Entanglement detection

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ABSTRACT

How can one prove that a given quantum state is entangled? In this paper we review different methods that have been proposed for entanglement detection. We first explain the basic elements of entanglement theory for two or more particles and then entanglement verification procedures such as Bell inequalities, entanglement witnesses, the determination of nonlinear properties of a quantum state via measurements on several copies, and spin squeezing inequalities. An emphasis is given to the theory and application of entanglement witnesses. We also discuss several experiments, where some of the presented methods have been implemented.

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1. Introduction

Entanglement was first described by Einstein, Podolsky, and Rosen [1] and Schrödinger [2] as a strange phenomenon of quantum mechanics, questioning the completeness of the theory. Later, Bell recognized that entanglement leads to experimentally testable deviations of quantum mechanics from classical physics [3]. Finally, with the advent of quantum information theory, entanglement was recognized as a resource, enabling tasks like quantum cryptography [4], quantum teleportation [5] or measurement based quantum computation [6]. Together with the rapid experimental progress on quantum control, this led to a rapidly growing interest in entanglement theory and many experiments nowadays aim at the generation of entanglement.

Indeed, in the last years enormous progress on the generation of entanglement has been achieved. For instance, six or eight ions have been entangled [7,8], photons have been used to demonstrate entanglement between six particles or ten qubits [9,10] and in diamond nuclear and electronic spins have been entangled [11]. These systems allow for individual addressing of the parties. In other types of systems only collective measurements are possible. Through spin squeezing, entanglement of 10^7 atoms were created in cold atomic clouds [12] and large scale entangling operations were realized in optical lattices of 10^5 two-state atoms [13]. As the underlying techniques of quantum control improve continuously, it can be expected that in the near future even larger systems can be entangled.

In any of these experiments, typical questions arise: How can one be sure that entanglement was indeed produced? How can one detect the presence of entanglement? Can we quantify the entanglement in the experiment? These questions are difficult to answer and many possible ways to tackle this problem have been proposed. These range from Bell inequalities, entanglement witnesses and spin squeezing inequalities to entropic inequalities, the measurement of nonlinear properties of the quantum state and the approximation of positive maps.

In general, one could imagine several desirable properties for an entanglement verification procedure. First, a crucial requirement is that the scheme under consideration must be easy to implement. Depending on the type of experiment, some measurements are easy to carry out, and some are not. In experiments using photons, the count rates are often not very high, making it impossible to characterize the experimental state completely. Then, one has to find the measurements that allow one to conclude much about the entanglement content of the given state. In addition, the scheme should be robust against noise, and it should also detect weakly entangled states.

Second, most experiments nowadays aim at the generation of entanglement between more than two particles and for this case different entanglement classes exist. Therefore, an entanglement detection scheme has to be capable of distinguishing between these different classes. In an N-qubit experiment it is not sufficient to say that some qubits are entangled, one has to prove that all of the N qubits are entangled.

Third, the conclusion that the experimentally generated state was entangled should not depend on some assumptions about the state. Most experiments aim at the generation of a special state, and clearly one can use this to design appropriate

measurements. But the final conclusion that the state was entangled, must not rely on assumptions concerning a special form of the state or its purity.

In this paper, we review different methods for the characterization of entanglement in experiments. This includes the mere detection of entanglement, but also its experimental quantification, the estimation of the state fidelity and the characterization whether a prepared state is useful for some task or not. We also explain several experiments, where these methods have been used. Due to their higher experimental relevance, we especially explain methods that can be used for the characterization of entanglement between more than two particles. A special emphasis is given on the method of entanglement witnesses, as they are the most frequently used tool nowadays, moreover, they have also applications in other areas of quantum information theory.

Throughout this paper we restrict our attention to the case of discrete systems; mainly we consider the case of qubits. It should be stressed that entanglement in continuous variable systems like harmonic oscillators or light modes is also under intensive research. However, as the underlying theory, e.g. the description of Gaussian states via covariance matrices of canonical conjugate variables, is significantly different from the case of qubits, we have not addressed these systems here. Moreover, for the entanglement theory of continuous variable systems several good reviews already exist [14–17].

As entanglement is a central topic in quantum information theory, there are some other excellent review articles that are related to some questions addressed here. For the case of bipartite entanglement, a recent exhaustive review was written by the Horodecki family [18] and entanglement measures have been reviewed in detail by Virmani and Plenio [19]. Some older proposals for entanglement detection schemes have been summarized by Terhal [20] and a survey about conceptual differences and possible problems of entanglement verification schemes has been given by van Enk, Lütkenhaus and Kimble [21]. Moreover, several other overview articles concerning bipartite and multipartite entanglement [22–26], entanglement measures [27,28] or Bell inequalities [29–31] have been published. Concerning the experimental techniques, photon experiments have been reviewed in Refs. [32–34] and ion trap experiments in Refs. [35–39].

This review is organized as follows: The Sections 2–4 give an overview about theoretical aspects of entanglement, while the Sections 5–8 explain different entanglement detection methods in detail.

In Section 2, we give an introduction into entanglement theory of bipartite systems. We introduce the notions of entanglement and separability, explain several separability criteria and the phenomenon of bound entanglement. We also introduce entanglement witnesses as a theoretical concept and explain how to construct and optimize them. In Section 3, we discuss entanglement between more than two particles. We explain the different classes of multipartite entanglement and introduce the most relevant families of multiparticle entangled states, such as *GHZ* states, cluster states or Dicke states. In Section 4, we introduce entanglement measures. We explain their basic properties and give the definitions of the most popular measures.

In Section 5, we review Bell inequalities as the oldest tool to verify entanglement. We explain how Bell inequalities bound the set of correlations originating from local hidden variable models and how the violation of a Bell inequality implies the quantum phenomenon of entanglement. We also explain parametric down-conversion as a process to generate entangled photons, leading to a violation of Bell inequalities and also an experiment, where this process has been used to demonstrate violation of a Bell inequality under locality conditions.

In Section 6, we review entanglement witnesses as a tool for the detection of entanglement. We show how entanglement witnesses can be implemented with simple local measurements and that they are especially useful for the detection of multipartite entanglement. We also explain several experiments, where witnesses have been used in various experimental situations. Finally, we explain how entanglement witnesses can be useful in other parts of quantum information theory, such as quantum cryptography.

In Section 7, we review other entanglement detection schemes that have been implemented or proposed. This includes entropic inequalities, estimates of the concurrence and positive maps via measurements on several copies, variance based criteria, and nonlinear entanglement witnesses.

Finally, in Section 8 we give an overview over entanglement detection with collective observables. This concerns spin squeezing inequalities and optical lattices, but also entanglement verification schemes in spin models, where global observables such as the magnetic susceptibility of the energy can be used to derive statements about the entanglement in the system.

2. Bipartite entanglement

In this section, we will explain the basic notions of bipartite entanglement. As many experiments nowadays aim at the generation of multiparticle entangled states, we will explain the theory of multipartite entanglement in the following section. However, the study of bipartite entangled states will already enable us to introduce the central concepts of entanglement detection.

2.1. Entanglement of pure states

Let us assume that we are given two quantum systems. The first one is owned by one physicist, called Alice, and the second one by another one, called Bob. The physical states of Alice's system may be described by states in a Hilbert space

$\mathcal{H}_A$ of dimension d_A, and in Bob's system in a Hilbert space $\mathcal{H}_B$ of dimension d_B. The composite system of both parties is then described by vectors in the tensor-product of the two spaces $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Thus, any vector in $\mathcal{H}_A \otimes \mathcal{H}_B$ can be written as

$$|\psi\rangle = \sum_{i,j=1}^{d_A,d_B} c_{ij}|a_i\rangle \otimes |b_j\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B, \tag{1}$$

with a complex $d_A \times d_B$ matrix $C = (c_{ij})$. To keep the notation simple, we often write tensor products of vectors as $|a\rangle \otimes |b\rangle \equiv |a\rangle|b\rangle \equiv |ab\rangle$. Now one can define separability and entanglement for pure states.

Definition 1 (*Entanglement for Pure States*). A pure state $|\psi\rangle \in \mathcal{H}$ is called a *product state* or *separable* if we can find states $|\phi^A\rangle \in \mathcal{H}_A$ and $|\phi^B\rangle \in \mathcal{H}_B$ such that

$$|\psi\rangle = |\phi^A\rangle \otimes |\phi^B\rangle \tag{2}$$

holds. Otherwise the state $|\psi\rangle$ is called *entangled*.

Physically, the definition of product states means that the state is uncorrelated. A product state can thus easily be prepared in a local way: Alice produces the state $|\phi^A\rangle$ and Bob produces independently the state $|\phi^B\rangle$. If Alice measures any observable A and Bob measures B, then the probabilities of the different outcomes factorize. Thus, the measurement outcomes for Alice do not depend on the outcomes on Bob's side.

Before proceeding to the definition of entanglement for mixed states we shall mention a very useful tool in the description of entanglement for bipartite systems. This is the so-called *Schmidt decomposition.*

Lemma 2 (*Schmidt Decomposition*). *Let $|\psi\rangle = \sum_{i,j=1}^{d_A,d_B} c_{ij}|a_i b_j\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ be a vector in the tensor product of two Hilbert spaces. Then there exists an orthonormal basis $|\alpha_i\rangle$ of $\mathcal{H}_A$ and an orthonormal basis $|\beta_j\rangle$ of $\mathcal{H}_B$ such that*

$$|\psi\rangle = \sum_{k=1}^{R} \lambda_k |\alpha_k \beta_k\rangle \tag{3}$$

holds, with positive real coefficients λ_k. The λ_k are uniquely determined as the square roots of the eigenvalues of the matrix $CC^\dagger$, where $C = (c_{ij})$ is the matrix formed by the coefficients in Eq. (1). The number $R \leq \min\{d_A, d_B\}$ is called the Schmidt rank of $|\psi\rangle$. If the λ_k are pairwise different, then also the $|\alpha_k\rangle$ and $|\beta_k\rangle$ are unique up to a phase.

The proof of this statement can be found in many textbooks [40–43]. Note that pure product states correspond to states of Schmidt rank one. Therefore, the Schmidt decomposition can be used to decide whether a given pure state is entangled or separable.

2.2. Entanglement of mixed states

In a more general situation one does not know the exact state of a quantum system. It is only known that it is, with some probability p_i, in one of some states $|\phi_i\rangle \in \mathcal{H}$. This situation is described by a density matrix

$$\varrho = \sum_i p_i |\phi_i\rangle\langle\phi_i|, \quad \text{with } \sum_i p_i = 1 \text{ and } p_i \geq 0. \tag{4}$$

In a given basis this density matrix or state is represented by a complex matrix. This matrix is positive semidefinite[1] and hermitian, since all the operators $|\phi_i\rangle\langle\phi_i|$ are positive and hermitian. Due to the condition on the p_i also $\text{Tr}(\varrho) = 1$ holds. Conversely, any positive semidefinite matrix of trace one can be interpreted as a density matrix of some state. This leads to a geometrical picture of the set of all states as a convex set. This means that given two states ϱ_1 and ϱ_2, their convex combination $\varrho = \alpha\varrho_1 + (1-\alpha)\varrho_2$ with $\alpha \in [0; 1]$ is again a state. This property holds also for combinations of more than two states. Given some $p_i \geq 0$ with $\sum_i p_i = 1$ then the convex combination $\sum_i p_i \varrho_i$ of some states is again a state. We call coefficients $p_i \geq 0$ with the property $\sum_i p_i = 1$ often *convex weights.*

Now one can define separability and entanglement for mixed states according to Ref. [44]. The idea is the same as for the pure state case: A state is separable, if it can be produced locally, otherwise it is entangled.

Definition 3 (*Entanglement for Mixed States*). Let ϱ be a density matrix for a composite system. We say that ϱ is a *product state* if there exist states ϱ^A for Alice and ϱ^B for Bob such that

$$\varrho = \varrho^A \otimes \varrho^B. \tag{5}$$

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[1] A hermitian matrix M is called positive semidefinite ($M \geq 0$), iff the eigenvalues are non-negative. Equivalently, $M \geq 0$ if $\langle\psi|M|\psi\rangle \geq 0$ for all $|\psi\rangle$.
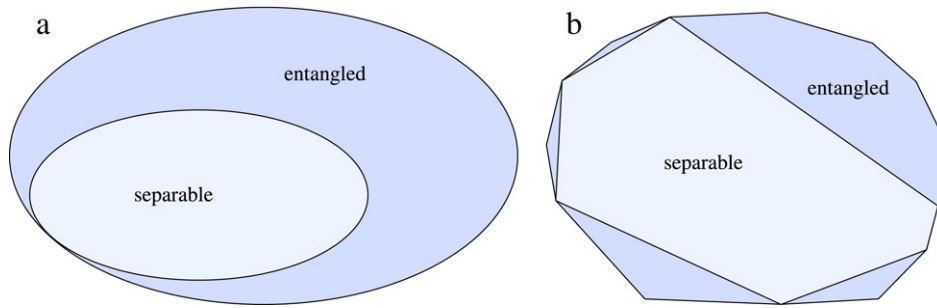
Fig. 1. (a) Schematic picture of the set of all states as a convex set and the set of separable states as a convex subset. (b) Different schematic picture of the same sets. Here, it is stressed that the extremal points of the separable states (the pure product states), are also extremal points of the set of all states, hence they are located on the border of the total set.

The state is called *separable*, if there are convex weights p_i and product states $\varrho_i^A \otimes \varrho_i^B$ such that

$$\varrho = \sum_i p_i \varrho_i^A \otimes \varrho_i^B \tag{6}$$

holds. Otherwise the state is called *entangled*.

Physically, this definition discriminates between three scenarios. First, a product state is an uncorrelated state, where Alice and Bob own each a separate state. For non-product states there are two different kinds of correlation. Separable states are classically correlated. This means that for the production of a separable state only local operations and classical communication (LOCC) are necessary. Alice and Bob can, by classical communication, share a random number generator that produces the outcomes i with probabilities p_i. For each of the outcomes, they can agree to produce the state $\varrho_i^A \otimes \varrho_i^B$ locally. By this procedure they produce the state $\varrho = \sum_i p_i \varrho_i^A \otimes \varrho_i^B$. This procedure is not specific for quantum theory, which justifies the notion of *classical* correlations. Otherwise, if a state is entangled, the correlations cannot originate from the classical procedure described above. In this sense entangled states are a typical feature of quantum mechanics.

For our later discussion it is very important to note that the set of separable states is a convex set. This is clear from the definition of separability, obviously a convex combination of two separable states is again separable. Furthermore, the definition of separability implies that any separable state can be written as a convex combination of pure product states. Thus, the set of separable states is the so called convex hull of the pure product states. Further, any separable state can be written as a convex combination of maximally $d_A^2 d_B^2$ pure product states. This follows from Carathéodory's theorem, which states that any element of a d-dimensional convex set can be written as a convex combination of $d + 1$ extremal points of this set [45]. For the special case of two qubits, however, this bound can be improved and any separable state can be written as a convex combination of four product states only [46,47]. The set of separable states is also shown in Fig. 1.[2]

Given the definition of entanglement and separability, it is very natural to ask whether a given density matrix is separable or entangled. This is the so-called *separability problem*. There are several criteria known that imply separability or entanglement of a state. However, up to now, no general solution for the separability problem is known.

2.3. Separability criteria

In this section, we will present some criteria for bipartite entanglement. As it is not our aim to discuss all of them in detail, we focus our discussion on the most important ones.

2.3.1. The PPT criterion

Let us start with the criterion of the *partial transposition*. In order to formulate this, note that we can expand any density matrix of a composite quantum system in a chosen product basis as

$$\varrho = \sum_{i,j}^{N} \sum_{k,l}^{M} \varrho_{ij,kl} |i\rangle \langle j| \otimes |k\rangle \langle l|. \tag{7}$$

Given this decomposition, one defines the partial transposition of ϱ as the transposition with respect to one subsystem. Thus, there are two partial transpositions: The partial transposition with respect to Alice is given by

$$\varrho^{T_A} = \sum_{i,j}^{N} \sum_{k,l}^{M} \varrho_{ji,kl} |i\rangle \langle j| \otimes |k\rangle \langle l| \tag{8}$$

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[2] A quantitative study on the shape of these sets in space where the coordinates are the density matrix elements is given in Ref. [48].

and similarly we can define ϱ^{T_B} by exchanging k and l instead of the i and j. Note that the partial transposition is related to the usual transposition by $\varrho^T = (\varrho^{T_A})^{T_B}$ and thus $\varrho^{T_B} = (\varrho^{T_A})^T$.

It is worth mentioning that the partial transposition depends on the product basis in which it is performed. But one can show that its spectrum does not depend on the basis.[3] We say a density matrix ϱ has a *positive partial transpose* (or: the matrix is PPT) if its partial transposition has no negative eigenvalues, i.e., it is positive semidefinite:

$$\varrho^{T_A} \geq 0 \Leftrightarrow \varrho^{T_B} \geq 0. \tag{9}$$

If a matrix is not PPT, we call it NPT. Now we can formulate the PPT criterion (also called Peres–Horodecki criterion), originally introduced in Ref. [49].

Theorem 4 (*PPT Criterion*). *Let ϱ be a bipartite separable state. Then ϱ is PPT.*

Proof. This fact follows directly from the definition of separability in Eq. (6) since for a separable $\varrho = \sum_k p_k \varrho_k^A \otimes \varrho_k^B$ we have $\varrho^{T_A} = \sum_k p_k (\varrho_k^A)^T \otimes \varrho_k^B = \sum_k p_k \tilde{\varrho}_k^A \otimes \varrho_k^B \geq 0.$ □

This theorem provides a very strong criterion for the detection of entanglement. For a given density matrix one can easily calculate the partial transpose and compute its spectrum. If one finds negative eigenvalues one can conclude that the state is entangled. Given this result, the question arises if this criterion is also sufficient for separability, i.e., whether $\varrho^{T_A} \geq 0$ implies separability. As it was shown already shortly after the discovery of the PPT criterion, this is the case only in low dimensional systems:

Theorem 5 (*Horodecki Theorem*). *If ϱ is a state in a 2×2 or 2×3 system, then $\varrho^{T_A} \geq 0$ implies that ϱ is separable. In other dimensions this is not the case.*

Proof. For the proof of the 2×2 or 2×3 case, see Ref. [50]. For the first counterexample in a 2×4- or 3×3-system see Ref. [51]. We will give an example in the context of bound entanglement later in Section 2.4. □

Although the PPT criterion does not constitute a necessary and sufficient criterion, it is the most popular criterion. This is, beside its simplicity, due to several reasons. First, the fact that it provides a complete characterization of entanglement for two-qubit systems makes it very appealing, since two-qubit systems are the most studied bipartite systems. Second, it has been shown that the amount of violation of the PPT condition can be used to *quantify* entanglement [52,53] (see also Section 4.2.4). As we will see in Section 3.2.2, however, the PPT criterion is of limited use for the investigation of multipartite entanglement.

2.3.2. The CCNR criterion

Since the PPT criterion does not detect all states, the question arises, how one can prove that a state is entangled, if the PPT criterion fails. For this problem, many criteria have been proposed. Here, we want to explain the *computable cross norm* or *realignment* criterion (CCNR criterion) [54,55], since it is simple and strong.

In order to formulate the CCNR criterion, we utilize the Schmidt decomposition in operator space. For a density matrix ϱ, such a decomposition is

$$\varrho = \sum_k \lambda_k G_k^A \otimes G_k^B, \tag{10}$$

where the $\lambda_k \geq 0$ and G_k^A and G_k^B are orthonormal bases of the observable spaces of $\mathcal{H}_A$ and $\mathcal{H}_B$. Such a basis consists of d^2 hermitian observables which have to fulfill $\text{Tr}(G_k^A G_l^A) = \text{Tr}(G_k^B G_l^B) = \delta_{kl}$. Such observables are often called *local orthogonal observables* (LOOs) [56]. For instance, for qubits the (appropriately normalized) Pauli matrices together with the identity form a set of LOOs.

Eq. (10) is nothing but the Schmidt decomposition from Section 2.1, but applied to the space of observables.[4] Therefore, the λ_k are (up to a permutation) unique and if the λ_k are pairwise different, the G_k^A and G_k^B are also unique (up to a sign). The λ_k can be computed as in the Schmidt decomposition: First, one decomposes $\varrho = \sum_{kl} \mu_{kl} \tilde{G}_k^A \otimes \tilde{G}_l^B$ with arbitrary LOOs $\tilde{G}_k^A$ and $\tilde{G}_l^B$, then, by performing the singular value decomposition of μ_{kl} one arrives at Eq. (10), the λ_k are the roots of the eigenvalues of the matrix $\mu\mu^\dagger$. One can formulate:

Theorem 6 (*CCNR Criterion*). *If the state ϱ is separable, then the sum of all λ_k in Eq. (10) is smaller than one:*

$$\sum_k \lambda_k \leq 1. \tag{11}$$

Hence, if $\sum_k \lambda_k > 1$ the state must be entangled.

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[3] Note that this also holds for the full transposition.

[4] Note that one can also define a Schmidt rank for density matrices [57], this is, however, not related to Eq. (10).

Proof. For a pure product state, $\varrho = |a\rangle\langle a| \otimes |b\rangle\langle b|$ the Schmidt decomposition is directly given, and the statement is trivial. For the general case $\varrho = \sum_k p_k |a_k\rangle\langle a_k| \otimes |b_k\rangle\langle b_k|$, note that the sum over the Schmidt coefficients defines a norm on the set of density matrices. Then we have due to the triangle inequality $\|\varrho\| \leq \sum_k p_k \||a_k\rangle\langle a_k| \otimes |b_k\rangle\langle b_k|\| = 1$. Note that the question, whether other functions of the Schmidt coefficients can be used for entanglement detection, has been studied in Ref. [58]. $\square$

Let us discuss this criterion. First, the two different names come from the fact that this criterion has been discovered in two different forms. It has been first found while investigating cross norms of density matrices [59], the above formulation was taken from [54]. Independently, it has been formulated in a way that one realigns the entries of a density matrix and calculates then the trace norm of this new matrix [55]. This second formulation is somehow more complicated, it allows, however, one to see formal similarities with the PPT criterion [60] or to generalize it into different directions [61–63].

The remarkable fact is that the CCNR criterion allows one to prove the entanglement for many states where the PPT criterion fails. Combined with its simplicity, this makes it a useful tool for the analysis of entanglement. However, it does not detect all entangled states of two-qubits [64]. Therefore, one may view it as complementary to the PPT criterion.

2.3.3. Other criteria

Beside the PPT and CCNR criterion, there are many other approaches to derive separability criteria. Not all of them will be needed for future discussion. However, in order to give the reader a coherent overview we would like to give a small list:

(i) The *range criterion* was one of the first criteria for the detection of states for which the PPT criterion fails [51]. It states that if a state ϱ is separable, then there is a set of product vectors $|a_i b_i\rangle$ such that the set $\{|a_i b_i\rangle\}$ spans the range of ϱ as well as the set $\{|a_i^* b_i\rangle\}$[5] spans the range of ϱ^{T_A}. This condition allowed one to discover many states which are entangled, but not detected by the PPT criterion [51,65,66]. However, it cannot be used if some state is affected by noise: Then, the density matrix and its partial transpose will usually have full rank, hence the condition in the range criterion is automatically fulfilled.

(ii) The PPT criterion is an example of a criterion using a *positive, but not completely positive map*. These objects are defined as follows. Let $\mathcal{H}_B$ and $\mathcal{H}_C$ be Hilbert spaces and let $\mathcal{B}(\mathcal{H}_i)$ denote the linear operators on it. A linear map $\Lambda : \mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_C)$ is called positive if it maps hermitian operators onto hermitian operators, fulfilling $\Lambda(X^\dagger) = \Lambda(X)^\dagger$ and it preserves the positivity, i.e., if $X \geq 0$ then $\Lambda(X) \geq 0$. Note that the second condition implies that it maps valid density matrices onto density matrices, up to normalization. A positive map Λ is called *completely* positive when for an arbitrary Hilbert space $\mathcal{H}_A$ the map $\mathbb{I}_A \otimes \Lambda$ is positive, otherwise, Λ is positive, but not completely positive. Here, $\mathbb{I}_A$ denotes the identity on $\mathcal{B}(\mathcal{H}_A)$. For example, the transposition map T is positive, but not completely positive: while $X \geq 0$ implies $X^T \geq 0$, the *partial* transposition does not necessarily preserve the positivity of the state.

From this, other entanglement criteria similar to the PPT criterion can be formulated from other positive, but not completely positive maps. For any separable state ϱ and any positive map Λ we have

$$(\mathbb{I}_A \otimes \Lambda)(\varrho) \geq 0. \tag{12}$$

Furthermore, it has been shown in Ref. [50] that a state ϱ is separable if and only if for all positive maps Λ the relation in Eq. (12) holds. In this sense, the separability problem is equivalent to the classification of all positive maps. Of course, in order to develop a separability criterion, in Eq. (12) only the positive, but not completely positive maps are of interest.

This problem was considered in the mathematical literature for a longer time [67–70], however, it has not been solved yet. From the perspective of quantum information theory, the classification of positive maps was under intensive research and has led to many new positive, but not completely positive maps, resulting in strong separability tests [71–75].

Another example of a positive, but not completely positive map is the *reduction map* [76]. This is on one system defined as

$$\Lambda^R(X) = \mathrm{Tr}(X) \cdot \mathbb{1} - X. \tag{13}$$

Consequently a separable state has to fulfill $\mathbb{I}_A \otimes \Lambda^R(\varrho) = \varrho_A \otimes \mathbb{1} - \varrho \geq 0$. This is the *reduction criterion for separability*. In Ref. [76], it has been shown that the reduction map is a so-called decomposable map, that is, it can be written as

$$\Lambda^R = P_1 + P_2 \circ T, \tag{14}$$

where P_1, P_2 are completely positive maps, and T is the transposition. This implies, that $\mathbb{I}_A \otimes \Lambda^R$ can never detect entanglement, unless $\mathbb{I}_A \otimes T$ detects it.[6] Therefore, the reduction criterion is weaker than the PPT criterion. Nevertheless, the reduction criterion is interesting for several reasons: For two qubits, it also provides a necessary and sufficient criterion for separability [76], and it is further closely connected to the process of distillation (see Section 2.4) and

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[5] The symbol $|a^*\rangle$ denotes the vector resulting when all coefficients of $|a\rangle$ in a certain basis are complex conjugated. Note that $|a^*\rangle\langle a^*| = |a\rangle\langle a|^T$.

[6] In fact, all positive maps with dim $\mathcal{H}_B = 2$ and dim $\mathcal{H}_B \leq 3$ are decomposable [68,70], from which it follows, that the PPT criterion is necessary and sufficient for low dimensions.

certain entanglement witnesses (see Section 7.2). Furthermore, one can extend the reduction map in the following way [72,77]: If U is a unitary matrix with $U^{\mathrm{T}} = -U$, then the map $\Lambda(X) = \mathrm{Tr}(X) \cdot \mathbb{1} - X - UX^{\mathrm{T}}U^{\dagger}$ is a positive, but not completely positive map. Such a unitary transformation can only exist in even dimensions. But then, the map clearly improves the reduction criterion and can detect states that are not detected by the PPT criterion.

For our purposes, the theory of positive maps will become important for the following reason. We will see that entanglement witnesses are good tools for the experimental detection of entanglement. Any entanglement witness, however, gives rise to a positive, but not completely positive map via the Choi–Jamiołkowski isomorphism, as will be explained in Section 2.5.3. Furthermore, the theory of positive maps will later enable us to construct nonlinear witnesses and to derive further entanglement tests (see Sections 7.3.2 and 7.4).

(iii) The *majorization criterion* relates the eigenvalues of the total state with reduced states [78]. For a general state ϱ one takes $\varrho_A = \mathrm{Tr}_B(\varrho)$ as the reduced state with respect to Alice and denotes by $\mathcal{P} = (p_1, p_2, \ldots)$ the decreasingly ordered eigenvalues of ϱ and by $\mathcal{Q} = (q_1, q_2, \ldots)$ the decreasingly ordered eigenvalues of ϱ_A. The majorization criterion states that if ϱ is separable, then

$$\sum_{i=1}^{k} p_i \le \sum_{i=1}^{k} q_i \tag{15}$$

holds for all k. The same inequality holds, when ϱ_A is replaced by the reduced density matrix of the second system $\varrho_B = \mathrm{Tr}_A(\varrho)$. Physically, this criterion means that for a separable state ϱ is more disordered than ϱ_A, and this criterion also results in inequalities for the entropies of ϱ and ϱ_A [79–82]. It was shown in Ref. [83], however, that the majorization criterion follows from the reduction criterion hence any state which can be detected by the majorization criterion, can also be detected by the reduction criterion and, consequently, by the PPT criterion. Similarly, one can derive criteria for entropies from other positive, but not completely positive maps, leading to a generalization of the majorization criterion [84–86].

(iv) There are further *algorithmic approaches* to the separability problem. These methods formulate separability in terms of (convex) optimization problems or semidefinite programs, in order to derive numerical algorithms for separability testing.

Let us explain the method of the *symmetric extensions* [87,88], since this is one of the most powerful separability criteria. The idea of this algorithm comes from the following observation. If a state $\varrho = \sum_k p_k |a_k\rangle\langle a_k| \otimes |b_k\rangle\langle b_k|$, is separable, we can find a symmetric extension to three parties (A, B and A_1) as

$$\varrho^{(ABA_1)} = \sum_k p_k |a_k\rangle\langle a_k| \otimes |b_k\rangle\langle b_k| \otimes |a_k\rangle\langle a_k|. \tag{16}$$

The state $\varrho^{(ABA_1)}$ has now the following properties: (a) it is PPT with respect to each partition, (b) the reduced state on the first two parties is given by ϱ and (c) it is symmetric under the exchange of the parties A, A_1. Similarly, one can define for a separable symmetric extensions to more parties $A, B, A_1, \ldots, A_k$.

The key observation in Ref. [87] is that the question "Is there a symmetric extension with the properties (a)–(c)?" can be directly formulated as a feasibility problem in semidefinite programming. Semidefinite programs are a family of optimization problems that can not only be solved efficiently, but under weak conditions the global optimality of the found solution can also be proven (see Ref. [89] for a review). Practically this means that the above mentioned feasibility problem can be directly tackled with standard numerical packages, and if no extension is found, the algorithm can also prove that no extension exists, consequently the state must be entangled.

In fact, the method of symmetric extensions delivers a hierarchy of separability criteria, the criterion looking for the extension to $k + 1$ parties is always stronger than the criterion with the extension to k parties. It was then shown in Ref. [90] (see also Ref. [88]) that this hierarchy is complete in the sense that any entangled state is detected in some step of the hierarchy. In addition, due to symmetry requirements, the number of parameters in the semidefinite program increases only polynomially in the number of extensions. Still, however, the numerical effort is considerable: while the first step of the hierarchy is, for system up to 5×5 feasible on a standard computer,[7] the higher order steps are difficult to perform.

Beside the above mentioned algorithms, there are several other algorithmic approaches. First, the method of symmetric extensions was extended to the multipartite case [91]. Then, in Ref. [92] it was shown that one can view the separability problem as an optimization problem with polynomial constraints, which can, in principle, be tackled with a hierarchy of semidefinite programs [93]. Other algorithms for separability testing have been proposed by making use of convex geometry [94–100], see also Ref. [101] for an overview. In addition, it should be noted that semidefinite programs can also be used to tackle other problems in quantum information theory, such as the existence of local hidden variable models [102], distillability of states [103] and the estimation of localizable information [104].

Finally, these algorithms pose the question concerning the algorithmic complexity of the separability problem, that is, the question how much time is needed to decide whether a given state is entangled or not. It has been shown that

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[7] Numerical code for this first step is available at http://www.iqi.caltech.edu/documents/spedalieri/pptsetest1.m.

the separability problem, considered as a weak membership problem, is an NP hard problem [105]. That is, deciding separability with an accuracy of ε in an $d \times d$ system requires an effort exponentially in d, if the inverse accuracy $1/\varepsilon$ should increase exponentially in d [101,105]. As recently shown, this result also holds if the inverse accuracy $1/\varepsilon$ is required to increase only polynomially [106]. It should be noted, however, that this statement does not exclude the possibility of a simple solution for any fixed dimension. In addition, as this statement concerns only the computational complexity, it does not exclude possible physical insights into the separability problem.

(v) Another approach to the separability problem makes use of criteria based on *covariance matrices* (CMs) [107–110]. Here, one considers for some observables M_k the covariance matrix γ of these observables in some state, given by its entries

$$\gamma_{ij} = \langle M_i M_j \rangle - \langle M_i \rangle \langle M_j \rangle. \tag{17}$$

Then, one tries to formulate separability criteria from this representation. The motivation for this approach comes from the theory of infinite-dimensional systems. There, an important family of states consists of Gaussian states, which are described by such a CM, using the canonical conjugate variables X and P (for reviews on this subject see Refs. [14–17]). Moreover, for these Gaussian states the separability problem has been solved [111–113].

With the help of these covariance matrices one can formulate a general *covariance matrix criterion* (CMC) [109,114]. For this, one chooses the observables in Eq. (17) to be the set $\{G_i^A \otimes \mathbb{1}, \mathbb{1} \otimes G_j^B\}$ with local orthogonal observables G_i^A and G_i^B. Then, γ has a 2×2 block structure, and if ϱ is separable, one can find matrices $\kappa_{A/B} = \sum_k p_k \gamma(|\psi_k^{A/B}\rangle)$ which are convex combinations of CMs of pure states on the subsystems, such that

$$\gamma = \begin{bmatrix} A & C \\ C^{\mathrm{T}} & B \end{bmatrix} \geq \begin{bmatrix} \kappa_A & 0 \\ 0 & \kappa_B \end{bmatrix}. \tag{18}$$

On the one hand, this criterion is very strong, as is detects many bound entangled states, and, with the help of filtering operations (see Sections 2.5.3 and 6.8), it can detect all entangled states for two qubits, just as the PPT criterion.[8] Moreover, the CCNR criterion can be shown to be a corollary of the CMC. For instance, from the CMC it follows that for separable states

$$\|C\|_1 \leq \sqrt{[1 - \mathrm{Tr}(\varrho_A^2)][1 - \mathrm{Tr}(\varrho_B^2)]}, \tag{19}$$

where $\|C\|_1$ is the trace norm of the matrix $C_{ij} = \langle G_i^A \otimes G_j^B \rangle - \langle G_i^A \rangle \langle G_j^B \rangle$, which is a systematic improvement of the CCNR criterion [58,114,115]. In addition, the CMC can be related to other known separability criteria [116,117].

It is not clear, how to use the CMC for entanglement detection in the multipartite case. The closely connected criterion of the *local uncertainty relations* (LURs) [107] is, however, experimentally relevant and will be discussed in Section 7.3.1.

(vi) A similar approach uses the *expectation value matrix* (EVM) of a quantum state to investigate its entanglement properties [118–121]. For a set of operators $\{M_k\}$ one defines the positive semidefinite EVM χ as

$$\chi_{ij} = \langle M_i^\dagger M_j \rangle. \tag{20}$$

Typically, one chooses the observables to be of the type $\{M_k\} = \{A_k \otimes B_l\}$. Then it is straightforward to see that for a separable state $\varrho = \sum_k p_k \varrho_k^A \otimes \varrho_k^B$ one has $\chi(\varrho) = \sum_k p_k \chi(\varrho_k^A) \otimes \chi(\varrho_k^B)$, hence $\chi(\varrho)$ is also separable and its separability can be investigated by criteria such as the PPT criterion. Especially, if $B_l^{\mathrm{T}} = B_l^\dagger$ one has $\chi^{\mathrm{T}_B} = \chi(\varrho^{\mathrm{T}_B})$. This has two interesting consequences: First, a violation of the PPT criterion for χ may be indicated by the negativity of some subdeterminants of χ, leading to nonlinear or variance-like criteria for separability [118,122]. Second, if the PPT criterion for χ is violated, there is a vector $|x\rangle = (x_i)$ such that $\langle x | \chi^{\mathrm{T}_B} | x \rangle < 0$ and hence a positive operator $P = X^\dagger X$ with $X = \sum_i x_i M_i$ such that $\mathrm{Tr}(\varrho P^{\mathrm{T}_B}) < 0$ while $\mathrm{Tr}(\varrho P^{\mathrm{T}_B}) = \langle x | \chi^{\mathrm{T}_B} | x \rangle \geq 0$ for all PPT states. The observable P^{T_B} will later be called an entanglement witness (see Section 2.5) and therefore the EVM allows to investigate for which states a violation of the PPT criterion can be proved with a given, restricted set of observables $\{M_k\} = \{A_k \otimes B_l\}$ [120]. Furthermore, the EVM can be used to investigate entanglement and separability for qubit-mode systems [123,124].

(vii) A different line of developing separability criteria uses *linear contractions* and *permutations* [60]. The main idea behind this approach is the following: given a density matrix $\varrho = \sum_{ij,kl} \varrho_{ijkl} |i\rangle \langle j| \otimes |k\rangle \langle l|$ expanded in a product basis, all information concerning the state is encoded in the coefficients ϱ_{ijkl}, which build a $(d_A \times d_A) \times (d_B \times d_B)$ tensor. For pure states, it is normalized, $\|\varrho_{ijkl}\|_1 = 1$, where $\|X\|_1 = \mathrm{Tr}(\sqrt{XX^\dagger})$ denotes the trace norm, and ϱ_{ijkl} is considered as an $(d_A^2) \times (d_B^2)$ matrix. It is easy to see that for a product state, any permutation $\pi(ijkl)$ of the indices i, j, k, l also yields a tensor with $\|\varrho_{\pi(ijkl)}\|_1 = 1$, where, depending on the permutation, $\varrho_{\pi(ijkl)}$ might be considered as a $(d_A d_B) \times (d_A d_B)$ matrix. Since the trace norm obeys the triangle inequality, this means that for any separable state $\|\varrho_{\pi(ijkl)}\|_1 \leq \sum_\alpha p_\alpha \|\varrho_{\pi(ijkl)}^{(\alpha)}\|_1 = 1$, where $\varrho = \sum_\alpha p_\alpha \varrho^{(\alpha)}$ It turns out, that in the end for two parties only two permutations yield independent separability criteria, resulting in the conditions

$$\|\varrho_{(ijlk)}\|_1 \leq 1 \quad \text{and} \quad \|\varrho_{(ikjl)}\|_1 \leq 1. \tag{21}$$

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[8] For general systems, however, there are NPT states which are not detected by the CMC and filtering; moreover, there are entangled PPT states which not detected by CMC.

It can be directly seen that the first condition is nothing but the PPT criterion, while the second one corresponds to the CCNR criterion. Thus, this approach delivers a way to formulate these two criteria in a unified way. Further, it can be extended to the multipartite case, where many new separability criteria arise [62,63], see also Section 3.5.

Other approaches to detect entanglement use special observables, they are called either Bell inequalities or entanglement witnesses. Bell inequalities will be discussed in Section 5 and entanglement witnesses in Section 2.5. Before doing that, we would like to discuss the phenomenon of bound entanglement.

2.4. Bound entanglement

For many tasks in quantum information theory, like teleportation or cryptography one ideally needs maximally entangled two-qubit states, i.e., singlet states. In the real world, however, noise is unavoidable, thus only mixed states are available. So one has to deal with the question, if and how one can create a singlet state out of some given mixed state. This leads to the problem of the so-called *distillation of entanglement.* It can be posed as follows: Assume that there is an arbitrary, but finite, number of copies of an entangled quantum state ϱ distributed between Alice and Bob. Can they perform local operations on the states, assisted by classical communication, such that at the end they share a singlet state,

$$\underbrace{\varrho \otimes \varrho \otimes \cdots \otimes \varrho}_{k \text{ copies}} \xrightarrow{\text{LOCC}} |\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)? \tag{22}$$

If this is the case, the state ϱ is called distillable, otherwise we call it undistillable or *bound entangled.*

It is not clear from the beginning that one can distill quantum states at all. However, the first distillation protocols were derived in Refs. [125,126], showing that it is in principle possible. The question whether a given state is distillable is, of course, difficult to decide, since there is no restriction to a specific kind of local operations, as well as to the number of copies. The failure of a special protocol is not enough to conclude that a state is undistillable. However, some simple criteria are known:

Theorem 7. *If a bipartite state is PPT, then the state is undistillable. If a state violates the reduction criterion (e.g., due to a violation of the majorization criterion) then the state is distillable.*

Proof. For the sufficient condition to be undistillable, see Ref. [127]. For the sufficient condition to be distillable, see Refs. [76,83]. □

By definition, bound entangled states require some entanglement for its creation, however, afterwards the entanglement cannot be distilled back. Naturally, such states are interesting, and the above theorem implies that the search for PPT entangled states is of great interest in studying bound entanglement. The first example of a PPT entangled state was given in Ref. [51] and in the meantime many other examples have been found [65,66,74,128].

In order to demonstrate how such states may be constructed, let us discuss a simple example from Ref. [65]. In a 3×3-system, one can consider the vectors

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}|0\rangle(|0\rangle - |1\rangle), \qquad |\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|2\rangle, \qquad |\psi_2\rangle = \frac{1}{\sqrt{2}}|2\rangle(|1\rangle - |2\rangle),$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)|0\rangle, \qquad |\psi_4\rangle = \frac{1}{3}(|0\rangle + |1\rangle + |2\rangle)(|0\rangle + |1\rangle + |2\rangle). \tag{23}$$

These five product vectors form a so-called *unextendible product basis:* they are all pairwise orthogonal and there is not another product vector orthogonal to all of them, as can be seen by direct inspection. Therefore, the mixed state

$$\varrho_{\text{BE}} = \frac{1}{4}\left(\mathbb{1} - \sum_{i=0}^{4} |\psi_i\rangle\langle\psi_i|\right) \tag{24}$$

has no product vector in its range, and must be entangled due to the range criterion. Since $\varrho_{\text{BE}} = \varrho_{\text{BE}}^{\text{T}_B}$ it is also PPT, and thus bound entangled. This state has also the minimal possible rank, since the minimal rank of a PPT entangled state in a $d_A \times d_B$-system is $\max\{d_A, d_B\} + 1$ [129].

Beside the mere existence, there are several results and questions, which make the phenomenon of bound entanglement interesting:

(i) At first sight, entangled states that cannot be distilled seem to be not very useful, since they cannot be used for certain tasks like quantum key distribution via the Ekert protocol [4]. It has been proved, however, that some bound entangled states can indeed be used for secure quantum key distribution [130,131], showing that entanglement distillation and secure key distillation are not equivalent.
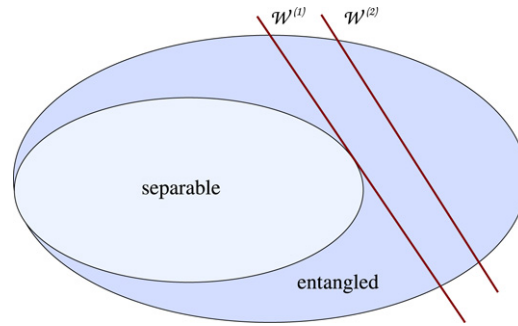
Fig. 2. Schematic picture of the set of all states and the set of separable states as nested convex sets and two witnesses, $\mathcal{W}^{(1)}$ and $\mathcal{W}^{(2)}$. The red lines represent the hyperplanes, where $\mathrm{Tr}(\mathcal{W}\varrho) = 0$. The witness $\mathcal{W}^{(1)}$ is finer than the witness $\mathcal{W}^{(2)}$ [see also Eq. (32)].

 (ii) Bound entangled states require entanglement for their creation, but the entanglement can then not be distilled again. Similarly, one may ask, whether there are classical probability distributions between two or several parties that require secret communication for their creation, but they do not allow generating a secret key [132]. It has been shown for the tripartite case, that such *bound information* exists [133]. Interestingly, the proof is based on a bound entangled tripartite quantum state, from which the classical probability distribution is derived.
(iii) It has already been conjectured by Peres in 1999, that bound entangled states admit a local hidden variable model and thus do not violate any Bell inequality [29,134]. We will discuss this open question in more detail in Section 5.3.
(iv) Since PPT entangled states are bound entangled, the question arises, whether NPT states are always distillable. It has been conjectured that this is not the case [135,136]. Examples of states have been found, which cannot be distilled, if only a few copies are available [135,136]. However, despite several partial results [137] the question remains open. For $2 \times d_B$-systems, any NPT state is distillable [135].
 (v) The given example of a bound entangled state may seem artificial. An interesting question is whether bound entangled states occur under natural conditions, e.g., as thermal states in spin models. For multipartite bound entanglement, this is indeed the case [138,139].

To conclude, all these results indicate that bound entangled states are an interesting object of study. Their construction, characterization and detection is a challenging and important task in entanglement theory.

2.5. Entanglement witnesses

Let us now come to a completely different type of separability criteria. The criteria from above have all something in common: at first sight they all assume that the density matrix is already known. They all require applying certain operations to a density matrix, to decide whether the state is entangled or not.[9] There is, however, a necessary and sufficient entanglement criterion in terms of directly measurable observables. These are the so called entanglement witnesses [24,50, 140,141], which we introduce now.

Definition 8. An observable $\mathcal{W}$ is called an *entanglement witness* (or witness for short), if

$$\mathrm{Tr}(\mathcal{W}\varrho_s) \geq 0 \quad \text{for all separable } \varrho_s,$$
$$\mathrm{Tr}(\mathcal{W}\varrho_e) < 0 \quad \text{for at least one entangled } \varrho_e \tag{25}$$

holds. Thus, if one measures $\mathrm{Tr}(\mathcal{W}\varrho) < 0$ one knows for sure that the state ϱ is entangled. We call a state with $\mathrm{Tr}(\mathcal{W}\varrho) < 0$ to be detected by $\mathcal{W}$.

The fact that entanglement witnesses are directly measurable quantities makes them a very useful tool for the analysis of entanglement in experiment. As we will see in the further course of this review, entanglement witnesses are one of the main methods to detect entanglement experimentally.

For the further understanding, it is crucial to note that entanglement witnesses have a clear geometrical meaning. The expectation value of an observable depends linearly on the state. Thus, the set of states where $\mathrm{Tr}(\mathcal{W}\varrho) = 0$ holds is a hyperplane in the set of all states, cutting this set into two parts. In the part with $\mathrm{Tr}(\mathcal{W}\varrho) > 0$ lies the set of all separable states, the other part [with $\mathrm{Tr}(\mathcal{W}\varrho) < 0$] is the set of states detected by $\mathcal{W}$. This scheme is shown in Fig. 2. From this geometrical interpretation it follows that all entangled states can be detected by witnesses:

Theorem 9 (*Completeness of Witnesses*). *For each entangled state ϱ_e there exists an entanglement witness detecting it.*

---

[9] Note that in principle one can decide whether a state violates the PPT criterion or not without knowing the state completely. For schemes to do that see Section 7.4.

Proof. This theorem was proved in Ref. [50]. The idea for the proof relies on the fact that the set of separable states is convex and closed. Thus, for a point outside this set, there is a hyperplane separating this point from the convex set. For finite-dimensional spaces with a scalar product this is easy to see, but the statement holds also for infinite-dimensional Banach spaces, where the proof requires a corollary of the Hahn–Banach-theorem [142].   □

Although this theorem ensures that any entangled state can in principle be detected with an entanglement witness, the task remains to construct witnesses. This is not an easy problem, since solving this problem would also solve the separability problem. Indeed, a large part of this review is concerned with the proper construction and evaluation of witnesses for the multipartite case.

2.5.1. Construction of witnesses

Let us discuss in three simple examples, how witnesses for bipartite entanglement can be constructed. Typically, if a state violates some criterion for separability, an entanglement witness detecting the state can be written down.

As the first example, let us take a state ϱ_e which is NPT. Then there is a negative eigenvalue $\lambda_- < 0$ of $\varrho_e^{T_A}$ and a corresponding eigenvector $|\eta\rangle$. Now

$$\mathcal{W} = |\eta\rangle\langle\eta|^{T_A} \tag{26}$$

is a witness detecting ϱ_e: Due to the fact that for general matrices X, Y the relation $\mathrm{Tr}(XY^{T_A}) = \mathrm{Tr}(X^{T_A}Y)$ holds,[10] we have $\mathrm{Tr}(\mathcal{W}\varrho_e) = \mathrm{Tr}(|\eta\rangle\langle\eta|^{T_A}\varrho_e) = \mathrm{Tr}(|\eta\rangle\langle\eta|\varrho_e^{T_A}) = \lambda_- < 0$ and $\mathrm{Tr}(\mathcal{W}\varrho_s) = \mathrm{Tr}(|\eta\rangle\langle\eta|\varrho_s^{T_A}) \geq 0$ for all separable ϱ_s, since they are PPT.

More generally, if ϱ_e is entangled and detected by some positive map Λ, then $\mathbb{I} \otimes \Lambda(\varrho_e)$ has again a negative eigenvalue $\lambda_- < 0$ for the eigenvector $|\eta\rangle$. Then, $\mathcal{W} = \mathbb{I} \otimes \Lambda^+(|\eta\rangle\langle\eta|)$ is a witness detecting ϱ_e. Here, Λ^+ denotes the adjoint map, i.e., the map fulfilling $\mathrm{Tr}[\Lambda^+(X)Y] = \mathrm{Tr}[X\Lambda(Y)]$ for all X, Y.

As a second example, let us consider the case when ϱ_e violates the CCNR criterion. Then, by definition, we have in the Schmidt decomposition in Eq. (10) $\sum_k \lambda_k > 1$. A witness is now given by [56,110]

$$\mathcal{W} = \mathbb{1} - \sum_k G_k^A \otimes G_k^B, \tag{27}$$

where the $G_k^{A/B}$ are the observables from the Schmidt decomposition. Clearly, for ϱ_e we have $\mathrm{Tr}(\mathcal{W}\varrho_e) = 1 - \sum_k \lambda_k < 0$. A general separable state $\varrho_s = \sum_{kl} \mu_{kl} G_k^A \otimes G_l^B$ can be expressed in the basis of $G_k^{A/B}$. Then, $\mathrm{Tr}(\mathcal{W}\varrho_s) = 1 - \sum_k \mu_{kk} \geq 0$, where the last estimate comes from the CCNR criterion and the fact, that a trace of a matrix is always smaller than the sum of its singular values [143]. The witness in Eq. (27) can also be derived from the realignment map [110,144].

As a third example, witnesses can also be constructed from the consideration that states close to an entangled state must also be entangled. Therefore, one may try for a given entangled pure state $|\psi\rangle$ to write down a projector-based witness like

$$\mathcal{W} = \alpha\mathbb{1} - |\psi\rangle\langle\psi|. \tag{28}$$

This witness can be interpreted as follows: The quantity $\mathrm{Tr}(\varrho|\psi\rangle\langle\psi|) = \langle\psi|\varrho|\psi\rangle$ is the fidelity of the state $|\psi\rangle$ in the mixed state ϱ, and if this fidelity exceeds a critical value α, then the expectation value of the witness is negative and the state ϱ must be entangled.

The problem remains to calculate the smallest possible value for α, in order to ensure that $\mathcal{W}$ is still positive on all separable states. This can be done by taking

$$\alpha = \max_{\varrho \text{ is separable}} \mathrm{Tr}(\varrho|\psi\rangle\langle\psi|) = \max_{|\phi\rangle=|a\rangle\otimes|b\rangle} |\langle\psi|\phi\rangle|^2. \tag{29}$$

Here, we used that a linear function takes its maximum on a convex set in one of the extremal points, and for the convex set of the separable states these extremal points are just the pure product states. Then, as shown in Ref. [145] the maximum can be directly computed and is given by the square of the maximal Schmidt coefficient of $|\psi\rangle$. Finally it is worth mentioning that the witness $\mathcal{W}$ in Eq. (28) does only detect states which are NPT therefore the construction is less general than the one in Eq. (26).[11] It has the advantage, however, that it can be generalized to the multipartite case, as we will see later.

Finally, let us mention other ways of constructing entanglement witnesses. Most of them are connected to special separability criteria.

(i) As already mentioned, any positive, but not completely positive map gives rise to a construction of entanglement witnesses. As we will see soon, also a connection in the other direction holds.

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[10] This is obvious for the case $X = A \otimes B$ and $Y = C \otimes D$, for the general case it follows from linearity.

[11] This follows from the fact that the witness $\mathcal{W} = \mathbb{1}/d - |\psi\rangle\langle\psi|$, where $|\psi\rangle$ is maximally entangled, does not detect NPT states [146], combined with an argument as in Theorem 3 of Ref. [147].

(ii) Many bound entangled states violate the range criterion in a way that for any product vector $|a, b\rangle \in R(\varrho)$ in the range of ϱ we have already $|a, b^*\rangle \notin R(\varrho^{T_B})$. States with such an extremal violation are called *edge states*, and if an edge state ϱ is PPT, then it lies on the boundary between the PPT and the NPT states. For such a state, one can write down a witness of the type

$$\mathcal{W} = \alpha \mathbb{1} - P - Q^{T_B}, \tag{30}$$

where P (Q) denotes the projector onto the range of ϱ (ϱ^{T_B}) and α is defined as $\alpha = \max_{|\phi\rangle = |a,b\rangle} \langle\phi|(P - Q^{T_B})|\phi\rangle < 2$ [141]. This construction guarantees that $\mathcal{W}$ is a valid witness, and ϱ is clearly detected. The parameter α can either be estimated analytically [71], or calculated numerically [92,148,149].

(iii) The semidefinite program in the criterion of the symmetric extensions [87] gives as a byproduct an entanglement witness, if no symmetric extension was found.

(iv) Entanglement witnesses can also be constructed from geometrical considerations: If ϱ_e is an entangled state, and σ is the closest separable state to ϱ_e [in Hilbert–Schmidt norm, $\|X\| := \sqrt{\text{Tr}(X^\dagger X)}$], then

$$\mathcal{W} = \frac{1}{\mathcal{N}} \left(\sigma - \varrho_e + \text{Tr}[\sigma(\varrho_e - \sigma)] \cdot \mathbb{1}\right) \tag{31}$$

with $\mathcal{N} = \|\varrho_e - \sigma\|$ is an entanglement witness, since the set of states with $\text{Tr}(\mathcal{W}\varrho) = 0$ is the hyperplane orthogonal to the line from ϱ_e to σ [150,151]. The normalization is chosen in such a way that its expectation value for ϱ_e equals (up to the sign) the distance to the separable states [151,152]. Such geometric entanglement witnesses have been applied to different situations [153–156].

(v) The criteria based on variances and covariance matrices also result in certain types of entanglement witnesses. These witnesses, however, are nonlinear, we will discuss them in Section 7.3. Further, special constructions of witnesses have derived from estimations of entanglement measures [157] (see also Section 7.2). Also, from a given witness detecting some state further witnesses can be constructed [158].

2.5.2. Optimization of witnesses

Every entanglement witness can, by definition, detect some entangled states. However, as can directly be seen in Fig. 2, some witnesses are better in this task than others. In this sense, one can *optimize* entanglement witnesses [141]. First, one calls a witness $\mathcal{W}^{(1)}$ *finer* than another witness $\mathcal{W}^{(2)}$ if $\mathcal{W}^{(1)}$ detects all the states detected by $\mathcal{W}^{(2)}$ and also some states in addition. This is the case, if

$$\mathcal{W}^{(2)} = \mathcal{W}^{(1)} + P, \tag{32}$$

where P is a positive operator. Consequently, for any state $\text{Tr}(\varrho\mathcal{W}^{(1)}) \le \text{Tr}(\varrho\mathcal{W}^{(2)})$ holds.

Furthermore, a witness $\mathcal{W}$ is called *optimal*, if there is no other witness, which is finer than $\mathcal{W}$. This implies that for any positive operator P the observable $X = \mathcal{W} - P$ is no longer a witness. From this it can be easily seen that $\mathcal{W}$ is optimal, if and only if the product vectors $|\phi_i\rangle = |a_i b_i\rangle$ with $\langle\phi_i|\mathcal{W}|\phi_i\rangle = 0$ span the whole space. A necessary condition for a witness to be optimal is that it "touches" the set of separable states (see Fig. 2), i.e., there must be a separable ϱ with $\text{Tr}(\varrho\mathcal{W}) = 0$. This is, however, not a sufficient condition, witnesses which fulfill this condition are sometimes also called weakly optimal.

The optimization of a given witness can be done as follows [141]: If one finds a positive operator P, which fulfills $\langle a_i b_i|P|a_i, b_i\rangle = 0$ for all product vectors with $\langle a_i b_i|\mathcal{W}|a_i, b_i\rangle = 0$, then $\tilde{\mathcal{W}} = \mathcal{W} - \lambda P$ is an entanglement witness finer than $\mathcal{W}$ if

$$\lambda \le \lambda_0 := \inf_a \min_{\text{eigenvalues}} \left[\frac{1}{\sqrt{P_a}} \mathcal{W}_a \frac{1}{\sqrt{P_a}}\right] \tag{33}$$

with the notation $X_a = \langle a|X|a\rangle \in \mathcal{B}(\mathcal{H}_B)$ for $|a\rangle \in \mathcal{H}_A$ and $X \in \{\mathcal{W}, P\}$. This can be iterated, until the product vectors with $\langle a_i b_i|\mathcal{W}|a_i b_i\rangle = 0$ span the complete space. However, the optimization in Eq. (33) is not straightforward, as it is equivalent to checking whether $\inf_{|a\rangle|b\rangle} \langle a, b|[\mathcal{W} - \lambda P]|a, b\rangle \ge 0$. In fact, such an optimization for general observables is equivalent to the separability problem [159], and it can be shown that the question whether a given state is separable or not is equivalent to the question whether some witness in a higher dimensional space is weakly optimal or not [160].

2.5.3. The Choi–Jamiołkowski isomorphism

Now we explain a close connection between witnesses and positive maps. This connection is given by an isomorphism between quantum states on a space of matrices, and maps between such spaces. In the literature, this isomorphism is known as the Choi–Jamiołkowski isomorphism [69,161–163].[12]

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[12] To our knowledge, an isomorphism of this type was first investigated by de Pillis [161], who used a map as in Eq. (36) without the transposition on the first subsystem and proved that for this map ε is positive if E is positive semidefinite. In Ref. [162], it was proved for the same map that ε is positive, iff E is positive on all product states, i.e., it is an entanglement witness, and the explained version and the proof of main properties in Theorem 10 are given in Refs. [69,163]. See also Ref. [164].

According to this, an operator E in $\mathcal{B}(\mathcal{H}_B) \otimes \mathcal{B}(\mathcal{H}_C)$ corresponds to a map ε that maps states from $\mathcal{B}(\mathcal{H}_B)$ to an element of $\mathcal{B}(\mathcal{H}_C)$. This map is given by

$$\varepsilon(\varrho) = \text{Tr}_B(E\varrho^{\text{T}} \otimes \mathbb{1}_C). \tag{34}$$

The inverse relation reads

$$E = (\mathbb{I}_{B'} \otimes \varepsilon)(|\phi^+\rangle\langle\phi^+|), \tag{35}$$

where $\mathcal{H}_{B'} \cong \mathcal{H}_B$ and $|\phi^+\rangle = \sum_i |ii\rangle$ is a non-normalized maximally entangled state on $\mathcal{H}_{B'} \otimes \mathcal{H}_B$. Using the fact that $|\phi^+\rangle\langle\phi^+| = \sum_k (G_k^{B'})^{\text{T}} \otimes G_k^B$ (see, e.g., Ref. [110]) for any orthogonal basis $G_k^{B'/B}$, this may be expressed as

$$E = \sum_k (G_k^{B'})^{\text{T}} \otimes \varepsilon(G_k^B) \tag{36}$$

from which it is also easy to see that Eq. (34) is the inverse of Eq. (35). Physically, these relations can be interpreted as follows. If the map ε can be realized, then the state E can be prepared by applying the map on one part of a maximally entangled state. On the other hand (due to the relation $\text{Tr}_{B'B}[X_{B'} \otimes Y_B|\phi^+\rangle\langle\phi^+|] = \text{Tr}_{B'B}[(X_{B'})^{\text{T}} \otimes Y_B|\phi^+\rangle\langle\phi^+|^{\text{T}_{B'}}] = \text{Tr}_B[X^{\text{T}}Y])$ one can view the map ε as generated via teleportation with the state E.

The Choi–Jamiołkowski isomorphism is useful for several purposes. For instance, it can be used to define a distance measure for maps as quantum operations by looking at the distance of the corresponding states. This distance has then several natural properties [165]. It can also serve to investigate the entanglement generating capacity of positive maps on a bipartite system [166]. For the theory of witnesses, the following theorem concerning the relation between the map ε and the state E is relevant (for a proof see Refs. [69,163,167]):

Theorem 10 (*Properties of the Choi–Jamiołkowski Isomorphism*). *The Choi–Jamiołkowski isomorphism has the following properties:*

(i) *The map ε is completely positive, iff E is a positive semidefinite operator.*
(ii) *The map ε is positive but not completely positive, iff E is an entanglement witness*
(iii) *The map ε is a decomposable map, iff E is a decomposable entanglement witness.*

Concerning the last point, note that an entanglement witness is called decomposable, if it can be written as $\mathcal{W} = P_1 + P_2^{\text{T}_B}$, where P_1, P_2 are positive operators [141]. Clearly, such a witness cannot detect any PPT entangled state.

Due to this isomorphism, any witness, which detects some PPT entangled state gives rise to a positive, but not completely positive map, which is non-decomposable. In this way, entanglement witnesses can be used to construct new and interesting positive maps.

The question arises, whether the map $\varepsilon(\mathcal{W})$ detects more states than the witness $\mathcal{W}$. Indeed, one can directly see that if $\Lambda : \mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_{A'})$ is a map detecting the state ϱ_{AB}, then it detects all the states $(F_A \otimes \mathbb{1}_B)\varrho_{AB}(F_A^\dagger \otimes \mathbb{1}_B)$, where F_A is an arbitrary invertible matrix [167]. On the other hand, if $\mathcal{W}_{AB}$ is a witness with $\text{Tr}(\mathcal{W}_{AB}\varrho_{AB}) < 0$, then the adjoint map $[\varepsilon(\mathcal{W})]^+ : \mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_{A'})$ detects the state, from which it follows that the map ε^+ detects all states which are detected by the witness

$$\tilde{\mathcal{W}} = [(F_A)^{-1} \otimes \mathbb{1}_B]\mathcal{W}[(F_A^\dagger)^{-1} \otimes \mathbb{1}_B]. \tag{37}$$

This is a larger set of states, for examples see Refs. [168,169]. The operator F_A corresponds to local filters [170–172]. Such transformations do not change the fact whether a state is entangled or not, hence they may be used to simplify entanglement detection. For instance, it has been shown that any state ϱ of full rank can be simplified via filters to (assuming $d_A \leq d_B$)

$$\varrho \mapsto \tilde{\varrho} = (F_A \otimes F_B)\varrho(F_A \otimes F_B)^\dagger = \frac{\mathbb{1}}{d_A d_B} + \sum_{k=1}^{d_A^2-1} \xi_i G_k^A \otimes G_k^B, \tag{38}$$

where the $G_k^{A/B}$ are traceless orthogonal observables [171,172]. Such normal forms under filtering operations have been used to improve given separability criteria [109,114] and we will see in Section 6.8 how filters can be used to improve witnesses systematically.

Finally, another interesting consequence from the Choi–Jamiołkowski isomorphism is that any witness can be written in the form Eq. (35). This form is similar to Eq. (26), but with a different positive map. This idea will be later used for the construction of nonlinear witnesses.

3. Multipartite entanglement

In this section we discuss the structure of entanglement when more than two parties are involved. It will turn out that this structure is much richer than the structure of entanglement in the bipartite case. Especially, it will turn out that several inequivalent classes of entanglement exist. We first discuss different notions of entanglement and separability for the case of three qubits. Then, we discuss the case of general multipartite systems. After that, we introduce the main families of multipartite entangled states, such as *GHZ* states, Dicke states and graph states, which are interesting because of their entanglement properties or their usefulness for applications. Finally, we discuss the construction of entanglement witnesses for the multipartite case.

3.1. Entanglement of three qubits

3.1.1. Pure states

Let us first consider pure three-qubit states. There are two different types of separability: the *fully separable* states that can be written as

$$|\phi^{\text{fs}}\rangle_{A|B|C} = |\alpha\rangle_A \otimes |\beta\rangle_B \otimes |\gamma\rangle_C, \tag{39}$$

and the *biseparable* states that can be written as a product state in the bipartite system. A biseparable state can be created, if two of the three qubits are grouped together to one party. There are three possibilities of grouping two qubits together, hence there are three classes of biseparable states. One example is

$$|\phi^{\text{bs}}\rangle_{A|BC} = |\alpha\rangle_A \otimes |\delta\rangle_{BC}. \tag{40}$$

The other possibilities read $|\phi^{\text{bs}}\rangle_{B|AC} = |\beta\rangle_B \otimes |\delta\rangle_{AC}$ and $|\phi^{\text{bs}}\rangle_{C|AB} = |\gamma\rangle_C \otimes |\delta\rangle_{AB}$. Here, $|\delta\rangle$ denotes a two-party state that might be entangled. Finally, a pure state is called *genuine tripartite entangled* if it is neither fully separable nor biseparable. Examples of such states are the Greenberger–Horne–Zeilinger (*GHZ*) state [173,174]

$$|GHZ_3\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle), \tag{41}$$

and the so-called *W* state [175,176]

$$|W_3\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle). \tag{42}$$

From a physical point of view, the generation of fully separable or biseparable states does not require interaction of all parties. Only for the creation of genuine tripartite entangled state do all three parties have to interact.[13]

The genuine entangled three-qubit states can, however, be further divided into two inequivalent classes in the following way: Given two three-qubit states, $|\phi\rangle$ and $|\psi\rangle$, one can ask whether it is possible to transform a single copy of $|\phi\rangle$ into $|\psi\rangle$ with local operations and classical communication, without requiring that this can be done with certainty. These operations are called stochastic local operations and classical communication (SLOCC). It turns out [176] that $|\phi\rangle$ can be transformed into $|\psi\rangle$ iff there exist invertible operators A, B, C, acting on the space of one qubit with

$$|\psi\rangle = A \otimes B \otimes C|\phi\rangle. \tag{43}$$

Since the operators A, B, C are invertible, this defines an equivalence relation with a clear physical meaning.

Surprisingly, it was proved in Ref. [176] that there are two different equivalence classes of genuine tripartite entangled states, which cannot be transformed into another by SLOCC. One class, the class of *GHZ* states is represented by the *GHZ* state $|GHZ_3\rangle$ defined in Eq. (41). The other class, the class of *W* states can be transformed via SLOCC into $|W_3\rangle$ given in Eq. (42). In this sense there are two different classes of tripartite entanglement. There are many more pure *GHZ* class states than *W* class states: By local unitary operations one can transform any pure three-qubit state into[14]

$$|\psi\rangle = \lambda_0|000\rangle + \lambda_1 e^{i\theta}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle, \tag{44}$$

where $\lambda_i \geq 0$, $\sum_i \lambda_i^2 = 1$ and $\theta \in [0; \pi]$, see Ref. [182]. Thus, six real parameters are necessary to characterize the nonlocal properties of a pure state. For the *W* class states, however, $\theta = \lambda_4 = 0$ holds, which shows that they are a set of measure zero in the set of all pure states.

Physically, there are also differences between the two classes: On one hand, the *GHZ* state is maximally entangled and a generalization of the Bell states of two qubits. For instance, for the most known Bell inequalities the violation is maximal for *GHZ* states [183]. On the other hand, the entanglement of the *W* state is more robust against particle losses: If one particle is lost in the *GHZ* state, the state $\varrho_{AB} = \text{Tr}_C(|GHZ_3\rangle\langle GHZ_3|)$ is separable, for the *W* state the resulting reduced density matrix $\varrho_{AB} = \text{Tr}_C(|W\rangle\langle W|)$ is entangled. Indeed, it can be shown that the *W* state is the state with the maximal possible bipartite entanglement in the reduced two-qubit states [176,184].

3.1.2. Mixed states

The classification of mixed states according to Refs. [185,186] is similar to the definition of bipartite entanglement via convex combinations. We define a mixed state ϱ^{fs} as fully separable if ϱ^{fs} can be written as a convex combination of fully separable pure states, i.e., if there are convex weights p_i and fully separable states $|\phi_i^{\text{fs}}\rangle$ such that we can write

$$\varrho^{\text{fs}} = \sum_i p_i |\phi_i^{\text{fs}}\rangle\langle\phi_i^{\text{fs}}|. \tag{45}$$

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[13] The notion of genuine multipartite entanglement was (in the context of Bell inequalities) already anticipated in Ref. [173], see also Section 5.2.3.

[14] This is a generalization of the Schmidt decomposition to three qubits, see Refs. [171,177–181] for other results on this problem.
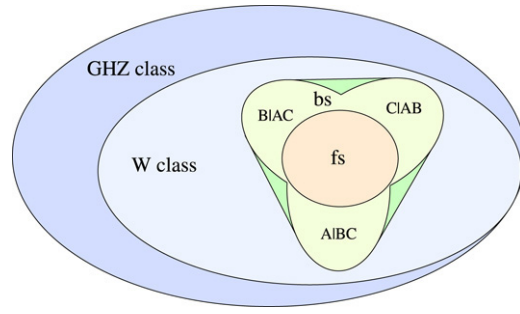
Fig. 3. Schematic picture of the structure of mixed states for three qubits. The convex set of all fully separable states (fs) is a subset of the set of all biseparable states (bs). The biseparable states are the convex combinations of the biseparable states with respect to fixed partitions, sketched by the three different leafs. Outside are the genuine tripartite entangled states, the *W* class and the *GHZ* class.

A state ϱ^{bs} that is not fully separable is called biseparable if it can be written as a convex combination of biseparable pure states:

$$\varrho^{\mathrm{bs}} = \sum_i p_i |\phi_i^{\mathrm{bs}}\rangle\langle\phi_i^{\mathrm{bs}}|. \tag{46}$$

Note that the biseparable states $|\phi_i^{\mathrm{bs}}\rangle$ might be biseparable with respect to different partitions. We will see in Section 3.2.2 why it is reasonable to exclude mixtures of biseparable states with respect to different partitions from the genuine multipartite entangled states. Of course, one may define three classes of biseparable mixed states that are biseparable with respect to a fixed partition.

Finally, ϱ is fully entangled if it is neither biseparable nor fully separable. There are again two classes of fully entangled mixed states: A fully entangled state belongs to the *W* class if it can be written as a convex combination of *W*-type pure states

$$\varrho^{\mathrm{w}} = \sum_i p_i |\phi_i^{\mathrm{w}}\rangle\langle\phi_i^{\mathrm{w}}|, \tag{47}$$

otherwise it belongs to the *GHZ* class. One can show that the *W* class forms a convex set inside the *GHZ* class. Also, in contrast to the case of pure states, the set of mixed *W* class states is not of measure zero compared to the *GHZ* class [185].

Again, this classification can be represented in a schematic picture of nested convex sets (see Fig. 3). Here, it is important to note that this picture is, of course, only a schematic picture, which does not take all properties into account. For instance, it has been shown that there exist states that are biseparable with respect to each fixed partition, however, they are not fully separable (for some examples see Refs. [65,138,185]).[15]

The question to which class a given mixed state belongs, is, as the separability problem for the bipartite case, not easy to decide. Some conditions will be discussed in Section 3.5. Methods to distinguish between the mixed *W* class and *GHZ* class are, however, very rare. As we will see later, witnesses give the possibility to prove that a state belongs to the *GHZ* class. However, it is not clear how one can show that a state is tripartite entangled and belongs to the *W* class. This cannot be done with witnesses, since they are designed to show that a state lies *outside* a convex set, they fail to prove that a state is *inside* a convex set.

3.2. Multipartite entanglement distillation

3.2.1. Multipartite distillability

It is also possible to define distillability for three and more qubits. For instance, one may aim at the distillation (also called purification) of the *GHZ* state $|GHZ_3\rangle$, that is, one asks whether the transformation

$$\underbrace{\varrho \otimes \varrho \otimes \cdots \otimes \varrho}_{k \text{ copies}} \xrightarrow{\mathrm{LOCC}} |GHZ_3\rangle \tag{48}$$

is possible. If this is possible, the state ϱ is called multipartite distillable. The fact that the target state is a *GHZ* state, is not important for the question, whether or not ϱ is distillable, since it is always possible to transform a *GHZ* state in a *W* state and vice versa, if many copies of the state are available.

Many protocols for the distillation of different multipartite states have been proposed (for a review see Ref. [188]). Also, multipartite bound entangled states exist. For instance, the above mentioned states that are separable with respect to each bipartition, but not fully separable, are multipartite bound entangled.

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[15] This counterintuitive behavior can lead to surprising applications, like the possibility of distributing entanglement using separable states [187].

3.2.2. Multipartite distillability vs. multipartite entanglement

Let us compare the notion of multipartite distillability with the notion of genuine multipartite entanglement and their importance for the analysis of experiments. First, one has to stress that both notions are inequivalent. There are genuine multipartite entangled states that are not distillable. Examples of such states were given in Ref. [128].

To investigate the relation better, consider the mixed three-qubit state

$$\varrho = \frac{1}{3} \left(|\phi^+\rangle\langle\phi^+|_{AB} \otimes |0\rangle\langle0|_C + |\phi^+\rangle\langle\phi^+|_{AC} \otimes |0\rangle\langle0|_B + |\phi^+\rangle\langle\phi^+|_{BC} \otimes |0\rangle\langle0|_A \right), \tag{49}$$

where $|\phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ is a two-qubit Bell state. ϱ is a mixture of biseparable states with respect to different partitions, and therefore biseparable. However, as can be easily checked, ϱ is entangled with respect to each bipartition, and also any reduced two-qubit state is NPT and hence entangled and two-qubit distillable. So, if many copies of ϱ are available, the three parties can first distill singlets between $A - B$ and $A - C$. Then, Alice may prepare a *GHZ* state locally in her lab, and teleport two of its parties via the existing singlets to Bob and Charlie. By this simple procedure one sees that ϱ is multipartite distillable, showing that multipartite distillability and genuine multipartite entanglement are really inequivalent.

The important point for our discussion in the above example is that the state ϱ is extremely easy to prepare experimentally. First, one prepares a product state $|000\rangle$ and then one entangles *randomly* two of the three qubits. Equivalently, one can also prepare the state $|\phi^+\rangle\langle\phi^+|_{AB} \otimes |0\rangle\langle0|$, and then one *forgets* which pair of qubits was the pair A, B.

From this example, we can conclude three things. First, multipartite distillability is not a good criterion for claiming success of an experiment, since some multipartite distillable states are very easy to prepare. Second, since ϱ is also NPT with respect to each bipartition, it is not sufficient to investigate all bipartitions, and the PPT criterion is a very limited tool to investigate multipartite entanglement. Finally, this example justifies that we did not define biseparability with respect to a fixed partition, and considered also mixtures of biseparable states that are biseparable with respect to different partitions as not highly entangled. Due to these facts, we will mainly investigate how genuine multipartite entanglement can be verified in experiments.

3.3. Entanglement classes for the general case

Here we shortly discuss the classification of entanglement for general multipartite systems. This is a simple generalization of the three-qubit case: First, we distinguish different types of entanglement for pure states. Then we extend this definition to mixed states by considering convex combinations of pure states.

Let us assume that a pure N-partite state $|\psi\rangle$ is given. We call this state *fully separable* if it is a product state of all parties, that is if

$$|\psi\rangle = \bigotimes_{i=1}^{N} |\phi_i\rangle \tag{50}$$

holds. A mixed state is called fully separable if it can be written as a convex combination of pure fully separable states, that is, it can be written as

$$\varrho = \sum_k p_k \varrho_k^{(1)} \otimes \varrho_k^{(2)} \otimes \cdots \otimes \varrho_k^{(N)}. \tag{51}$$

If a pure state is not fully separable, it contains some entanglement. Again, as in the three-qubit case, this does not have to be true N-partite entanglement. Thus, we call a pure state m-separable, with $1 < m < N$, if there exists a splitting of the N parties into m parts $P_1, \ldots, P_m$ such that

$$|\psi\rangle = \bigotimes_{i=1}^{m} |\phi_i\rangle_{P_i} \tag{52}$$

holds. There are $(m^N)/m!$ possible partitions of the N parties into m parts. Note that an m-separable state still may contain some entanglement. Again, we call mixed states m-separable, if they can be written as convex combinations of pure m-separable states, which might belong to different partitions. One can refine these definitions, by considering separability with respect to fixed bipartitions [186,189,190]. Finally, we call a state truly N-partite entangled when it is neither fully separable, nor m-separable, for any $m > 1$.

The above classification distinguished different classes, according to the question: "How many partitions are separable?". Alternatively, one may ask the question "How many particles are entangled?" A classification in this sense was proposed in Refs. [191–193]. According to that, a pure state $|\psi\rangle$ is said to contain only m-party entanglement, if $|\psi\rangle$ can be written as

$$|\psi\rangle = \bigotimes_{i=1}^{K} |\phi_i\rangle, \tag{53}$$

where the $|\phi_i\rangle$ are states of maximally m qubits. Consequently, $K \geq N/m$ has to hold. If $|\psi\rangle$ is not of this form, it contains (at least) $m + 1$-party entanglement. Again, this definition can be extended to mixed states via convex combinations. A mixed state that is k-producible requires only the generation of k-party pure entangled states and mixing for its production. Consequently, a mixed state ϱ contains k-party entanglement, iff the density matrix cannot be obtained by mixing pure states that are $(k - 1)$-producible. This classification of separability classes does not play a significant role in the analysis of experiments, since there typically only few particles are considered, and the aim is to prove genuine multipartite entanglement, which coincides with the statement that ϱ contains N-partite entanglement. However, for spin models with a large number of particles, this notion can be a reasonable property to study [192,193] (see also Section 8.3).

Another characterization is possible based on the number of unentangled particles. An N-qubit pure product state contains n unentangled particles if it can be written as $|\Psi\rangle = \bigotimes_{k=1}^{n} |\phi_k\rangle \otimes |\Psi_{n+1..N}\rangle$ where $|\phi_k\rangle$ are single-qubit states, while $|\Psi_{n+1..N}\rangle$ is a $(N - n)$-qubit state vector. Consequently, a mixed state contains at most n unentangled spins, if it can be constructed mixing pure states with n or less unentangled spins. There are entanglement conditions that can provide upper bounds on the number of unentangled spins (see Section 8.1.2). If there are no unentangled spins, the state is not necessarily genuine multi-partite entangled: It can be a tensor product of two-qubit entangled states. However, creating very many (e.g., 10^5) two-qubit singlets can be a goal of quantum control in itself.

Finally, it is worth mentioning that a classification of the pure truly N-partite entangled states like in the three-qubit case is not straightforward. At least a classification via equivalence classes under SLOCC is not simple, since it has been shown that already for four qubits there are infinitely many equivalence classes under SLOCC [194,195]. This continuous set can then be separated into nine or eight different classes. Other approaches try to identify different classes via a generalization of the Schmidt rank [196] or tensor ranks [197], but there it is difficult to determine in which class a pure state lies.

3.4. Families of multipartite entangled states

In order to further investigate multipartite entangled states, it is useful to identify interesting families of multipartite pure states. Here, one might be interested in maximally entangled states, or states that are useful for certain applications. For the three-qubit case, we already know the *GHZ* state and the *W* state as such examples. There are, however, several families of interesting multi-qubit states with an arbitrary number of qubits:

3.4.1. GHZ states
The *GHZ* state for N qubits is defined as

$$|GHZ_N\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle^{\otimes N} + |1\rangle^{\otimes N} \right) \tag{54}$$

and is a simple generalization of the three-qubit *GHZ* state. It is a superposition of two maximally distinct states, and therefore also sometimes called a "Schrödinger cat state". *GHZ* states have been studied intensively for a long time and it turned out that they are useful for many applications. These include entanglement enhanced spectroscopy and quantum metrology [198,199], quantum secret sharing [200], open destination teleportation [201], quantum computation [202] and cryptographic protocols [203,204]. Furthermore, it has been shown that *GHZ* states lead to an extremal violation of local realism [174,205,206]. In fact, it can be shown that for many Bell inequalities, the unique state leading to a maximal violation is the *GHZ* state [183].

These properties make the *GHZ* states interesting from many perspectives. In some sense they may be considered as the maximally entangled multi-qubit states. Therefore, many experiments aim at the generation of *GHZ* states, and *GHZ* states with up to ten qubits have already been prepared using ions [7,207,208] or photons [9,10,201,209–211].

3.4.2. Dicke states and W states
These states have been investigated already in 1954 by R.H. Dicke, while studying light emission from a cloud of atoms [212]. Dicke found that, if the atoms are in certain highly entangled states, then the intensity of radiation is much larger than if the atoms were emitting light independently, starting all from an excited state. In particular, the intensity scales quadratically with the number of atoms, while for independent atoms it scales linearly.

Dicke states, in general, are simultaneous eigenstates of the collective angular momentum operators $J_z = 1/2 \sum_k \sigma_z^{(k)}$ and J^2.[16] Typically, for a certain eigenvalue for J_z and J^2 there are several eigenstates and additional parameters are needed to distinguish them from each other. Since very often many-particle systems start from a symmetric initial state and they evolve under symmetric dynamics, perhaps the most important examples for quantum information are the symmetric Dicke states. Such states are all eigenstates of J^2 with the maximal eigenvalue $\frac{N}{2}(\frac{N}{2} + 1)$ and they can be characterized uniquely

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[16] We will discuss these operators and their properties later in Section 8.1 in more detail.

by their eigenvalues j_z for J_z. Beside j_z, it is also common to use for their characterization $k = N/2 - j_z$. With this, an (N, k) symmetric Dicke state on N qubits is defined as an equal superposition of k excitations, i.e.,

$$|D_{k,N}\rangle = \binom{N}{k}^{-\frac{1}{2}} \sum_j P_j \Big\{ |1\rangle^{\otimes k} \otimes |0\rangle^{\otimes N-k} \Big\}, \tag{55}$$

where $\sum_j P_j\{\ldots\}$ denotes the sum over all possible permutations of the qubits.

The W state corresponds to the Dicke state $|D_{1,3}\rangle$ and is therefore an example of a Dicke state. Another important Dicke state is the four-qubit state $|D_{2,4}\rangle = (|0011\rangle + |0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle + |1100\rangle)/\sqrt{6}$. In general, the state $|D_{\frac{N}{2},N}\rangle$ is the state for which the superradiance is the strongest. Also, its overlap with biseparable states approaches $\frac{1}{2}$ for large N [213]. Since the overlap is determined by the maximal squared Schmidt coefficient and for N-qubit states the $A|BC\ldots N$ bipartition has always one squared Schmidt coefficient not smaller than $\frac{1}{2}$, this is the maximal value, and for an experimental investigation of genuine multipartite entanglement only a fidelity of $\frac{1}{2}$ is required. Finally, it should be noted that the entanglement in the Dicke state $|D_{2,4}\rangle$ is relatively robust against decoherence [214]. Dicke states and W states have been prepared in many experiments [8,208,215–217]. We will discuss two of them in Sections 6.8 and 8.2.

3.4.3. Graph states and cluster states

Let us come now to the family of graph states [218–220]. These states are defined in a mathematical way as follows: First one considers a graph, i.e., a set of N vertices and some edges connecting them. Examples of graphs are shown in Fig. 4. For each vertex i the neighborhood $\mathcal{N}(i)$ denotes the vertices that are connected with i. Then one can associate to each vertex i a correlation operator or stabilizing operator g_i by

$$g_i = X_i \bigotimes_{j \in \mathcal{N}(i)} Z_j. \tag{56}$$

Here X_i, Y_i, Z_i denote the Pauli matrices $\sigma_x, \sigma_y, \sigma_z$, acting on the i-th qubit; we will use this notation often when working in the graph state formalism. For instance, for the three vertex graph No. 2 in Fig. 4(a), the stabilizing operators are $g_1 = Z_1 X_2 \mathbb{1}_3, g_2 = X_1 Z_2 Z_3$ and $g_3 = Z_1 \mathbb{1}_2 X_3$. The graph state $|G\rangle$ associated with the graph G is the then the N-qubit state fulfilling

$$g_i|G\rangle = |G\rangle, \quad \text{for } i = 1, \ldots, N. \tag{57}$$

Clearly, $|G\rangle$ is then also an eigenstate of all possible products of the g_k, the commutative group of this operators is called the stabilizer [221]. Since the g_i form a maximal set of commuting observables, the graph state $|G\rangle$ is uniquely determined by these eigenvalue equations.[17]

The physical meaning of the graph is that it describes the perfect correlations in the state $|G\rangle$, since $\langle g_i \rangle = \langle X_i \bigotimes_{j \in N(i)} Z_j \rangle = 1$. At the same time, the graph denotes a possible interaction history leading to $|G\rangle$: the state $|G\rangle$ can be produced from the product state $[(|0\rangle + |1\rangle)/\sqrt{2}]^{\otimes N}$ by an Ising type interaction $U_{kl} = \exp\{-it(\mathbb{1}_k - Z_k)(\mathbb{1}_l - Z_l)\}$ acting for the time $t = \pi/4$ between the connected qubits.[18] Finally, the graph state can also directly be written in the stabilizer elements as

$$|G\rangle\langle G| = \prod_{k=1}^{N} \frac{\mathbb{1} + g_k}{2}. \tag{58}$$

There are several interesting properties of graph states. First, it should be noted that the GHZ states belong to this class. If we consider the state in Eq. (54), it can be easily checked that this is an eigenstate of

$$g_1 = X_1 X_2 \ldots X_N; \qquad g_k = Z_1 \mathbb{1}_2 \ldots Z_k \ldots \mathbb{1}_N; \quad \text{for } k = 2, \ldots, N. \tag{59}$$

Up to some local rotations, these are nothing but the stabilizing operators of the star graphs [Nos. 2, 3, and 5 in Fig. 4(a)]. Therefore, graph states can be seen as an extension of the GHZ states.

Second, it is important to note that different graphs may lead to graph states, which can be transformed into each other by local unitary transformations, implying that the entanglement properties of the states are the same. In general, it is not known yet, under which conditions two graphs yield equivalent graph states [223,224]. However, it is known that some graph transformations (called local complementation) do not change the graph state, and for a small number of qubits ($N \leq 8$) a full classification has been achieved [220,225].

This leads directly to the question, what other graph states exists beside the GHZ state. For 2 qubits, there is only one graph [no. 1 in Fig. 4(a)], corresponding to a Bell state. For three qubits, there is again only one independent graph state,

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[17] For a given graph, one can consider a whole family of 2^N states that are eigenstates of the g_i with eigenvalues ± 1. This orthogonal basis is the so-called graph state basis, we will use it in Section 6.6 in more detail.

[18] If the interaction time depends on k and l, the resulting state is a so-called weighted graph state, see Ref. [222] for discussions and applications.
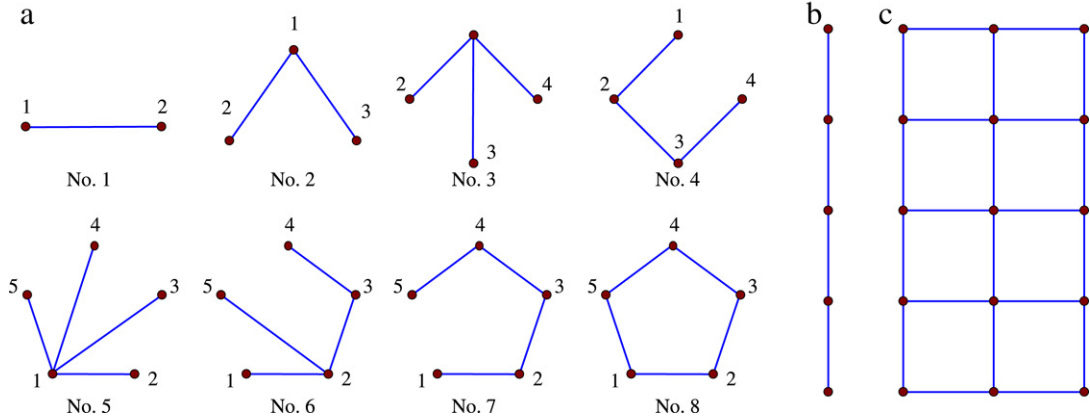
Fig. 4. (a) Several types of graphs for up to five qubits. The shown graphs represent all inequivalent graph states up to five qubits [220]. (b) The graph for a five qubit linear cluster state. (c) The graph for a 2D cluster state (for the case of 15 qubits). This type of graph states is a universal resource for measurement based quantum computation.

corresponding to the three-qubit *GHZ* state (No. 2). For four qubits, there are already two independent graph states: the four-qubit *GHZ* state (No. 3), and the four-qubit cluster state, given by

$$|CL_4\rangle = \frac{1}{2}\left(|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle\right),\tag{60}$$

which corresponds (up to a local change of the basis) to the graph no. 4. For five qubits there are already four different graph states (nos. 5–8), and with increasing number of qubits, the number of inequivalent graph states increases rapidly.

Third, it should be stressed that graph states are of eminent importance for applications and protocols of quantum information processing. First, a special graph state, the 2D cluster state [see Fig. 4(c)] can be used as a resource for measurement based quantum computation [6,226]. In addition, graph states occur as codewords of quantum error correcting codes [227], violate certain Bell inequalities maximally [147,228,229], and are also robust against decoherence [218]. Finally, graph states can be used for the simulation of anyonic statistics [230] and can be related to the computation of the partition function of classical spin models [231].

Due to this importance, it is not surprising that graph states have been prepared in many experiments [9,232–236]. We will discuss in Section 6.7 an experiment for the generation of the four-qubit cluster state.

3.4.4. Multi-qubit singlet states

Another interesting family of multi-qubit states are multi-qubit singlet states. These are pure states $|\psi\rangle$, that are invariant under a simultaneous unitary rotation on all qubits, i.e., we have $(U)^{\otimes N}|\psi\rangle = e^{i\phi}|\psi\rangle$ (or, equivalently $[|\psi\rangle\langle\psi|, (U)^{\otimes N}] = 0$). Such states exist only for the case that N is even. For two qubits, the only state of this form is the two-qubit singlet state $|\psi^-\rangle$. For four qubits, these states form a two-dimensional subspace of the total Hilbert space, spanned by the vectors $|\Psi_1\rangle = |\psi^-\rangle \otimes |\psi^-\rangle$ and

$$|\Psi_2\rangle = \frac{1}{\sqrt{3}}\left[|0011\rangle + |1100\rangle - \frac{1}{2}(|01\rangle + |10\rangle) \otimes (|01\rangle + |10\rangle)\right].\tag{61}$$

This is the so-called four-qubit singlet state [237]. In general, there are $d_N = N!/[(N/2)!(N/2 + 1)!]$ linearly independent states in this family [238,239].

The importance of these states comes from the fact that they are by definition robust against certain types of noise, where all qubits undergo the same (random or unknown) unitary transformation. They lie in a so-called decoherence free subspace [240,241]. This can be used to encode quantum information, e.g., one qubit can be encoded in superpositions of the four-qubit states $|\Psi_1\rangle$ and $|\Psi_2\rangle$.

Experimentally, it is remarkable that the state $|\Psi_2\rangle$ can simply be prepared by a second order process in parametric down-conversion [237] (see also Section 6.5). This has been done in Ref. [242], also protocols like secret sharing or the encoding of quantum information as described above have been performed [243,244].

The expression *singlet* is also used in another context, denoting states with zero total angular momentum. That is, these are the states that are the eigenstates of the total angular momentum components J_x, J_y and J_z with eigenvalue zero. Such states appear very often in condensed matter physics, for example, the ground state of the anti-ferromagnetic Heisenberg chain is such a multi-qubit singlet state. We will discuss entanglement detection in the vicinity of such states in Section 8.1.

The connection between the two definitions can be seen as follows.[19] One can rewrite a simultaneous multilateral unitary transformation as

$$U \otimes U \otimes \cdots \otimes U = \exp(i\vec{\alpha} \cdot \vec{J}), \tag{62}$$

where $\vec{J} = (J_x, J_y, J_z)$ and $\vec{\alpha}$ is a three-element real vector. Hence, we can see that if a pure state is an eigenstate of $\exp(i\vec{\alpha}\vec{J})$ for any $\vec{\alpha}$, then it is an eigenstate of J_k for $k = x, y, z$, and vice versa. Thus, the two definitions coincide for pure states.

3.4.5. Werner states

In the general case, N-qudit mixed states invariant under $U \otimes U \otimes \cdots \otimes U$ are called Werner states [44]. They do not necessarily have zero total angular momentum variance. Such multi-qudit states naturally arise as the result of the twirling operation

$$\text{Twirl}[\varrho] := \int U^{\otimes N} \varrho (U^{\dagger})^{\otimes N} \mu(\text{d}U), \tag{63}$$

where the integral is over the unitary group of $d \times d$ matrices $U(d)$, and μ is the Haar measure. Multi-qudit twirling can efficiently be realized experimentally [245]. The resulting Werner states are the linear combination of relatively few basis matrices [44,246,247]. For example, for $N = 2$ there are two such matrices for any dimension d, while for $N = 3$ there are 5 for $d = 2$ and 6 for any $d > 2$. Thus, such states can be characterized with fewer and simpler measurements than the measurements needed for full tomography. For example, for the two-qubit case the single free parameter of a two-qubit Werner state can be obtained by measuring $\langle \sigma_x^{(1)} \sigma_x^{(2)} \rangle$. Twirling has also been suggested for simplifying process tomography [248].

3.4.6. Maximally entangled states for comb monotones

As already mentioned, it is a natural question to consider the maximally entangled states for N qubits. This, however, depends on the entanglement measure chosen: As we will discuss later, there are several inequivalent measures of multipartite entanglement, and the maximally entangled states differ for different measures.

In Refs. [249,250] a class of entanglement measures has been introduced, which are based on anti-linear operators and combs. There are three different measures of this type for four qubits, resulting in three different maximally entangled states. The first one is the *GHZ* state, the second one the cluster state, and the third one is the state

$$|\chi\rangle = \frac{1}{\sqrt{6}} \left(\sqrt{2}|1111\rangle + |0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle \right). \tag{64}$$

This state has also been mentioned in Ref. [251], where it has been noticed that it is the symmetric four-qubit state that maximizes certain bipartite entanglement properties. Also, $|\chi\rangle$ is probably the most entangled symmetric state for the geometric measure of entanglement [252] (see also Section 4.2.6).

As it was shown in Ref. [250] one can generalize this state to five and six qubits, where it is also a maximally entangled state for some comb measure. It is not yet clear, which physical properties follow from this fact. Also, the generation of this state is still an experimental challenge.

Finally, it should be added that there are further types of states, which deserve attention. For example, there are states for optimal cloning via teleportation [253], or the family of multipartite states which can be maximally entangled with local auxiliary systems [254].

3.5. Separability criteria for the multipartite case

As for the bipartite case, one may ask how to decide whether a given mixed multipartite state is entangled or not. This problem is not so well understood as for the bipartite case, however, several criteria are known:

(i) *Permutation criteria.* As discussed already for the bipartite case, one can view the PPT as well as the CCNR criterion as norm conditions on a density matrix where some indices are permuted [see Eq. (21)]. This approach can be generalized to more parties [60,62,63]. One first writes the density matrix in some product basis, $\varrho = \sum_{i_1 j_1, \ldots, i_N j_N} \varrho_{i_1 j_1, \ldots, i_N j_N} |i_1\rangle\langle j_1| \otimes \cdots \otimes |i_N\rangle\langle j_N|$, and then for separable states we have

$$\|\varrho_{\pi(i_1 j_1, \ldots, i_N j_N)}\|_1 \leq 1, \tag{65}$$

where $\pi(\ldots)$ is an arbitrary permutation of the indices. This approach generalizes the CCNR and PPT criterion to the multipartite case.[20] Clearly, not all permutations π result in different criteria. As we have already seen, for two parties there are only two inequivalent permutations, resulting in the PPT and CCNR criterion. Further, it has been shown in Refs. [62,63] that for the case of three parties there are six independent permutations, and for four parties there are 22 independent permutation criteria. All these criteria can only rule out full separability.

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[19] We thank Sofyan Iblisdir for discussions about this point.

[20] For a different generalization of the CCNR criterion see Ref. [61].

(ii) *Quadratic Bell-type inequalities.* It was already demonstrated in Ref. [255] that quadratic Bell-type inequalities can be a useful tool for the investigation of multipartite entanglement. Recently, a general method to derive quadratic inequalities for the investigation of the different classes of multipartite entanglement has been found [190,256]. This method can be sketched as follows: for two qubits and $i = 0, 1$ one considers the families of observables $A_i^{(2)} = (A_1 \otimes A_2 \mp B_1 \otimes B_2)/2, B_i^{(2)} = (B_1 \otimes A_2 \pm A_1 \otimes B_2)/2, C_i^{(2)} = (C_1 \otimes \mathbb{1}_2 \pm \mathbb{1}_1 \otimes C_2)/2$, and $I_i^{(2)} = (\mathbb{1}_1 \otimes \mathbb{1}_2 \pm C_1 \otimes C_2)/2$, where A_k, B_k, C_k denote orthogonal spin observables, i.e., Pauli matrices in arbitrary directions. The two-qubit observables $A_i^{(2)}, \ldots, I_i^{(2)}$ fulfill similar relations as the Pauli matrices, and for separable states it can be shown that

$$\max_i \left\{ \left\langle A_i^{(2)} \right\rangle^2 + \left\langle B_i^{(2)} \right\rangle^2 \right\} \leq \min_i \left\{ \left\langle I_i^{(2)} \right\rangle^2 - \left\langle C_i^{(2)} \right\rangle^2 \right\} \tag{66}$$

holds. This is in fact a necessary and sufficient condition for two-qubit separability [257]. This construction can then be iterated: One defines four sets of three-qubit observables $A_i^{(3)}, \ldots, I_i^{(3)}$ as $A_i^{(3)} = (A_1 \otimes A_k^{(2)} \mp B_1 \otimes B_k^{(2)})/2$ etc., and it is possible to prove similar relations as above for the different classes of separability. These conditions can be translated into conditions on the density matrix elements by fixing A_i, B_i and C_i. For instance, for three qubits it can be shown that the matrix elements of biseparable states fulfill

$$|\varrho_{18}| \leq \sqrt{\varrho_{22}\varrho_{77}} + \sqrt{\varrho_{33}\varrho_{66}} + \sqrt{\varrho_{44}\varrho_{55}}. \tag{67}$$

These and the other conditions improve some already known conditions [258]. In addition, they are relatively easy to check in experiments.

(iii) *Algorithmic approaches.* Also some of the computational criteria can be extended to the multipartite scenario. The method of the symmetric extensions can be formulated for the multipartite case [91], and it can be shown again that the hierarchy is complete. However, this method is practically not very feasible. Also, the method of Brandão and Vianna [98] can be applied to show the entanglement of some multipartite bound entangled states.

(iv) *Further criteria.* There is a series of other approaches that tries to extend bipartite criteria to the multipartite case. These include positive maps [259] and conditions on the generalization of the Bloch vector [260,261]. There is also a variety of entanglement witness like criteria for full separability, these will be discussed in Section 3.6.3. Finally, it should be noted that any bipartite separability criterion can also be used to rule out full separability, since a fully separable state is also separable with respect to each partition.

Finally, it should be added that also criteria for the distillability of a multipartite quantum state are known. First, in a similar argument as in Section 3.2 one can conclude that a mixed state is distillable, if enough reduced two-qubit states are NPT, allowing the distillation of singlet pairs between them. For some special families of states (i.e., states that are diagonal in a graph state basis or the *GHZ* basis) more elaborate criteria have been worked out [189,262].

3.6. Entanglement witnesses for multipartite entanglement

Let us now discuss entanglement witnesses for multipartite entanglement. Clearly, witnesses can be used to distinguish the different classes of multipartite entanglement. This is due to the fact that all the classes defined above are defined via convex combinations, thus they are convex sets. Contrary to the bipartite case, one has several types of witnesses, for the different types of multipartite entanglement.

3.6.1. Three qubits

First, there are *GHZ* class witnesses $\mathcal{W}_{GHZ}$ that allow to detect mixed states belonging to the *GHZ* class. Thus, their expectation value is positive on all fully separable, biseparable, and *W*-type states:

$$\text{Tr}(\mathcal{W}_{GHZ}\varrho) < 0 \Rightarrow \varrho \text{ is in the } GHZ \text{ class,}$$
$$\text{Tr}(\mathcal{W}_{GHZ}\varrho) \geq 0 \Rightarrow \varrho \text{ is not detected.} \tag{68}$$

An example for such a witness is

$$\mathcal{W}_{GHZ} = \frac{3}{4}\mathbb{1} - |GHZ_3\rangle\langle GHZ_3|. \tag{69}$$

This witness is constructed as in Eqs. (28) and (29), and the constant 3/4 is the maximal overlap between the state $|GHZ_3\rangle$ and the pure *W* class states [185]. In other words, this witness expresses the fact that if the fidelity of the *GHZ* state is larger than 3/4, then the state belongs to the *GHZ* class.

In a similar fashion, other witnesses can be defined: Witnesses for genuine tripartite entanglement, denoted by $\mathcal{W}_3$, have a positive expectation value on all biseparable states thus a negative expectation value indicates the presence of true tripartite entanglement. Witnesses for biseparable entanglement $\mathcal{W}_e$ have a positive expectation value on all fully separable states, thus a negative expectation value is still a signature of entanglement, which might be only biseparable entanglement.
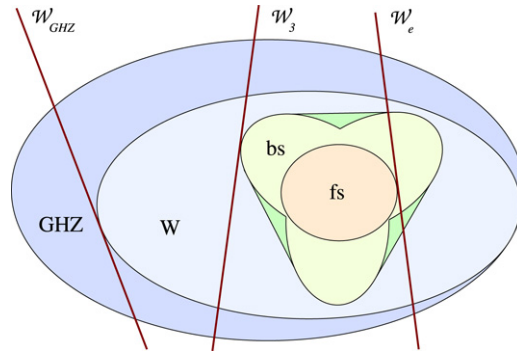
Fig. 5. Schematic picture of the set of mixed three-qubit states with three witnesses: The witness $\mathcal{W}_{GHZ}$ detects *GHZ*-type states, the witness $\mathcal{W}_3$ detects three-qubit entanglement, and the witness $\mathcal{W}_e$ rules out full separability.

Such witnesses can be constructed in a similar manner as above: one makes the ansatz

$$\mathcal{W} = \alpha \mathbb{1} - |\psi\rangle\langle\psi|, \tag{70}$$

where α is the maximal overlap between $|\psi\rangle$ and the biseparable (or fully separable, respectively) states. Note that it suffices to consider pure biseparable states, as they are the extremal points of the convex set of mixed biseparable states. The calculation of α depends on the type of states, one wishes to exclude. All the types of witnesses can be cast into a schematic picture as shown in Fig. 5.

The computation of the maximal overlap between a given state $|\psi\rangle$ and the pure biseparable states is straightforward [145]: Any pure biseparable state is separable with respect to some fixed bipartition, and for this case we know already that this overlap is given by the maximal squared Schmidt coefficient [see Eq. (28)]. So one can take for the construction of witnesses for genuine tripartite entanglement

$$\alpha = \max_{\text{bipartitions } bp} \left\{ \max_{\text{Schmidt coefficients } s_k(bp)} \{[s_k(bp)]^2\} \right\} \tag{71}$$

as the maximal squared Schmidt coefficient over all bipartitions. The computation of other overlaps, e.g., the overlap with fully separable states is, in general, not so simple. However, for many cases, results on this problem are known, especially in the context of the so-called geometric measure of entanglement [263], which will be explained in Section 4.2.6.[21] To give an example, a witness for genuine multipartite entanglement around the state $|W_3\rangle$ would be $\mathcal{W}_3 = 2/3 \cdot \mathbb{1} - |W_3\rangle\langle W_3|$ and a witness excluding full separability would be $\mathcal{W}_3 = 4/9 \cdot \mathbb{1} - |W_3\rangle\langle W_3|$.

3.6.2. The general case

For the case of more than three qubits similar methods can be directly applied. Especially for the detection of genuine multipartite entanglement the method from above provides a standard method for the construction of witnesses. In addition, the required overlaps are for many classes of states known.

For instance, it has been shown in Ref. [264] that the maximal overlap between graph states and biseparable states is always 1/2, hence

$$\mathcal{W}_{G_N} = \frac{\mathbb{1}}{2} - |G_N\rangle\langle G_N| \tag{72}$$

is for any graph state $|G_N\rangle$ a suitable witness. Especially,

$$\mathcal{W}_{GHZ_N} = \frac{\mathbb{1}}{2} - |GHZ_N\rangle\langle GHZ_N| \quad \text{and} \quad \mathcal{W}_{CL_N} = \frac{\mathbb{1}}{2} - |CL_N\rangle\langle CL_N| \tag{73}$$

are suitable witnesses for *GHZ* states and cluster states, respectively. For some symmetric Dicke states, the maximal overlap can also be computed straightforwardly [213], e.g., a witness for the $|W_N\rangle$ state reads

$$\mathcal{W}_{W_N} = \frac{N-1}{N}\mathbb{1} - |W_N\rangle\langle W_N| \tag{74}$$

and for the $|D_{\frac{N}{2},N}\rangle$ state it is

$$\mathcal{W}_{D_N} = \frac{1}{2}\frac{N}{N-1}\mathbb{1} - |D_{\frac{N}{2},N}\rangle\langle D_{\frac{N}{2},N}|, \tag{75}$$

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[21] However, such witnesses are not so relevant for this review, as the detection of genuine multipartite entanglement has more experimental relevance.

Other witnesses for the states in Eqs. (61) and (64) read

$$\mathcal{W}_{\Psi_2} = \frac{3}{4}\mathbb{1} - |\Psi_2\rangle\langle\Psi_2| \quad \text{and} \quad \mathcal{W}_{\chi} = \frac{1}{2}\mathbb{1} - |\chi\rangle\langle\chi|. \tag{76}$$

Note that these are only the basic constructions for witnesses for multipartite states, more sophisticated constructions and their experimental implementation will be discussed in great detail in Section 6.

3.6.3. Witnesses for full separability

Let us finally mention some other ways of constructing witnesses (or witness like inequalities) for multipartite systems which result in conditions excluding full separability.

First, there are witnesses for multipartite entanglement that are derived in a similar way as Bell inequalities [255,265]. Then, one can use geometrical arguments to construct entanglement witnesses [266,267] or use properties of correlation functions [268]. Also, similarly as demonstrated above, one can construct witnesses, if the expectation value for a given observable can be bounded for product states. To compute such bounds, linear programming can be used [269–271].

Finally, other ways of constructing witnesses will be discussed in the other parts of this review. Bell inequalities will be discussed in Section 5. In Section 6.6 we explain the construction of witnesses using stabilizer theory. In Section 8.1 we will discuss separability criteria based on collective angular momentum observables (spin squeezing inequalities). In Section 8.3 we will review the possibility to use the Hamiltonian of a spin model as an entanglement witness.

4. Entanglement measures

In this section, we explain the basic notions of entanglement measures. We restrict our attention to the basic definitions and requirements of entanglement measures that are needed for the methods presented later to analyze entanglement in experiments. For a deeper discussion, we can recommend several excellent general reviews on entanglement measures [18,19,28].

4.1. General properties of entanglement measures

4.1.1. Requirements for entanglement measures

By definition, entanglement measures (or entanglement monotones) should quantify the amount of entanglement in a given state. As such, a general entanglement measure $E(\varrho)$ should have several desired properties [272], which are listed below. It should be noted, however, that not all of the following properties are fulfilled by all discussed entanglement quantifiers.

(i) First, a natural requirement is that the entanglement measure $E(\varrho)$ vanishes, if ϱ is separable.
(ii) Second, an entanglement measure should be invariant under a local change of the basis. This means that it should be invariant under local unitary transformations,

$$E(\varrho) = E(U_A \otimes U_B \varrho U_A^\dagger \otimes U_B^\dagger). \tag{77}$$

(iii) As entanglement cannot be created by LOCC, it is reasonable to require that $E(\varrho)$ does not increase under such transformations. That is, if Λ^{LOCC} is a positive map that can be implemented by LOCC, then

$$E[\Lambda^{\text{LOCC}}(\varrho)] \le E(\varrho). \tag{78}$$

Often, this condition is replaced by a different requirement, namely that $E(\varrho)$ should not increase on average under LOCC. That is, if a LOCC transformation maps ϱ to states ϱ_k with probabilities p_k, then

$$\sum_k p_k E(\varrho_k) \le E(\varrho). \tag{79}$$

This condition is stronger than Eq. (78), but many entanglement measures also fulfill this stronger condition. The monotonicity under LOCC in Eq. (78), implies invariance under local unitary transformations.
(iv) A further property that is often demanded and that is fulfilled by most entanglement measures is convexity. That is, one requires that entanglement decreases under mixing of two or more states,

$$E\left(\sum_k p_k \varrho_k\right) \le \sum_k p_k E(\varrho_k). \tag{80}$$

This inequality expresses the fact that if one starts from an ensemble ϱ_k of states, and loses information about the single instance ϱ_k, then the entanglement decreases.

Not all measures fulfill this property, and it has been argued that the convexity condition may be relaxed by requiring only that $E(\varrho)$ should not increase if locally distinguishable states are mixed (see Ref. [273] for a discussion).

(v) Further questions arise, if more than two copies of a state are available. For instance, if Alice and Bob share n copies of the same state ϱ, it may be reasonable to require additivity, that is

$$E(\varrho^{\otimes n}) = nE(\varrho). \tag{81}$$

Even stronger, one may require full additivity. This means, that if Alice and Bob share two different states, ϱ_1 and ϱ_2, then

$$E(\varrho_1 \otimes \varrho_2) = E(\varrho_1) + E(\varrho_2). \tag{82}$$

Such additivity requirements are not fulfilled for some measures, or they are difficult to prove [19].

4.1.2. Convex roof constructions

There are several strategies to define an entanglement monotone. A first choice is to take the usefulness of a state for a certain task as a measure of entanglement. For instance, one may define for a state ϱ the optimal distillation rate $E_D(\varrho)$ as a measure of entanglement, as we will discuss in Section 4.2.1. Clearly, such quantities are difficult to compute, especially for mixed states, as their computation involves a complicated optimization over all distillation protocols. Only for special cases, exact results are known (see Refs. [18,19] for details).

Another strategy for defining entanglement measures uses the so-called convex roof construction. For that, one first defines the measure $E(|\psi\rangle)$ for pure states, and then defines for mixed states

$$E(\varrho) = \inf_{p_k, |\phi_k\rangle} \sum_k p_k E(|\phi_k\rangle), \tag{83}$$

where the infimum is taken over all possible decompositions of ϱ, i.e., over all p_k and $|\phi_k\rangle$ with $\varrho = \sum_k p_k |\phi_k\rangle\langle\phi_k|$. In other words, $E(\varrho)$ is defined as the largest convex function smaller than $E(|\psi\rangle)$.

The advantage of the convex roof construction lies in the fact that the resulting entanglement measure has by construction some desirable properties, as $E(\varrho)$ is convex. Moreover, from the properties of $E(|\psi\rangle)$ for pure states one can often directly see, whether the measure for mixed states fulfills the conditions for entanglement monotones, e.g., whether it is non-increasing under LOCC [274–276].

Clearly, the optimization in Eq. (83) is also not straightforward to compute. Again, only for special cases results are known. There are however, some general recipes to give lower bounds on the convex roof [277,278]. The main idea is to take an easily computable convex function $F(\varrho)$ and to derive a lower bound in terms of F for pure states. Then, as the convex roof is the largest convex function smaller than E for pure states, the same lower bound also holds for mixed states. For instance, the trace norm of the partially transposed state $F(\varrho) = \|\varrho^{T_A}\|_1$ is such a convex function, and from this the lower bound on the concurrence [see Eq. (89)] in a $d \times d$-system

$$C(\varrho) \geq \sqrt{\frac{2}{d(d-1)}} \|\varrho^{T_A}\|_1 \tag{84}$$

can be established. The resulting lower bounds are often surprisingly good [277,278]. In the Sections 6.9 and 7.2 we will investigate how entanglement measures can be evaluated or estimated in experiments.

4.2. Examples of entanglement measures

Now we discuss some examples of entanglement measures. We start with measures for the bipartite case, and then explain some measures for the multipartite case.

4.2.1. Entanglement cost and entanglement of distillation

The entanglement cost $E_C(\varrho)$ is defined as the minimal rate of singlets that have to be used to create many copies of the state ϱ via LOCC [279]. More formally, it is given by

$$E_C(\varrho) = \inf_{\text{LOCC}} \lim_{n_{\text{out}} \to \infty} \frac{n_{\text{in}}}{n_{\text{out}}}. \tag{85}$$

Here, $\inf_{\text{LOCC}}$ denotes the minimization over all LOCC protocols that map n_{in} input singlets onto n_{out} output copies of the state ϱ.[22]

Conversely, the entanglement of distillation $E_D(\varrho)$ is defined as the optimal singlet distillation rate from many copies of ϱ,

$$E_D(\varrho) = \sup_{\text{LOCC}} \lim_{n_{\text{in}} \to \infty} \frac{n_{\text{out}}}{n_{\text{in}}}, \tag{86}$$

where the LOCC protocols map now n_{in} input copies of ϱ onto n_{out} output singlets.

---

[22] To be more precise, one should also include in this definition that the LOCC protocols do not generate prefect copies of the state ϱ, but states arbitrarily close to it [19].

With these definitions, $E_D(\varrho) \leq E_C(\varrho)$ has to hold. Since there are bound entangled states that require entanglement for their generation ($E_C > 0$), but from which no entanglement can be distilled ($E_D = 0$), it can happen that $E_D(\varrho) \neq E_C(\varrho)$. For pure states, however, it was shown in Ref. [279] that the entanglement cost equals the entanglement of distillation and they are given by the von Neumann entropy of the reduced state ϱ_A,

$$E_C(|\psi\rangle) = E_D(|\psi\rangle) = S(\varrho_A) = -\mathrm{Tr}[\varrho_A \log_2(\varrho_A)]. \tag{87}$$

Therefore, pure states can be transformed reversibly into singlet states.

4.2.2. Entanglement of formation

The entanglement of formation is defined as the convex roof of the von Neumann entropy,

$$E_F(\varrho) = \inf_{p_k, |\phi_k\rangle} \sum_k p_k S[(\varrho_A)_k], \tag{88}$$

where $(\varrho_A)_k$ is Alice's reduced state of the state $|\psi_k\rangle$, and the optimization is defined as in Eq. (83) [279]. Physically, the entanglement of formation may be interpreted as a minimal number of singlets that is required to build a single copy of the state.

A central problem in the study of entanglement is whether or not the entanglement of formation is fully additive. This question is not solved yet, but its solution would have wide ranging consequences also on other topics, like the additivity of classical capacity of quantum channels (see Ref. [19] for a discussion).

4.2.3. Concurrence

A very popular measure for the quantification of bipartite quantum correlations is the concurrence [280,281]. This quantity can be defined for pure states as

$$C(|\psi\rangle) = \sqrt{2[1 - \mathrm{Tr}(\varrho_A^2)]}, \tag{89}$$

where ϱ_A the reduced state of $|\psi\rangle$ for Alice. For mixed states this definition is extended via the convex roof construction.

The popularity of the concurrence stems from the fact that for two qubits the convex roof can analytically be computed [46]. Namely, one has

$$C(\varrho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \tag{90}$$

where the λ_i are the decreasingly ordered eigenvalues of the matrix $X = \sqrt{\sqrt{\varrho}(\sigma_y \otimes \sigma_y)\varrho^*(\sigma_y \otimes \sigma_y)\sqrt{\varrho}}$ and the complex conjugation ϱ^* is taken in the standard basis. Moreover, for two qubits the entanglement of formation can be expressed in terms of the concurrence as

$$E_F(\varrho) = h\left(\frac{1 + \sqrt{1 - C^2(\varrho)}}{2}\right), \tag{91}$$

where $h(p) = -p \log(p) - (1 - p) \log(1 - p)$ is the binary entropy function. For other dimensions, however, such a relation does not hold and the physical interpretation of the concurrence is not so clear. Moreover, the concurrence is not additive.

4.2.4. Negativity

The negativity is just given as the violation of the PPT criterion [52,53]

$$N(\varrho) = \frac{\|\varrho^{T_B}\|_1 - 1}{2}, \tag{92}$$

where $\|\ldots\|_1$ denotes the trace norm (i.e., the sum of all singular values) of the partially transposed state. Two of the main advantages are that negativity is very easy to compute and it is convex. In order to make this quantity additive, one can consider the logarithmic negativity $E_N(\varrho) = \log_2 \|\varrho^{T_B}\|_1$. This gives an upper bound on the entanglement of distillation, $E_N(\varrho) \geq E_D(\varrho)$ [53]. The logarithmic negativity is, however, not convex anymore [273]. By construction, the negativity fails to recognize entanglement in PPT states.

4.2.5. Distance measures

Another class of entanglement quantifiers uses the distance to the separable states as a measure for entanglement. As there are several possible notions of a distance, several entanglement parameters arise.

One of the first distance measures is the relative entropy of entanglement [282],

$$E_R(\varrho) = \inf_{\sigma} S(\varrho \| \sigma), \tag{93}$$

where the infimum is taken over all separable states, and the distance $S(\varrho \| \sigma)$ is the relative entropy, $S(\varrho \| \sigma) = \mathrm{Tr}[\varrho \log(\varrho) - \varrho \log(\sigma)]$.

Another class of distance-like measures that are relevant in our future discussion are the so-called robustness measures. Roughly speaking, these measures quantify, how much noise must be added, in order to make the state separable. For instance, the robustness of entanglement $R(\varrho)$ is defined as the minimal s, such that the state

$$\chi = \frac{1}{1+s}(\varrho + s\sigma) \tag{94}$$

is separable [283]. Here, σ denotes an arbitrary separable state. One can study variations of this measure, by allowing σ to be an arbitrary state (the generalized robustness $R_g(\varrho)$ [284,285]) or by fixing it to be the maximally mixed state (the random robustness $R_g(\varrho)$ [283]). The interesting point for our purpose is, that the negative mean value of an entanglement witness can be directly turned into a lower bound on the robustness (see Section 6.9).

4.2.6. The geometric measure of entanglement

A further interesting entanglement measure for multipartite systems is the geometric measure of entanglement E_G [263,286,287]. This also quantifies the distance to the separable states. The geometric measure is defined for pure states as

$$E_G(|\psi\rangle) = 1 - \sup_{|\phi\rangle = |a\rangle|b\rangle|c\rangle \ldots} |\langle \phi | \psi \rangle|^2, \tag{95}$$

i.e., as one minus the maximal squared overlap with pure fully separable states, and for mixed states via the convex roof construction. In many cases, the geometric measure is similarly defined as $E_G^{\log}(|\psi\rangle) = -2 \log_2(\sup_{|\phi\rangle = |a\rangle|b\rangle|c\rangle \ldots} |\langle \phi | \psi \rangle|)$. This is then a lower bound on the relative entropy of $|\psi\rangle$, however, it is still not additive [288] and its convex roof is no longer an entanglement monotone [289].

The geometric measure is a *multipartite* entanglement measure, as it is not only a summation over bipartite entanglement properties. Despite its abstract definition, it has turned out that E_G can be used to quantify the distinguishability of multipartite states by local means [290]. Furthermore, the optimization involved in Eq. (95) can be performed in many cases, e.g. for W-type states [263,291] or certain graph states [292,293]. Also the convex roof can be computed in important cases [214,263]. We will see in Section 6.9 how the geometric measure can be estimated from the mean values of entanglement witnesses.

4.2.7. The three-tangle and comb monotones

The three-tangle τ, introduced in Ref. [294], is an entanglement measure for three-qubit states. For states in the form of Eq. (44) it is given by

$$\tau(|\psi\rangle) = 4\lambda_0^2 \lambda_4^2. \tag{96}$$

Interestingly, an arbitrarily pure three-qubit state fulfills the monogamy relation $C_{A|BC}^2(|\psi\rangle) = C_{AB}^2(\varrho_{AB}) + C_{AC}^2(\varrho_{AC}) + \tau(|\psi\rangle)$ where $C_{A|BC}^2(|\psi\rangle) = 2\sqrt{\det(\varrho_A)}$ is the concurrence between A and the other two qubits, and $C_{AB}^2(\varrho_{AB})$ is the concurrence between A and B [294,295]. For mixed states, the tangle is defined via the convex roof construction, which can, in important cases, be performed analytically [296,297]. For more than three qubits, the tangle can be generalized via combs and filters [249,250].

5. Bell inequalities

In this section, we review Bell inequalities as the oldest tool to detect entanglement. Originally, Bell inequalities were designed to rule out local hidden variable (LHV) models. We will therefore first explain LHV theories and then present different Bell inequalities for various situations. While Bell inequalities were violated in many experiments, several loopholes make it still possible that *in principle* LHV models could describe the measurement results. We will discuss the detection efficiency loophole and the locality loophole. Finally, we will discuss a two-photon experiment, where the locality loophole has been closed.

5.1. Bipartite systems

The first Bell inequality was published by John Bell in 1964 in Ref. [3]. The goal of this paper is to capture the Einstein–Podolsky–Rosen paradox in a quantitative way. The basic idea behind Bell inequalities is the following. Let us
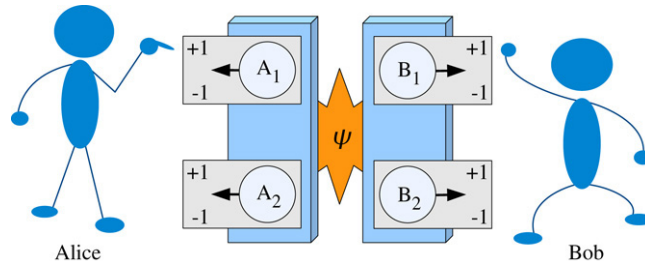
Fig. 6. The basic setup for the CHSH inequality: In a bipartite system Alice can choose to measure two quantities, denoted by A_1 and A_2, each giving the results $a_i \in \{-1, +1\}$. Similarly, Bob can measure B_1 and B_2 on the right party.

consider a bipartite system, and perform measurements on both parties. If we assume that measurement results existed locally at the parties before the measurement then it is possible to obtain bounds on certain quantities composed from two-body correlation terms. In quantum mechanics, it is possible to design experiments in which a higher value is measured than these bounds. This, on one hand, shows that quantum physics violates *local realism*: The measurement is not simply reading out a pre-existing local quantity and its results cannot be described by a LHV model. On the other hand, as we will see later, if measurements on a quantum state violate a Bell inequality, it implies that the state is entangled.

5.1.1. Local hidden variable models and CHSH-inequality

Let us consider a simple example of a bipartite system. We assume that Alice can measure two quantities at her party, called A_1 and A_2, while Bob can also measure two quantities at his party, called B_1 and B_2, as shown in Fig. 6. The results of these experiments are a_1, a_2 and b_1, b_2 and we assume that these results can take the values $+1$ or -1. Now let us perform simultaneous measurements of these four quantities M times. Expectation values can be obtained simply by averaging the measurement results, $\langle A_i B_j \rangle = \frac{1}{M} \sum_{k=1}^{M} a_i(k) b_j(k)$. If the probabilities of the outcomes are given by expressions like $p(a_1^+, b_2^-) := p(a_1 = +1, b_2 = -1)$ this can be rewritten as

$$\langle A_i B_j \rangle = p(a_i^+, b_j^+) - p(a_i^-, b_j^+) - p(a_i^+, b_j^-) + p(a_i^-, b_j^-). \tag{97}$$

Now one can ask whether these probabilities and expectation values can be described by a *local hidden variable (LHV) model*. The latter assumes that the probabilities of the measurement results are independent of whether they are indeed measured or not (the assumption of reality) and that Alice's probabilities do not depend on the choice of the observables by Bob and vice versa (locality assumption). So one makes the ansatz

$$p(a_i^\alpha, b_j^\beta) = \int d\lambda \, p(\lambda) \mathcal{A}_\lambda(a_i^\alpha) \mathcal{B}_\lambda(b_j^\beta), \tag{98}$$

where α, β denote the possible outcomes ± 1 and λ is the hidden variable, occurring with a probability $p(\lambda)$. The response function $\mathcal{A}_\lambda(a_i^\alpha) \mathcal{B}_\lambda(b_j^\beta)$ factorizes due to the locality assumption, as for a fixed λ Alice's probabilities should not depend on Bob's choice of observables. Furthermore, one can always assume the LHV model to be deterministic (i.e., for a fixed λ the $\mathcal{A}_\lambda(a_i^\alpha)$ and $\mathcal{B}_\lambda(b_j^\beta)$ take only the values 0 or 1) as a nondeterministic model corresponds to a deterministic one, where the hidden variable λ is not known [29,30]. Finally, it should be noted that if the probabilities are of the form as in Eq. (98), then the expectation values can be written as $\langle A_i B_j \rangle = \int d\lambda \, p(\lambda) \mathfrak{A}_\lambda(A_i) \mathfrak{B}_\lambda(B_j)$ with $\mathfrak{A}_\lambda(A_i) = \mathcal{A}_\lambda(a_i^+) - \mathcal{A}_\lambda(a_i^-)$ etc.

From the fact that the probabilities are described by a LHV model, bounds on the correlations follow. These are the Bell inequalities. On the level of probabilities, the Clauser–Horne (CH) inequality [298] reads[23]

$$p(a_1^-, b_1^-) + p(a_1^+, b_2^-) + p(a_2^-, b_1^+) - p(a_2^-, b_2^-) \geq 0. \tag{99}$$

Eq. (99) reflects the fact that the last event (a_2^-, b_2^-) can only occur if at least one of the three other possibilities also occurs, and follows directly from the structure of the response function.

Using the normalization of the probabilities, one can derive a similar inequality for the mean values, which is the Clauser–Horne–Shimony–Holt (CHSH) inequality [299,300]. It reads

$$\langle A_1 B_1 \rangle + \langle A_2 B_1 \rangle + \langle A_1 B_2 \rangle - \langle A_2 B_2 \rangle \leq 2. \tag{100}$$

Eq. (100) can also be directly understood by considering all possible measurement results a_1, a_2, b_1, b_2, as for all possible of them $a_1 b_1 + a_1 b_2 + a_2 b_1 - a_2 b_2 \leq 2$.

Naturally, inequality equation (100) is satisfied if there is a hidden variable model for the probabilities of the measurement results. Such an inequality is called *Bell inequality*. Note at this point that for defining Bell inequalities quantum physics is not used at all, only simple basic assumptions lying behind classical physics were used.

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[23] This is the simplest possible form from Ref. [29], the original CH inequality is equivalent, but contains six terms.

Let us connect these ideas to quantum physics. Assuming that Alice and Bob both have a quantum mechanical spin-1/2 particle one can identify A_i and B_j with measurement results of the Pauli spin operators $X = \sigma_x$ and $Y = \sigma_y$, respectively. Then, the left-hand side of Eq. (100) defines a quantum mechanical observable $\mathscr{B} = A_1 \otimes B_1 + A_2 \otimes B_1 + A_1 \otimes B_2 - A_2 \otimes B_2$, the so-called Bell operator [301] that can be measured. One finds that there are quantum states that violate the inequality and give $2\sqrt{2}$ for the left-hand side. Taking the observables $A_1 = -\sigma_x, A_2 = -\sigma_y, B_1 = (\sigma_x + \sigma_y)/\sqrt{2}$ and $B_2 = (\sigma_x - \sigma_y)/\sqrt{2}$ the quantum state with the highest violation is the eigenstate corresponding to the largest eigenvalue of this Bell operator. It is given by

$$|\Psi_{\text{CHSH}}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), \tag{101}$$

which is just the two-qubit singlet state. It can be shown that quantum mechanics does not allow for violations larger than $2\sqrt{2}$, a fact that is known as the Tsirelson bound [302]. Furthermore, the question whether a given two-qubit state violates the CHSH inequality for some optimized settings can directly be answered [303]: If one considers the 3×3 matrix $\tau = \langle \sigma_i \otimes \sigma_j \rangle$ with $i, j \in \{x, y, z\}$ then the state violates the CHSH inequality iff the sum of the two largest eigenvalues of the matrix $\tau^{\mathsf{T}}\tau$ is larger than one. For general states one can show that any entangled state violates the CHSH inequality, after local filterings and adding an auxiliary system that does not violate the CHSH inequality itself [304]. In general, the problem whether a given state violates violates a Bell inequality for some choice of settings can be tackled using semidefinite programming [305].

How is a violation of the Bell inequality possible? The reason is that the simple assumptions we made for the Bell inequality are not valid for quantum mechanics. Indeed, the violation of Bell inequalities in experiments shows that either locality or realism must be abandoned, however it is not clear which one of them. Interestingly, one can derive Bell-like inequalities that rely only on realism and on some assumptions about the correlation functions $\langle A_i B_j \rangle$ [306] and are also found to be violated in experiments [307] (for generalizations see Ref. [308]). This might be interpreted as if the assumption of realism must be dropped, but it should be noted that there are realistic (but non-local) theories such as Bohmian mechanics that can describe these experiments. Another interesting non-local realistic model is described in Ref. [309]. It is based on simulating the correlations arising from quantum mechanics without communication between the parties, using Popescu–Rohrlich non-local boxes [310].

5.1.2. Violation of the CHSH inequality and entanglement

The violation of a Bell inequality implies non-locality, i.e., that the measurement probabilities cannot be described by a LHV model. In quantum mechanics, the violation implies that the state is entangled. This can be seen as follows. By construction, a separable state in a bipartite system can be written in the form $\varrho = \sum_k p_k \varrho_k^A \otimes \varrho_k^B$. Hence, one can write

$$\langle A_i B_j \rangle = \sum_k p_k \text{Tr}(A_i \varrho_k^A) \text{Tr}(B_j \varrho_k^B), \tag{102}$$

which is, if we define $\mathfrak{A}_k(A_i) = \text{Tr}(A_i \varrho_k^A)$ and $\mathfrak{B}_k(B_j) = \text{Tr}(B_j \varrho_k^B)$, directly a LHV model. This means that if some measurements on a state violate a Bell inequality then it cannot be separable, i.e., it must be entangled. Note, however, that the converse is not true: there are entangled states for which a local realistic model for all von Neumann measurements can be written down explicitly (see also below) [44].[24] Therefore, they do not violate any Bell inequality. The same statement holds even if one allows positive-operator-valued measurements (POVMs) [311]. For pure states, however any bipartite entangled state violates a Bell inequality [312–314]. Due to that, Bell inequalities may be viewed as non-optimal entanglement witnesses [315]. (Concerning the comparison of Bell inequalities and witnesses see Section 6.4.)

Finally, before considering further Bell inequalities, let us mention ways to characterize them from the point of view of their usefulness in experiments. We have seen that the CHSH inequality is maximally violated by the state Eq. (101). However, in a real experiment one can never prepare the state Eq. (101) perfectly, it is always mixed with noise. This can be modeled by a singlet state mixed with white noise

$$\varrho_{\text{noisysinglet}}(p) := p|\Psi_{\text{CHSH}}\rangle\langle\Psi_{\text{CHSH}}| + (1-p)\frac{\mathbb{1}}{4}, \tag{103}$$

which is a two-qubit Werner state [44]. The CHSH inequality is violated if $p > 1/\sqrt{2} \approx 0.71$ and, using the PPT criterion, the state is entangled for $p > 1/3$. Moreover, it is known that this state allows for a LHV model (for all von Neumann measurements) for $p \leq 0.6595$ [44,316] and it violates a Bell inequality (different from the CHSH one) for $p \geq 0.7056$ [317].[25]

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[24] The question whether a state allows for a LHV model or not can also be tackled by looking for symmetric extensions of the state using semidefinite programming [102].

[25] See also Problem 19 on http://www.imaph.tu-bs.de/qi/problems/.

The bound for the CHSH inequality characterizes the minimum *visibility* that the CHSH inequality requires. In general, the visibility is the ratio between the maximum of the Bell operator $\mathcal{B}$ for quantum states and the maximum for LHV models

$$\mathcal{V}(\mathcal{B}) := \frac{\max_{\Psi} |\langle \mathcal{B} \rangle_{\Psi}|}{\max_{LHV} |\langle \mathcal{B} \rangle_{LHV}|}. \tag{104}$$

Clearly, a large visibility is an advantage from the experimental point of view.

5.2. Multipartite systems

5.2.1. Multipartite LHV models

In the previous subsection we discussed Bell inequalities for bipartite systems. The ideas can straightforwardly be generalized to multipartite systems. Let us consider an N-partite system, in which each party can measure some observables. Then the measurement results fit a local hidden variable model if for any choice of measured variables we have

$$\langle A_i B_j C_k D_l \ldots \rangle = \int d\lambda\, p(\lambda)\mathfrak{A}_\lambda(A_i)\mathfrak{B}_\lambda(B_j)\mathfrak{C}_\lambda(C_k)\mathfrak{D}_\lambda(D_l)\ldots, \tag{105}$$

where $-1 \le \mathfrak{A}_\lambda(A_i) \le +1$ etc. If a multipartite Bell inequality is violated then there is no local hidden variable model that could explain the measurement results.

Similarly to the definition of biseparability one can also ask, whether it is possible to explain our measurement results by bipartite models in which several of the parties join [173,318,319]. For example, for the three-partite case this would mean that any expectation values could be explained by

$$\langle A_i B_j C_k \rangle = p_1 \int d\lambda\, p(\lambda)\mathfrak{A}_\lambda(A_i)\mathfrak{X}_\lambda(B_j C_k) + p_2 \int d\lambda\, p'(\lambda)\mathfrak{B}_\lambda(B_j)\mathfrak{Y}_\lambda(A_i C_k) + p_3 \int d\lambda\, p''(\lambda)\mathfrak{C}_\lambda(C_k)\mathfrak{Z}_\lambda(A_i B_j), \tag{106}$$

where $p_1 + p_2 + p_3 = 1$. If there is no such description for the measurement results, we say that there is genuine three-partite non-locality in the system.

5.2.2. Mermin and Ardehali inequalities

In the previous subsection we discussed the CHSH inequality for bipartite systems. It needed the measurement of two quantities for each party and these measurements had two outcomes. Similar *two-setting* inequalities exist also for multipartite systems. The most famous one is the Mermin inequality [206]. For N qubits it is given by

$$\sum_\pi \langle X_1 X_2 X_3 X_4 X_5 \ldots X_N \rangle - \sum_\pi \langle Y_1 Y_2 X_3 X_4 X_5 \ldots X_N \rangle + \sum_\pi \langle Y_1 Y_2 Y_3 Y_4 X_5 \ldots X_N \rangle - \cdots + \cdots \le L_{\text{Mermin}}, \tag{107}$$

where X_i and Y_i represent the Pauli matrices σ_x, σ_y on the i-th qubit,[26] $\sum_\pi$ represents the sum of all possible permutations of the qubits that give distinct terms and L_{Mermin} is the maximum for local states. It is $L_{\text{Mermin}} = 2^{N/2}$ for even N and $L_{\text{Mermin}} = 2^{(N-1)/2}$ for odd N.

The Mermin inequality is maximally violated by the *GHZ* state defined in Eq. (54).[27] The Bell operator is the sum of $2^{(N-1)}$ terms that all give $+1$ for the *GHZ* state. Hence the maximum of the Bell operator is $2^{(N-1)}$ for quantum states. Thus, the visibility is $\mathcal{V}_{\text{Mermin}} = 2^{N/2-1}$ for even N and $2^{(N-1)/2}$ for odd N. Note that the visibility increases exponentially with the number of qubits. It can be proved that for odd number of parties the Mermin inequality has the largest visibility possible among two-setting inequalities with full correlation terms. Latter means that the Bell operator is the sum of correlation terms that contain a variable for each spin. For example, for $N = 4$, $X_1 Y_2 X_3 Y_4$ is a full correlation term, however, $X_1 Y_2 \mathbb{1}_3 X_4$ is not. The $\mathbb{1}_3$ indicates that on the third particle no measurement has been done.

The Mermin inequality consists of stabilizing operators of the *GHZ* state (see Section 3.4.3). This implies that there is a quantum state that gives $+1$ for all terms, while LHV models have to result in -1 for at least one term. This is a form of the *GHZ* argument [174], which leads to an obvious contradiction to local realism. Similarly, one can derive Mermin-like inequalities for other graph states [147,228,229,320–322].

Let us now discuss the Ardehali inequality [323]. For an even number of parties this inequality is superior to the Mermin inequality. It is defined as

$$\left\langle \left(A_1^{(+)} - A_1^{(-)} \right) \left(-\sum_\pi X_2 X_3 X_4 X_5 \cdot\cdot X_N + \sum_\pi Y_2 Y_3 X_4 X_5 \cdot\cdot X_N - \sum_\pi Y_2 Y_3 Y_4 Y_5 X_6 \cdot\cdot X_N + \cdots - \cdots \right) \right\rangle$$

$$+ \left\langle \left(A_1^{(+)} + A_1^{(-)} \right) \left(\sum_\pi Y_2 X_3 X_4 X_5 \cdot\cdot X_N - \sum_\pi Y_2 Y_3 Y_4 X_5 \cdot\cdot X_N + \sum_\pi X_2 Y_3 Y_4 Y_5 Y_6 X_7 \cdot\cdot X_N - \cdots + \cdots \right) \right\rangle$$

$$\le L_{\text{Ardehali}}, \tag{108}$$

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[26] Of course, in the sense of LHV models, they can be replaced by arbitrary observables with two outcomes ± 1.

[27] Note that the original Mermin inequality is slightly different and is maximally violated by the state $(|000\ldots\rangle + i|111\ldots\rangle)/\sqrt{2}$.

where $A_1^{(\pm)}$ are operators corresponding to measuring the first spin along directions corresponding to the quantum operators $A_1^{(\pm)} = (\mp X_1 - Y_1)/\sqrt{2}$. Note that on the other spins X or Y is measured as in the case of the Mermin inequality. The constant L_{Ardehali} appearing in Eq. (108) is the maximum for local states. It is $2^{N/2}$ for even N and $2^{(N+1)/2}$ for odd N. It is again maximally violated by the GHZ state.[28] The Bell operator is the sum of 2^N terms that all give $1/\sqrt{2}$ for this state. Thus, the visibility $\mathcal{V}_{\text{Mermin}} = 2^{N/2}$ for even N and $2^{(N-1)/2}$ for odd N.

An interesting feature of the Ardehali inequality is that the Bell operator is essentially the same as for the Mermin inequality, but the rewriting with $A_1^{(\pm)}$ leads to a higher violation of local realism. This shows that the violation of local realism of GHZ states with an even number of qubits can increase, if non-stabilizer observables are considered. This also holds for some other graph states [320,324].

Note that so far all the inequalities needed the measurement of two operators for each qubit. Also, they were the sum of full correlations terms. Among such inequalities, it can be shown that for any N the Mermin–Ardehali construction, also discovered independently by Klyshko [325,326] provides Bell inequalities with an optimal violation. It turns out that the full set of such Bell inequalities can be generated and written down concisely in the form of a single nonlinear inequality [327,328]. Surprisingly, there are multi-qubit pure entangled states that do not violate any of these Bell inequalities [329].

The situation is more complicated with inequalities that are not sums of full correlation terms. Then, it has been shown that such inequalities can detect any pure entangled multi-qubit state [330]. Also, inequalities of this type can be constructed such that they are maximally violated by cluster states and graph states [147,228,320] (see Section 3.4.3 for the definition of these states). In particular, for the four-qubit cluster state this inequality looks like [228,331][29]

$$\langle X_1 \mathbb{1}_2 X_3 Z_4 \rangle + \langle Z_1 Y_2 Y_3 Z_4 \rangle + \langle X_1 \mathbb{1}_2 Y_3 Y_4 \rangle - \langle Z_1 Y_2 X_3 Y_4 \rangle \le 2. \tag{109}$$

Note that on all of the qubits two operators are measured except for the second qubit for which only Y_2 is measured. One can show that for a large class of graph states, e.g., for linear cluster states, it is possible to construct two-setting Bell inequalities that have a visibility increasing exponentially with N [320].

5.2.3. Bell inequalities detecting genuine multipartite non-locality

So far all the Bell inequalities presented could be used to confirm that a state is nonlocal, that is they ruled out local hidden variable models as in Eq. (105), but did not give any information on whether or not the state possesses genuine multipartite non-locality [see Eq. (106)]. However, some Bell inequalities exist that can rule out such hybrid LHV models. Let us consider the three-particle Svetlichny inequality as an example [173]. It reads

$$\langle A_1 B_1 C_2 \rangle + \langle A_1 B_2 C_1 \rangle + \langle A_2 B_1 C_1 \rangle - \langle A_2 B_2 C_2 \rangle + \langle A_2 B_2 C_1 \rangle + \langle A_2 B_1 C_2 \rangle + \langle A_1 B_2 C_2 \rangle - \langle A_1 B_1 C_1 \rangle \le 4, \tag{110}$$

where the A_i, B_j and C_k are two outcome measurements (e.g., Pauli matrices) on each particle. This Bell inequality is a sum of two three-particle Mermin inequalities. However, one can directly see that for hybrid LHV models (e.g., for an 1|23 case, where the particles 2 and 3 are combined, the Svetlichny inequality can be written as a sum of two CHSH inequalities) the same bound holds. For GHZ states, the Bell operator can take values of $4\sqrt{2}$ [319,332], however, not all genuine multipartite entangled states lead to a violation of the Svetlichny inequality [256], even if they are pure [333].

A generalization of this inequality for an arbitrary number of parties is presented in Ref. [318]. Ref. [319] gives bounds for Bell inequalities both for genuine multi-qubit non-locality and genuine multi-qubit entanglement.

5.2.4. Further Bell inequalities

There are Bell inequalities that use more than two operators for each particle (i.e., more than two settings) or need the measurement of observables with more than two outcomes. Ref. [334] considers bipartite systems with qudits having dimension $d > 2$, and studies two-setting Bell inequalities with observables with d outcomes. It finds that with the increase of d, it is possible to find Bell inequalities with increasing maximal violation. Ref. [335] presents a family of Bell inequalities, called the Collins–Gisin–Linden–Massar–Popescu (CGLMP) inequalities for bipartite quantum systems of arbitrarily high dimensionality, which are strongly resistant to noise. They are for systems with two measurement settings per site, with more than two measurement outcomes. Ref. [336] presents Bell inequalities stronger than the CHSH inequality for isotropic states of two three-state particles.

Systematic study of such inequalities were also presented for small number of outcomes. Such an inequality can be characterized by four numbers $ijmn$, where Alice measures one of i operators that have m outcomes. Similarly, Bob measures one of j operators that have n outcomes. In the bipartite case, for two two-outcome observables the CHSH inequality is the only Bell inequality apart from simple transformations [337]. That is, it is the only tight Bell inequality, where tight means that the inequality is a facet of the polytope of correlations allowed by LHV models [338]. Non-tight inequalities also exist, but they do not detect more states than the tight inequalities.

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[28] Note that the original Ardehali inequality has a different definition for $A_1^{(\pm)}$ and is maximally violated by the state $(|000\ldots\rangle - |111\ldots\rangle)/\sqrt{2}$.

[29] This is the inequality for the cluster state in the basis as defined via Eqs. (56) and (57), for the cluster state as in Eq. (60) it reads $\langle Z_1 \mathbb{1}_2 X_3 X_4 \rangle + \langle X_1 Y_2 Y_3 X_4 \rangle - \langle Z_1 \mathbb{1}_2 Y_3 Y_4 \rangle + \langle X_1 Y_2 X_3 Y_4 \rangle \le 2$.

For the 3322 case, that is, when three two-outcome observables are measured at each party, then there is a single new inequality [339–341]. It reads

$$\langle A_1 \mathbb{1} \rangle - \langle A_2 \mathbb{1} \rangle + \langle \mathbb{1} B_1 \rangle - \langle \mathbb{1} B_2 \rangle - \langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle - \langle A_2 B_2 \rangle + \langle A_1 B_3 \rangle + \langle A_2 B_3 \rangle + \langle A_3 B_1 \rangle + \langle A_3 B_2 \rangle \le 4.$$
(111)

This is a relevant inequality in the sense that it detects two-qubit states that cannot be detected by the CHSH inequality [340]. For the 4322 case there are three new inequalities [340]. For systems with measurements with more outcomes the numerical search is exceedingly difficult. A set of 26 inequalities for the 4422 case are presented in Ref. [342]. Ref. [343] determines the quantum violation of 241 tight bipartite Bell inequalities with up to five two-outcome measurement settings per party, for up to eight dimensional complex and six dimensional real Hilbert spaces, using semidefinite programming. The violation of bipartite inequalities with arbitrary number of two-outcome observables is bounded by the Grothendieck constant [316].

For three parties and for two-outcome observables, inequalities with arbitrarily large violation can be designed [344]. Ref. [345] presents three-setting two-outcome inequalities for *GHZ* states that have a higher violation than the Mermin inequality. A set of multipartite three-setting Bell inequalities were studied, with two-outcome observables in Refs. [346,347]. Finally, Ref. [339] presents the full list of Bell inequalities for three parties, two measurements and two outcomes.

5.3. Consequences of a Bell inequality violation

We have seen already that for the bipartite case a violation of a Bell inequality implies entanglement. For the multipartite case similarly non-locality implies entanglement, however, only ruling out LHV models as in Eq. (106) can prove genuine multipartite entanglement. As in the bipartite case, entanglement does not imply non-locality. For example, there are three-qubit states with genuine multipartite entanglement that are local if von Neumann measurements are considered [348].

Interestingly, while Bell inequalities are primarily designed to rule out LHV models (as described in Section 5.2.1) they can sometimes be used to detect genuine multipartite entanglement. For example, the three-qubit Mermin inequality used in the photonic experiments of Refs. [210,349] is

$$\langle X_1 X_2 X_3 \rangle - \langle Y_1 Y_2 X_3 \rangle - \langle X_1 Y_2 Y_3 \rangle - \langle Y_1 X_2 Y_3 \rangle \le 2.$$
(112)

As a Bell inequality, violation of this inequality does not imply three-particle nonlocality. In quantum mechanics, however, it can be proved that, if X_k and Y_k refer to the measurement of Pauli spin matrices at qubit k, then for biseparable states the maximum is also two [191,350]. Thus, the violation of this inequality implies genuine multi-qubit entanglement and in the experiment of Ref. [210] genuine multi-qubit entanglement was detected.[30] In another experiment creating a four-qubit *GHZ* state [211], a bound for multipartite entanglement for the Ardehali inequality was used to detect four-qubit entanglement. In general, the detection of multi-qubit entanglement with many-qubit Bell inequalities was studied in Refs. [319,352]. These papers presented bounds for various forms of multipartite entanglement. Refs. [255,353] present nonlinear Bell inequalities for the same aim.

Besides, there are some other interesting connections between the violation of a Bell inequality and the properties of a state:

(i) *Quantum communication.* It has been observed that in the BB84 protocol the error thresholds where a Bell inequality can be violated and where the mutual information between Alice and Bob and Alice and Eve are the same, coincide [354–356]. Similar results have been found in the multipartite scenario [357,358]. This suggests a connection between Bell inequality violation and the security of quantum cryptography. Moreover, the Ekert protocol is based on distributing entangled particles [4]. Its security is directly related to the violation of a Bell inequality. Note that Bell inequality violation can imply security even in the device independent case when no assumption is made on the quantum system we are using for communication (e.g., dimension of the system) [359].

(ii) *Communication complexity.* The violation of a Bell inequality means that measurement results on a quantum state cannot be replaced by shared classical randomness. Because of that, using the quantum state as a resource has an advantage in quantum communication. Indeed, for any state violating a Bell inequality one can write down a communication complexity problem, for which the quantum state improves the solution compared with classical resources [360–362].

(iii) *Distillability.* Quantum states violating two-setting Bell inequalities are bipartite distillable [363,364]. This fact, however, relies on the spectral decomposition of the Bell operator, and not on the nonexistence of a LHV model. In the multipartite case, this means distillability at least with respect to some bipartitioning of the qubits, which arises if two groups of parties join. On the other hand, if the distillation protocol has to be local with respect to each party, there exist multipartite non-distillable states which violate a Bell inequality [365].

This directly leads to the conjecture of Peres, stating that bipartite states with a positive partial transposition allow a LHV description [29]. While no proof or counterexample has been found, in several special cases it is shown that such states do not violate large classes of Bell inequalities, e.g., for the Mermin–Klyshko inequalities [366] and for a wide range of multi-setting inequalities [134,367].

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[30] This has also been shown in Ref. [351] based on a different argument.

5.4. Loopholes

While Bell inequalities have been violated many times in experiments, several loopholes make it still possible *in principle* that the experimental results can be explained with a LHV model. There are mainly two loopholes:

(i) *The fair sampling assumption or detector efficiency loophole.* In many setups based on photons, photo-detectors detect photons only in part of the experiments, while those experiments in which there are no photons detected (no "clicks") go unnoticed. We assume, naturally, that the unnoticed experiments have the same statistical properties as the noticed ones. However, a so far unknown mechanism could cheat us in this respect. It could happen, that all the experiments together would not violate a Bell inequality, however, the subset for which our detectors click, violate it [368,369].

The question, for which detector efficiency the loophole becomes closed depends on the Bell inequality, and is, in general, not straightforward to answer. For several important experimental situations, results have been obtained in Refs. [370–374]. The detection loophole was eliminated in experiments with trapped ions with very large detection efficiency in Ref. [375]. In a recent experiment, also eliminating the detection loophole, the ions were 1 m apart [376].

(ii) *The locality loophole.* Ideally, before each correlation measurement the circuits connected to the detectors should decide locally which operator to measure. This has to be done such that the detectors cannot communicate their choice if we assume that the speed of this communication cannot be larger than the speed of light. In practice this means that the event of deciding the measurement direction for one of the detectors must be space-like separated from event of reading out the measurement on the other detector. This requirement is very challenging. In Ref. [377], an experiment is described in which the directions for detection are determined after the photons left the source. In Ref. [378], a photonic experiment is described that eliminates the locality loophole, however, in this realization the detection loophole is still present. We will discuss it in the following section.

Another assumption in the derivation of Bell inequalities is that Alice's and Bob's choices of the measurements do not depend on the hidden variables. This corresponding loophole is sometimes called the freedom-of-choice-loophole [379,380] and has been addressed experimentally in Ref. [380].

5.5. Experimental violation of a Bell inequality

In this section we describe as a first experiment the violation of Bell inequalities in a two-photon experiment by Weihs et al. [378].

5.5.1. Generation of entangled photons

First, we review briefly the fundamentals of the experimental techniques, since there are numerous similar many-photon experiments. An excellent and exhaustive review of this topic can be found in Ref. [32]. As is well known, a photon can be used to store a single qubit information since it can be in a superposition of the horizontally and vertically polarized states, denoted by $|H\rangle$ and $|V\rangle$, respectively, as

$$|\Psi\rangle = \alpha|H\rangle + \beta|V\rangle, \tag{113}$$

where $|\alpha|^2 + |\beta|^2 = 1$. Other basis states can be also chosen. The left and right circularly polarized states can be expressed as $|L\rangle := \frac{1}{\sqrt{2}}(|H\rangle + \mathrm{i}|V\rangle)$ and $|R\rangle := \frac{1}{\sqrt{2}}(|H\rangle - \mathrm{i}|V\rangle)$.

The basic linear optical elements for the manipulation of a single photon are as follows:

(i) *Wave plates* are devices that delay one of the polarization components with respect to the other, and from the state Eq. (113) they lead to

$$|\Psi\rangle = \alpha|H\rangle + \mathrm{e}^{\mathrm{i}\phi}\beta|V\rangle, \tag{114}$$

where ϕ describes the delay. For half-wave plates (HWP) and quarter-wave plates (QWP) the delays are $\phi = \pi = 180°$ and $\phi = \pi/2 = 90°$, respectively.

(ii) A *beam splitter* (BS) is a device that has two *spatially* separated input modes. The state of the two spatially separated output modes can be obtained through linear transformations. That is, let us denote the two spatially separated modes $\{|T\rangle$ and $|B\rangle\}$ referring to Top and Bottom. Then a 50%–50% beam splitter that does not introduce any additional phase shift makes the transformation

$$|T\rangle \rightarrow \frac{1}{\sqrt{2}}\left(|T\rangle + |B\rangle\right),$$

$$|B\rangle \rightarrow \frac{1}{\sqrt{2}}\left(|T\rangle - |B\rangle\right). \tag{115}$$

(iii) *Polarizing beam splitters.* If photons carry also a polarization degree of freedom, then a polarizing beam splitter (PBS) can be used to direct photons with H polarization into one direction, while photons with V polarization into the other direction. Using polarizing beam splitters is typical before detecting the state of the photons. That is, after the polarizing beam splitter two detectors detect the photons of the two different polarizations.
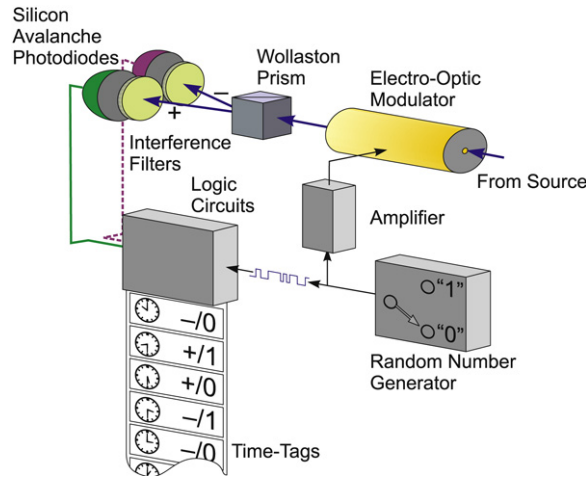
Fig. 7. One of the two observer stations. All alignments and adjustments were pure local operations that did not rely on a common source or on communication between the observers. Figure from Ref. [378].

Next we discuss, how to realize multi-photon experiments. The difficulty of using photons, rather than atoms or ions, for creating entangled states is that they do not interact with each other. Thus, special techniques are needed to create entangled states. The most often used method is using spontaneous parametric down-conversion and post-selection.

 (i) *Spontaneous parametric down-conversion (SPDC)* is a process in which a photon with frequency ν is converted to two photons with lower frequencies ν_1 and ν_2 [381]. Due to conservation of energy, $\nu = \nu_1 + \nu_2$. In a typical experiment, by filtering out the unwanted frequencies one sets the frequencies of the outgoing photons equal, that is, $\nu_1 = \nu_2 = \frac{\nu}{2}$. Due to conservation of momentum, the sum of the momenta of the outgoing photons equals the momentum of the incoming photon. The down-conversion is collinear, if the outgoing photons proceed in the same direction as the incoming one, while it is non-collinear if this is not the case. Non-collinear down-conversion allows the spatial separation of the two photons right after the down-conversion. In case of type-I down-conversion the two photons have the same polarization, while for type-II down-conversion they have perpendicular polarizations.
(ii) *Conditional detection* or *post-selection*. Since the SPDC happens with *some small probability*, e.g., when a laser beam is directed to a β-Barium-Borate (BBO) crystal, and the photon detectors do not have perfect efficiency, only the events where two photons are detected are relevant. In experiments with more than two photons, the photons are usually generated by higher order emissions of SPDC (see Section 6.5) or by interaction of entangled photon pairs via beam splitters. Then, it happens that the N photons are not equally distributed over the N spatial modes, i.e., with some probability one mode may contain more than one photon and some modes may contain none. In this case, the observation of a multipartite state is usually based on conditional detection (or post-selection): only if each spatial mode contains one photon, i.e., all detectors register one photon, the desired state is observed (for an example see Section 6.5).

5.5.2. The experiment

Now we are in the position to review the experiment of Ref. [378]. In the experiment they created a two-qubit singlet state [Eq. (101)] and used a CHSH Bell's inequality [Eq. (100)] to prove the non-locality of the state. The key point of the experiment is that the two detection events are space-like separated. Thus, what is measured at one of the qubits cannot influence the results of the measurement of the other qubits, even if we allow for a physical mechanism, so far unknown, that could make communication possible between the two detectors with the speed of light. Thus, this experiment closed the locality loophole discussed in Section 5.4. Note, however, that it did not close the detection loophole since the detection efficiency for photonic experiments is relatively low (in the present case it was 5%).

In the experiment, type-II parametric down-conversion was used to generate an entangled photon pair in state Eq. (101) [381]. Then the two photons were led to the two observers stations (see Fig. 7) with optical fibers. In order to achieve space-like separated detector events ("clicks"), the observer stations were 400 m away from each other.

It was decided locally, which one of the two operators to measure at each qubit. This was done using a quantum random generator, rather than a pseudo-random number generator. The latter would be based on a deterministic process and in principle one party could predict what the other measures knowing the algorithm of the generator. Moreover, the timing of the generation of the random decision for the detectors was such that, again, communication that is not faster than light, could not bring information on the choice of what to measure at one detector to the photon at the other detector. This means that the generation of a random bit, the choice between the two possible measurements and even detection of the result was realized locally at the two parties, as can be seen in Fig. 7. The operator to be measured was set by an electro-optical modulator. The result was recorded in a logic circuit together with a time tag and the information on which one of the two operators were measured.
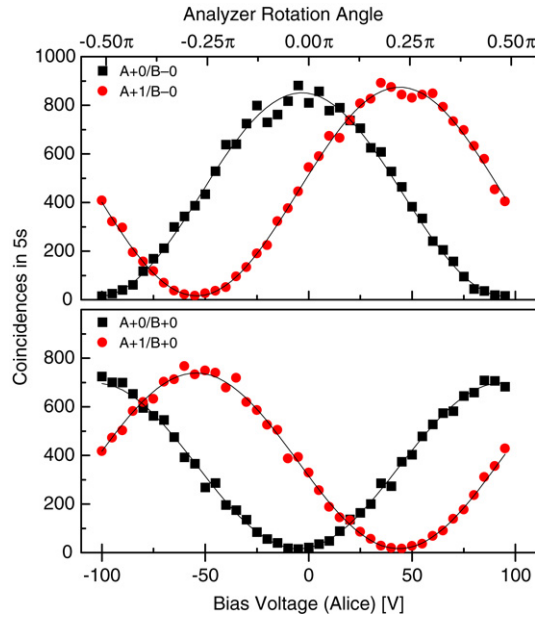
Fig. 8. Four out of sixteen coincidence rates between various detection channels as functions of bias voltage (analyzer rotation angle) on Alice's modulator. $A + 1/B - 0$ for example are the coincidences between Alice's "+" detector with switch having been in position "1" and Bob's "−" detector with switch position "0". The difference in height can be explained by different efficiencies of the detectors. Figure from Ref. [378].

If the two polarization analyzers are set into directions α and β, respectively, then the coincidence rates show a sinusoidal dependence on $\alpha - \beta$, which can be seen in Fig. 8. The solid lines are the theoretical curves, while the dots correspond to the measurement values. The experimental results fit the theoretical calculations very well. The mean value of the Bell operator was 2.73 ± 0.02 and the visibility is 97%. For that 14 700 coincidence events were collected in 10 s. Note that the coincidence rates in multi-qubit experiments based on down-conversion and post-selection are much lower, and the experimental time is much longer. Such experiments are presented in Sections 6.5 and 8.2.

6. Entanglement witnesses

In this section, we will explain how entanglement witnesses can be used for the *experimental* detection of entanglement. The theoretical concept of entanglement witnesses has already been introduced in the Sections 2.5 and 3.6, and some major construction methods were shown. Now we will discuss how witnesses can be implemented in experiments. Furthermore, we will describe advanced methods to construct witnesses for special experimental situations and will also discuss several recent experiments from the point of view of entanglement detection. Finally, we will discuss how entanglement witnesses can be used to derive *quantitative* statements about the entanglement in a quantum state and will explain applications of witnesses in quantum key distribution.

6.1. Witnesses for two qubits

Let us start our discussion with a simple two-qubit example [148,382]. From an experimental point of view, the investigation of two qubits seems trivial, as most experiments nowadays aim at the generation of multiparticle entangled states. However, the investigation of the two-qubit case will show already the main features of entanglement detection via witnesses in a simple setting.

6.1.1. Construction of the witness

Let us consider a setup that is intended to produce a particular pure state $|\psi\rangle\langle\psi|$, but due to the imperfections some noise is added. As a simple approximation, one may consider a mixed state ϱ of the form

$$\varrho(p) := p|\psi\rangle\langle\psi| + (1-p)\frac{\mathbb{1}}{4}, \tag{116}$$

where $\mathbb{1}/4$ is the maximally mixed state of two qubits. The aim is to give a scheme to decide whether $\varrho(p)$ is entangled or not. Clearly, as the form of the state in Eq. (116) is only a rough approximation, the conclusion that the state is entangled should not depend on any assumption on the form of the state. However, it is reasonable to choose the detection method such that it detects states of the form Eq. (116) even for small p. The optimal entanglement witness for an entangled $\varrho(p)$

is easy to construct from the results of Section 2.5. First, one has to compute the eigenvector corresponding to the negative eigenvalue of $\varrho(p)^{T_B}$, then the witness is given by the partially transposed projector onto this eigenvector.

If the Schmidt decomposition of $|\psi\rangle$ is $|\psi\rangle = a|01\rangle + b|10\rangle$ the eigenvalues of $\varrho(p)^{T_B}$ are given by $\{(1-p)/4+pa^2; (1-p)/4+pb^2; (1-p)/4+pab; (1-p)/4-pab\}$. Therefore $\varrho(p)$ is entangled iff $p > 1/(1+4ab)$. The eigenvector corresponding to the minimal eigenvalue λ_- is given by

$$|\phi_-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \tag{117}$$

and thus the witness $\mathcal{W}$ can be written in several ways as

$$\mathcal{W} = |\phi_-\rangle\langle\phi_-|^{T_B} = \frac{1}{2}\mathbb{1} - |\psi^+\rangle\langle\psi^+| = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{118}$$

with $|\psi_+\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$. Note that this witness does neither depend on p, nor on the Schmidt coefficients a, b. Due to the special assumption about the state, it detects $\varrho(p)$ iff it is entangled, since we have $\text{Tr}[|\phi_-\rangle\langle\phi_-|^{T_B}\varrho(p)] = \text{Tr}[|\phi_-\rangle\langle\phi_-|\varrho(p)^{T_B}] = \lambda_-$.

6.1.2. Local decomposition of the witness

In the form of Eq. (118) the witness $\mathcal{W}$ cannot easily be implemented in an experiment. The reason is that $\mathcal{W}$ is up to a constant a projector onto one of the Bell states and such a nonlocal observable is not straightforward to measure in practice (for a theoretical proposal to do that see Ref. [383]). The observables that can easily be measured in any experiment are local observables like $\langle\sigma_z \otimes \sigma_z\rangle$ or projectors like $\langle|0\rangle\langle0| \otimes |0\rangle\langle0|\rangle$. Therefore, for the experimental implementation it is necessary to decompose the witness into operators that can be measured locally [20,382]. Thus, we need a decomposition into projectors onto product vectors of the form

$$\mathcal{W} = \sum_{i=1}^{k} c_i |e_i\rangle\langle e_i| \otimes |f_i\rangle\langle f_i|. \tag{119}$$

Such a decomposition can be measured locally: Alice and Bob measure the expectation values of the $|e_i\rangle\langle e_i| \otimes |f_i\rangle\langle f_i|$ and add their results with the weights c_i. One can construct such a decomposition in many ways, but it is reasonable to do it in a way that corresponds to the smallest possible experimental effort for Alice and Bob.

The experimental "cost" that Alice and Bob have to pay when measuring the witness $\mathcal{W}$ via such a decomposition is the number of measurements they have to perform. As already mentioned, local measurements are easily possible, and by measurements we always mean von Neumann (or projective) measurements. One measurement on Alice's side in this sense consists of a choice of one orthonormal basis for Alice's Hilbert space. For qubits, this corresponds to a polarization measurement in a certain direction. For a particle with spin s one may interpret this as the choice of a direction for a Stern–Gerlach-like apparatus. With this setup, Alice can measure the probabilities for all the outcomes.

A similar argument holds for Bob, thus, if one considers a $d \times d$ system a term of the form

$$\mathcal{M} = \sum_{k,l=1}^{d} c_{kl} |a_k\rangle\langle a_k| \otimes |b_l\rangle\langle b_l| \tag{120}$$

with $\langle a_s|a_t\rangle = \langle b_s|b_t\rangle = \delta_{st}$ can be measured with one collective setting of measurement devices of Alice and Bob. Alice and Bob can discriminate between the states $|a_k b_l\rangle$, measure the probabilities of these states and add their results with the weights c_{kl} using one collective setting and some classical communication. See also Fig. 12 for a four-qubit example of such measurement results.

It is therefore reasonable to find a decomposition of the form

$$\mathcal{W} = \sum_{i=1}^{m} \mathcal{M}_i = \sum_{i=1}^{m} \sum_{k,l=1}^{d} c_{kl}^i |a_k^i\rangle\langle a_k^i| \otimes |b_l^i\rangle\langle b_l^i| \tag{121}$$

with $\langle a_s^i|a_t^i\rangle = \langle b_s^i|b_t^i\rangle = \delta_{st}$ and minimal number of measurements, i.e., a minimal m. Note that a decomposition of the form (121) is more general than a decomposition into a sum of tensor products of operators $\mathcal{W} = \sum_{i=1}^{m} \gamma_i A_i \otimes B_i$. The difference is that a measurement as in Eq. (120) consist of a choice of observables (say, $\sigma_z \otimes \sigma_z$ for definiteness), but then all expectation values like $\langle\sigma_z \otimes \sigma_z\rangle$, $\langle\sigma_z \otimes \mathbb{1}\rangle$ and $\langle\mathbb{1} \otimes \sigma_z\rangle$ can be determined from the same data. These terms would occur as different $A_i \otimes B_i$ in a decomposition like the above.[31]

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[31] One can also ask for a decomposition of $\mathcal{W}$ that minimizes the number of product vectors in Eq. (119) [47], although this is not optimal from an experimental point of view. This has been discussed in Refs. [148,382].

The optimal decomposition of the witness can be obtained as follows: $\mathcal{W}$ is a partially transposed projector $|\psi\rangle\langle\psi|^{T_B}$, and we consider the Schmidt decomposition of $|\psi\rangle = \alpha|00\rangle + \beta|11\rangle$. The actual $\mathcal{W}$ is the special case $\alpha = 1/\sqrt{2} = -\beta$, but one can directly deal with the most general $|\psi\rangle$. We define as usual the spin directions by $|z^+\rangle = |0\rangle$, $|z^-\rangle = |1\rangle$, $|x^\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$, $|y^\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle)$ and obtain the decomposition

$$|\psi\rangle\langle\psi|^{T_B} = \alpha^2|z^+z^+\rangle\langle z^+z^+| + \beta^2|z^-z^-\rangle\langle z^-z^-| + \alpha\beta\left(|x^+x^+\rangle\langle x^+x^+| + |x^-x^-\rangle\langle x^-x^-|\right.$$
$$\left. - |y^+y^-\rangle\langle y^+y^-| - |y^-y^+\rangle\langle y^-y^+|\right) \tag{122}$$

$$= \frac{1}{4}\left[\mathbb{1}\otimes\mathbb{1} + \sigma_z\otimes\sigma_z + (\alpha^2 - \beta^2)(\sigma_z\otimes\mathbb{1} + \mathbb{1}\otimes\sigma_z) + 2\alpha\beta(\sigma_x\otimes\sigma_x + \sigma_y\otimes\sigma_y)\right]. \tag{123}$$

This decomposition into six product vectors requires only a measurement of three settings: Alice and Bob only have to measure the coincidence probabilities of $\sigma_x\otimes\sigma_x$, $\sigma_y\otimes\sigma_y$, and $\sigma_z\otimes\sigma_z$ to measure $|\psi\rangle\langle\psi|^{T_B}$. We will discuss an implementation of this witness in Section 6.2.

The question arises, whether this decomposition is really optimal. Indeed, this is the case. It can be shown (see Refs. [148,382]) that the witness $\mathcal{W}$ cannot be decomposed in two measurements, hence the decomposition is optimal.

Finally, it should be noted that for the case that the target state is given by $|\psi\rangle = (|01\rangle + |10\rangle)/\sqrt{2} = |\psi^+\rangle$, the witness $\mathcal{W}$ also allows for a determination of the fidelity of the state. This quantity is defined as $F_\psi = \langle\psi|\varrho|\psi\rangle$ and quantifies to which extent the desired state was produced. Due to Eq. (118) we have

$$F_{\psi^+} = \frac{1}{2} - \langle\mathcal{W}\rangle \tag{124}$$

and the decomposition of $\mathcal{W}$ in three local measurements just means that the fidelity of $|\psi^+\rangle$ can be determined with three measurements. This connection between mean values of witnesses and the fidelity of the target state will also occur in the multipartite setting.

6.1.3. Discussion

We have seen that with the help of a local decomposition entanglement witnesses can be used for the experimental detection of entanglement. For two qubits, this required only three local measurements, which is less than quantum state tomography that would require $3^2 = 9$ measurements, namely the measurement of all correlations $\langle\sigma_i\otimes\sigma_j\rangle$. In general, two criteria determine the efficiency of an entanglement detection scheme:

(i) *Robustness to noise:* The robustness to noise is given by the minimal p in Eq. (116) that has to be reached for the state to be detected. Equivalently, one can define a $p_{\text{noise}} = 1 - p$ and ask for the maximal p_{noise} that can be tolerated. This robustness quantifies the noise that can be allowed in an experiment, and if some amount of noise is expected one can estimate, whether there is a chance to confirm the success of the experiment at all.

Clearly, in an experimental setting the noise does not have to be white and the state may not be of the form as in Eq. (116). However, this is often a reasonable approximation, hence the robustness to noise is a good indicator to compare different entanglement witnesses or different entanglement detection methods.

(ii) *Number of measurement settings:* Another crucial point is the number of local measurement settings that are required for an implementation. To use as few measurements as possible is important from several points of view: First, in multi-photon experiments the count rates (i.e., the number of successfully generated states per time unit) is often low. Therefore, the measurement of a single setting with a given accuracy needs a certain time, and only a restricted set of measurements can be done while keeping the setup stable. But even if measurements can be repeated fast, and a high rate of states is available (as it is in ion trap experiments), state tomography would require an exponentially increasing effort, making it practically impossible for ten or more qubits [8]. Finally, for experiments using hyper-entangled photon states, the switching of one measurement setting to the other requires the adjustment of an interferometer, which needs a considerable amount of time and work [234,235] (see also Section 6.7).

In the example above, three measurement settings suffice, which is a modest improvement compared with full state tomography. However, the difference becomes large, if the number of particles increases. For tomography of an N-qubit state, 3^N measurements are necessary, which amounts to an exponentially growing effort. For the measurement of the witness, typically only a few measurements are required. As we will see in Section 6.6, for important states, even for arbitrarily many qubits, only two measurements are required for the detection of entanglement and an estimation of the fidelity.

6.1.4. Further results for bipartite systems

Let us finally mention further results on the decomposition of witnesses for bipartite systems. As a first question, one may ask what happens if the witness is of the form $\mathcal{W} = |\psi\rangle\langle\psi|^{T_B}$, where

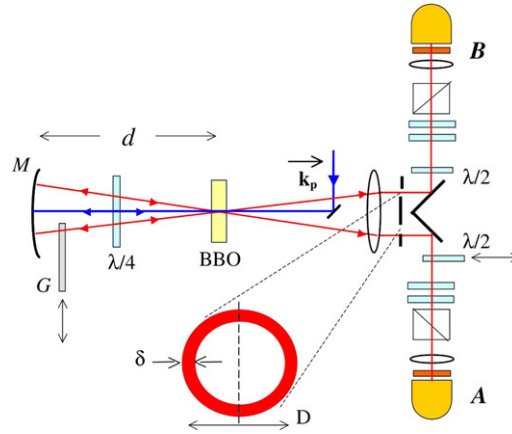$$|\psi\rangle = \sum_i \lambda_i|ii\rangle \tag{125}$$

Fig. 9. Layout of polarization entangled photon source. For details see the text. Figure taken from Ref. [385].

is a state on a $d \times d$-system. It has been shown in Ref. [148] that if $|\psi\rangle$ has a Schmidt rank of k, then at least $k+1$ measurements are necessary. Also, if k is even (odd), an explicit decomposition into $2k$ $(2k-1)$ measurements has been found. For the case that $|\psi\rangle$ is a projector onto a maximally entangled state in a $d \times d$-system and d is prime, it has been shown that $\mathcal{W}$ can indeed be decomposed into $d+1$ measurements [384], hence for this special case the problem of the optimal decomposition is solved.

A different interesting problem, which is somehow reverse to the original problem is the following: If we assume that some correlation measurements like $\sigma_x \otimes \sigma_x$, $\sigma_x \otimes \sigma_z$, $\sigma_z \otimes \sigma_x$ and $\sigma_x \otimes \sigma_x$ have been done, which states can then be detected with that data? This question is important in the context of quantum key distribution, and we will discuss it in Section 6.10.

6.2. Implementation of a two-qubit witness

In this section we describe the experiment by M. Barbieri et al. [385], which was one of the first implementations of entanglement witnesses.[32] In the experiment they created two-qubit Werner states (see Section 3.4.5) of photons using parametric down-conversion and linear optics (see Section 5.5).

The experimental setup can be seen in Fig. 9. The incoming photon with wavelength λ_P and V-polarization arrives from the top from the laser. Then, after a mirror, it propagates to the $-k_p$ direction towards the BBO crystal. Here with some probability it is converted down (type I) into two photons with $\lambda := \lambda_P/2$ and H polarization. The two photons must have total momentum $-k_p$ due to momentum conservation. Thus, if one of them is propagating in a direction slightly up, the other has to propagate into a direction slightly down. Both of these photons are reflected on the mirror M, go back through the crystal and a lens. Another beam of photon pairs arises from the photons that went through the crystal without down-conversion, got reflected on the mirror and then converted down on the BBO crystal when propagating from the left to right. These photons also arrive at the same lens. Due to the wave plate between the mirror M and the BBO crystal, the photon pair from the first down-conversion arrived in $|HH\rangle$ state to the lens, while the other arrives in the $|VV\rangle$ state. As the contributions are indistinguishable, finally the setup generates photons in the state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|HH\rangle + e^{i\phi}|VV\rangle). \tag{126}$$

There is a $\lambda/2$ wave plate inserted before detector A, which leads to a $|\Psi_{\text{singlet}}\rangle = \frac{1}{\sqrt{2}}(|HV\rangle - |VH\rangle)$ state.

It is also possible to "spoil" the entangled state Eq. (126) and create a mixture

$$\varrho(p) = p|\Psi_{\text{singlet}}\rangle\langle\Psi_{\text{singlet}}| + (1-p)\frac{\mathbb{1}}{4}. \tag{127}$$

This can be done using the coated glass plate G and the wave plate before the detector A. This partially spoils the indistinguishability and leads to the Werner state. For large values of p the state created is entangled, while for small values it is separable. The entanglement of the state is detected by the following witness operator [see Eq. (123)]

$$\mathcal{W} = \frac{\mathbb{1}}{2} - |\Psi_{\text{singlet}}\rangle\langle\Psi_{\text{singlet}}| = \frac{1}{4}\left(\mathbb{1} + \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z\right), \tag{128}$$

that can efficiently be measured locally. The experimental results are shown in Fig. 10. Note that the witness detects the states also for $p \leq 0.65$, where no Bell inequality can detect the entanglement of that state.

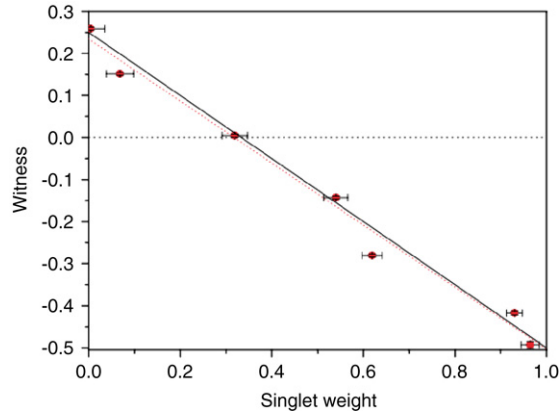[32] For a similar experiment see Ref. [386].

Fig. 10. Experimental results of the entanglement witness for Werner states: Theoretical curve (solid), experimental results (dots), and experimental best fit (dotted). The horizontal dashed line indicates the transition between separable and entangled states. Figure taken from Ref. [385].

6.3. Witnesses for multi-qubit systems

Let us now discuss witnesses for multipartite systems. Here, the general strategy is quite similar to the two-qubit case: The experimenter aims to prepare a state that is affected by some noise. The problem is to find a witness for this scenario that is robust against noise, and can be measured with a moderate effort. In this section we will mainly discuss the "standard" witnesses for genuine multipartite entanglement as described in Eq. (70). Different witness constructions will be discussed in the following sections.

6.3.1. Three-qubit GHZ state

Let us discuss first the three-qubit *GHZ* state $|GHZ_3\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$ as a simple example. The experimentally created state may be of the form

$$\varrho = (1 - p_{\text{noise}})|GHZ_3\rangle\langle GHZ_3| + p_{\text{noise}}\frac{\mathbb{1}}{8}. \tag{129}$$

As we already know from Eq. (73) the standard witness would read $\mathcal{W} = \mathbb{1}/2 - |GHZ_3\rangle\langle GHZ_3|$ and this detects the state for $p_{\text{noise}} < 4/7 \approx 0.57$.

For this witness, a decomposition into local measurements can be found as follows: first one decomposes $\mathcal{W}$ into tensor products of Pauli matrices, then, one optimizes this decomposition in order to minimize the measurement settings. So one can write [387]:

$$\mathcal{W} = \frac{1}{8}\big(3 \cdot \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} - \mathbb{1} \otimes \sigma_z \otimes \sigma_z - \sigma_z \otimes \mathbb{1} \otimes \sigma_z - \sigma_z \otimes \sigma_z \otimes \mathbb{1} - \sigma_x \otimes \sigma_x \otimes \sigma_x + \sigma_x \otimes \sigma_y \otimes \sigma_y$$

$$+ \sigma_y \otimes \sigma_y \otimes \sigma_x + \sigma_y \otimes \sigma_x \otimes \sigma_y\big).$$

$$= \frac{1}{8}\Big(3 \cdot \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} - \mathbb{1} \otimes \sigma_z \otimes \sigma_z - \sigma_z \otimes \mathbb{1} \otimes \sigma_z - \sigma_z \otimes \sigma_z \otimes \mathbb{1} - 2 \cdot \sigma_x^{\otimes 3}$$

$$+ \sqrt{2} \cdot \left(\frac{\sigma_x + \sigma_y}{\sqrt{2}}\right)^{\otimes 3} + \sqrt{2} \cdot \left(\frac{\sigma_x - \sigma_y}{\sqrt{2}}\right)^{\otimes 3}\Big). \tag{130}$$

This decomposition requires only the measurement of the four correlations $\sigma_z^{\otimes 3}, \sigma_x^{\otimes 3}$ and $(\sigma_x \pm \sigma_y/\sqrt{2})^{\otimes 3}$. These are simple measurements on the Bloch sphere that can be implemented experimentally. From the data, the fidelity of the *GHZ* state can also be determined, as in Eq. (124). It can be further shown that this decomposition is optimal, i.e., it is not possible, to determine $\langle \mathcal{W} \rangle$ with three local measurement settings [387,388].

6.3.2. General states

Let us discuss now general states. Using the presented methods one can calculate the witness and determine its robustness to noise straightforwardly for an arbitrary state $|\psi\rangle$. To obtain a good local decomposition requires often some effort, especially proving that a given decomposition is optimal, is often very difficult [388]. However, for many interesting states results on this problem are known, and we summarize them in Table 1. Interestingly, one can also make some general statements about the required measurement settings: For any pure state there exists a witness that requires $2N - 1$ measurements [389,390], but the robustness to noise may be small. Furthermore, there exist observables, for which the local decomposition requires $2 \cdot 3^{N-1}/(N + 1)$ local measurements, which means that a local measurement of these observables requires nearly the same effort as state tomography, but specific examples of such observables are not known [388].

Table 1
Results on local decompositions of different entanglement witnesses for different states.

# of qubits	State	Witness	Maximal p_{noise}	Local measurements	References	Remarks
3	$|GHZ_3\rangle$	$\frac{1}{2}\mathbb{1} - |GHZ_3\rangle\langle GHZ_3|$	4/7	4 (optimal)	[387]	a
3	$|W_3\rangle$	$\frac{2}{3}\mathbb{1} - |W_3\rangle\langle W_3|$	8/21	5 (optimal)	[387]	b
4	$|CL_4\rangle$	$\frac{1}{2}\mathbb{1} - |CL_4\rangle\langle CL_4|$	8/15	9 (optimal)	[388,391]	a
4	$|\Psi_2\rangle$	$\frac{3}{4}\mathbb{1} - |\Psi_2\rangle\langle \Psi_2|$	4/15	15	[145,390]	c
4	$|D_{2,4}\rangle$	$\frac{2}{3}\mathbb{1} - |D_{2,4}\rangle\langle D_{2,4}|$	16/45	21	[213]	d
N	$|GHZ_N\rangle$	$\frac{1}{2}\mathbb{1} - |GHZ_N\rangle\langle GHZ_N|$	$1/2 \cdot [1/(1-1/2^N)]$	$N+1$	[390]	a
N	$|W_N\rangle$	$\frac{N-1}{N}\mathbb{1} - |W_N\rangle\langle W_N|$	$1/N \cdot [1/(1-1/2^N)]$	$2N-1$	[8,390]	b
N	$|G_N\rangle$	$\frac{1}{2}\mathbb{1} - |G_N\rangle\langle G_N|$	$1/2 \cdot [1/(1-1/2^N)]$	Depends on the graph	[264]	a
N	$|D_{\frac{N}{2},N}\rangle$	$\frac{N}{2N-2}\mathbb{1} - |D_{\frac{N}{2},N}\rangle\langle D_{\frac{N}{2},N}|$	$1/2 \cdot (N-2)/[(N-1)(1-1/2^N)]$	Not known	[213]	

a Witnesses that tolerate less noise but require fewer settings exist. See Section 6.6.1.
b Witnesses that tolerate more noise with the same measurements exist. See Sections 6.8.2 and 8.2.
c Witnesses that tolerate more noise and require fewer settings exist [190,390].
d For witnesses that tolerate less noise fewer settings see Section 8.2, for witnesses which tolerate more noise see Ref. [190].

6.4. Comparing witnesses with Bell inequalities

Let us now compare the entanglement detection via Bell inequalities and entanglement witnesses in some more detail. This will also shed light on the underlying assumptions of different entanglement verification methods (for a detailed treatment on this see Ref. [21]).

When comparing different entanglement verification schemes, two main questions are of interest. First, one may ask for the power of the method in an experiment, i.e., to which extent it gives useful and relevant information about the state. Here, also the question for the experimental effort becomes relevant. A second main question concerns the underlying assumption for the scheme, and to which extent these assumptions are realistic in a given experimental situation.

6.4.1. Usefulness

Concerning the first question, we have seen in Section 5.2.3 that a violation of most of the Bell inequalities for multipartite systems only excludes full separability of the state. But, as seen in the same Section, there are some Bell inequalities that can be used for proving genuine multipartite entanglement. However, up to now such constructions work only for few states (like the *GHZ* states) and there is no construction known for other important states, such as cluster states. Furthermore, the existing Bell inequalities for the detection of genuine multipartite non-locality require a high fidelity of the target state and an exponentially growing number of measurement settings. In contrast to that, we will see in Section 6.6.1 that one can write down witnesses for the *GHZ* state that tolerate more noise and need only two measurement settings, independently from the number of qubits.

6.4.2. Underlying assumptions

A further interesting question concerns the underlying assumptions of the different entanglement detection methods. As the loopholes are in most experiments not closed, Bell inequalities require the assumption that the parties act like as if they were space-like separated and that the detector efficiencies do not play a role. Apart from that, however, no more assumptions are needed. Especially, the measurement directions do not have to be fixed, a misalignment of the measurement directions does not invalidate the conclusion that the state was entangled.

For entanglement witnesses, more assumptions are needed. First, loopholes can also arise for entanglement witnesses [392].[33] Then, the validity of quantum mechanics is assumed. Most importantly, entanglement witnesses require that the experimentalist knows what he is doing: The measurements have to be made in the proper direction and the measurement apparatuses must work as they are intended to work.[34]

A further question is whether the use of entanglement witnesses requires an assumption on the dimension of the underlying Hilbert space [395,396]. In many experiments one works with multi-level systems like ions in a trap, however, for the theoretical description one considers it as a two level system and disregards the levels that are not occupied. The question arises, whether this might lead to false detection of entanglement. To discuss this in a simple setting, consider two qubits and the witness from Eq. (122). A measurement of this witness consists of the determination of the outcome probabilities for the eigenvectors $|x^\pm\rangle\langle x^\pm|$, $|y^\pm\rangle\langle y^\pm|$, and $|z^\pm\rangle\langle z^\pm|$ for Alice and Bob. These probabilities are defined on an arbitrary dimensional Hilbert space, and if these probabilities violate the witness inequality, the corresponding reduced two-qubit state must be entangled. This implies entanglement of the total state.[35]

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[33] However, since quantum mechanics is assumed, these loopholes are less relevant as in the Bell inequality case.

[34] For some schemes to deal with non-ideal detectors see Refs. [393,394].

[35] To be precise, if ϱ_{tot} is the total state on a $d \times d$ system, violation of the inequality implies entanglement of the (not normalized) state $\varrho_{red} = \mathcal{P}_A^{(2)} \otimes \mathcal{P}_B^{(2)} \varrho_{tot} \mathcal{P}_A^{(2)} \otimes \mathcal{P}_B^{(2)}$ with $\mathcal{P}^{(2)} = |0\rangle\langle 0| + |1\rangle\langle 1|$. This implies entanglement of ϱ_{tot}, as the projection is local.

Finally, it should be noted that due to the finite number of copies available in any experiment, Bell inequalities and entanglement witnesses can only detect entanglement with a finite statistical significance. This is quantified by the standard deviation of the experimentally measured mean value.[36]

6.5. Implementation of a multi-qubit witness

Now we discuss an experiment by M. Bourennane et al. [145], which was the first experiment, where a multi-qubit witness has been implemented. In this experiment, the four-qubit singlet state $|\Psi_2\rangle$ [Eq. (61)] has been generated using the second order emission of parametric down-conversion and postselection as proposed in Ref. [237].

One possibility to generate entangled states of polarized photons in a laboratory uses type-II spontaneous parametric down-conversion (SPDC) [381]. In this process, an ultraviolet (UV) laser pumps a nonlinear β-Barium-Borate (BBO) crystal (see Fig. 11). Multiple emission events during one pump pulse lead to the emission of the state

$$|\psi\rangle \sim \exp[-i\alpha(a_{0,H}^\dagger b_{0,V}^\dagger + a_{0,V}^\dagger b_{0,H}^\dagger)]|0\rangle \tag{131}$$

distributed onto two modes, a_0 and b_0. Here, $a_{0,H}^\dagger$ denotes a creation operator, creating a horizontally polarized photon in mode a_0; $b_{0,V}^\dagger$ creates a vertically polarized photon in mode b_0 and $|0\rangle$ denotes the vacuum. In the second order of this process only the quartic terms are relevant, leading to $|\psi\rangle \sim (a_{0,H}^\dagger b_{0,V}^\dagger + a_{0,V}^\dagger b_{0,H}^\dagger)^2|0\rangle$. This results in the state

$$|\psi\rangle \sim |2H\rangle_{a_0}|2V\rangle_{b_0} + |2V\rangle_{a_0}|2H\rangle_{b_0} + 2|1H, 1V\rangle_{a_0}|1H, 1V\rangle_{b_0}, \tag{132}$$

where $|2H\rangle_{a_0}$ denotes a state of two horizontally polarized photons in mode a_0, and $|1H, 1V\rangle_{b_0}$ denotes one horizontally and one vertically polarized photon in mode b_0, etc.

Two beam-splitters are used to distribute the photons in the four modes a, b, c and d. Then postselection is used and one can directly calculate that fourfold coincidences in all the four modes a, b, c, and d can only occur when the state

$$|\psi\rangle \sim |H\rangle_a|H\rangle_b|V\rangle_c|V\rangle_d + |V\rangle_a|V\rangle_b|H\rangle_c|H\rangle_d + \frac{1}{2}\left(|V\rangle_a|H\rangle_b + |H\rangle_a|V\rangle_b\right)\left(|V\rangle_c|H\rangle_d + |H\rangle_c|V\rangle_d\right) \tag{133}$$

is produced. This state is, up to a sign which can be removed by a local change of the basis the four-qubit singlet state $|\Psi_2\rangle$. In the experiment, this state was created with a count rate of around five four-photon events per minute.

The witness for this state, is given by Eq. (76) as $\mathcal{W}^{\Psi_2} = \frac{3}{4}\mathbb{1} - |\Psi_2\rangle\langle\Psi_2|$. This witness is equivalent to a measurement of the fidelity and requires fifteen measurement settings. It should be noted, that in the meantime witnesses have been found that require significantly fewer measurements and tolerate even more noise [190,390]. The measurement results are given in Fig. 12. The fact that the coincidence probability results are similar for different settings is a signature of the $U \otimes U \otimes U \otimes U$-symmetry of the singlet state. From these data, one obtains

$$\langle \mathcal{W}^{\Psi_2}\rangle = -0.151 \pm 0.01 \quad \text{and} \quad F_{\Psi_2} = 0.901 \pm 0.01 \tag{134}$$

which clearly proves the four-partite entanglement. The state $|\Psi_2\rangle$ can then be further used to encode a qubit in a decoherence free subspace [244] or for secret sharing [243]. With a similar setup, three-qubit W states can be generated [216].

6.6. Stabilizer witnesses

In this section we will explain a method for constructing witnesses that are very convenient for experiments, since they can be implemented with very few, often only two measurements.

6.6.1. The main idea and GHZ states

The main idea of the stabilizer witnesses is the following: If an existing witness $\mathcal{W}$ requires many measurement settings, one can look for witnesses that might be less powerful, but easier to measure. The relationship between the old witness that is difficult to measure and the new one is demonstrated in Fig. 13.

How can such a new witness be constructed? Let us first consider a general observable $\widetilde{\mathcal{W}}$. If one can find an $\alpha > 0$ such that

$$\widetilde{\mathcal{W}} - \alpha\mathcal{W} \geq 0, \tag{135}$$

then $\text{Tr}(\varrho\widetilde{\mathcal{W}}) < 0$ implies $\text{Tr}(\varrho\mathcal{W}) < 0$.[37] Therefore any state $\widetilde{\mathcal{W}}$ detects is also detected by $\mathcal{W}$. Moreover, if we know that $\mathcal{W}$ detects states with genuine multipartite entanglement, so does $\widetilde{\mathcal{W}}$. The question remains, how $\widetilde{\mathcal{W}}$ can be found. It turns out that stabilizer theory is a well suited tool for this task, but it should be noted that it works also for non-stabilizer states.

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[36] For an interesting proposal to determine the probability of entanglement via Bayesian updating see Section 6.9.2

[37] The condition in Eq. (135) is indeed equivalent to the "finer" relation in the sense of Section 2.5.2, i.e., $\mathcal{W}$ is finer as $\widetilde{\mathcal{W}}$ iff a relation as in Eq. (135) holds.
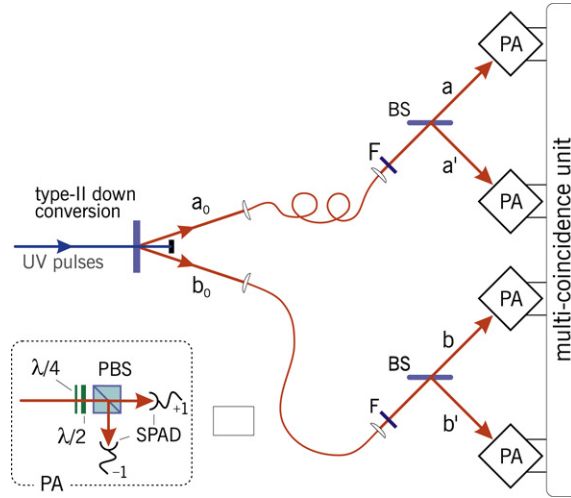
Fig. 11. Setup for the generation of the four-photon singlet state $|\Psi_2\rangle$. See the text for details. The figure is taken from Ref. [242].
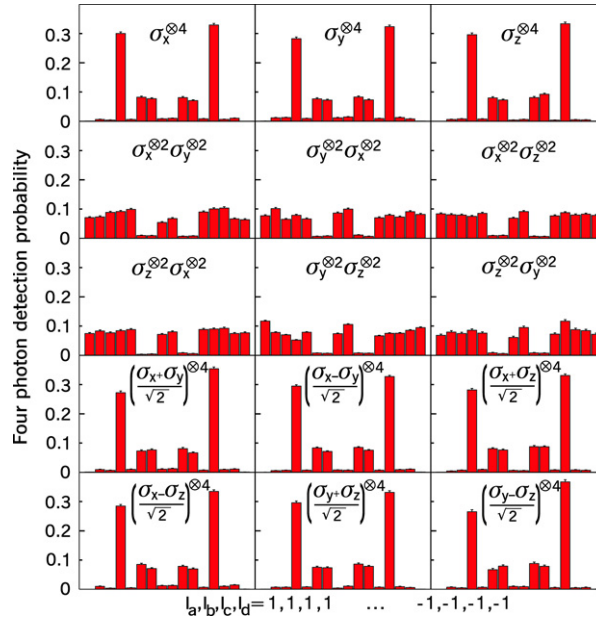

Fig. 12. Fourfold coincidence probabilities for the fifteen measurement settings, needed for the fidelity measurement of the $|\psi_2\rangle$-state. The figure is taken from Ref. [145].

Let us shortly recall some facts about it, see also Section 3.4.3. Stabilizing operators are locally measurable operators that have the state $|\Psi\rangle$ as an eigenstate with eigenvalue $+1$,

$$|\Psi\rangle = S_k|\Psi\rangle. \tag{136}$$

It is common to express Eq. (136) saying that operator S_k stabilizes the state $|\Psi\rangle$. If operators S_k and S_l stabilize $|\Psi\rangle$, then so does their product $S_k S_l$. It is easy to see that these stabilizing operators form a commutative group. This group was named stabilizer in Ref. [221]. If $|\Psi\rangle$ is an N-qubit quantum state then the group of stabilizing operators has 2^N elements. The group has N generators g_k from which all elements are possible products of the S_k. Finally, the projector to $|\Psi\rangle$ can be written as

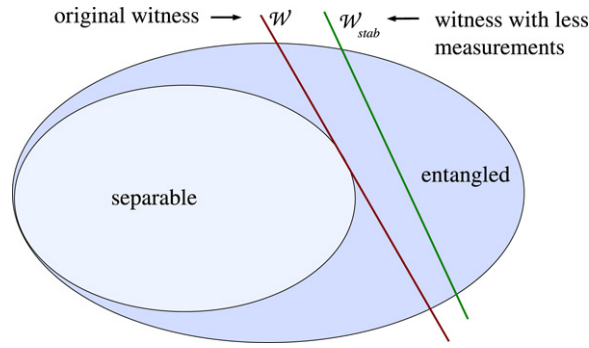$$|\Psi\rangle\langle\Psi| = \frac{1}{2^N}\sum_{k=1}^{2^N} S_k = \prod_{k=1}^{N}\frac{1+g_k}{2}. \tag{137}$$

Fig. 13. Schematic view of the construction of stabilizer witnesses: Instead of measuring the original witness $\mathcal{W}$, (which might be difficult) one considers the witness $\mathcal{W}_{stab}$. This witness detects fewer states, however, it is often much easier to measure than the original witness. The witness $\mathcal{W}_{stab}$ can be constructed from the stabilizing observables for the target state.
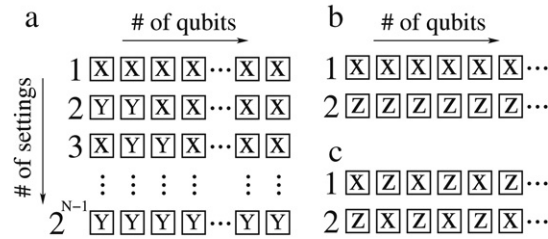

Fig. 14. Measurement settings needed for detecting genuine multi-qubit entanglement close to *GHZ* states with the Mermin inequality [see Eqs. (107) and (141)]. For each qubit the measured spin component is indicated. (b) Settings needed for the entanglement witnesses detecting entangled states close to *GHZ* states [Eq. (139)] and (c) cluster states [Eq. (143)].

Let us show a simple example and consider the witness operator (Eq. (73)) detecting genuine multipartite entanglement around a *GHZ* state. The elements of the stabilizer for the *GHZ* state are products of the generating operators

$$g_1^{(GHZ_N)} := \prod_{k=1}^{N} X_k, \qquad g_k^{(GHZ_N)} := Z_{k-1} Z_k \quad \text{for } k = 2, 3, \ldots, N. \tag{138}$$

Then, one considers the following observable:

$$\widetilde{\mathcal{W}}_{GHZ_N} := 3\mathbb{1} - 2\left[\frac{g_1^{(GHZ_N)} + \mathbb{1}}{2} + \prod_{k=2}^{N} \frac{g_k^{(GHZ_N)} + \mathbb{1}}{2} \right]. \tag{139}$$

Knowing that $\mathcal{W}_{GHZ_N} = 1/2 \cdot \mathbb{1} - |GHZ_N\rangle\langle GHZ_N|$ is a valid witness, one can verify Eq. (135) for $\alpha = 2$: The left-hand side of Eq. (135) is then diagonal in the *GHZ*-basis, and it suffices to prove that the diagonal elements are positive, which can be directly checked [264,397]. Therefore, $\widetilde{\mathcal{W}}_{GHZ_N}$ is a valid witness for the detection of genuine multipartite entanglement around the *GHZ* state. For the special case of three qubits, it reads $\widetilde{\mathcal{W}}_{GHZ_3} = \frac{3}{2}\mathbb{1} - X_1 X_2 X_3 - \frac{1}{2}(Z_1 Z_2 \mathbb{1}_3 + Z_1 \mathbb{1}_2 Z_3 + \mathbb{1}_1 Z_2 Z_3)$.

The first remarkable thing is that $\widetilde{\mathcal{W}}_{GHZ_N}$ requires only two measurement settings: The first term in the square bracket can be measured with a $\{X, X, X, \ldots, X\}$ setting, the second with a $\{Z, Z, Z, \ldots, Z\}$ setting. This is shown in Fig. 14(b). Second, the witness (Eq. (139)) is quite robust against noise. It detects the *GHZ* state as entangled if the fraction of the white noise fulfills

$$p_{\text{noise}} < \frac{1}{3 - 2^{2-N}}. \tag{140}$$

Note that for large N the right-hand side of Eq. (140) converges to $1/3$.[38] Therefore, *GHZ* states with a fixed amount of noise can always be detected with two measurement settings, *independently* from the number of qubits. This shows that entanglement detection does not become necessarily more complicated when increasing the number of qubits.

If more than two settings can be measured then criteria with a higher noise tolerance can be obtained. At this point it is interesting to show a connection between stabilizer witnesses and the Mermin inequality (Eq. (107)). Let us consider the following witness

$$\mathcal{W}_{\text{Mermin}} := 2^{N-2}\mathbb{1} - \sum_{\pi} X_1 X_2 X_3 X_4 \ldots X_N + \sum_{\pi} Y_1 Y_2 X_3 X_4 \ldots X_N - \sum_{\pi} Y_1 Y_2 Y_3 Y_4 X_5 \ldots X_N + \cdots - \cdots, \tag{141}$$

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[38] It can be shown, that the witness in Eq. (139) is optimal within a certain class of witnesses with two measurements [397].

where $\sum_\pi$ indicates that every term represents the sum of all of its permutations. This witness is just constructed from the Bell operator of the Mermin inequality given in Eq. (107). We used the bound for genuine multipartite entanglement for this inequality presented in Refs. [353,397]. Thus, $\mathcal{W}_{\text{Mermin}}$ detects genuine multipartite entanglement. Note that the witness is also a stabilizer witness since all the terms in Eq. (141) are stabilizing operators of the GHZ state. This witness detects the noisy GHZ state as entangled if $p_{\text{noise}} < 1/2$.

6.6.2. Linear cluster states and other graph states

Similar methods can be applied to create witnesses with few measurements for cluster states and in general, graph states, introduced in Section 3.4.3. The generators for the group of stabilizing operators of the cluster state are

$$g_1^{(C_N)} := X_1 Z_2, \qquad g_k^{(C_N)} := Z_{k-1} X_k Z_{k+1} \quad \text{for } k = 2, 3, \ldots, N-1, \qquad g_N^{(C_N)} := Z_{N-1} X_N. \tag{142}$$

For detecting genuine multipartite entanglement we have already presented a witness Eq. (139). Based on arguments similar to creating witnesses for the GHZ state, one can construct a witness that also detects genuine multipartite entanglement, however, it is easier to measure

$$\widetilde{\mathcal{W}}_{C_N} := 3\mathbb{1} - 2\left[\prod_{\text{even } k} \frac{g_k^{(C_N)} + \mathbb{1}}{2} + \prod_{\text{odd } k} \frac{g_k^{(C_N)} + \mathbb{1}}{2}\right]. \tag{143}$$

It detects the cluster state as entangled if

$$p_{\text{noise}} < \begin{cases} (4 - 4/2^{\frac{N}{2}})^{-1} & \text{for even } N, \\ [4 - 2(1/2^{\frac{N+1}{2}} + 1/2^{\frac{N-1}{2}})]^{-1} & \text{for odd } N. \end{cases} \tag{144}$$

For large N the limit is $p_{\text{noise}} = \frac{1}{4}$. The measurement settings needed are shown in Fig. 14(c).

The ideas presented for GHZ and cluster states can be straightforwardly generalized for graph states. This requires the notion of the colorability of a graph. A graph is called k-colorable, if there exist k disjoint subsets $M_1, \ldots, M_k$ of the set of vertices, such that for any j there are no edges between any pair of vertices in M_j. For instance, the star graph (see Fig. 4 in Section 3.4.3) and the cluster graph are two-colorable, while a three-vertex ring graph is three-colorable.

Using this notation, for a k-colorable graph state $|G_N\rangle$ the following witness detects genuine multi-partite entanglement.

$$\widetilde{\mathcal{W}}_{G_N} := 3\mathbb{1} - 2\left[\sum_{j=1}^{k}\left(\prod_{i \in M_j} \frac{g_i^{(G_N)} + \mathbb{1}}{2}\right)\right]. \tag{145}$$

This witness requires the measurement of k settings, since each of the terms in the sum can be measured with a single setting, namely with a setting measuring $\{X_i\}_{i \in M_j}$ and $\{Z_i\}_{i \notin M_j}$. The robustness to noise depends on k. It should be noted, however, that the notion of k-colorability is a property of the graph (and not the state) and if a graph state can be represented (up to local unitaries) by different graphs, the colorability properties of these graphs may be different [220].

Such witnesses have been used in many experiments. That is, the two-setting witness for cluster states (Eq. (143)) was used first in Ref. [232], later in Refs. [234,235]. In another experiment, six photon graph states were created in a photonic experiment and genuine multipartite entanglement was detected with a witness using six measurement settings [9]. The advantage of using six, rather than only two settings is the larger robustness to noise of the witness. We will discuss in Section 6.7 an experiment for the generation of the four-qubit cluster state in detail.

6.6.3. Non-stabilizer states

Similar ideas can also be used for the detection close to states that are not stabilizer states. For such states it is not possible to find 2^N operators such that they stabilize the state and also they are the tensor-products of single-qubit operators. An example for such a state is the three-qubit W-state $|W_3\rangle$. If nonlocal operators are also considered, however, the following operators, which are the sum of several operator products, stabilize this state:

$$g_1^{(W_3)} := \frac{1}{3}(Z_1 + 2Y_1 Y_2 Z_3 + 2X_1 Z_2 X_3), \qquad g_2^{(W_3)} := \frac{1}{3}(Z_2 + 2Z_1 Y_2 Y_3 + 2X_1 X_2 Z_3),$$

$$g_3^{(W_3)} := \frac{1}{3}(Z_3 + 2Y_1 Z_2 Y_3 + 2Z_1 X_2 X_3). \tag{146}$$

There is only a single state, the W-state, that gives $+1$ for all the three operators, as in the case of stabilizer states. These operators commute with each other. Multiplying these operators gives further operators that stabilize the W state thus these operators form an eight-element group. The $g_k^{(W_3)}$ operators are the generators of the group. However, now the choice of these eight stabilizing operators is not unique. A simple method for finding such operators is given in Ref. [397]. Using these operators, and the strategy from above, one can find now the witness [397]

$$\widetilde{\mathcal{W}}_W = \frac{11}{3}\mathbb{1} + 2Z_1 Z_2 Z_3 - \frac{1}{3}\sum_{k \neq l}(2X_k X_l + 2Y_k Y_l - Z_k Z_l), \tag{147}$$

which requires only three measurement settings. For another application of similar ideas see Ref. [398].

6.6.4. Fidelity estimation

Detecting genuine multipartite entanglement is closely related to measuring how close the experimental state is to a given multipartite quantum state. This can be seen noting that the projector-based entanglement witnesses $\mathcal{W}_{GHZ_N}$ and $\mathcal{W}_{C_N}$ of Eq. (73) were simply based on measuring the projector, that is, the fidelity $F = \langle\psi|\varrho_{\exp}|\psi\rangle$ with respect to a highly entangled quantum state $|\psi\rangle$. Thus, methods similar to the ones for detecting genuine multipartite entanglement with few measurements also work for estimating the fidelity with few measurements [391,397,399]. In particular, one can consider the relation

$$|GHZ_N\rangle\langle GHZ_N| \geq \left[\frac{g_1^{(GHZ_N)} + \mathbb{1}}{2} + \prod_{k=2}^{N}\frac{g_k^{(GHZ_N)} + \mathbb{1}}{2}\right] - \mathbb{1}. \tag{148}$$

which is essentially equivalent to Eq. (135) for the stabilizer witness for the *GHZ* state. This relation implies that measurement of the right-hand side of Eq. (148) can be used to give a lower bound on the fidelity of the *GHZ* state. Equivalently, the stabilizer witness allows one to give a lower bound on the fidelity via $F_{GHZ_N} \geq (1 - \langle\tilde{\mathcal{W}}_{GHZ_N}\rangle)/2$.

There are other ways to estimate the fidelity, similar to Eq. (148) [390,391]. To see the main idea, let us consider the example of a four qubit cluster state $|CL_4\rangle = (|0000\rangle + |1100\rangle + |0011\rangle - |1111\rangle)/2$, for which the usual fidelity-based witness is given by $\mathcal{W} = \mathbb{1}/2 - |CL_4\rangle\langle CL_4|$. Then, the observable

$$\tilde{\mathcal{W}} = \frac{1}{2}\mathbb{1} - |CL_4\rangle\langle CL_4| + |CL'_4\rangle\langle CL'_4|$$

$$= \frac{1}{2}[\mathbb{1} - |00\rangle\langle 00| \otimes (|00\rangle\langle 11| + |11\rangle\langle 00|) + (|00\rangle\langle 11| + |11\rangle\langle 00|) \otimes |11\rangle\langle 11|] \tag{149}$$

with $|CL'_4\rangle = (-|0000\rangle + |1100\rangle + |0011\rangle + |1111\rangle)/2$ is, due to Eq. (135), also an entanglement witness and allows one to estimate the fidelity via $F \geq 1/2 - \langle\tilde{\mathcal{W}}\rangle$. As the off-diagonal terms of $\tilde{\mathcal{W}}$ in the computational basis are of a simple form, $\tilde{\mathcal{W}}$ can be measured with four measurements only [391], which is significantly less than the nine measurements for $\mathcal{W}$. Similar ideas can be applied to other states [390] and have been used in several six-photon experiments [9,236].

6.6.5. Witnesses for full separability

Up to now, we considered stabilizer witnesses that detect only genuine multi-qubit entanglement. Typically, the noise in an experiment must be relatively low if we want to detect this form of entanglement. On the other hand, if we want to detect entanglement in general, including partial or biseparable entanglement, then typically in an experiment much higher noise is allowed.

The basic idea behind detecting entanglement with stabilizing operators is that a product state cannot give $+1$ for two stabilizing operators that do not commute locally [397]. Based on this idea, and after simple algebra, a witness detecting entanglement around an N-qubit *GHZ* state can be obtained as

$$\mathcal{W}_m^{(GHZ_N)} := \mathbb{1} - g_1^{(GHZ_N)} - g_m^{(GHZ_N)}, \tag{150}$$

where $m = 2, 3, \ldots, N$. The proof is based on the Cauchy-Schwarz inequality. Using this and $\langle X_i\rangle^2 + \langle Z_i\rangle^2 \leq 1$, we obtain for pure product states

$$\langle g_1^{(GHZ_N)}\rangle + \langle g_m^{(GHZ_N)}\rangle = \langle X_1\rangle\langle X_2\rangle\cdots\langle X_N\rangle + \langle Z_{m-1}\rangle\langle Z_m\rangle$$

$$\leq |\langle X_{m-1}\rangle| \cdot |\langle X_m\rangle| + |\langle Z_{m-1}\rangle| \cdot |\langle Z_m\rangle| \leq \sqrt{\langle X_{m-1}\rangle^2 + \langle Z_{m-1}\rangle^2}\sqrt{\langle X_m\rangle^2 + \langle Z_m\rangle^2} \leq 1. \tag{151}$$

By convexity, the bound is also valid for mixed separable states. This proof can be straightforwardly generalized for arbitrary two locally non-commuting elements of the stabilizer of any graph state. Witnesses can be constructed with more than two elements of the stabilizer as

$$\widehat{\mathcal{W}}_m^{(GHZ_N)} := \mathbb{1} - g_1^{(GHZ_N)} - g_m^{(GHZ_N)} - g_1^{(GHZ_N)}g_m^{(GHZ_N)} \tag{152}$$

for $m = 2, 3, \ldots, N$ that rule out full separability. The proof is similar to the proof of Eq. (150). These witnesses tolerate noise if $p_{\text{noise}} < \frac{1}{2}$ and $\frac{2}{3}$, respectively. As can be seen, more terms make it possible to have higher noise tolerance. One can even construct a witness from the Bell operator of the Mermin inequality

$$\mathcal{W}'_{\text{Mermin}} := 2^{\frac{N-2}{2}}\mathbb{1} - \sum_\pi X_1X_2X_3X_4\ldots X_N + \sum_\pi Y_1Y_2X_3X_4\ldots X_N - \sum_\pi Y_1Y_2Y_3Y_4\ldots X_N + \cdots - \cdots. \tag{153}$$

This differs from Eq. (141) in having a smaller constant term corresponding to the relaxed conditions for biseparable entanglement compared to genuine multipartite entanglement. The constant comes from the calculations presented in Refs. [190,265,267]. While the number of terms increases with the number of qubits for $\mathcal{W}'_{\text{Mermin}}$, the noise tolerance also increases.

For the cluster state similar witnesses are [397,400] $\mathcal{W}_k^{(C_N)} := \mathbb{1} - g_k^{(C_N)} - g_{k+1}^{(C_N)}$ for $k = 1, 2, \ldots, N-1$, and, with a better noise tolerance $\widehat{\mathcal{W}}_k^{(C_N)} := \mathbb{1} - g_k^{(C_N)} - g_{k+1}^{(C_N)} - g_k^{(C_N)}g_{k+1}^{(C_N)}$ for $k = 1, 2, \ldots, N-1$. Both witnesses involve only the qubits of a quadruplet and tolerate noise if $p_{\text{noise}} < \frac{1}{2}$ and $\frac{2}{3}$, respectively. It is also possible to obtain a series of witnesses for which the noise tolerance approaches 1 for increasing N [397].

6.6.6. Mixed stabilizer states

So far, entanglement conditions were presented that detect entanglement in the vicinity of pure multipartite states. It is also possible to use witnesses made out of stabilizing operators for detecting entanglement in the vicinity of mixed stabilizer states [397]. The general idea is considering witnesses of the from

$$\mathcal{W}_{\text{mixed}} := \text{const.} - \sum_{k=1}^{M} g_k, \tag{154}$$

where g_k are *some* of the generators of the stabilizer for some graph state. If $M = N$ then there is only a single quantum state giving the minimum for $\langle \mathcal{W}_{\text{mixed}} \rangle$. However, if $M < N$ then the minimum is degenerate and the states giving the minimum make up a subspace of dimension $(N - M + 1)$. Then, if one can determine the constant in Eq. (154) such that $\langle \mathcal{W}_{\text{mixed}} \rangle$ is positive on product states, yet it is still negative for some states, then $\mathcal{W}_{\text{mixed}}$ is a valid witness.

As an example, let us consider the biseparable (but not fully separable) mixed quantum state

$$\varrho_3 := \frac{1}{2} \left(|\xi^+\rangle\langle\xi^+| + |\xi^-\rangle\langle\xi^-| \right), \qquad |\xi^+\rangle := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \otimes |0\rangle, \qquad |\xi^-\rangle := \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \otimes |1\rangle, \tag{155}$$

which occurs if the fourth qubit of a cluster state is traced out. The states $|\xi^\pm\rangle$ are stabilizer states. Any mixture of states $|\xi^\pm\rangle$ is stabilized by the common elements of $\mathscr{S}^{(\xi^\pm)}$. The generators of the stabilizer of these mixed states are given by $\mathscr{S}^{(\varrho_3)} = \{Z_1 Z_2, X_1 X_2 X_3\}$. Based on that, the following witness can be constructed

$$\mathcal{W}_{\varrho_3} := \mathbb{1} - Z_1 Z_2 - X_1 X_2 X_3. \tag{156}$$

The constant term in Eq. (156) can be obtained via methods similar to the ones used for obtaining a bound for product states for the sum of two stabilizing operators used for Eqs. (150).

6.7. Experimental generation of cluster states

Now we describe an experiment from K. Chen et al. [235], where a four-qubit cluster state has been observed and a stabilizer witness has been used.[39] This experiment used hyper-entanglement of photons for the creation of the cluster state [401]. This means that several degrees of freedom of each photon are used to carry the entanglement. In the given experiment, each photon represented two qubits, one polarization qubit and one spatial qubit. This technique is becoming more and more important in multi-photon experiments as it allows one to extend the Hilbert space significantly [402–404] and also complete Bell state analysis becomes possible [405]. Quite recently, this technique has been used to realize a ten-qubit *GHZ* state with five photons [10].

The setup of the experiment is shown in Fig. 15. A UV laser pulse passes two times through two combined BBO crystals. This leads either to emission of $|H_A H_B\rangle + |V_A V_B\rangle$ in the upward direction (modes L_A, L_B) or to emission of $|H_A H_B\rangle - |V_A V_B\rangle$ in the downwards direction, i.e., modes R_A, R_B. Here, H and V denote horizontal and vertical polarization. If the photons overlap perfectly at the beamsplitters BS_1 and BS_2, the state will be a superposition

$$|\psi\rangle = \frac{1}{2} \left[(|H_A H_B\rangle + |V_A V_B\rangle)|L_A L_B\rangle + e^{i\phi}(|H_A H_B\rangle - |V_A V_B\rangle)|R_A R_B\rangle \right]. \tag{157}$$

By adjusting the concave mirror, $\phi = 0$ can be obtained. Then the state is equivalent to the cluster state $|\psi\rangle = (|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle)$, if the replacements $|H/V\rangle \mapsto |0/1\rangle$ and $|L/R\rangle \mapsto |0/1\rangle$ are done. Since only two photons are required, the count rate in the present experiment is around 1.2×10^4 per second, which is about four orders of magnitude higher than in experiments that produced the same cluster state using four photons [232,233].

In order to measure the state, the elements as in Fig. 15(b) have been used. A measurement without any beamsplitter as in (i) first reads out whether the photon is in L/R and also the polarization of the photon. Hence it can be used to measure $Z_{\text{spat}} X_{\text{pol}}$ at one photon. Using a beamsplitter as in (ii) leads to a Hadamard transformation on the spatial modes, such that $X_{\text{spat}} Z_{\text{pol}}$ can be measured. The problem is that such a setup is sensitive to small path length changes, hence it requires a significant amount of adjustment and is only stable for relatively short times.[40] Therefore, despite the high counting rate, an analysis of the state should require only a few measurements.

One can use the witness

$$\mathcal{W} = 2\mathbb{1} - \frac{1}{2}(X_1 X_2 \mathbb{1}_3 Z_4 + X_1 X_1 Z_3 \mathbb{1}_4 + \mathbb{1}_1 \mathbb{1}_2 Z_3 Z_4 + \mathbb{1}_1 Z_2 X_3 X_4 + Z_1 \mathbb{1}_2 X_3 X_4 + Z_1 Z_2 \mathbb{1}_3 \mathbb{1}_4) \tag{158}$$

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[39] At the same time, a similar experiment has been performed by Vallone et al. [234].

[40] Only recently, using a Sagnac-like interferometer, stable interferometers for this problem have been introduced [10].
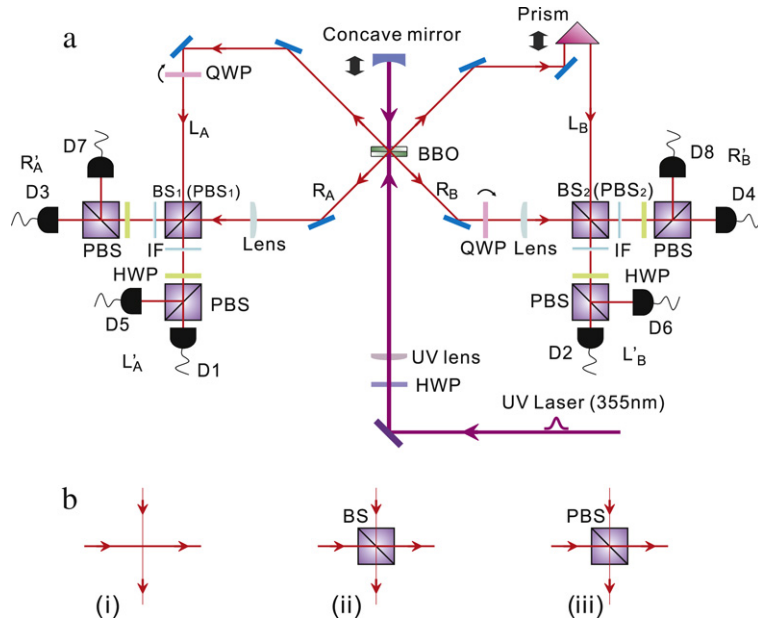
Fig. 15. (a) Setup for the generation of four-qubit cluster states using two photons and hyper-entanglement. (b) Required setups for the measurement of the polarization qubit and the spatial qubit of one photon. See the text for further details. Figure is taken from Ref. [235].

which is just the witness from Eq. (143) adapted to the form of the cluster state used here. This witness can be measured with the settings $X_1X_2Z_3Z_4$ and $Z_1Z_2X_3X_4$. This corresponds to the setups in Fig. 15b (i) and (ii). The experimental data give

$$\langle \mathcal{W} \rangle = -0.766 \pm 0.004 \quad \text{and} \quad F \geq \frac{1 - \langle \mathcal{W} \rangle}{2} = 0.883 \pm 0.002, \tag{159}$$

clearly confirming the four-qubit entanglement. Furthermore, this cluster state has then been used to demonstrate two-qubit gates.

The nature of multipartite entanglement in experiments with hyper-entangled states deserves some more comments. First, it should be stressed that due to the hyper-entanglement not all qubits are spatially separated, hence the entanglement refers to different degrees of freedom, and not different particles. Second, in multiphoton experiments as in Ref. [10] the process of postselection has a different effect now. Ideally, in any experiment, one aims to generate some entangled state ϱ which can then be distributed and used it as a resource for some task to be determined later. In usual postselection experiments, a state ϱ_{ini} is prepared, which contains different numbers of photons. Postselection results in a "state" ϱ_{ps}, which, however, is destroyed and cannot be stored. However, as usual postselection is a local projection, entanglement of ϱ_{ps} implies entanglement of ϱ_{ini}, which may therefore be considered as the (in principle) distributable and storable resource [21]. When hyper-entanglement is used, the postselection acts collectively on two qubits at the same time, and there is no direct connection between the multi-qubit entanglement of ϱ_{ps} and the one of ϱ_{ini}. For many applications, however, this does not matter.

Finally, it is interesting that if one wants to detect entanglement only between the different degrees of freedom of hyper-entangled particles, stabilizer witnesses can again be used [406].

6.8. Eight-qubit W states in ion traps

In this section we explain an experiment in which eight qubits have been entangled using trapped ions [8]. Apart from being the most advanced experiment to generate multi-particle entanglement in ion traps, it has also some interesting aspects from a theoretical point of view. First, a witness different from the projector-based one was used, second, the usage of local filters enabled the detection of entanglement even in a very noisy environment.

6.8.1. Description of the experiment

Let us first shortly describe the setup. Detailed overviews on quantum information processing with ion traps and their physical principles can be found in Refs. [36–39] and for a more detailed description of the present experiment see Refs. [39,407].

In the experiment strings of up to eight $^{40}\text{Ca}^+$ ions were trapped in a linear Paul trap. The $D_{5/2}$ ($= |0\rangle$) and the $S_{1/2}$ ($= |1\rangle$) levels of the ion are used as the basis states of the internal qubit [see Fig. 16(a)]. The ions can be individually addressed via laser pulses on the $|0\rangle \leftrightarrow |1\rangle$ transition.
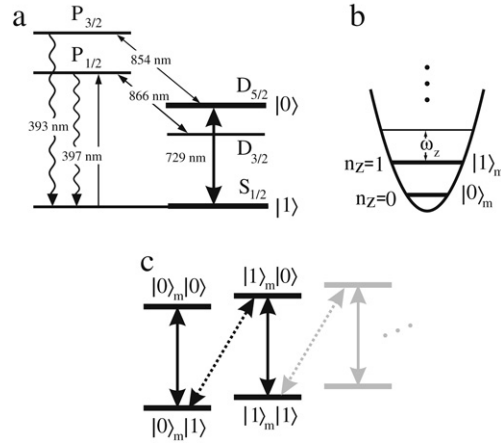
Fig. 16. (a) Level scheme of the ^{40}Ca$^+$ ion. The $D_{5/2}$ and the $S_{1/2}$ levels are used as the basis states of the qubit. (b) Schematic view of the motional levels describing the bus mode. (c) Combined level scheme of the motional qubit and the internal qubit. The figure is taken from Ref. [407].

For the interaction between the qubits, the vibrational excitations of the ion string are used. These excitations arise from a harmonic oscillator potential [see Fig. 16 (b)], and the two lowest levels $|0\rangle_m$ and $|1\rangle_m$ are referred to as the motional qubit. Via driving the $|0\rangle_m|1\rangle \leftrightarrow |1\rangle_m|0\rangle$ transition (see Fig. 16 (c)), each ion can individually interact with the motional qubit, allowing finally two-qubit gates [408–412].

The idea to prepare the N-qubit W states

$$|W_N\rangle = \frac{1}{\sqrt{N}}(|00\ldots01\rangle + |00\ldots10\rangle + \cdots + |10\ldots00\rangle) \tag{160}$$

can be sketched as follows. First, the state $|0\rangle_m|10\ldots0\rangle$ is created, where all but the first internal qubit are in the $|0\rangle$ state. Then, the motional qubit is partly excited and distributed over all the ions. That is, the state is changed according to the sequence

$$
\begin{aligned}
|0\rangle_m|10\ldots0\rangle &\mapsto \frac{1}{\sqrt{N}}|0\rangle_m|100\ldots0\rangle + \frac{N-1}{\sqrt{N}}|1\rangle_m|000\ldots0\rangle \\
&\mapsto \frac{1}{\sqrt{N}}|0\rangle_m|100\ldots0\rangle + \frac{1}{\sqrt{N}}|0\rangle_m|010\ldots0\rangle + \frac{N-2}{\sqrt{N}}|1\rangle_m|000\ldots0\rangle \\
&\mapsto \cdots \\
&\mapsto \frac{1}{\sqrt{N}}|0\rangle_m|100\ldots0\rangle + \frac{1}{\sqrt{N}}|0\rangle_m|010\ldots0\rangle + \cdots + \frac{1}{\sqrt{N}}|0\rangle_m|000\ldots1\rangle. \tag{161}
\end{aligned}
$$

Details on the required pulse sequences are given in Refs. [8,407]. This scheme allows to generate W states in a time of 500–1000 μs, for comparison, the life time of the $D_{5/2}$ level is $\tau = 1.16$ s. It should be noted that W states in ion traps can also be prepared in different ways [413].

In order to characterize the experimentally generated state ϱ_{exp}, state tomography has been performed. For that, all possible correlations of the type $\sigma_i \otimes \sigma_j \otimes \cdots \otimes \sigma_k$ have been measured. Repeating each correlation measurement at least a hundred times requires $100 \cdot 3^N$ repetitions of the state generation, for $N = 8$ qubits this amounts to a total measurement time of 10 h.

Due to the statistical nature of the measurement, the raw data from the correlation measurements do not result directly in a density matrix that is positive semidefinite. Therefore, using the results of Ref. [414] a maximum-likelihood approximation to the data has been performed. In Ref. [414], it was shown that the maximum-likelihood estimation is a fixed point of an iterative map, where the iteration depends on the measurement data. This fixed point was numerically determined by iterating the map, starting from a maximally mixed state until convergence was reached. Due to the exponential growing size of the density matrix this iteration requires a considerable effort, for the eight qubit case it requires more than one day. As a result, valid density matrices have been obtained, for the case $N = 4$ the absolute values of it are plotted in Fig. 17.[41] From these matrices, the fidelity of the W states can be calculated, see Table 2. In addition, given the estimated density matrix one can generate with a Monte Carlo simulation further data sets, which were used to calculate the error bars.

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[41] The numerical values of the density matrices are given in the online material of Ref. [8].
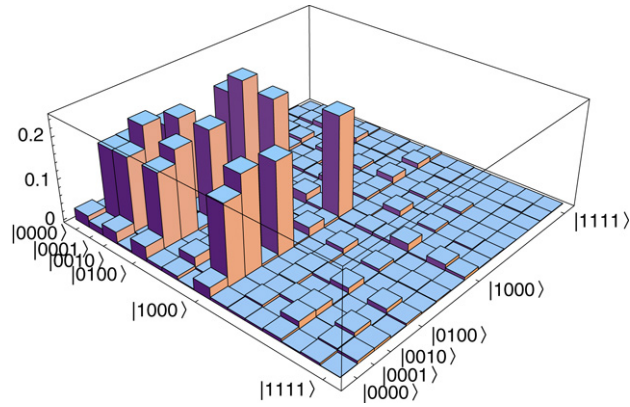
Fig. 17. Absolute values of the reconstructed density matrix for the four-qubit W state.

Table 2
Fidelities and mean values for the witnesses of the N-qubit W states for different values of N.

N	4	5	6	7	8
F	0.846 ± 0.011	0.759 ± 0.007	0.788 ± 0.005	0.763 ± 0.003	0.722 ± 0.001
$\langle \mathcal{W} \rangle$	-0.460 ± 0.031	-0.202 ± 0.027	-0.271 ± 0.031	-0.071 ± 0.032	-0.029 ± 0.008

6.8.2. Analysis of the states with witnesses and local filters

A first tool for the analysis of the states is the fidelity-based witness $\mathcal{W} = \frac{N-1}{N}\mathbb{1} - |W_N\rangle\langle W_N|$. However, as can be seen from the fidelities in Table 2, this witness does not detect any entanglement for $N \geq 5$. To improve this witness, one starts with the following observation: The projector onto the W state is a matrix that is acting on the subspace with one excitation ($= |1\rangle$) only. Therefore, if one calculates the overlap with a biseparable state $|\phi\rangle = |a\rangle|b\rangle$ it suffices to assume that $|a\rangle$ and $|b\rangle$ have maximally one excitation, as any contribution with two ore more $|1\rangle$ in $|a\rangle$ or $|b\rangle$ has a vanishing overlap with the state $|W_N\rangle$. From this consideration, one can directly see that

$$\mathcal{W} = \frac{N-1}{N}\mathcal{P}_{\leq 2} - |W_N\rangle\langle W_N|, \tag{162}$$

where $\mathcal{P}_{\leq 2}$ is the projector onto the space with maximally two excitations, is a valid witness.[42]

For larger N, the witness in Eq. (162) is already a significant improvement, as the difference between $\mathcal{P}_{\leq 2}$ and $\mathbb{1}$ increases exponentially. However, one can make the more general ansatz

$$\mathcal{W} = \alpha \mathcal{P}_0 + \beta \mathcal{P}_1 + \gamma \mathcal{P}_2 - |W_N\rangle\langle W_N|, \tag{163}$$

where the $\mathcal{P}_i$ are now projectors onto the subspaces with exactly i excitations. It remains to determine the allowed values for α, β and γ in order to guarantee that $\mathcal{W}$ is indeed non-negative on all biseparable states $|\phi\rangle = |a\rangle|b\rangle$. From the symmetry of the W state, one can conclude that it suffices to prove non-negativity for $|a\rangle$ of the form $|a\rangle = a_0|00...00\rangle + a_1(|00...01\rangle + \cdots + |10...00\rangle)$ and similarly for $|b\rangle$ [8]. Therefore, the resulting conditions on α, β and γ are given by simple optimization problems of a polynomial in four variables with two normalization constraints, which can numerically simply be solved for any N.

For given experimental data, one can choose then the tuples (α, β, γ) that yield the most negative value. In practice, the further condition $\alpha = \gamma$ was fixed.[43] In order to make the values comparable, all witnesses have been normalized such that $\text{Tr}[\mathcal{W}(\mathbb{1}/2^N)] = 1$, i.e., the mean value for the maximally mixed state equals one. The witness values in Table 2 for $4 \leq N \leq 7$ have been obtained in this way.

For the case $N = 8$, however, this construction still does not yield a witness with a negative expectation value. In order to prove the presence of entanglement, one can use the help of local filtering operations [170–172]. Such a local filter is a map

$$\varrho \mapsto \tilde{\varrho} = \mathcal{N} \cdot (\mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \cdots \otimes \mathcal{F}_8)\varrho(\mathcal{F}_1^\dagger \otimes \mathcal{F}_2^\dagger \otimes \cdots \otimes \mathcal{F}_8^\dagger), \tag{164}$$

where the $\mathcal{F}_i$ are arbitrary invertible 2×2 matrices and $\mathcal{N}$ denotes the normalization (see also Section 2.5.3). As the $\mathcal{F}_i$ are invertible, the state ϱ is entangled (or biseparable) if and only if $\tilde{\varrho}$ is entangled (or biseparable) and it suffices to detect the entanglement in the filtered state. Equivalently, one may also consider the class of witnesses

$$\tilde{\mathcal{W}} = \tilde{\mathcal{N}}(\mathcal{F}_1^\dagger \otimes \mathcal{F}_2^\dagger \otimes \cdots \otimes \mathcal{F}_8^\dagger)\mathcal{W}(\mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \cdots \otimes \mathcal{F}_8) \tag{165}$$

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[42] For example, for $N = 3$ we have $\mathcal{P}_{\leq 2} = \mathbb{1} - |111\rangle\langle 111|$.

[43] In Ref. [8] the witness was found in a different way, leading to a different notation. However, it is equivalent to the construction in Eq. (163) with $\alpha = \gamma$. For the experimental data of Ref. [8], a possible additional variation over α, γ would not yield a big improvement.

and try to find filters such that the mean value $\langle \tilde{W} \rangle$ gets negative. Here, one also has to renormalize the filtered witness with a factor $\tilde{\mathcal{N}}$ [e.g., by fixing $\text{Tr}(\tilde{W}) = \text{Tr}(W)$] in order to make the witnesses' values comparable.

The optimization over all filters $\mathcal{F}_i$ can be performed by first optimizing $\mathcal{F}_1$, then $\mathcal{F}_2$, etc., until a negative mean value can be found. With this technique, the mean value for the eight-qubit witness could be decreased significantly, and finally a negative value has been found.

It should be noted that the filter technique is not restricted to the case where complete information about the state via state tomography is available. As shown in Refs. [10,390] some witnesses that require only a few measurements can be improved by special filters that leave the required measurement settings invariant. Finally, it should be added that Ref. [415] gave an interesting scheme to implement witnesses as in Eqs. (74) and (162) in linear optics experiments (see also Ref. [416] and Section 7.3.1).

6.9. Estimation of entanglement measures

In the previous Sections we introduced methods for the detection of entanglement in experiments. More generally, one can ask how entanglement in experiments can be *quantified.* This leads to the problem, how to quantify entanglement, if only incomplete information about the state is given [417].

In this section, we will explain methods to quantify entanglement using entanglement witnesses. It should be noted that there are other proposals for the determination or estimation of entanglement measures in experiments, they are described in Sections 7.2 and 7.4. We will first describe a general method to derive a lower bound on a generic entanglement measure from the mean value of a witness [418–420]. Then we will discuss results for specific witnesses and entanglement measures.

6.9.1. A general method using the Legendre transform

Let us consider the following situation: In an experiment, an entanglement witness W has been measured, and the mean value $\langle W \rangle = \text{Tr}(\varrho W) = w$ has been found. The task is to derive from this single value a quantitative statement about the entanglement present in the quantum state. That is, one aims to derive statements like

$$\text{Tr}(\varrho W) = w \Rightarrow E(\varrho) \geq f(w), \tag{166}$$

where $E(\varrho)$ denotes some convex entanglement measure or another convex figure of merit, which one is interested in (e.g. the fidelity). Clearly, one aims to derive an optimal bound $f(w)$ and an estimate is optimal, if there is a state ϱ_0 with $\text{Tr}(\varrho_0 W) = w$ and $E(\varrho_0) = f(w)$.

In order to derive such lower bounds, one can use the Legendre transform [421,422] of E for the witness W, defined via the maximization

$$\hat{E}(W) = \sup_{\varrho}\{\text{Tr}(W\varrho) - E(\varrho)\}. \tag{167}$$

As this is defined as the maximum over all ϱ, we have for any fixed ϱ that $\hat{E}(W) \geq \text{Tr}(W\varrho) - E(\varrho)$, hence $E(\varrho) \geq \text{Tr}(W\varrho) - \hat{E}(W)$. The point of this rewriting is that the first term on the right hand side is the given measurement data, while the second term can be computed. Therefore, a measurable bound on $E(\varrho)$ has been obtained.

More generally, one can use the fact that $\text{Tr}(\varrho W) = w$ is equivalent to knowing $\text{Tr}(\varrho \lambda W) = \lambda w$ for any λ. Therefore, optimizing over all λ yields

$$E(\varrho) \geq \sup_{\lambda}\{\lambda \text{Tr}(W\varrho) - \hat{E}(\lambda W)\}. \tag{168}$$

This is already the optimal bound in Eq. (166). This follows from the geometrical interpretation, of the Legendre transform (see Fig. 18): For a given convex function $f(w)$ an affine lower bound $g(w) = \lambda w - \mu$ with a given fixed slope λ is optimal, if $\mu_{\text{opt}} = \sup_w[\lambda w - f(w)]$, which is exactly the Legendre transform. Such a lower bound is optimal for one w_0. To obtain the optimal bound for any given w_1 one optimizes over the slope λ, corresponding to Eq. (168). Since f is convex, we obtain for each w the tight linear bound.

Note that it is not needed that W be an entanglement witness, we may consider an arbitrary observable instead. Also one may consider a set of observables $\vec{W} = \{W_1, \ldots, W_n\}$ at the same time, by introducing a vector $\vec{\lambda} = \{\lambda_1, \ldots, \lambda_n\}$ and replacing λW by $\sum_k \lambda_k W_k$.

In any case, the main problem in this scheme lies in the calculation of the Legendre transform in Eq. (167). The difficulty of this task clearly depends on the witness W and on the measure $E(\varrho)$ chosen. For this problem, the following results have been obtained:

(i) For entanglement measures defined via the convex roof extension [see Eq. (83)] the optimization can be simplified to

$$\hat{E}(W) = \sup_{|\psi\rangle}\{\langle\psi|W|\psi\rangle - E(|\psi\rangle)\}, \tag{169}$$

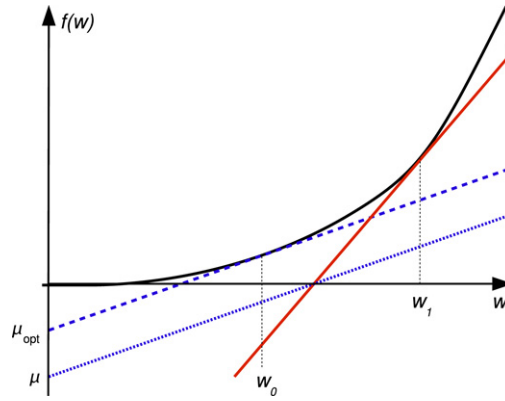i.e., it suffices to optimize over pure states only [418].

Fig. 18. Geometrical interpretation of the Legendre transform. The dotted line is a general affine lower bound, and the dashed line is an optimized bound with the minimal μ. By varying the slope, one finds an affine lower bound, which is tight for a given w_1 (solid line). This figure is taken from Ref. [420].

(ii) For the entanglement of formation [see Eq. (88)] one can design a simple iterative algorithm that can perform the calculation numerically [418]. For special witnesses of the type $\mathcal{W} = |\psi\rangle\langle\psi|^{T_B}$ the Legendre transform can be determined analytically [419]. Concerning the concurrence, one can find a similar algorithm as for the entanglement of formation [420].

(iii) For the geometric measure of entanglement, one can also construct an iterative algorithm for general witnesses. Further one can compute the Legendre transform for a witness like $\mathcal{W} = \alpha\mathbb{1} - |\psi\rangle\langle\psi|$ analytically [418]. From this one can show that if an experimentally generated state has a fidelity F with the target state $|\psi\rangle$, and the geometric measure of $|\psi\rangle$ is E, then for $F \geq 1 - E$ the geometric measure is bounded by[44]

$$E_G \geq 1 - F - E + 2EF + \frac{1}{2}\sqrt{\frac{E(E-1)}{F(F-1)}}[(2F-1)^2 - 1]. \tag{170}$$

For some other witnesses one can analytically estimate the Legendre transform [419,420].

6.9.2. Other relationships between witnesses and entanglement measures

In order to investigate the relationship between witnesses and entanglement measures from a different viewpoint, one can ask whether the negative expectation value can directly serve for an entanglement quantification. Therefore, one considers the quantity [423]

$$E(\varrho) = \max\{0, \max_{\mathcal{W}\in\mathcal{C}}[-\mathrm{Tr}(\varrho\mathcal{W})]\}, \tag{171}$$

where the second optimization is restricted over $\mathcal{C}$, which is a certain compact subset of all witnesses. This $E(\varrho)$ is by definition already convex.

Then, by choosing $\mathcal{C}$ appropriately, one can obtain different $E(\varrho)$ that can be used for the entanglement quantification: If $\mathcal{C}$ is the set of witnesses with $\mathcal{W} \leq \mathbb{1}$ then $E(\varrho)$ is the generalized robustness (see Section 4.2.5) and if $\mathcal{C}$ is the set of witnesses with $\mathrm{Tr}(\mathcal{W}) = d$, then $E(\varrho)$ is the random robustness [423,424]. Also the distance to the set of separable states in Hilbert–Schmidt norm, the negativity and the concurrence can be expressed in a similar way [151,423]. All these connections allow for a simple estimation of entanglement quantifiers in experiments, as any appropriately normalized witness gives a lower bound [425]. Experimentally, this has been used in Ref. [426] to probe the boundary of the set of separable states.

Let us add that witnesses as in Eqs. (26) and (27) can also be used to estimate the violation of the PPT or CCNR criterion, which can further be used to estimate the concurrence or the entanglement of formation as in Eq. (84) [277,278]. Also other special witnesses can be used: One can relate the violation of a CHSH inequality for two qubits to the entanglement of formation [427], use witnesses for the reduction criterion to estimate the concurrence [157] (see Section 7.2) and can also use the extended reduction map to estimate the concurrence [428]. For special cases one can also estimate the negativity from incomplete tomographical information [429,430]. A comparison of some of these methods can be found in Ref. [420].

Finally, a completely different approach tries to estimate entanglement from incomplete information via Bayesian updating. If one assumes some *a priori* probability distribution $p(\varrho)$ for the quantum states and measures an observable with the result d [with a priori probability $p(d) = \int_{\varrho} p(\varrho)p(d|\varrho)$], one can update the probability distribution via the Bayesian rule,

$$p(\varrho|d) = \frac{p(d|\varrho)p(\varrho)}{p(d)}. \tag{172}$$

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[44] Clearly, for $F \leq 1 - E$ the fidelity is compatible with a separable state, hence the only lower bound is the trivial bound $E \geq 0$.

This can then be iterated. Consequently, one can obtain for the negativity a probability distribution $p[N(\varrho)]$ from incomplete data [431]. Similar methods can be used for state tomography [432].

6.10. Entanglement witnesses in quantum key distribution

Let us finally discuss another application of entanglement witnesses, namely the security analysis of quantum key distribution (QKD) [433–436]. There are many different protocols for quantum key distribution. They can be divided into two classes: First, in *entanglement based* (EB) schemes an untrusted source distributes an entangled state ϱ, and Alice and Bob make measurements on it. From these measurement data they obtain the secret key, using classical post-processing of the data (e.g., key distillation or advantage distillation). An example is the Ekert protocol, where the source ideally distributes two-qubit singlet states $|\psi\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ [4].

In *prepare & measure* (P&M) schemes like the BB84 protocol [437–439], Alice prepares randomly states from a certain set $\{|\psi_i\rangle\}$ and sends them to Bob, who makes measurements. Then, using classical communication they obtain the secret key. At first sight, these two types of protocol do not seem to be closely related, but, as shown in Ref. [440] one can always view a P&M scheme as an EB scheme: The state preparation on Alice's side can be interpreted as if a bipartite entangled state

$$|\Psi\rangle = \sum_i |i\rangle_A |\psi_i\rangle_B \tag{173}$$

is prepared in Alice's lab, then one part is sent to Bob, and Alice measures in the computational basis $|i\rangle_A$. This is practically equivalent to an EB scheme, however, the possibilities of an eavesdropper are slightly more limited: In the usual EB scheme, Eve may have access to the source and can hence affect the complete ϱ, while in the P&M scheme Eve can only access the part sent to Bob, i.e., the reduced state ϱ_A of Alice is fixed.

What is the role of entanglement in these protocols? In the EB protocols, it is natural to ask whether the presence of entanglement is necessary for the possibility to create a secret key from the measurement data. Indeed, Ref. [441] showed the following result:

Theorem. *If the measurement data obtained in an EB scheme are compatible with a separable state, then no secret key can be extracted from that data and the QKD scheme is not secure. The same holds for P&M schemes, if one considers the corresponding EB scheme.*

Concerning the contrary implication, it has been shown in Ref. [442] that a state is entangled iff there exist measurements which result in probabilities which contain secret correlations. Hence, if Alice and Bob can verify entanglement, their correlations contain secrecy, however, as there might be bound information, they might not be able to distill a secret key.

Note that the theorem requires that the entanglement can be proven with the *measured* data. Since in many protocols Alice and Bob do not perform state tomography of the distributed state, this leads to the question, how one can prove the entanglement of a state, if only the expectation values of some observables are given.

It is not difficult to see that if mean values $\langle A_1 \rangle, \ldots, \langle A_k \rangle$ of some observables are given, then entanglement of the underlying state can be proven, if and only if there is a witness that is a linear combination of the A_i that detects it. This can be seen from the fact that in the space of the mean values of the A_i the separable states form again a convex set and this set is characterized by linear functions of the $\langle A_1 \rangle, \ldots, \langle A_k \rangle$. Therefore, the task remains to characterize the optimal witnesses for a given set of observables.

As an example let us consider the Ekert protocol. There, Alice and Bob measure randomly the polarization in the *x*- and *z*-direction. As they can communicate via classical communication, one can assume that the mean values of

$$\begin{aligned} A_1 &= \sigma_x \otimes \mathbb{1}; & A_2 &= \mathbb{1} \otimes \sigma_x; & A_3 &= \sigma_x \otimes \sigma_x; & A_4 &= \sigma_x \otimes \sigma_z; \\ A_5 &= \sigma_z \otimes \mathbb{1}; & A_6 &= \mathbb{1} \otimes \sigma_z; & A_7 &= \sigma_z \otimes \sigma_z; & A_8 &= \sigma_z \otimes \sigma_x; \end{aligned} \tag{174}$$

are known, while other correlations like $\sigma_y \otimes \sigma_x$ are unknown.[45] Using the decomposition of a witness into Pauli matrices, one can first prove that all entanglement witnesses $\mathcal{W}$ that can be measured with these observables fulfill $\mathcal{W} = \mathcal{W}^T = \mathcal{W}^{T_A} = \mathcal{W}^{T_B}$.

Then, it has been shown in Refs. [441,443] that the optimal witnesses for entanglement detection in this class are given by

$$\mathcal{W} = |\psi\rangle\langle\psi| + |\psi\rangle\langle\psi|^{T_B}, \tag{175}$$

where $|\psi\rangle$ is an entangled state with real coefficients. Therefore, it suffices to check these witnesses, and if they do not detect any entanglement, the QKD implementation is provably not secure.

Similar ideas have then be considered for other QKD protocols, often using the EVM matrix [see Eq. (20)]. This includes the B92 protocol [443], QKD with continuous variables [124,444] and entanglement detection using Stokes parameters [123].

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[45] For the P&M protocol, the reduced density matrix of Alice is fixed, as Eve cannot access it. Therefore, one can assume that also $\langle \sigma_y \otimes \mathbb{1} \rangle$ is known. This does, however, not affect the optimal witnesses obtained later. See Ref. [443] for a discussion.
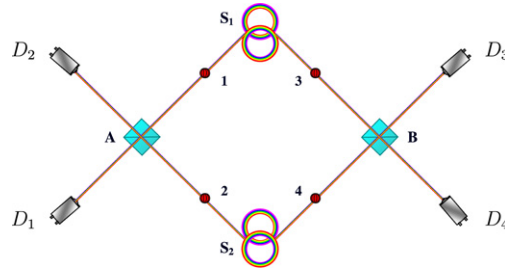
Fig. 19. Schematic view of the setup of the experiment in Ref. [446]. S_1 and S_2 denote the sources of the entangled photons, A and B two beamsplitters and $D_1, \ldots, D_4$ the four output ports. The figure is taken from Ref. [446].

7. Further methods of entanglement detection

In this section, we explain other methods for entanglement detection and quantification beside witnesses and Bell inequalities. We first discuss methods that have already been implemented in experiments, then we consider theoretical proposals that are still waiting for an implementation.

7.1. Entanglement criteria using entropies

Here, we describe an entanglement detection method that requires the measurement of some nonlinear properties of a quantum state. To start, let us recall the majorization criterion for separability [see Eq. (15)]. It states that for a separable bipartite state the global state is more disordered than the local (reduced) states. If we measure the disorder by the linear (or Tsallis) entropy $S(\varrho) = 1 - \text{Tr}(\varrho^2)$, this implies that for separable states

$$\text{Tr}(\varrho_A^2) \geq \text{Tr}(\varrho^2) \text{ and } \text{Tr}(\varrho_B^2) \geq \text{Tr}(\varrho^2) \tag{176}$$

has to hold [80,81,445].[46] For two qubits, it can be shown that these inequalities detect strictly more states than the CHSH inequality [445], however, there are some entangled states that are not detected by it.

From an experimental point of view, it is not obvious how Eq. (176) can be tested, as it is a nonlinear function of the quantum state. However, in Ref. [446] an elegant measurement scheme has been implemented that we will describe now.[47]

Let us first discuss the determination of $\text{Tr}(\varrho^2)$. This can be written on two copies of the state as $\text{Tr}(\varrho^2) = \text{Tr}(\varrho \otimes \varrho V)$, where $V = \sum_{ij} |ij\rangle \langle ji|$ is the swap operator between the two copies. Using the fact that $V = P_S - P_A = \mathbb{1} - 2P_A$, where P_S (P_A) is the projector onto the symmetric (antisymmetric) subspace of the two-copy system, we have $\text{Tr}(\varrho^2) = \text{Tr}[\varrho \otimes \varrho(P_S - P_A)]$. Decomposing P_S and P_A into local projectors onto the symmetric (or antisymmetric) subspaces of Alice's or Bob's local two-copy system, one can write it further as

$$\text{Tr}(\varrho^2) = \text{Tr}([\varrho \otimes \varrho]X) \quad \text{with } X = P_S^a \tilde{\otimes} P_S^b - P_A^a \tilde{\otimes} P_S^b - P_S^a \tilde{\otimes} P_A^b + P_A^a \tilde{\otimes} P_A^b, \tag{177}$$

where P_S^a denotes the projector onto the symmetric space of Alice's two copies, etc., $\otimes$ denotes the tensor product between the two copies, and $\tilde{\otimes}$ denotes the tensor product between Alice and Bob. Clearly, $\text{Tr}(\varrho_A^2) = \text{Tr}([\varrho \otimes \varrho][(P_S^a - P_A^a)\tilde{\otimes}\mathbb{1}])$ can be written in a similar fashion as an expectation value on a two-copy state.

For the experimental measurement of these quantities, two beamsplitters are required (see Fig. 19). Consider first the left beamsplitter (BS) A. If two identical photons 1 and 2 arrive at the BS the photon statistics of the BS will result in a coalescence of the photons, i.e., they will either be both in the output port D_1 or D_2 [32]. More precisely, if the two photons are in a symmetric polarization state, they will coalesce, while if they are in an antisymmetric polarization state, they will anti-coalesce and there will be one photon in each of the output ports. Therefore, the probability of coalescence equals the mean value of P_S^a, and the probability of anti-coalescence corresponds to P_A^a.

In the experiment, the coincidence probabilities of coalescence and anti-coalescence for Alice and Bob, p_{cc}, p_{ac}, p_{ca} and p_{aa} have been measured. In this language, the inequality $\text{Tr}(\varrho_A^2) \geq \text{Tr}(\varrho^2)$ is equivalent to $p_{ca} \geq p_{aa}$. Experimentally, $p_{ac} = 0.0255 \pm 0.008$ and $p_{aa} = 0.2585 \pm 0.008$ have been found, clearly violating the inequalities for separable states.

There are, however, some caveats with the interpretation of such data. First, in the generation of entangled photon pairs in the setup of Ref. [446] it happens that the source S_1 emits four photons, while S_2 emits none. This is not a rare event (it has the same probability as the case that each source emits two photons) and may alter the probabilities p_{cc}, p_{ac}, p_{ca} and p_{aa}. However, these events could be taken into account by varying the relative phase in the pump beams of the sources S_1 and S_2.

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[46] For generalizations of such inequalities see Refs. [84,85].

[47] For a general scheme to measure nonlinear properties without state tomography see Refs. [447,448].

A more fundamental problem lies in the usage of two copies of the state ϱ [449]. From a theoretical point there is no problem to write the separability conditions as inequalities for two copies. Experimentally, however, it is not clear that the two instances of the state really are the same and uncorrelated. This can not be verified without state tomography [21]. Especially, classical correlations between the two systems may lead to a systematic error in some entanglement detection schemes, see Refs. [449,450] for a discussion.

A scheme similar to the one described can also be used for the investigation of entanglement in optical lattices [451,452]. Namely, any fully separable state fulfills inequalities similar to Eq. (176), and also a transformation acting as a beamsplitter can, in principle be implemented. It should be noted, however, that the resulting scheme is unable to detect genuine multipartite entanglement, only full separability can be excluded.

7.2. Estimating the concurrence by using several copies

Similar ideas as in the previous subsection can be used to investigate the entanglement in experiments quantitatively, by giving lower bounds on the concurrence. To start, recall that the squared concurrence of a pure state is given by

$$C^2(|\psi\rangle) = 2[1 - \text{Tr}(\varrho_A^2)]. \tag{178}$$

In a similar way as in the previous section one can write therefore the concurrence as

$$C^2(|\psi\rangle) = 4\langle\psi|\otimes\langle\psi|[P_A^a\tilde{\otimes}\mathbb{1}]|\psi\rangle\otimes|\psi\rangle = 4\langle\psi|\otimes\langle\psi|[P_A^a\tilde{\otimes}P_A^b]|\psi\rangle\otimes|\psi\rangle, \tag{179}$$

where the second equality comes from the fact that $P_A^a\tilde{\otimes}P_S^b$ acts on the global antisymmetric subspace only, and its expectation value should therefore vanish, if the two copies of $|\psi\rangle$ are indeed identical [453].

Therefore, by measuring the (local) projector onto the antisymmetric subspace on Alice's part only, one can get a quantitative statement about the entanglement contained in the state. This has been implemented using hyper-entangled photons [453,454] and a proposal for the implementation using ion traps or atoms in cavities exists [455]. It should be noted, however, that this approach requires the assumption that the generated state is pure, which is difficult to guarantee in any experiment without doing state tomography [450].[48]

Interestingly, however, expressions similar to Eq. (179) could be used to obtain lower bounds on the concurrence for mixed states. As shown in Ref. [456], a lower bound on the concurrence can be obtained by

$$C(\rho)^2 \geq \text{Tr}(\rho\otimes\rho V) \quad \text{with } V = 4(P_A^a\tilde{\otimes}P_A^b - P_A^a\tilde{\otimes}P_S^b). \tag{180}$$

The difference to the previous expression in Eq. (179) is the sign of the term $P_A^a\tilde{\otimes}P_S^b$, which therefore, could be interpreted as taking account of the mixedness of the states. For an experimental implementation, assumptions about the measurements on two copies still have to be discussed [449,450]. In theory, however, Eq. (180) already allows to estimate the convex roof of the concurrence, and the lower bounds are reasonable for states that are close to pure states, for investigations of the tightness see Refs. [420,453,456,457]. Also upper bounds on the concurrence can be achieved from measurements on two copies [458].

Moreover, Eq. (180) can be generalized to

$$C(\rho)C(\sigma) \geq \text{Tr}(\rho\otimes\sigma V), \tag{181}$$

where ϱ and σ are different states [157]. This inequality has been experimentally investigated using photons [459]. In this experiment, two two-photon states ϱ and σ have been generated via down-conversion, then the nonlocal observable V has been measured using a controlled-phase gate between two of the four photons. Using $C(\varrho) \leq 1$ one obtains bounds on $C(\sigma)$ and vice versa, however, assumptions about the fidelity of the phase gate have to be discussed. Furthermore, Eq. (180) shows that

$$\mathcal{W} = -\frac{1}{C(\sigma)}\text{Tr}_2(\mathbb{1}\otimes\sigma V) = -\frac{2}{C(\sigma)}(\sigma - \mathbb{1}\otimes\sigma_B) \tag{182}$$

is a quantitative entanglement witness that can be used for the estimation of the concurrence. This witness corresponds to the reduction criterion [see Eq. (13) and the discussion after Eq. (26)], hence a lower bound on the concurrence can be obtained, iff the state violates this criterion.

Finally, note that one may define a multipartite generalization of the concurrence C_m as the sum of the bipartite concurrence over all bipartitions,

$$C_m^2(|\psi\rangle) = \frac{1}{2^{N-1}}\sum_{bp} C_{bp}^2(|\psi\rangle), \tag{183}$$

where the sum goes over all $2^N - 2$ bipartitions [460]. This quantity does not measure genuine multipartite correlations, but again expressions similar to Eqs. (179) and (180) can be derived, since bounds for each $C_{bp}^2(|\psi\rangle)$ are available from Eq. (180) [461,462].

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[48] Moreover, since one explicitly assumes that the generated state is pure, one could also just determine ϱ_A by state tomography on Alice's side, and then compute the concurrence or the entanglement of formation.

7.3. Variance based criteria and nonlinear witnesses

Entanglement witnesses and Bell inequalities can be viewed as inequalities for mean values of observables, where a violation indicates entanglement. Therefore, it is natural to ask whether it is possible to derive similar conditions using the variances of observables. These conditions would then also be nonlinear in the state ϱ, hence one may consider them as nonlinear witnesses. One one hand, it might be interesting to use nonlinear witnesses as an improvement of a given linear witness. On the other hand, nonlinear witnesses are also interesting from a geometrical point of view, as they approximate the convex set of separable states better.

In the literature, entanglement criteria using variances were first known for the case of continuous variables and infinite dimensional systems ([118,463–467], for reviews see Refs. [14–17]). For the case of discrete systems, the first variance based criteria have been derived in Refs. [255,468]. A very interesting approach was then proposed by Hofmann and Takeuchi [107]. These, so-called *local uncertainty relations* (LURs) turned later out to be a very interesting tool to investigate the separability problem, so we will explain them now.

7.3.1. Local uncertainty relations

First, recall that the variance of an observable M in the state ϱ is given by $(\Delta M)^2_\varrho := \left\langle (M - \langle M \rangle_\varrho)^2 \right\rangle_\varrho = \left\langle M^2 \right\rangle_\varrho - \langle M \rangle^2_\varrho$ and is concave in the state [107], i.e., if $\varrho = p\varrho_1 + (1 - p)\varrho_2$ is a convex combination of two states, then $(\Delta M)^2_\varrho \geq p(\Delta M)^2_{\varrho_1} + (1 - p)(\Delta M)^2_{\varrho_2}$. If $\varrho = |\psi\rangle\langle\psi|$ describes a pure state, the variance of M is zero iff $|\psi\rangle$ is an eigenstate of M.

Let us assume that we have observables A_i for Alice and B_i for Bob, so that the observables in each of these two sets do not share a common eigenstate. Then, there must be positive numbers U_A and U_B such that

$$\sum_{i=1}^n (\Delta A_i)^2 \geq U_A \quad \text{and} \quad \sum_{i=1}^n (\Delta B_i)^2 \geq U_B \tag{184}$$

holds, for all states on Alice's (resp. Bob's) system. Under this condition, it was shown in Ref. [107] that for separable states and for the observables $M_i := A_i \otimes \mathbb{1} + \mathbb{1} \otimes B_i$ the inequality

$$\sum_{i=1}^n (\Delta M_i)^2_\varrho \geq U_A + U_B \tag{185}$$

holds and violation of it implies entanglement. Physically, this may be interpreted as if the separable states inherit the uncertainty relations from the reduced states.

From an experimental point of view it is interesting, that the LURs can be viewed as nonlinear witnesses improving an optimal linear witness. To see this, let us consider an example [469]. For a single-qubit system is is easy to see that for the Pauli matrices the uncertainty relation $\sum_{i=x,y,z}(\Delta\sigma_i)^2 \geq 2$ holds. Defining $M_i = \sigma_i \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_i$ this yields the LUR $\sum_{i=x,y,z}(\Delta M_i)^2 \geq 4$. Writing down this explicitly, one obtains the condition

$$\left\langle \mathbb{1} \otimes \mathbb{1} + \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z \right\rangle - \frac{1}{2} \sum_{i=x,y,z} \langle \sigma_i \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_i \rangle^2 \geq 0, \tag{186}$$

which has to hold for all separable states. This is a quite remarkable inequality for the following reason: The first part, which is linear in the expectation values is known to be an *optimal* entanglement witness [see Eq. (123)]. From this witness some quadratic terms are subtracted. Note that an experimental measurement of these terms would require just the same measurements as the measurements that are required anyway for the measurement of the witness in Eq. (123). Thus, in this case, the LUR can be viewed as a nonlinear witness that improves an optimal linear witness.[49]

In general, the entanglement criteria resulting from the LURs are very strong: they can detect bound entanglement [470], indeed, it can be shown that they are stronger than the CCNR criterion [110].[50] Finally, it turned out, that they are equivalent to the covariance matrix criterion described above [109,114] (see Eq. (18)).

Also, statements similar to the LURs can be extended to nonlocal observables. For instance, if one takes an orthonormal basis of entangled vectors $|\psi_i\rangle$ (e.g. the *GHZ* basis) then there must be a $C > 0$ such that for all product states the bound

$$\sum_i (\Delta[|\psi_i\rangle\langle\psi_i|])^2_\varrho \geq C \tag{187}$$

holds [108,416,471]. The $|\psi_i\rangle$, however, are common eigenstates of all these observables, hence for them the sum of all variances vanishes and they are detected as entangled. This method has been worked out in Ref. [416] to detect the genuine

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[49] It is further interesting that for any pure product state $|\psi\rangle = |a\rangle|b\rangle$ equality holds in Eq. (186), which shows that the nonlinear witness is curved around the whole set of separable states.

[50] This can be easily seen from the fact that LURs always allow one to improve the witnesses for the CCNR criterion, see Eq. (27).

multipartite entanglement in four-qubit W states and has been applied to an experiment studying excitations in atomic ensembles.

Finally, methods like the LURs can be extended to other formulations of the uncertainty principle, like entropic uncertainty relations [471,472] or the Landau-Pollak uncertainty relations [473]. The local uncertainty relations and the variance based criteria can also be used to give lower bounds on entanglement measures, similarly to Section 6.9 [116,474–476]. Concerning an experimental implementation, some of the variance based criteria have been implemented in Ref. [477].

7.3.2. Nonlinear entanglement witnesses

The LURs show that in some cases it is possible to construct nonlinear witnesses. However, it is easy to see that the recipe of the LURs cannot be applied to arbitrary witnesses. Therefore, the question arises whether it is possible to find a general recipe to improve a witness by some nonlinear terms.

Indeed, as shown in Ref. [478], for bipartite witnesses this is the case. The recipe is the following: Let us consider the standard witness from the PPT criterion

$$\mathcal{W} = |\psi\rangle\langle\psi|^{T_B}. \tag{188}$$

One defines $X = |\psi\rangle\langle\phi|$ with an arbitrary $|\phi\rangle$ and considers the functional

$$\mathcal{F}(\varrho) = \langle\mathcal{W}\rangle - \frac{1}{s(\phi)}\langle X^{T_B}\rangle\langle(X^{T_B})^{\dagger}\rangle, \tag{189}$$

where $s(\phi)$ denotes the maximal squared Schmidt coefficient of $|\phi\rangle$, which is a bound on the maximal overlap between $|\phi\rangle$ and product states. Then one has for a pure product state $|\eta\rangle = |a\rangle|b\rangle$ that $\mathcal{F}(|\eta\rangle\langle\eta|) = |\langle\tilde{\eta}|\psi\rangle|^2[1 - |\langle\tilde{\eta}|\phi\rangle|^2/s(\phi)] \geq 0$, where $|\tilde{\eta}\rangle = |a\rangle|b^*\rangle$ is the vector corresponding to the projector $|\eta\rangle\langle\eta|^{T_B} = |\tilde{\eta}\rangle\langle\tilde{\eta}| = |ab^*\rangle\langle ab^*|$. By convexity, this positivity holds also for all separable states, proving that $\mathcal{F}(\varrho)$ is indeed a nonlinear entanglement witness, which improves the witness $\mathcal{W}$.

There are also some other constructions of nonlinear witnesses bases on the same idea [478,479] and due to the Choi–Jamiołkowski isomorphism [see Section 2.5.3 and Eq. (35)] all these methods can be extended to *arbitrary* bipartite witnesses. This also has some consequences for the geometry of separable states, as it proves that the set of separable states has no facets [479].[51] Also, one can define sequences of nonlinear witnesses by an iteration procedure, such that the nonlinear witnesses become stronger and stronger in each step and finally detect everything that is detected by the corresponding positive, but not completely positive map [122].

At the moment, there are not many results known concerning nonlinear witnesses for multipartite entanglement. The above method cannot be extended to witnesses for genuine multipartite entanglement. Some nonlinear witnesses for multipartite entanglement have already been proposed in 2002 by Uffink [255] and recently some criteria have been developed, which can also be viewed as some nonlinear conditions on the state (see Ref. [190] and Section 3.5).

7.4. Estimation of positive maps

Given the importance of positive maps for the theoretical characterization of entanglement, one may wonder whether it is possible to use them directly for entanglement detection. Indeed, several proposals for this task exist. Although their implementation is beyond current experimental capabilities, they offer interesting insights in measuring certain nonlinear properties of the density matrix without doing state tomography.

As the transposition is not a completely positive map, it is not possible to perform it directly in a lab. However, as shown in Ref. [481] one can circumvent this problem by a so-called structural physical approximation (SPA) of the transposition: if a sufficient amount of white noise is added, the total map becomes completely positive. Especially, it was shown in Ref. [481] that in a $d \times d$ system the map

$$\Lambda(\varrho) = \frac{d}{d^3 + 1}\mathbb{1}_A \otimes \mathbb{1}_B + \frac{1}{d^3 + 1}\varrho^{T_B} \tag{190}$$

is completely positive and can be implemented. SPAs of other positive maps have been discussed in Ref. [482]. The noise term $d/(d^3+1)\cdot\mathbb{1}_A\otimes\mathbb{1}_B$ only shifts the spectrum of the partial transposition ϱ^{T_B}, hence it remains to measure the eigenvalues of $\sigma = \Lambda(\varrho)$ to evaluate the PPT criterion.

To do so, it suffices to measure $\text{Tr}(\sigma), \text{Tr}(\sigma^2), \ldots, \text{Tr}(\sigma^{d^2})$, as from these d^2 quantities all d^2 eigenvalues can be determined [447,448,483]. The quantity $\text{Tr}(\sigma^k)$ can be determined similarly as in Section 7.1 by measuring a multiple of the shift operator on a k-fold copy. Alternatively, one may use the spectrum estimation technique from [484]. Note that knowing the spectrum of σ, one can then determine the negativity as an entanglement measure.

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[51] A face of a convex set is a hyperplane at the boundary, and a facet is a face of maximal dimension. For example, a three-dimensional cube has its six surfaces as facets, while the edges are faces, but not facets [480]. Therefore, the set of separable states is not as simple as in Fig. 1(b).

As shown in Ref. [485] one can determine the concurrence in a similar way: As explained in Section 4.2.3 the concurrence for two qubits can be computed from the eigenvalues of $\varrho\tilde{\varrho}$ with

$$\tilde{\varrho} = (\sigma_y \otimes \sigma_y)\varrho^{\mathrm{T}}(\sigma_y \otimes \sigma_y). \tag{191}$$

To determine $\mathrm{Tr}[(\varrho\tilde{\varrho})^k]$ requires $2k$ copies, where on half of them first the transposition is implemented via the SPA, and then the unitary transformation $(\sigma_y \otimes \sigma_y)$ is performed. Then, the eigenvalues of $\varrho\tilde{\varrho}$ can be determined similarly as before.

The previous proposals try to characterize entanglement by implementing an interesting positive map *physically*. However, as noted in Ref. [486] this is not mandatory. The k-th moments $\mathrm{Tr}[(\varrho^{\mathrm{T}_B})^k]$ are invariants under local unitary transformations, which can be measured using quantum networks [487]. They can also be measured directly by making measurements on k copies of the state [488,489]. Similar ideas can be used to determine the concurrence or the 3-tangle of mixed states [490] or to evaluate the separability criteria given by arbitrary positive, but not completely positive maps [491].

Finally, for the case of two qubits, similar methods can be used to decide separability using a single, nonlinear entanglement witness. As shown in Ref. [492], for two qubits one has that

$$\varrho \text{ is entangled } \Leftrightarrow det(\varrho^{\mathrm{T}_B}) < 0. \tag{192}$$

This equivalence reflects the fact that for two-qubits the partially transposed state can have maximally one negative eigenvalue [47]. The quantity $det(\varrho^{\mathrm{T}_B})$ is a fourth order polynomial of the matrix elements of ϱ and can hence be measured by a single witness on a fourfold copy of ϱ [492]. Similarly, for rotationally invariant states on a $2 \times d$-system, the PPT criterion is necessary and sufficient for separability [493], and if d is even, Eq. (192) can again be used for entanglement detection [494].

8. Entanglement detection with collective measurements

In the following section we discuss how to detect entanglement in a many-qubit physical system if the qubits cannot be individually accessed. For such systems entanglement criteria are needed that are based on the measurement of collective observables.[52] This includes spin squeezing inequalities as well as measurements of the magnetic susceptibility or the energy in spin models. We also discuss entanglement detection in optical lattices of cold atoms. In future, when larger and larger coherent quantum systems will be prepared, such criteria will surely gain large importance. It should be noted, however, that several criteria in this section do not detect genuine multipartite entanglement, they only rule out full separability.

8.1. Spin squeezing inequalities

In this section we describe entanglement detection with collective spin-$\frac{1}{2}$ observables, which is relevant to multi-qubit systems in which the qubits cannot be accessed individually. Even for systems in which the qubits can be accessed individually, such criteria can be advantageous since they need few measurements.

In a multi-qubit system, the quantities that can be measured collectively are the components of the collective angular momentum

$$J_l := \frac{1}{2} \sum_{k=1}^{N} \sigma_l^{(k)}, \tag{193}$$

where $l = x, y, z$ and $\sigma_l^{(k)}$ are the Pauli spin matrices, and their moments $\langle J_l^m \rangle$ where $m = 2, 3, 4, \ldots$. This includes also the variances $(\Delta J_k)^2 = \langle J_k^2 \rangle - \langle J_k \rangle^2$. Clearly, measuring only the expectation values of collective observables is not enough for entanglement detection: Such measurements do not give information on entanglement between the spins. The minimum requirement for detecting entanglement is measuring the first and second order moments of collective angular momenta [400].

8.1.1. Spin squeezing and entanglement

The first criterion based on such ideas was the spin squeezing criterion. On one hand, Ref. [495] defined spin squeezing in analogy with squeezing in quantum optics. Spin squeezing, according to the definition of Ref. [495], means the following: In general, the variances of the angular momentum components are bounded by the uncertainty relation

$$(\Delta J_y)^2(\Delta J_z)^2 \geq \frac{1}{4}|\langle J_x \rangle|^2. \tag{194}$$

If $(\Delta J_z)^2$ is smaller than the standard quantum limit $|\langle J_x \rangle|/2$ then the state is called spin squeezed.[53] In practice this means that a spin squeezed state has a large mean spin and a small variance in a direction orthogonal to the mean spin. On the other hand, in Refs. [496,497], spin squeezing was defined from the point of view of spectroscopy. States fulfilling this definition are useful for reducing spectroscopic noise or increasing the accuracy of atomic clocks.[54]

It has already been pointed out in Ref. [495] that the collective spin can be squeezed due to quantum correlations between the particles. Indeed, spin squeezing can be shown to be connected to entanglement: If an N-qubit state violates the inequality [500]

$$\frac{(\Delta J_z)^2}{\langle J_x \rangle^2 + \langle J_y \rangle^2} \geq \frac{1}{N}, \tag{195}$$

then the state is entangled, i.e., not fully separable. The degree of violation can also be used to characterize how strong the entanglement is [501]. It was found in Ref. [502] that if a symmetric state violates Eq. (195) then its two-qubit reduced density matrix is entangled, that is, the state is two-qubit entangled. However, in general, the violation of Eq. (195) does not indicate two-qubit entanglement, as one can find counterexamples [503].

A typical spin squeezing experiment starts with a fully polarized sample $|\Psi\rangle_{\text{init}} := |\frac{1}{2}\rangle_x^{\otimes N}$. Such a state saturates Eq. (195). During the entangling dynamics the large spin decreases only slightly, while the uncertainty in an orthogonal direction decreases considerably.[55] Thus, the resulting state violates Eq. (195). Entanglement can be generated by direct interaction between the particles, as in the proposal of Ref. [500]. In another scheme, the atomic ensemble interacts with light [504]. The light is then measured, projecting the atoms into an entangled state. For an ensemble strongly polarized into the x direction one has $\langle J_x \rangle \approx \frac{N}{2}$ and $\langle J_y \rangle \approx 0$. Thus, the entanglement condition (Eq. (195)) can be reformulated as $(\Delta J_z)^2 \geq \frac{N}{4}$.

8.1.2. Singlet criterion

After the discovery of the criterion in Eq. (195), it was realized that other generalized spin squeezing criteria can also be constructed that detect various types of entangled states and still need only collective measurements. Such an entanglement condition was presented in Ref. [400]. It contains the variances of all the three angular momentum components: For separable states

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq \frac{N}{2} \tag{196}$$

holds. States violating this condition have small angular momentum: Eq. (196) detects entanglement in the vicinity of many-qubit singlet states, which were considered in Section 3.4.4.[56] Such states give zero for the left-hand side of Eq. (196). One of such states is the chain of two-qubit singlet states. Another example is the ground state of the isotropic anti-ferromagnetic Heisenberg chain. Interestingly, all pure product states saturate the inequality. Eq. (196) has a good noise tolerance, it detects a singlet state mixed with white noise as entangled if $p_{\text{noise}} < \frac{2}{3}$. Again, violation of Eq. (196) does not imply two-qubit entanglement. Moreover, it can detect PPT entangled states (bound entangled states) in the thermal states of several spin models [138,503].

In addition, the violation of the criterion gives information about the number of spins that are unentangled with the rest [503,505]: Let us consider a pure state for which the first M qubits are not entangled with other qubits while the rest of the qubits are entangled with each other $|\Psi\rangle = (\otimes_{k=1}^{M} |\psi_k\rangle) \otimes |\psi\rangle_{M+1,...,N}$. For such a state, based on the theory of entanglement detection with uncertainties, one has [108]

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq \frac{M}{2}. \tag{197}$$

If a mixed state $\varrho := \sum_k p_k |\Psi_k\rangle\langle\Psi_k|$ violates Eq. (197) then at least one of the components $|\Psi_k\rangle\langle\Psi_k|$ must have M or more spins that are entangled with other spins. If the left-hand side of Eq. (197) is smaller than $\frac{1}{2}$ then the state cannot be created by mixing states that have one or more unentangled spins.

8.1.3. Inequalities detecting entanglement close to Dicke states

A third type of inequality for entanglement detection appeared in Ref. [213]: For separable states

$$\langle J_x^2 \rangle + \langle J_y^2 \rangle \leq \frac{N}{2} \left(\frac{N}{2} + \frac{1}{2} \right) \tag{198}$$

---

[53] Note that for Eq. (194) we assume that the mean spin points to the x direction. Without this requirement, even a pure state in which all spins point into the same direction could violate Eq. (194) [495].

[54] The definitions of spin squeezing in Ref. [495] and in Refs. [496,497] are slightly different. For a discussion about that, see Section IV.A in Ref. [498]. For an analysis concerning the two-qubit case, see Ref. [499].

[55] For an exhaustive survey of the topic and a list of references, see the excellent review in Refs. [498].

[56] Note that the criterion in Eq. (196) can straightforwardly be generalized for N spin-j particles. In this case on the right-hand side there must be Nj [400].

holds. This inequality detects entanglement close to the N-qubit Dicke state with two excitations. It detects such a state as entangled if for the added white noise $p_{\text{noise}} < \frac{1}{N}$. Thus, the robustness to noise decreases with N. Moreover, the violation of this inequality implies the presence of two-qubit entanglement. This can be seen as follows: The average two-qubit density matrix is

$$\varrho_{\text{av2}} := \frac{1}{N(N-1)} \sum_{k \neq l} \varrho_{kl}. \tag{199}$$

Here ϱ_{kl} represents the two-qubit reduced density matrix obtained from the original one after tracing out all qubits but qubits k and l. Eq. (198) can be re-expressed with ϱ_{av2} as

$$\langle \sigma_k^{(1)} \sigma_k^{(2)} + \sigma_l^{(1)} \sigma_l^{(2)} \rangle_{\varrho_{\text{av2}}} \leq 1. \tag{200}$$

Eq. (200) can be violated only if ϱ_{av2} is entangled [see Eq. (151)].

Criteria, similar to Eq. (198) can also be used to detect genuine multi-qubit entanglement. Such criteria for $N = 3$ and 4 qubits, respectively, are

$$\langle J_x^2 \rangle + \langle J_y^2 \rangle \leq 2 + \frac{\sqrt{5}}{2} \approx 3.12 \quad \text{for } N = 3, \tag{201}$$

$$\langle J_x^2 \rangle + \langle J_y^2 \rangle \leq \frac{7}{2} + \sqrt{3} \approx 5.23 \quad \text{for } N = 4. \tag{202}$$

Entanglement was detected based on Eq. (198) around a four-qubit Dicke state with two excitations in a four-qubit photonic system [215]. Although in such a system the qubits are individually accessible, using such a criterion was still advantageous since only two measurement settings (i.e., the $\{\sigma_x, \sigma_x, \sigma_x, \ldots, \sigma_x\}$ and $\{\sigma_y, \sigma_y, \sigma_y, \ldots, \sigma_y\}$ settings) were needed for the detection. The experiment will be explained in Section 8.2.

8.1.4. Detecting two- and three-qubit entanglement

Finally, in Refs. [407,506,507] spin squeezing criteria were constructed detecting two-qubit entanglement of the reduced two-qubit state. It was proven that if

$$\left[\langle J_x^2 \rangle + \frac{N(N-2)}{4} \right]^2 \geq \left[\langle J_y^2 \rangle + \langle J_z^2 \rangle - \frac{N}{2} \right]^2 + (N-1)^2 \langle J_x \rangle^2 \tag{203}$$

is violated then the state is two-qubit entangled. This criterion can be simplified to

$$1 - \frac{4\langle J_z \rangle^2}{N^2} \geq \frac{4(\Delta J_z)^2}{N} \tag{204}$$

for symmetric states. A symmetric state is two-qubit entangled iff it violates Eq. (204) for some choice of the z-axis.[57] Moreover, criteria detecting genuine three-qubit entanglement of the W and the GHZ classes, respectively, were also presented that need the measurement of third order moments of the collective angular momenta [407]. The previous ideas have already been applied to investigate the data of a ion trap experiment (see Section 6.8) [407]. Such criteria are especially applicable to W states, as W states maximize two-qubit concurrence among symmetric states [176,184].

8.1.5. Optimal spin squeezing inequalities

After presenting several generalized spin squeezing inequalities the questions arises: can one find a unifying framework for all these inequalities? In Ref. [138] this has been done. It turned out, that the first and second order moments of collective angular momenta for fully separable states fulfill simple constraints, namely

$$\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle \leq \frac{N(N+2)}{4}, \tag{205a}$$

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq \frac{N}{2}, \tag{205b}$$

$$\langle J_k^2 \rangle + \langle J_l^2 \rangle - \frac{N}{2} \leq (N-1)(\Delta J_m)^2, \tag{205c}$$

$$(N-1) \left[(\Delta J_k)^2 + (\Delta J_l)^2 \right] \geq \langle J_m^2 \rangle + \frac{N(N-2)}{4}. \tag{205d}$$

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[57] Ref. [508] conjectures a formula with collective observables for the two-qubit concurrence of symmetric states.
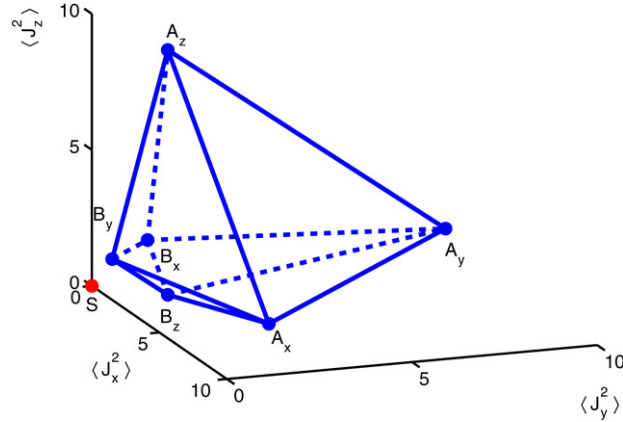
Fig. 20. The polytope of separable states corresponding to Eq. (205) for $N = 6$ and for $\vec{J} = 0$. The point S corresponds to a many body singlet state. The Figure is taken from Ref. [138].

While Eq. (205a) is valid for all quantum states, Eqs. (205b)–(205d) can be violated. Any state violating them is entangled. It was also proven, that this set of generalized spin squeezing inequalities is complete: For certain cases (e.g., $\langle J_x \rangle = \langle J_y \rangle = \langle J_z \rangle = 0$ and N is even) or for large N, a state that does not violate Eqs. (205b)–(205d) cannot be detected by any other entanglement condition based only on the expectation values of the first and second moments of J_k.

For any value of $\vec{J} := (\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle)$ these eight inequalities define a polytope in the three-dimensional ($\langle J_x^2 \rangle, \langle J_y^2 \rangle, \langle J_z^2 \rangle$)-space. Separable states lie inside this polytope. If one of the inequalities is violated, then the state is on the outside and is hence entangled. For the case $\vec{J} = 0$ and $N = 6$ the polytope is depicted in Fig. 20. Such a polytope is completely characterized by its extremal points. They are given by

$$A_x := \left[\frac{N^2}{4} - \kappa(\langle J_y \rangle^2 + \langle J_z \rangle^2), \frac{N}{4} + \kappa \langle J_y \rangle^2, \frac{N}{4} + \kappa \langle J_z \rangle^2 \right],$$

$$B_x := \left[\langle J_x \rangle^2 + \frac{\langle J_y \rangle^2 + \langle J_z \rangle^2}{N}, \frac{N}{4} + \kappa \langle J_y \rangle^2, \frac{N}{4} + \kappa \langle J_z \rangle^2 \right], \tag{206}$$

where $\kappa := (N-1)/N$. The points $A_{y/z}$ and $B_{y/z}$ can be obtained from these by permuting the coordinates.

Interestingly, violation of these spin squeezing inequalities does not necessarily imply entanglement of the reduced two-qubit states. As all the expectation values of first and second moments of J_k can be evaluated knowing the average two-qubit reduced density matrix ϱ_{av2} defined in Eq. (199), one could naively expect that these criteria detect the entanglement of the reduced two-qubit density matrix. However, somehow surprisingly these criteria are also able to detect entangled states for which the reduced states are not entangled.

This phenomenon is related to the representability problem [509], i.e., to the problem of finding multipartite quantum states that have a given set of states as reduced states. For a given ϱ_{av2}, one has to answer the question whether there is a separable N-qubit state that has ϱ_{av2} as the average two-qubit reduced state? If the answer is no then the system is entangled. Interestingly, it is also possible that ϱ_{av2} is separable, however, there is not an N-qubit separable quantum state that has it as a reduced state. In this case we can conclude that the system is an entangled state even if ϱ_{av2} is separable. If ϱ_{av2} is separable and of the form

$$\varrho_{av2} = \sum_k p_k \varrho_k \otimes \varrho_k \tag{207}$$

then there always can be found an N-qubit separable state that has this state as its reduced state $\varrho_N = \sum_k p_k \varrho_l^{\otimes N}$. Note that separable states in the symmetric subspace can always be written in the form of Eq. (207). On the other hand, if ϱ_{av2} is not symmetric then it is possible that it is separable, however, there is not an N-qubit symmetric separable state that has it as its reduced two-qubit state. One can write down the condition in Eq. (205b) with the operator expectation values of ϱ_{av2} as

$$\langle \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z \rangle_{\varrho_{av2}} \geq \frac{N}{N-1} \sum_{k=x,y,z} \langle \sigma_k \otimes \mathbb{1} \rangle_{\varrho_{av2}}^2 - \frac{1}{N-1}. \tag{208}$$

If this condition is violated then Eq. (205b) detects the state as entangled. However, Eq. (208) is not a condition for detecting the entanglement of ϱ_{av2}. That is, there are separable ϱ_{av2} that violate it.[58]

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[58] For example, for the $T = 0$ ground state of the Hamiltonian $H = J_x^2 + J_y^2 + J_z^2$ a simple calculation shows that $\langle \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z \rangle_{\varrho_{av2}} = -\frac{3}{4(N+1)}$ and $\langle \sigma_k \otimes \mathbb{1} \rangle_{\varrho_{av2}} = 0$, but ϱ_{av2} can be shown to be unentangled for $N \geq 4$ for even N [505].
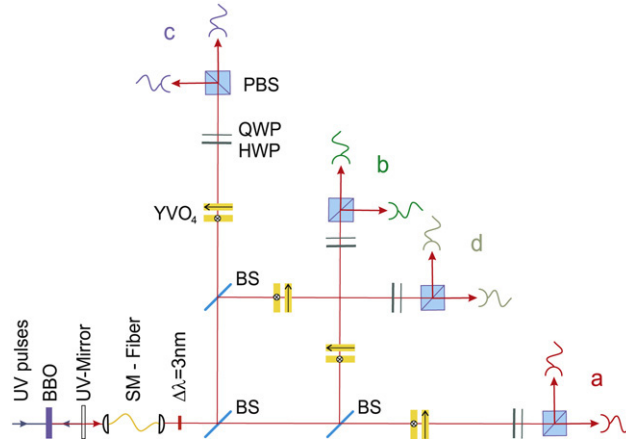
Fig. 21. Experimental setup for the analysis of the four-photon polarization-entangled state $|D_{2,4}\rangle$. The state is detected after the symmetric distribution of four photons onto the spatial modes a, b, c and d via non-polarizing beam splitters (BS). The Figure is taken from Ref. [215].

8.2. Experimental observation of a four-qubit symmetric Dicke state

In this section we explain an experiment aiming to create a four-photon symmetric Dicke state with two excitations, $|D_{2,4}\rangle$ in a photonic system [215]. On one hand, the experiment is interesting since the state $|D_{2,4}\rangle$ is highly entangled while the experimental setup is quite simple. On the other hand, the generalized spin squeezing criteria can be used for the detection of entanglement with few measurement settings.

The Dicke state $|D_{2,4}\rangle$ has the form

$$|D_{2,4}\rangle = \frac{1}{\sqrt{6}}(|HHVV\rangle + |HVHV\rangle + |VHHV\rangle + |HVVH\rangle + |VHVH\rangle + |VVHH\rangle),$$

where $|H\rangle$ and $|V\rangle$ denote linear horizontal (H) and vertical (V) polarization of a photon in the four spatial modes, denoted by a, b, c, and d. This is the symmetrization of a product state with two H and two V photons.

The setup can be seen in Fig. 21. The four photons are created by the second order emission of collinear type-II parametric down-conversion. The BBO crystal is used for the down-conversion, while the UV-mirrors separate the down-converted photons from the pump beam. As a result, in the single-mode fiber four photons appear, two in state $|V\rangle$, the other two in state $|H\rangle$. Now, the four photons must be distributed into four spatial modes. This is done by beam splitters. Birefringent effects of these non-polarizing beam splitters is compensated by pairs of perpendicularly oriented YVO$_4$ crystals in all the four modes. Note that the Dicke state can also be prepared with a different setup, which flexibly allows one to create an entire family of multi-photon states [217].

In the experiment, first the density matrix was determined by measuring all correlations of the type $\langle \sigma_k \otimes \sigma_l \otimes \sigma_m \otimes \sigma_n \rangle$ where $k, l, m, n \in \{0, x, y, z\}$. For that $3^4 = 81$ measurement settings are needed. This makes it possible to compute the fidelity of the prepared state: $F_{\text{exp}} = 0.844 \pm 0.008$. Note that the fidelity is high since the generation of the state was quite straightforward with parametric down-conversion and post-selection and no additional gates were needed. In comparison, for creating a cluster state, it is necessary to realize a phase gate. The larger complexity of the quantum circuit results in a decrease of the fidelity of the prepared state [232].

The fidelity was higher than the bound for biseparable states, 0.6 [see the witness Eq. (75)]. Thus, the fidelity computed from the density matrix already signals the presence of genuine multipartite entanglement. Instead of complete tomography, 21 settings would also be enough to get the fidelity and detect genuine multipartite entanglement [215]. However, in the experiment a more efficient criterion, namely Eq. (202) was also tested, which needed only two settings. For the left-hand side of Eq. (202) a value of 5.58 ± 0.02 was obtained that clearly demonstrated genuine multipartite entanglement.

In the second part of the experiment, one of the photons was measured. This leads to various symmetric states, in particular states of the type

$$|\psi\rangle = \alpha|W_3\rangle + \beta|\overline{W}_3\rangle, \tag{209}$$

where $|W_3\rangle$ is the three-qubit W state and $|\overline{W}_3\rangle$ is a state obtained from $|W_3\rangle$ by inverting all the qubits. In the experiment, the criterion (Eq. (201)) was used to detect the genuine multipartite entanglement of these states. After the loss of one of the photons, from the $|D_{2,4}\rangle$ state we get a mixture of $|W_3\rangle$ and $|\overline{W}_3\rangle$ states. This state is also genuine multipartite entangled, that indicates the persistency of the entanglement of the $|D_{2,4}\rangle$ state. Criterion Eq. (201) was then used to detect entanglement of this state.
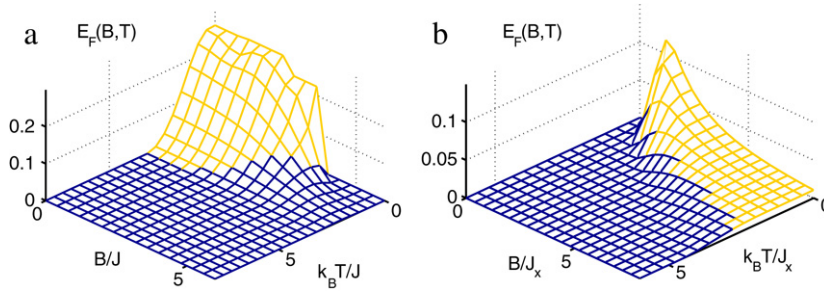
Fig. 22. (a) Heisenberg chain of 8 spins. Nearest-neighbor entanglement (quantified by the entanglement of formation) as a function of magnetic field B and temperature T. (b) The same for an Ising spin chain. Here k_B is the Boltzmann constant, J and J_x are coupling constants. Light color indicates the region where entanglement is detected by Hamiltonian as a witness. The Figure is taken from Ref. [511].

8.3. The Hamiltonian as a witness

In this and the following section we discuss how entanglement can be detected by measuring fundamental quantities of a physical system, e.g., the energy of a spin chain [505,510–512]. This allows one to connect thermodynamical quantities of condensed matter systems to their entanglement properties [26,513]. We will also discuss how these ideas can be extended to detect multipartite entanglement and how other fundamental observables beside energy can also be used. It should be noted, however, that in these approaches the conclusion that a given state was entangled very often depends on the assumption that the considered spin model indeed describes the physical system.

There are two basic approaches in entanglement detection: (i) One would like to detect entanglement in the vicinity of a given quantum state. Then some entanglement condition, for instance an entanglement witness, is designed for this aim. (ii) It is also possible to proceed in the opposite way and look for a fundamental observable of a physical system that is easy to measure. Then one can examine what quantum states can easily be detected by such an observable. When following the second path, one constructs witness operators of the form

$$\mathcal{W}_O := O - \inf_{\Psi \in S}[\langle \Psi | O | \Psi \rangle], \tag{210}$$

where S is the set of separable states, "inf" denotes infimum, and O is a fundamental quantum operator of a spin system that is easy to measure. Note that in the general case $\inf_{\Psi \in S}\langle \Psi | O | \Psi \rangle$ is difficult, if not impossible, to compute.

A natural observable for the second approach is the Hamiltonian of the system. First of all, it is typically easy to measure. In some systems it can be measured directly. In other systems the expectation value of the Hamiltonian can be measured indirectly by two-body correlation measurements since the Hamiltonian is typically the sum of few two-body correlation terms. Second, it turns out that it is simple to find the minimum for separable states for many spin model Hamiltonians. This makes it possible to investigate entanglement in systems in thermal equilibrium, which appear in many areas of physics. In thermal equilibrium the state of the system is given by

$$\varrho_T \propto \exp(-H/k_B T), \tag{211}$$

where T is the temperature and k_B is the Boltzmann constant. For simplicity we will set $k_B = 1$. Using Eq. (210) a temperature bound, T_E, can be found such that when $T < T_E$ then the system is shown to be entangled.

To give a simple example, consider the one-dimensional Heisenberg Hamiltonian on N spins with periodic boundary conditions,

$$H_H = \sum_k \sigma_x^{(k)} \sigma_x^{(k+1)} + \sigma_y^{(k)} \sigma_y^{(k+1)} + \sigma_z^{(k)} \sigma_z^{(k+1)}. \tag{212}$$

With an argument as in Eq. (151) one can directly see that for a fully separable state $\langle H_H \rangle \geq -N$ has to hold. As the ground state energy for large N for the Heisenberg model equals $E_0 = -[4\ln(2) - 1]N \approx -1.773N$ the thermal state at low temperatures (namely $k_B T \leq 3.18$) has to be entangled [193].

Similar ideas have been applied for the detection of entanglement in various spin models such as the XY-, Heisenberg- and Ising spin lattices with an external field in different various dimensions [192,193,505,510,511,514–518]. An example is given in Fig. 22. The figures show the entanglement of formation for the reduced two-qubit state as a function of the external field and the temperature. Light color indicates that the energy of the system is smaller than the minimum for separable states thus the Hamiltonian as a witness detects it as entangled. Note that if the entanglement is not very small, the Hamiltonian as a witness detects the thermal state as entangled. Thus, in such systems measurement of a single observable, i.e., the energy, seems to be sufficient for entanglement detection. Moreover, for systems in thermal equilibrium it is possible to find witnesses that are different from the Hamiltonian and detect all entangled thermal states [512].

Beside entanglement in general, one can also use similar ideas for detecting various forms of multipartite entanglement. As explained in Section 3.3 [Eq. (53)] one can ask whether multipartite entanglement is necessary to form a given state [190,192,193]. Then, a state $|\psi\rangle$ contains only k-party entanglement (or is k-producible) if one can write the state $|\psi\rangle$ as a

tensor product $|\psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \otimes \cdots \otimes |\phi_m\rangle$, where the states $|\phi_i\rangle$ are states on maximally k-qubits. In this definition, a two-producible state does not contain any multipartite entanglement, since it suffices to generate the two-qubit states $|\phi_i\rangle$ to arrive at the state $|\psi\rangle$. In addition, one says that a state contains genuine k-party entanglement if it is not producible by $(k-1)$-party entanglement. This definition can be extended to mixed states via convex combinations.

For spin chains of *macroscopic* size, it is in general very difficult to prove that the total state is genuine N-partite entangled via energy measurements. This is due to the fact that the notion of genuine N-partite entanglement is extremely sensitive to the properties of a single qubit. Indeed, in order to prove genuine multipartite entanglement, one has to exclude the possibility, that one single qubit can be separated from the remaining $N-1$ qubits. However, multipartite entanglement in the *reduced* states of small numbers of qubits can easily be detected. Moreover, if the reduced state is multipartite entangled then the state is not two-producible. Based on these ideas, in Refs. [192,193,519,520] energy bounds for two-, three- and four-producibility are presented for various spin models such as the Heisenberg- and the XY model, and also for two-dimensional models with frustration.

8.4. Entanglement detection with susceptibility measurements

Finally, let us consider the magnetic susceptibility as another thermodynamic quantity. In particular, the generalized spin squeezing inequality Eq. (196) can be reformulated using susceptibility measurements [521]. Let us consider a system with a Hamiltonian H and add a magnetic interaction $H_I := \sum_{l=x,y,z} B_l J_l$ to it, where $\vec{B}$ is the strength of the magnetic field. It has been shown in Ref. [521] that if we define the magnetic susceptibility along the directions $l = x, y, z$ as

$$\chi_l := \left(\frac{\partial \langle J_l \rangle}{\partial B_l} \right)\Big|_{\vec{B}=0}, \tag{213}$$

and if H commutes with J_k for $k = x, y, z$ then for separable states

$$\chi_x + \chi_y + \chi_z \geq \frac{N}{2kT}. \tag{214}$$

Here it is assumed that the system is in thermal equilibrium. As one can directly check [using $\chi_l = \frac{1}{kT}(\Delta J_l)^2$] Eq. (214) is equivalent to Eq. (196), giving the spin squeezing inequality a new physical interpretation.

There have been experiments for detecting entanglement with susceptibility measurements in solid state systems. In Ref. [522], it is explained that the magnetic susceptibility of copper nitrate measured in 1963 shows the presence of entanglement. Recently, experiments in low-dimensional spin systems have been carried out with pyroborate $MgMnB_2O_5$ and the warwickite $MgTiOBO_3$, systems with spin 5/2 and 1/2, respectively [523].

Eq. (214) has also been used for looking for entanglement in small molecular spin-clusters theoretically. Ref. [524] determined the bound temperature below which the system is entangled. The four spins interact via nearest-neighbor, diagonal, and four-spin Heisenberg interactions. Ref. [525] considered thermal entanglement in the nanotubular system $Na_2V_3O_7$ and determined also the bound temperature for entanglement.

8.5. Entanglement detection in optical lattices

Optical lattices of cold atoms are important examples for large-scale quantum information processing.[59] In particular, in a three-dimensional lattice of 10^5 two-state bosonic atoms an array of one-dimensional cluster states has been generated [13]. The entangling-disentangling dynamics, which created the cluster state and then restored the initial product state, was observed through the appearance and disappearance of the interference fringes. In this experiment the lattice sites could not be addressed individually, thus entanglement detection in such systems is a relevant example of entanglement detection with collective measurements.

The simplest model of such a one-dimensional array of trapped two-state bosonic atoms is the following. Let a_k and b_k be the destruction operators corresponding to the two species at site k. (Here, species "a" and "b" mean atoms in states $|a\rangle$ and $|b\rangle$, respectively.) A typical Hamiltonian for this system is [528]

$$H_{\text{lattice}} := -J_a \sum_k \left(a_k a_{k+1}^\dagger + a_k^\dagger a_{k+1} \right) - J_b \sum_k \left(b_k b_{k+1}^\dagger + b_k^\dagger b_{k+1} \right) - \Omega \sum_k \left(a_k b_k^\dagger + a_k^\dagger b_k \right)$$

$$+ \frac{1}{2} U_{aa} a_k^\dagger a_k^\dagger a_k a_k + \frac{1}{2} U_{bb} b_k^\dagger b_k^\dagger b_k b_k + U_{ab} a_k a_k^\dagger b_k b_k^\dagger, \tag{215}$$

where J_k are the tunnel couplings between the sites, Ω is the Rabi frequency for the transition $|a\rangle \to |b\rangle$, and U_k are the on-site interactions between atoms. The term $a_k a_{k+1}^\dagger + a_k^\dagger a_{k+1}$ denotes the tunneling of an atom in state $|a\rangle$ between sites k and $k+1$. Moreover, $a_k b_k^\dagger + a_k^\dagger b_k$ denotes a process in which an atom at site k turns from state $|a\rangle$ to state $|b\rangle$ and vice

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[59] For a review on quantum information processing in optical lattices of cold atoms see Refs. [526,527].

versa. This can be the result of manipulating atoms with a laser pulse. The sign of the on-site interactions U_a and U_b between atoms of the same specie control whether atoms favor to be at the same site or they tend to be distributed equally over the lattice. The sign of U_{ab} determines whether atoms with different internal states favor being at the same site.

In the realization of Ref. [13], the atoms cannot be individually addressed. In the experiment, after producing the desired state, one lets the cloud expand to many times its original size. Atoms with a large momentum can get further from their original position than atoms with a small momentum. Thus, from the distribution of the particles after the expansion, one can obtain information on the distribution in momentum space before the expansion. Knowing the distribution in the momentum space, one can obtain the distribution in real space by a Fourier transform. This way, in principle, one can obtain for the prepared state such quantities as

$$\langle Q_x^a \rangle := \sum_m \langle a_m^\dagger a_{m+x} \rangle \quad \text{and} \quad \langle Q_x^{ab} \rangle := \sum_m \langle a_m^\dagger a_{m+x} b_{m+x}^\dagger b_m \rangle. \tag{216}$$

In one of the realizations of quantum information processing, most of the lattice sites have a single atom. The qubit is encoded in the internal states of the atoms. Schemes can aim to detect specific states in the experiment:

(i) *Cluster states.* In a lattice of such two-state atoms, typically the atoms can interact with the nearest neighbors in a one-dimensional chain. While in some systems such an interaction is part of the Hamiltonian [529], in the experiment of Ref. [13] it was efficiently engineered by applying two different optical lattices for trapping the two species [530]. Thus, cluster states can be realized, since they could be created from product states with a nearest-neighbor Ising dynamics. However, first and second moments of the collective observables J_k are not sufficient for detecting the entanglement of cluster states [400]. One proposal in this case is to use dynamics before the detection of collective observables [400]. In this way, a simple scheme can be constructed that can be used to study the decohering cluster state and find a good lower bound for its entanglement lifetime. However, since now the dynamics is part of the entanglement detection, its accuracy must be included (or assumed) in the calculations.

(ii) *Singlet states.* Instead of cluster states, two-particle singlets can be generated in a lattice of double-wells [531]. Such states can be readily detected by the criterion (Eq. (205b)), based on second order moments, and its generalization for multi-qudit systems [400]. This is also true for various proposals for realizing multi-particle singlet states as ground states of Heisenberg spin lattices with spins $\frac{1}{2}$ or higher [532].

Entanglement detections schemes not designed for particular states are also available. Such proposals are as follows:

(i) In Ref. [533], an approach is described using two copies of a multi-qubit pure state to compute a multi-particle entanglement measure based on the single-qubit purities. The two copies can be in two one-dimensional optical lattices that interact with each other during the entanglement detection.

(ii) In Ref. [451], a method is suggested that works if one possesses two copies of the mixed multi-qubit state of the lattice (see also Section 7.1). It is based on the fact that for separable states the purity of reduced states is larger than the purity of the state. If for a quantum state this is not the case, then it is entangled.

(iii) In Ref. [534], it is shown how to obtain a bound for the fidelity of the two-qubit states with respect to some maximally entangled state, measuring correlations between the momentum distributions, that is, measuring $\langle Q_x^{kl} \rangle$ for $k, l = a, b$. Knowing the fidelity with respect to a maximally entangled state allows to obtain a bound on the two-qubit entanglement of formation.

In the other realization of quantum information processing one does not employ the internal states for storing information but the presence or absence of atoms at the lattice sites encodes the single qubits [535]. That is $|0\rangle$ corresponds to an empty site, while $|1\rangle$ describes a site with a single atom. The system can be described by the Bose-Hubbard model, that is, Eq. (215) with only specie "a". In the limit of very strong repulsive interaction between the particles, there is at most a single particle per site. Entanglement conditions for such systems are the following:

(i) In Ref. [505], a simple entanglement condition is described that needs the measurement of two-site correlations of the type $\langle a_k a_{k+1}^\dagger + a_k^\dagger a_{k+1} \rangle$, where a_k is the destruction operator corresponding to the site k. This criterion detects entanglement close to the ground state of the system.

(ii) In Ref. [534], an approach is presented that exploits particle number superselection rules. These prohibit having single lattice site states that are the superposition of states with different particle numbers. Under such conditions, all states that can be prepared locally are diagonal in the basis of $|0\rangle$ and $|1\rangle$. Thus, if a state is nondiagonal, then it implies the presence of a resource, which is, however, less powerful than entanglement defined independently from superselection rules. Nevertheless, the state detected cannot be prepared locally with atoms, thus if the state is nondiagonal, the presence of quantum correlations are demonstrated. Any non-flat momentum distribution indicates entanglement in this sense, which has been observed in several experiments [534]. Moreover, knowing $\langle Q_x \rangle$ makes it possible to obtain a lower bound on the entanglement of formation of the two-qubit reduced state.

9. Conclusion

As we have seen, many methods for the detection of entanglement have been proposed, ranging from Bell inequalities to spin squeezing inequalities. Each of them has its advantages and disadvantages and the question which of them is preferable depends on the given experimental implementation. But one has always to be careful that the entanglement detection scheme used really detects the desired type of entanglement. Furthermore, possible hidden assumptions in an entanglement verification procedure have to be taken into account.

For the future, it can be foreseen that due to the progress in the experimental techniques more and more qubits can be entangled. Especially in ion traps and in photonic experiments using hyper-entanglement significant progress can be expected. In these experiments the creation of genuine multipartite entanglement will probably remain the main goal, as this ensures that the experiment presents something qualitatively new compared to previous ones. However, it will become more and more important to design entanglement verification schemes that require less experimental effort, as measurements become more demanding.

On the other hand, as such systems become larger, it can be expected that in many of these experiments the qubits will no longer be individually accessible. Thus, entanglement detection schemes based only on collective measurements will be needed. In such experiments, the detection of genuine multipartite entanglement of the total state is not in all cases realistic. However, as we have seen, one can aim to investigate how many qubits are entangled, or to quantify the entanglement in the state.

In this way, quantum information science and in particular entanglement theory can play a crucial role in the technological development of quantum control and quantum engineering. Finally, quantum control over larger and larger systems might also help to answer fundamental questions concerning quantum theory, such as the appearance of a classical macroscopic world based on a quantum microworld.

Note added in proof

After submission of this paper, entanglement detection methods for Dicke states have been improved in Refs. [536,537]. These methods have been used in two experiments creating six-photon Dicke states [538,539].

Acknowledgements

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