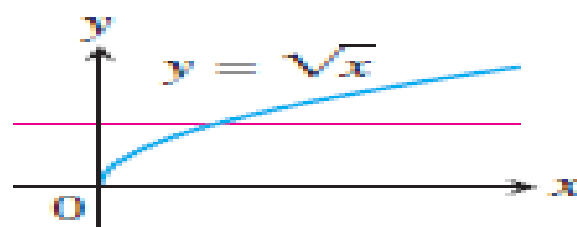
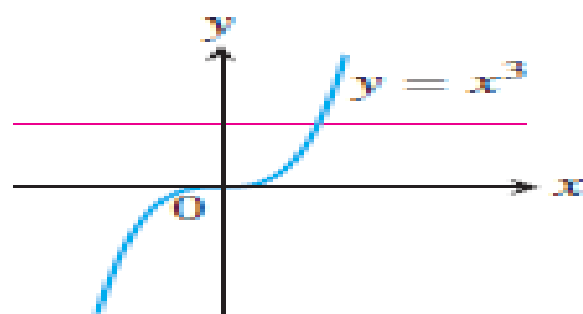


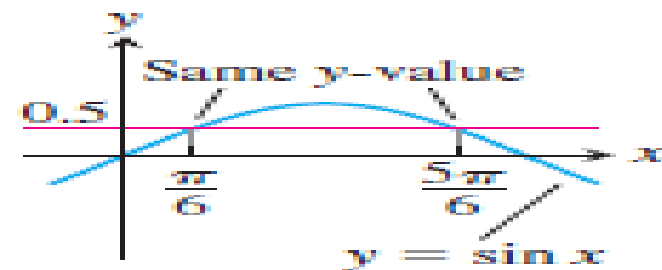
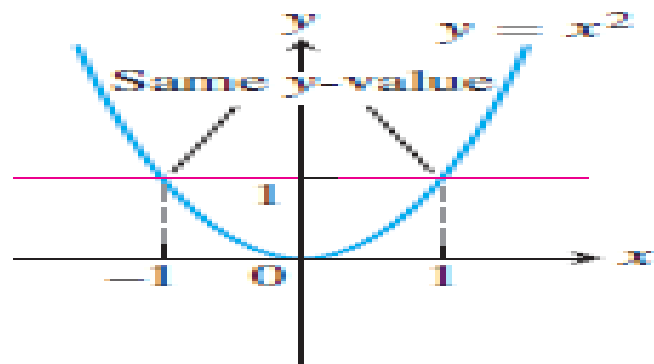
One-to-One Functions

DEFINITION A function $f(x)$ is one-to-one on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D .

- (a) $f(x) = \sqrt{x}$ is one-to-one on any domain of nonnegative numbers because $\sqrt{x_1} \neq \sqrt{x_2}$ whenever $x_1 \neq x_2$.
- (b) $g(x) = \sin x$ is *not* one-to-one on the interval $[0, \pi]$ because $\sin(\pi/6) = \sin(5\pi/6)$. In fact, for each element x_1 in the subinterval $[0, \pi/2)$ there is a corresponding element x_2 in the subinterval $(\pi/2, \pi]$ satisfying $\sin x_1 = \sin x_2$, so distinct elements in the domain are assigned to the same value in the range. The sine function *is* one-to-one on $[0, \pi/2]$, however, because it is an increasing function on $[0, \pi/2]$ giving distinct outputs for distinct inputs.



(a) One-to-one: Graph meets each horizontal line at most once.



(b) Not one-to-one: Graph meets one or more horizontal lines more than once.

The Horizontal Line Test for One-to-One Functions

A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.

One-to-One/Onto Functions

Here are the definitions:

- f is one-to-one (injective) if f maps every element of A to a unique element in B . In other words no element of B are mapped to by two or more elements of A .
 - $(\forall a, b \in A) f(a) = f(b) \Rightarrow a = b$.
- f is onto (surjective) if every element of B is mapped to by some element of A . In other words, nothing is left out.
 - $(\forall b \in B) (\exists a \in A) f(a) = b$.
- f is one-to-one onto (bijective) if it is both one-to-one and onto. In this case the map f is also called a *one-to-one correspondence*.

Example-1

Classify the following functions $f_j : \mathbb{N} \rightarrow \mathbb{N}$ between natural numbers as one-to-one and onto.

f_j	One-to-One?	Onto?
$f_1(n) = n^2$	Yes	No
$f_2(n) = n + 3$	Yes	No
$f_3(n) = \lfloor \sqrt{n} \rfloor$	No	Yes
$f_4(n) = \begin{cases} n - 1, & \text{odd } n \\ n + 1, & \text{even } n \end{cases}$	Yes	Yes

Reasons

- f_1 is not onto because it does not have any element n such that $f_1(n) = 3$, for instance.
- f_2 is not onto because no element n such that $f_2(n) = 0$, for instance.
- f_3 is not one-to-one since $f_3(2) = f_3(1) = 1$.

Example-2

Prove that the function $f(n) = n^2$ is one-to-one.

Proof: We wish to prove that whenever $f(m) = f(n)$ then $m = n$. Let us assume that $f(m) = f(n)$ for two numbers $m, n \in \mathbb{N}$. Therefore, $m^2 = n^2$. Which means that $m = \pm n$. Splitting cases on n , we have

- For $n \neq 0$, $-n \notin \mathbb{N}$, therefore $m = n$ for this case.
- For $n = 0$, we have $m = n = 0$. Therefore, it follows that $m = n$ for both cases.

Example-3

Prove that the function $f(n) = \lfloor \sqrt{n} \rfloor$ is onto.

Proof

Given any $m \in \mathbb{N}$, we observe that $n = m^2 \in \mathbb{N}$ is such that $f(n) = m$. Therefore, all $m \in \mathbb{N}$ are mapped onto.

LET US REMEMBER

Definition If $f : A \rightarrow B$ and $g : B \rightarrow C$, and if $R(f) \subseteq D(g) = B$, then the **composite function** $g \circ f$ (note the order!) is the function from A into C defined by

$$(g \circ f)(x) := g(f(x)) \quad \text{for all } x \in A.$$

Examples (a) The order of the composition must be carefully noted. For, let f and g be the functions whose values at $x \in \mathbb{R}$ are given by

$$f(x) := 2x \quad \text{and} \quad g(x) := 3x^2 - 1.$$

Since $D(g) = \mathbb{R}$ and $R(f) \subseteq \mathbb{R} = D(g)$, then the domain $D(g \circ f)$ is also equal to \mathbb{R} , and the composite function $g \circ f$ is given by

$$(g \circ f)(x) = 3(2x)^2 - 1 = 12x^2 - 1.$$

On the other hand, the domain of the composite function $f \circ g$ is also \mathbb{R} , but

$$(f \circ g)(x) = 2(3x^2 - 1) = 6x^2 - 2.$$

Thus, in this case, we have $g \circ f \neq f \circ g$.

(b) In considering $g \circ f$, some care must be exercised to be sure that the range of f is contained in the domain of g . For example, if

$$f(x) := 1 - x^2 \quad \text{and} \quad g(x) := \sqrt{x},$$

then, since $D(g) = \{x : x \geq 0\}$, the composite function $g \circ f$ is given by the formula

$$(g \circ f)(x) = \sqrt{1 - x^2}$$

only for $x \in D(f)$ that satisfy $f(x) \geq 0$; that is, for x satisfying $-1 \leq x \leq 1$.

DEFINITION Suppose that f is a one-to-one function on a domain D with range R . The inverse function f^{-1} is defined by

$$f^{-1}(b) = a \text{ if } f(a) = b.$$

The domain of f^{-1} is R and the range of f^{-1} is D .

The symbol f^{-1} for the inverse of f is read “ f inverse.” The “ -1 ” in f^{-1} is *not* an exponent; $f^{-1}(x)$ does not mean $1/f(x)$. Notice that the domains and ranges of f and f^{-1} are interchanged.

EXAMPLE Suppose a one-to-one function $y = f(x)$ is given by a table of values

x	1	2	3	4	5	6	7	8
$f(x)$	3	4.5	7	10.5	15	20.5	27	34.5

A table for the values of $x = f^{-1}(y)$ can then be obtained by simply interchanging the values in the columns (or rows) of the table for f :

y	3	4.5	7	10.5	15	20.5	27	34.5
$f^{-1}(y)$	1	2	3	4	5	6	7	8

$$(f^{-1} \circ f)(x) = x, \quad \text{for all } x \text{ in the domain of } f$$

$$(f \circ f^{-1})(y) = y, \quad \text{for all } y \text{ in the domain of } f^{-1} \text{ (or range of } f)$$

EXAMPLE Find the inverse of $y = \frac{1}{2}x + 1$, expressed as a function of x .

Solution

1. Solve for x in terms of y : $y = \frac{1}{2}x + 1$

$$2y = x + 2$$

$$x = 2y - 2.$$

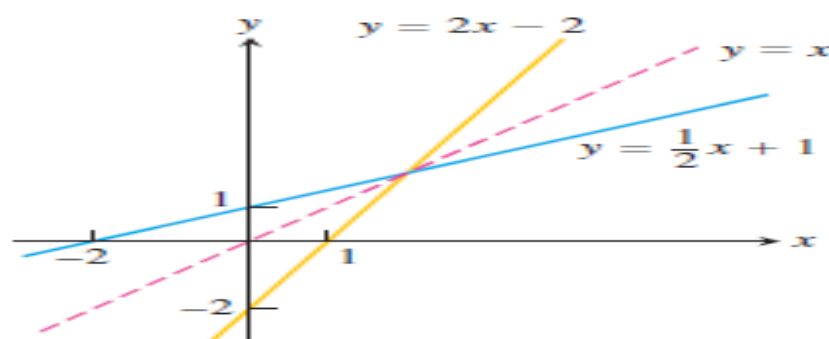
2. Interchange x and y : $y = 2x - 2$.

The inverse of the function $f(x) = (1/2)x + 1$ is the function $f^{-1}(x) = 2x - 2$.

To check, we verify that both composites give the identity function:

$$f^{-1}(f(x)) = 2\left(\frac{1}{2}x + 1\right) - 2 = x + 2 - 2 = x$$

$$f(f^{-1}(x)) = \frac{1}{2}(2x - 2) + 1 = x - 1 + 1 = x.$$



Graphing
 $f(x) = (1/2)x + 1$ and $f^{-1}(x) = 2x - 2$
 together shows the graphs' symmetry with
 respect to the line $y = x$

EXAMPLE Find the inverse of the function $y = x^2, x \geq 0$, expressed as a function of x .

Solution We first solve for x in terms of y :

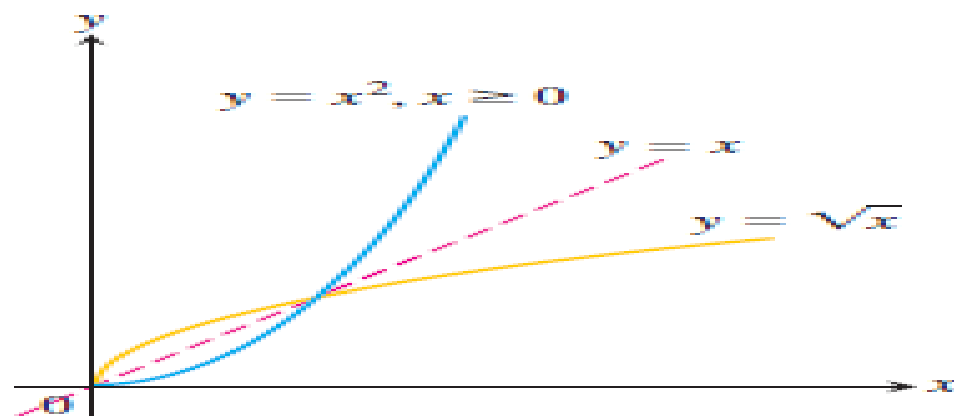
$$\begin{aligned} y &= x^2 \\ \sqrt{y} &= \sqrt{x^2} = |x| = x \quad |x| = x \text{ because } x \geq 0 \end{aligned}$$

We then interchange x and y , obtaining

$$y = \sqrt{x}.$$

The inverse of the function $y = x^2, x \geq 0$, is the function $y = \sqrt{x}$ (

Notice that the function $y = x^2, x \geq 0$, with domain *restricted* to the nonnegative real numbers, *is* one-to-one and has an inverse. On the other hand, the function $y = x^2$, with no domain restrictions, *is not* one-to-one and therefore has no inverse.



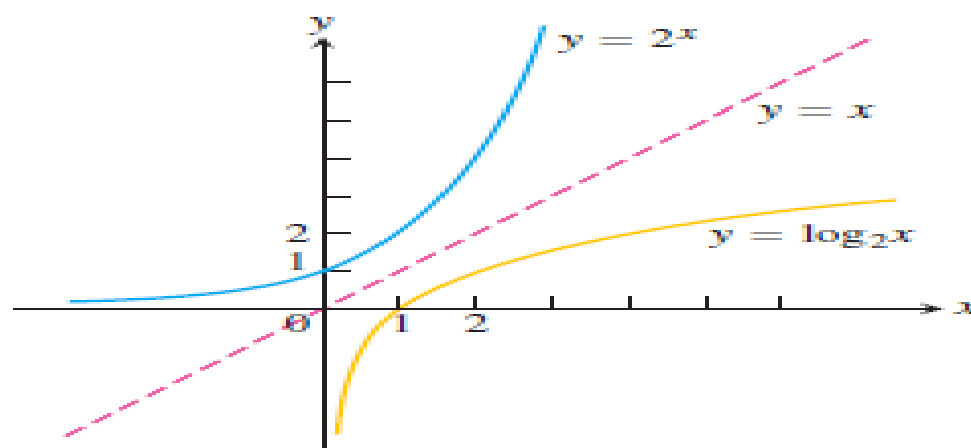
The functions $y = \sqrt{x}$ and $y = x^2, x \geq 0$, are inverses of one another

DEFINITION The logarithm function with base a , $y = \log_a x$, is the inverse of the base a exponential function $y = a^x$ ($a > 0, a \neq 1$).

The domain of $\log_a x$ is $(0, \infty)$, the range of a^x . The range of $\log_a x$ is $(-\infty, \infty)$, the domain of a^x .
Let us consider $y = 2^x$. The inverse of it $y = \log_2 x$.

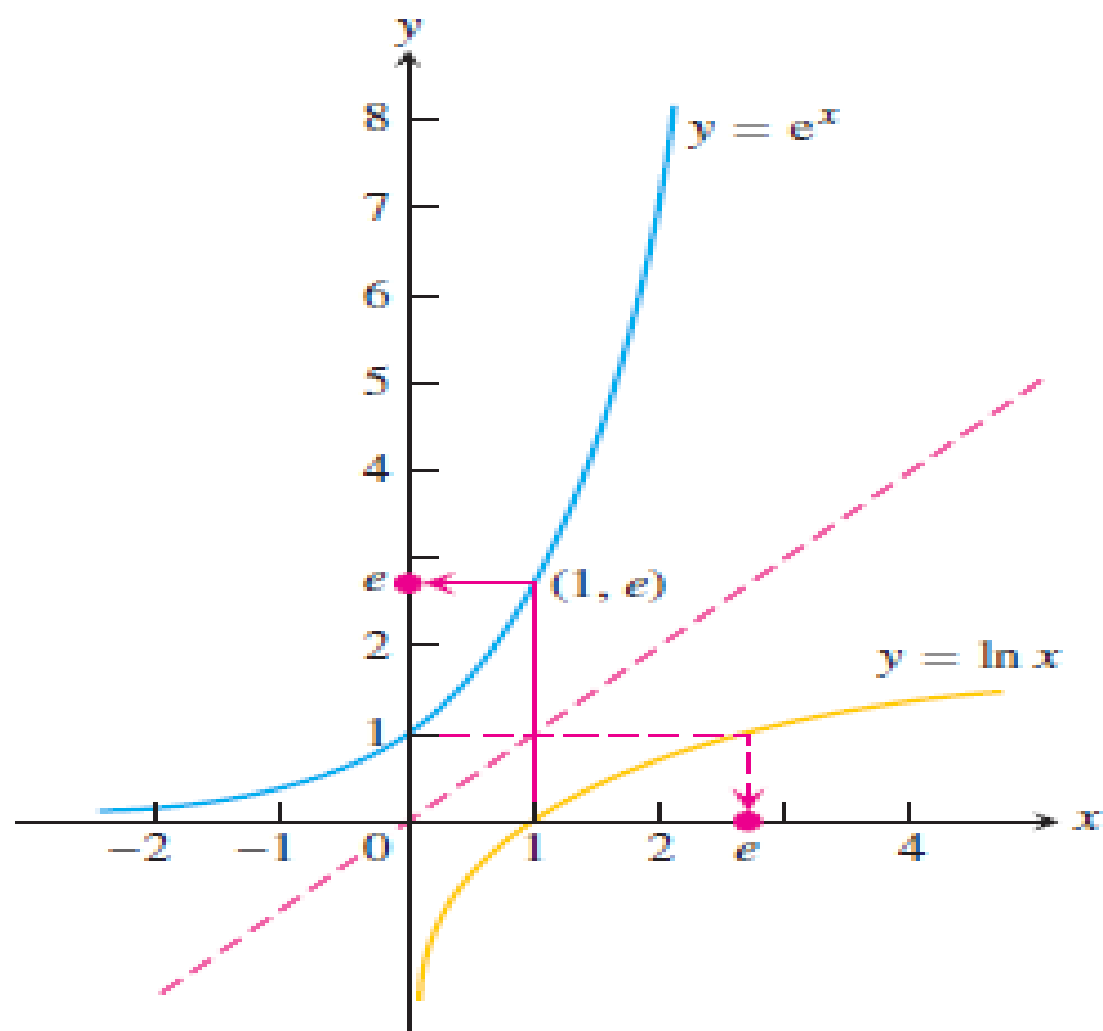
$\log_e x$ is written as $\ln x$.

$\log_{10} x$ is written as $\log x$.



$$\ln x = y \Leftrightarrow e^y = x.$$

$$\ln e = 1$$



THEOREM 1—Algebraic Properties of the Natural Logarithm For any numbers $b > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1. *Product Rule:* $\ln bx = \ln b + \ln x$ $\ln 4 + \ln \sin x = \ln(4 \sin x)$
2. *Quotient Rule:* $\ln \frac{b}{x} = \ln b - \ln x$ $\ln \left(\frac{x+1}{2x-3} \right) = \ln(x+1) - \ln(2x-3)$
3. *Reciprocal Rule:* $\ln \frac{1}{x} = -\ln x$ $\ln \frac{1}{8} = -\ln 8$
 $= -\ln 2^3 = -3 \ln 2$ *Power Rule*
Rule 2 with $b = 1$
4. *Power Rule:* $\ln x^r = r \ln x$

Inverse Properties for a^x and $\log_a x$

1. Base a : $a^{\log_a x} = x$, $\log_a a^x = x$, $a > 0, a \neq 1, x > 0$
2. Base e : $e^{\ln x} = x$, $\ln e^x = x$, $x > 0$

Substituting a^x for x in the equation $x = e^{\ln x}$ enables us to rewrite a^x as a power of e :

$$\begin{aligned} a^x &= e^{\ln(a^x)} && \text{Substitute } a^x \text{ for } x \text{ in } x = e^{\ln x}. \\ &= e^{x \ln a} && \text{Power Rule for logs} \\ &= e^{(\ln a)x}. && \text{Exponent rearranged} \end{aligned}$$

Every exponential function is a power of the natural exponential function.

$$a^x = e^{x \ln a}$$

That is, a^x is the same as e^x raised to the power $\ln a$: $a^x = e^{kx}$ for $k = \ln a$.

For example,

$$2^x = e^{(\ln 2)x} = e^{x \ln 2}, \quad \text{and} \quad 5^{-3x} = e^{(\ln 5)(-3x)} = e^{-3x \ln 5}.$$

Returning once more to the properties of a^x and $\log_a x$, we have

$$\begin{aligned} \ln x &= \ln(a^{\log_a x}) && \text{Inverse Property for } a^x \text{ and } \log_a x \\ &= (\log_a x)(\ln a). && \text{Power Rule for logarithms, with } r = \log_a x \end{aligned}$$

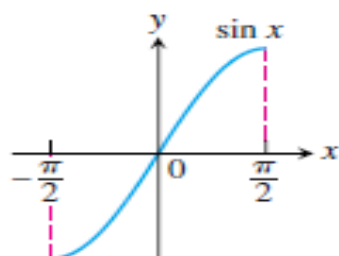
Rewriting this equation as $\log_a x = (\ln x)/(\ln a)$ shows that every logarithmic function is a constant multiple of the natural logarithm $\ln x$. This allows us to extend the algebraic properties for $\ln x$ to $\log_a x$. For instance, $\log_a bx = \log_a b + \log_a x$.

Change of Base Formula

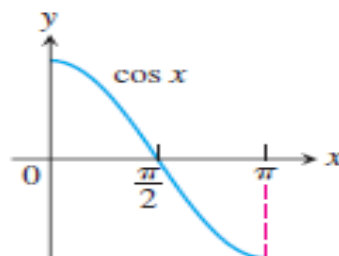
Every logarithmic function is a constant multiple of the natural logarithm.

$$\log_a x = \frac{\ln x}{\ln a} \quad (a > 0, a \neq 1)$$

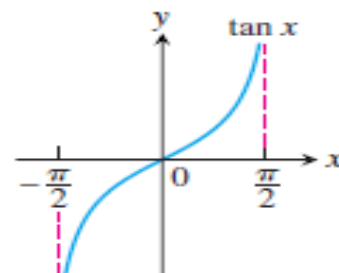
Domain restrictions that make the trigonometric functions one-to-one



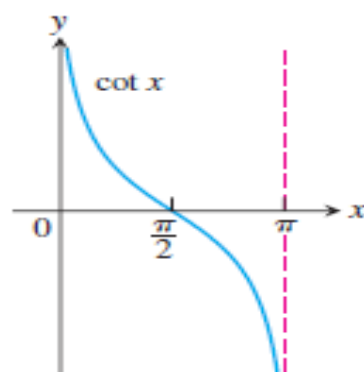
$y = \sin x$
Domain: $[-\pi/2, \pi/2]$
Range: $[-1, 1]$



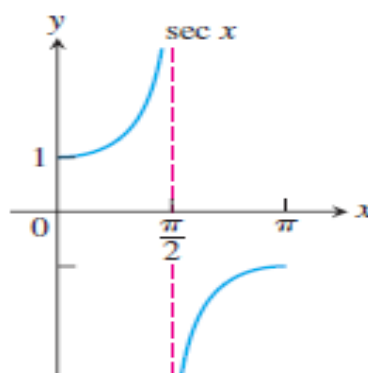
$y = \cos x$
Domain: $[0, \pi]$
Range: $[-1, 1]$



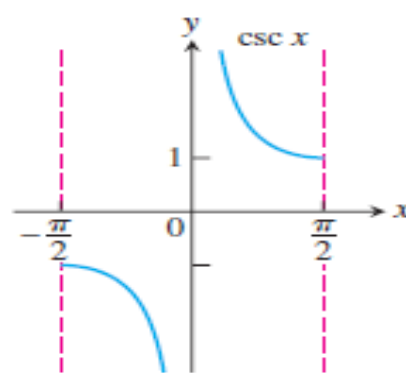
$y = \tan x$
Domain: $(-\pi/2, \pi/2)$
Range: $(-\infty, \infty)$



$y = \cot x$
Domain: $(0, \pi)$
Range: $(-\infty, \infty)$



$y = \sec x$
Domain: $[0, \pi/2) \cup (\pi/2, \pi]$
Range: $(-\infty, -1] \cup [1, \infty)$



$y = \csc x$
Domain: $[-\pi/2, 0) \cup (0, \pi/2]$
Range: $(-\infty, -1] \cup [1, \infty)$

Since these restricted functions are now one-to-one, they have inverses, which we denote by

$$y = \sin^{-1} x \quad \text{or} \quad y = \arcsin x$$

$$y = \cos^{-1} x \quad \text{or} \quad y = \arccos x$$

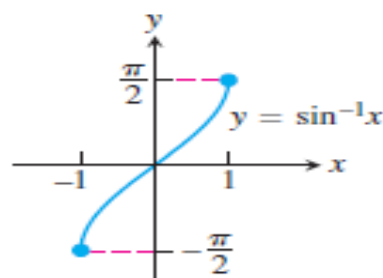
$$y = \tan^{-1} x \quad \text{or} \quad y = \arctan x$$

$$y = \cot^{-1} x \quad \text{or} \quad y = \operatorname{arccot} x$$

$$y = \sec^{-1} x \quad \text{or} \quad y = \operatorname{arcsec} x$$

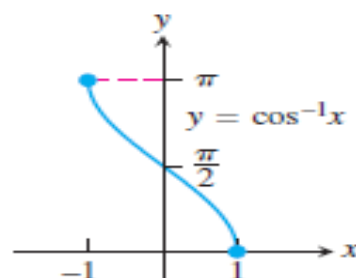
$$y = \csc^{-1} x \quad \text{or} \quad y = \operatorname{arccsc} x$$

Domain: $-1 \leq x \leq 1$
Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$



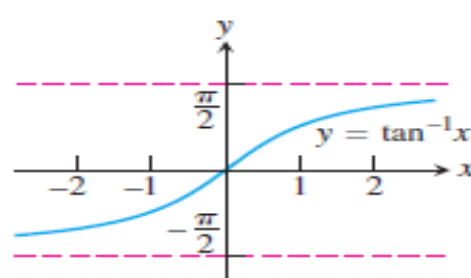
(a)

Domain: $-1 \leq x \leq 1$
Range: $0 \leq y \leq \pi$



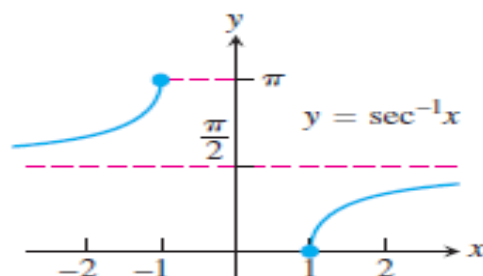
(b)

Domain: $-\infty < x < \infty$
Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$



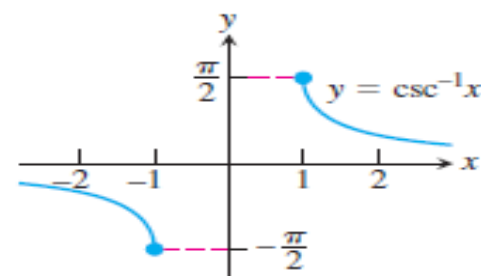
(c)

Domain: $x \leq -1$ or $x \geq 1$
Range: $0 \leq y \leq \pi, y \neq \frac{\pi}{2}$



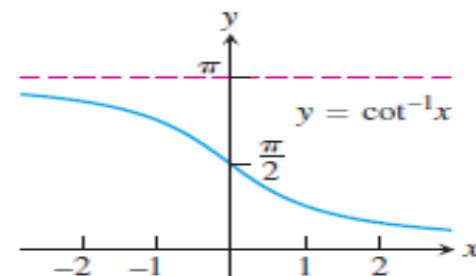
(d)

Domain: $x \leq -1$ or $x \geq 1$
Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$



(e)

Domain: $-\infty < x < \infty$
Range: $0 < y < \pi$

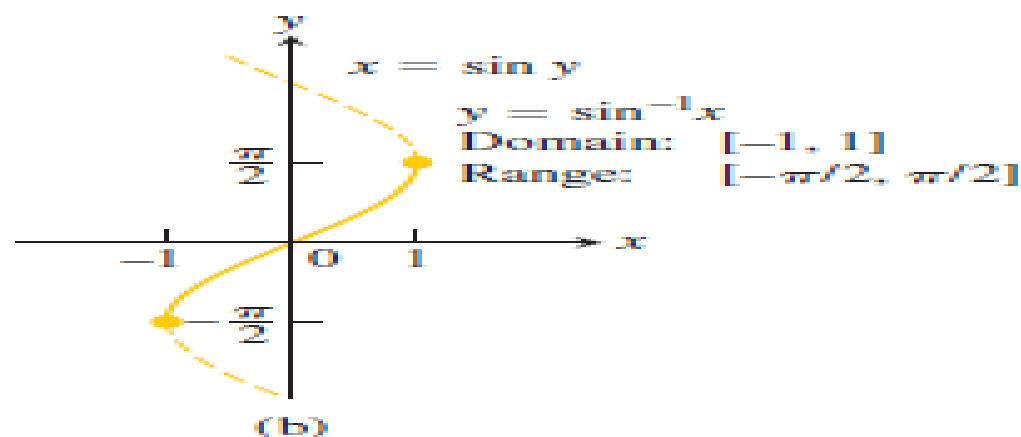
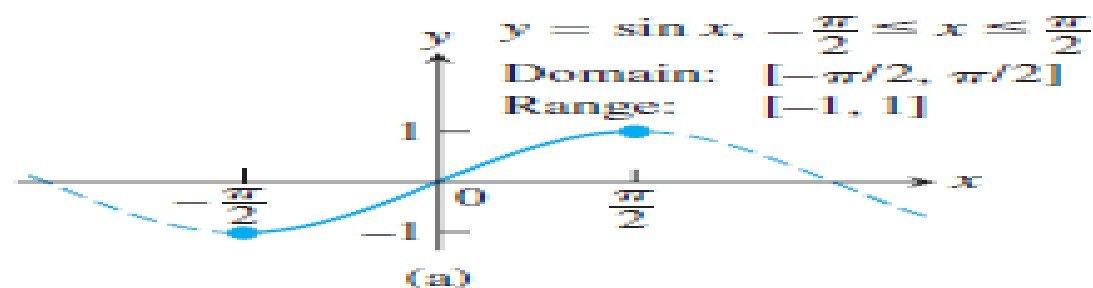


(f)

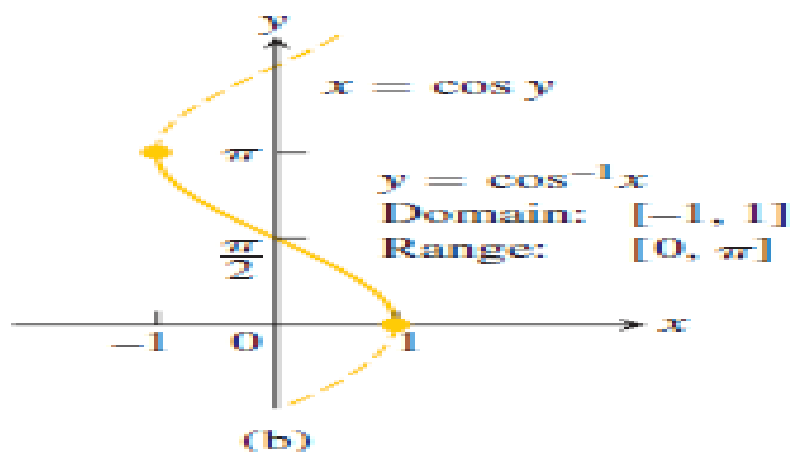
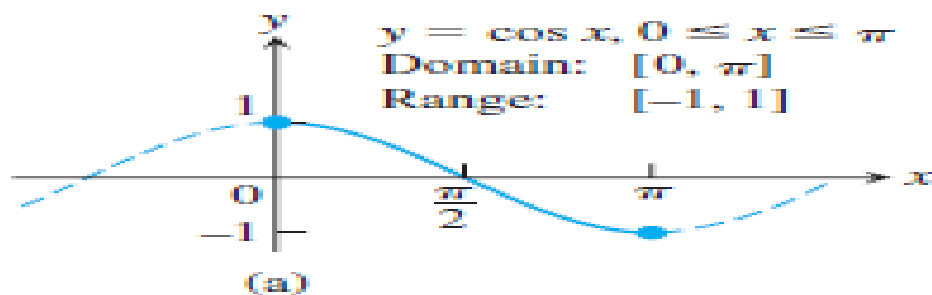
DEFINITION

$y = \sin^{-1} x$ is the number in $[-\pi/2, \pi/2]$ for which $\sin y = x$.

$y = \cos^{-1} x$ is the number in $[0, \pi]$ for which $\cos y = x$.



The graphs of
(a) $y = \sin x, -\pi/2 \leq x \leq \pi/2$, and
(b) its inverse, $y = \sin^{-1} x$. The graph of $\sin^{-1} x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \sin y$.

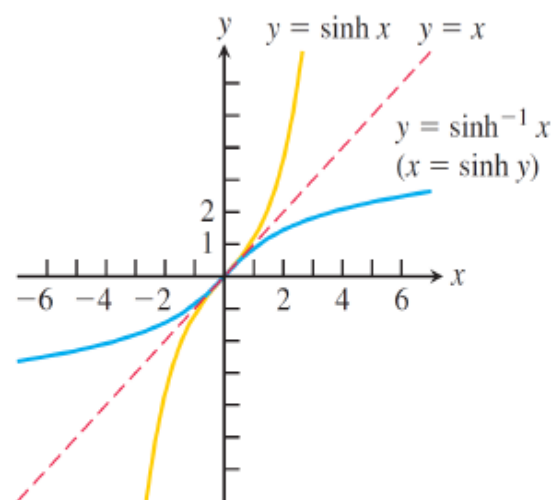


The graphs of
 (a) $y = \cos x, 0 \leq x \leq \pi$, and
 (b) its inverse, $y = \cos^{-1} x$. The graph of $\cos^{-1} x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \cos y$.

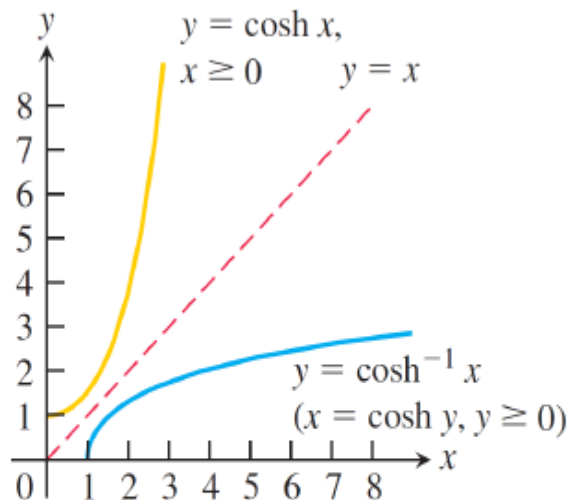
Inverse Hyperbolic Functions

The function $y = \cosh x$ is not one-to-one because its graph does not pass the horizontal line test. The restricted function $y = \cosh x, x \geq 0$, however, is one-to-one and therefore has an inverse, denoted by

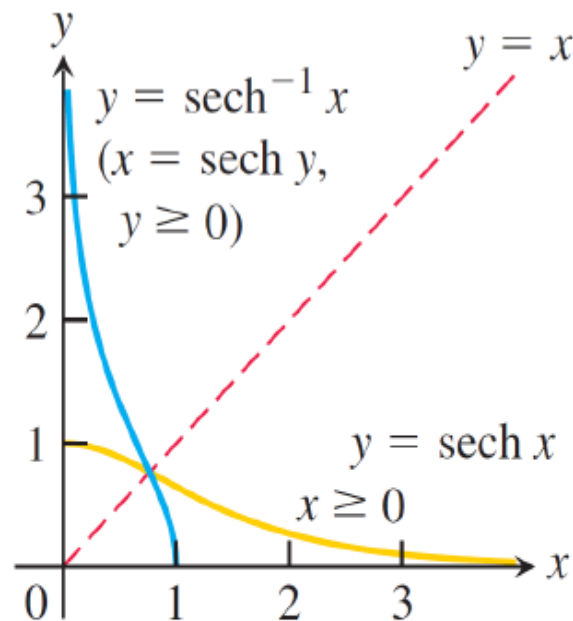
$$y = \cosh^{-1} x.$$



(a)



(b)



(c)

(a) Graphs of $y = \sinh x$ and $y = \sinh^{-1} x$.

(b) Graphs of $y = \cosh x$ and $y = \cosh^{-1} x$.

(c) Graphs of $y = \operatorname{sech} x$ and $y = \operatorname{sech}^{-1} x$.

The graphs of the inverse hyperbolic sine, cosine, and secant of x . Notice the symmetries about the line $y = x$.

For every value of $x \geq 1$, $y = \cosh^{-1} x$ is the number in the interval $0 \leq y < \infty$ whose hyperbolic cosine is x .

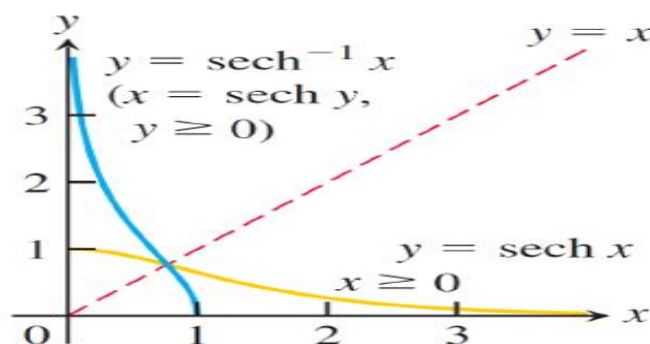
Like $y = \cosh x$, the function $y = \operatorname{sech} x = \frac{1}{\cosh x}$ fails to be one-to-one, but its restriction to nonnegative values of x does have an inverse, denoted by

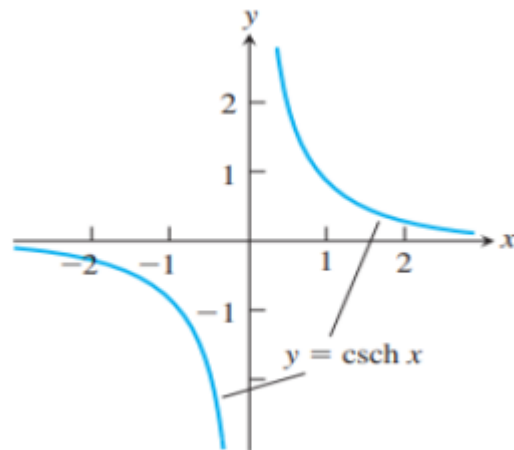
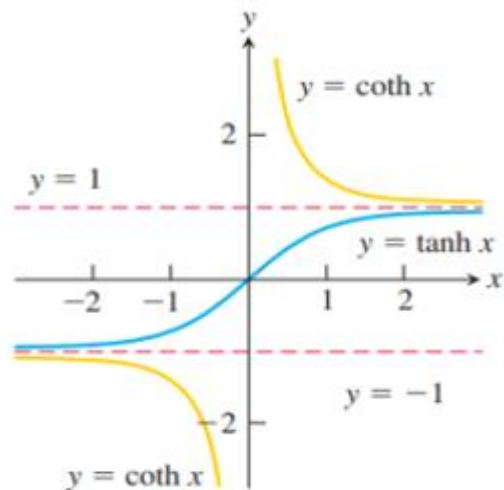
$$y = \operatorname{sech}^{-1} x.$$

For every value of x in the interval $(0, 1]$, $y = \operatorname{sech}^{-1} x$ is the nonnegative number whose hyperbolic secant is x .

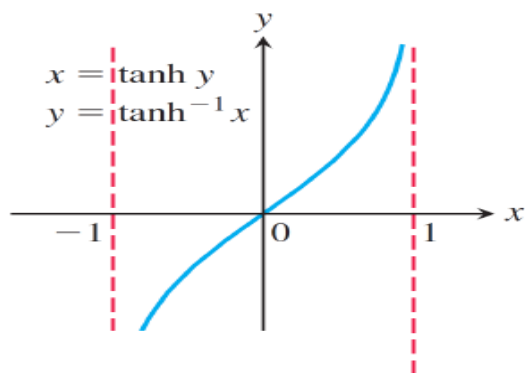
The hyperbolic tangent, cotangent, and cosecant are one-to-one on their domains and therefore have inverses, denoted by

$$y = \tanh^{-1} x, \quad y = \coth^{-1} x, \quad y = \operatorname{csch}^{-1} x.$$

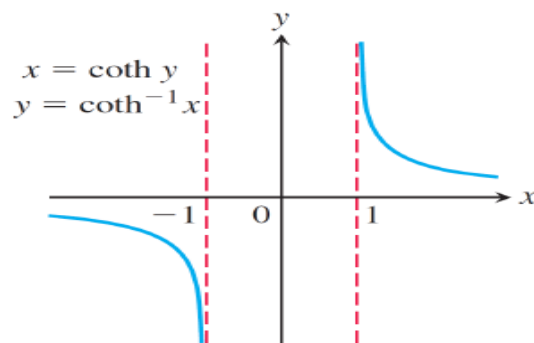




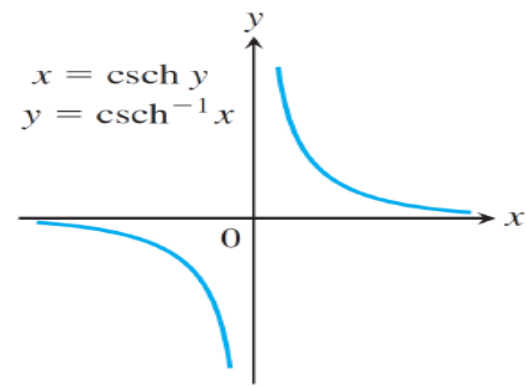
These functions are graphed as follows:



(a)



(b)



(c)

(a) Graphs of $x = \tanh y$ and $y = \tanh^{-1} x$.

(b) Graphs of $x = \coth y$ and $y = \coth^{-1} x$.

(c) Graphs of $x = \operatorname{csch} y$ and $y = \operatorname{csch}^{-1} x$.

The graphs of the inverse hyperbolic tangent, cotangent, and cosecant of x .

$$\begin{aligned}\operatorname{sech}^{-1} x &= \cosh^{-1} \left(\frac{1}{x} \right) \\ \operatorname{csch}^{-1} x &= \sinh^{-1} \left(\frac{1}{x} \right) \\ \operatorname{coth}^{-1} x &= \tanh^{-1} \left(\frac{1}{x} \right)\end{aligned}$$

Identities for inverse hyperbolic functions

Useful Identities We can use the identities in table 1 to express $\operatorname{sech}^{-1} x$, $\operatorname{csch}^{-1} x$, and $\operatorname{coth}^{-1} x$ in terms of $\cosh^{-1} x$, $\sinh^{-1} x$, and $\tanh^{-1} x$. These identities are direct consequences of the definitions. For example, if $0 < x \leq 1$, then

$$\operatorname{sech} \left(\cosh^{-1} \left(\frac{1}{x} \right) \right) = \frac{1}{\cosh \left(\cosh^{-1} \left(\frac{1}{x} \right) \right)} = \frac{1}{\frac{1}{x}} = x.$$

We also know that $\operatorname{sech} (\operatorname{sech}^{-1} x) = x$, so because the hyperbolic secant is one-to-one on $(0, 1]$, we have

$$\cosh^{-1} \left(\frac{1}{x} \right) = \operatorname{sech}^{-1} x.$$

Simplify the expression $\cos(\tan^{-1} x)$.

Solution. Let $y = \tan^{-1} x$. In this case, $\tan y = x$ and $-\pi/2 < y < \pi/2$. To find $\cos y$, we use the trigonometric identity involving $\tan y$:

$$\sec^2 y = 1 + \tan^2 y = 1 + x^2.$$

Thus:

$$\sec y = \sqrt{1 + x^2} \quad (\text{since } -\pi/2 < y < \pi/2 \text{ ensures } \sec y > 0).$$

Now, since $\cos y = \frac{1}{\sec y}$, we have:

$$\cos(\tan^{-1} x) = \cos y = \frac{1}{\sec y} = \frac{1}{\sqrt{1 + x^2}}.$$

Find $\sec\left(\tan^{-1}\frac{x}{3}\right)$.

Solution. Let $\theta = \tan^{-1}\frac{x}{3}$ (just to assign a name to the angle), and note that:

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{x}{3}.$$

We represent θ in a right triangle. The hypotenuse of the triangle is:

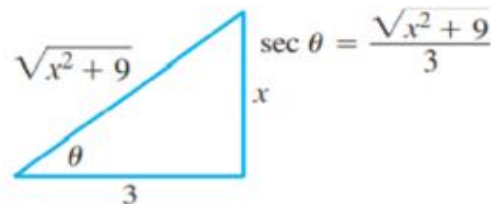
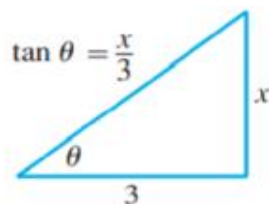
$$\sqrt{x^2 + 3^2} = \sqrt{x^2 + 9}.$$

Using the triangle:

$$\sec \theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{\sqrt{x^2 + 9}}{3}.$$

Thus:

$$\sec\left(\tan^{-1}\frac{x}{3}\right) = \sec \theta = \frac{\sqrt{x^2 + 9}}{3}.$$



Summary:

$$\sec\left(\tan^{-1}\frac{x}{3}\right) = \frac{\sqrt{x^2 + 9}}{3}.$$

Find the domain of the function:

$$f(x) = 5 + \log_{\frac{1}{2}}((x^2 - 4)^3).$$

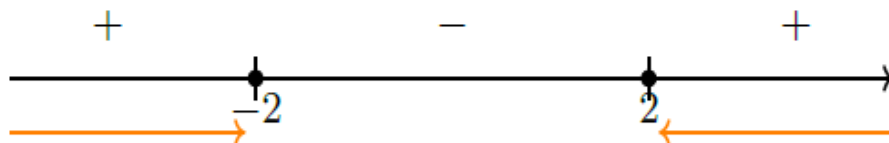
Solution. For the logarithmic function to be defined, the argument must be positive:

$$x^2 - 4 > 0.$$

Solving the inequality:

$$x^2 > 4 \implies x > 2 \quad \text{or} \quad x < -2.$$

Using a sign chart for $x^2 - 4$:



Thus, the domain of $f(x)$ is:

$$\text{Domain: } (-\infty, -2) \cup (2, \infty).$$

Alternatively:

$$\text{Domain: } \mathbb{R} \setminus [-2, 2].$$

Find the domain and range of each of the following functions:

Solution. (a) $f(x) = \sqrt{x-2} + 3$, $2 \leq x \leq 4$

To satisfy the square root's domain, $x - 2 \geq 0$, so:

$$x \geq 2.$$

Since $x \leq 4$ is also given:

$$\text{Domain: } D_f = [2, 4].$$

For the range:

$$f(x) = \sqrt{x-2} + 3, \quad \text{where } x - 2 \text{ varies from } 0 \text{ to } 2.$$

Thus:

$$\text{Range: } R_f = [3, 3 + \sqrt{2}].$$

(b) $f(x) = \frac{x}{1-x}$

The function is undefined when $1 - x = 0$, i.e., $x = 1$. Thus:

$$\text{Domain: } D_f = (-\infty, 1) \cup (1, \infty).$$

To find the range, solve $f(x) = y$:

$$\frac{x}{1-x} = y \implies x = y - xy \implies x(1+y) = y \implies x = \frac{y}{1+y}.$$

The function $x = \frac{y}{1+y}$ is undefined when $y = -1$. Thus:

$$\text{Range: } R_f = (-\infty, -1) \cup (-1, \infty).$$

(c) $f(x) = \frac{1}{\sqrt{x-2}}$

For the square root to be defined, $x - 2 > 0$, so:

$$x > 2.$$

Thus:

$$\text{Domain: } D_f = (2, \infty).$$

For the range:

$$f(x) = \frac{1}{\sqrt{x-2}}.$$

As $x \rightarrow \infty$, $f(x) \rightarrow 0^+$, and as $x \rightarrow 2^+$, $f(x) \rightarrow \infty$. Thus:

$$\text{Range: } R_f = (0, \infty).$$

Let $f(x) = \frac{1}{x-2}$ and $g(x) = \frac{x}{1-x}$.

1. Compute $(f \circ g)(x)$ and its domain.

2. Compute $(g \circ f)(x)$ and its domain.

Solution. (a) Compute $f(g(x))$:

$$f(g(x)) = \frac{1}{g(x) - 2} = \frac{1}{\frac{x}{1-x} - 2}.$$

Simplify the denominator:

$$\frac{x}{1-x} - 2 = \frac{x - 2(1-x)}{1-x} = \frac{x - 2 + 2x}{1-x} = \frac{3x - 2}{1-x}.$$

Thus:

$$f(g(x)) = \frac{1}{\frac{3x-2}{1-x}} = \frac{1-x}{3x-2}.$$

The domain of $f(g(x))$ is determined by ensuring that $g(x) \neq 2$ and $f(g(x))$ remains defined:

$$g(x) = \frac{x}{1-x} \neq 2$$

Additionally, $g(x)$ is undefined at $x = 1$. Therefore:

$$D_{f \circ g} = (-\infty, \frac{2}{3}) \cup (\frac{2}{3}, 1) \cup (1, \infty).$$

(b) Compute $g(f(x))$:

$$g(f(x)) = \frac{f(x)}{1 - f(x)} = \frac{\frac{1}{x-2}}{1 - \frac{1}{x-2}}.$$

Simplify the denominator:

$$1 - \frac{1}{x-2} = \frac{x-2-1}{x-2} = \frac{x-3}{x-2}.$$

Thus:

$$g(f(x)) = \frac{\frac{1}{x-2}}{\frac{x-3}{x-2}} = \frac{1}{x-3}.$$

The domain of $g(f(x))$ is determined by ensuring $f(x) \neq 1$ and $g(f(x))$ remains defined:

$$f(x) = \frac{1}{x-2} \neq 1 \implies \frac{1}{x-2} \neq 1 \implies x \neq 3.$$

Additionally, $f(x)$ is undefined at $x = 2$. Therefore:

$$D_{g \circ f} = (-\infty, 2) \cup (2, 3) \cup (3, \infty).$$

Given the functions $f(x) = e^{3x-1}$ and $g(x) = \ln(x-1)$:

(a) Find the domain of $g \circ f$.

(b) Find the domain of $\frac{f}{g}$.

Solution. (a) To find the domain of $g \circ f$, we first note that:

$$g \circ f(x) = g(f(x)) = \ln(e^{3x-1} - 1).$$

For $\ln(e^{3x-1} - 1)$ to be defined, we require:

$$e^{3x-1} - 1 > 0 \implies e^{3x-1} > 1.$$

Taking the natural logarithm of both sides:

$$3x - 1 > 0 \implies x > \frac{1}{3}.$$

Therefore, the domain of $g \circ f$ is:

$$D_{g \circ f} = \left(\frac{1}{3}, \infty\right).$$

(b) To find the domain of $\frac{f}{g}$, we start with the definition:

$$\frac{f}{g}(x) = \frac{e^{3x-1}}{\ln(x-1)}.$$

For $\frac{f}{g}(x)$ to be defined, both e^{3x-1} and $\ln(x-1)$ must exist, and $\ln(x-1) \neq 0$.

- The function $\ln(x-1)$ is defined for $x-1 > 0$, i.e., $x > 1$.
- $\ln(x-1) \neq 0$ implies:

$$\ln(x-1) \neq 0 \implies x-1 \neq e^0 = 1 \implies x \neq 2.$$

Combining these restrictions:

$$x > 1 \quad \text{and} \quad x \neq 2.$$

Thus, the domain of $\frac{f}{g}$ is:

$$D_{f/g} = (1, 2) \cup (2, \infty).$$

Given the functions $f(x) = \arcsin(x)$ and $g(x) = \sin(x)$:

- (a) Find the domain of f and g .
- (b) Find the domain of $f + g$.
- (c) Find the domain of $g \circ f$.
- (d) Find the domain of $\frac{f}{g}$.

Solution. (a) Finding the domain of f and g :

- For $f(x) = \arcsin(x)$: The function $\arcsin(x)$ is defined only when $-1 \leq x \leq 1$. Therefore, the domain of f is:

$$D_f = [-1, 1].$$

- For $g(x) = \sin(x)$: The sine function is defined for all real numbers. Therefore, the domain of g is:

$$D_g = (-\infty, \infty).$$

- (b) **Finding the domain of $f + g$:** The sum $f + g$ is defined only when both $f(x)$ and $g(x)$ are defined. Thus, the domain of $f + g$ is the intersection of the domains of f and g :

$$D_{f+g} = D_f \cap D_g.$$

Substituting the domains:

$$D_f = [-1, 1], \quad D_g = (-\infty, \infty).$$

The intersection is:

$$D_{f+g} = [-1, 1].$$

Final Result:

- Domain of f : $[-1, 1]$.
- Domain of g : $(-\infty, \infty)$.
- Domain of $f + g$: $[-1, 1]$.

(c) Finding the domain of $g \circ f$: The composition $g \circ f(x)$ is defined as:

$$(g \circ f)(x) = g(f(x)) = \sin(\arcsin(x)).$$

Since $\arcsin(x)$ is the inverse of $\sin(x)$, we have:

$$\sin(\arcsin(x)) = x.$$

Therefore, $g \circ f(x)$ is defined wherever $\arcsin(x)$ is defined. From part (a), we know that $\arcsin(x)$ is defined for $x \in [-1, 1]$. Thus:

$$D_{g \circ f} = [-1, 1].$$

(d) **Finding the domain of $\frac{f}{g}$:** The function $\frac{f}{g}(x)$ is defined as:

$$\frac{f}{g}(x) = \frac{\arcsin(x)}{\sin(x)}.$$

For $\frac{f}{g}(x)$ to be defined, both $\arcsin(x)$ and $\sin(x)$ must be defined, and $\sin(x) \neq 0$. From part (a), we know:

$$D_f = [-1, 1], \quad D_g = (-\infty, \infty).$$

The intersection of D_f and D_g is:

$$D_f \cap D_g = [-1, 1].$$

Next, exclude the points where $\sin(x) = 0$. The sine function is zero at $x = 0$. Therefore:

$$D_{f/g} = [-1, 1] \setminus \{0\} = [-1, 0) \cup (0, 1].$$

Final Results:

- Domain of $g \circ f$: $[-1, 1]$.
- Domain of $\frac{f}{g}$: $[-1, 0) \cup (0, 1]$.

Absolute value function:

Let $A \subset \mathbb{R}$ and $f: A \rightarrow \mathbb{R}$

$$|f(x)| = \begin{cases} f(x) & , f(x) \geq 0 \\ -f(x) & , f(x) < 0 \end{cases}$$

$$f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$$f(x) = \frac{|x|}{x} = \begin{cases} \frac{x}{x} & x \geq 0 \\ -\frac{x}{x} & x < 0 \end{cases}$$

$$= \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

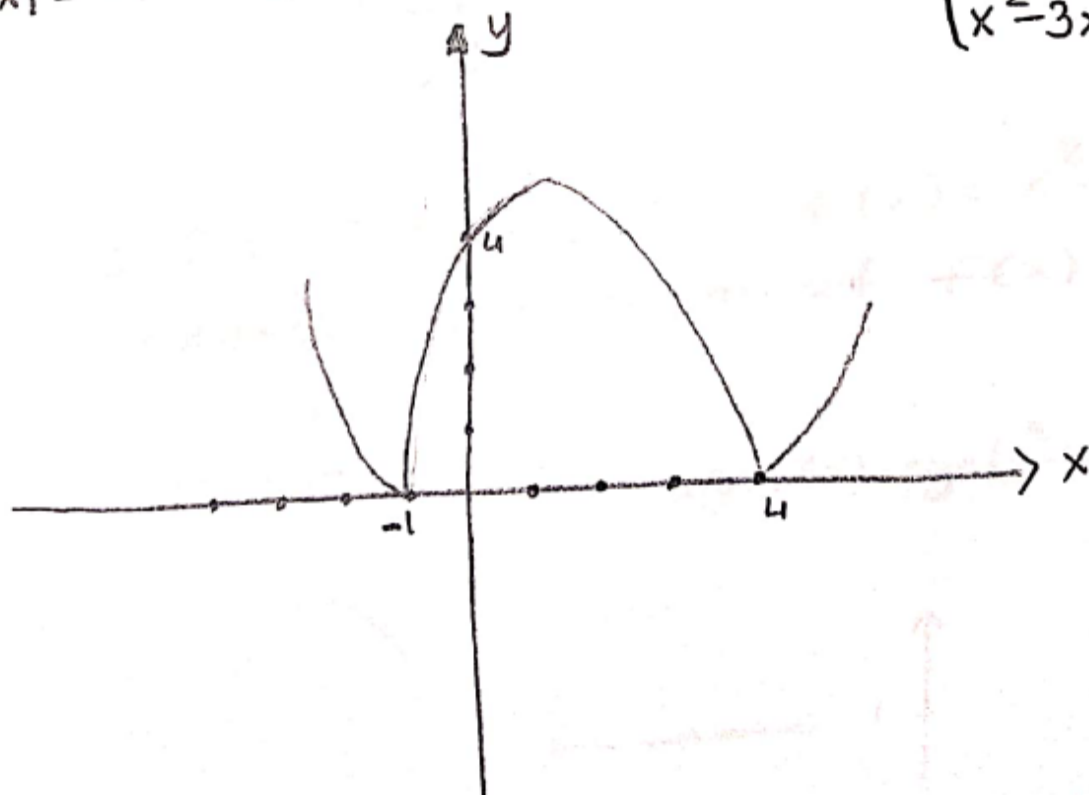
Let

$$f(x) = | -x^2 + 3x + 4 | \quad ; \quad f: \mathbb{R} \rightarrow \mathbb{R}$$

$$-x^2 + 3x + 4 = 0$$

$$x_1 = -1 \quad x_2 = 4$$

$$f(x) = \begin{cases} x^2 - 3x - 4 & x < -1 \\ -x^2 + 3x + 4 & -1 \leq x \leq 4 \\ x^2 - 3x - 4 & x > 4 \end{cases}$$



$$y = -x^2 + 3x + 4$$

Solve the inequality

$$|x-3| + |x+2| < 11$$

$$|x-3| = \begin{cases} x-3 & \text{if } x-3 \geq 0 \\ -(x-3) & \text{if } x-3 < 0 \end{cases} = \begin{cases} x-3 & \text{if } x \geq 3 \\ -x+3 & \text{if } x < 3 \end{cases}$$

$$|x+2| = \begin{cases} x+2 & \text{if } x \geq -2 \\ -x-2 & \text{if } x < -2 \end{cases}$$

$$x < -2 \quad -2 \leq x < 3 \quad x \geq 3$$

Case 1: if $x < -2$

$$|x-3| + |x+2| < 11$$

$$-x+3 -x-2 < 11$$

$$x > -5$$

Case 2: if $-2 \leq x < 3$
 $|x-3| + |x+2| < 11$

$$-x+3 + x+2 < 11$$

$$5 < 11 \text{ always true}$$

Case 3: if $x \geq 3$
 $|x-3| + |x+2| < 11$

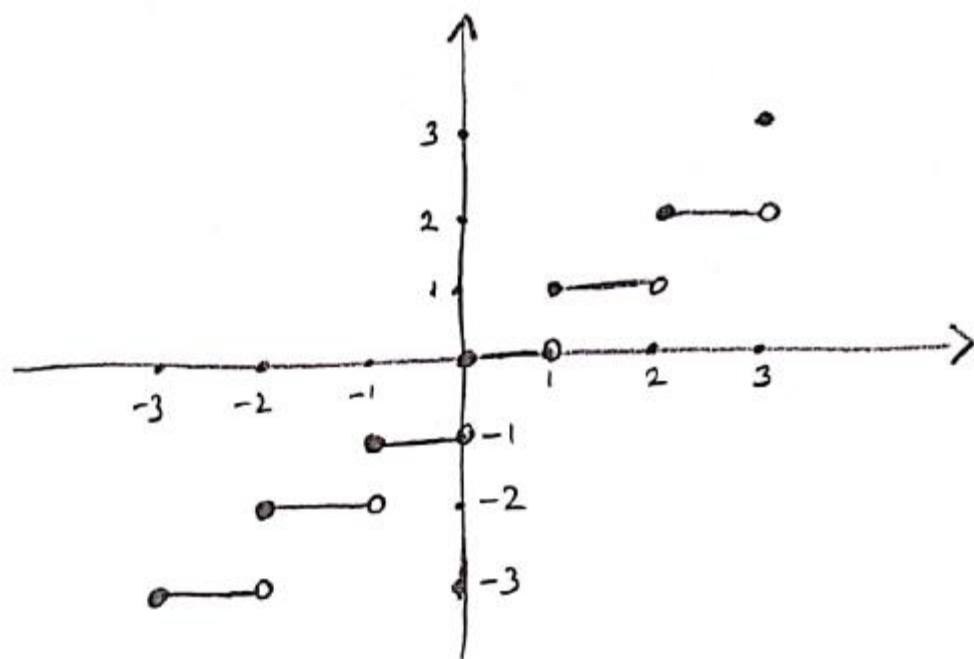
$$x-3 + x+2 < 11$$

$$x < 6$$

Combining cases I, II and III,
the inequality^{is} satisfied when $-5 < x < 6$

So the solution is in the interval $(-5, 6)$

Ex: $y = \lfloor x \rfloor$, $f: [-3, 3] \rightarrow \mathbb{R}$



$$0 \leq x < 1 \Rightarrow \lfloor x \rfloor = 0$$

$$1 \leq x < 2 \Rightarrow \lfloor x \rfloor = 1$$

$$2 \leq x < 3 \Rightarrow \lfloor x \rfloor = 2$$

$$-3 \leq x < -2 \Rightarrow \lfloor x \rfloor = -3$$

$$-2 \leq x < -1 \Rightarrow \lfloor x \rfloor = -2$$

$$-1 \leq x < 0 \Rightarrow \lfloor x \rfloor = -1$$

(Sgn) function:

Let $f: A \rightarrow \mathbb{R}$ ($A \subset \mathbb{R}$)

$$g(x) = \begin{cases} \frac{|f(x)|}{f(x)} & f(x) \neq 0 \\ 0 & f(x) = 0 \end{cases}$$

$$\operatorname{sgn} f(x) = \begin{cases} 1 & f(x) > 0 \\ 0 & f(x) = 0 \\ -1 & f(x) < 0 \end{cases}$$

Ex: Let $f: [-3, 3] \rightarrow \mathbb{R}$, $f(x) = x^2 - x - 2$
sketch the graph of $f(x)$.

x	-3	-1	2	3
$x^2 - x - 2$	+	0	-	+

$$\operatorname{sgn} f(x) = \operatorname{sgn}(x^2 - x - 2) = \begin{cases} 1 & x \in [-3, -1) \cup (2, 3] \\ 0 & x \in \{-1, 2\} \\ -1 & x \in (-1, 2) \end{cases}$$

$$\begin{aligned} x^2 - x - 2 &= 0 \\ x_1 &= -1 \quad x_2 = 2 \end{aligned}$$

