

Exercise 2.2.7 (not in book, but needed). Let $\Delta = \det G$ be the **determinant of G** , where $G = (g_{ab})$ is the metric tensor matrix. The **minor**, m^{ab} , for element g_{ab} is the determinant of the submatrix that excludes row a and column b . The **cofactor** for element g_{ab} is $c^{ab} = (-1)^{a+b} m^{ab}$. Show that

$$\partial_c \Delta = c^{ab} \partial_c g_{ab} = \Delta g^{ab} \partial_c g_{ab}.$$

Solution By definition, Δ is the sum of all possible products of the form $(-1)^{a_1+b_1} g_{a_1 b_1} \cdot \dots \cdot g_{a_N b_N}$ where the factors $g_{a_k b_k}$ are in distinct rows and columns. This definition is somewhat complicated to state and even more complicated to express mathematically, which we do now.

There are two formulas for Δ that we use here. The first one is to sum the products $g_{ab} c^{ab}$ across row a . However, we have to be careful on how to express this in Einstein notation. In non-Einstein notation, we can write

$$\Delta = \sum_{b=1}^N g_{ab} c^{ab}.$$

However, in Einstein notation, $g_{ab} c^{ab}$ sums over not just row a , but over all rows, which results in $\Delta + \dots + \Delta$, N times. Thus, in Einstein notation we express this sum as

$$g_{ab} c^{ab} = N \Delta. \quad (a)$$

A related formula is:

$$g_{ab} g^{ab} = N \quad (b)$$

$$g_{ab} g^{ab} = g_{ab} g^{ad} \delta_d^b \stackrel{(1.78)}{=} \delta_b^d \delta_d^b = N.$$

Putting these two formulas together we get:

$$c^{ab} = \Delta g^{ab} \quad :$$

$$g_{ab} c^{ab} \stackrel{(a)}{=} N \Delta \stackrel{(b)}{=} g_{ab} g^{ab} \Delta \Rightarrow c^{ab} = g^{ab} \Delta \quad \checkmark \quad (c)$$

The second definition of Δ is more convoluted. We express Δ mathematically exactly as stated in words, above:

$$\Delta \equiv \epsilon^{b_1 \dots b_N} g_{1 b_1} \dots g_{N b_N}, \quad (d)$$

by taking advantage of the **Levi-Civita symbol**, $\epsilon^{b_1 \dots b_N}$, which we now explain.

This formula for Δ uses Einstein notation to sum over all of the different combinations of indices b_k . The indices b_k range over the integers $1 - N$ for all k . The symbol $\epsilon^{b_1 \dots b_N}$ is defined to be zero unless each integer appears exactly once; that is, unless b_1, \dots, b_N is a permutation of $1, 2, \dots, N$. For permutations, its value is defined as $+1$ for **even permutations** and -1 for **odd permutations**. A permutation is even if the number of pairwise swaps required to restore the natural order is even, and odd, otherwise. For example, $1, 3, 2, 4$ is an odd permutation because it requires just one swap. $1, 3, 4, 2$ is an even permutation. So, in the determinant for $N = 3$, the product $g_{11} g_{22} g_{33}$ is preceded by a plus sign, represented by $\epsilon^{b_1 b_2 b_3} = \epsilon^{123}$, and the product $g_{11} g_{23} g_{32}$ is preceded by a negative sign, ϵ^{132} .

We call $\epsilon^{b_1 \dots b_N}$ the *Levi-Civita symbol* because it is a pseudo-tensor, almost a tensor except that it reverses sign under Jacobian transformations having negative determinant. It is not quite proper to call it the Levi-Civita tensor.

The cofactor, also, can be defined in Levi-Civita terminology since it is a determinant. Consider the following attempt at a cofactor expression:

$$c^{ab} \equiv \epsilon^{k_1 \dots k_{a-1} b k_{a+1} \dots k_N} g_{1 k_1} \dots g_{a-1 k_{a-1}} g_{a+1 k_{a+1}} \dots g_{N k_N}.$$

Every row appears in $g_{1 k_1} \dots g_{a-1 k_{a-1}} g_{a+1 k_{a+1}} \dots g_{N k_N}$ except row a . The exponent of ϵ ranges over all strings $k_1 \dots k_{a-1} b k_{a+1} \dots k_N$, but ϵ is non-zero only for strings that are permutations of $1 - N$, and so none of the non-zero terms have any k_a equal to b . So, when the string is a permutation, every column appears in $k_1 \dots k_{a-1} b k_{a+1} \dots k_N$ and, so, every column except b appears in $g_{1 k_1} \dots g_{a-1 k_{a-1}} g_{a+1 k_{a+1}} \dots g_{N k_N}$. So far, so good. (Note that b must appear in the superscript of ϵ because the Levi-Civita symbol is only defined for strings of length N).

Is b in a superscript position that will produce the correct sign for the term $g_{1 k_1} \dots g_{a-1 k_{a-1}} g_{a+1 k_{a+1}} \dots g_{N k_N}$? Consider the case where $k_1=1, k_2=2, \dots, k_N=N$. Then we have $\epsilon^{1 \dots a-1 b a+1 \dots N} g_{11} \dots g_{a-1 a-1} g_{a+1 a+1} \dots g_{NN}$, and the correct sign should be $+1$. Note that b is in the a th position of the superscript. If we put it in the b th position, then we would have $\epsilon^{1 \dots b-1 b b+1 \dots N} = +1$. It requires $|b-a|$ index swaps to move b from position a to position b . So, $\epsilon^{1 \dots a-1 b a+1 \dots N} = (-1)^{b-a}$. In order to give $g_{11} \dots g_{a-1 a-1} g_{a+1 a+1} \dots g_{NN}$ a coefficient of $+1$, we must

precede it by $(-1)^{b-a} \epsilon^{1 \dots a-1 \ b \ a+1 \dots N}$. So, we must modify our first attempt. We try

$$c^{ab} \equiv (-1)^{b-a} \epsilon^{k_1 \dots k_{a-1} \ b \ k_{a+1} \dots k_N} g_{1 k_1} \dots g_{a-1 k_{a-1}} g_{a+1 k_{a+1}} \dots g_{N k_N}.$$

This expression has the correct sign as well as the correct rows and columns for the cofactor c^{ab} . However, in proper tensor notation, when a and b are superscripts on LHS then they must be superscripts on RHS, but index a appears in the RHS subscripts as well as the RHS superscripts. We can try to remedy this with a Kronecker delta, like

$$(-1)^{b-a} \delta_a^s \epsilon^{k_1 \dots k_{a-1} \ b \ k_{a+1} \dots k_N} g_{1 k_1} \dots g_{s-1 k_{s-1}} g_{s+1 k_{s+1}} \dots g_{N k_N}.$$

In this attempt, index a no longer appears among the subscripts of g , but it does appear as a subscript in the Kronecker delta. To solve that problem, we toss in a summation sign:

$$c^{ab} \equiv (-1)^{b-a} \sum_{s=1}^N \delta_s^a \epsilon^{k_1 \dots k_{s-1} \ b \ k_{s+1} \dots k_N} g_{1 k_1} \dots g_{s-1 k_{s-1}} g_{s+1 k_{s+1}} \dots g_{N k_N}. \quad (e)$$

In this summation we throw away all terms except when $s = a$, making this identical to the prior expression while adhering to the Einstein summation convention.

Example To illustrate cofactor definition (e), let $a = 3$, $b = 5$, and $N = 7$. We can replace the sum by letting $s = 3$:

$$c^{3 \cdot 5} \equiv (-1)^{5-3} \epsilon^{k_1 \ k_2 \ 5 \ k_4 \dots k_7} g_{1 k_1} g_{2 k_2} g_{4 k_4} \dots g_{7 k_7}.$$

Einstein summation occurs over every k_a except k_3 and so includes sequences with repeated numbers, but the only non-zero values of ϵ are when the k_a 's include one each of the values 1, 2, 3, 4, 6, 7 in any order. Thus, as desired, $g_{1 k_1} g_{2 k_2} g_{4 k_4} \dots g_{7 k_7}$ pairs every possible column with every possible row, excluding row 3 and column 5.

This expression is preceded by $(-1)^2 = +1$ because the number of pairwise swaps for $b=5$ to be the fifth superscript differs by two swaps from it being the 3rd superscript. ■

Finally, we can compute $\partial_c \Delta$ from equation (d), which we re-express here:

$$\Delta \equiv \epsilon^{k_1 \dots k_N} g_{1 k_1} \dots g_{a-1 k_{a-1}} g_{a k_a} g_{a+1 k_{a+1}} \dots g_{N k_N}. \quad (f)$$

Then

$$\begin{aligned}
 \partial_c \Delta &= \partial_c g_{1b} (-1)^{b-1} \epsilon^{b k_2 \dots k_N} g_{2 k_2} \dots g_{N k_N} \\
 &+ \dots \\
 &+ \partial_c g_{ab} (-1)^{b-a} \epsilon^{k_1 \dots k_{a-1} b k_{a+1} \dots k_N} g_{1 k_1} \dots g_{a-1 k_{a-1}} g_{a+1 k_{a+1}} \dots g_{N k_N} \\
 &+ \dots \\
 &+ \partial_c g_{Nb} (-1)^{b-N} \epsilon^{k_1 \dots k_{N-1} b} g_{1 k_1} \dots g_{N-1 k_{N-1}} \\
 &= \sum_{s=1}^N \delta_s^a (-1)^{b-s} \partial_c g_{sb} \epsilon^{k_1 \dots k_{s-1} b k_{s+1} \dots k_N} g_{1 k_1} \dots g_{s-1 k_{s-1}} g_{s+1 k_{s+1}} \dots g_{N k_N}. \quad (g)
 \end{aligned}$$

Terms with factor $(-1)^{b-s}$ in equation (g) are non-zero only when $s = a$. To see that we need the factor $(-1)^{b-s} = (-1)^{b-a}$ in equation (g), again consider $k_1=1, k_2=2, \dots, k_N=N$. Then

$$\begin{aligned}
 &\epsilon^{k_1 \dots k_{a-1} b k_{a+1} \dots k_N} g_{1 k_1} \dots g_{a-1 k_{a-1}} g_{a k_a} g_{a+1 k_{a+1}} \dots g_{N k_N} \\
 &= \epsilon^{1 \dots a-1 b a+1 \dots k_N} g_{11} \dots g_{a-1 a-1} g_{aa} g_{a+1 a+1} \dots g_{NN}.
 \end{aligned}$$

The correct sign for this term is $+1$, so, in order to move superscript b to position a from position b , the factor $(-1)^{b-a}$ is needed.

We can simplify equation (g) by moving $\partial_c g_{sb} (-1)^{b-s}$ to the left of the summation sign. It becomes $\partial_c g_{ab} (-1)^{b-a}$ since the summations are all zero except when $s = a$, and this completes the solution:

$$\begin{aligned}
 \partial_c \Delta &= \partial_c g_{ab} (-1)^{b-a} \sum_{s=1}^N \delta_s^a \epsilon^{k_1 \dots k_{b-1} b k_{b+1} \dots k_N} g_{1 k_1} \dots g_{s-1 k_{s-1}} g_{s+1 k_{s+1}} \dots g_{N k_N} \\
 &\stackrel{(e)}{=} \partial_c g_{ab} c^{ab} \quad \checkmark \\
 &\stackrel{(c)}{=} \Delta g^{ab} \partial_c g_{ab} \quad \checkmark
 \end{aligned}$$