Exercise 2.2.7 (not in book, but needed). Let Δ = det G be the **determinant of** ***G***, where $G = (g_{ab})$ is the metric tensor matrix. The **minor**, m^{ab}, for element g_{ab} is the determinant of the submatrix that excludes row a and column b. The **cofactor** for element g_{ab} is $c^{ab} = (-1)^{a+b}\, m^{ab}$. Show that

$$\boxed{\partial_c \Delta = c^{ab}\, \partial_c g_{ab} = \Delta g^{ab}\, \partial_c g_{ab}}\,.$$

Solution By definition, Δ is the sum of all possible products of the form $(-1)^{a_1+b_1}\, g_{a_1 b_1} \cdot \ldots \cdot g_{a_N b_N}$ where the factors $g_{a_k b_k}$ are in distinct rows and columns. This definition is somewhat complicated to state and even more complicated to express mathematically, which we do now.

There are two formulas for Δ that we use here. The first one is to sum the products $g_{ab} c^{ab}$ across row a. However, we have to be careful on how to express this in Einstein notation. In non-Einstein notation, we can write

$$\Delta = \sum_{b=1}^{N} g_{ab} c^{ab}.$$

However, in Einstein notation, $g_{ab}\, c^{ab}$ sums over not just row a, but over all rows, which results in $\Delta + \ldots + \Delta$, N times. Thus, in Einstein notation we express this sum as

$$g_{ab}\, c^{ab} = N\,\Delta. \tag{a}$$

A related formula is:

$$g_{ab}\, g^{ab} = N \tag{b}$$

$$g_{ab}\, g^{ab} = g_{ab}\, g^{ad} \delta^b_d \overset{(1.78)}{=} \delta^d_b\, \delta^b_d = N.$$

Putting these two formulas together we get:

$$c^{ab} = \Delta\, g^{ab} \qquad : \tag{c}$$

$$g_{ab}\, c^{ab} \overset{(a)}{=} N\,\Delta \overset{(b)}{=} g_{ab}\, g^{ab} \Delta \quad \Rightarrow \qquad c^{ab} = g^{ab} \Delta \qquad ✓$$

The second definition of Δ is more convoluted. We express Δ mathematically exactly as stated in words, above:

$$\Delta \equiv \epsilon^{b_1 \ldots b_N}\, g_{1\, b_1} \cdots g_{N\, b_N}, \tag{d}$$

by taking advantage of the **Levi-Civita symbol,** $\epsilon^{b_1 \ldots b_N}$**,** which we now explain.

This formula for Δ uses Einstein notation to sum over all of the different combinations of indices b_k. The indices b_k range over the integers 1 – N for all k. The symbol $\epsilon^{b_1 \dots b_N}$ is defined to be zero unless each integer appears exactly once; that is, unless $b_1, \dots, b_N$ is a permutation of 1, 2, ..., N. For permutations, its value is defined as +1 for **even permutations** and -1 for **odd permutations**. A permutation is even if the number of pairwise swaps required to restore the natural order is even, and odd, otherwise. For example, 1, 3, 2, 4 is an odd permutation because it requires just one swap. 1, 3, 4, 2 is an even permutation.

For example, for $N = 3$, $\Delta = \epsilon^{b_1 b_2 b_3} g_{1 b_1}\, g_{2 b_2}\, g_{3 b_3}$. Consider $b_1=1$, $b_2=2$, $b_3=3$. That term in equation (d) is $\epsilon^{123} g_{11}\, g_{22}\, g_{33} = (-1)^0\, g_{11}\, g_{22}\, g_{33}$ because zero permutations are required to restore the natural order. But, for $b_1=1$, $b_2=3$, $b_3=2$, the term is $\epsilon^{132} g_{11}\, g_{23}\, g_{32} = (-1)^1 g_{11}\, g_{23}\, g_{32}$ because one permutation is needed.

We call $\epsilon^{b_1 \dots b_N}$ the Levi-Civita *symbol* because it is a pseudo-tensor, almost a tensor except that it reverses sign under Jacobian transformations having negative determinant. It is not quite proper to call it the Levi-Civita tensor.

The cofactor, also, can be defined in Levi-Civita terminology since it is a determinant. Consider the following attempt at a cofactor expression:

$$c^{a\,b} \equiv \epsilon^{k_1 \dots k_{a-1}\, b\, k_{a+1} \dots k_N}\, g_{1\,k_1} \cdots g_{a-1\,k_{a-1}}\, g_{a+1\,k_{a+1}} \cdots g_{N\,k_N}. \qquad \text{(e)}$$

Every row appears in $g_{1\,k_1} \dots g_{a-1\,k_{a-1}}\, g_{a+1\,k_{a+1}} \dots g_{N\,k_N}$ except row a. The exponent of ϵ ranges over all strings $k_1 \dots k_{a-1}\, b\;\, k_{a+1} \dots k_N$, and ϵ is non-zero only for strings that are permutations of 1 – N. So, none of the non-zero terms have any k_a equal to b. Thus, when a string is a permutation, every column except b appears in $g_{1\,k_1} \dots g_{a-1\,k_{a-1}}\, g_{a+1\,k_{a+1}} \dots g_{N\,k_N}$. So far, so good.

Index b is in the ath superscript position. Will that produce the correct sign for the term $g_{1\,k_1} \dots g_{a-1\,k_{a-1}}\, g_{a+1\,k_{a+1}} \dots g_{N\,k_N}$? It requires $|b–a|$ index swaps to move b from position b to position a. So, $\epsilon^{1 \dots a-1\, b\; a+1 \dots N} = (-1)^{b-a}$. The term $g_{11} \dots g_{a-1\,a-1}\, g_{a+1\,a+1} \dots g_{N\,N}$ in the minor $m^{a\,b}$ should have a coefficient of +1 because it is the diagonal term of a matrix. Since $b - a$ is even or odd according to whether $a + b$ is even or odd, the product

$$\epsilon^{1\,\ldots\,a-1\;b\;a+1\,\ldots\,N}\, g_{11}\,\cdots\, g_{a-1\,a-1}\, g_{a+1\,a+1}\,\cdots\, g_{N\,N}$$
$$= (-1)^{b-a}\, g_{11}\,\ldots\, g_{a-1\,a-1}\, g_{a+1\,a+1}\,\cdots\, g_{N\,N}$$
$$= (-1)^{a+b}\, g_{11}\,\ldots\, g_{a-1\,a-1}\, g_{a+1\,a+1}\,\cdots\, g_{N\,N}$$

has the correct sign for this term in the cofactor $c^{a\,b} = (-1)^{a+b}\, m^{a\,b}$, and similarly all other terms have the correct sign.

Thus, expression (e) has the correct sign as well as the correct rows and columns for the cofactor $c^{a\,b}$. However, in proper tensor notation, when a and b are superscripts on LHS then they must be superscripts on RHS, but index a appears in the RHS subscripts (as well as in the RHS superscripts). We can try to remedy this with a Kronecker delta, like

$$\delta_a^s\; \epsilon^{k_1\,\ldots\,k_{a-1}\;b\;k_{a+1}\,\ldots\,k_N}\, g_{1\,k_1}\,\ldots\, g_{s-1\,k_{s-1}}\, g_{s+1\,k_{s+1}}\,\cdots\, g_{N\,k_N}.$$

In this attempt, index a no longer appears among the subscripts of g, but it now appears as a subscript in the Kronecker delta. To solve this new problem, we toss in a summation sign:

$$c^{a\,b} \equiv \sum_{s=1}^{N} \delta_s^a\; \epsilon^{k_1\,\ldots\,k_{s-1}\;b\;k_{s+1}\,\ldots\,k_N}\, g_{1\,k_1}\,\ldots\, g_{s-1\,k_{s-1}}\, g_{s+1\,k_{s+1}}\,\cdots\, g_{N\,k_N}. \tag{f}$$

In this summation we throw away all terms except when $s = a$, making this identical to the prior expression (e) while adhering to the Einstein summation convention.

Finally, we can compute $\partial_c\Delta$ from equation (d), which we re-express here:

$$\Delta \equiv \epsilon^{k_1\,\ldots\,k_N}\, g_{1\,k_1}\cdots g_{a-1\,k_{a-1}}\, g_{a\,k_a}\, g_{a+1\,k_{a+1}}\cdots\, g_{N\,k_N}. \tag{g}$$

Using the product rule to compute $\partial_c\Delta$, consider the ath term:

$$\epsilon^{k_1\,\ldots\,k_{a-1}\,k_a\,k_{a+1}\,\ldots\,k_N}\, \partial_c g_{a\,k_a}\, g_{1\,k_1}\cdots g_{a-1\,k_{a-1}}\, g_{a+1\,k_{a+1}}\cdots\, g_{N\,k_N}.$$

Replace the sum over k_a with a sum over b:

$$\epsilon^{k_1\,\ldots\,k_{a-1}\,b\,k_{a+1}\,\ldots\,k_N}\, \partial_c g_{a\,b}\, g_{1\,k_1}\cdots g_{a-1\,k_{a-1}}\, g_{a+1\,k_{a+1}}\cdots\, g_{N\,k_N}.$$

So,

$$\partial_c\Delta \;= \sum_{s=1}^{N}\, \delta_s^a\; \partial_c g_{s\,b}\, \epsilon^{k_1\ldots\,k_{s-1}\;b\;k_{s+1}\ldots\,k_N}\, g_{1\,k_1}\cdots g_{s-1\,k_{s-1}}\, g_{s+1\,k_{s+1}}\cdots\, g_{N\,k_N}. \tag{h}$$

We can simplify equation (h) by moving $\partial_c g_{s\,b}$ to the left of the summation sign. It becomes $\partial_c g_{a\,b}$ since the summations are all zero except when $s = a$, and this completes the solution:

$$\partial_c \Delta = \partial_c g_{a\,b} \sum_{s=1}^{N} \delta_s^a \; \epsilon^{k_1 \ldots k_{b-1}\, b\, k_{b+1} \ldots k_N} \, g_{1\,k_1} \ldots g_{s-1\,k_{s-1}} \, g_{s+1\,k_{s+1}} \cdots g_{N\,k_N}$$

$$\overset{(f)}{=} \partial_c g_{a\,b} \, c^{a\,b} \qquad ✓$$

$$\overset{(c)}{=} \Delta \, g^{a\,b} \, \partial_c g_{a\,b} \qquad ✓$$