Exercise 2.2.7 (not in book, but needed). Let Δ = det G be the **determinant of *G*,** where $G = (g_{ab})$ is the metric tensor matrix. The **minor**, m^{ab}, for element g_{ab} is the determinant of the submatrix that excludes row a and column b. The **cofactor** for element g_{ab} is $c^{ab} = (-1)^{a+b}\, m^{ab}$. Show that

$$\boxed{\partial_c \Delta = c^{ab}\, \partial_c g_{ab} = \Delta g^{ab}\, \partial_c g_{ab}}\,.$$

Solution Δ is defined as a sum of signed products of permutations of the elements g_{ab}. G has N rows. If $b_1, b_2, \ldots, b_n$ are the row numbers 1–N in any random order, this constitutes a **permutation** of the rows 1, 2, ... , N. Let $\mathcal{P}$ denote the set of all such permutations. **A permutation is even (odd)** if it takes an even (odd) number of pairwise swaps of adjacent numbers to restore the natural order. For example, 1, 3, 2, 4 is an odd permutation because it requires just one swap. 1, 3, 4, 2 is an even permutation. If π is a permutation, define **sign(π)** as +1 if π is even, and -1 when π is odd.

The definition of **determinant of *G*** is

$$\Delta = \sum_{\pi \in \mathcal{P}} \text{sign}(\pi)\, g_{1,\pi(1)}\, g_{2,\pi(2)} \cdots g_{N,\pi(N)}$$

There are various formulas for computing the determinant. We use two formulas here. The first one is to sum the products $g_{ab}c^{ab}$ across row a. However, we have to be careful on how to express this in Einstein notation. In non-Einstein notation, we can write

$$\Delta = \sum_{b=1}^{N} g_{ab}c^{ab}.$$

However, in Einstein notation, $g_{ab}\, c^{ab}$ sums over not just row a, but over all rows, which results in $\Delta + \ldots + \Delta$, N times. Thus, in Einstein notation we express this sum as

$$g_{ab}\, c^{ab} = N\, \Delta. \qquad \text{(a)}$$

A related formula is:

$$g_{ab}\, g^{ab} = N \qquad \text{(b)}$$

$$g_{ab}\, g^{ab} = g_{ab}\, g^{ad} \delta^b_d \overset{(1.78)}{=} \delta^d_b\, \delta^b_d = N.$$

Putting these two formulas together we get:

$$c^{ab} = \Delta\, g^{ab} \qquad : \qquad \text{(c)}$$

$$g_{a\,b}\,c^{a\,b} \overset{(a)}{=} N\,\Delta \overset{(b)}{=} g_{a\,b}\,g^{a\,b}\Delta \quad \Rightarrow \qquad c^{a\,b} = g^{a\,b}\Delta \qquad ✓$$

The second formula for Δ is expressed in terms of permutations,

$$\Delta \equiv \epsilon^{b_1 \ldots b_N}\, g_{1\,b_1} \ldots g_{N\,b_N}, \tag{d}$$

by taking advantage of the **Levi-Civita symbol, $\epsilon^{b_1 \ldots b_N}$,** which we now explain.

This formula for Δ uses Einstein notation to sum over all of the different combinations of indices b_k. The indices b_k range over the integers 1 – N for all k. The symbol $\epsilon^{b_1 \ldots b_N}$ is defined to be zero unless each integer appears exactly once; that is, unless b_1, ..., b_N is a permutation of 1, 2, ..., N. For permutations, its value is defined as +1 for **even permutations** and -1 for **odd permutations**.

For example, for $N = 3$, $\Delta = \epsilon^{b_1\,b_2\,b_3} g_{1\,b_1}\; g_{2\,b_2}\; g_{3\,b_3}$. Consider b_1=1, b_2=2, b_3=3. That term in equation (d) is $\epsilon^{123} g_{11}\; g_{22}\; g_{33} = (-1)^0\; g_{11}\; g_{22}\; g_{33}$ because zero permutations are required to restore the natural order. But, for b_1=1, b_2=3, b_3=2, the term is $\epsilon^{132} g_{11}\; g_{23}\; g_{32} = (-1)^1 g_{11}\; g_{23}\; g_{32}$ because one permutation is needed.

We call $\epsilon^{b_1 \ldots b_N}$ the Levi-Civita *symbol* because it is a pseudo-tensor, almost a tensor except that it reverses sign under Jacobian transformations having negative determinant. It is not quite proper to call it the Levi-Civita tensor.

The cofactor, also, can be defined in Levi-Civita terminology since it is a determinant. Consider the following attempt at a cofactor expression:

$$c^{a\,b} \equiv \epsilon^{k_1 \ldots k_{a-1}\, b\; k_{a+1} \ldots k_N}\, g_{1\,k_1} \ldots g_{a-1\,k_{a-1}}\; g_{a+1\,k_{a+1}} \ldots g_{N\,k_N} \tag{e}$$

Every row appears in $g_{1\,k_1} \ldots g_{a-1\,k_{a-1}}\; g_{a+1\,k_{a+1}} \ldots g_{N\,k_N}$ except row a. The exponent of ϵ ranges over all strings $k_1 \ldots k_{a-1}\; b\; k_{a+1} \ldots k_N$, and ϵ is non-zero only for strings that are permutations of 1 – N. So, none of the non-zero terms have any k_a equal to b. Thus, when a string is a permutation, every column except b appears in $g_{1\,k_1} \ldots g_{a-1\,k_{a-1}}\; g_{a+1\,k_{a+1}} \ldots g_{N\,k_N}$. So far, so good.

Index b is in the ath superscript position. Will that produce the correct sign for the term $g_{1\,k_1} \ldots g_{a-1\,k_{a-1}}\; g_{a+1\,k_{a+1}} \ldots g_{N\,k_N}$? It requires $|b-a|$ index swaps to move b from position b to position a. So, $\epsilon^{1 \ldots a-1\; b\; a+1 \ldots N} = (-1)^{b-a}$. The term

$g_{11} \cdots g_{a-1\,a-1}\, g_{a+1\,a+1} \cdots g_{N\,N}$ in the minor $m^{a\,b}$ should have a coefficient of +1 because it is the diagonal term of a matrix. Since $b - a$ is even or odd according to whether $a + b$ is even or odd, the product

$$\begin{aligned}
&\epsilon^{1 \ldots a-1\; b\; a+1 \ldots N}\, g_{11} \cdots g_{a-1\,a-1}\, g_{a+1\,a+1} \cdots g_{N\,N} \\
&= (-1)^{b-a}\, g_{11} \cdots g_{a-1\,a-1}\, g_{a+1\,a+1} \cdots g_{N\,N} \\
&= (-1)^{a+b}\, g_{11} \cdots g_{a-1\,a-1}\, g_{a+1\,a+1} \cdots g_{N\,N}
\end{aligned}$$

has the correct sign for this term in the cofactor $c^{a\,b} = (-1)^{a+b}\, m^{a\,b}$, and similarly all other terms have the correct sign.

Thus, expression (e) has the correct sign as well as the correct rows and columns for the cofactor $c^{a\,b}$. However, in proper tensor notation, when a and b are superscripts on LHS then they must be superscripts on RHS, but index a appears in the RHS subscripts (as well as in the RHS superscripts). We can try to remedy this with a Kronecker delta, like

$$\delta_a^s\; \epsilon^{k_1 \ldots k_{a-1}\; b\; k_{a+1} \ldots k_N}\, g_{1\,k_1} \cdots g_{s-1\,k_{s-1}}\, g_{s+1\,k_{s+1}} \cdots g_{N\,k_N}.$$

In this attempt, index a no longer appears among the subscripts of g, but it now appears as a subscript in the Kronecker delta. To solve this new problem, we toss in a summation sign:

$$c^{a\,b} \equiv \sum_{s=1}^{N} \delta_s^a\; \epsilon^{k_1 \ldots k_{s-1}\; b\; k_{s+1} \ldots k_N}\, g_{1\,k_1} \cdots g_{s-1\,k_{s-1}}\, g_{s+1\,k_{s+1}} \cdots g_{N\,k_N}. \qquad \text{(f)}$$

In this summation we throw away all terms except when $s = a$, making this identical to the prior expression while adhering to the Einstein summation convention.

Now we can compute $\partial_c \Delta$ from equation (d), which we re-express here:

$$\Delta \equiv \epsilon^{k_1 \ldots k_N}\, g_{1\,k_1} \cdots g_{a-1\,k_{a-1}}\, g_{a\,k_a}\, g_{a+1\,k_{a+1}} \cdots g_{N\,k_N}. \qquad \text{(g)}$$

Using the product rule to compute $\partial_c \Delta$:

$$\begin{aligned}
\partial_c \Delta = \;& \partial_c g_{1\,k_1}\, \epsilon^{b\, k_2 \ldots k_N} g_{2\,k_2} \cdots g_{N\,k_N} \\
&+ \;\ldots \\
&+ \;\partial_c g_{a\,k_a}\, \epsilon^{k_1 \ldots k_{a-1}\, k_a\, k_{a+1} \ldots k_N}\, g_{1\,k_1} \cdots g_{a-1\,k_{a-1}}\, g_{a+1\,k_{a+1}} \cdots g_{N\,k_N} \qquad \text{(h)} \\
&+ \;\ldots \\
&+ \partial_c g_{N\,k_N}\, \epsilon^{k_1 \ldots k_{N-1}\, b} g_{1\,k_1} \cdots g_{N-1\,k_{N-1}}
\end{aligned}$$

Consider the *a*th term. If we replace k_a by *b,* we get

$$\partial_c g_{a\,b}\,\epsilon^{k_1\,\ldots\,k_{a-1}\,b\,k_{a+1}\,\ldots\,k_N}\;g_{1\,k_1}\ldots g_{a-1\,k_{a-1}}\;g_{a+1\,k_{a+1}}\cdots\,g_{N\,k_N}.$$

This represents a sum over *b* instead of a sum over k_a. We wish to make this summation imitate equation (f), so we first insert a Kronecker delta to replace *a* by *s*:

$$\delta^a_s\;\partial_c g_{s\,b}\,\epsilon^{k_1\ldots\,k_{s-1}\;b\;k_{s+1}\ldots\,k_N}\;g_{1\,k_1}\ldots g_{s-1\,k_{s-1}}\;g_{s+1\,k_{s+1}}\cdots\,g_{N\,k_N}.$$

This does not change the value of the *a*th term. Next we can sum over *s* and it still will not change the value of the *a*th term:

$$\sum_{s=1}^{N}\delta^a_s\;\partial_c g_{s\,b}\,\epsilon^{k_1\ldots\,k_{s-1}\;b\;k_{s+1}\ldots\,k_N}\;g_{1\,k_1}\ldots g_{s-1\,k_{s-1}}\;g_{s+1\,k_{s+1}}\cdots\,g_{N\,k_N}. \qquad \text{(i)}$$

We can simplify equation (i) by moving $\partial_c g_{s\,b}$ to the left of the summation sign. It becomes $\partial_c g_{a\,b}$ since the summations are all zero except when $s = a$. However, in so doing , we are now summing over *a*, resulting in RHS of equation (h), not just the *a*th term. That is,

$$\partial_c\Delta = \partial_c g_{a\,b}\sum_{s=1}^{N}\delta^a_s\;\epsilon^{k_1\ldots\,k_{b-1}\;b\;k_{b+1}\ldots\,k_N}\;g_{1\,k_1}\ldots g_{s-1\,k_{s-1}}\;g_{s+1\,k_{s+1}}\cdots\,g_{N\,k_N} \qquad \text{(j)}$$

We can now complete the solution. We are summing over both *a* and *b*.

$$\begin{aligned}\partial_c\Delta &\overset{\text{(j)}}{=} \partial_c g_{a\,b}\sum_{s=1}^{N}\delta^a_s\;\epsilon^{k_1\ldots\,k_{b-1}\;b\;k_{b+1}\ldots\,k_N}\;g_{1\,k_1}\ldots g_{s-1\,k_{s-1}}\;g_{s+1\,k_{s+1}}\cdots\,g_{N\,k_N}\\ &\overset{\text{(f)}}{=} \partial_c g_{a\,b}\;c^{a\,b} \qquad \checkmark\\ &\overset{\text{(c)}}{=} \Delta\;g^{a\,b}\;\partial_c g_{a\,b} \qquad \checkmark\end{aligned}$$