4$ by using ε-δ (epsilon-delta) technique.

$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

$f(4) = 2 \cdot 4 + 6 = 14$

$|x - 4| < \delta \Rightarrow |(2x + 6) - 14| < \varepsilon$

$|2x - 8| < \varepsilon$

$2|x - 4| < \varepsilon \qquad |x - 4| < \frac{\varepsilon}{2} \; \delta(\varepsilon)$

$|x - 4| < \frac{\varepsilon}{2} \Rightarrow 2|x - 4| < \varepsilon \Rightarrow |(2x + 6) - 14| < \varepsilon$

eg. $f(x) = 2x^2+1$ for $x \in \mathbb{R}$. Prove f is continuous on $\mathbb{R}$ by using definition

a) $\lim_{x \to a} f(x) = f(a)$ $\qquad 2a^2+1 = 2a^2+1$

b) epsilon-delta (ε-δ)

$$|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$$

$$2x^2+1 - (2a^2+1)$$

$$2|(x^2-a^2)| < \varepsilon$$

$$2.|x-a||x+a| < \varepsilon$$

$$|x| - |a| < |x-a| < 1$$

$$|x| < 1 + |a|$$

$$|x+a| \leq |x| + |a| < 1 + |a| + |a|$$

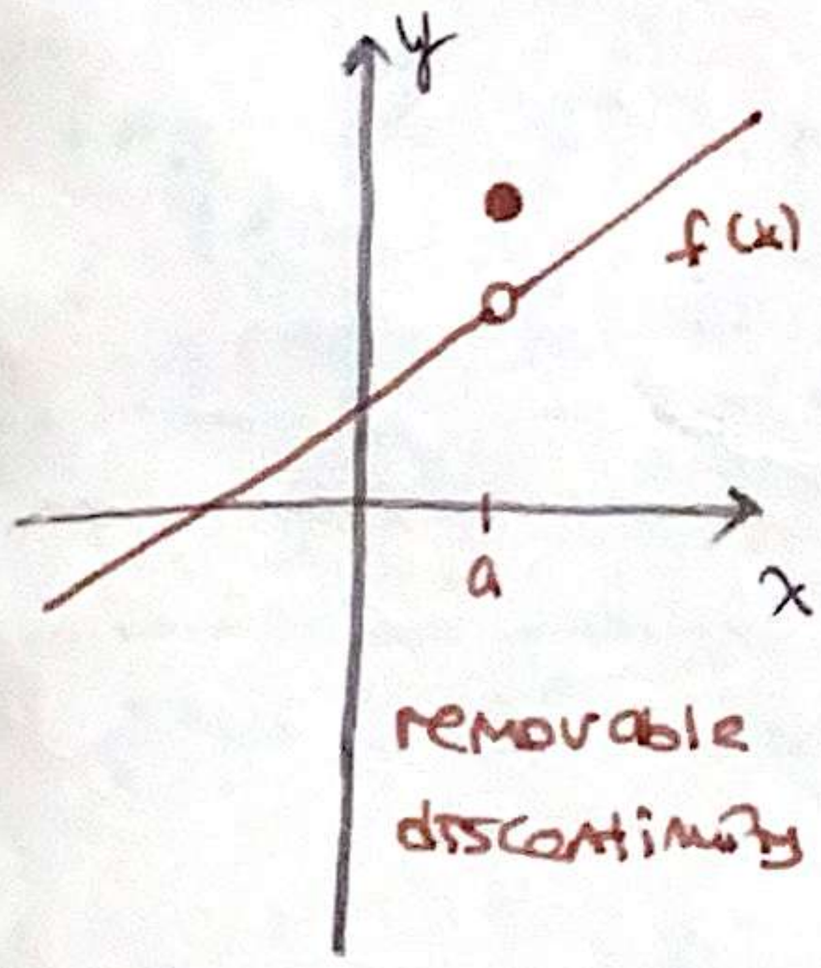
$$|x+a| < 2|a|+1$$

$$2.|x-a|.|2|a|+1| < \varepsilon$$

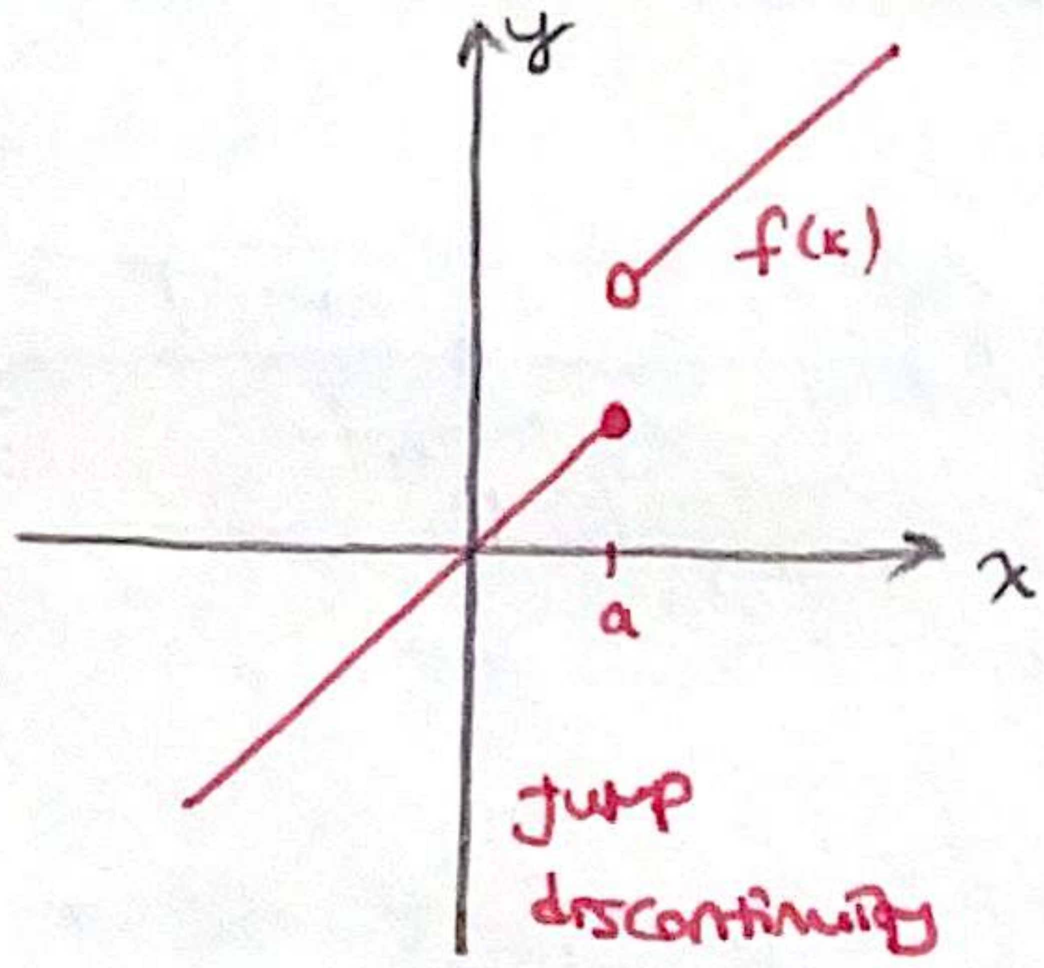
$$|x-a| < \frac{\varepsilon}{2.(2|a|+1)} \qquad \delta : \min\left\{1, \ \varepsilon/2(2|a|+1)\right\}$$

Note: If g is continuous at the point b and $\lim_{x \to c} f(x) = b$, then

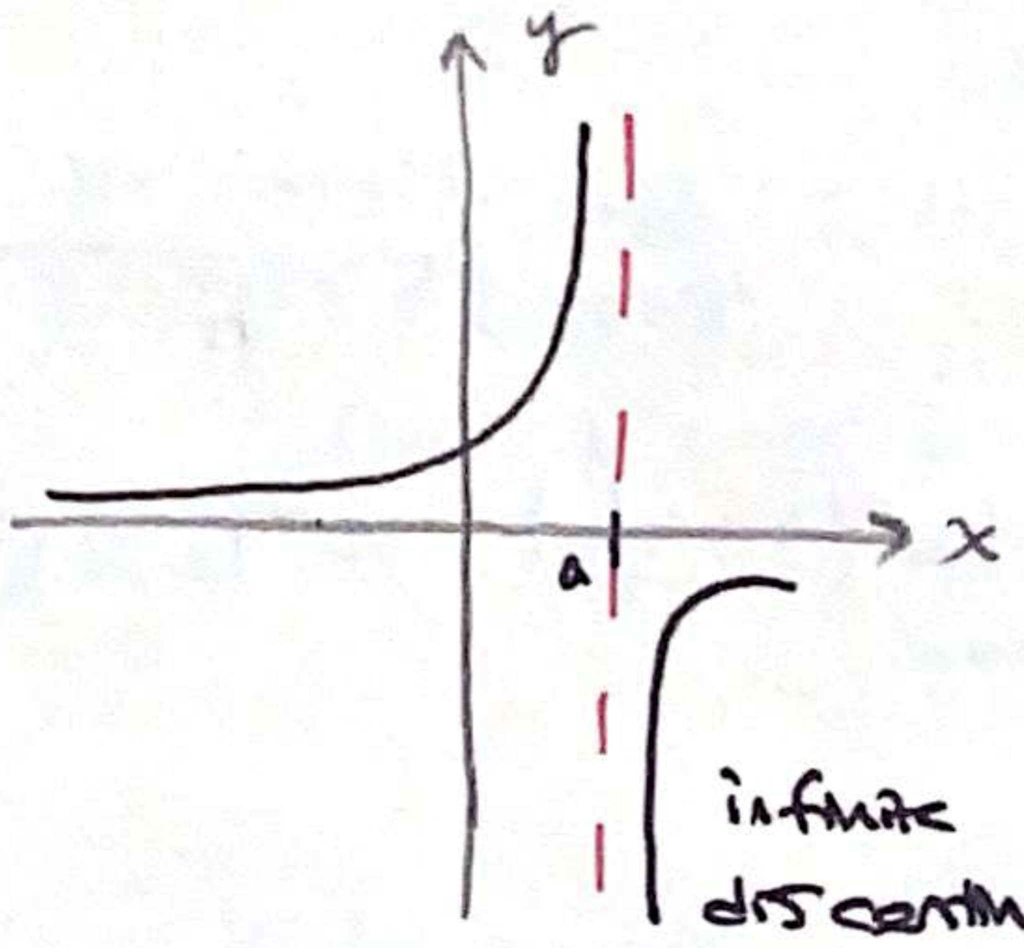
$$\lim_{x \to c} g(f(x)) = g(b) = g\left(\lim_{x \to c} f(x)\right)$$



removable discontinuity



jump discontinuity



infinite discontinuity

Indeterminent forms & L'Hospital

1)	2)	3)	4)	5)
$\frac{0}{0}$	$\frac{\infty}{\infty}$	$\infty - \infty$	$0 \cdot \infty$	$1^{\infty} / 0^{0} / \infty^{0}$

$\underbrace{1), 2)}_{\text{l'hospital (directly)}}$ $\underbrace{3), 4), 5)}_{\text{make them similar to the first two (indirectly)}}$

1) $\lim_{x \to 3} \left(\frac{x^3 - 27}{2x - 6} \right) \Rightarrow \frac{3x^2}{2} = \frac{27}{2}$ $\qquad \frac{\cancel{(x-3)}(x^2+3x+9)}{2\cancel{(x-3)}}$

optional.

2) $\bullet \lim_{x \to +\infty} \left(\frac{x^2 - 3x + 4}{2x^2 - 5x + 1} \right) \Rightarrow \frac{2x - 3}{4x - 5} = \frac{2 - \cancel{\frac{3}{x}}}{4 - \cancel{\frac{5}{x}}} = \frac{1}{2}$

$= \frac{2}{4} = \frac{1}{2}$

$\bullet \lim_{x \to +\infty} \left(\frac{\ln x}{x} \right) \Rightarrow \frac{\frac{1}{x}}{1} \Rightarrow \lim_{x \to \infty} \frac{1}{x} = \frac{1}{\infty} = 0$

$\bullet \lim_{x \to -\infty} \left(\frac{x^3 - 4x^2 + 5x - 7}{x^2 + 6x - 1} \right) \Rightarrow \frac{3x^2 - 8x + 5}{2x + 6} \Rightarrow \frac{6x - 8}{2} = \frac{-\infty}{2} = -\infty$

3) $\bullet \lim_{x \to 2} \frac{4}{x^2 - 4} - \frac{1}{x - 2} \Rightarrow \frac{2 - x}{x^2 - 4} \Rightarrow \frac{-1}{2x} = -\frac{1}{4}$

$\bullet \lim_{x \to 0} \frac{1}{\sin x} - \frac{1}{x} \Rightarrow \frac{x - \sin x}{x \cdot \sin x} = \frac{1 - \cos x}{1 \cdot \sin x + x \cos x}$

$\Rightarrow \frac{\sin x}{\cos x + \cos x - x \sin x} = \frac{\sin x}{2\cos x - x \sin x} = \frac{0}{2} = 0$

$\infty - \infty = \frac{\text{num.}}{0} - \frac{\text{num2.}}{0}$

4) $0 \cdot \infty \Rightarrow \dfrac{\infty}{\frac{1}{0}} = \dfrac{\infty}{\infty}$ $\quad 0 \cdot \infty \Rightarrow \dfrac{0}{\frac{1}{\infty}} = \dfrac{0}{0}$

• $\lim\limits_{x \to 0^+} x \ln x \Rightarrow \dfrac{\ln x}{\frac{1}{x}} \left[\dfrac{\infty}{\infty}\right] \Rightarrow \dfrac{\frac{1}{x}}{-\frac{1}{x^2}} \Rightarrow \lim\limits_{x \to 0^+} (-x) = 0$

• $\lim\limits_{x \to \frac{\pi}{2}} \left(x - \frac{\pi}{2}\right) \cdot \tan x \Rightarrow \dfrac{\left(x - \frac{\pi}{2}\right)}{\underbrace{\frac{1}{\tan x}}_{} \cot x} \left[\dfrac{0}{0}\right] \Rightarrow \dfrac{1}{-\csc^2 x} = \dfrac{-1}{\frac{1}{\sin^2 x}} = -1$

5) • $\lim\limits_{x \to 0^+} (x+1)^{\frac{1}{x}} = L$

$e^{\left(\lim\limits_{x \to 0^+}\left[\frac{1}{x} \cdot \ln(x+1)\right]\right)} = L \Rightarrow L = \exp\left(\dfrac{[\ln(x+1)]'}{x'}\right) \Rightarrow \dfrac{\frac{1}{x+1}}{1}$

$\frac{0}{0}$

$L = e^1 = e$

• $\lim\limits_{x \to \infty} \left(\dfrac{x+1}{x-1}\right)^x = M$ $\quad e^{\left(\lim\limits_{x \to \infty}\left[\dfrac{\ln\left(\frac{x+1}{x-1}\right)}{\left(\frac{1}{x}\right)}\right]\right)} = M$

$\frac{\infty}{\infty}$

$e^{\left(\lim\limits_{x \to \infty}\left[\dfrac{\left(\cancel{\frac{x-1}{x+1}}\right) \cdot \frac{(x-1) - (x+1)}{(x-1)^2}}{+\frac{1}{x^2}}\right]\right)} = M \quad \Bigg| \quad e^{\left(\lim\limits_{x \to \infty}\left[\dfrac{2x^2}{x^2-1}\right]\right)} = M$

$e^{\left(\lim\limits_{x \to \infty} \frac{4x}{2x}\right)} = e^2 \qquad M = e^2$

eg.) $\lim\limits_{x\to 0^+} x.(\ln x)^2 \to 0.\infty$ $\quad \lim\limits_{x\to 0^+}\left(\dfrac{[\ln(x)]^2}{\frac{1}{x}}\right) \to \dfrac{\infty}{\infty}$

$$\lim_{x\to 0^+} \frac{2.\ln(x).\cancel{\frac{1}{x}}}{-\frac{1}{\cancel{x}x}} \Rightarrow -2\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \to \frac{\infty}{\infty}$$

$$\frac{0}{0}$$

$$-2\lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \Rightarrow 2\lim_{x\to 0^+} \frac{1}{\frac{1}{x}} = 2.\lim_{x\to 0^+} x = 0$$

eg.) $\lim\limits_{x\to 0^+} (1+2x)^{\frac{1}{3x}} = L$ $\to 1^\infty$

$$\exp\left[\lim_{x\to 0^+}\left(\frac{1}{3x}.\ln(1+2x)\right)\right] = L \to \frac{0}{0} \qquad \exp.\left[\lim_{x\to 0^+} \frac{\frac{1}{1+2x}.2}{3}\right] = L = \sqrt[3]{e^2}$$

eg.) $\lim\limits_{x\to +\infty}\left(1+\frac{1}{x}\right)^x = L \to 1^\infty$ $\qquad \exp.\left[\lim\limits_{x\to +\infty} x.\ln\left(1+\frac{1}{x}\right)\right] = L$ $\to \infty.0$

$$\frac{1}{\frac{1}{\infty}}.0$$

$$\exp\left(\lim_{x\to +\infty}\left[\frac{\ln\left(1+\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}\right]\right) \to \frac{0}{0}$$

$$\Rightarrow \exp\left(\lim_{x\to +\infty} \frac{\frac{1}{1+\frac{1}{x}}.\frac{-1}{x^2}}{\frac{-1}{x^2}}\right) \Rightarrow L = e$$

e.g.) $\lim\limits_{x\to+\infty}\left(\frac{3x+4}{3x-1}\right)^{2x+5} = M \longrightarrow \sqrt[3]{e^{10}}$

$\longrightarrow 1^{\infty}$

$$\exp\left(\lim_{x\to+\infty}\left[(2x+5)\cdot\ln\left(\frac{3x+4}{3x-1}\right)\right]\right) = M$$

$\to \infty \cdot 0 = \frac{0}{\frac{1}{\infty}} = \frac{0}{0}$

$$\exp\left[\lim_{x\to+\infty}\left(\frac{\ln\left(\frac{3x+4}{3x-1}\right)}{\left(\frac{1}{2x+5}\right)}\right)\right] = M$$

$\to \frac{0}{0}$

$$\underbrace{\exp\left[\lim_{x\to+\infty}\left(\frac{\left[\ln\left(\frac{3x+4}{3x-1}\right)\right]'}{\left(\frac{1}{2x+5}\right)'}\right)\right] = M}_{\text{l'hospital}}$$

$$\exp\left(\lim_{x\to+\infty}\left[\frac{\frac{\cancel{3x-1}}{3x+4}\cdot\frac{\cancel{3(3x-1)-3(3x+4)}}{(3x-1)^{\cancel{2}}}}{\frac{+2}{(2x+5)^2}}\right]\right) = M$$

$\cancel{3(3x-1-3x-4)}\ {+15}$

$$\exp\left[\lim_{x\to+\infty}\left(\frac{60x^2+300x+375}{18x^2+18x-8}\right)\right] = \exp\left(\lim_{x\to+\infty}\frac{120x+300}{36x+18}\right) = \exp\left[\lim_{x\to+\infty}\left(\frac{120}{36}\right)\right]$$

e.g.) $\lim\limits_{x\to+\infty}\left(\cos\left(\frac{1}{x}\right)\right)^{x} = L$ $\quad \exp\left[\lim\limits_{x\to+\infty}\left(x\cdot\ln\left[\cos\left(\frac{1}{x}\right)\right]\right)\right] = L$

$\to \infty\cdot 0 \Rightarrow \frac{0}{\frac{1}{\infty}} = \frac{0}{0}$

$$= \exp\left(\lim_{x\to+\infty}\left[\frac{\ln\left(\cos\left(\frac{1}{x}\right)\right)}{\left(\frac{1}{x}\right)}\right]\right) \Rightarrow \frac{\sec\left(\frac{1}{x}\right)\cdot -\sin\left(\frac{1}{x}\right)\cdot \cancel{-\frac{1}{x^2}}}{\cancel{-\frac{1}{x^2}}}$$

$$= \exp\left[\lim_{x\to+\infty}\left(\sec\left(\frac{1}{x}\right)\cdot -\sin\left(\frac{1}{x}\right)\right)\right] = 1$$

e.g.) $\lim_{x \to +\infty} \left[x \cdot \arctan\left(\frac{2}{x}\right) \right]$

$\hookrightarrow \infty \cdot 0 \Rightarrow \frac{0}{\frac{1}{\infty}} = \frac{0}{0}$

$$\lim_{x \to +\infty} \left[\frac{\arctan\left(\frac{2}{x}\right)}{\left(\frac{1}{x}\right)} \right] \Rightarrow \frac{\frac{1}{\sec^2\left(\arctan\left(\frac{2}{x}\right)\right)} \cdot \frac{+2}{\cancel{x^2}}}{+\cancel{\frac{1}{x^2}}}$$

$\hookrightarrow \frac{0}{0}$

---

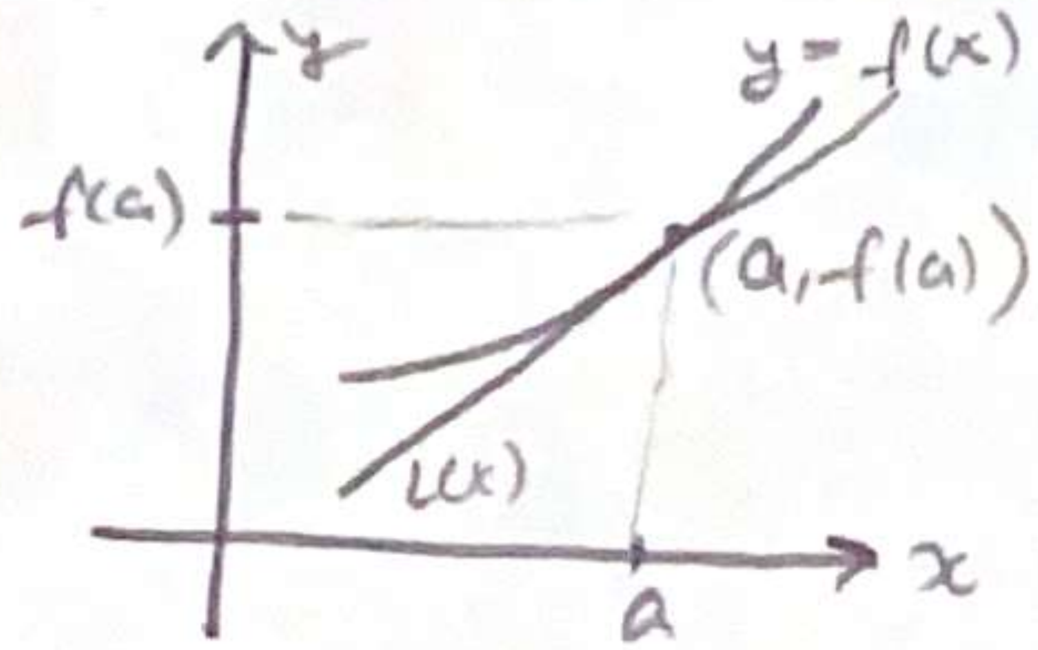
let $f(x) = \tan x$ $\quad \frac{d}{dx} f(f^{-1}(x)) = \frac{d}{dx} x \Rightarrow (f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$

$$(\arctan x)' = \frac{1}{\sec^2(\arctan(x))}$$

$$\lim_{x \to +\infty} \left[\frac{2}{\sec^2\left(\arctan\left(\frac{2}{x}\right)\right)} \right] = 2$$

$\frac{\pi}{4}$, $\frac{\pi}{4}$ the domain $\boxed{c = \frac{\pi}{4}}$

Linearization



$y' = f'(x)$

$L(x) - f(a) = f'(a)(x-a)$

e.g) $f(x) = \sqrt{1+x}$ find the linearization of the function at $x=0$

$(0,1)$ $f'(0) = \frac{1}{2}$ $L(x) - f(0) = f'(0)(x-0)$

$f'(x) = \dfrac{1}{2\sqrt{1+x}}$

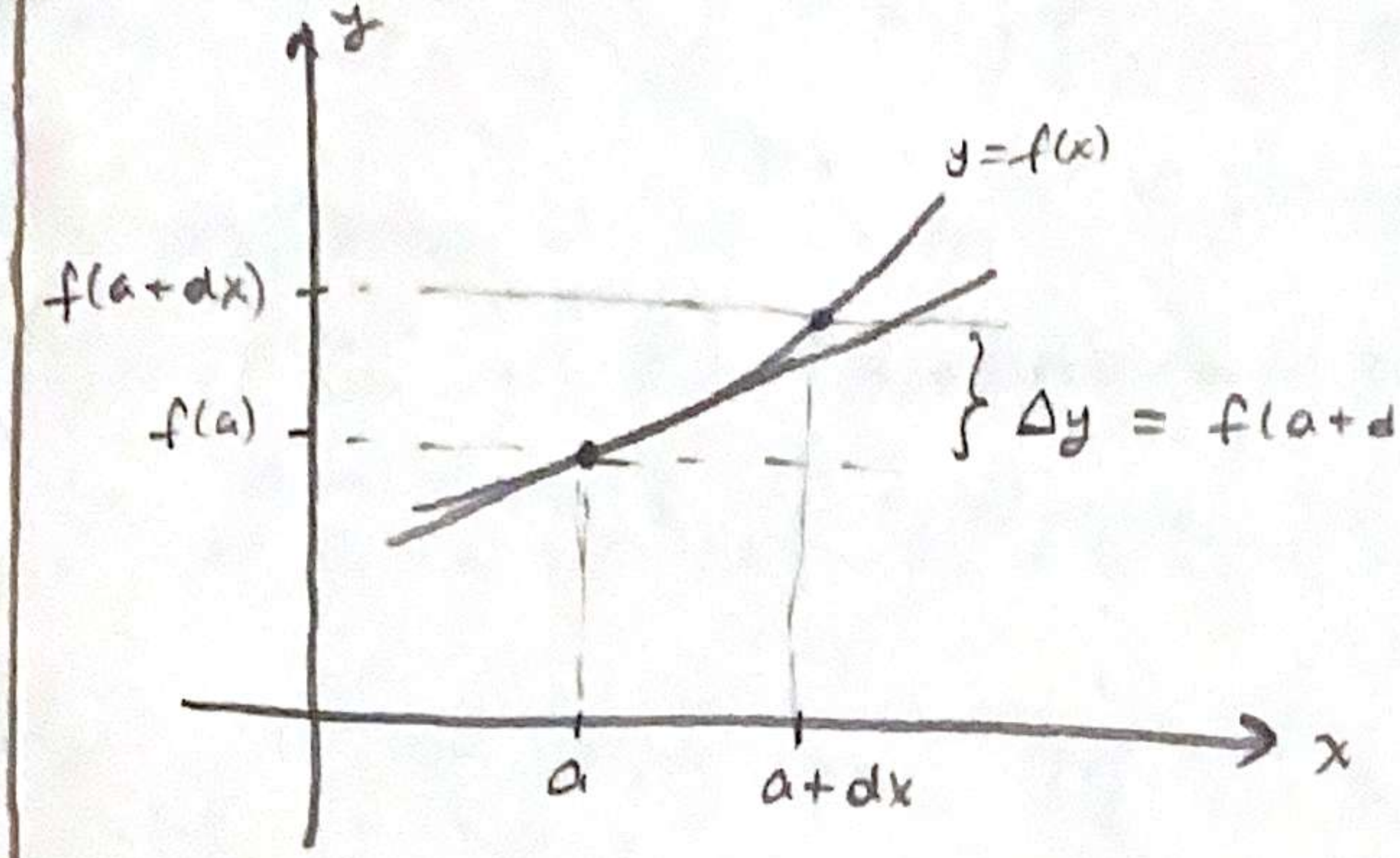
① $L(x) = \frac{1}{2}x + 1$

at $x = 3$ $(3,2)$ $f'(3) = \frac{1}{4}$ $L(x) = \frac{1}{4}x - \frac{3}{4}$

② $L(x) = \frac{1}{2} \cdot 0.2 + 1 = 1.1$

real

④ $\sqrt{1.2} \approx \sqrt{1+0.2} = 1 + \frac{1}{2} \cdot 0.2 = 1.10$ $1.0954\ldots$

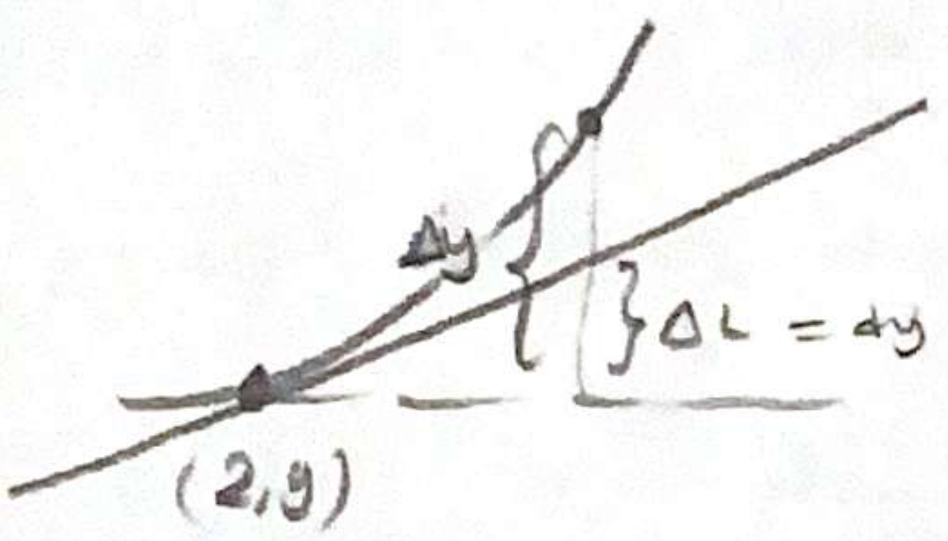


$$\Delta L = L(a+dx) - L(a)$$

$$= f(a) + f'(a)((a+dx)-a) - f(a)$$

$$\Delta L = f'(a)\,dx$$

e.g.) $y = f(x) = x^3 + x^2 - 2x + 1$ 2 to 2.05



$f(2) = 9$ $f(2.05) = 9.717625$

$\Delta y = f(2.05) - f(2) = 0.717625$

$dy = f'(x)\,.dx$

$dy = (3\underset{2}{x}^2 + 2x - 2)\underset{0.05}{dx}$

$dy = 0.7$

e.g.) $f(x) = \sqrt{x+3}$ $\underset{x=0.98}{3.98}$ $\underset{x=1.05}{4.05}$

$f'(x) = \dfrac{1}{2\sqrt{x+3}}$ $f'(1) = \dfrac{1}{2.2} = \dfrac{1}{4}$ $f(1) = 2$ $L(x) = 2 + \frac{1}{4}(x-1)$

$L(x) = \frac{1}{4}x + \frac{7}{4}$

$\sqrt{3.98} = \sqrt{3+0.98} \approx \frac{1}{4}\cdot 0.98 + \frac{7}{4} = 1.995$

$\sqrt{4.05} = \sqrt{3+1.05} \approx \frac{1}{4}\cdot 1.05 + \frac{7}{4} = 2.0125$

Sigma notation

e.g.) $$\sum_{m=1}^{n} 1 = \underbrace{1+1 \cdots +1}_{n \text{ terms}}$$

$f(i) = a_i$

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_n$$

Infinite series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 \ldots$

$$\sum_{i=m}^{n} (A f(i) + B g(i)) = A \sum_{i=m}^{n} f(i) + B \sum_{i=m}^{n} g(i)$$

$$\sum_{j=m}^{m+n} f(j) = \sum_{i=0}^{n} f(i+m)$$

e.g.) $\sum_{j=3}^{17} \sqrt{1+j^2} = \sum_{i=1} f(i)$ $\quad j = i+2$ $\quad \boxed{\sum_{i=1}^{15} \sqrt{1+(i+2)^2}}$

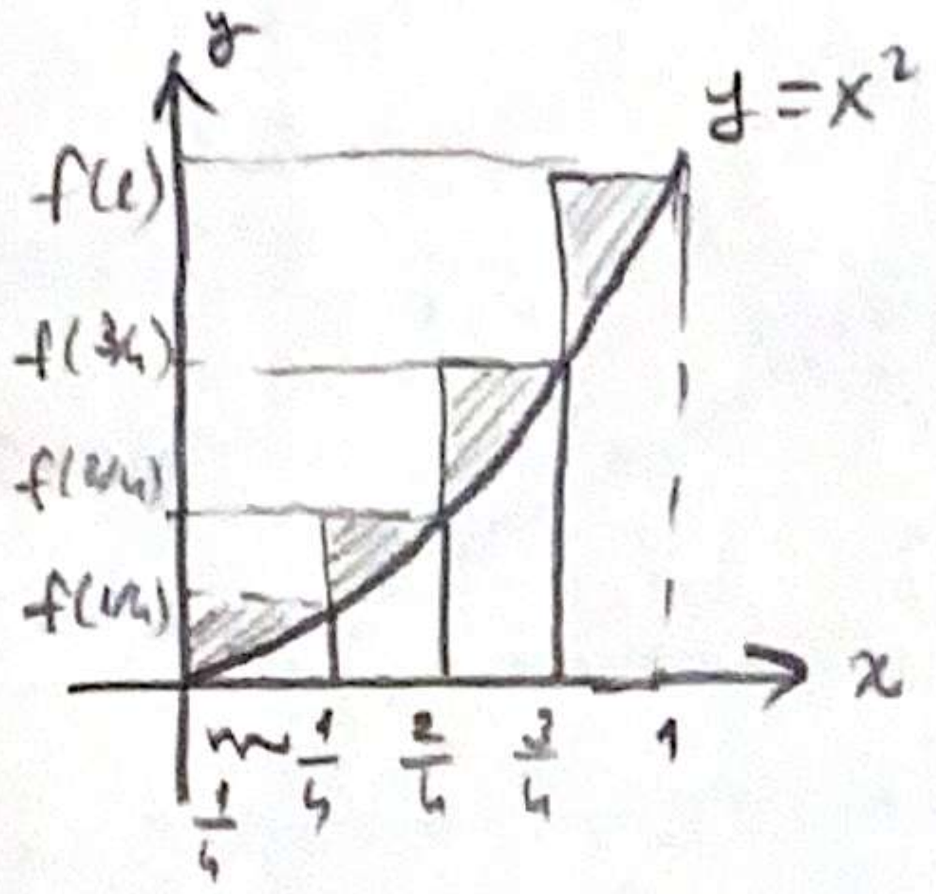
Evaluating sums

$$S = \sum_{i=1}^{n} i = 1 + 2 + \ldots + n = \frac{n \cdot (n+1)}{2}$$

e.g.) $\sum_{k=m+1}^{n} (6k^2 - 4k + 3)$ when $1 \leq m < n$

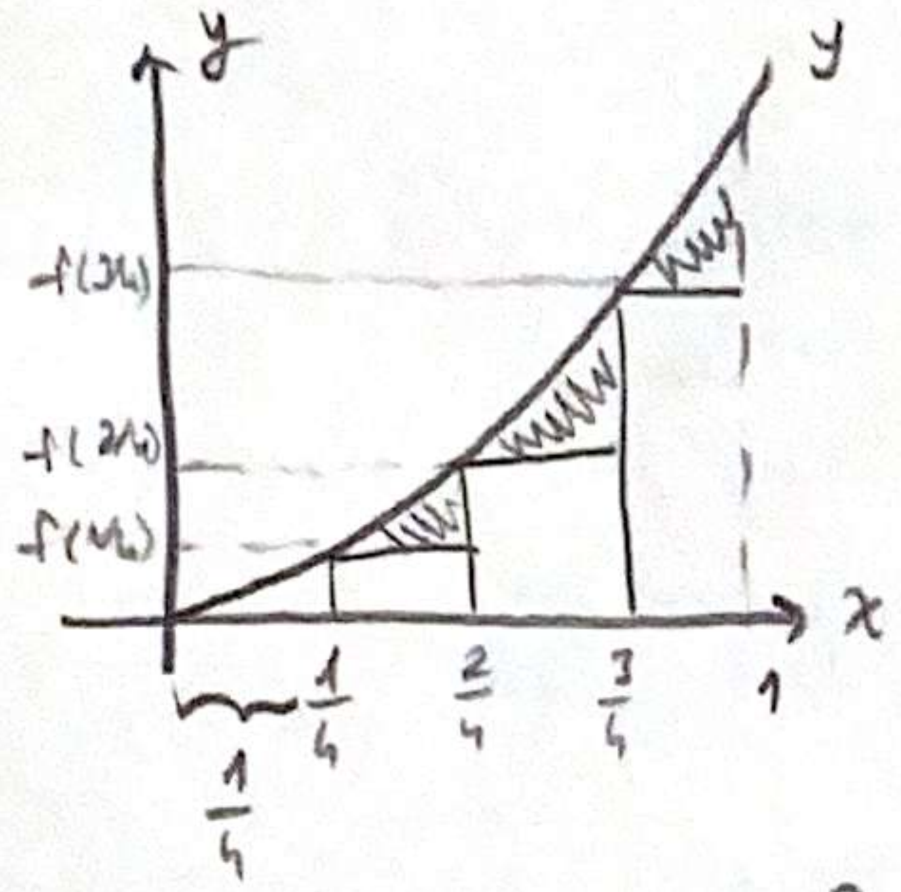
$$\sum_{k=1}^{n} (6k^2 - 4k + 3) - \sum_{k=1}^{m} (6k^2 - 4k + 3)$$

The Area Problem



$$R_4 = \frac{1}{4}\left(f(1/4) + f(2/4) + f(3/4) + f(1)\right)$$

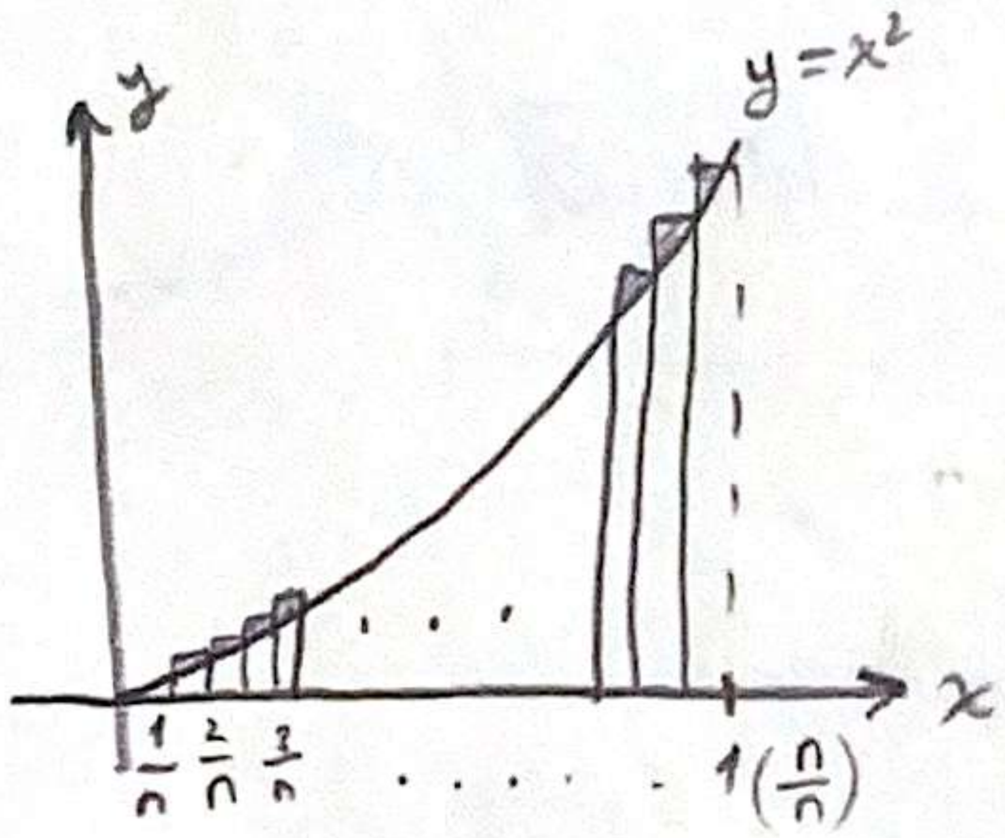
$$R_4 = 0.46875$$



$$L_4 = \frac{1}{4}\left(f(1/4) + f(2/4) + f(3/4)\right)$$

$$L_4 = 0.21875$$

$$\underset{0.21875}{L_4} < A < \underset{0.46875}{R_4}$$



$$f\left(\tfrac{1}{n}\right) = \left(\tfrac{1}{n}\right)^2$$

$$f\left(\tfrac{2}{n}\right) = \left(\tfrac{2}{n}\right)^2$$

$$\vdots$$

$$f\left(\tfrac{n}{n}\right) = \left(\tfrac{n}{n}\right)^2$$

$$R_n = \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \dots + \frac{1}{n}\left(\frac{n}{n}\right)^2$$

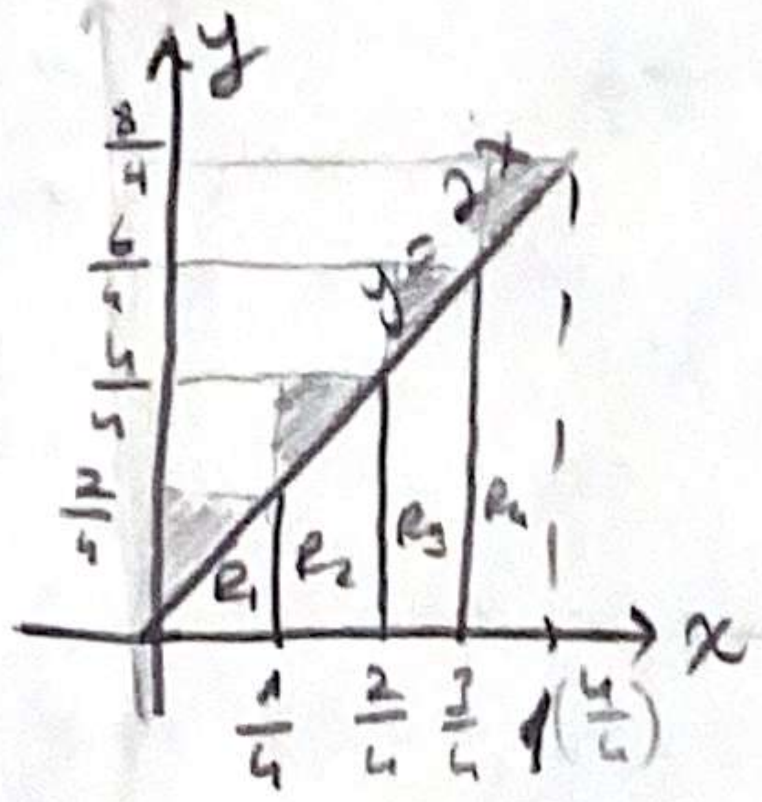
$$R_n = \frac{1}{n}\cdot\frac{1}{n^2}\left(1^2 + 2^2 + \dots + n^2\right) = \frac{1}{n^3}\underbrace{\left(1^2 + 2^2 + \dots + n^2\right)}_{\frac{n(n+1)(2n+1)}{6}}$$

$$R_n = \frac{(n+1)(2n+1)}{6 \cdot n^2}$$

$$\lim_{n\to\infty} R_n = \frac{1}{6}\lim_{n\to\infty}\left[\frac{(n+1)}{n}\cdot\frac{(2n+1)}{n}\right] \Rightarrow R_n = \frac{1}{3}$$

$$\left(1+\cancel{\tfrac{1}{n}}\right)\left(2+\cancel{\tfrac{1}{n}}\right)$$

e.g.)

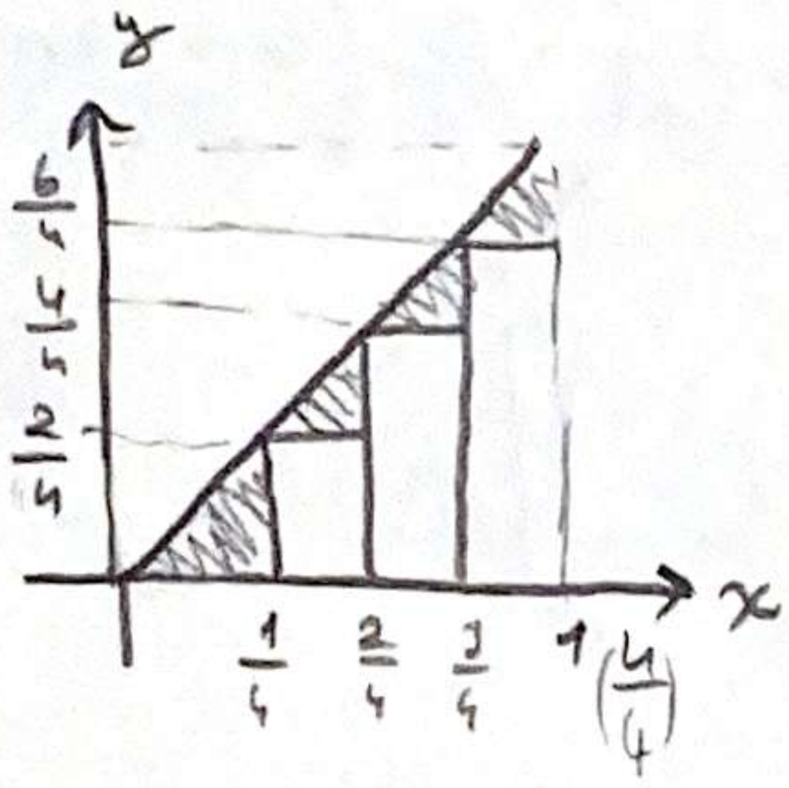


$$\Delta x = \frac{1-0}{4} = \frac{1}{4}$$

$$R_4 = \frac{1}{4}\cdot 2\cdot\frac{1}{4} + \frac{1}{4}\cdot 2\cdot\frac{2}{4} + \frac{1}{4}\cdot 2\cdot\frac{3}{4} + \frac{1}{4}\cdot 2\cdot\frac{4}{4}$$

$$R_4 = \frac{1}{4}\cdot 2\left(\frac{1}{4}+\frac{2}{4}+\frac{3}{4}+\frac{4}{4}\right)$$

$$R_4 = \frac{1}{2}\cdot\frac{10}{4} = \frac{5}{4}$$



$$L_4 = \frac{1}{4}\cdot 2\cdot 0 + \frac{1}{4}\cdot 2\cdot\frac{1}{4} + \frac{1}{4}\cdot 2\cdot\frac{2}{4} + \frac{1}{4}\cdot 2\cdot\frac{3}{4}$$

$$L_4 = \frac{1}{4}\cdot 2\left(0+\frac{1}{4}+\frac{2}{4}+\frac{3}{4}\right)$$

$$L_4 = \frac{1}{2}\cdot\frac{6}{4} = \frac{3}{4}$$

$$\frac{3}{4} < \underset{\text{Real Area}}{1} < \frac{5}{4}$$

$$R_n = \sum_{k=1}^{n} \underbrace{f\left(\frac{k}{n}\right)}_{2\cdot\frac{k}{n}} \cdot \underbrace{\Delta x}_{\frac{1}{n}} = \frac{n+1}{n}$$

$$\frac{2}{n^2}\underbrace{\sum_{k=1}^{n} k}$$

$$\frac{2}{n^2}\cdot\frac{n(n+1)}{2} = \frac{n+1}{n}$$

$$L_n = \sum_{k=1}^{n-1} \underbrace{f\left(\frac{k}{n}\right)} \cdot \underbrace{\Delta x} = \frac{n-1}{n}$$

$$\sum_{k=1}^{n} 2\,\frac{k-1}{n}\quad\frac{1}{n}$$

$$\frac{2}{n^2}\cdot\sum_{k=1}^{n} k-1$$

$$\frac{2}{n^2}\cdot\frac{(n-1)\cdot n}{2} = \frac{n-1}{n}$$

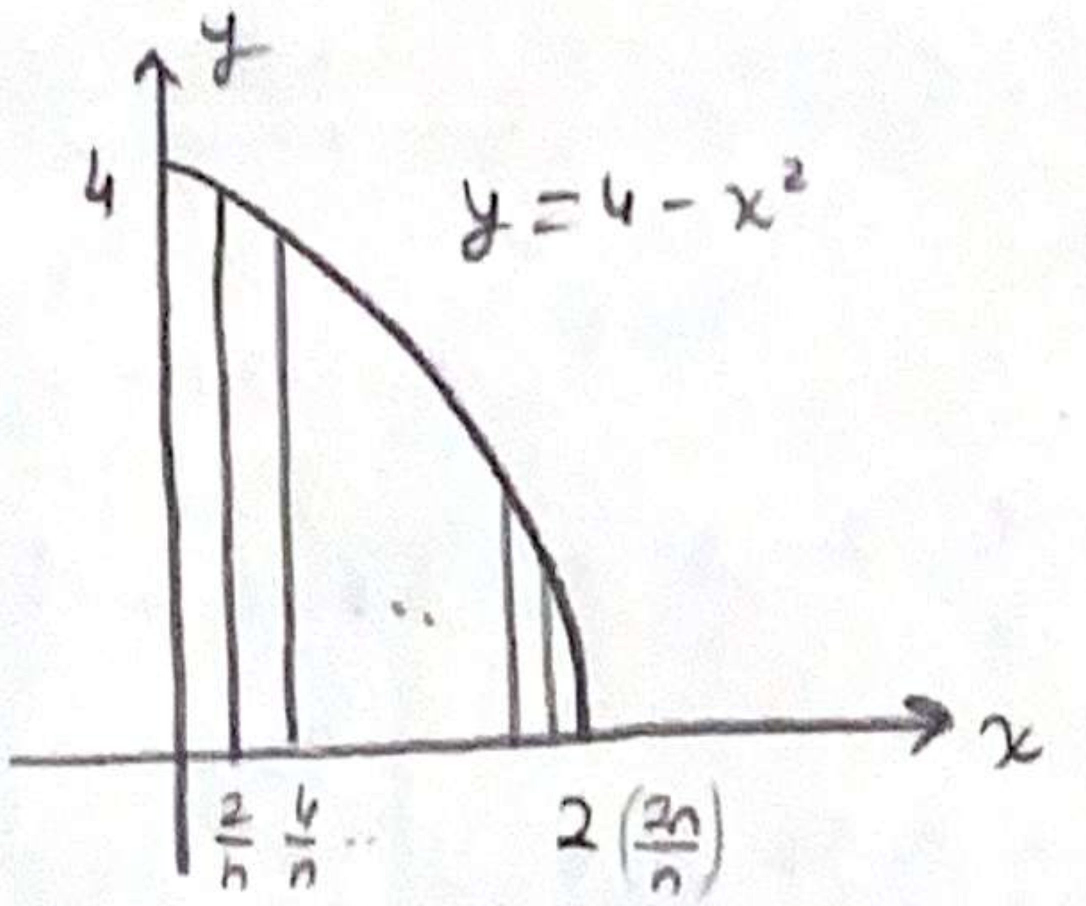
$$\lim_{n\to\infty} R_n = \lim_{n\to\infty}\frac{n+1}{n} = 1 \qquad \lim_{n\to\infty} L_n = \lim_{n\to\infty}\frac{n-1}{n} = 1$$

$$A = 1$$

$[a,b]$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=0}^{n} f(x_i)\,dx_i$$

e.g.)



$f\left(\frac{2}{n}\right) = 4 - \left(\frac{2}{n}\right)^2$

$f\left(\frac{4}{n}\right) = 4 - \left(\frac{4}{n}\right)^2$

$\vdots$

$f\left(\frac{2n}{n}\right) = 4 - \left(\frac{2n}{n}\right)^2$

$$R_n = \frac{2}{n} \cdot \left(4 - \left(\frac{2}{n}\right)^2\right) + \frac{2}{n}\left(4 - \left(\frac{2.2}{n}\right)^2\right) \cdots\cdots + \frac{2}{n}\left(4 - \left(\frac{2n}{n}\right)^2\right)$$

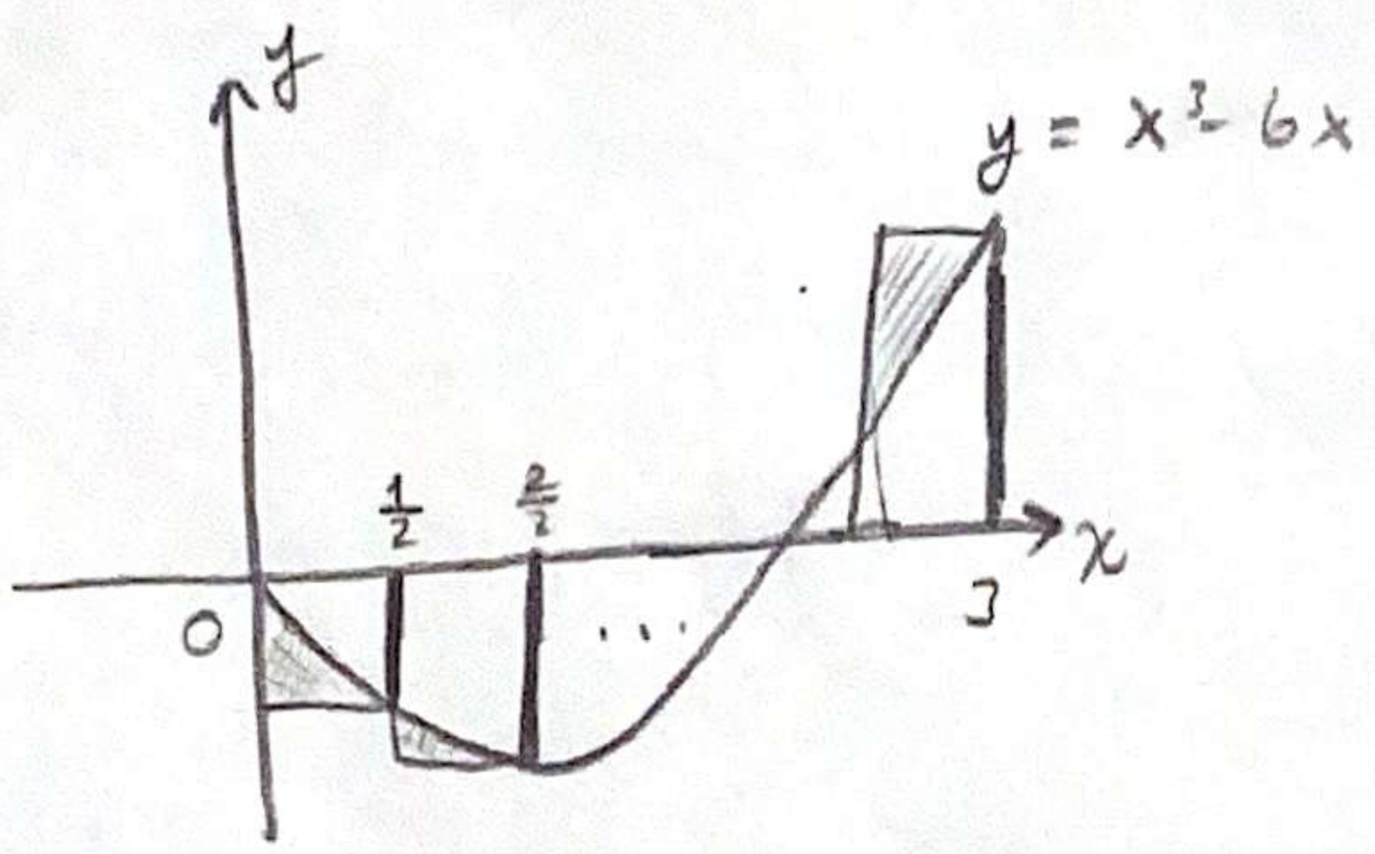
$$R_n = \frac{2}{n} \cdot \left[4n - \left(\frac{2}{n}\right)^2 \cdot \left(1^2 + 2^2 + \cdots + n^2\right)\right]$$

$$8 - \frac{\cancel{8}^{4}}{\cancel{n}\,n^2} \cdot \frac{\cancel{n}(n+1)(2n+1)}{\cancel{6}_{3}}$$

$$R_n = 8 - \frac{4}{3n^2}(2n^2 + 3n + 1)$$

$$\lim_{n \to \infty} R_n = \lim_{n \to \infty}\left(8 - \frac{4}{3n^2}(2n^2 + 3n + 1)\right) = 8 - \frac{8}{3} = \frac{16}{3}$$

e.g.)



$n = 6$ $\quad \frac{3-0}{6} = \Delta x = \frac{1}{2}$

$$R_6 = \frac{1}{2} \cdot f\left(\frac{1}{2}\right) + \frac{1}{2} f\left(\frac{2}{2}\right) + \frac{1}{2} f\left(\frac{3}{2}\right) + \cdots + \frac{1}{2} f\left(\frac{6}{2}\right)$$

$$= \frac{1}{2}\left(\left[\left(\frac{1}{2}\right)^3 - 6\left(\frac{1}{2}\right)\right] + \left[\left(\frac{2}{2}\right)^3 - 6 \cdot \left(\frac{2}{2}\right)\right] + \cdots + \left[\left(\frac{6}{2}\right)^3 - 6\left(\frac{6}{2}\right)\right]\right)$$

$$= -3.9375$$

Note: $\sum_{i=1}^{n} i^3 = \left[\frac{n.(n+1)}{2}\right]^2$

Notes:

① $\sum_{i=m}^{n} a \cdot f(i) = a \cdot \sum_{i=m}^{n} f(i)$

② $\sum_{k=m}^{n} (k^2 - 5k + 6) = \sum_{k=m}^{n} k^2 - 5 \sum_{k=m}^{n} k + \sum_{k=m}^{n} 6$

③ $\sum_{k=m}^{n} a = a \cdot (n - m + 1)$

④ $\sum_{k=1}^{n} k = \frac{n \cdot (n+1)}{2}$

eg.) $\sum_{k=4}^{20} 5k = \sum_{k=1}^{17} 5(k+3) = 5 \sum_{k=1}^{17} k + \sum_{k=1}^{17} 15$

$= \sum_{k=1}^{20} 5k - \sum_{k=1}^{3} 5k$

⑤ $\sum_{k=1}^{n} k^2 = \frac{n \cdot (n+1)(2n+1)}{6}$

⑥ $\sum_{k=1}^{n} k^3 = \left[\frac{n \cdot (n+1)}{2} \right]^2$

Right Riemann

$\Delta x = \frac{b-a}{n}$

$x_i = a + \Delta x \, i$

$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x$

Left Riemann

$\Delta x = \frac{b-a}{n}$

$x_i = a + \Delta x (i-1)$

$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x$

eg.) $\int_0^3 x \, dx$ $\quad f(x) = x \quad 0,3 \quad \frac{3-0}{n} = \Delta x$

$x_i = 0 + \Delta x \cdot i \quad \frac{3}{n} \cdot i$

$$\lim_{n\to\infty} \sum_{i=0}^{n} f(x_i) \cdot \Delta x = \lim_{n\to\infty} \sum_{i=0}^{n} f\left(\frac{3}{n} i\right) \cdot \frac{3}{n} = \lim_{n\to\infty} \sum_{i=0}^{n} \frac{9i}{n^2}$$

$$\Rightarrow \lim_{n\to\infty} \frac{9}{n^2} \cdot \sum_{i=1}^{n} i = \lim_{n\to\infty} \frac{9}{n^2} \cdot \frac{n(n+1)}{2} = \lim_{n\to\infty} \frac{9}{2}\left(1 + \underbrace{\cancel{\frac{1}{n}}}_{0}\right) = \frac{9}{2}$$

$$\left[\frac{n\cdot(n+1)}{2} - 0\right]$$

---

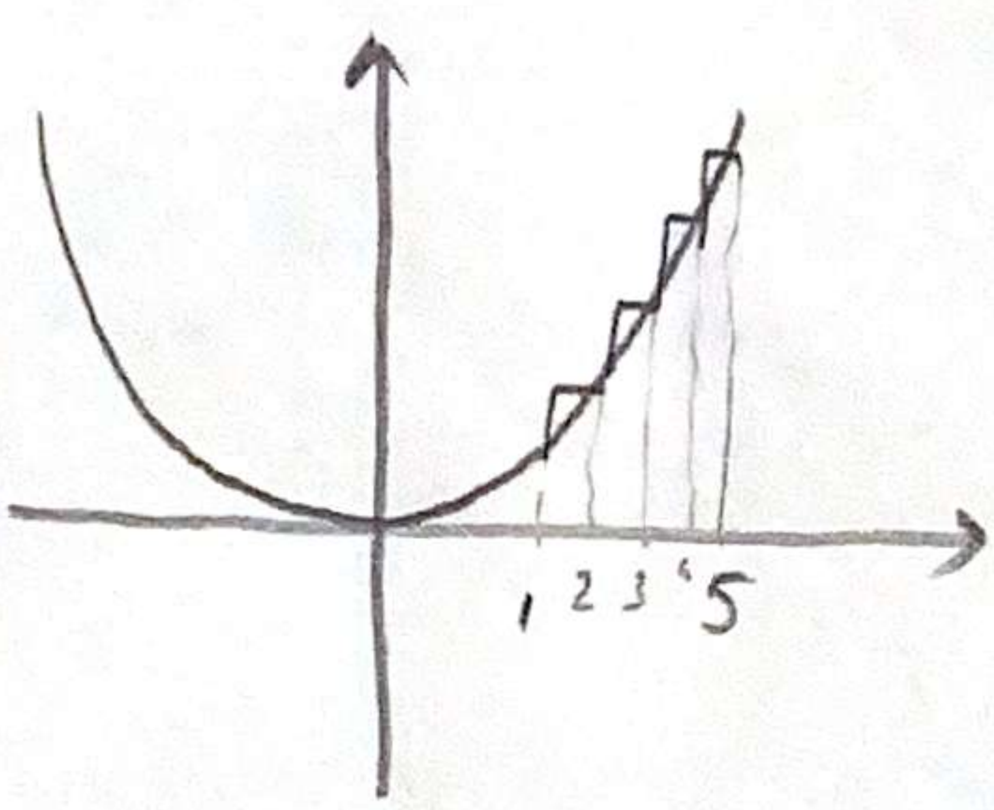
eg.) $f(x) = x^2 \quad n = 4 \quad [1, 5]$

a) $R_4 = ?$

$\Delta x = \frac{5-1}{4} = 1$

$f(2)\cdot 1 + f(3)\cdot 1 + f(4)\cdot 1 + f(5)\cdot 1$

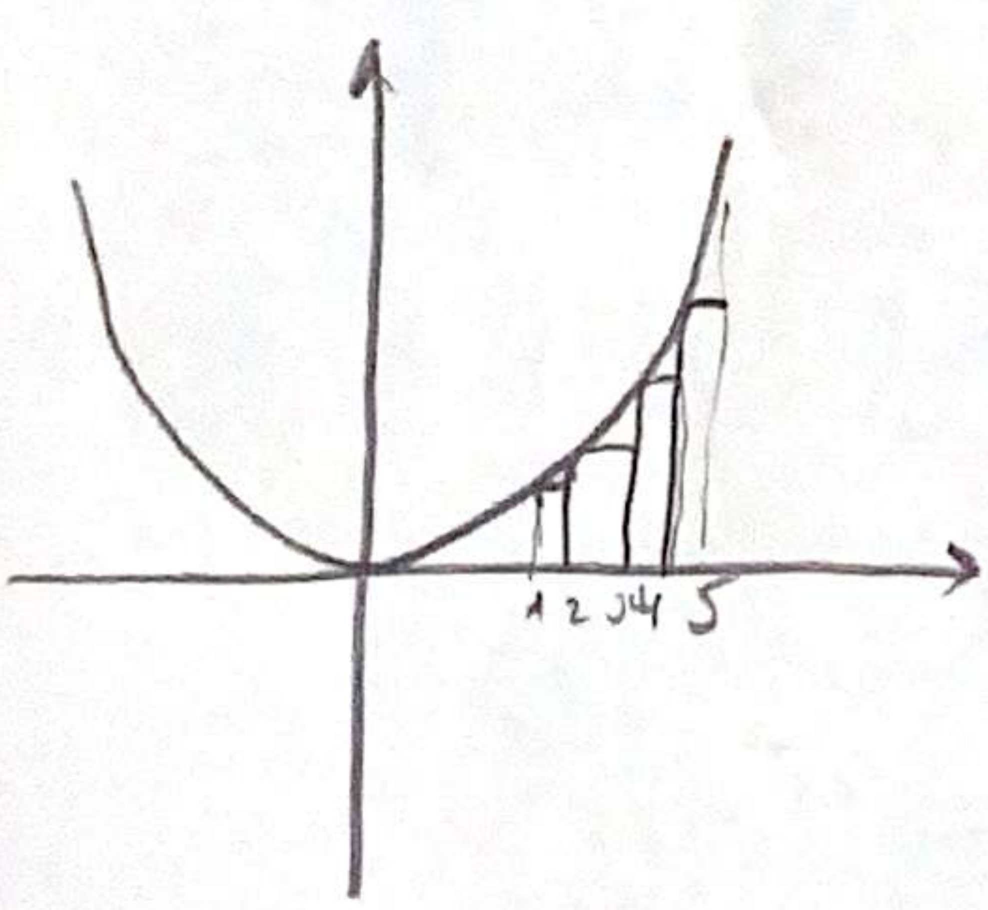
$4 + 9 + 16 + 25 = 54$



b) $L_4 = ?$

$f(1)\cdot 1 + f(2)\cdot 1 + f(3)\cdot 1 + f(4)\cdot 1$

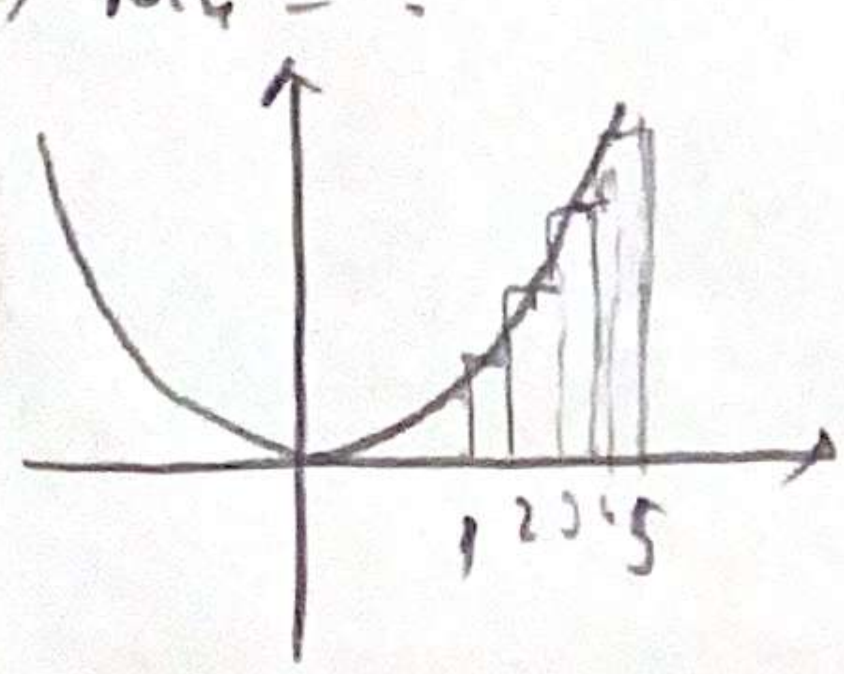
$1 + 4 + 9 + 16 = 30$



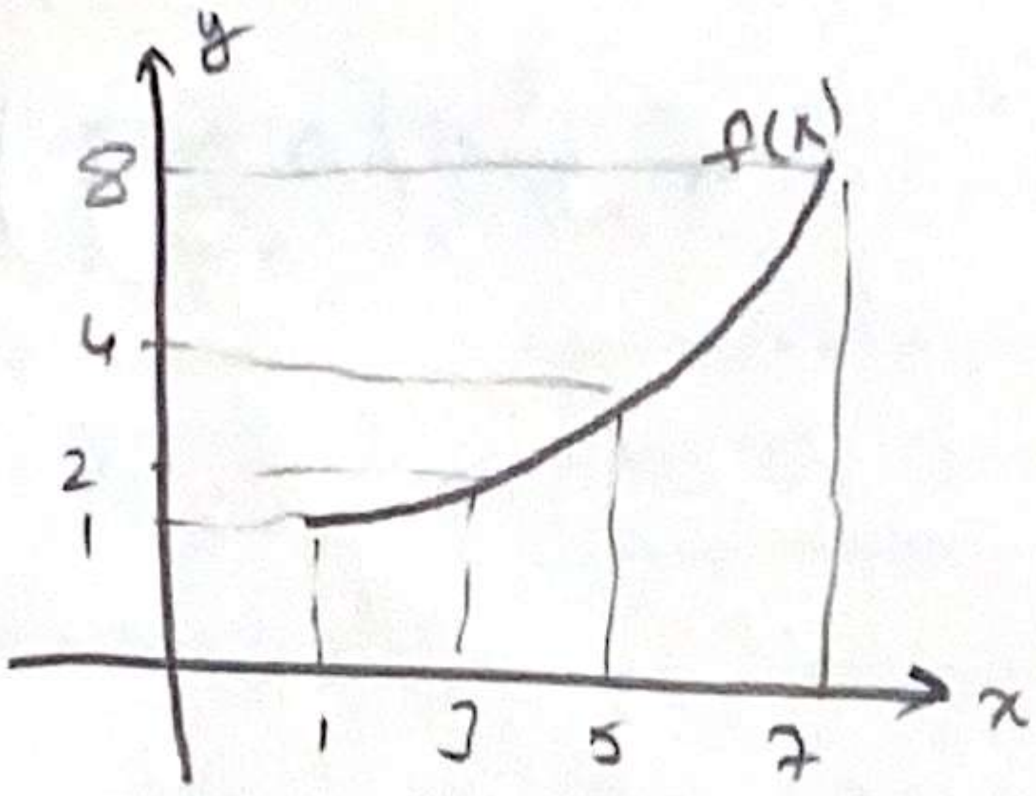
$f\left(\frac{3}{2}\right)\cdot 1 + f\left(\frac{5}{2}\right)\cdot 1 + f\left(\frac{7}{2}\right)\cdot 1 + f\left(\frac{9}{2}\right)\cdot 1$

c) $M_4 = ?$

$\frac{9}{4} + \frac{25}{4} + \frac{49}{4} + \frac{81}{4} = \frac{164}{4} = 41$



eg.)



$\int_1^7 f(x)\,dx$ max, min ?

$14 < \int_1^7 f(x)\,dx < 28$

$\underset{15}{\downarrow} \qquad \underset{27}{\downarrow}$

eg.) $\lim_{n\to\infty} \frac{3}{n} \sum_{k=1}^{n} \left(2 + \frac{3k}{n}\right)$ $\qquad \Delta x = \frac{3}{n}$ $\qquad \frac{3}{n} = \frac{b-a}{n}$ $\boxed{5=b}$ $\boxed{2=a}$

$x_k = a + \Delta x \cdot k$

$\int_2^5 x\,dx = \frac{x^2}{2}\Big|_2^5 = \frac{21}{2}$ $\qquad \underbrace{2}_{a} + \underbrace{\frac{3k}{n}}_{\Delta x}$

eg.) $\lim_{n\to\infty} \left(\frac{1^5+2^5+3^5+\dots+n^5}{n^6}\right) = \lim_{n\to\infty} \frac{1}{n}\left(\frac{1^5+2^5+3^5+\dots n^5}{n^5}\right)$

$\lim_{n\to\infty} \underbrace{\frac{1}{n}}_{\Delta x} \left[\sum_{k=1}^{n} \left(\frac{k}{n}\right)^5\right] = \int_1^5 x^5\,dx$

$x_k = \underset{0}{a} + \underset{\frac{1}{n}}{\Delta x} \cdot k$

$\boxed{b=1}$ $\boxed{a=0}$ $\boxed{f(x)=x^5}$

eg.) $\lim_{n\to\infty} \left(\frac{\sin\left(\frac{1}{n}\right) + \sin\left(\frac{2}{n}\right) + \dots + \sin\left(\frac{n}{n}\right)}{n}\right)$

$\lim_{n\to\infty} \underbrace{\frac{1}{n}}_{\Delta x} \cdot \left[\sum_{k=1}^{n} \sin\left(\frac{k}{n}\right)\right]$

$x_k = \boxed{\underset{0}{a}} + \underset{\frac{1}{n}}{\Delta x} \cdot k$ $\qquad \boxed{f(x) = \sin x}$ $\qquad \boxed{b=1}$

$\int_0^1 (\sin x)\,dx = -\cos x\Big|_0^1 = (-\cos 1) - (-1) = 1 - \cos(1)$

eg.) $$\lim_{n\to\infty} \underbrace{\frac{1}{n}}_{\Delta x}\left(\cos\left(\frac{\pi}{n}\right) + \cos\left(\frac{2\pi}{n}\right) + \dots + \cos\left(\frac{n\pi}{n}\right)\right)$$

$$\lim_{n\to\infty} \underbrace{\frac{1}{n}}_{\Delta x}\left[\sum_{k=1}^{n} \cos\left(\frac{\pi k}{n}\right)\right]$$

$\longrightarrow x_k = a + \underbrace{\Delta x}_{\frac{1}{n}} k \qquad f(x) = \cos(\pi x)$

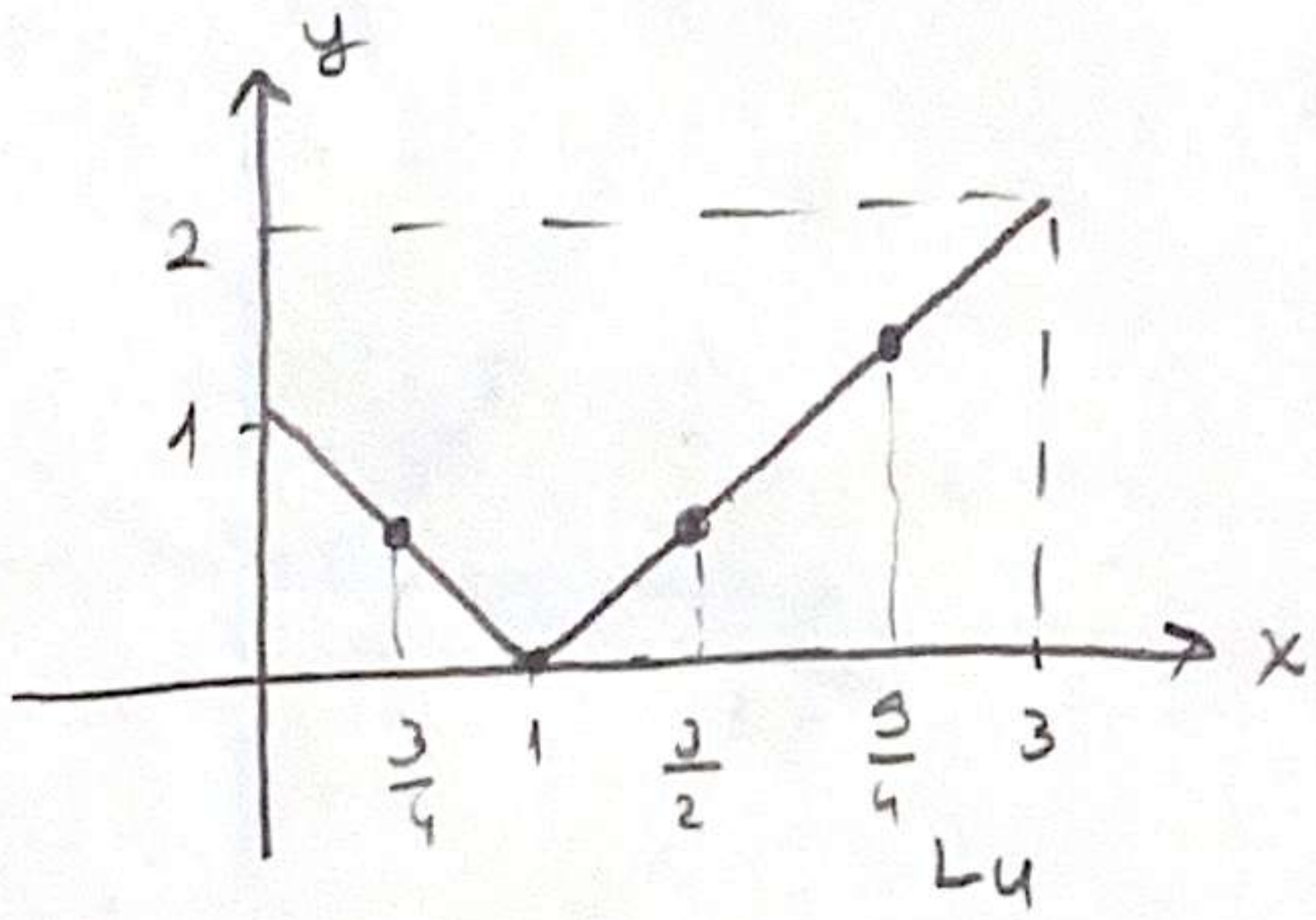
$a=0 \quad b=1$

$$\Rightarrow \int_0^1 \cos(\pi x)\,dx = \left.\frac{\sin(\pi x)}{\pi}\right|_0^1 = 0 - 0 = 0$$

eg.) $f(x) = |x-1| \quad [0,3]$

a) Lower Riemann $(n=4)$

b) Upper Riemann $(n=4)$



$$\Delta x = \frac{3-0}{4} = \frac{3}{4}$$

	L_4	U_4
$0 \le x \le \frac{3}{4}$	$\longrightarrow \frac{3}{4}$	$\longrightarrow 0$
$\frac{3}{4} \le x \le \frac{3}{2}$	$\longrightarrow 1$	$\longrightarrow \frac{3}{2}$
$\frac{3}{2} \le x \le \frac{9}{4}$	$\longrightarrow \frac{3}{2}$	$\longrightarrow \frac{9}{4}$
$\frac{9}{4} \le x \le 3$	$\longrightarrow \frac{9}{4}$	$\longrightarrow 3$

e.g.) $\lim_{n\to\infty}\left(\frac{1}{1+9n}+\frac{1}{3+9n}+\frac{1}{6+9n}+\cdots+\frac{1}{3n+9n}\right)$

$\boxed{\Delta x = \frac{b-a}{n}}$ $\boxed{x_i = a + \Delta x \cdot i}$

$\lim_{n\to\infty}\frac{1}{1+9n}+\lim_{n\to\infty}\left(\frac{1}{3+9n}+\frac{1}{6+9n}+\cdots+\frac{1}{3n+9n}\right)$

$\underbrace{\lim_{n\to\infty}\cancel{\frac{1}{1+9n}}}_{0}+\lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{1}{3i+9n}\right)$

$\Rightarrow \lim_{n\to\infty}\frac{1}{n}\left[\sum_{i=1}^{n}\frac{1}{\frac{3i}{n}+9}\right]$

$\Delta x = \frac{b-a}{n} = \frac{1}{n}$

$x_i = \underbrace{a}_{0} + \underbrace{\Delta x}_{\frac{1}{n}} \cdot i$ $a=0$ $b=1$

$f(x) = \frac{1}{3x+9}$

$\Rightarrow \int_0^1\left(\frac{1}{3x+9}\right)dx$

e.g.) $\lim_{n\to\infty}\left(\frac{1}{1+n}+\frac{1}{2+n}+\cdots+\frac{1}{n+n}\right)$

$\Delta x = \frac{b-a}{n}$ $x_i = a + \Delta x \cdot i$

$\lim_{n\to\infty}\left[\sum_{i=1}^{n}\left(\frac{1}{i+n}\right)\right] = \lim_{n\to\infty}\left[\underbrace{\frac{1}{n}}_{\Delta x}\left(\sum_{i=1}^{n}\frac{1}{\frac{i}{n}+1}\right)\right]$

$x_i = \underbrace{a}_{1} + \underbrace{\Delta x}_{\frac{1}{n}} \cdot i$

$a=1$ $b=2$ $f(x)=\frac{1}{x}$

$\Rightarrow \int_1^2 \frac{1}{x}\,dx$

e.g.) $\lim_{n\to\infty} \sum_{k=1}^{n} \ln\left(\sqrt[n]{2+\frac{k}{n}}\right)$ Express as Riemann sum

$\Delta x = \frac{b-a}{n}$ $\quad x_k = a + \Delta x . k$

$\Rightarrow \lim_{n\to\infty} \underbrace{\frac{1}{n}}_{\Delta x} \sum_{k=1}^{n} \underbrace{\ln\left(2+\frac{k}{n}\right)}$

$f(x_k) = \ln\left(2+\frac{k}{n}\right)$ $\quad x_k = 2 + \frac{1}{n}.k$

$f(x) = \ln x$ $\quad a=2 \quad b=3$

$\Rightarrow \int_2^3 \ln x \, dx$

e.g.) $\lim_{n\to\infty} \frac{\pi}{n}\left(\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \cdots + \sin\frac{n\pi}{n}\right)$

$\Delta x = \frac{\overset{1}{b} - \overset{0}{a}}{n}$ $\quad x_i = \cancel{a} + \Delta x . i$ ($a \to 0$) $\quad \Delta x = \frac{1}{n}$

$\lim_{n\to\infty} \pi . \frac{1}{n} \sum_{i=1}^{n} \sin\left(\frac{i\pi}{n}\right) = \pi \int_0^1 \sin(\pi x) \, dx$

e.g.) $\lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{n+jc}$, $c>1$ $\quad \Delta x = \frac{b-a}{n}$ $\quad x_j = \cancel{a} + \Delta x . j$ ($a \to 0$, $\Delta x \to \frac{c}{n}$)

$\Delta x = \frac{c}{n}$ $\quad f(x_j) = \frac{1}{1+\frac{jc}{n}}$

$\frac{1}{c} \lim_{n\to\infty} \underbrace{c.\frac{1}{n}}_{\Delta x} \left[\sum_{j=1}^{n}\left(\frac{1}{1+\frac{jc}{n}}\right)\right]$

$f(x) = \frac{1}{1+x}$

$\Rightarrow \frac{1}{c}\int_0^c \frac{1}{1+x}\,dx = \frac{\ln(1+x)}{c}\Big|_0^c = \frac{\ln(1+c)}{c}$

Let $F(t) = \int_a^b f(x,t)\,dx$

$f(x,t)$ and $\dfrac{\partial f(x,t)}{\partial t} = f_t(x,t)$ are cont. at $a \leq t \leq b$

$c \leq t \leq d$

Δt increment

$F(t+\Delta t) = \int_a^b f(x, t+\Delta t)\,dx$

$F(t+\Delta t) - F(t) = \int_a^b [f(x,t+\Delta t) - f(x,t)]\,dx$

$\dfrac{F(t+\Delta t) - F(t)}{\Delta t} = \int_a^b f_t(x+\theta\Delta t)\,dx$

$\int_a^b f_t(x,t)\,dx + \int_a^b f_t(x+\theta\Delta t)\,dx - \int_a^b f_t(x,t)\,dx$

$\dfrac{F(t+\Delta t) - F(t)}{\Delta t} - \int_a^b f_t(x,t)\,dx = \int_a^b [f_t(x+\theta\Delta t) - f_t(x,t)]\,dx$

for $\varepsilon > 0$, since $f_t(x,t)$ is continuous, there is a $\delta > 0$

$|f_t(x, t+\theta\Delta t) - f_t(x,t)| < \varepsilon$ $\mathcal{M}$

Note: $\left|\int_a^b f(t)\,dt\right| \leq \int_a^b |f(t)|\,dt$

$\left|\int_a^b [f_t(x,t+\theta\Delta t) - f_t(x,t)]\,dx\right| < \int_a^b |f_t(x,t+\theta\Delta t) - f_t(x,t)|\,dx = \varepsilon(b-a)$

$\left|\dfrac{F(t+\Delta t) - F(t)}{\Delta t} - \int_a^b f_t(x,t)\,dx\right| \leq \varepsilon(b-a)$

$\lim\limits_{\Delta t \to 0} \dfrac{F(t+\Delta t) - F(t)}{\Delta t} = \int_a^b f_t(x,t)\,dx$ since $|\Delta t| \leq \delta$

$\#$

$$\frac{dF}{dt} = \frac{d}{dt}\int_a^b f(x,t)\,dx = \int_a^b \frac{\partial f(x,t)}{\partial t}\,dx = \int_a^b f_t(x,t)\,dx$$

$$F(t) = \int_{a(t)}^{b(t)} f(x,t)\,dx$$

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial b}\cdot\frac{\partial b}{\partial t} + \frac{\partial F}{\partial a}\cdot\frac{\partial a}{\partial t}$$

$$F(t) = \int_{a(t)}^{b} f(x,t)\,dx = -\int_{b}^{a(t)} f(x,t)\,dt \Rightarrow \frac{\partial F}{\partial a} - f(a(t),t)$$

$$\frac{\partial F}{\partial b} = f(b(t),t)$$

$$\frac{dF}{dt} = \int_{a(t)}^{b(t)} \frac{\partial F(x,t)}{\partial t}\cdot dx + f(b(t),t)\cdot b'(t) - f(a(t),t)\cdot a'(t)$$

$$\frac{d}{dx}F(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

$$\int_{u(x)}^{v(x)} f(t)\,dt = \int_{u(x)}^{c} f(t)\,dt + \int_c^{v(x)} f(t)\,dt$$

$$\frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(t)\,dx\right] = f(v(x))\cdot v'(x) - f(u(x))\cdot u'(x)$$

eg.) $F(t) = \int_{t}^{t^2} \sin x^2 \, dx$ $\qquad F'(t) = ?$

$$F'(t) = \frac{d}{dt} \int_{t}^{t^2} \sin x^2 dx$$

$$= \int_{t}^{t^2} 0 . dx + \sin(t^2)^2 . 2t - \sin(t)^2 . 1$$

$$= 2t \sin(t^2)^2 - \sin(t^2)$$

---

eg.) $y = \int_{t\to 0}^{t\to\sqrt{x}} e^{t^2} dt$ $\qquad \frac{dy}{dx} = ?$

$$y' = e^{(\sqrt{x})^2} . \frac{1}{2\sqrt{x}} - e^0 . 0 = \frac{e^x}{2\sqrt{x}}$$

---

eg.) $$\lim_{x\to\pi} \frac{\int_{\pi}^{x} \frac{dt}{2+\cos t}}{x-\pi} = \lim_{x\to\pi} \frac{\frac{1}{2+\cos x} . 1 - 0}{1} = 1$$

---

eg.) $F(x) = \int_{u(x)}^{v(x)} f(x,y) \, dy$ $\qquad F'(x) = ?$

$$F'(x) = \int_{u(x)}^{v(x)} \frac{\partial F}{\partial x} dy + f(x, v(x)) . v'(x) - f(x, u(x)) . u'(x)$$

eg.)

$$F(x) = \int_{1-x}^{2-x} e^{t^2} dt \Rightarrow F'(x) = e^{(2-x)^2} . (-1) - e^{(1-x)^2} . (-1)$$

$$F'(1) = -e + 1 = \underline{1-e}$$