Math 731 – Algebraic Topology

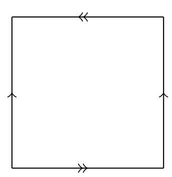
Instructor: Matthias Nagel — due in class 9:30, October 24

Homework 5

Exercise 5.1. Consider the space $S = \mathbb{R}^2$ together with the action of the group $G = \langle s, t | stst^{-1} \rangle$ given by:

$$\begin{aligned} G \times S &\to S \\ s \cdot (x, y) &= \big(x + 1, y\big) \\ t \cdot (x, y) &= \big(-x, y + 1\big) \end{aligned}$$

Show that these formulas give a well-defined group action. Show that the action is free. The quotient $K = S/\sim$ is called the *Klein bottle*. Show that K is homeomorphic to the quotient of the disk:



Exercise 5.2. Show that

(1) $yx^{-1} = x^{-1}y$ holds for the generators x, y in the group $\langle x, y | xyx^{-1}y^{-1} \rangle$.
(2) $\langle x, y | xy^2 \rangle \cong \langle x, y | xy \rangle$.

Exercise 5.3. Compute the fundamental group of the surface of genus 2; see Exercise 2.2. Is it abelian?

Exercise 5.4. Consider the chain complex

$$C = \big(\dots \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \dots\big).$$

Is the identity $\mathrm{id}\colon C \to C$ chain-homotopic to the zero map 0?