



Ostero: a didactic finite element code for solid mechanics

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Abstract

Ostero intends to be a didactic finite element code for the numerical simulation of solid deformable bodies. It was developed as part of a PhD thesis currently done at Barcelona Supercomputing Center ([BSC](#)), to use it as a test framework for several mechanical models and problems. It was a very useful tool to perform proof-of-concept evaluations of contact mechanics algorithms before implementing them in the BSC code called Alya, which is a bigger, powerful but more complex code. As a general fact, Ostero was designed in order to first: provide the user a clear idea of the structure that usually a finite element code for linear and non-linear mechanics has and second: to allow the user to use Ostero as a workbench for personal tests. So far Ostero has only support for 2D problems, using triangles or quadrilateral elements and is limited to quasi-static analysis. Ostero allows to solve geometrically linear and non-linear problems, using elastic material models. For the non-linear case, Kirchhoff and hyperelastic (neo-Hookean) material models are available. Ostero was also designed to interact seamlessly with open source meshing and post-processing tool. To this aim, Ostero reads Gmsh mesh files and writes Vtk files, which can be easily visualized using the ParaView post-processing tool.

Revision history

Rev.	Date	Author	
v0.1	July 18, 2016	Matias Rivero	First issue - Some examples still missing

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1 Description

Ostero is a finite element code that solves the equilibrium equation which governs the mechanics of a deformable body subjected to external forces, Dirichlet and/or Neumann boundary conditions. In other words, Ostero allows to determine the response of a deformable solid body to an applied external load or displacement. Ostero allows to solve a mechanical problem considering geometric linearity or non-linearity. For the specific case of the geometrical non-linear model, Ostero uses an implicit scheme and the update of tangent matrix is performed at each time step. When solving using a geometrically linear setting, the equations of equilibrium are formulated in the undeformed state, and are not updated with the deformation. In some engineering problems, as the deformations are considered small and the deviation from the original geometry is not perceptible, the use of a geometrically linear setting is a very good approximation to the non-linear model. The mathematical complexity generated by a more realistic theory and the associated increment of the computation time doesn't compensate the small error introduced by ignoring the update of deformations in the equilibrium equations. But, in engineering there are also a number of problems where the deformation (large strains and/or large rotations) can not be ignored. In those cases a geometrically non-linear model should be used in Ostero to account for the large deformations. It's part of the engineering criteria to choose which model, linear or non-linear, could be use.

Ostero intends to be a didactic code, and its main objective is to allow the user to understand the very basic structure of a linear and non-linear mechanics finite element code. Also intends to provide a framework for beta testing of different models such as elastic material models, contact, plasticity, fracture, etc.

Ostero is based on the *solidz* module of the [Alya](#) code, which is a multi-physics parallel simulation code developed at Barcelona Supercomputing Center ([BSC](#)). So far it can solve 2D quasi-static problems using triangles or quadrilateral elements. The non-linear module include the isolinear (or Kirchhoff) material model and several formulations of the hyperelastic neo-Hookean material model. As many engineering applications involve small strains but large rotations, the effect of large deformation are primarily due to rotations. In these cases the response of the material may then be modeled by a simple extension of the linear elastic laws with small modifications (see [1]), such as the isolinear material model. For those elastic materials subjected to large strains for which the work is independent of the load path (no dissipation of energy), the hyperelastic material model is the most suitable choice.

Ostero was designed to read mesh files generated using the open source meshing tool [Gmsh](#), with no need of additional conversions to adapt the mesh file format to the input requested by Ostero. Ostero writes outputs in [Vtk](#) ASCII format, which can be easily postprocessed using the open source post-processing tool [ParaView](#).

Ostero was written in Python and Fortran to exploit the main advantages and properties of each programming language. The parsing of the input parameters, boundary conditions and other options is done in the main program, coded in Python. The main program also includes the main execution loop, which call the subroutines that performs the elementary matrix calculations and the assembly operations from the Fortran module. This external Fortran module is imported in the main program as an external library.

2 How to get Ostero

Ostero is hosted in [Bitbucket](#). Bitbucket is a web-based hosting service for projects that use [Mercurial](#) or [Git](#) revision control systems. Bitbucket is similar to [GitHub](#) (which primarily uses Git), but the main difference is that Bitbucket allows free private repositories, while in GitHub only

public repositories are available for free users. Besides being hosted in Bitbucket server, Ostero is under Git revision control, which provides a perfect framework for collaborative development. Git is a free and open source distributed version control system, which allows to manage changes in the code in a very efficient way. Is an essential tool for collaborative projects, but also very useful for individual programmers. Git takes a peer-to-peer approach to version control, as opposed to the client-server approach of centralized systems, as SVN. Rather than a single, central repository on which clients synchronize, in Git each peer's working copy is the complete repository which includes the complete history information of the codebase.

There are two main ways to get ostero:

1. The easiest alternative is to download the source code from the main page of Ostero <https://bitbucket.org/matrivero/ostero>, in *Downloads* section. The source code is available to download in compressed zip form, so just uncompress the downloaded file using the unzip tool or any other zip decompressor tool.
2. If you want to get a copy of Ostero with Git support (i.e. to keep track of your own changes or to be able to update Ostero without losing your modifications) you have to clone the repository to your home system. To do this, first you need to check if Git is installed on your system by executing the following command in a bash terminal:

```
1 :~$ git --version
2 git version 1.7.10.4
```

If Git is not installed on your system and you are using a Debian or Ubuntu Linux distribution (or any Linux distribution that has support for the aptitude package manager), you can install Git by doing:

```
1 :~$ sudo apt-get install git-all
```

If on the contrary, Git is installed on your system, run the following command to clone the repository in your computer:

```
1 :~$ git clone https://bitbucket.org/matrivero/ostero.git
2 Cloning into 'ostero'...
3 remote: Counting objects: 852, done.
4 remote: Compressing objects: 100% (790/790), done.
5 remote: Total 852 (delta 271), reused 231 (delta 61)
6 Receiving objects: 100% (852/852), 35.84 MiB | 5.80 MiB/s, done.
7 Resolving deltas: 100% (271/271), done.
```

You can check that Ostero was correctly cloned by doing:

```
1 :~$ cd ostero
2 :~/ostero$ ls
3 drwxr-xr-x  7 user user   4096 Jun 18 18:38 alya_benchmark
4 drwxr-xr-x 13 user user   4096 Jun 22 10:54 examples
5 -rw-r--r--  1 user user 29073 Jun 20 14:53 external.f90
6 -rwxr-xr-x  1 user user 22932 Jul 12 15:43 finite_strain.py
7 -rw-r--r--  1 user user 34602 Jun 18 18:38 LICENCE.txt
8 -rw-r--r--  1 user user  3812 Jun 18 18:38 README.md
9 -rw-r--r--  1 user user   297 Jun 20 20:28 TODO
10 -rw-r--r--  1 user user  5487 Jun 18 18:38 writeout.py
```

3 First steps

In order to execute Ostero you need a [Python](#) interpreter for 2.X version, [Numpy](#) and the [f2py](#) package, which is a Fortran to Python interface generator. Since 2007, f2py is part of Numpy.

First, if you are not sure if your system fulfill the requirements to run Ostero, you can check if Python, Numpy and f2py are installed in your system (and which versions you are using).

The latest version of most Linux distribution come with Python 2.7 out of the box. To see which version of Python you have installed, simply execute the following command in a bash terminal:

```
1 :~$ python --version
2 Python 2.7.3
```

To check if Numpy and f2py is installed in your system, simply execute:

```
1 :~$ python -c "import numpy; print(numpy.__version__)"
2 1.6.2
3 :~$ f2py -v
4 2
```

To install Numpy, in case is not installed in your system, you can do it using the aptitude packet manager, getting Numpy from repositories:

```
1 :~$ sudo apt-get install python-numpy
```

Once you have checked the system requirements, to start using Ostero the first step is to compile the external Fortran library by doing:

```
1 :~/ostero$ f2py -c -m external external.f90
```

If the external library was generated ok (check for external.so in the installation root directory), you are ready to execute Ostero. Let's start with an example. First, change the directory to the example folder i.e.:

```
1 :~/ostero$ cd examples/square
```

Then, execute the example by doing:

```
1 :~/ostero/examples/square$ ../../finite_strain.py input_file.dat boundary_file.dat
2 ISOLINEAL MATERIAL MODEL / NONLINEAR FORMULATION / PLANE STRESS APPROXIMATION
3
4 Solving time step 1 ...
5 Newton-Raphson iteration: 1
6 Displacement increment error: 1.0
7 Newton-Raphson iteration: 2
8 Displacement increment error: 0.000556623737174
9 Newton-Raphson iteration: 3
10 Displacement increment error: 2.96846401262e-10
11
12 Solving time step 2 ...
13 Newton-Raphson iteration: 1
14 Displacement increment error: 0.500137516494
15 Newton-Raphson iteration: 2
16 Displacement increment error: 0.00027907811949
17 Newton-Raphson iteration: 3
18 Displacement increment error: 1.4971842011e-10
19
20 Solving time step 3 ...
21 Newton-Raphson iteration: 1
```

```
22 Displacement increment error: 0.333578181727
23 ...
```

The execution should finish successfully and one output file should be created for each time step. The output format is [Vtk](#), which can be postprocessed using the open-source application [ParaView](#).

The main program of Ostero is *finite_strain.py*. This main program receives two additional arguments which need to be specified by the user. The first argument is the input file and the second is the boundaries file. The name of these files can be user-defined, but the order in which they are called must be respected. The first argument must be the input file while the second must be the boundaries file.

In the input file you must specify:

- The case name. Is just a user-specified name for the case under consideration (i.e. 2DBeam). If nothing is specified the default will be **NONAME**. The output files will be titled using this name. (keyword *case_name*)
- The mesh path. Is the relative or absolute path to the mesh file. If the mesh file is located in the execution folder, is enough to specify the name of the mesh. The mesh must be in [Gmsh](#) format. (keyword *mesh_path*)
- The geometrical treatment. In other words, if the calculation will be performed taking into account the finite strain theory or the linear theory. The accepted keywords for this option are **LINEAR** or **NONLINEAR**. (keyword *geometrical_treatment*)
- The material constitutive model for the non-linear treatment. Is a mandatory option if the **NONLINEAR** geometrical treatment was selected. For the **LINEAR** geometrical treatment this keyword is ignored. (keyword *constitutive_model*, options **ISOL** for isolinear (Kirchhoff) model or **BELY**, **ZIEN** or **LAUR** for neo-Hookean hyperelastic model as described in Belytschko's [1], Zienkiewicz's [2] and Laursen's [3] books respectively)
- The material 2D approximation for the **ISOL** option in nonlinear geometrical treatment or for the geometrical linear model. If the **LINEAR** or **NONLINEAR** and **ISOL** options are selected, the user must specify if the 2D approximation will be plane stress or plane strain. (keyword *submodel*, options **PLANE_STRESS** or **PLANE_STRAIN**)
- The size of each time step. Is a unique number that will define the length of each simulation time step. All time steps will be equal. (keyword *time_step_size*)
- The total number of time steps. Is a unique number which defines the total number of time steps. The multiplication between the total number of time steps and the size of each time step will give the total simulation time. (keyword *total_steps*)

Any new line introduced in the input file, at any location, that doesn't start with any of the keywords described before will be interpreted as a general comment.

In the boundaries file you must specify:

- The volume definition and its mechanical properties. The order of parameters is: the name of the physical entity which defines the surface to which the material properties will be applied; the Young modulus; the Poisson modulus and the density. As Ostero solves in quasi-static mode and using an implicit scheme, the density value is used only if Gravity is activated, but for parsing reasons, this value is always needed. (keyword *\$VolumeDefinition*)

- The definition of Dirichlet boundary conditions. The order of parameters is: the name of the physical entity which defines the line, group of points or individual point to which the Dirichlet boundary condition will be applied; pair of tags which indicate if the condition is active for x_1 and/or x_2 direction (1 active, 0 unactive); the value of displacement for x_1 and x_2 and the start and end time steps, which defines when the boundary condition will be active. (keyword *\$BoundaryConditionsDisplacement*)
- The definition of Neumann boundary conditions. The order of parameters is: the name of the physical entity which defines the line or lines (Neumann boundary) to which the boundary condition will be applied and the pressure value, which will be applied in the normal direction to the Neumann boundary. Negative values mean compressive pressure while positive values represent an imposed traction pressure. (keyword *\$BoundaryConditionsPressure*)
- The gravity effect (if present). The order of parameters is: the magnitude of gravity and its direction in x_1 - x_2 plane. (keyword *\$Gravity*)

To add comments in the boundaries file, they must be in a new line and start with the special character “#” or “!”.

4 Examples

To demonstrate how Ostero works, in the next chapter we will analyze several examples which show all the functionalities included in the code. Also this examples intends to show the user how to define the physical problem and create the input files.

4.1 ostero/examples/square

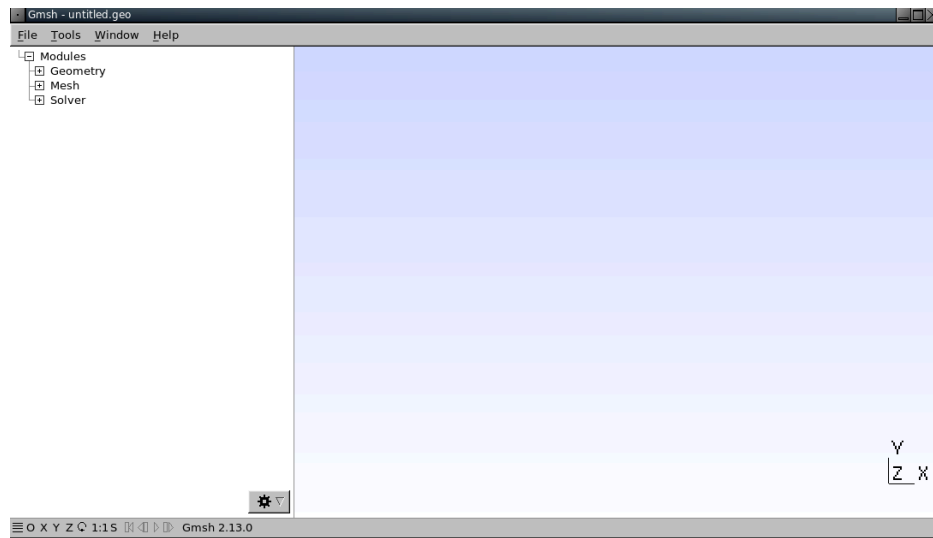
In this example we are going to model the non-linear deformation of a square geometry composed of an isilinear (Kirchhoff) material, which is subjected to a completely fixed Dirichlet boundary condition on its bottom and a compressive Neumann boundary condition on its top. In this first example we will start from the very beginning of any finite elements analysis, that is the generation of the triangular mesh. The mesh generation procedure is the same for every mesh used in these examples, so the methodology explained in this section can be easily applied to generate the meshes of the remaining examples (despite there is no need to do that because their are included in the distribution) or the meshes for other 2D analysis using Ostero.

As mentioned before, Ostero reads meshes generated with the open source meshing tool Gmsh, which can be downloaded in binary form [here](#). Using Gmsh we will generate a square as the union of 4 lines of size 5 meters (to be consistent with the units of measure, we will express all the physical quantities in the International System of Units).

Once Gmsh is on your system, simply execute the program by writing:

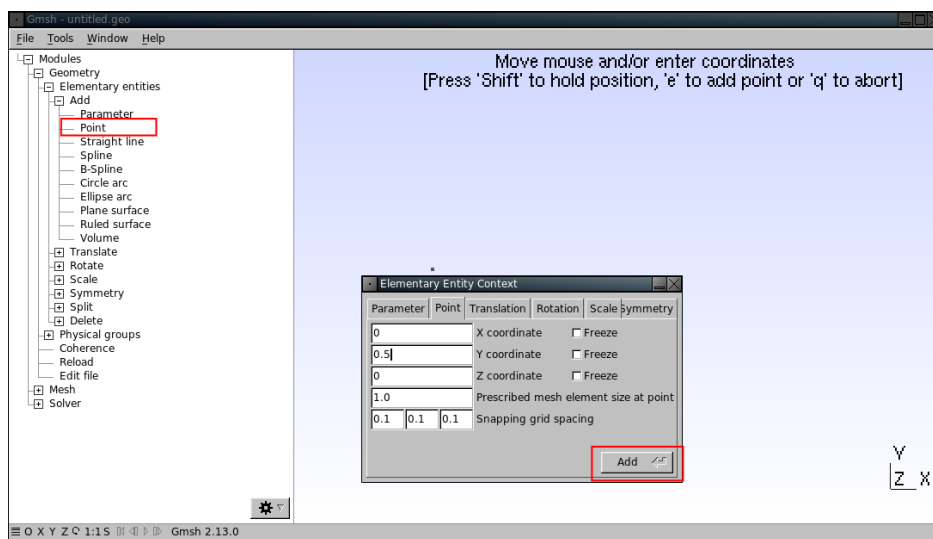
```
1 :~/gmsh-2.13.0-Linux/bin$ ./gmsh
```

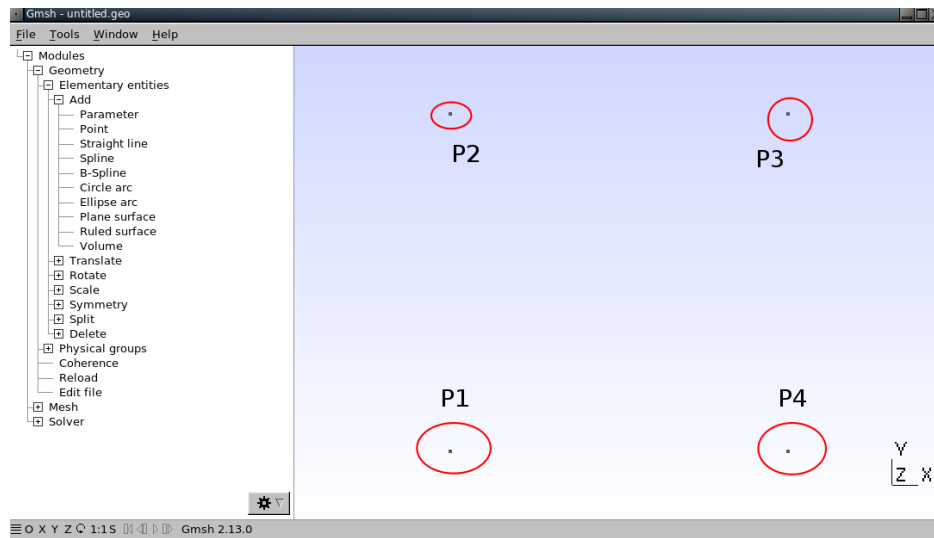
and the graphical user interface (GUI) of Gmsh should appear:



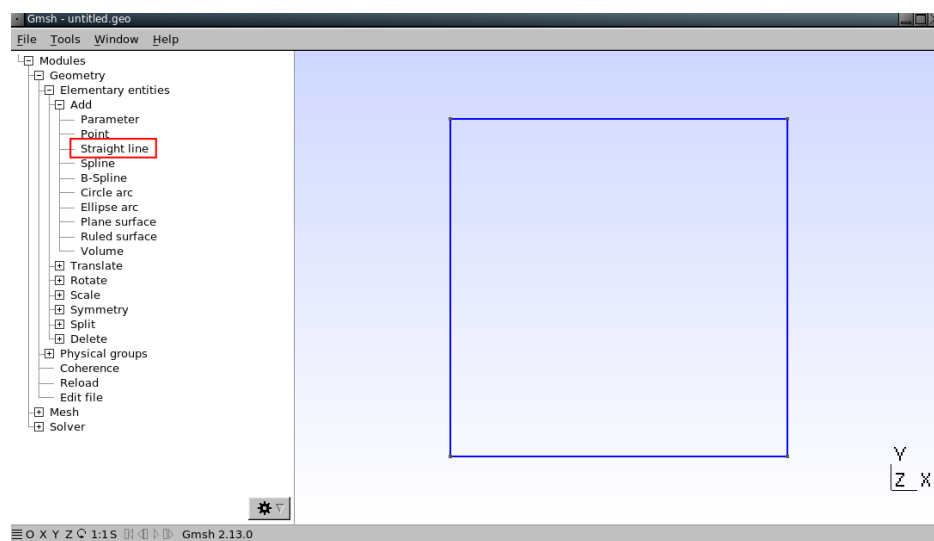
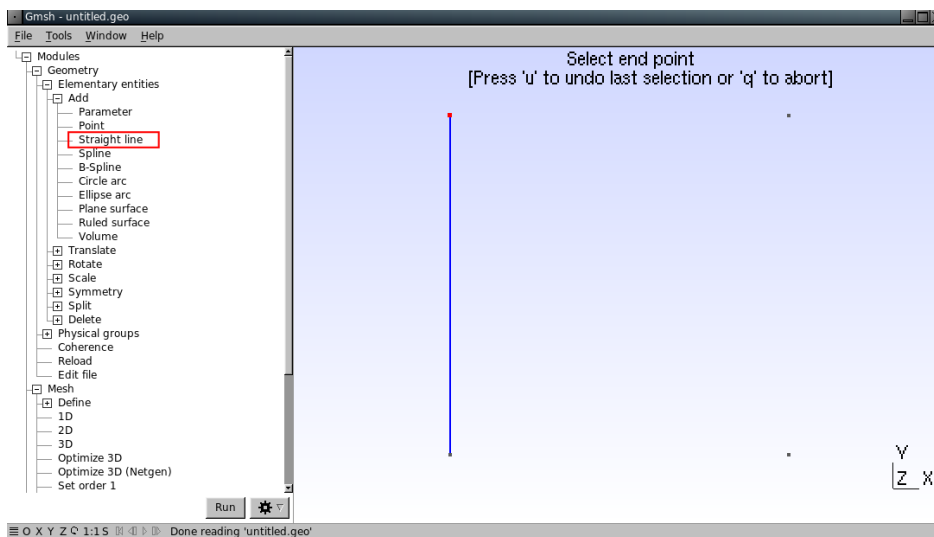
The first step is to add 4 points, which will be the 4 vertex of the square. Its locations are:

- Point 1: (0.0 ; 0.0)
- Point 2: (0.0 ; 5.0)
- Point 3: (5.0 ; 5.0)
- Point 4: (5.0 ; 0.0)

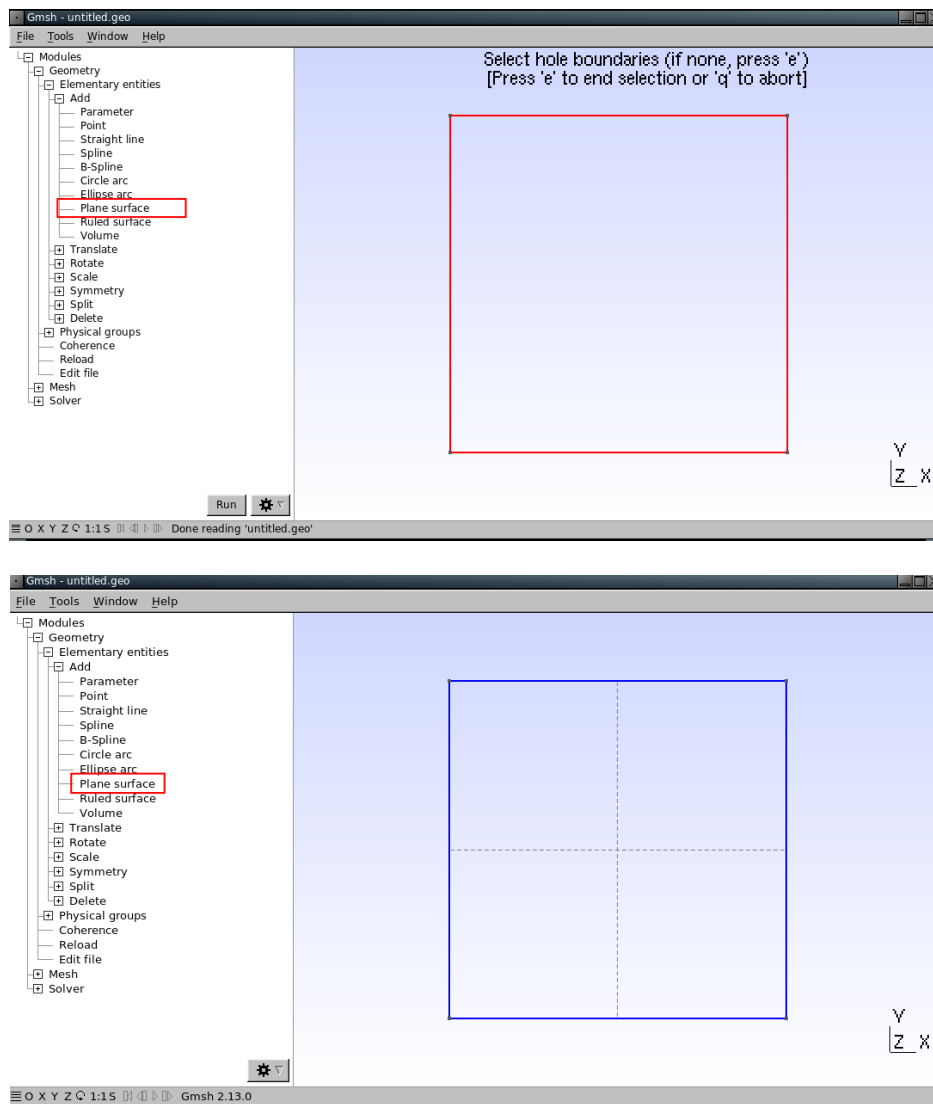




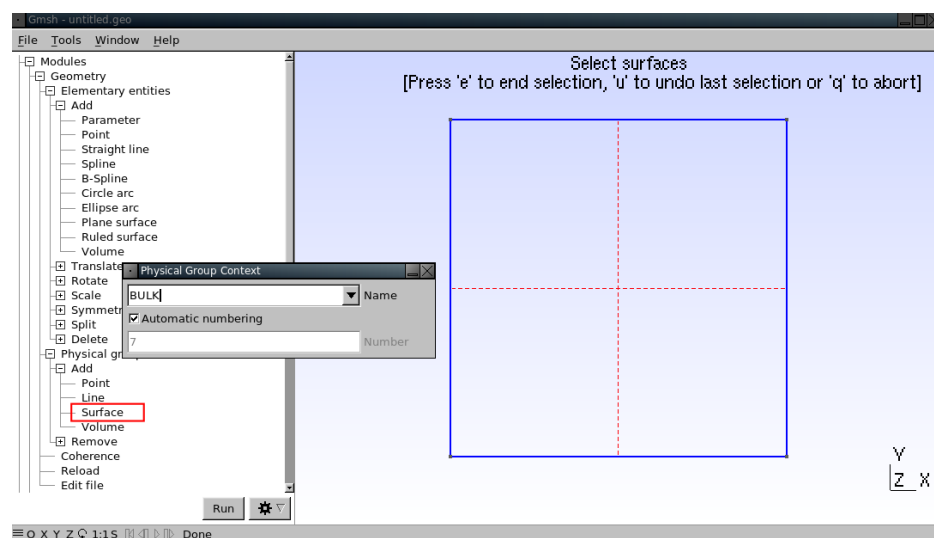
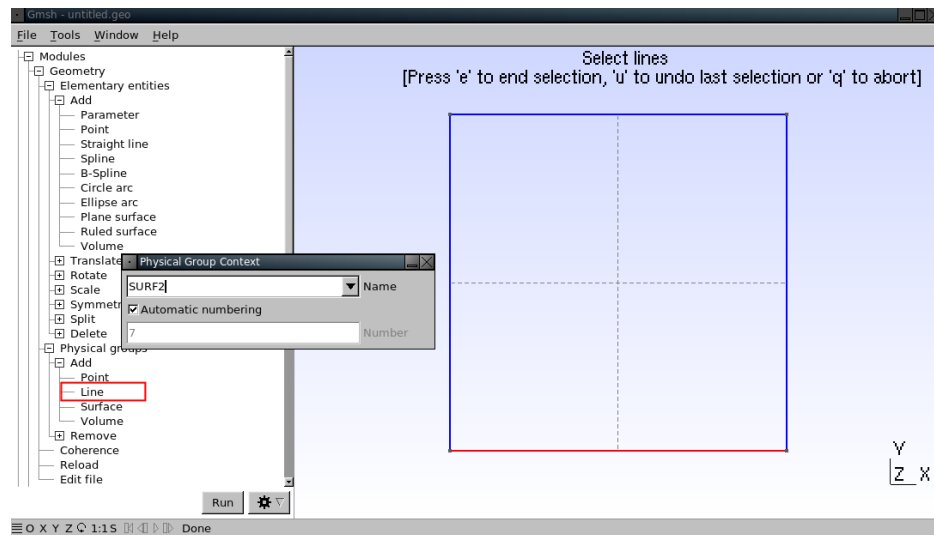
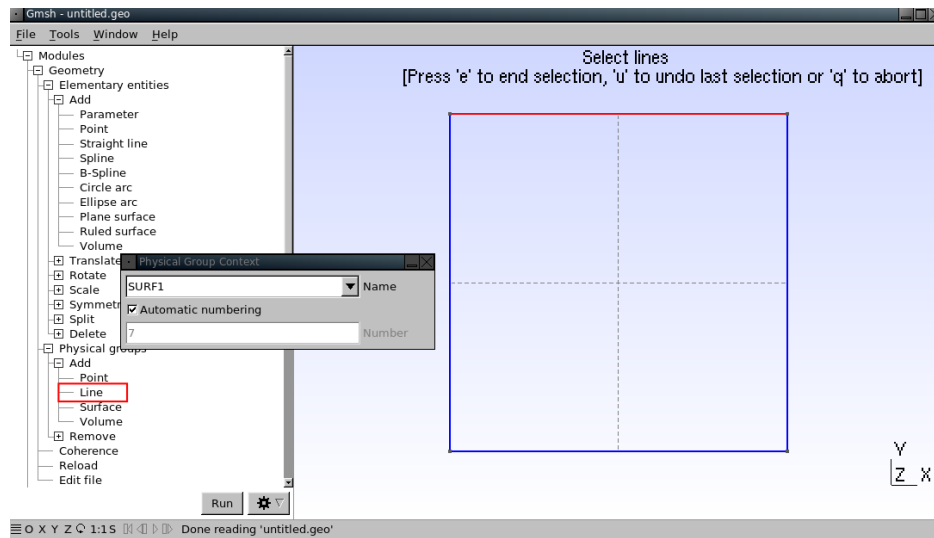
Now, we are ready to connect the points with straight lines to create the wireframe:



Once we have created the wireframe, it is time to construct the 2D plane surface. To do this we must select the wireframe that will be the boundary of the domain:



Up to now we have created the geometry. Now we need to add tags that will tell Ostero which will be the simulation domain where the material will be defined, and which will be the boundaries where the different boundary conditions will be applied. As in this example we are interested on applying boundary condition on the bottom and top part of the geometry, we will apply tags to those regions. In Gmsh these tags are added by means of the Physical Groups option, which allows to define tags for points, lines, surfaces or volumes. In this case we will add two different physical groups for each line and one physical group for the hole surface:



Once you have reached this point, you can close the Gmsh GUI. By default Gmsh saves all the commands executed in the GUI in a file called *untitled.geo*, located in the execution path of Gmsh. This file can be renamed and opened later from inside the Gmsh GUI for future uses.

If everything went fine, the *untitled.geo* (or *_any_name_the_user_want.geo*) file will look as the following:

```

1 Point(1) = {0, 0, 0, 1.0};
2 Point(2) = {0, 5, 0, 1.0};
3 Point(3) = {5, 5, 0, 1.0};
4 Point(4) = {5, 0, 0, 1.0};
5 Line(1) = {1, 2};
6 Line(2) = {2, 3};
7 Line(3) = {3, 4};
8 Line(4) = {4, 1};
9 Line Loop(5) = {1, 2, 3, 4};
10 Plane Surface(6) = {5};
11 Physical Line("SURF1") = {2};
12 Physical Line("SURF2") = {4};
13 Physical Surface("BULK") = {6};

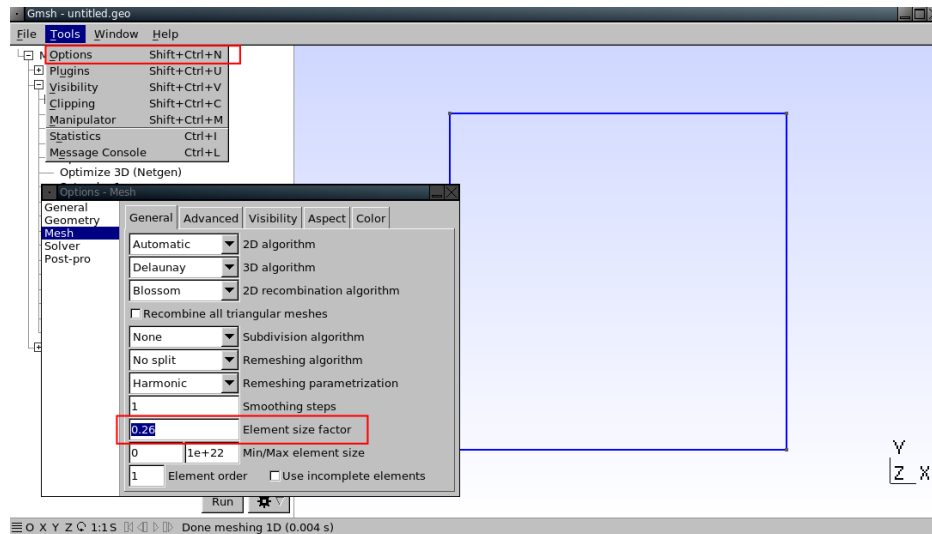
```

If in your *.geo* version the name of the physical entities are missing, just add them by hand.

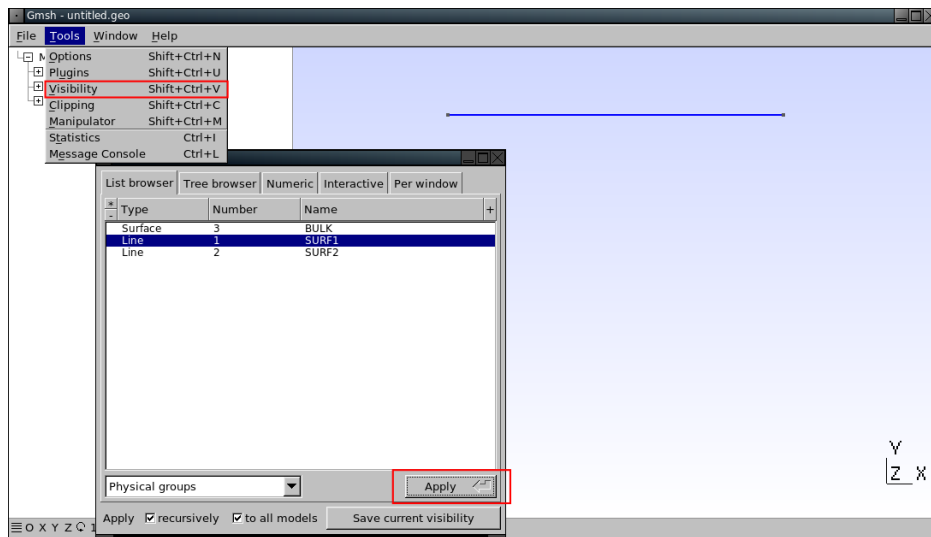
The next and final step is to generate the mesh. To do this we load the *untitled.geo* file in Gmsh by doing:

```
1 :~/gmsh-2.13.0-Linux/bin$ ./gmsh untitled.geo
```

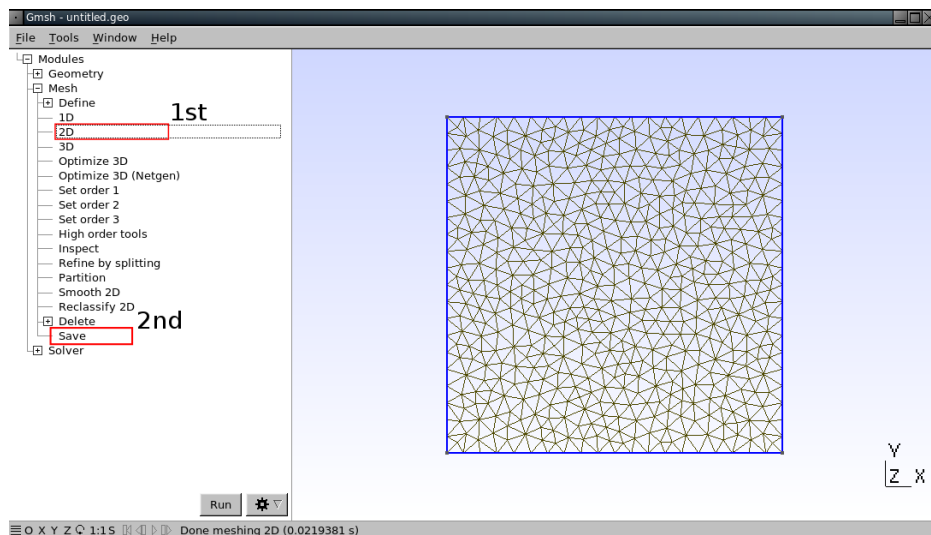
and set the element size factor to define the size of the mesh to 0.26.



Optionally, we can visualize the created physical entities to check that they are ok:

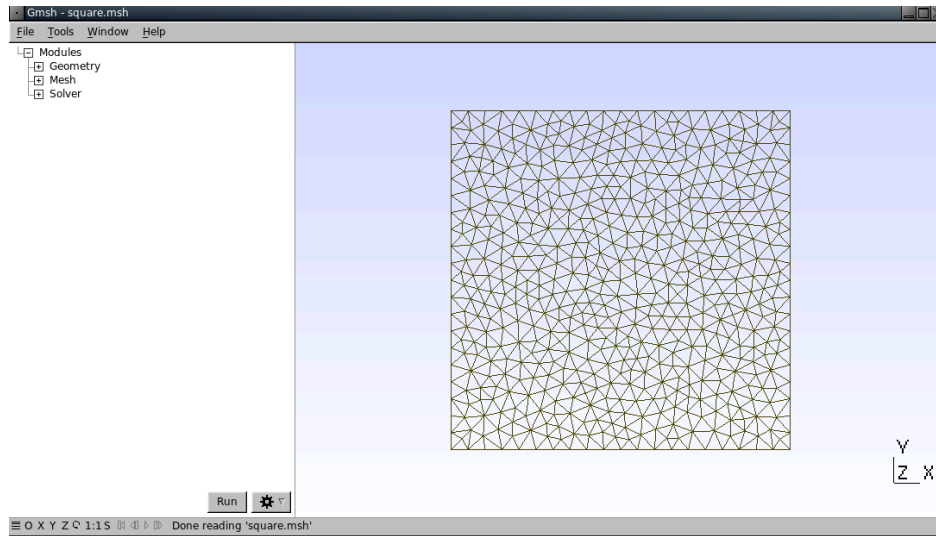


Finally, we generate and save the mesh:



The mesh will be written in a file with extension .msh and name equal to the one assigned to the .geo file. In our case, as we didn't change the name to the .geo, the mesh will be written as untitled.msh. We will rename the mesh to square.msh and check if it was correctly saved:

```
1 ~/gmsh-2.13.0-Linux/bin$ mv untitled.msh square.msh
2 ~/gmsh-2.13.0-Linux/bin$ ./gmsh square.msh
```



Now that we have generated the mesh, it is time to set up the input and boundaries file for the finite element analysis. Let's start from scratch creating a new folder called *square2* inside the examples folder of Ostero, and coping the recently created mesh to that location:

```
1 ~/ostero/examples$ mkdir square2
2 ~/ostero/examples$ cd square2
3 ~/ostero/examples/square2$ cp /gmsh-2.13.0-Linux/bin/square.msh .
```

Next step is to set the input file. As we said before, we would like to use the square.msh mesh. Also we want to solve considering geometrical non-linearities, using the isolinear material model and supposing we are in a plane stress assumption. We would like to run a simulation analysis of 0.05 seconds of total time, in 5 time steps of 0.01 seconds each one. With all this in mind, let's create the input file and name it *input.dat*, which should have the following content:

```
1 case_name SQUARE
2 mesh_path square.msh
3 geometrical_treatment NONLINEAR
4 constitutive_model ISOL
5 sub_model PLANE_STRESS
6 time_step_size 0.01
7 total_steps 5
```

Now it is time to specify the material parameters and boundary conditions for the analysis. First, we want to apply the material properties to the hole domain, which was named *BULK* during the mesh generation. To this material we will assign a Young modulus of 6.89 N/m^2 , a Poisson modulus of 0.32 and a density equal to 1 kg/m^3 . Then we want to impose a Neumann boundary condition over the top surface, which was named *SURF1*, equal to -100000.0 Pa (is negative because is a compressive pressure). Finally, we want to fix the bottom part of the body (*SURF2*) in both directions x_1 and x_2 , during the total simulation. To fix, we impose a null displacement (Dirichlet boundary condition). The boundaries file, which will be named as *boundaries.dat*, should look like the following:

```
1 $VolumeDefinition
2 BULK 6.896E7 0.32 100
3 $BoundaryConditionsDisplacement
4 SURF2 1 1 0.0 0.0 1 5
5 $BoundaryConditionsPressure
6 SURF1 -100000.0
```

The final step is to run the simulation by executing Ostero with the input file and boundaries file as command arguments:

```
1  :~/ostero/examples/square2$ ../../finite_strain input.dat boundaries.dat
2  ISOLINEAL MATERIAL MODEL / NONLINEAR FORMULATION / PLANE STRESS APPROXIMATION
3
4  Solving time step 1 ...
5  Newton-Raphson iteration: 1
6  Displacement increment error: 1.0
7  Newton-Raphson iteration: 2
8  Displacement increment error: 0.000439386037081
9  Newton-Raphson iteration: 3
10 Displacement increment error: 9.02152198392e-11
11
12 Solving time step 2 ...
13 Newton-Raphson iteration: 1
14 Displacement increment error: 0.500109778173
15 Newton-Raphson iteration: 2
16 Displacement increment error: 0.000220112630595
17 Newton-Raphson iteration: 3
18 Displacement increment error: 4.53296713812e-11
19
20 Solving time step 3 ...
21 Newton-Raphson iteration: 1
22 Displacement increment error: 0.333528700297
23 Newton-Raphson iteration: 2
24 Displacement increment error: 0.000147022482619
25 Newton-Raphson iteration: 3
26 Displacement increment error: 3.03647367894e-11
27
28 Solving time step 4 ...
29 Newton-Raphson iteration: 1
30 Displacement increment error: 0.250247531434
31 Newton-Raphson iteration: 2
32 Displacement increment error: 0.000110478153067
33 Newton-Raphson iteration: 3
34 Displacement increment error: 2.28900053569e-11
35
36 Solving time step 5 ...
37 Newton-Raphson iteration: 1
38 Displacement increment error: 0.200281948004
39 Newton-Raphson iteration: 2
40 Displacement increment error: 8.85521539567e-05
41
42 EXECUTION FINISHED SUCCESSFULLY!
```

For each time step, including the initial mesh, Ostero writes the output in a Vtk file. Once the simulation is finished, we can check that the output was generated ok by listing all the contents of the case directory:

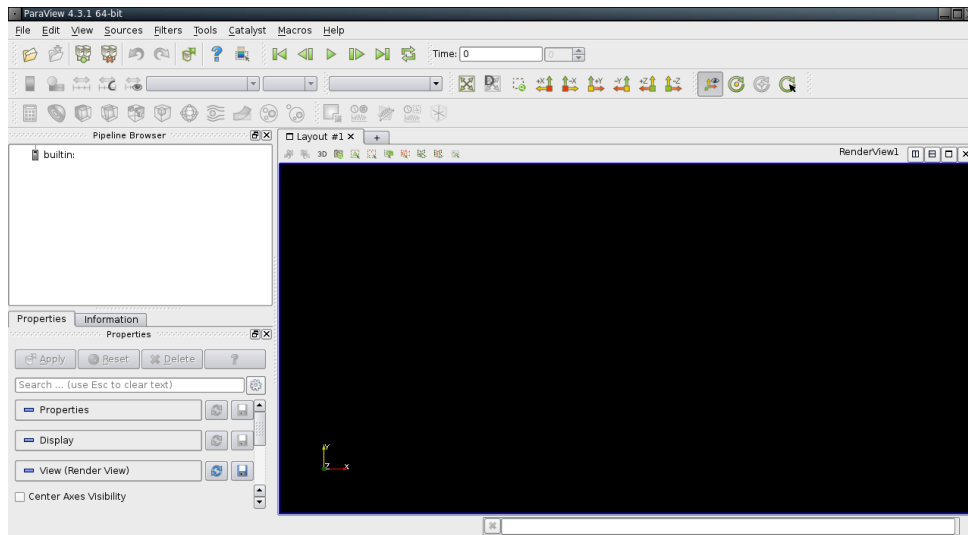
```
1  :~/ostero/examples/square2$ ls
2  -rw-r--r-- 1 user user Jul 15 18:53 boundaries.dat
3  -rw-r--r-- 1 user user Jul 16 18:50 input.dat
4  -rw-r--r-- 1 user user Jul 16 14:51 square.msh
5  -rw-r--r-- 1 user user Jul 16 18:50 SQUARE_NONLINEAR_ISOLIN_PLANE_STRESS_out.0.vtk
6  -rw-r--r-- 1 user user Jul 16 18:50 SQUARE_NONLINEAR_ISOLIN_PLANE_STRESS_out.1.vtk
7  -rw-r--r-- 1 user user Jul 16 18:50 SQUARE_NONLINEAR_ISOLIN_PLANE_STRESS_out.2.vtk
8  -rw-r--r-- 1 user user Jul 16 18:50 SQUARE_NONLINEAR_ISOLIN_PLANE_STRESS_out.3.vtk
9  -rw-r--r-- 1 user user Jul 16 18:50 SQUARE_NONLINEAR_ISOLIN_PLANE_STRESS_out.4.vtk
10 -rw-r--r-- 1 user user Jul 16 18:50 SQUARE_NONLINEAR_ISOLIN_PLANE_STRESS_out.5.vtk
```


Vtk files can be opened with the open source postprocessing tool ParaView, which can be downloaded from [here](#). ParaView can be downloaded as a binary package, so there is no need of compilation from sources. If ParaView is not in your system, simply download, uncompress if necessary, and execute the binary file.

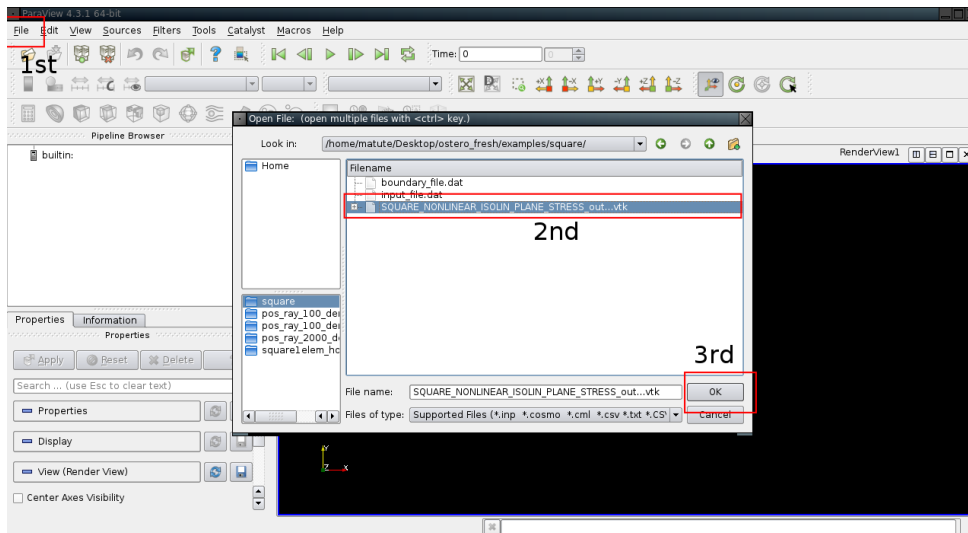
To launch ParaView, change directory to the bin folder and simply execute:

```
1 :~/ParaView-4.3.1-Linux-64bit/bin$ ./paraview
```

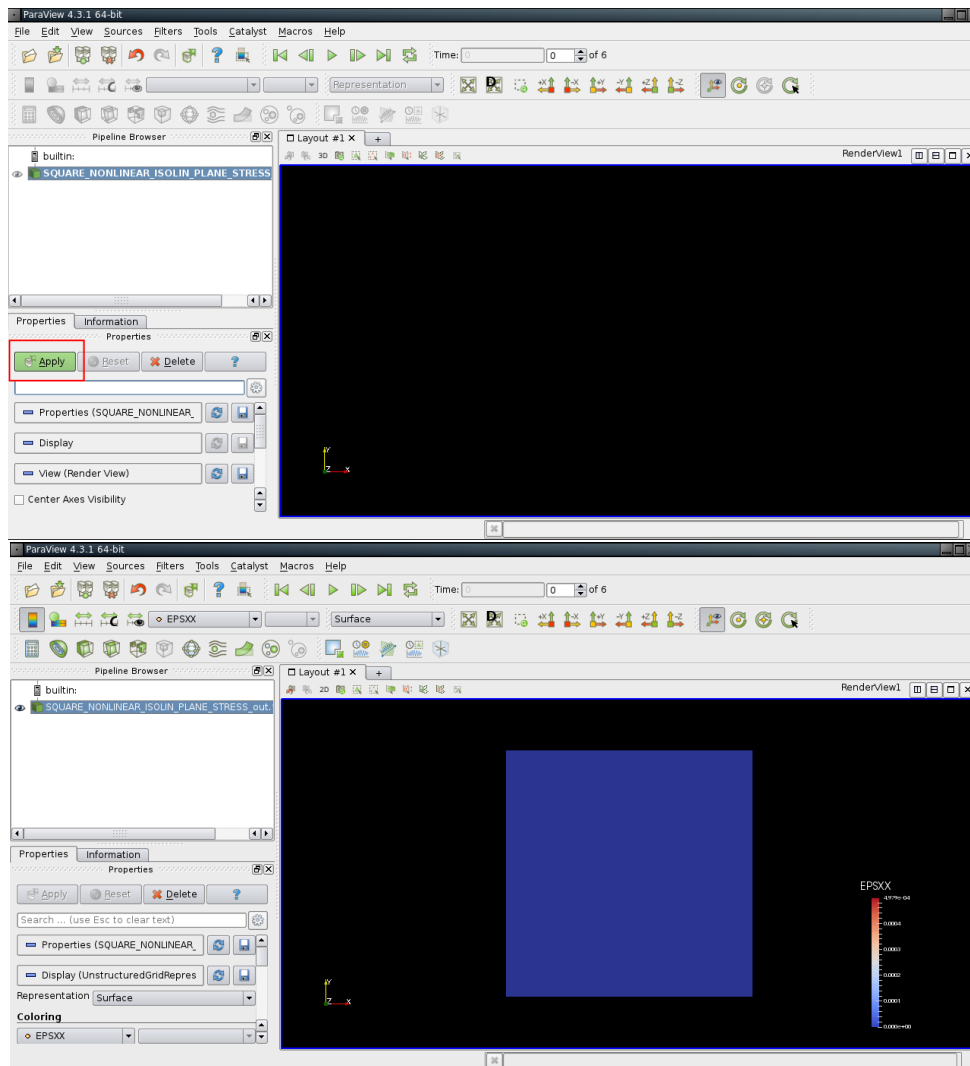
The main screen of ParaView should appear:



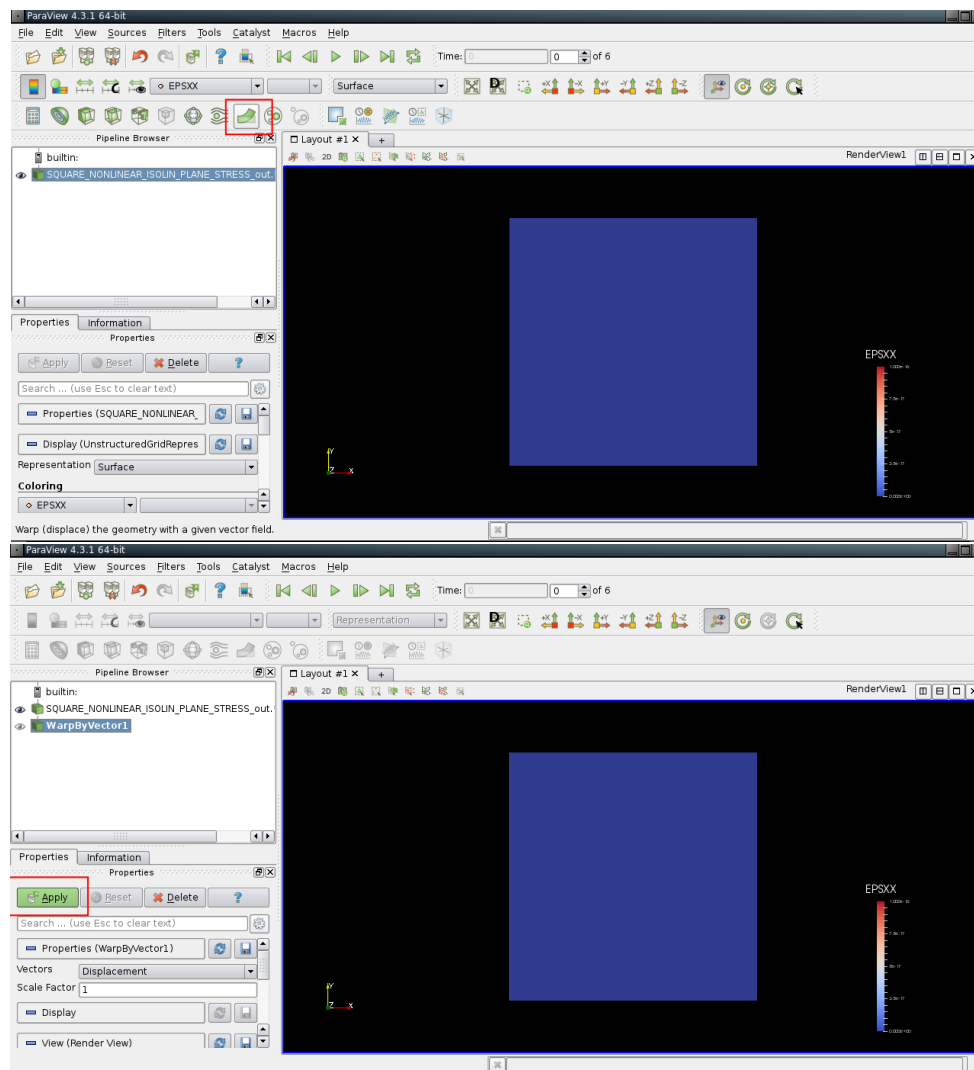
To open a set of results, go to File -> Open, navigate to the results path and select the hole bunch of Vtk files generated by Ostero, as shown in the following picture:



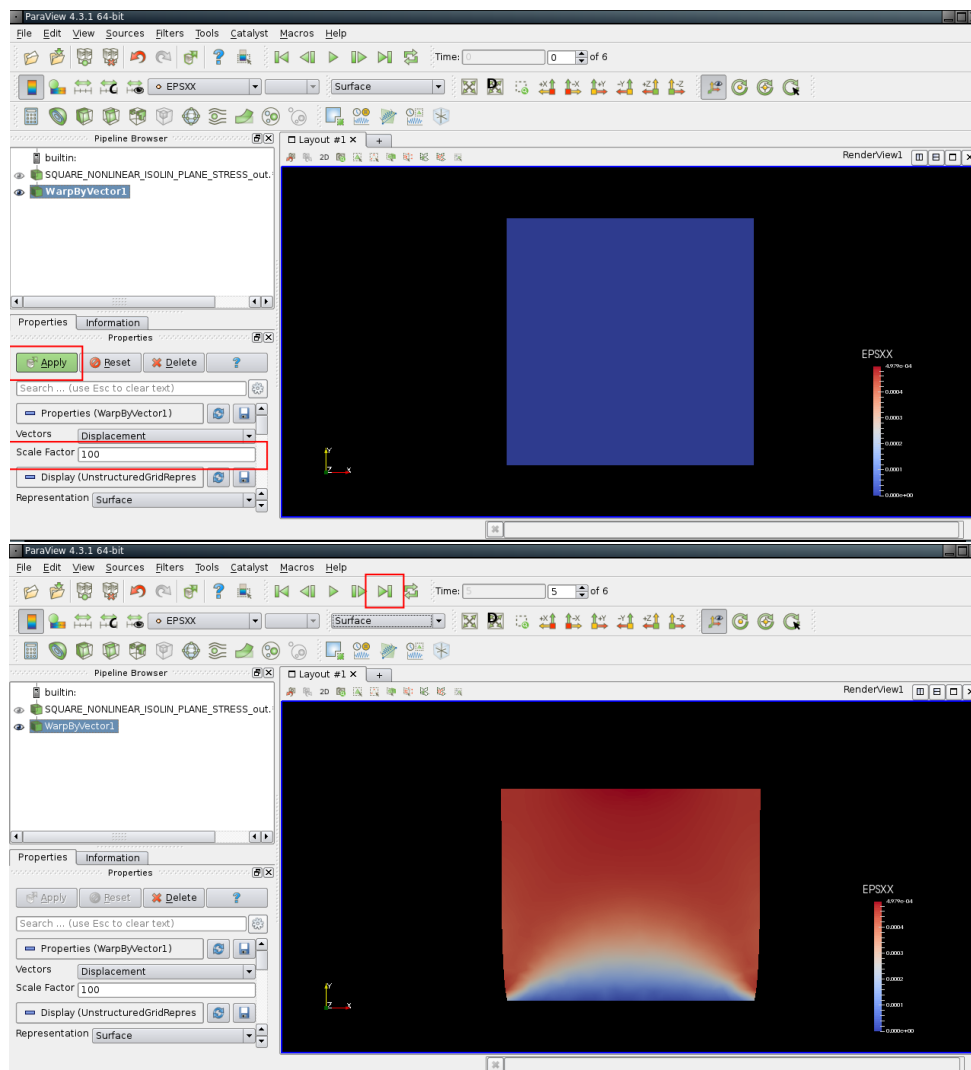
Once the results were loaded, simply press the apply button to visualize them on screen:



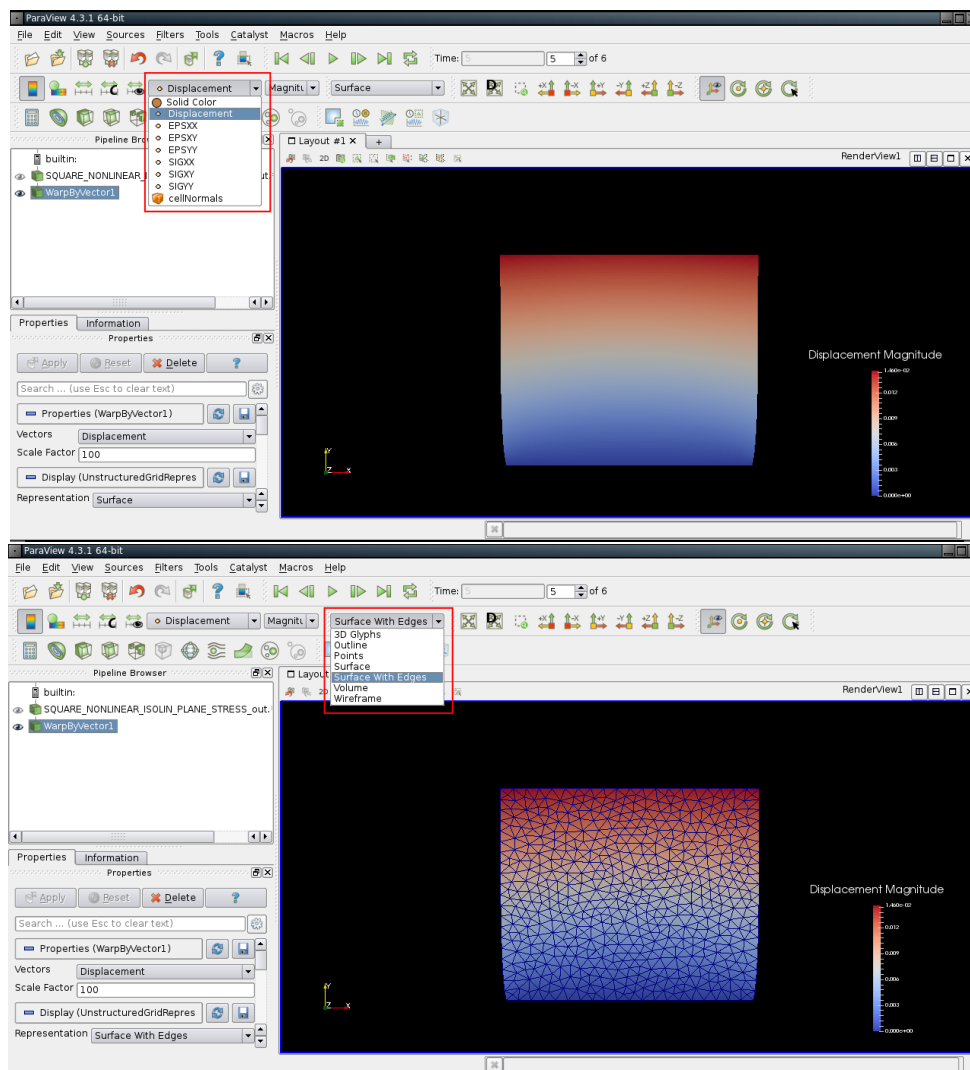
To see the mesh deformation, first select and then apply the *Wrap by Vector* filter:



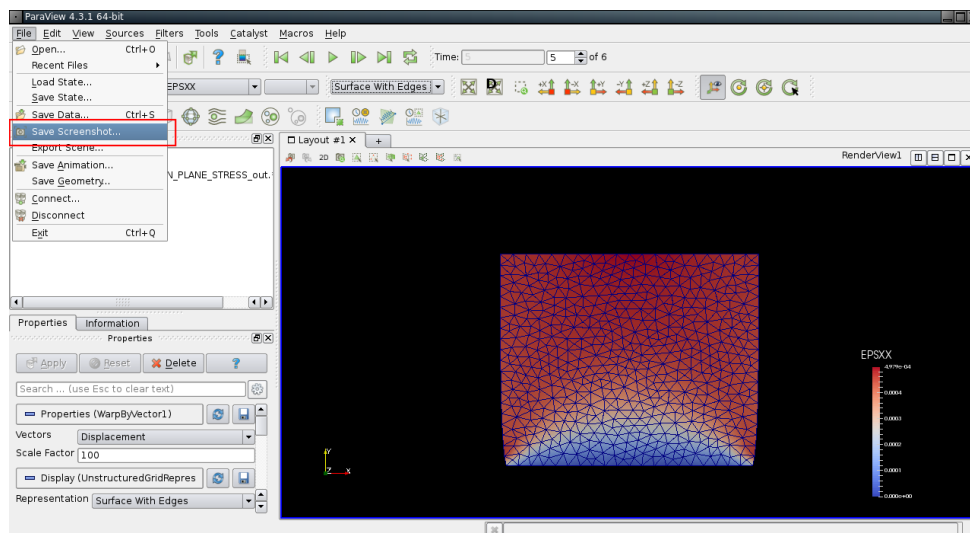
Then apply a scale factor of 100 to magnify the results and advance to the last time step:



Also is possible to change the variable to postprocess and the geometry features that will be displayed on screen:



You can also save a screenshot of a given time step of the simulation by doing File -> Save Screenshot:



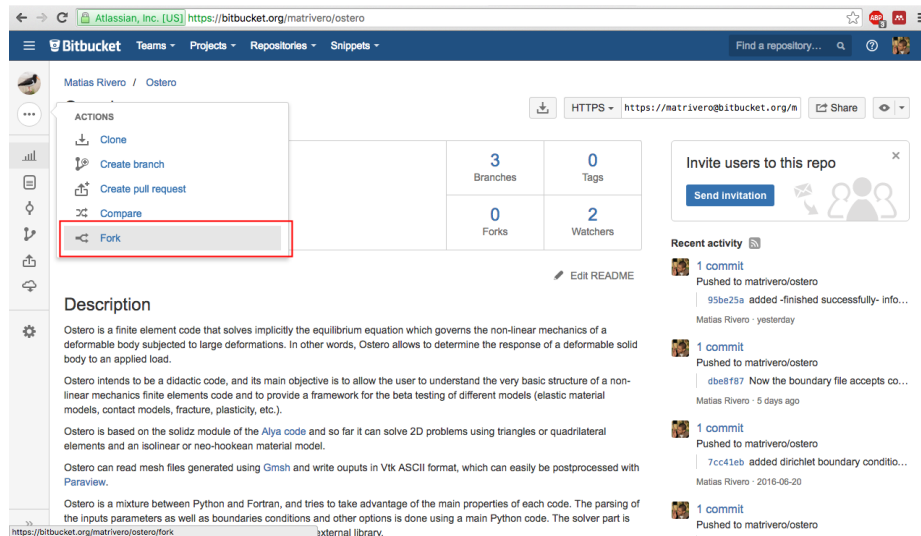
4.2 ostero/examples/complete

More examples coming up soon...

5 How to collaborate with Ostero

You will have to create a Bitbucket account and have Git installed on your system.

First, sign in to your Bitbucket account, navigate to the [Ostero's home page](#) and fork the repository, by clicking in the Fork button:



This will create a server-side copy of Ostero's repository.

Once you have forked Ostero, clone your forked repository to have a working copy of the project on your local machine. Then create a new branch and start editing the code. Once you have finished your implementations and commits, push the new feature or modification you have implemented to your Bitbucket repository (the forked one). Finally, create a *Pull request* through your Bitbucket account. This will create a request to merge your new feature located in your forked repository to the main codebase of Ostero.

The pull request you sent will be reviewed by the main developer of Ostero and he/she will approve the request, reject it or comment on the request asking for fixes.

To see more in detail how to collaborate with Ostero or any other open source project, see this excellent example from this Git tutorial:

<https://www.atlassian.com/git/tutorials/making-a-pull-request/example>

References

- [1] Ted Belytschko, Wing Kam Liu, and Brian Moran. *Nonlinear Finite Elements for Continua and Structures*. 1st. Wiley, 2006.
- [2] Olgierd C. Zienkiewicz and Robert Leroy Taylor. *The Finite Element Method: Solid mechanics*. Butterworth-Heinemann, 2000.
- [3] Tod A. Laursen. *Computational Contact and Impact Mechanics*. 1st. Springer, 2003.