Ex 2.1.5 Show for a sphere of radius a parameterized w/ $u=\theta$, $v=\phi$

that $(g_{ab}) = \begin{pmatrix} a^2 & 0 \\ 0 & a^2\sin^2\theta \end{pmatrix}$ and only non-zero connection coefficients are

$\Gamma^1_{22} = -\sin\theta\cos\theta$ and $\Gamma^2_{12} = \Gamma^2_{21} = \cot\theta$

From Example (p.8 of Notes): $\boxed{g_{11}=a^2 \quad g_{22}=a^2\sin^2\theta}$ and $g_{12}=g_{21}=0$

$g^{11}=\frac{1}{a^2} \quad g^{22}=\frac{1}{a^2\sin^2\theta}$ and $g^{12}=g^{21}=0$

The connection coefficients are $\Gamma^a_{bc} \overset{(2.13)}{=} \frac{1}{2}g^{ad}(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc})$

The only non-zero $\partial_a g_{bc}$ term is $\boxed{\partial_1 g_{22} = 2a^2\sin\theta\cos\theta}$:

$g_{12}=g_{21}=0$ and $\partial_\theta g_{11} = \partial_\phi g_{11} = 0$. and $\partial_2 g_{22} = \frac{\partial}{\partial\phi}(a^2\sin^2\theta) = 0$

Consider Γ^1_{bc}: $\frac{1}{2}g^{1d} \Rightarrow d=1$ to get $\frac{1}{2}g^{11}$. so $g_{dc}=g_{bd}=0$, leaving $\partial_d g_{bc} = \partial_1 g_{bc}$

$\Rightarrow b=c=2$

So $\Gamma^1_{22} = \frac{1}{2}g^{1d}(-\partial_d g_{bc}) = -\frac{1}{2}g^{11}(\partial_1 g_{22}) = -\frac{2a^2\sin\theta\cos\theta}{2a^2} = -\sin\theta\cos\theta$ ✓

Consider Γ^2_{bc}: $\frac{1}{2}g^{2d} \Rightarrow d=2$ to get $\frac{1}{2}g^{22}$. so $\Gamma^2_{bc} = \frac{1}{2}g^{22}(\partial_b g_{2c} + \partial_c g_{b2} - \partial_2 g_{bc})$

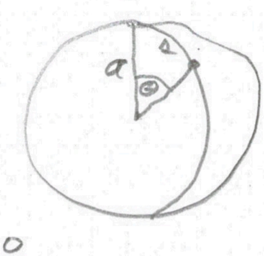
So to be non-zero, $\underset{b=1 \,\&\, c=2}{\partial_b g_{2c} = \partial_1 g_{22}}$; $\underset{c=1 \text{ and } b=2}{\partial_c g_{b2} = \partial_1 g_{22}}$; but $\partial_2 g_{bc} = 0$

$\therefore \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2}g^{22}\partial_1 g_{22} = \frac{2a^2\sin\theta\cos\theta}{2a^2\sin^2\theta} = \cot\theta$ ✓

Exercise 2.1.6 Show that all longitude lines on a sphere are geodesics

A line of longitude runs from the north pole to the south pole. Such a line has eq $\phi = \phi_0$.

It is parameterized by $u^1 = u^\theta = \theta = \frac{s}{a}$ where $r=a$, and $u^2 = u^\phi = \phi_0$



We can write this as $u^A = \frac{s}{a}\delta^A_1 + \phi_0\delta^A_2$ for $A=1,2$.

Thus $\dot{u}^A \equiv \frac{du^A}{ds} = \frac{1}{a}\delta^A_1$ and $\ddot{u}^A = 0$. The geodesic equations are

$\ddot{u}^A + \Gamma^A_{BC}\dot{u}^B\dot{u}^C = \frac{1}{a^2}\delta^B_1\delta^C_1\Gamma^A_{BC} \overset{(2.17)}{=} 0$ for $A=1,2$ $\Leftrightarrow$ $\delta^B_1\delta^C_1\Gamma^A_{BC} = 0$

From Exercise 2.1.5, the only non-zero connection coefficients are $\Gamma^1_{22} = -\sin\theta\cos\theta$ and $\Gamma^2_{12} = \Gamma^2_{21} = \cot\theta$

So, for $A=1$ the only possible non-zero term is $\delta^B_1\delta^C_1\Gamma^1_{BC} = \delta^2_1\delta^2_1\Gamma^1_{22} = 0$ and for $A=2$

$\delta^B_1\delta^C_1\Gamma^2_{BC} = \delta^2_1\delta^1_1\Gamma^2_{21} + \delta^1_1\delta^2_1\Gamma^2_{12} = 0$. Since both eqs are always true, the longitude line $\phi = \phi_0$ is a geodesic. ∎