Exercise 3.1 (a) Show that in a 2-dim'l Riemannian manifold all cpnts of R_{ABCD} are either zero or $\pm R_{1212}$

(b) Show that in spherical coordinates for a sphere of radius a that $R_{1212} = a^2 \sin^2\theta$ and $R = -\frac{2}{a^2}$

(a) $R_{aacd} \overset{(3.16)}{=} -R_{aacd} \Rightarrow R_{aacd} = 0$
$R_{abcc} \overset{(3.17)}{=} -R_{abcc} \Rightarrow R_{abcc} = 0$ $\Bigg\} \Rightarrow$ if $R_{abcd} \neq 0$ then $a \neq b$ and $c \neq d$

The only remaining candidates are R_{1212}, R_{1221}, R_{2112}, and R_{2121}

$R_{1221} \overset{(3.17)}{=} -R_{1212}$ ✓ $\quad R_{2112} \overset{(3.16)}{=} -R_{1212}$ ✓ $\quad R_{2121} \overset{(3.17)}{=} -R_{2112} \overset{(3.16)}{=} R_{1212}$ ✓ $\quad$ (I)

(b) The sphere of radius a is parameterized by θ and ϕ

The natural basis is

$\vec{e}_1 = \vec{e}_\theta = a\cos\theta\cos\phi\,\vec{i} + a\cos\theta\sin\phi\,\vec{j} - a\sin\theta\,\vec{k}$

$\vec{e}_2 = e_\phi = -a\sin\theta\sin\phi\,\vec{i} + a\sin\theta\cos\phi\,\vec{j}$

The dual basis is

$\vec{e}^{\,1} = \vec{e}^{\,\theta} = \frac{1}{a}\cos\theta\cos\phi\,\vec{i} + \frac{1}{a}\cos\theta\sin\phi\,\vec{j} - \frac{1}{a}\sin\theta\,\vec{k}$

$\vec{e}^{\,2} = \vec{e}^{\,\phi} = -\frac{\sin\phi}{a\sin\theta}\vec{i} + \frac{\cos\phi}{a\sin\theta}\vec{j}$

$g_{11} = \vec{e}_1 \cdot \vec{e}_1 = a^2 \quad g_{12} = g_{21} = 0 \quad g_{22} = g_{\phi\phi} = a^2\sin^2\theta$ $\quad$ (II)

$g^{11} = \vec{e}^{\,1} \cdot \vec{e}^{\,1} = \frac{1}{a^2} \quad g^{12} = g^{21} = 0 \quad g^{22} = g^{\phi\phi} = \frac{1}{a^2\sin^2\theta}$ $\quad$ (III)

$R_{ABCD} = R_{1212} \overset{(3.15)}{=} \frac{1}{2}(\partial_2\partial_1 \cancel{g_{21}}^{\,0} - \cancel{\partial_2\partial_2 g_{11}} + \partial_1\partial_2 \cancel{g_{12}}^{\,0} - \partial_1\partial_1 g_{22})$

$- g^{EF}[\Gamma_{EAC}\Gamma_{FBD} - \Gamma_{EAD}\Gamma_{FBC}]$

$\overset{(II)}{=} -\frac{1}{2}\partial^2_\theta(a^2\sin^2\theta) - \Gamma^F{}_{AC}\Gamma_{FBD} + \Gamma^F{}_{AD}\Gamma_{FBC}$ $\quad$ (IV)

$-\frac{1}{2}\partial^2_\theta(a^2\sin^2\theta) = -a^2\partial_\theta(\sin\theta\cos\theta) = -a^2(\cos^2\theta - \sin^2\theta)$ $\quad$ (V)

In Exercise 2.15 we showed that the only non-zero connection coefficients are

$\Gamma^1_{22} = -\sin\theta\cos\theta$ and $\Gamma^2_{12} = \Gamma^2_{21} = \cot\theta$ $\quad$ (VI)

$\Gamma^F_{AC}\Gamma_{FBD} + \Gamma^F_{AD}\Gamma_{FBC} = \cancel{\Gamma^1_{11}}\Gamma_{122} + \cancel{\Gamma^1_{12}}\Gamma_{121} + \cancel{\Gamma^2_{11}}\Gamma_{222} + \Gamma^2_{12}\Gamma_{221} \overset{(VI)}{=} \Gamma_{221}\cot\theta$ $\quad$ (VII)

$\Gamma_{221} \overset{(2.33)}{=} \frac{1}{2}(\cancel{\partial_2 g_{21}} + \partial_1 g_{22} - \partial_2 \cancel{g_{21}}^{\,0}) = a^2\sin\theta\cos\theta$

$\Gamma_{221}\cot\theta = a^2\sin\theta\cos\theta\frac{\cos\theta}{\sin\theta} = a^2\cos^2\theta$ $\quad$ (VIII)

$R_{1212} \overset{(IV, V, VIII)}{=} -a^2(\cos^2\theta - \sin^2\theta) + a^2\cos^2\theta = a^2\sin^2\theta$ ✓ $\quad$ (IX)

Exercise 3.1 (p.2)

$$R_{ba} \overset{(3.21)}{=} R^{c}{}_{bac} = g^{dc} R_{dbac} = g^{11} R_{1ba1} + g^{22} R_{2ba2}$$

and

$$R \overset{(3.22)}{=} g^{ab} R_{ba} = g^{1b} R_{b1} + g^{2b} R_{b2} = g^{11} R_{11} + \cancel{g^{12}}^{0} R_{21} + \cancel{g^{21}}^{0} R_{12} + g^{22} R_{22} \overset{(\mathrm{III})}{=} \frac{1}{a^2} R_{11} + \frac{1}{a^2 \sin^2\theta} R_{22} \quad (\mathrm{X})$$

$$R_{11} = g^{11} \cancel{R_{1111}}^{0} + g^{22} R_{2112} \overset{(\mathrm{I})}{=} -g^{22} R_{1212} \overset{(\mathrm{III},\mathrm{IX})}{=} -\frac{1}{a^2 \sin^2\theta} (a^2 \sin^2\theta) = -1$$

$$R_{22} = g^{11} R_{1221} + g^{22} \cancel{R_{2222}}^{0} \overset{(\mathrm{I})}{=} -g^{11} R_{1212} \overset{(\mathrm{III},\mathrm{IX})}{=} -\frac{1}{a^2} (a^2 \sin^2\theta) = -\sin^2\theta$$

$$R \overset{(\mathrm{X})}{=} \frac{1}{a^2}(-1) + \frac{1}{a^2 \sin^2\theta} (-\sin^2\theta) = -\frac{2}{a^2} \checkmark$$