Exercise 2.1.1.

(a) If $t = f(s)$ is used to parameterize a straight line in Euclidean space, show that the geodesic equation (2.11) takes the form

$$\frac{d^2 u^i}{dt^2} + \Gamma^i_{jk}\frac{du^i}{dt}\frac{du^j}{dt} = h(s)\frac{du^i}{dt} \quad \text{where} \quad h(s) = -\frac{d^2 t}{ds^2}\left(\frac{dt}{ds}\right)^{-2}.$$

(b) Show that this formula reduces to the simple form (2.11) iff $t = A\,s + B$ where A and B are constants and $A \neq 0$.

(a)

$$\frac{d}{ds} = \frac{dt}{ds}\frac{d}{dt} \quad \text{(i)}$$

$$\Rightarrow \frac{du^i}{ds} = \frac{dt}{ds}\frac{du^i}{dt} \quad \text{(ii), and}$$

$$\frac{d^2 u^i}{ds^2} = \frac{d}{ds}\frac{du^i}{ds} \overset{\text{(ii)}}{=} \frac{d}{ds}\left(\frac{dt}{ds}\frac{du^i}{dt}\right) \overset{\text{(Prdt Rule)}}{=} \frac{dt}{ds}\frac{d}{ds}\frac{du^i}{dt} + \frac{du^i}{dt}\frac{d}{ds}\frac{dt}{ds}$$

$$\overset{\text{(i)}}{=} \frac{dt}{ds}\frac{dt}{ds}\frac{d}{dt}\frac{du^i}{dt} + \frac{du^i}{dt}\frac{d^2 t}{ds^2} = \left(\frac{dt}{ds}\right)^2\frac{d^2 u^i}{dt^2} + \frac{du^i}{dt}\frac{d^2 t}{ds^2} \quad \text{(iii)}$$

$$\therefore\ 0 \overset{(2.4)}{=} \frac{d^2 u^i}{ds^2} + \Gamma^i_{jk}\frac{du^j}{ds}\frac{du^k}{ds} \overset{\text{(iii, ii)}}{=} \left(\frac{dt}{ds}\right)^2\frac{d^2 u^i}{dt^2} + \frac{du^i}{dt}\frac{d^2 t}{ds^2} + \Gamma^i_{jk}\frac{dt}{ds}\frac{du^j}{dt}\frac{dt}{ds}\frac{du^k}{dt}$$

$$= \left(\frac{dt}{ds}\right)^2\left[\frac{d^2 u^i}{dt^2} + \Gamma^i_{jk}\frac{du^j}{dt}\frac{du^k}{dt}\right] + \frac{du^i}{dt}\frac{d^2 t}{ds^2}$$

$$= \left(\frac{dt}{ds}\right)^2\left[\frac{d^2 u^i}{dt^2} + \Gamma^i_{jk}\frac{du^j}{dt}\frac{du^k}{dt}\right] + \left[\frac{d^2 t}{ds^2}\left(\frac{dt}{ds}\right)^{-2}\right]\left(\frac{dt}{ds}\right)^2\frac{du^i}{dt}$$

$$= \left(\frac{dt}{ds}\right)^2\left[\frac{d^2 u^i}{dt^2} + \Gamma^i_{jk}\frac{du^j}{dt}\frac{du^k}{dt} - h(s)\frac{du^i}{dt}\right],$$

or

$$\frac{d^2 u^i}{dt^2} + \Gamma^i_{jk}\frac{du^j}{dt}\frac{du^k}{dt} = h(s)\frac{du^i}{dt} \qquad ✓$$

(b) $h(s)\dfrac{du^i}{dt} = 0 \quad \Leftrightarrow \quad h(s) = 0 \quad \Leftrightarrow \quad \dfrac{d^2 t}{ds^2} = 0 \quad \Leftrightarrow \quad \dfrac{dt}{ds} = A \quad \Leftrightarrow \quad t = A\,s + B$ ✓