Exercise 2.2.7 (not in book, but needed). Let Δ = det G be the **determinant of *G*,** where $G = (g_{ab})$ is the metric tensor matrix. The **minor**, m^{ab}, for element g_{ab} is the determinant of the submatrix that excludes row a and column b. The **cofactor** for element g_{ab} is $c^{ab} = (-1)^{a+b}\, m^{ab}$. Show that

$$\boxed{\partial_c \Delta = c^{ab}\,\partial_c g_{ab} = \Delta g^{ab}\,\partial_c g_{ab}}\,.$$

Solution Δ is defined as a sum of signed products of permutations of the elements g_{ab} in the following way. Rows are labeled "a" and columns are labeled "b". G has N rows and N columns. If $b_1, b_2, \ldots, b_n$ are the row numbers 1–N in any random order, this constitutes a **permutation** of the rows 1, 2, ... , N. Let $\mathcal{P}$ denote the set of all such permutations. A **permutation is even (odd)** if it takes an even (odd) number of pairwise swaps of adjacent numbers to restore the natural order. For example, 1, 3, 2, 4 is an odd permutation because it requires just one swap. 1, 3, 4, 2 is an even permutation. If π is a permutation, define **sign(π)** as +1 if π is even, and -1 when π is odd.

The definition of **determinant of *G*** is

$$\begin{aligned}\Delta &= \textstyle\sum_{\pi\in\mathcal{P}} \operatorname{sign}(\pi)\, g_{1,\pi(1)}\, g_{2,\pi(2)} \cdots g_{N,\pi(N)} \\ &= \textstyle\sum_{\pi\in\mathcal{P}} \operatorname{sign}(\pi)\, g_{1,b_1}\, g_{2,b_2} \cdots g_{N,b_N},\end{aligned}$$

where $\pi = b_1, b_2, \ldots, b_N$, and so we have denoted $b_1 = \pi(1), \ldots, b_N = \pi(N)$.

There are various formulas for computing the determinant. We use two formulas here. The first one is to sum the cofactors across row a. However, we have to be careful on how to express this in Einstein notation. In non-Einstein notation, we can write

$$\Delta = \sum_{b=1}^{N} g_{ab} c^{ab}.$$

However, in Einstein notation, $g_{ab}\, c^{ab}$ sums across not just row a, but across all rows, which results in $\Delta + \ldots + \Delta$, N times. Thus, in Einstein notation we express this sum as

$$g_{ab}\, c^{ab} = N\,\Delta. \qquad \text{(a)}$$

A related formula is:

$$g_{ab}\, g^{ab} = N: \qquad \text{(b)}$$

$$g_{ab}\, g^{ab} = g_{ab}\, g^{ad} \delta^b_d \overset{(1.78)}{=} \delta^d_b\, \delta^b_d = N. \quad ✓$$

Putting these two formulas together we get

$$c^{ab} = \Delta\, g^{ab}: \tag{c}$$

$$g_{ab}\, c^{ab} \overset{(a)}{=} N\,\Delta \overset{(b)}{=} g_{ab}\, g^{ab}\Delta \quad \Rightarrow \quad c^{ab} = g^{ab}\Delta \qquad ✓$$

The second formula we use for Δ is the definition of determinant as given above but expressed using Einstein notation to sum over all of the different combinations of indices $b_1, b_2, \ldots, b_n$:

$$\Delta \equiv \epsilon^{b_1 \ldots b_N}\, g_{1\, b_1} \ldots g_{N\, b_N}, \tag{d}$$

Equation (d) utilizes the **Levi-Civita symbol,** $\epsilon^{b_1 \ldots b_N}$**,** which we now explain.

Each index b_k range over the integers 1 – N. The symbol $\epsilon^{b_1 \ldots b_N}$ is defined to be zero unless each integer appears exactly once; that is, unless $b_1, \ldots, b_N$ is a permutation of 1, 2, ..., N. For permutations, its value is defined as +1 for even permutations and -1 for odd permutations. That is, for a permutation $\pi = b_1, b_2, \ldots, b_n$, the Levi-Civita symbol is $\epsilon^{b_1 \ldots b_N} = \text{sign}(\pi)$. For example, for $N = 3$, $\Delta = \epsilon^{b_1 b_2 b_3} g_{1\, b_1}\, g_{2\, b_2}\, g_{3\, b_3}$. Consider b_1=1, b_2=2, b_3=3. That term in equation (d) is $\epsilon^{123} g_{11}\, g_{22}\, g_{33} = (-1)^0\, g_{11}\, g_{22}\, g_{33}$ because zero permutations are required to restore the natural order. But, for b_1=1, b_2=3, b_3=2, the term is $\epsilon^{132} g_{11}\, g_{23}\, g_{32} = (-1)^1 g_{11}\, g_{23}\, g_{32}$ because one permutation is needed.

We call $\epsilon^{b_1 \ldots b_N}$ the Levi-Civita *symbol* because it is a pseudo-tensor, almost a tensor except that it reverses sign under Jacobian transformations having negative determinant. It is not quite proper to call it the Levi-Civita tensor.

The cofactor, also, can be expressed in Levi-Civita terminology since it is a determinant. Consider the following proposal for the cofactor expression:

$$c^{ab} \equiv \epsilon^{k_1 \ldots k_{a-1}\, b\, k_{a+1} \ldots k_N}\, g_{1\, k_1} \ldots g_{a-1\, k_{a-1}}\, g_{a+1\, k_{a+1}} \ldots g_{N\, k_N} \tag{e}$$

There are several things to confirm.

1. Only permutations have non-zero values
2. Every row except row a appears
3. Every column except column b appears
4. $\epsilon^{k_1 \ldots k_{a-1}\, b\, k_{a+1} \ldots k_N}$ has the correct sign for the permutation $k_1 \ldots k_{a-1}\, k_{a+1} \ldots k_N$

#1 is true by definition of the Levi-Civita symbol. ✓

2 is true since the rows, represented by the first subscripts of g, are
$1, ..., a–1, a+1, ... , N$. ✓

#3 The exponent of ϵ for a permutation $k_1 \ ... \ k_{a-1} \ b \ k_{a+1} \ ... \ k_N$ means that the k_a's equal every column except b. Thus, the 2nd subscripts of g represent every column except b. ✓

#4 Index b is in the ath superscript position. It requires $|b–a|$ index swaps to move b from position b to position a. So, $\epsilon^{1 \ ... \ a-1 \ b \ a+1 \ ... \ N} = (-1)^{b-a}$. We show that this is the correct sign to append to the minor $m^{a\,b} = g_{1\,k_1} \cdots g_{a-1\,k_{a-1}} \, g_{a+1\,k_{a+1}} \cdots g_{N\,k_N}$ in three steps.

#4a The term $g_{11} \ g_{22} \ ... \ g_{a-1\,a-1} \ g_{a+1\,a+1} \ ... \ g_{N\,N}$ in the minor $m^{a\,b}$ is the term having natural order, so it should has a sign of +1. Thus, the corresponding cofactor term must have sign $(-1)^{a+b}$. Observe that $(-1)^{b-a} = (-1)^{a+b}$ since $b - a$ is even or odd according to whether $a + b$ is even or odd. Hence, the cofactor term is

$$\epsilon^{1 \ ... \ a-1 \ b \ a+1 \ ... \ N} \, g_{11} \cdots g_{a-1\,a-1} \, g_{a+1\,a+1} \cdots g_{N\,N}$$
$$= (-1)^{b-a} \, g_{11} \ ... \ g_{a-1\,a-1} \, g_{a+1\,a+1} \cdots g_{N\,N}$$
$$= (-1)^{a+b} \, g_{11} \ ... \ g_{a-1\,a-1} \, g_{a+1\,a+1} \cdots g_{N\,N} \quad ✓$$

#4b If a permutation involves a pairwise swap of adjacent numbers from another permutation, then we want the sign to change:

If the adjacent numbers are to the left of b, the the Levi-Civita symbol implements one swap, which is a sign change. ✓

Ditto if the adjacent numbers are to the right of b. ✓

If one number is just left of b and the other just right of b, then the Levi-Civita symbol implements 3 swaps (swap the left number with b, swap the two numbers, then swap right right number with b), which also is a change of sign. ✓

#4c Every permutation can be built up from the natural order by a series of single swaps. Since each swap implements the correct sign, the permutation has the correct sign. ✓

Thus, expression (e) has the correct sign as well as the correct rows and columns for the cofactor $c^{a\,b}$. However, in proper tensor notation, when indices a and b are superscripts on LHS then they must be superscripts on RHS, but index a appears in the RHS subscripts. (Note: don't confuse index k_a with index a. Indices k_{a-1} and k_{a+1} are dummy variables and are allowed to appear anywhere.) We try to remedy this problem with a Kronecker delta:

$$\delta^s_a\ \epsilon^{k_1\,\ldots\,k_{a-1}\,b\ k_{a+1}\,\ldots\,k_N}\ g_{1\,k_1}\ \cdots\ g_{s-1\,k_{a-1}}\ g_{s+1\,k_{a+1}}\ \cdots\ g_{N\,k_N}.$$

In this attempt, index a no longer appears among the subscripts of g, but it now appears as a subscript in the Kronecker delta. To solve this new problem, we toss in a summation sign:

$$c^{a\,b} \equiv \sum_{s=1}^{N} \delta^a_s\ \epsilon^{k_1\,\ldots\,k_{a-1}\,b\ k_{a+1}\,\ldots\,k_N}\ g_{1\,k_1}\ \cdots\ g_{s-1\,k_{a-1}}\ g_{s+1\,k_{a+1}}\ \cdots\ g_{N\,k_N}. \tag{f}$$

In this summation we throw away all terms except when $s = a$, making this identical to the prior expression while adhering to the Einstein summation convention. This is the cofactor equation using Levi-Civita terminology.

Next, we compute $\partial_c\Delta$ from equation (d), which we re-express here:

$$\Delta \equiv \epsilon^{k_1\,\ldots\,k_N}\ g_{1\,k_1}\cdots g_{a-1\,k_{a-1}}\ g_{a\,k_a}\ g_{a+1\,k_{a+1}}\cdots\ g_{N\,k_N}. \tag{g}$$

Using the product rule to compute $\partial_c\Delta$, we get

$$\begin{aligned}\partial_c\Delta = &\ \partial_c g_{1\,k_1}\ \epsilon^{b\,k_2\ldots\,k_N} g_{2\,k_2}\cdots\ g_{N\,k_N}\\ &+\ \ldots\\ &+\ \partial_c g_{a\,k_a}\ \epsilon^{k_1\,\ldots\,k_{a-1}\,k_a\,k_{a+1}\,\ldots\,k_N}\ g_{1\,k_1}\cdots g_{a-1\,k_{a-1}}\ g_{a+1\,k_{a+1}}\cdots\ g_{N\,k_N}\\ &+\ \ldots\\ &+ \partial_c g_{N\,k_N}\ \epsilon^{k_1\ldots\,k_{N-1}\,b} g_{1\,k_1}\cdots\ g_{N-1\,k_{N-1}}\end{aligned} \tag{h}$$

Consider the ath term. If we replace k_a by b, we get

$$\partial_c g_{a\,b}\ \epsilon^{k_1\,\ldots\,k_{a-1}\,b\,k_{a+1}\,\ldots\,k_N}\ g_{1\,k_1}\cdots g_{a-1\,k_{a-1}}\ g_{a+1\,k_{a+1}}\cdots\ g_{N\,k_N}.$$

This represents a sum over b instead of a sum over k_a. We wish to make this summation imitate equation (f), so we first insert a Kronecker delta to replace a by s:

$$\delta^a_s\ \partial_c g_{s\,b}\ \epsilon^{k_1\ldots\,k_{a-1}\,b\ k_{a+1}\ldots\,k_N}\ g_{1\,k_1}\cdots g_{s-1\,k_{a-1}}\ g_{s+1\,k_{a+1}}\cdots\ g_{N\,k_N}.$$

This does not change the value of the ath term. Next we can sum over s and it still will not change the value of the *a*th term:

$$\sum_{s=1}^{N} \delta_s^a \; \partial_c g_{s\,b} \; \epsilon^{k_1 \ldots\, k_{a-1}\, b\; k_{a+1} \ldots\, k_N} \; g_{1\,k_1} \ldots g_{s-1\,k_{a-1}} \; g_{s+1\,k_{a+1}} \cdots \; g_{N\,k_N}. \tag{i}$$

We can replace $\partial_c g_{s\,b}$ by $\partial_c g_{a\,b}$ since the summations are all zero except when $s = a$. Then we can move it to the left of the summation sign. However, in so doing, in Einstein notation we are now summing over *a*, resulting in RHS of equation (h), no longer just the *a*th term. That is,

$$\partial_c \Delta = \partial_c g_{a\,b} \sum_{s=1}^{N} \delta_s^a \; \epsilon^{k_1 \ldots\, k_{a-1}\, b\; k_{a+1} \ldots\, k_N} \; g_{1\,k_1} \ldots g_{s-1\,k_{a-1}} \; g_{s+1\,k_{a+1}} \cdots \; g_{N\,k_N} \tag{j}$$

We can now complete the solution. We are summing over both *a* and *b*.

$$\begin{aligned}
\partial_c \Delta &\overset{\text{(j)}}{=} \partial_c g_{a\,b} \sum_{s=1}^{N} \delta_s^a \; \epsilon^{k_1 \ldots\, k_{a-1}\, b\; k_{a+1} \ldots\, k_N} \; g_{1\,k_1} \ldots g_{s-1\,k_{a-1}} \; g_{s+1\,k_{a+1}} \cdots \; g_{N\,k_N} \\
&\overset{\text{(f)}}{=} \partial_c g_{a\,b} \, c^{a\,b} \qquad \checkmark \\
&\overset{\text{(c)}}{=} \Delta \, g^{a\,b} \, \partial_c g_{a\,b} \qquad \checkmark
\end{aligned}$$