Ex 1.2.4. Coord sys is orthogonal means $g_{ij} \overset{defn}{=} e_i \cdot e_j = K_{ij}\delta^i_j$. $\therefore G = \begin{pmatrix} K_{11} & & 0 \\ & K_{22} & \\ 0 & & K_{33} \end{pmatrix}$

Ex 1.2.6 In Example 1.2.1, $\mu_i = v\delta^1_i - u\delta^2_i + \delta^3_i$. Find μ^i.

$$M^* = (\mu_i) = \begin{pmatrix} v \\ -u \\ 1 \end{pmatrix}$$

$$(\mu^i) = M = \hat{G} M^* = \begin{pmatrix} \frac{1}{2} & 0 & -v \\ 0 & \frac{1}{2} & -u \\ -v & -u & 2u^2+2v^2+1 \end{pmatrix} \begin{pmatrix} v \\ -u \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}v \\ -\frac{3}{2}u \\ v^2+3u^2+1 \end{pmatrix}$$

$$\Rightarrow \mu^i = -\tfrac{1}{2}v\delta^1_i - \tfrac{3}{2}u\delta^2_i + (v^2+3u^2+1)\delta^3_i$$

Ex 1.2.7 (a) $\delta^i_i = \delta^1_1 + \delta^2_2 + \delta^3_3 = 3$ (b) $\delta^A_A = \delta^1_1 + \delta^2_2 = 2$

(c) $\delta^a_a = \delta^{a_1}_{a_1} + \cdots + \delta^{a_n}_{a_n} = n$ (d) $\delta^\mu_\mu = \delta^1_1 + \cdots + \delta^4_4 = 4$

Ex 1.4.1 Show $U^k_{i'} U^{i'}_j = \delta^k_j$ and $U^{k'}_i U^i_{j'} = \delta^{k'}_{j'}$

$$\delta^k_j = \frac{\partial u^k}{\partial u^j} \underset{\text{Rule}}{\overset{\text{ch.}}{=}} \frac{\partial u^k}{\partial u^{i'}} \frac{\partial u^{i'}}{\partial u^j} = U^k_{i'} U^{i'}_j$$

Ex 1.6.2 Find line element ds^2 for surfaces

(a) sphere of radius a: $u=\theta$, $v=\phi$

From Example 1.1

$\vec{e}_\theta = a\cos\theta\cos\phi\,\vec{i} + a\cos\theta\sin\phi\,\vec{j} - a\sin\theta\,\vec{k}$

$\vec{e}_\phi = -a\sin\theta\sin\phi\,\vec{i} + a\sin\theta\cos\phi\,\vec{j}$

$$d\vec{r} \overset{(1.37a)}{=} du^A\vec{e}_A = d\theta\,\vec{e}_\theta + d\phi\,\vec{e}_\phi$$

$$ds^2 \overset{(1.65)}{=} d\vec{r}\cdot d\vec{r} = a^2[(\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta)d\theta^2 + (\sin^2\theta\sin^2\phi + \sin^2\theta\cos^2\phi)d\phi^2]$$

$$= a^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

(b) cylinder of radius a: $u=\phi$, $v=z$

From Exercise 1.1.1

$\vec{e}_\phi = a\sin\phi\,\vec{i} + a\cos\phi\,\vec{j}$

$\vec{e}_z = \vec{k}$

$$d\vec{r} = \vec{e}_\phi d\phi + \vec{e}_z dz$$

$$ds^2 = d\vec{r}\cdot d\vec{r} = a^2(\sin^2\phi + \cos^2\phi)d\phi^2 + dz^2 = a^2 d\phi^2 + dz^2$$

(c) Hyperbolic paraboloid of Example 1.6.1:

$$\begin{cases} \vec{e}_u = \vec{i} + \vec{j} + 2v\vec{k} \\ \vec{e}_v = \vec{i} - \vec{j} + 2u\vec{k} \end{cases}$$

$$d\vec{r} = \vec{e}_u\,du + \vec{e}_v\,dv = (du+dv)\vec{i} + (du-dv)\vec{j} + 2(v\,du + u\,dv)\vec{k}$$

$$ds^2 = d\vec{r}\cdot d\vec{r} = 2(1+2v^2)du^2 + 2(1+2u^2)dv^2 + 8uv\,du\,dv$$