Exercise A.6.3 Confirm equation (A.36) for Compton scattering frequency of the photon and also the resulting velocity of the electron.

Solution We are given the equations for conservation of system momentum:

$$(1)\quad \frac{h\nu}{c} + mc = \frac{h\bar{\nu}}{c} + \gamma mc$$

$$(2)\quad \frac{h\nu}{c} = \frac{h\bar{\nu}}{c}\cos\theta + \gamma m v\cos\phi \quad\Leftrightarrow\quad \cos\phi = \frac{h}{\gamma m v c}(\nu - \bar{\nu}\cos\theta)$$

$$(3)\quad 0 = \frac{h\bar{\nu}}{c}\sin\theta - \gamma m v\sin\phi \quad\Leftrightarrow\quad \sin\phi = \frac{h\bar{\nu}\sin\theta}{\gamma m v c}.$$

Notice that this constitutes 3 equations in 2 unknowns [$\bar{\nu}$ and v (or γ)]. Presumably these equations are consistent and lead to correct values for v and $\bar{\nu}$.

We can replace equations (2) and (3) by a new equation (4) that does not involve ϕ:

$$1 = \sin^2\phi + \cos^2\phi = \left(\frac{h}{\gamma m v c}\right)^2 [\,\bar{\nu}^2(\sin^2\theta + \cos^2\theta) - 2\,\nu\,\bar{\nu}\cos\theta + \nu^2\,]$$

$$\Rightarrow\quad \gamma^2 m^2 v^2 = \frac{h^2}{c^2}(\nu^2 - 2\,\nu\bar{\nu}\cos\theta + \bar{\nu}^2)\,. \qquad (4)$$

Equations (1) and (4) constitute two equations in two unknowns. We next wish to eliminate v. Equation (1) has γ (which is a function of v) and equation (4) has v. We first express v in terms of γ:

$$\gamma^2 v^2 = \frac{v^2}{1-\frac{v^2}{c^2}} = \frac{c^2\,\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}} = -\,c^2\,\frac{\left(1-\frac{v^2}{c^2}\right)-1}{1-\frac{v^2}{c^2}}$$

$$= -\,c^2\left(1 - \frac{1}{1-\frac{v^2}{c^2}}\right) = c^2(\gamma^2 - 1) \qquad (a)$$

$$\Rightarrow\quad \gamma^2 m^2 v^2 = m^2c^2(\gamma^2 - 1) \qquad (b)$$

We can rearrange equation (1) to solve for $m^2c^2(\gamma^2 - 1)$:

$$\gamma mc = \frac{h}{c}(\nu - \bar{\nu}) + mc \quad\Rightarrow\quad \gamma^2 m^2 c^2 = \frac{h^2}{c^2}(\nu - \bar{\nu})^2 + 2\,hm\,(\nu - \bar{\nu}) + m^2c^2$$

$$\Rightarrow \quad m^2c^2\,(\gamma^2-1) = \frac{h^2}{c^2}(\nu-\bar{\nu})^2 + 2\,hm\,(\nu-\bar{\nu})$$
$$= \frac{h^2}{c^2}(\nu^2+\bar{\nu}^2) - 2\,\frac{h^2}{c^2}\nu\,\bar{\nu} + 2\,hm\,(\nu-\bar{\nu})$$
$$= \left(\frac{h\nu}{c} + mc - \frac{h\,\bar{\nu}}{c}\right)^2 - m^2c^2 \qquad (1')$$

Now, combining (1′) and (4) we can eliminate v (and γ):

$$\frac{h^2}{c^2}\,(\nu^2 - 2\,\nu\bar{\nu}\cos\theta + \bar{\nu}^2) \overset{(4)}{=} \gamma^2 m^2 v^2 \overset{(b)}{=} m^2c^2\,(\gamma^2-1)$$
$$\overset{(1')}{=} \left(\frac{h\nu}{c} + mc - \frac{h\,\bar{\nu}}{c}\right)^2 - m^2c^2 = \frac{h^2}{c^2}(\nu^2+\bar{\nu}^2) - 2\,\frac{h^2}{c^2}\nu\,\bar{\nu} + 2\,hm\,(\nu-\bar{\nu})$$

We solve this equation for $\bar{\nu}$ by moving all of the $\bar{\nu}$ terms to the LHS:

$$\bar{\nu}^2\left(\frac{h^2}{c^2} - \frac{h^2}{c^2}\right) + \bar{\nu}\left(-2\,\frac{h^2}{c^2}\nu\cos\theta + 2\,\frac{h^2}{c^2}\nu + 2hm\right) = -\frac{h^2}{c^2}\nu^2 + \frac{h^2}{c^2}\nu^2 + 2\,hm\nu$$
$$\Rightarrow \quad \bar{\nu}\left(-2\,\frac{h^2}{c^2}\nu\cos\theta + 2\,\frac{h^2}{c^2}\nu + 2hm\right) = 2\,hm\nu$$
$$\Rightarrow \quad \bar{\nu} = \frac{2\,hm\,\nu}{2\,hm + 2\,\frac{h^2\,\nu}{c^2}\,(1-\cos\theta)}$$
$$\Rightarrow \quad \boxed{\bar{\nu} = \frac{\nu}{1 + \frac{h\nu}{mc^2}\,(1-\cos\theta)}} \qquad ✓$$

We can now solve for v using equation (3):

$$\frac{c^2\,v^2}{c^2-v^2} = \frac{v^2}{1-\frac{v^2}{c^2}} = v^2\gamma^2 \overset{(3)}{=} \frac{h^2\,\bar{\nu}^2\sin^2\theta}{m^2\,c^2\sin^2\phi}.$$

Cross multiplying yields

$$m^2c^4v^2\sin^2\phi = c^2h^2\bar{\nu}^2\sin^2\theta - v^2h^2\bar{\nu}^2\sin^2\theta$$
$$\Rightarrow \quad v^2(\,m^2c^4\sin^2\phi + h^2\bar{\nu}^2\sin^2\theta) = c^2h^2\bar{\nu}^2\sin^2\theta$$
$$\Rightarrow \quad \boxed{v^2 = \frac{c^2\,h^2\,\bar{\nu}^2\sin^2\theta}{m^2\,c^4\sin^2\phi + h^2\,\bar{\nu}^2\sin^2\theta}} \qquad ✓$$

We could also have solved for v using just equation (4), yielding and equation for v that does not depend on ϕ.