Exercise 2.2.7 (not in book, but needed). Let **Δ** = det G, where **G = (g_{ab})** is the metric tensor matrix. The **minor**, $\boldsymbol{m^{ab}}$, for element g_{ab} is the determinant of the submatrix that excludes row a and column b. The **cofactor** for element g_{ab} is $\boldsymbol{c^{ab}} = (-1)^{a+b}\, m^{ab}$. Show that

$$\boxed{\partial_c \Delta = c^{ab}\,\partial_c g_{ab} = \Delta g^{ab}\,\partial_c g_{ab}}\,.$$

Solution By definition, Δ is the sum of all possible products of the form $(-1)^{a_1+b_1}\, g_{a_1 b_1} \cdot \ldots \cdot g_{a_N b_N}$ where every factor is in a distinct row and column. This definition is somewhat complicated to state and even more complicated to express mathematically, which we do now.

There are two formulas for Δ that we use here. The first one is to sum the products $g_{ab}c^{ab}$ across row a. However, there is a problem on how to express this. In non-Einstein notation, we can write

$$\Delta = \sum_{b=1}^{N} g_{ab}c^{ab}.$$

However, in Einstein notation, $g_{ab}\,c^{ab}$ sums over all rows and we end up with Δ + ... + Δ, N times:

$$g_{ab}\,c^{ab} = N\,\Delta. \tag{a}$$

A similar formula is:

$$g_{ab}\,g^{ab} = N \tag{b}$$

$$g_{ab}\,g^{ab} = g_{ab}\,g^{ad}\delta^b_d \overset{(1.78)}{=} \delta^d_b\,\delta^b_d = N.$$

Putting these two formulas together we get:

$$c^{ab} = \Delta\, g^{ab} \quad : \tag{c}$$

$$g_{ab}\,c^{ab} \overset{(a)}{=} N\,\Delta \overset{(b)}{=} g_{ab}\,g^{ab}\Delta \quad \Rightarrow \qquad c^{ab} = g^{ab}\Delta \qquad ✓$$

The second definition of Δ is more convoluted. We express Δ mathematically exactly as stated in words, above:

$$\Delta \equiv \epsilon^{b_1 \ldots b_N}\, g_{1\,b_1} \ldots g_{N\,b_N}, \tag{d}$$

by taking advantage of the **Levi-Civita symbol, $\boldsymbol{\epsilon^{b_1 \ldots b_N}}$,** that we now explain.

The indices b_k range over the integers 1 – N for all k. The symbol $\epsilon^{b_1 \ldots b_N}$ is defined to be zero unless each integer appears exactly once; that is, unless $b_1, \ldots, b_N$ is a permutation of 1, 2, ..., N. For permutations, its value is defined as +1 for **even permutations** and -1 for **odd permutations**. A permutation is even if the number of pairwise swaps required to restore the natural order is even, and odd, otherwise. For example, 1, 3, 2, 4 is an odd permutation because it requires just one swap. 1, 3, 4, 2 is an even permutation. So, in the determinant for $N = 3$, $g_{11}\, g_{22}\, g_{33}$ would (correctly) be preceded by a plus sign, represented by $\epsilon^{b_1 b_2 b_3} = \epsilon^{123}$ and $g_{11}\, g_{23}\, g_{32}$ would be preceded by a negative sign, ϵ^{132}.

We call it the Levi-Civita *symbol* because it is a pseudo-tensor, almost a tensor except that it reverses sign under Jacobian transformations having negative determinant. It is not quite proper to call it the Levi-Civita tensor.

The cofactor, also, can be defined in Levi-Civita terminology since it is a determinant. Consider the following attempt at a cofactor expression:

$$c^{a\,b} \equiv \epsilon^{k_1 \ldots k_{a-1}\, b\, k_{a+1} \ldots k_N}\, g_{1\, k_1} \cdots g_{a-1\, k_{a-1}}\, g_{a+1\, k_{a+1}} \cdots g_{N\, k_N}.$$

Every row appears in $g_{1\, k_1} \ldots g_{a-1\, k_{a-1}}\, g_{a+1\, k_{a+1}} \ldots g_{N\, k_N}$ except row a. The exponent of ϵ ranges over all strings $k_1 \ldots k_{a-1}\, b\, k_{a+1} \ldots k_N$, but ϵ is non-zero only for strings that are permutations of 1 – N; that is, only when none of the k_a's equal b. So, when the string is a permutation, every column appears in $k_1 \ldots k_{a-1}\, b\, k_{a+1} \ldots k_N$ and, so, every column except b appears in $g_{1\, k_1} \cdots g_{a-1\, k_{a-1}}\, g_{a+1\, k_{a+1}} \cdots g_{N\, k_N}$. So far, so good. (Note that b must appear in the superscript of ϵ because the Levi-Civita symbol is only defined for strings of length N).

Is b in a superscript position that will produce the correct sign for the term $g_{1\, k_1} \ldots g_{a-1\, k_{a-1}}\, g_{a+1\, k_{a+1}} \ldots g_{N\, k_N}$? Consider the case where k_1=1, k_2=2, ... k_N=N. Then we have $\epsilon^{1 \ldots a-1\, b\, a+1 \ldots k_N}\, g_{11} \ldots g_{a-1\, a-1}\, g_{a+1\, a+1} \ldots g_{N\, N}$, and the correct sign should be +1. Note that b is in the ath position of the superscript. If we put it in the bth position, then we would have $\epsilon^{\, 1 \times 2 \ldots b-1\, b\, b+1 \ldots N} = +1$. It requires $|b–a|$ index swaps to move b from position a to position b. So, $\epsilon^{1 \ldots a-1\, b\, a+1 \ldots N} = (-1)^{b-a}$. In order to give $g_{11} \ldots g_{a-1\, a-1}\, g_{a+1\, a+1} \cdots g_{N\, N}$ a coefficient of +1, we must precede it by $(-1)^{b-a}\, \epsilon^{1 \ldots a-1\, b\, a+1 \ldots N}$. So, we must modify our first attempt. We try

$$c^{a\,b} \equiv (-1)^{b-a}\, \epsilon^{k_1 \ldots k_{a-1}\, b\, k_{a+1} \ldots k_N}\, g_{1\, k_1} \cdots g_{a-1\, k_{a-1}}\, g_{a+1\, k_{a+1}} \cdots g_{N\, k_N}.$$

This expression has the correct sign as well as the correct rows and columns for the cofactor $c^{a\,b}$. However, in proper tensor notation, when a and b are superscripts on LHS then they must be superscripts on RHS, but a is only a subscript on RHS. We can't remedy this with just a Kronecker delta, but a summation sign along with a Kronecker delta works as follows:

$$c^{a\,b} \equiv (-1)^{b-a} \sum_{s=1}^{N} \delta_s^a \; \epsilon^{k_1 \ldots k_{s-1}\, b\; k_{s+1} \ldots k_N}\, g_{1\,k_1} \ldots g_{s-1\,k_{s-1}}\, g_{s+1\,k_{s+1}} \cdots g_{N\,k_N}. \tag{e}$$

In this summation we throw away all terms except when $s = a$, making this identical to the prior expression while adhering to the Einstein summation convention.

Example To illustrate cofactor definition (e), let a = 3, b = 5, and N = 7. Then we replace the sum by letting s = 3:

$$c^{\,3\times5} \equiv (-1)^{5-3}\, \epsilon^{k_1\; k_2\; 5\;\; k_4 \ldots\, k_7}\, g_{1\,k_1}\, g_{2\,k_2}\, g_{4\,k_4} \cdots g_{7\,k_7}.$$

Einstein summation occurs over every k_a except k_3 and so includes sequences with repeated numbers, but the only non-zero values of ϵ are when the k_a's include one each of the values 1, 2, 3, 4, 6, 7 in any order. Thus, as desired, $g_{1\,k_1}\, g_{2\,k_2}\, g_{4\,k_4} \ldots g_{7\,k_7}$ pairs every possible column with every possible row, excluding row 3 and column 5.

This expression is preceded by $(-1)^2 = +1$ because the number of pairwise swaps for b=5 to be the fifth superscript differs by two swaps from it being the 3rd superscript. ■

Finally, we can compute $\partial_c \Delta$ from equation (d), which we re-express here:

$$\Delta \equiv \epsilon^{k_1 \ldots k_N}\, g_{1\,k_1} \cdots g_{a-1\,k_{a-1}}\, g_{a\,k_a}\, g_{a+1\,k_{a+1}} \cdots g_{N\,k_N}. \tag{f}$$

Then

$$\begin{aligned}
\partial_c \Delta \;=\; & \partial_c g_{1\,b}\, (-1)^{b-1}\, \epsilon^{b\,k_2 \ldots k_N} g_{2\,k_2} \ldots g_{N\,k_N} + \ldots \\
& + \partial_c g_{a\,b}\, (-1)^{b-a}\, \epsilon^{k_1 \ldots k_{a-1}\, b\; k_{a+1} \ldots k_N}\, g_{1\,k_1} \cdots g_{a-1\,k_{a-1}}\, g_{a+1\,k_{a+1}} \cdots g_{N\,k_N} + \ldots \\
& + \partial_c g_{N\,b}\, (-1)^{b-N}\, \epsilon^{k_1 \ldots k_{N-1}\, b} g_{1\,k_1} \cdots g_{N-1\,k_{N-1}} \\
& = \sum_{s=1}^{N} \delta_s^a\, (-1)^{b-s}\, \partial_c g_{s\,b}\, \epsilon^{k_1 \ldots k_{s-1}\, b\; k_{s+1} \ldots k_N}\, g_{1\,k_1} \cdots g_{s-1\,k_{s-1}}\, g_{s+1\,k_{s+1}} \cdots g_{N\,k_N}.
\end{aligned} \tag{g}$$

To see that we need the factor $(-1)^{b-a}$ in equation (g), again consider k_1=1, k_2=2, ..., k_N=N. Then

$$\epsilon^{k_1 \ldots k_{a-1}\, b\, k_{a+1} \ldots k_N}\, g_{1\, k_1} \cdots g_{a-1\, k_{a-1}}\, g_{a\, k_a}\, g_{a+1\, k_{a+1}} \cdots g_{N\, k_N}$$
$$= \epsilon^{1 \ldots a-1\, b\, a+1 \ldots k_N}\, g_{11} \cdots g_{a-1\, a-1}\, g_{a\, a}\, g_{a+1\, a+1} \cdots g_{N\, N}.$$

The correct sign for this term is +1, so, in order to move superscript b to position a from position b, the factor $(-1)^{b-a}$ is needed.

We can simplify equation (g) by moving $\partial_c g_{s\, b}\, (-1)^{b-s}$ to the left of the summation sign. It becomes $\partial_c g_{a\, b}\, (-1)^{b-a}$ since the summations are all zero except when $s = a$, and this completes the solution:

$$\partial_c \Delta = \partial_c g_{a\, b}\, (-1)^{b-a} \sum_{s=1}^{N} \delta_s^a\, \epsilon^{k_1 \ldots k_{b-1}\, b\, k_{b+1} \ldots k_N}\, g_{1\, k_1} \cdots g_{s-1\, k_{s-1}}\, g_{s+1\, k_{s+1}} \cdots g_{N\, k_N}$$
$$\overset{(e)}{=} \partial_c g_{a\, b}\, c^{a\, b} \qquad \checkmark$$
$$\overset{(c)}{=} \Delta\, g^{a\, b}\, \partial_c g_{a\, b} \qquad \checkmark$$