Exercise 1.2.2 Show $\mathbf{e}_i = g_{ij}\,\mathbf{e}^j$ and $\mathbf{e}^i = g^{ij}\,\mathbf{e}_j$

Since $\{\mathbf{e}^j\}$ is a basis, $\forall i \;\; \exists\, a_{ij} \ni$

$$\mathbf{e}_i = a_{ij}\,\mathbf{e}^j \qquad (1)$$

Thus,

$$\forall k \quad g_{ik} \equiv \mathbf{e}_i \cdot \mathbf{e}_k \overset{(1)}{=} a_{ij}\,\mathbf{e}^j \cdot \mathbf{e}_k = a_{ij}\,\delta^j_k = a_{ik}.$$

$$\therefore \quad \mathbf{e}_i \overset{(1)}{=} a_{ij}\,\mathbf{e}^j = g_{ij}\,\mathbf{e}^j \qquad ✓$$

Proof of 2nd equation is similar. ■