Exercise A.8.1 Show that the Lorentz gauge condition (A.49) can be written as $A^{\mu}{}_{,\mu} = 0$

Soln: $A^{\mu}{}_{,\mu} = \frac{\partial A^{\mu}}{\partial x^{\mu}} \overset{(A.52)}{=} \frac{\partial(\phi/c)}{\partial ct} + \frac{\partial \vec{A}}{\partial x^{i}} = \left(\frac{1}{c^2}\frac{\partial \phi}{\partial t} + \frac{\partial A^1}{\partial x} + \frac{\partial A^2}{\partial y} + \frac{\partial A^3}{\partial z}\right) = \left(\frac{1}{c^2}\frac{\partial \phi}{\partial t} + \nabla\cdot\vec{A}\right)$

$= \nabla\cdot\vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = \nabla\cdot\vec{A} + \varepsilon_0\mu_0\frac{\partial \phi}{\partial t} \overset{(A.49)}{=} 0$ ∎

Exercise A.8.2. Confirm equation (A.54).

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & -\frac{1}{c}E^1 & -\frac{1}{c}E^2 & -\frac{1}{c}E^3 \\ \frac{1}{c}E^1 & 0 & B^3 & -B^2 \\ \frac{1}{c}E^2 & -B^3 & 0 & B^1 \\ \frac{1}{c}E^3 & B^2 & -B^1 & 0 \end{pmatrix}$$

Soln: $F_{\mu\nu} \overset{(A.53)}{=} A_{\mu,\nu} - A_{\nu,\mu} = \frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu}$ where $\vec{A}$ is a vector potential.

If $\mu=\nu$, $F_{\mu\mu} = A_{\mu,\mu} - A_{\mu,\mu} = 0$ ✓ (diagonal ✓)

Example A.0.1

$F_{01} = \frac{\partial A_0}{\partial x} - \frac{\partial A_1}{\partial ct} \overset{(A.52)}{=} \frac{1}{c}\frac{\partial \phi}{\partial x} + \frac{1}{c}\frac{\partial A^1}{\partial t}$ since $A^{\mu} \overset{(A.52)}{=} \left(\frac{\phi}{c}, \vec{A}\right)$ and $A_\mu = \left(\frac{\phi}{c}, -\vec{A}\right)$

and $\vec{A} = (A^1, A^2, A^3) = -(A_1, -A_2, -A_3)$

$\vec{E} \overset{(A.48)}{=} -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \Rightarrow E^1 = -\frac{\partial \phi}{\partial x} - \frac{\partial A^1}{\partial t}$. So $F_{01} = -\frac{1}{c}E^1$ ✓

Also, $\vec{B} \overset{(A.48)}{=} \nabla\times\vec{A} = (\nabla_2 A^3 - \nabla_3 A^2)\vec{i} + (\nabla_3 A^1 - \nabla_1 A^3)\vec{j} + (\nabla_1 A^2 - \nabla_2 A^1)\vec{k}$

$= \left(\frac{\partial A^3}{\partial y} - \frac{\partial A^2}{\partial z}\right)\vec{i} + \left(\frac{\partial A^1}{\partial z} - \frac{\partial A^3}{\partial x}\right)\vec{j} + \left(\frac{\partial A^2}{\partial x} - \frac{\partial A^1}{\partial y}\right)\vec{k}$

$F_{12} = \frac{\partial A_1}{\partial y} - \frac{\partial A_2}{\partial x} \overset{(A.52)}{=} -\frac{\partial A^1}{\partial y} + \frac{\partial A^2}{\partial x} \overset{(A.48)}{=} B^3$ ✓ ∎

Exercise A.8.3

$$(F^{\mu}{}_{\nu}) = (\eta^{\mu\rho})(F_{\rho\nu}) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}\begin{pmatrix} 0 & -\frac{1}{c}E^1 & -\frac{1}{c}E^2 & -\frac{1}{c}E^3 \\ \frac{1}{c}E^1 & 0 & B^3 & -B^2 \\ \frac{1}{c}E^2 & -B^3 & 0 & B^1 \\ \frac{1}{c}E^3 & B^2 & -B^1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{c}E^1 & -\frac{1}{c}E^2 & -\frac{1}{c}E^3 \\ -\frac{1}{c}E^1 & 0 & -B^3 & B^2 \\ -\frac{1}{c}E^2 & B^3 & 0 & -B^1 \\ -\frac{1}{c}E^3 & -B^2 & B^1 & 0 \end{pmatrix}$$

$$(F^{\mu\nu}) = (F^{\mu}{}_{\sigma})(\eta^{\sigma\nu}) = \begin{pmatrix} 0 & -\frac{1}{c}E^1 & -\frac{1}{c}E^2 & -\frac{1}{c}E^3 \\ -\frac{1}{c}E^1 & 0 & -B^3 & B^2 \\ -\frac{1}{c}E^2 & B^3 & 0 & -B^1 \\ -\frac{1}{c}E^3 & -B^2 & B^1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{c}E^1 & \frac{1}{c}E^2 & \frac{1}{c}E^3 \\ -\frac{1}{c}E^1 & 0 & B^3 & -B^2 \\ -\frac{1}{c}E^2 & -B^3 & 0 & B^1 \\ -\frac{1}{c}E^3 & B^2 & -B^1 & 0 \end{pmatrix}$$

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