Math 731 – Algebraic Topology

Instructor: Matthias Nagel — due in class 9:30, October 27

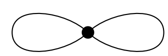
Homework 3

Exercise 3.1. Let X be a topological space and $e \in X$ a point. Let $\mu\colon X \times X \to X$ be a map with $\mu(e,x) = \mu(x,e) = x$ for every point $x \in X$. Prove that $\pi_1(X,e)$ is abelian.

Exercise 3.2. Let Z be a path-connected space, and $p\colon Y \to X$ a cover. Let $z_0 \in Z$ be a point and $f\colon Z \to X$ a map. Let $y_0 \in Y$ be a point with $p(y_0) = x_0 = f(z_0)$. Let $z \in Z$ be another point, and $\gamma_0, \gamma_1 \in \Omega(Z, z_0, z)$ paths from z_0 to z. Denote $\gamma_i' = f \circ \gamma_i\colon I \to Y$ for $i = 0, 1$. Let $\widetilde{\gamma}_i$ be lifts of the paths γ_i' that start at $y_0 \in Y$. Show that if the image of $\pi_1(f)\colon \pi_1(Z, z_0) \to \pi_1(X, f(z_0))$ is contained in the image of $\pi_1(p)\colon \pi_1(Y, y_0) \to \pi_1(X, f(z_0))$, then $\widetilde{\gamma}_0(1) = \widetilde{\gamma}_1(1)$.

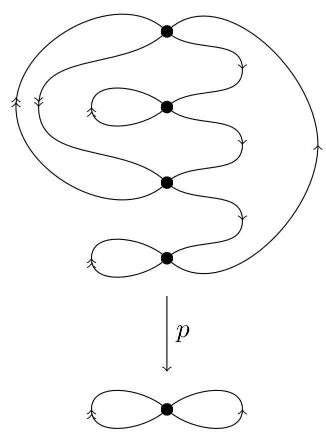
Exercise 3.3.

(1) Let X be the following graph:



Find all covers of degree 3 up to homeomorphism over X.

(2) Compute the group of deck transformations $G(Y)$ of the cover $p\colon Y \to X$:



Exercise 3.4. Let X be path-connected and $x \in X$. Show that $\pi_1(X,x)$ is trivial if and only if every map $S^1 \to X$ extends to a map $D^2 \to X$.