Exercise 3.1.1 Show that $T = \begin{pmatrix} ec^2_P & 0 \\ 0 & p_P \end{pmatrix}$ for a Cartesian coord sys which brings the velocity to rest momentarily at the point P for a perfect fluid

$T^{\mu\nu} \overset{(3.2)}{=} (e + \frac{p}{c^2}) u^\mu u^\nu - p\eta^{\mu\nu}$. For a fluid at rest, $u^\mu \overset{(h)}{=} (c, 0, 0, 0)$.

$\Rightarrow u^0u^0 = c^2$ and $u^{\mu i} = 0 \ \forall \mu, i$. So, $(e + \frac{p}{c^2}) u^\mu u^\nu = \begin{cases} ec^2 + p & \text{if } \mu=\nu=0 \\ 0 & \text{o.w.} \end{cases}$

Since $\eta^{\mu\nu} = \begin{cases} 1 & \text{if } \mu=\nu=0 \\ -1 & \text{if } \mu=\nu=i \\ 0 & \text{o.w.} \end{cases}$, $-p\eta^{\mu\nu} = \begin{cases} -p & \text{if } \mu=\nu=0 \\ p & \text{if } \mu=\nu=i \\ 0 & \text{o.w.} \end{cases}$

$\therefore T^{00} = ec^2 + p - p = ec^2$, $T^{ii} = -(-p) = p$ and $T^{\mu\nu} = 0$ o.w. ∎

Exercise 3.1.2 An observer with world velocity U^μ encounters a particle with 4-momentum p^μ.
Show that ~~he~~ ~~show~~ he assigns an energy $p_\mu U^\mu$ to the particle

We use the observer's rest frame at the moment of the encounter:

Observer's 4-velocity is $U^\mu \overset{(h)}{=} (c, \vec{0})$

Particle's 4-momentum is $p^\mu \overset{(3.1)}{=} (\frac{E}{c}, \vec{p}) \Rightarrow p_\mu = (\frac{E}{c}, -\vec{p})$ by Example A.0.1

So $p_\mu U^\mu = \frac{E}{c}(c) - \vec{p}\cdot\vec{0} = E$. Since energy is conserved, $E = p_\mu u^\mu$ at all times. ∎

Note that the inner prdt $p_\mu U^\mu$ uses the particle's momentum but the observer's velocity.

Exercise 3.1.3 Show that the stress tensor units are compatible in eq (3.2)

$T^{\mu\nu} = (e + \frac{p}{c^2}) U^\mu u^\nu - p\eta^{\mu\nu}$ Let m = mass, l = length and t = time, Let a = accel

$e = \frac{m}{l^3}$ $\frac{p}{c^2} = \frac{F}{Ac^2} = \frac{ma}{Ac^2} = \frac{m\,l}{l^2\,\frac{l^2}{t^2}\,t^2} = \frac{m}{l^3}$ ✓ $\eta^{\mu\nu}$ is dimensionless $\Rightarrow$

$(e + \frac{p}{c^2}) u^\mu u^\nu = \frac{m}{l^3}\frac{l^2}{t^2} = \frac{m}{lt^2}$ $p\eta^{\mu\nu} = \frac{ma}{A} = \frac{ml}{l^2t^2} = \frac{m}{lt^2}$ ✓ ∎