Exercise 2.7.4 Show that $g^{\mu\nu} \approx \eta^{\mu\nu} - h^{\mu\nu}$

Since $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $h_{\mu\nu}$ is small, we assume there is a small $f^{\mu\nu}$ such that

$$g^{\mu\nu} = \eta^{\mu\nu} + f^{\mu\nu}.$$

Observe that

$$\begin{aligned}\delta^{\mu}_{\nu} = g^{\mu\sigma} g_{\sigma\nu} &= (\eta^{\mu\sigma} + f^{\mu\sigma})\,(\eta_{\sigma\nu} + h_{\sigma\nu}) \\ &= \eta^{\mu\sigma}\eta_{\sigma\nu} + \eta^{\mu\sigma}h_{\sigma\nu} + f^{\mu\sigma}\eta_{\sigma\nu} + f^{\mu\sigma}h_{\sigma\nu} \\ &\approx \delta^{\mu}_{\nu} + \eta^{\mu\sigma}h_{\sigma\nu} + f^{\mu\sigma}\eta_{\sigma\nu} \quad \text{since } f^{\mu\sigma}h_{\sigma\nu} \text{ is 2nd order.}\end{aligned}$$

This implies that

$$f^{\mu\sigma}\eta_{\sigma\nu} \approx - \eta^{\mu\sigma}h_{\sigma\nu}.$$

Changing dummy variable ν to ρ, we get

$$f^{\mu\sigma}\eta_{\mu\rho} \approx - \eta^{\mu\sigma}h_{\sigma\rho}.$$

So,

$$f^{\mu\nu} = f^{\mu\sigma}\delta^{\nu}_{\sigma} \approx f^{\mu\sigma}\eta_{\sigma\rho}\eta^{\nu\rho} \approx - \eta^{\mu\sigma}h_{\sigma\rho}\eta^{\nu\rho} \approx - h^{\mu\nu}$$

which gives

$$g^{\mu\nu} = \eta^{\mu\nu} + f^{\mu\nu} \approx \eta^{\mu\nu} - h^{\mu\nu}. \qquad \blacksquare$$

But ... why not just use the (exact) equality $g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}$?

$$g^{\mu\nu} = g^{\mu\sigma}\,\delta^{\nu}_{\sigma} = g^{\mu\sigma}\,g^{\nu\rho}\,g_{\sigma\rho} = g^{\mu\sigma}\,g^{\nu\rho}(\eta_{\sigma\rho} + h_{\sigma\rho}) = g^{\mu\sigma}(\eta_{\sigma}{}^{\nu} + h_{\sigma}{}^{\nu}) = \eta^{\mu\nu} + h^{\mu\nu} \checkmark$$

Since $h^{\mu\nu}$ is small, the two expressions are approximately equal.