Exercise 2.2.6

Let P be a pt of manifold Σ! There is an open nbd of P that is Euclidean and has a Cartesian coords sys. Let (x^a) and $(x^{a'})$ be alternate coord systems at P. a curve γ thru P can be represented parametrically by $x^a(u)$ and $x^{a'}(u)$. Let Q be a nearly pt on γ. Denote $P = x^a(u_0) = x^{a'}(u)$ and $Q = x^a(u+\delta u) = x^{a'}(u+\delta u)$.

Let $\vec{\lambda}(u)$ be a vector generated at Q by the parallel transport of $\vec{\lambda}_0 = \vec{\lambda}(u_0)$ along γ. In the unprimed coord sys this means that $\vec{\lambda}(u)$ satisfies

$$\dot{\lambda}^a(u) + \Gamma^a_{bc}(u)\,\lambda^b(u)\,\dot{x}^c(u) = 0 \qquad (2.23)$$

where

$$\Gamma^a_{bc}(u) = \tfrac{1}{2} g^{ad}(u)\left[\partial_b g_{dc}(u) + \partial_c g_{bd}(u) - \partial_d g_{bc}(u)\right] \qquad (2.13)$$

In the primed coord sys this means that $\vec{\lambda}(u)$ satisfies

$$\dot{\lambda}^{a'} + \Gamma^{a'}_{b'c'}\lambda^{b'}\dot{x}^{c'} = 0 \qquad (2.29)$$

where $\Gamma^{a'}_{b'c'} = \frac{1}{2} g^{a'd'}\left[\partial_{b'} g_{d'c'} + \partial_{c'} g_{b'd'} - \partial_{d'} g_{b'c'}\right]$ (2.30)

For defn (2.23) to be coord indep, we require

$$\lambda^{a'} = X^{a'}_b \lambda^b \qquad (1.45)$$

where $X^{a'}_b$ is the Jacobian

$$X^{a'}_b = \frac{\partial x^{a'}}{\partial x^b}.$$

$$\lambda^{a'} = X^{a'}_f \lambda^f \Rightarrow \dot{\lambda}^{a'} = X^{a'}_f \dot{\lambda}^f + X^{a'}_{ef}\dot{x}^e\lambda^f = -\Gamma^{a'}_{b'c'}\lambda^{b'}\dot{x}^{c'}$$

$$\frac{d}{dt}X^{a'}_f = \frac{\partial X^{a'}_f}{\partial x^e}\frac{dx^e}{dt} = X^{a'}_{ef}\dot{x}^e$$