Math 731 – Algebraic Topology

Instructor: Matthias Nagel   due in class 9:30, September 22

Homework 1

Exercise 1.1 (Continuity)**.** Let $(X, d), (Y, d)$ be two metric spaces. Let $f\colon X \to Y$ be a function. Show the following are equivalent:

(1) f is continuous in the ε-δ-sense.
(2) for all $x \in X$ and an open set $V \ni f(x)$, there exists an open set $U \ni x$ such that $f(U) \subset V$.
(3) for all open sets $V \subset Y$, the preimage $f^{-1}(V)$ is open.

Exercise 1.2 (Subspace topology)**.**   (1) Let $(X, \mathcal{O})$, $(Y, \mathcal{S})$ be two topological space, and a continuous function $f\colon X \to Y$. Let $A \subset X$ and $B \subset Y$ two subsets such that $f(A) \subset B$. Show the restriction $f|_A\colon A \to B$ is continuous with respect to the subspace toplogies on A and B.

(2) Let $(X, \mathcal{O})$ be a topological space and A a subset of X. Let τ be a topology on A with the following property: for every topological spaces Y and every continuous map $f\colon X \to Y$, also the restriction $f|_A\colon A \to Y$ is continuous. Show that the subspace topology $\mathcal{O}_A$ is coarser than τ; i.e. $\mathcal{O}_A \subset \tau$.

Exercise 1.3.   (1) Show that S^n is path-connected for $n \geq 1$. Hint: you can use the stereographic projection.

(2) Show that the upper hemisphere $D^n_+ = \left\{ (x_1, \dots, x_{n+1}) \in S^n \colon x_{n+1} \geq 0 \right\}$ is homeomorphic to the n–disk D^n for $n \geq 0$.

Exercise 1.4. Recall that we have considered the set of path components

$$\pi_0(X) = \{\text{path components of } X\}$$

for a topological space X, and constructed a map $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$ for every continuous function $f\colon X \to Y$. Show that $\pi_0\colon \mathsf{Top} \to \mathsf{Set}$ is a *functor*; i.e. the following two properties hold:

(1) $\pi_0(f \circ g) = \pi_0(f) \circ \pi_0(g)$ for every $f\colon X \to Y$ and $g\colon Y \to Z$, and every space X, Y, Z.
(2) $\pi_0(\mathrm{id}_X) = \mathrm{id}_{\pi_0(X)}$ for all topological spaces X.