2.1.2 Let $\gamma = \{x^a(u)\}$ be a curve parameterized by u.

Suppose γ is an affinely parametized geodesic:

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0 \quad (2.12)$$

where $\dot{x}^a \equiv \frac{dx^a}{du}$ and

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}) \quad (2.13)$$

where $\partial_a \equiv \frac{\partial}{\partial x^a}$ and $g^{ab} = g^{ab}(u)$, $g_{ab} = g_{ab}(u)$, and so $\Gamma^a_{bc} = \Gamma^a_{bc}(u)$

Let L = length of the tangent vector $\dot{x}^a$. That is, $L(u)$ = length of $\dot{x}^a(u)$.

(a) Show $\pm L^2 = g_{ab}\dot{x}^a\dot{x}^b$

$$L \overset{(1.80)}{=} \sqrt{|g_{ab}\dot{x}^a\dot{x}^b|} \iff \pm L^2 = g_{ab}\dot{x}^a\dot{x}^b \checkmark$$

(b) Differentiate both sides

LHS: $\pm\frac{dL^2}{du} = \pm 2L\dot{L}$ ✓

RHS: $\frac{d}{du}(g_{ab}\dot{x}^a\dot{x}^b) = \dot{g}_{ab}\dot{x}^a\dot{x}^b + g_{ab}[\dot{x}^a\ddot{x}^b + \dot{x}^b\ddot{x}^a]$

Substitute $a\to b$ & $b \to a$ into: $g_{ab}\dot{x}^a\ddot{x}^b = g_{ba}\dot{x}^b\ddot{x}^a \overset{(1.30)}{=} g_{ab}\dot{x}^b\ddot{x}^a$

So $g_{ab}[\dot{x}^a\ddot{x}^b + \dot{x}^b\ddot{x}^a] = 2g_{ab}\dot{x}^b\ddot{x}^a$ ✓

$$\pm 2L\dot{L} = \dot{g}_{ab}\dot{x}^a\dot{x}^b + 2g_{ab}\dot{x}^b\ddot{x}^a \quad (1)$$

(c) Put $\dot{g}_{ab} = \partial_c g_{ab}\dot{x}^c$ and use (2.12) to express $\ddot{x}^a$ in terms of Γ^a_{bc} and $\dot{x}^a$

First $\dot{g}_{ab} = \frac{dg_{ab}}{du} = \frac{\partial g_{ab}}{\partial x^c}\frac{dx^c}{du} = (\partial_c g_{ab})\dot{x}^c$ ✓ Next we remove $\ddot{x}^a$ in (1).

From (2.12), $\ddot{x}^a = -\Gamma^a_{cd}\dot{x}^c\dot{x}^d$. So $2g_{ab}\dot{x}^b\ddot{x}^a = -2g_{ab}\Gamma^a_{cd}\dot{x}^c\dot{x}^d\dot{x}^b$

$$\therefore \pm 2L\dot{L} = \partial_c g_{ab}\dot{x}^c\dot{x}^a\dot{x}^b - 2g_{ab}\Gamma^a_{cd}\dot{x}^c\dot{x}^d\dot{x}^b \quad (2)$$

(d) Use (2.13) to replace Γ^a_{cd} in (2)

sending $b\to c$, $c \to d$, and $d \to e$ in (2.13): $\Gamma^a_{cd} = \frac{1}{2}g^{ae}(\partial_c g_{ed} + \partial_d g_{ce} - \partial_e g_{cd})$

$$\therefore \pm 2L\dot{L} \overset{(2)}{=} \partial_c g_{ab}\dot{x}^c\dot{x}^a\dot{x}^b - g_{ab}g^{ae}(\partial_c g_{ed} + \partial_d g_{ce} - \partial_e g_{cd})\dot{x}^c\dot{x}^d\dot{x}^b \quad (3)$$

(e) Simplify using $g_{ab}g^{ae} = \delta^e_b$

$$\pm 2L\dot{L} \overset{(3)}{=} \partial_c g_{ab}\dot{x}^c\dot{x}^a\dot{x}^b - (\partial_c g_{bd} + \partial_d g_{cb} - \partial_b g_{cd})\dot{x}^c\dot{x}^d\dot{x}^b$$

$$= \underbrace{(\partial_c g_{ab}\dot{x}^c\dot{x}^a\dot{x}^b - \partial_c g_{ab}\dot{x}^c\dot{x}^a\dot{x}^b)}_{=0} + (\partial_b g_{cd}\dot{x}^c\dot{x}^b\dot{x}^d - \partial_b g_{cd}\dot{x}^c\dot{x}^d\dot{x}^b) \xrightarrow{} 0$$

$\Rightarrow \underset{= \frac{dL}{du}}{\dot{L}} = 0 \Rightarrow L$ is constant ✓