Exercise 2.2.7 (not in book, but needed). Let Δ = det G be the **determinant of *G*,** where $G = (g_{ab})$ is the metric tensor matrix. The **minor**, m^{ab}, for element g_{ab} is the determinant of the submatrix that excludes row a and column b. The **cofactor** for element g_{ab} is $c^{ab} = (-1)^{a+b}\, m^{ab}$. Show that

$$\boxed{\partial_c \Delta = c^{ab}\, \partial_c g_{ab} = \Delta g^{ab}\, \partial_c g_{ab}}\,.$$

Solution Note: This derivation is valid for any matrix g_{ab}, not just the metric tensor.

By definition, Δ is the sum of all possible products of the form $(-1)^{a_1+b_1}\, g_{a_1 b_1} \cdot \ldots \cdot g_{a_N b_N}$ where the factors $g_{a_k b_k}$ are in distinct rows and columns. This definition is somewhat complicated to state and even more complicated to express mathematically, which we do now.

There are two formulas for Δ that we use here. The first one is to sum the products $g_{ab}c^{ab}$ across row a. However, we have to be careful on how to express this in Einstein notation. In non-Einstein notation, we can write

$$\Delta = \sum_{b=1}^{N} g_{ab}c^{ab}.$$

However, in Einstein notation, $g_{ab}\, c^{ab}$ sums over not just row a, but over all rows, which results in $\Delta + \ldots + \Delta$, N times. Thus, in Einstein notation we express this sum as

$$g_{ab}\, c^{ab} = N\,\Delta. \tag{a}$$

A related formula is:

$$g_{ab}\, g^{ab} = N \tag{b}$$

$$g_{ab}\, g^{ab} = g_{ab}\, g^{ad}\delta^b_d \overset{(1.78)}{=} \delta^d_b\, \delta^b_d = N.$$

Putting these two formulas together we get:

$$c^{ab} = \Delta\, g^{ab} \qquad\qquad : \tag{c}$$

$$g_{ab}\, c^{ab} \overset{(a)}{=} N\,\Delta \overset{(b)}{=} g_{ab}\, g^{ab}\Delta \quad \Rightarrow \qquad c^{ab} = g^{ab}\Delta \qquad ✓$$

The second definition of Δ is more convoluted. We express Δ mathematically exactly as stated in words, above:

$$\Delta \equiv \epsilon^{b_1 \ldots b_N}\, g_{1\,b_1} \ldots g_{N\,b_N}, \tag{d}$$

by taking advantage of the **Levi-Civita symbol,** $\epsilon^{b_1 \ldots b_N}$**,** which we now explain.

This formula for Δ uses Einstein notation to sum over all of the different combinations of indices b_k. The indices b_k range over the integers 1 – N for all k. The symbol $\epsilon^{b_1 \ldots b_N}$ is defined to be zero unless each integer appears exactly once; that is, unless b_1, ..., b_N is a permutation of 1, 2, ..., N. For permutations, its value is defined as +1 for **even permutations** and -1 for **odd permutations**. A permutation is even if the number of pairwise swaps required to restore the natural order is even, and odd, otherwise. For example, 1, 3, 2, 4 is an odd permutation because it requires just one swap. 1, 3, 4, 2 is an even permutation. So, in the determinant for $N = 3$, the product $g_{11}\, g_{22}\, g_{33}$ is preceded by a plus sign, represented by $\epsilon^{b_1 b_2 b_3} = \epsilon^{123}$, and the product $g_{11}\, g_{23}\, g_{32}$ is preceded by a negative sign, ϵ^{132}.

We call $\epsilon^{b_1 \ldots b_N}$ the Levi-Civita *symbol* because it is a pseudo-tensor, almost a tensor except that it reverses sign under Jacobian transformations having negative determinant. It is not quite proper to call it the Levi-Civita tensor.

The cofactor, also, can be defined in Levi-Civita terminology since it is a determinant. Consider the following attempt at a cofactor expression:

$$c^{a\,b} \equiv \epsilon^{k_1 \ldots k_{a-1}\, b\, k_{a+1} \ldots k_N}\, g_{1\,k_1} \ldots g_{a-1\,k_{a-1}}\, g_{a+1\,k_{a+1}} \ldots g_{N\,k_N}.$$

Every row appears in $g_{1\,k_1} \ldots g_{a-1\,k_{a-1}}\, g_{a+1\,k_{a+1}} \ldots g_{N\,k_N}$ except row a. The exponent of ϵ ranges over all strings $k_1 \ldots k_{a-1}\; b\; k_{a+1} \ldots k_N$, and ϵ is non-zero only for strings that are permutations of 1 – N. So, none of the non-zero terms have any k_a equal to b. Thus, when a string is a permutation, every column except b appears in $g_{1\,k_1} \ldots g_{a-1\,k_{a-1}}\, g_{a+1\,k_{a+1}} \ldots g_{N\,k_N}$. So far, so good.

Index b is in the ath superscript position. Will that produce the correct sign for the term $g_{1\,k_1} \ldots g_{a-1\,k_{a-1}}\, g_{a+1\,k_{a+1}} \ldots g_{N\,k_N}$? It requires $|b–a|$ index swaps to move b from position b to position a. So, $\epsilon^{1 \ldots a-1\; b\; a+1 \ldots N} = (-1)^{b-a}$. The term $g_{11} \ldots g_{a-1\,a-1}\, g_{a+1\,a+1} \ldots g_{N\,N}$ should have a coefficient of +1. Since ϵ contains a factor of $(-1)^{b-a}$, we must precede this term by another $(-1)^{b-a}$. We therefore modify our first attempt and try

$$c^{a\,b} \equiv (-1)^{b-a}\, \epsilon^{k_1 \ldots k_{a-1}\, b\, k_{a+1} \ldots k_N}\, g_{1\,k_1} \cdots g_{a-1\,k_{a-1}}\, g_{a+1\,k_{a+1}} \cdots g_{N\,k_N}.$$

This expression has the correct sign as well as the correct rows and columns for the cofactor $c^{a\,b}$. However, in proper tensor notation, when a and b are superscripts on LHS then they must be superscripts on RHS, but index a appears in the RHS subscripts as well as the RHS superscripts. We can try to remedy this with a Kronecker delta, like

$$(-1)^{b-a} \delta_a^s\, \epsilon^{k_1 \ldots k_{a-1}\, b\, k_{a+1} \ldots k_N}\, g_{1\,k_1} \cdots g_{s-1\,k_{s-1}}\, g_{s+1\,k_{s+1}} \cdots g_{N\,k_N}.$$

In this attempt, index a no longer appears among the subscripts of g, but it now appears as a subscript in the Kronecker delta. To solve that problem, we toss in a summation sign:

$$c^{a\,b} \equiv (-1)^{b-a} \sum_{s=1}^{N} \delta_s^a\, \epsilon^{k_1 \ldots k_{s-1}\, b\, k_{s+1} \ldots k_N}\, g_{1\,k_1} \cdots g_{s-1\,k_{s-1}}\, g_{s+1\,k_{s+1}} \cdots g_{N\,k_N}. \tag{e}$$

In this summation we throw away all terms except when $s = a$, making this identical to the prior expression while adhering to the Einstein summation convention.

Since this is a little confusing, we give one example.

Example To illustrate cofactor definition (e), let $a = 3$, $b = 5$, and $N = 7$. We can replace the sum by letting $s = 3$:

$$c^{\,3\times 5} \equiv (-1)^{5-3}\, \epsilon^{k_1\; k_2\; 5\;\; k_4 \ldots\; k_7}\, g_{1\,k_1}\, g_{2\,k_2}\, g_{4\,k_4} \cdots g_{7\,k_7}.$$

Einstein summation occurs over every k_a except k_3 and so includes sequences with repeated numbers, but the only non-zero values of ϵ are when the k_a's include one each of the values 1, 2, 3, 4, 6, 7 in any order. Thus, as desired, $g_{1\,k_1}\, g_{2\,k_2}\, g_{4\,k_4} \ldots g_{7\,k_7}$ pairs every possible column with every possible row, excluding row 3 and column 5.

This expression is preceded by $(-1)^2 = +1$ because the number of pairwise swaps for b=5 to be the fifth superscript differs by two swaps from it being the 3rd superscript. ■

Finally, we can compute $\partial_c \Delta$ from equation (d), which we re-express here:

$$\Delta \equiv \epsilon^{k_1 \ldots k_N}\, g_{1\,k_1} \ldots g_{a-1\,k_{a-1}}\, g_{a\,k_a}\, g_{a+1\,k_{a+1}} \cdots g_{N\,k_N}. \tag{f}$$

Then

$$\begin{aligned}
\partial_c\Delta \;=\;& (-1)^{b-1}\, \partial_c g_{1\,b}\, \epsilon^{b\,k_2 \ldots\, k_N} g_{2\,k_2} \ldots\, g_{N\,k_N} \\
&+ \ldots \\
&+ \;(-1)^{b-a}\, \partial_c g_{a\,b}\, \epsilon^{k_1 \ldots\, k_{a-1}\, b\;\, k_{a+1} \ldots\, k_N}\, g_{1\,k_1} \ldots g_{a-1\,k_{a-1}}\, g_{a+1\,k_{a+1}} \cdots\, g_{N\,k_N} \\
&+ \ldots \\
&+ \;(-1)^{b-N}\, \partial_c g_{N\,b}\, \epsilon^{k_1 \ldots\, k_{N-1}\, b} g_{1\,k_1} \ldots\, g_{N-1\,k_{N-1}} \\
&= \sum_{s=1}^{N} \delta_s^a\, (-1)^{b-s}\, \partial_c g_{s\,b}\, \epsilon^{k_1 \ldots\, k_{s-1}\, b\;\, k_{s+1} \ldots\, k_N}\, g_{1\,k_1} \ldots g_{s-1\,k_{s-1}}\, g_{s+1\,k_{s+1}} \cdots\, g_{N\,k_N}. \qquad \text{(g)}
\end{aligned}$$

Observe that we needed to toss in $(-1)^{b-a}$ terms again because when index b is in position a, we have to again compensate for the fact that ϵ contains a $(-1)^{b-a}$ factor.

We can simplify equation (g) by moving $(-1)^{b-s}\, \partial_c g_{s\,b}$ to the left of the summation sign. It becomes $\partial_c g_{a\,b}\, (-1)^{b-a}$ since the summations are all zero except when $s = a$, and this completes the solution:

$$\begin{aligned}
\partial_c\Delta \;&= \partial_c g_{a\,b}\, (-1)^{b-a} \sum_{s=1}^{N} \delta_s^a\; \epsilon^{k_1 \ldots\, k_{b-1}\, b\;\, k_{b+1} \ldots\, k_N}\, g_{1\,k_1} \ldots g_{s-1\,k_{s-1}}\, g_{s+1\,k_{s+1}} \cdots\, g_{N\,k_N} \\
&\overset{(e)}{=}\; \partial_c g_{a\,b}\; c^{a\,b} \qquad \checkmark \\
&\overset{(c)}{=}\; \Delta\, g^{a\,b}\, \partial_c g_{a\,b} \qquad \checkmark
\end{aligned}$$