HW01 for ECE 9343

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1 Question 1: Prove the Symmetry property

$f(n) = \Theta(g(n)) \rightarrow \exists c_1, c_2, n_0, \forall n > n_0, 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$
$\leftrightarrow \forall n > n_0, 0 \leq \frac{f(n)}{c_2} \leq g(n) \leq \frac{f(n)}{c_1}$
$\leftrightarrow g(n) = \Theta(f(n))$

2 Question 2: Problem 3-2

A	B	O	o	Ω	ω	Θ
$lg^k n$	n^ϵ	yes	yes	no	no	no
n^k	c^n	yes	yes	no	no	no
$n^{\frac{1}{2}}$	n^{sinn}	no	no	no	no	no
2^n	$2^{\frac{1}{2}n}$	no	no	yes	yes	no
n^{lgc}	c^{lgn}	yes	no	yes	no	yes
$lg(n!)$	$lg(n^n)$	yes	yes	no	no	no

3 Question 3: Problem 3-3-a

$2^{2^{n+1}} > 2^{2^n} > (n+1)! > n! > e^n > n2^n$
$> 2^n > \frac{3}{2}^n > n^{lglgn} = lgn^{lgn} > (lgn)! > n^3$
$> n^2 = 4^{lgn} > lg(n!) = nlgn > 2^{lgn} = n >$
$(2^{\frac{1}{2}})^{lgn} > 2^{(2lgn)^{1/2}} > lg^2 n > lnn > (lgn)^{\frac{1}{2}} > ln(lnn)$
$> 2^{lg^* n} > lg^* n > lg^* lgn > lglg^* n > n^{\frac{1}{lgn}} > 1$

Some procedure:
$n^n = 2^{nlgn} < 2^{2^n}$
$((2^{1/2})^{lgn}) = n^{1/2}$
$lg^2 n = 2^{2lglgn} < 2^{(2lgn)^{1/2}} < (2^{1/2})^{lgn}$
$n^{\frac{1}{lgn}} = 2^{\frac{lgn}{lgn}} = 2$
$4^{lgn} = n^{lg4} = n^2$
$n^{lglgn} = lgn^{lgn} = e^{lnnlglgn} = 2^{lgnlglgn} > 2^{(2lgn)^{(1/2)}}$
$n! > \frac{n^n}{e^n} = e^{nlnn-n} > e^{lnnlglgn}$

$lgn! = lgn^{1/2} \frac{lgn^{lgn}}{e^{lgn}}(1+\frac{1}{n}) < (lgn)^{lgn}$
$lnlnn = 2^{lglnlnn} > 2^{lg*n}$

4 Question 4: Problem 3-4-c-d-e-f

4.1 c

True.
$f(n) = O(g(n)) \rightarrow \exists c, n_0, \forall n > n_0, 1 \leq f(n) \leq cg(n)$
$\rightarrow 0 \leq lg(f(n)) \leq lg(cg(n)) = lgc + lg(g(n))$
$\rightarrow \exists c^{'}, lgc + lg(g(n)) \leq c^{'} lg(g(n))$
$\rightarrow lg(f(n)) = O(lg(g(n)))$

4.2 d

False.
Suppose that $f(n) = n, g(n) = \frac{1}{2}n$, never find $c, n_0, \forall n > n_0, 2^n \leq c2^{\frac{1}{2}n}$

4.3 e

False, consider any $f(x), \lim_{n\rightarrow\infty} f(x) < 1$, such as e^{-x}

4.4 f

True.
$f(n) = O(g(n)) \rightarrow \exists c, n_0, \forall n > n_0, 0 \leq f(n) \leq cg(n)$
$\rightarrow \exists c^{'} = \frac{1}{c}, 0 \leq c^{'} f(n) \leq g(n)$
$\rightarrow g(n) = \Omega(f(n))$

5 Question 5: verify

Proof: $T(n) = O(n)$
Suppose $\forall k < n, \exists c_2, T(k) \leq c_2 k - 10$
$\rightarrow T(n) = c_2 \alpha n + c_2(1-\alpha)n - 20 + 10 \leq c_2 n - 10$
$\rightarrow T(n) = O(c_2 n - 10)$
$\rightarrow T(n) = O(n)$

Proof: $T(n) = \Omega(n)$
Suppose $\forall k < n, \exists c_1, T(k) \geq c_1 k$
$\rightarrow T(n) = c_1 \aleph n + c_1(1-\alpha)n + 10 \geq c_1 n$
$\rightarrow T(n) = \Omega(n)$

$T(n) = O(n), T(n) = \Omega(n) \rightarrow T(n) = \Theta(n)$

6 Question 6: solve and verify

Notice that $TreeHeight = h = log_{\frac{3}{2}} n$

For branch $\Theta(n) = \sum_{1}^{h+1} n(\frac{4}{3})^h = n\frac{(\frac{4}{3})^h - 1}{\frac{4}{3} - 1} = \Theta(n^{\frac{ln2}{ln3 - ln2}})$

For leaf $\Theta(n) = 2^h = \Theta(n^{\frac{ln2}{ln3-ln2}})$

$\rightarrow T(n) = \Theta(n^{\frac{ln2}{ln3-ln2}}) = \Theta(n^{log_{\frac{3}{2}} 2}) = \Theta(2^{log_{\frac{3}{2}} n})$

Proof: $T(n) = O(2^{log_{\frac{3}{2}} n}) - 3n$

Suppose $\forall k < n, \exists c_2, T(k) \leq c_2 2^{log_{\frac{3}{2}} n} - 3n$

$\rightarrow T(n) = c_2 2 * 2^{log_{\frac{3}{2}} \frac{2}{3} n} - 4n + n \leq c_2 2^{log_{\frac{3}{2}} n} - 3n$

$\rightarrow T(n) = O(2^{log_{\frac{3}{2}} n} - 3n)$

$\rightarrow T(n) = O(2^{log_{\frac{3}{2}} n})$

Proof: $T(n) = \Omega(n^{log_{\frac{3}{2}} 2})$

Suppose $\forall k < n, \exists c_1, T(k) \geq c_1 n^{log_{\frac{3}{2}} 2}$

$\rightarrow T(n) = c_1 2(\frac{2}{3}n)^{log_{\frac{3}{2}} 2} + \frac{4}{3}n = c_1 2 * (\frac{3}{2})^{log_{\frac{3}{2}} 2} * n^{log_{\frac{3}{2}} 2} + \frac{4}{3}n = c_1 n^{log_{\frac{3}{2}} 2} + \frac{4}{3}n \geq c_1 n^{log_{\frac{3}{2}} 2} + n$

$\rightarrow T(n) = \Omega(n^{log_{\frac{3}{2}} 2})$

$T(n) = O(2^{log_{\frac{3}{2}} n}), T(n) = \Omega(n^{log_{\frac{3}{2}} 2}) \rightarrow T(n) = \Theta(n^{log_{\frac{3}{2}} 2})$

7 Question 7: solve and verify

Notice that for iterative tree:

$\Theta(n^2) = 2T(\frac{1}{4}n) + n^2 \leq T(n) \leq 2T(\frac{1}{2}n) + n^2 = \Theta(n^2)$

Proof: $T(n) = O(n^2)$, Suppose $\forall k < n, T(k) = O(k^2)$

$\rightarrow \exists c_2 > \frac{16}{11}, T(k) \leq c_2 n^2, T(n) \leq (\frac{5}{16}c_2 + 1)n^2 \leq c_2 n^2, c_2 > \frac{16}{11}$

Proof: $T(n) = \Omega(n^2)$, Suppose $\forall k < n, T(k) = \Omega(k^2)$

$\rightarrow \exists c_1 < \frac{16}{11}, T(k) \geq c_1 n^2, T(n) \geq (\frac{5}{16}c_1 + 1)n^2 \geq c_1 n^2, c_1 < \frac{16}{11}$

$\rightarrow T(n) = \Theta(n^2)$

8 Question 8: solve

Let $n = 2^m$, Then $T(2^m) = 9T(2^{\frac{m}{6}}) + m^2$

$\rightarrow S(m) = 9S(\frac{1}{6}m) + m^2$

From Branch: $S(m) = m^{log_6 \frac{3}{2} + 2}$

From Leave: $S(m) = m^{log_6 9}$

So, $S(m) = \Theta(m^{log_6 \frac{3}{2} + 2})$

$\rightarrow T(n) = T(2^m) = \Theta((lg(n))^{log_6 \frac{3}{2}+2})$

9 Question 9: solve and justify

9.1 a

For leaf $\Theta(n) = n^{log_3 2}$
For branch, Notice that $n^{\frac{1}{2}} < n^{\frac{1}{2}} lgn < n^{\frac{1}{2}+\epsilon}$
$\rightarrow 2S(\frac{1}{3}n) + n^{\frac{1}{2}} < T(n) < S(n) = 2S(\frac{1}{3}n) + n^{\frac{1}{2}+\epsilon}$
Notice that the branch complexity of $\Theta(n^{log_3 2}) \leq S(n) = n^{\frac{1}{2}+\epsilon} \leq \Theta(n^{log_3 2})$
$\rightarrow T(n)$ is equally dominated by branch and leaf, $T(n) = \Theta(n^{log_3 2})$

9.2 b

For branch $T(n) = \Theta(hn^2) = \Theta(lognn^2)$
For leaf $T(n) = \Theta(n^2)$
$\rightarrow T(n)$ is dominated by branch, $T(n) = \Theta(lognn^2)$

9.3 c

For leaf $T(n) = \Theta(4^{log_2 n}) = \Theta(n^2)$
For branch, notice that $4 * (\frac{1}{2})^{\frac{5}{2}} = 2^{-\frac{1}{2}} < 1, T(n) = \Theta(n^{\frac{5}{2}})$
$\rightarrow T(n)$ is dominated by branch, $T(n) = \Theta(n^{\frac{5}{2}})$

9.4 d

For branch $TreeHeight = h = \frac{n}{2}, T(n) = \frac{1}{2}\sum_1^{h+1} \frac{1}{n} = \frac{1}{2}(lnn - ln2) = \Theta(lnn)$
For leaf $T(n) = \Theta(c)$
$\rightarrow T(n)$ is dominated by branch, $T(n) = \Theta(lnn)$