HW01 for ECE 9343

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1 Question 1: Prove the Symmetry property

$f(n) = \Theta(g(n)) \rightarrow \exists c_1, c_2, n_0, \forall n > n_0, 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$
$\leftrightarrow \forall n > n_0, 0 \leq \frac{f(n)}{c_2} \leq g(n) \leq \frac{f(n)}{c_1}$
$\leftrightarrow g(n) = \Theta(f(n))$

2 Question 2: Problem 3-2

A	B	O	o	Ω	ω	Θ
$lg^k n$	n^ϵ	yes	yes	no	no	no
n^k	c^n	yes	yes	no	no	no
$n^{\frac{1}{2}}$	n^{sinn}	no	no	no	no	no
2^n	$2^{\frac{1}{2}n}$	no	no	yes	yes	no
n^{lgc}	c^{lgn}	yes	no	yes	no	yes
$lg(n!)$	$lg(n^n)$	yes	yes	no	no	no

3 Question 3: Problem 3-3-a

$2^{2^{n+1}} > 2^{2^n} > (n+1)! > n! > e^n > n2^n$
$> 2^n > \frac{3}{2}^n > n^{lglgn} = lgn^{lgn} > (lgn)! > n^3$
$> n^2 = 4^{lgn} > nlgn > 2^{lgn} = n > (2^{\frac{1}{2}})^{lgn}$
$> 2^{(2lgn)^{1/2}} > lg^2 n > lg(n!) > lnn > (lgn)^{\frac{1}{2}} > ln(lnn)$
$> 2^{lg^* n} > lg^* n > lg^* lgn > lglg^* n > n^{\frac{1}{lgn}} > 1$

Some procedure:
$n^n = 2^{nlgn} < 2^{2^n}$
$((2^{1/2})^{lgn}) = n^{1/2}$
$lg^2 n = 2^{2lglgn} < 2^{(2lgn)^{1/2}} < (2^{1/2})^{lgn}$
$n^{\frac{1}{lgn}} = 2^{\frac{lgn}{lgn}} = 2$
$4^{lgn} = n^{lg4} = n^2$
$n^{lglgn} = lgn^{lgn} = e^{lnnlglgn} = 2^{lgnlglgn} > 2^{(2lgn)^{(1/2)}}$
$n! > \frac{n^n}{e^n} = e^{nlnn-n} > e^{lnnlglgn}$

$lgn! = lgn^{1/2}\frac{lgn^{lgn}}{e^{lgn}}(1+\frac{1}{n}) < (lgn)^{lgn}$
$lnlnn = 2^{lglnlnn} > 2^{lg*n}$

4 Question 4: Problem 3-4-c-d-e-f

4.1 c

True.
$f(n) = O(g(n)) \rightarrow \exists c, n_0, \forall n > n_0, 1 \leq f(n) \leq cg(n)$
$\rightarrow 0 \leq lg(f(n)) \leq lg(cg(n)) = lgc + lg(g(n))$
$\rightarrow \exists c^{'}, lgc + lg(g(n)) \leq c^{'}lg(g(n))$
$\rightarrow lg(f(n)) = O(lg(g(n)))$

4.2 d

True.
$f(n) = O(g(n)) \rightarrow \exists c, n_0, \forall n > n_0, 0 \leq f(n) \leq cg(n)$
$\rightarrow 1 \leq 2^{f(n)} \leq 2^{cg(n)} = 2^c 2^{g(n)}$
$\rightarrow \exists c^{'} > 2^c, 2^{f(n)} \leq c^{'}2^{g(n)}$
$\rightarrow 2^{f(n)} = O(2^{g(n)})$

4.3 e

False, consider any $f(x), \lim_{n\rightarrow\infty} f(x) < 1$, such as e^{-x}

4.4 f

True.
$f(n) = O(g(n)) \rightarrow \exists c, n_0, \forall n > n_0, 0 \leq f(n) \leq cg(n)$
$\rightarrow \exists c^{'} = \frac{1}{c}, 0 \leq c^{'}f(n) \leq g(n)$
$\rightarrow g(n) = \Omega(f(n))$

5 Question 5: verify

Proof: $T(n) = O(n)$
Suppose $\forall k < n, \exists c_2, T(k) \leq c_2 k - 10$
$\rightarrow T(n) = c_2\aleph n + c_2(1-\alpha)n - 20 + 10 \leq c_2 n - 10$
$\rightarrow T(n) = O(c_2 n - 10)$
$\rightarrow T(n) = O(n)$

Proof: $T(n) = \Omega(n)$
Suppose $\forall k < n, \exists c_1, T(k) \geq c_1 k$
$\rightarrow T(n) = c_1\aleph n + c_1(1-\alpha)n + 10 \geq c_1 n$
$\rightarrow T(n) = \Omega(n)$

$T(n) = O(n), T(n) = \Omega(n) \to T(n) = \Theta(n)$

6 Question 6: solve and verify

Notice that $TreeHeight = h = log_{\frac{3}{2}} n$
For branch $\Theta(n) = \sum_1^{h+1} n(\frac{4}{3})^h = n\frac{(\frac{4}{3})^h - 1}{\frac{4}{3}-1} = \Theta(n^{\frac{ln2}{ln3-ln2}})$
For leaf $\Theta(n) = 2^h = \Theta(n^{\frac{ln2}{ln3-ln2}})$
$\to T(n) = \Theta(n^{\frac{ln2}{ln3-ln2}}) = \Theta(n^{log_{\frac{3}{2}} 2}) = \Theta(2^{log_{\frac{3}{2}} n})$

Proof: $T(n) = O(2^{log_{\frac{3}{2}} n} - 3n)$
Suppose $\forall k < n, \exists c_2, T(k) \le c_2 2^{log_{\frac{3}{2}} n} - 3n$
$\to T(n) = c_2 2 * 2^{log_{\frac{3}{2}} \frac{2}{3} n} - 4n + n \le c_2 2^{log_{\frac{3}{2}} n} - 3n$
$\to T(n) = O(2^{log_{\frac{3}{2}} n} - 3n)$
$\to T(n) = O(2^{log_{\frac{3}{2}} n})$

Proof: $T(n) = \Omega(n^{log_{\frac{3}{2}} 2})$
Suppose $\forall k < n, \exists c_1, T(k) \ge c_1 n^{log_{\frac{3}{2}} 2}$
$\to T(n) = c_1 2(\frac{2}{3}n)^{log_{\frac{3}{2}} 2} + \frac{4}{3}n = c_1 2 * (\frac{3}{2})^{log_{\frac{3}{2}} 2} * n^{log_{\frac{3}{2}} 2} + \frac{4}{3}n = c_1 n^{log_{\frac{3}{2}} 2} + \frac{4}{3}n \ge c_1 n^{log_{\frac{3}{2}} 2} + n$
$\to T(n) = \Omega(n^{log_{\frac{3}{2}} 2})$

$T(n) = O(2^{log_{\frac{3}{2}} n}), T(n) = \Omega(n^{log_{\frac{3}{2}} 2}) \to T(n) = \Theta(n^{log_{\frac{3}{2}} 2})$

7 Question 7: solve and verify

Notice that for iterative tree:
$\Theta(n^2) = 2T(\frac{1}{4}n) + n^2 \le T(n) \le 2T(\frac{1}{2}n) + n^2 = \Theta(n^2)$

Proof: $T(n) = O(n^2)$, Suppose $\forall k < n, T(k) = O(k^2)$
$\to \exists c_2 > \frac{16}{11}, T(k) \le c_2 n^2, T(n) \le (\frac{5}{16}c_2 + 1)n^2 \le c_2 n^2, c_2 > \frac{16}{11}$

Proof: $T(n) = \Omega(n^2)$, Suppose $\forall k < n, T(k) = \Omega(k^2)$
$\to \exists c_1 < \frac{16}{11}, T(k) \ge c_1 n^2, T(n) \ge (\frac{5}{16}c_1 + 1)n^2 \ge c_1 n^2, c_1 < \frac{16}{11}$
$\to T(n) = \Theta(n^2)$

8 Question 8: solve

Let $n = 2^m$, Then $T(2^m) = 9T(2^{\frac{m}{6}}) + m^2$
$\to S(m) = 9S(\frac{1}{6}m) + m^2$

From Branch: $S(m) = m^{log_6 \frac{3}{2}+2}$
From Leave: $S(m) = m^{log_6 9}$
So, $S(m) = \Theta(m^{log_6 \frac{3}{2}+2})$
$\rightarrow T(n) = T(2^m) = \Theta((lg(n))^{log_6 \frac{3}{2}+2})$

9 Question 9: solve and justify

9.1 a

For leaf $\Theta(n) = n^{log_3 2}$
For branch, Notice that $n^{\frac{1}{2}lgn} < n^{\frac{1}{2}+\epsilon}$
$\rightarrow T(n) < S(n) 2S(\frac{1}{3}n) + n^{\frac{1}{2}+\epsilon}$
Notice that the branch complexity of $S(n) = n^{\frac{1}{2}+\epsilon} < n^{log_3 2}$
$\rightarrow T(n)$ is dominated by leaf, $T(n) = \Theta(n^{log_3 2})$

9.2 b

For branch $T(n) = \Theta(hn^2) = \Theta(lognn^2)$
For leaf $T(n) = \Theta(n^2)$
$\rightarrow T(n)$ is dominated by branch, $T(n) = \Theta(lognn^2)$

9.3 c

For leaf $T(n) = \Theta(4^{log_2 n}) = \Theta(n^2)$
For branch, notice that $4 * (\frac{1}{2})^{\frac{5}{2}} = 2^{-\frac{1}{2}} < 1, T(n) = \Theta(n^{\frac{5}{2}})$
$\rightarrow T(n)$ is dominated by branch, $T(n) = \Theta(n^{\frac{5}{2}})$

9.4 d

For branch $TreeHeight = h = \frac{n}{2}, T(n) = \frac{1}{2}\sum_1^{h+1} \frac{1}{n} = \frac{1}{2}(lnn - ln2) = \Theta(lnn)$
For leaf $T(n) = \Theta(c)$
$\rightarrow T(n)$ is dominated by branch, $T(n) = \Theta(lnn)$