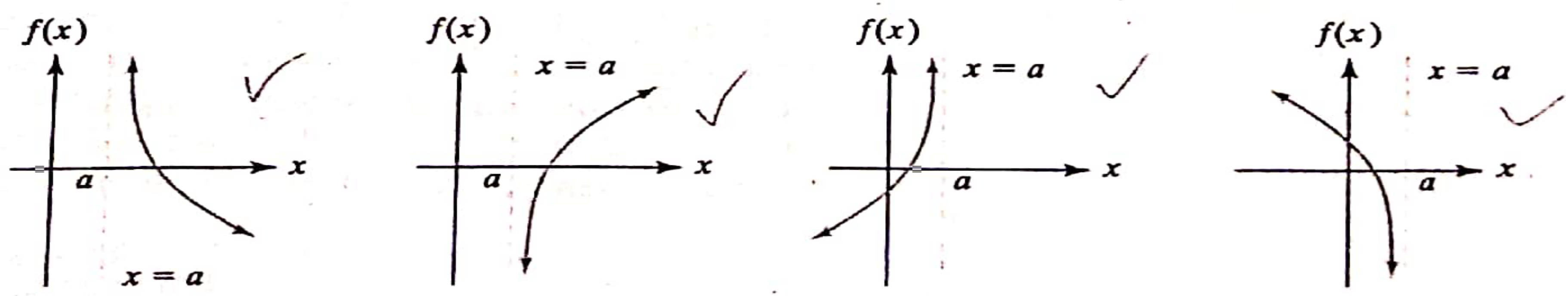
Vertical Asymptotes

In this section, we conclude our discussion of curve-sketching techniques by investigating functions having *asymptotes*. An asymptote is a line that a curve approaches arbitrarily closely. For example, in each part of Figure , the dashed line $x = a$ is an asymptote. But to be precise about it, we need to make use of infinite limits.



(a) (b) (c) (d)

In Figure (a), notice that as $x \to a^+$, $f(x)$ becomes positively infinite:

$$\lim_{x \to a^+} f(x) = \infty$$

In Figure (b), as $x \to a^+$, $f(x)$ becomes negatively infinite:

$$\lim_{x \to a^+} f(x) = -\infty$$

In Figures (c) and (d), we have

$$\lim_{x \to a^-} f(x) = \infty \quad \text{and} \quad \lim_{x \to a^-} f(x) = -\infty$$

The line $x = a$ is called a *vertical asymptote* for the graph. A vertical asymptote is not part of the graph but is a useful aid in sketching it because part of the graph approaches the asymptote. Because of the explosion around $x = a$, the function is *not* continuous at a.

Definition

The line $x = a$ is a ***vertical asymptote*** for the graph of the function f if and only if at least one of the following is true:

$$\lim_{x \to a^+} f(x) = \pm\infty$$

or

$$\lim_{x \to a^-} f(x) = \pm\infty$$

$$f(x) = \frac{3x - 5}{x - 2}$$

When x is 2, the denominator is 0, but the numerator is not. If x is slightly larger than 2, then $x - 2$ is both close to 0 and positive, and $3x - 5$ is close to 1. Thus, $(3x - 5)/(x - 2)$ is very large, so

$$\lim_{x \to 2^+} \frac{3x - 5}{x - 2} = \infty$$

This limit is sufficient to conclude that the line $x = 2$ is a vertical asymptote. Because we are ultimately interested in the behavior of a function around a vertical asymptote, it is worthwhile to examine what happens to this function as x approaches 2 from the left. If x is slightly less than 2, then $x - 2$ is very close to 0 but negative, and $3x - 5$ is close to 1. Hence, $(3x - 5)/(x - 2)$ is "very negative," so

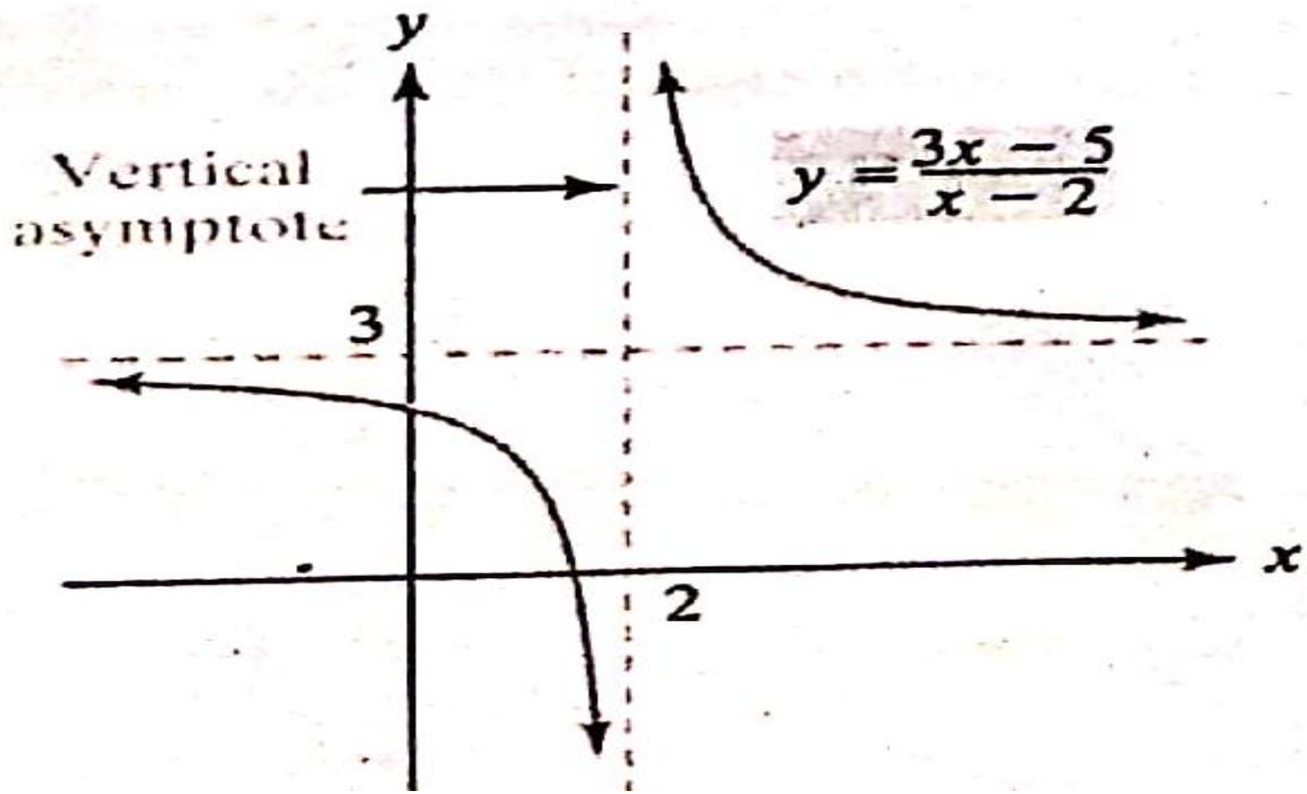
$$\lim_{x \to 2^-} \frac{3x - 5}{x - 2} = -\infty$$

We conclude that the function increases without bound as $x \to 2^+$ and decreases without bound as $x \to 2^-$.

To see that the proviso about *lowest terms* is necessary, observe that

$$f(x) = \frac{3x-5}{x-2} = \frac{(3x-5)(x-2)}{(x-2)^2} \text{ so}$$

that $x = 2$ is a vertical asymptote of $\frac{(3x-5)(x-2)}{(x-2)^2}$, and here 2 makes both the denominator *and* the numerator 0.



Vertical-Asymptote Rule for Rational Functions

Suppose that

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomial functions and the quotient is in lowest terms. The line $x = a$ is a vertical asymptote for the graph of f if and only if $Q(a) = 0$ and $P(a) \neq 0$.

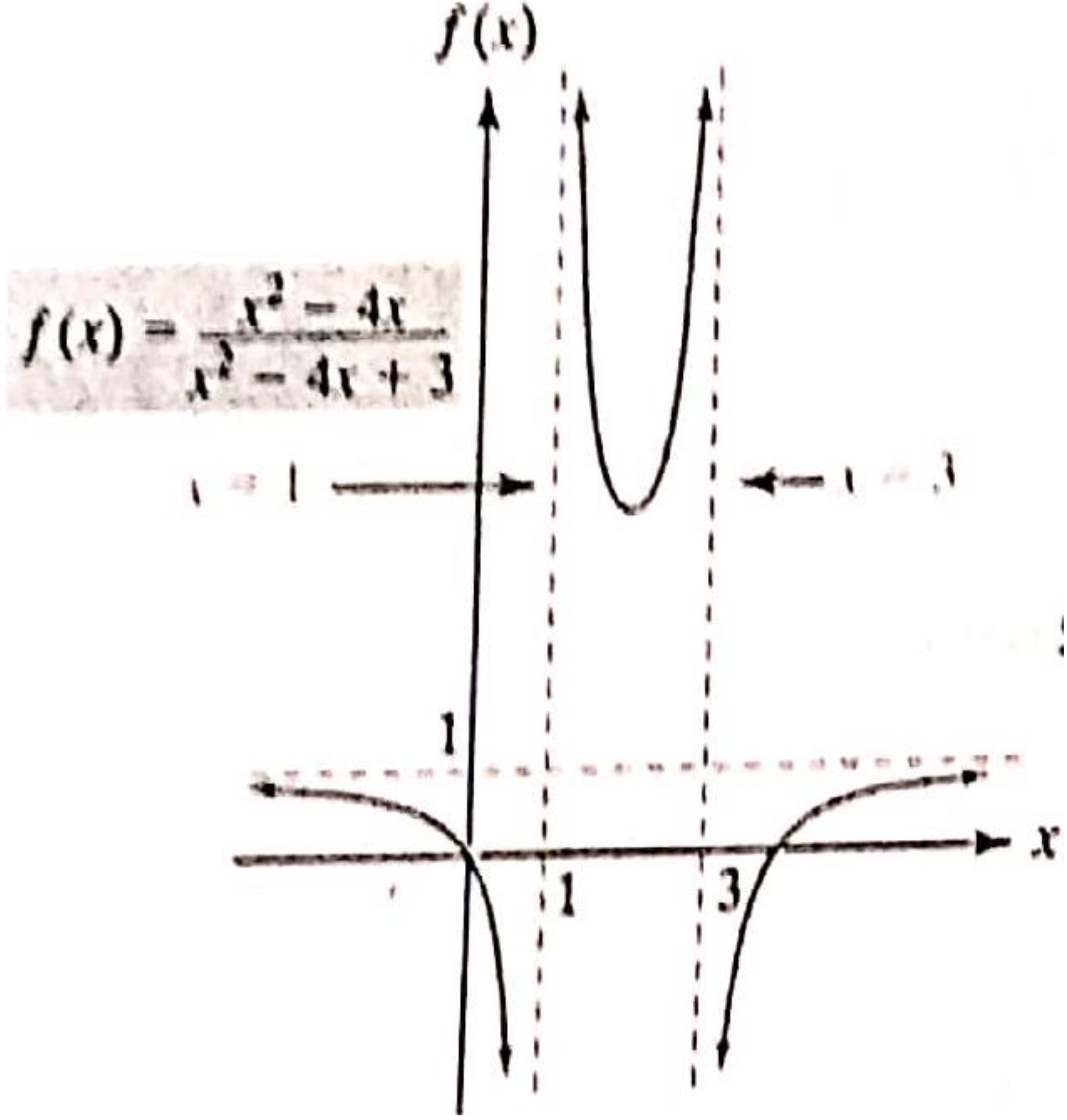
Determine vertical asymptotes for the graph of

$$f(x) = \frac{x^2 - 4x}{x^2 - 4x + 3}$$

Solution: Since f is a rational function, the vertical-asymptote rule applies. Writing

$$f(x) = \frac{x(x-4)}{(x-3)(x-1)} \quad \text{factoring}$$

makes it clear that the denominator is 0 when x is 3 or 1. Neither of these values makes the numerator 0. Thus, the lines $x = 3$ and $x = 1$ are vertical asymptotes.



$f(x)$
$f(x) = \frac{x^2 - 4x}{x^2 - 4x + 3}$
1
1
3
x

Horizontal and Oblique Asymptotes

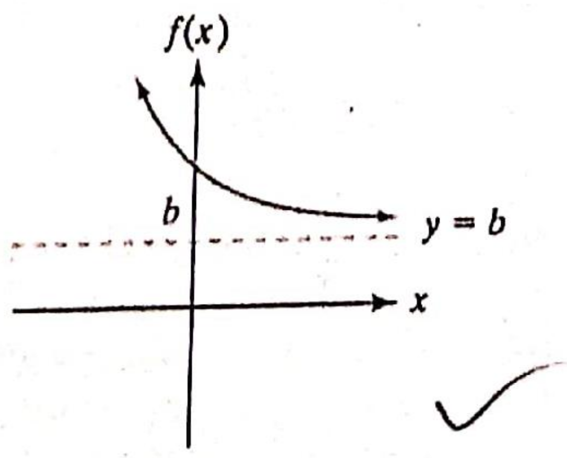
A curve $y = f(x)$ may have other kinds of asymptote. In Figure (a), as x increases without bound ($x \to \infty$), the graph approaches the horizontal line $y = b$. That is,

$$\lim_{x \to \infty} f(x) = b$$

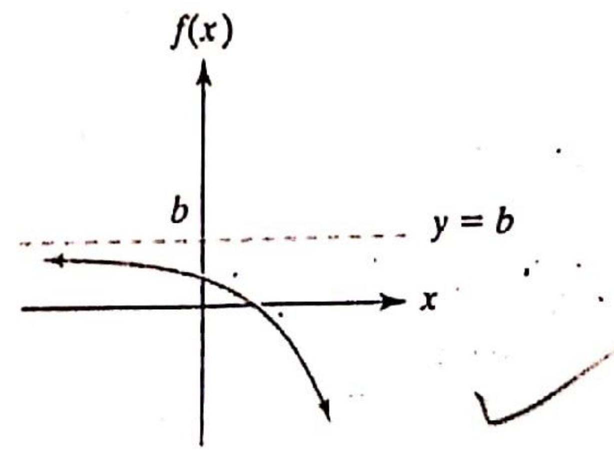
In Figure (b), as x becomes negatively infinite, the graph approaches the horizontal line $y = b$. That is,

$$\lim_{x \to -\infty} f(x) = b$$

In each case, the dashed line $y = b$ is called a *horizontal asymptote* for the graph. It is a horizontal line around which the graph "settles" either as $x \to \infty$ or as $x \to -\infty$.



(a)



(b)

Definition

Let f be a function. The line $y = b$ is a ***horizontal asymptote*** for the graph of f if and only if at least one of the following is true:

$$\lim_{x \to \infty} f(x) = b \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b$$

To test for horizontal asymptotes, we must find the limits of $f(x)$ as $x \to \infty$ and as $x \to -\infty$. To illustrate, we again consider

$$f(x) = \frac{3x - 5}{x - 2}$$

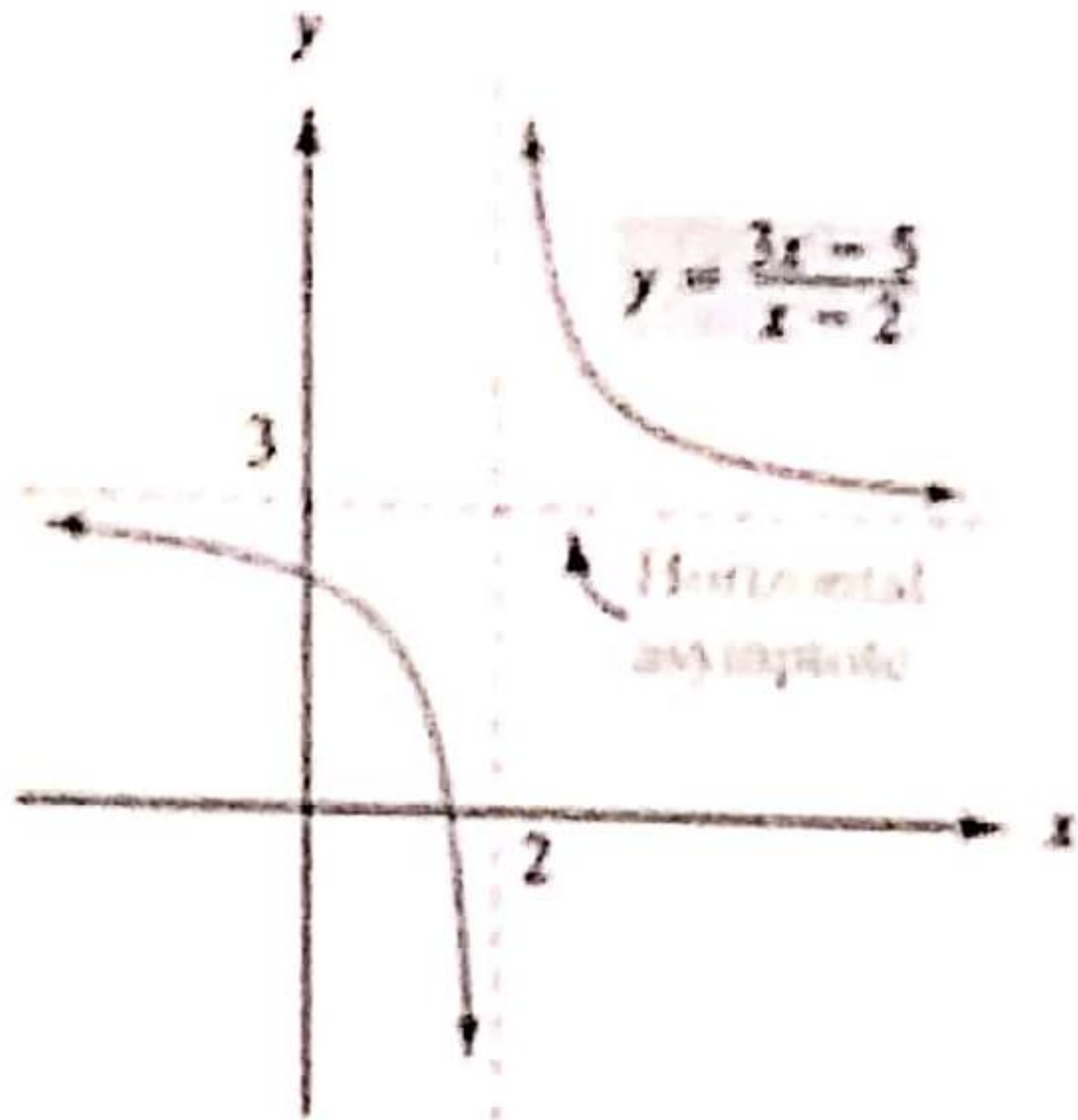
Since this is a rational function, we can use the procedures to find the limits. Because the dominant term in the numerator is $3x$ and the dominant term in the denominator is x, we have

$$\lim_{x \to \infty} \frac{3x - 5}{x - 2} = \lim_{x \to \infty} \frac{3x}{x} = \lim_{x \to \infty} 3 = 3$$

Thus, the line $y = 3$ is a horizontal asymptote. Also,

$$\lim_{x \to -\infty} \frac{3x - 5}{x - 2} = \lim_{x \to -\infty} \frac{3x}{x} = \lim_{x \to -\infty} 3 = 3$$

Hence, the graph settles down near the horizontal line $y = 3$ both as $x \to \infty$ and as $x \to -\infty$.



y
$y = \frac{3x - 5}{x - 2}$
3
Horizontal
asymptote
2
x

Definition

Let f be a function. The line $y = mx + b$ is a ***nonvertical asymptote*** for the graph of f if and only if at least one of the following is true:

$$\lim_{x \to \infty} (f(x) - (mx + b)) = 0 \quad \text{or} \quad \lim_{x \to -\infty} (f(x) - (mx + b)) = 0$$

Of course, if $m = 0$, then we have just repeated the definition of horizontal asymptote. But if $m \neq 0$, then $y = mx + b$ is the equation of a nonhorizontal (and nonvertical) line with slope m that is sometimes described as *oblique*. Thus to say that $\lim_{x \to \infty} (f(x) - (mx + b)) = 0$ is to say that for large values of x, the graph settles down near the line $y = mx + b$, often called an *oblique asympote* for the graph.

If $f(x) = \dfrac{P(x)}{Q(x)}$, where the degree of P is one more than the degree of Q, then long divison allows us to write $\dfrac{P(x)}{Q(x)} = (mx + b) + \dfrac{R(x)}{Q(x)}$, where $m \neq 0$ and where either

$R(x)$ is the zero polynomial or the degree of R is strictly less than the degree of Q.

Find the oblique asyptote for the graph of the rational function

$$y = f(x) = \frac{10x^2 + 9x + 5}{5x + 2}$$

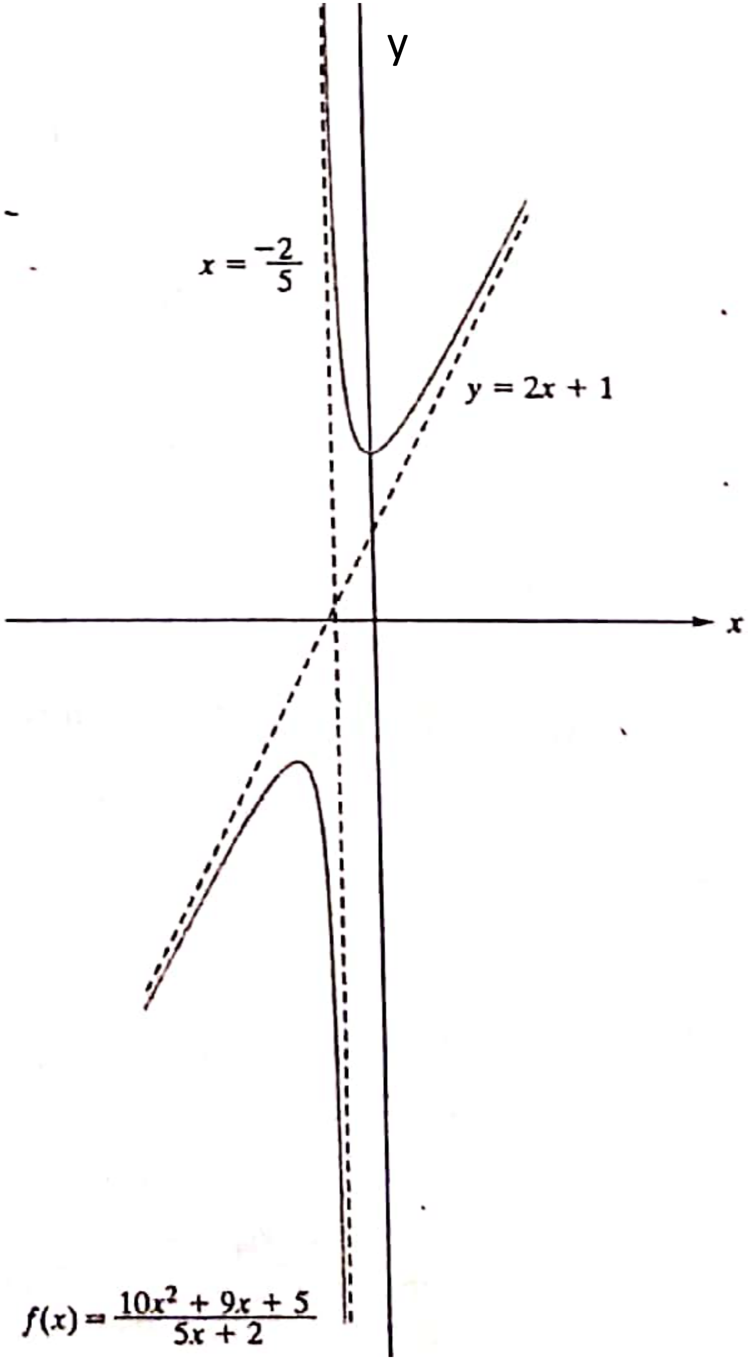
Solution: Since the degree of the numerator is 2, one greater than the degree of the denominator, we use long division to express

$$f(x) = \frac{10x^2 + 9x + 5}{5x + 2} = 2x + 1 + \frac{3}{5x + 2}$$

Thus

$$\lim_{x \to \pm\infty} (f(x) - (2x + 1)) = \lim_{x \to \pm\infty} \frac{3}{5x + 2} = 0$$

which shows that $y = 2x + 1$ is an oblique asymptote, in fact the only nonvertical asymptote, as we explain On the other hand, it is clear that $x = -\frac{2}{5}$ is a vertical asymptote—and the only one.


y
$x = \frac{-2}{5}$
$y = 2x + 1$
x
$f(x) = \frac{10x^2 + 9x + 5}{5x + 2}$

A polynomial function of degree greater than 1 has no asymptotes.

Find vertical and horizontal asymptotes for the graph of the polynomial function

$$y = f(x) = x^3 + 2x$$

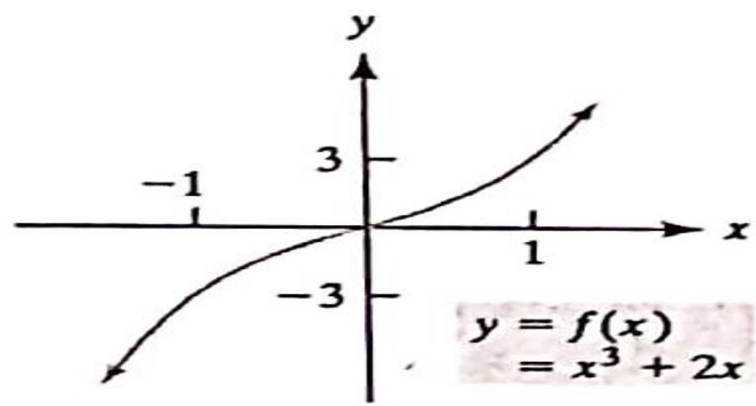
Solution: We begin with vertical asymptotes. This is a rational function with denominator 1, which is never zero. By the vertical-asymptote rule, there are no vertical asymptotes. Because the degree of the numerator (3) is greater than the degree of the denominator (0), there are no horizontal asymptotes. However, let us examine the behavior of the graph of f as $x \to \infty$ and $x \to -\infty$. We have

$$\lim_{x\to\infty} (x^3 + 2x) = \lim_{x\to\infty} x^3 = \infty$$

and

$$\lim_{x\to-\infty} (x^3 + 2x) = \lim_{x\to-\infty} x^3 = -\infty$$

Thus, as $x \to \infty$, the graph must extend indefinitely upward, and as $x \to -\infty$, the graph must extend indefinitely downward.



Graph of $y = x^3 + 2x$ has neither horizontal nor vertical asymptotes.

Find horizontal and vertical asymptotes for the graph of $y = e^x - 1$.

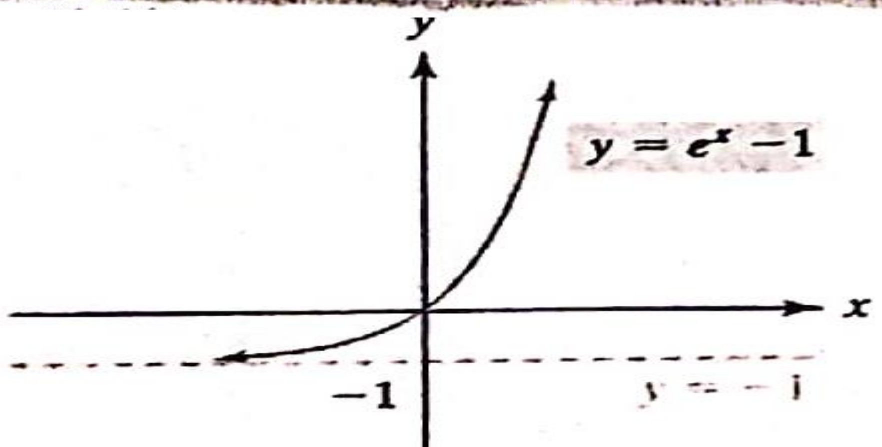
Solution: Testing for horizontal asymptotes, we let $x \to \infty$. Then e^x increases without bound, so

$$\lim_{x \to \infty} (e^x - 1) = \infty$$

Thus, the graph does not settle down as $x \to \infty$. However, as $x \to -\infty$, we have $e^x \to 0$, so

$$\lim_{x \to -\infty} (e^x - 1) = \lim_{x \to -\infty} e^x - \lim_{x \to -\infty} 1 = 0 - 1 = -1$$

Therefore, the line $y = -1$ is a horizontal asymptote. The graph has no vertical asymptotes because $e^x - 1$ neither increases nor decreases without bound around any fixed value of x.



(a) $\lim_{x\to\infty} \frac{5x^2+8x-3}{3x^2+2} = \lim_{x\to\infty} \frac{5+(8/x)-(3/x^2)}{3+(2/x^2)}$ Divide numerator and denominator by x^2.

$$= \frac{5+0-0}{3+0} = \frac{5}{3}$$

(b) $\lim_{x\to-\infty} \frac{11x+2}{2x^3-1} = \lim_{x\to-\infty} \frac{(11/x^2)+(2/x^3)}{2-(1/x^3)}$ Divide numerator and denominator by x^3.

$$= \frac{0+0}{2-0} = 0$$

The graph of the function

$$f(x) = \frac{5x^2+8x-3}{3x^2+2}$$

sketched in Figure has the line $y = 5/3$ as a horizontal asymptote on both the right and the left because

$$\lim_{x\to\infty} f(x) = \frac{5}{3} \quad \text{and} \quad \lim_{x\to-\infty} f(x) = \frac{5}{3}.$$

EXAMPLE Find the horizontal asymptotes of the graph of

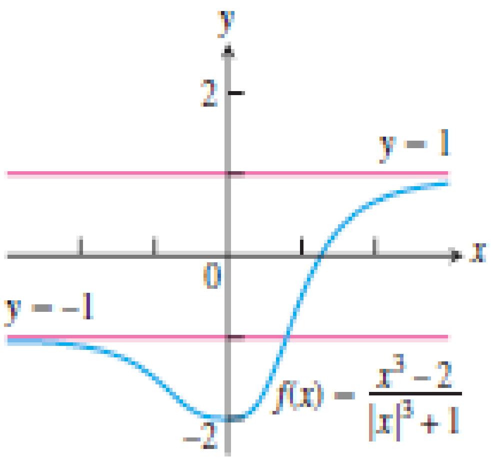
$$f(x) = \frac{x^3 - 2}{|x|^3 + 1}.$$

Solution We calculate the limits as $x \to \pm\infty$.

For $x \geq 0$: $\displaystyle \lim_{x\to\infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x\to\infty} \frac{x^3 - 2}{x^3 + 1} = \lim_{x\to\infty} \frac{1 - (2/x^3)}{1 + (1/x^3)} = 1.$

For $x < 0$: $\displaystyle \lim_{x\to-\infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x\to-\infty} \frac{x^3 - 2}{(-x)^3 + 1} = \lim_{x\to-\infty} \frac{1 - (2/x^3)}{-1 + (1/x^3)} = -1.$

The horizontal asymptotes are $y = -1$ and $y = 1$.
Notice that the graph crosses the horizontal asymptote $y = -1$ for a positive value of x.



Oblique Asymptotes

If the degree of the numerator of a rational function is 1 greater than the degree of the denominator, the graph has an **oblique** or **slant line asymptote**. We find an equation for the asymptote by dividing numerator by denominator to express f as a linear function plus a remainder that goes to zero as $x \to \pm\infty$.

EXAMPLE Find the oblique asymptote of the graph of

$$f(x) = \frac{x^2 - 3}{2x - 4}$$

Solution We are interested in the behavior as $x \to \pm\infty$. We divide $(2x - 4)$ into $(x^2 - 3)$:

$$\begin{array}{r} \frac{x}{2} + 1 \\ 2x - 4 \overline{) x^2 - 3} \\ \underline{x^2 - 2x} \\ 2x - 3 \\ \underline{2x - 4} \\ 1 \end{array}$$

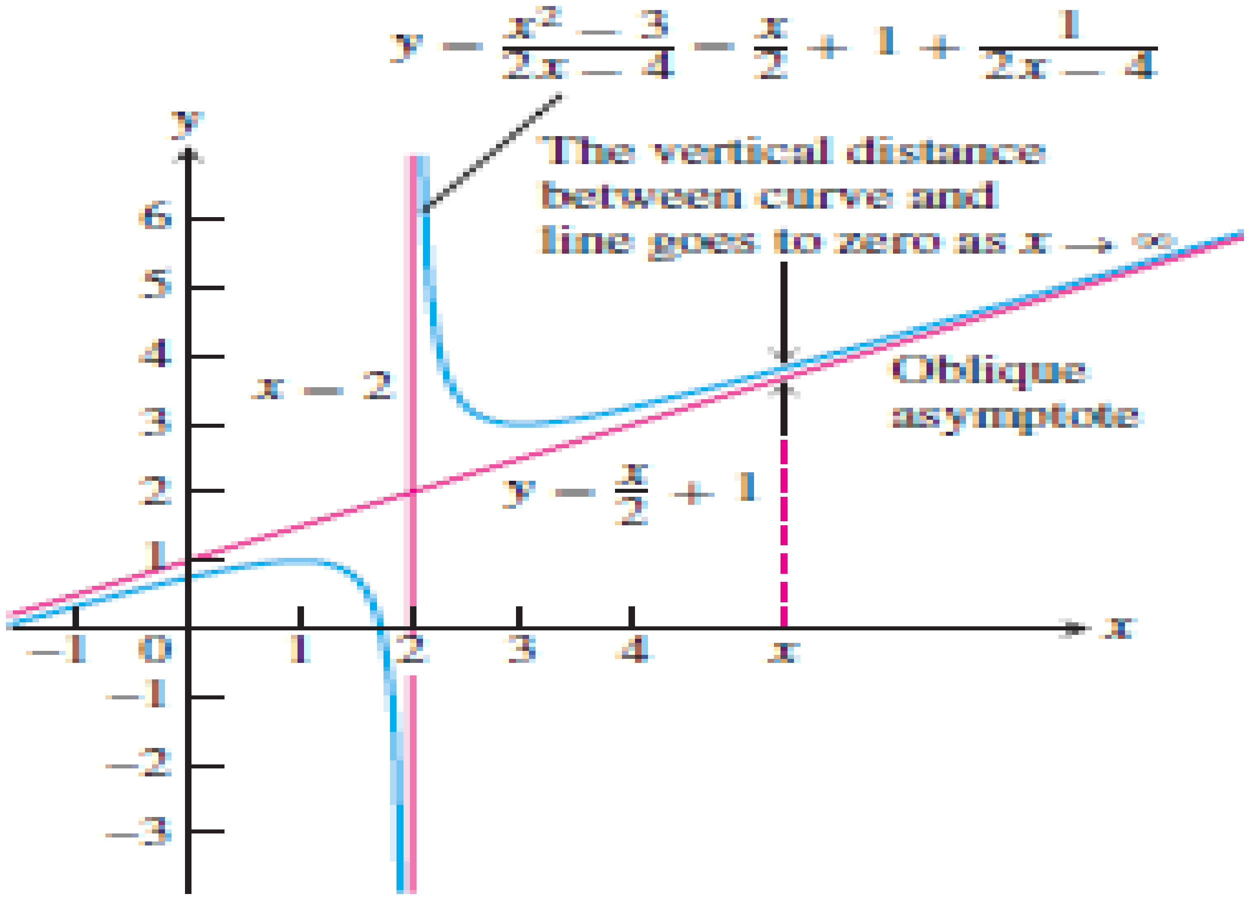
This tells us that

$$f(x) = \frac{x^2 - 3}{2x - 4} = \underbrace{\left(\frac{x}{2} + 1\right)}_{\text{linear } g(x)} + \underbrace{\left(\frac{1}{2x - 4}\right)}_{\text{remainder}}.$$

As $x \to \pm\infty$, the remainder, whose magnitude gives the vertical distance between the graphs of f and g, goes to zero, making the slanted line

$$g(x) = \frac{x}{2} + 1$$

an asymptote of the graph of f . The line $y = g(x)$ is an asymptote both to the right and to the left. The next subsection will confirm that the function $f(x)$ grows arbitrarily large in absolute value as $x \to 2$ (where the denominator is zero), as shown in the graph.



$y = \frac{x^2 - 3}{2x - 4} = \frac{x}{2} + 1 + \frac{1}{2x - 4}$
The vertical distance between curve and line goes to zero as $x \to \infty$
$x = 2$
Oblique asymptote
$y = \frac{x}{2} + 1$
y
x
6
5
4
3
2
1
−1
−2
−3
−1
0
1
2
3
4
x

EXAMPLE Find the horizontal and vertical asymptotes of the graph of

$$f(x) = -\frac{8}{x^2 - 4}.$$

Solution We are interested in the behavior as $x \to \pm\infty$ and as $x \to \pm 2$, where the denominator is zero. Notice that f is an even function of x, so its graph is symmetric with respect to the y-axis.

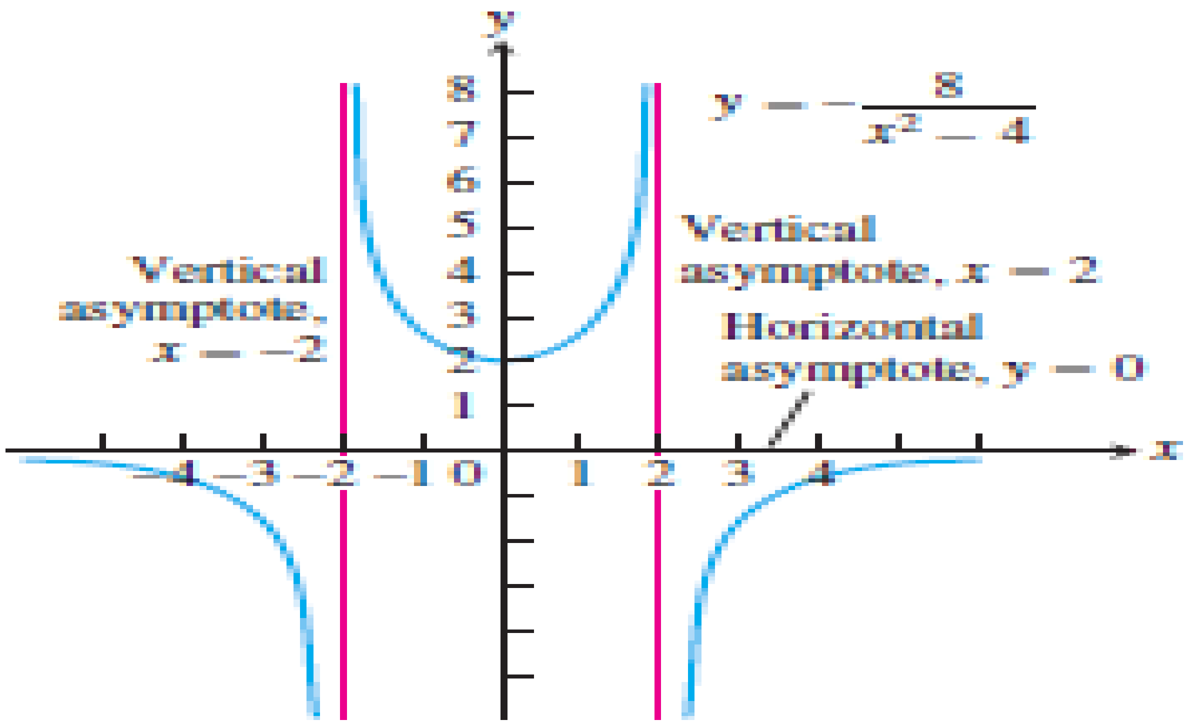
(a) *The behavior as* $x \to \pm\infty$. Since $\lim_{x\to\infty} f(x) = 0$, the line $y = 0$ is a horizontal asymptote of the graph to the right. By symmetry it is an asymptote to the left as well . Notice that the curve approaches the x-axis from only the negative side (or from below). Also, $f(0) = 2$.

(b) *The behavior as* $x \to \pm 2$. Since

$$\lim_{x\to 2^+} f(x) = -\infty \qquad \text{and} \qquad \lim_{x\to 2^-} f(x) = \infty,$$

the line $x = 2$ is a vertical asymptote both from the right and from the left. By symmetry, the line $x = -2$ is also a vertical asymptote.

There are no other asymptotes because f has a finite limit at every other point.



. Notice that the curve approaches the x-axis from only one side. Asymptotes do not have to be two-sided.

EXAMPLE Sketch a graph of the function

$$f(x) = x^4 - 4x^3 + 10$$

using the following steps.

(a) Identify where the extrema of f occur.

(b) Find the intervals on which f is increasing and the intervals on which f is decreasing.

(c) Find where the graph of f is concave up and where it is concave down.

(d) Sketch the general shape of the graph for f.

(e) Plot some specific points, such as local maximum and minimum points, points of inflection, and intercepts. Then sketch the curve.

The function f is continuous since $f'(x) = 4x^3 - 12x^2$ exists. The domain of f is $(-\infty, \infty)$, and the domain of f' is also $(-\infty, \infty)$. Thus, the critical points of f occur only at the zeros of f'. Since

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3),$$

the first derivative is zero at $x = 0$ and $x = 3$. We use these critical points to define intervals where f is increasing or decreasing.

Interval	$x < 0$	$0 < x < 3$	$3 < x$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing

(a) Using the First Derivative Test for local extrema and the table above, we see that there is no extremum at $x = 0$ and a local minimum at $x = 3$.

(b) Using the table above, we see that f is decreasing on $(-\infty, 0]$ and $[0, 3]$, and increasing on $[3, \infty)$.

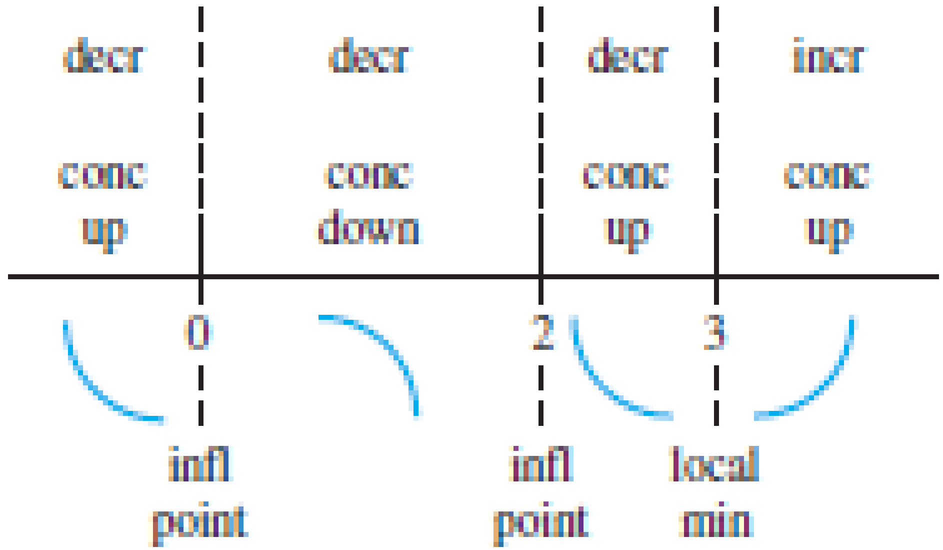
(c) $f''(x) = 12x^2 - 24x = 12x(x - 2)$ is zero at $x = 0$ and $x = 2$. We use these points to define intervals where f is concave up or concave down.

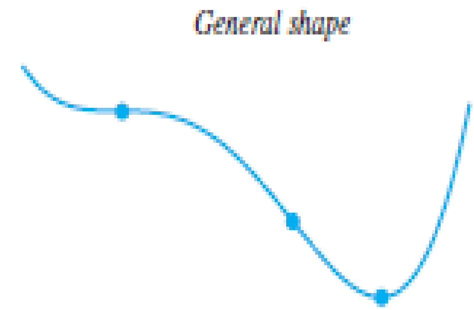
Interval	$x < 0$	$0 < x < 2$	$2 < x$
Sign of f''	$+$	$-$	$+$
Behavior of f	concave up	concave down	concave up

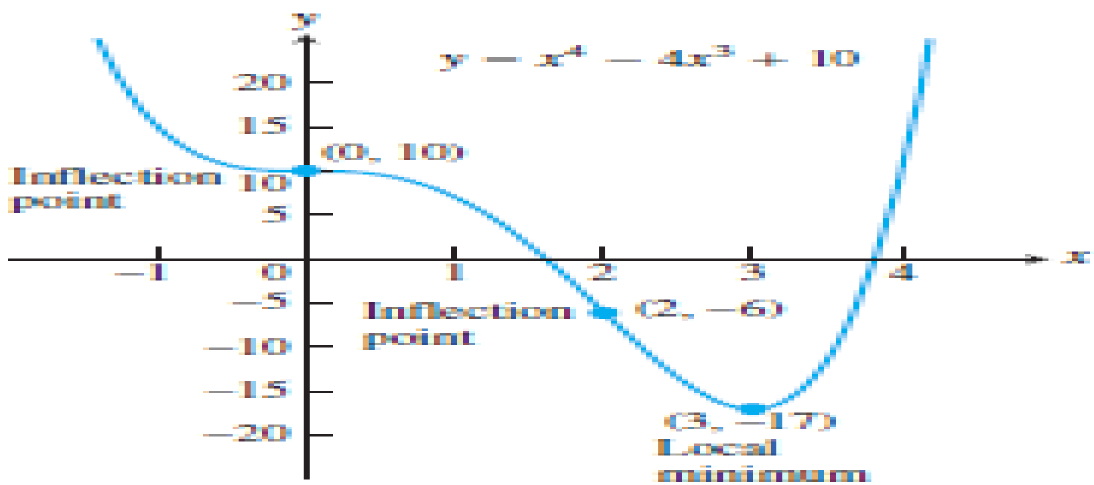
We see that f is concave up on the intervals $(-\infty, 0)$ and $(2, \infty)$, and concave down on $(0, 2)$.

(d) Summarizing the information in the last two tables, we obtain the following.

$x < 0$	$0 < x < 2$	$2 < x < 3$	$3 < x$
decreasing concave up	decreasing concave down	decreasing concave up	increasing concave up







(e) Plot the curve's intercepts (if possible) and the points where y' and y'' are zero. Indicate any local extreme values and inflection points.

Procedure for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.
2. Find the derivatives y' and y''.
3. Find the critical points of f, if any, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes that may exist
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve together with any asymptotes that exist.