Exercise 3.2.1 Prove $R^d_{abc} = 0$ is a necessary and sufficient condition for being able to interchange the order of covariant differentiation of fields

(a) For a contravariant vector field λ^a, show $\lambda^a{}_{;bc} - \lambda^a{}_{;cb} = - R^a_{dbc}\,\lambda^d$

$\lambda^a{}_{;b} \overset{(2.55)}{=} \partial_b\lambda^a + \Gamma^a_{eb}\,\lambda^e$. Set $\tau^a{}_b \equiv \lambda^a{}_{;b}$. Then

$$\lambda^a{}_{;bc} = \tau^a{}_{b;c} \overset{(2.59)}{=} \partial_c\tau^a{}_b + \Gamma^a_{ec}\,\tau^e{}_b - \Gamma^e_{bc}\tau^a{}_e = \partial_c\lambda^a{}_{;b} + \Gamma^a_{ec}\,\lambda^e{}_{;b} - \Gamma^e_{bc}\,\lambda^a{}_{;e}$$
$$= \partial_c\,\partial_b\lambda^a + (\partial_c\Gamma^a_{db})\,\lambda^d + \Gamma^a_{db}\,\partial_c\lambda^d$$
$$+ \Gamma^a_{ec}\,[\,\partial_b\lambda^e + \Gamma^e_{db}\,\lambda^d] - \Gamma^e_{bc}\,[\,\partial_e\lambda^a + \Gamma^e_{de}\,\lambda^d]$$

$$\lambda^a{}_{;cb} = \partial_b\,\partial_c\lambda^a + (\partial_b\Gamma^a_{dc})\,\lambda^d + \Gamma^a_{dc}\,\partial_b\lambda^d$$
$$+ \Gamma^a_{eb}\,[\,\partial_c\lambda^e + \Gamma^e_{dc}\,\lambda^d] - \Gamma^e_{cb}\,[\,\partial_e\lambda^a + \Gamma^e_{de}\,\lambda^d]$$

The colored items subtract out, leaving

$$\lambda^a{}_{;bc} - \lambda^a{}_{;cb} = (\partial_c\Gamma^a_{db} - \partial_b\Gamma^a_{dc} + \Gamma^a_{ec}\,\Gamma^e_{db} - \Gamma^a_{eb}\,\Gamma^e_{dc})\,\lambda^d$$

$$- R^a_{dbc}\,\lambda^d = - (\partial_b\Gamma^a_{dc} - \partial_c\Gamma^a_{db} + \Gamma^e_{dc}\,\Gamma^a_{eb} - \Gamma^e_{db}\,\Gamma^a_{ec})\,\lambda^d = \lambda^a{}_{;bc} - \lambda^a{}_{;cb} \quad ✓$$

(b) For a type (2,0) tensor field τ^{ab}, show $\tau^{ab}{}_{;cd} - \tau^{ab}{}_{;dc} = -R^a_{ecd}\,\tau^{eb} - R^b_{ecd}\,\tau^{ae}$

$$R^a_{ecd}\,\tau^{eb} = (\partial_c\Gamma^a_{ed} - \partial_d\Gamma^a_{ec} + \Gamma^f_{ed}\,\Gamma^a_{fc} - \Gamma^f_{ec}\,\Gamma^a_{fd})\,\tau^{eb}$$
$$R^b_{ecd}\,\tau^{ae} = (\partial_c\Gamma^b_{ed} - \partial_d\Gamma^b_{ec} + \Gamma^f_{ed}\,\Gamma^b_{fc} - \Gamma^f_{ec}\,\Gamma^b_{fd})\,\tau^{ae}$$
$$\tau^{ab}{}_{;c} \overset{(2.57)}{=} \partial_c\tau^{ab} + \Gamma^a_{ec}\,\tau^{eb} + \Gamma^b_{ec}\,\tau^{ae} \qquad \text{Define } \sigma^{ab}{}_c \equiv \tau^{ab}{}_{;c}$$
$$\tau^{ab}{}_{;cd} = \sigma^{ab}{}_{c;d} \overset{(2.59\,a)}{=} \partial_d\sigma^{ab}{}_c + \Gamma^a_{ed}\sigma^{eb}{}_c + \Gamma^b_{ed}\sigma^{ea}{}_c - \Gamma^e_{cd}\sigma^{ab}{}_e$$
$$= \partial_d\tau^{ab}{}_{;c} + \Gamma^a_{ed}\tau^{eb}{}_{;c} + \Gamma^b_{ed}\tau^{ea}{}_{;c} - \Gamma^e_{cd}\tau^{ab}{}_{;e}$$
$$= \partial_d(\partial_c\tau^{ab} + \Gamma^a_{ec}\,\tau^{eb} + \Gamma^b_{ec}\,\tau^{ae}) + \Gamma^b_{ed}\tau^{ea}{}_{;c} - \Gamma^e_{cd}\tau^{ab}{}_{;e}$$

$$\tau^{ab}{}_{;cd} = \partial_d\partial_c\tau^{ab} + (\partial_d\Gamma^a_{ec})\tau^{eb} + \Gamma^a_{ec}\,\partial_d\tau^{eb} + (\partial_d\Gamma^b_{ec})\tau^{ae} + \Gamma^b_{ec}\,\partial_d\tau^{ae}$$
$$+ \Gamma^a_{ed}\,[\partial_c\tau^{eb} + \Gamma^e_{fc}\,\tau^{fb} + \Gamma^b_{fc}\,\tau^{ef}] + \Gamma^b_{ed}\,[\partial_c\tau^{ea} + \Gamma^e_{fc}\,\tau^{fa} + \Gamma^a_{fc}\,\tau^{ef}]$$
$$- \Gamma^e_{cd}\,[\partial_e\tau^{ab} + \Gamma^a_{fe}\,\tau^{fb} + \Gamma^b_{fe}\,\tau^{af}]$$

$$\tau^{ab}{}_{;dc} = \partial_c\partial_d\tau^{ab} + (\partial_c\Gamma^a_{ed})\tau^{eb} + \Gamma^a_{ed}\,\partial_c\tau^{eb} + (\partial_c\Gamma^b_{ed})\tau^{ae} + \Gamma^b_{ed}\,\partial_c\tau^{ae}$$
$$+ \Gamma^a_{ec}\,[\partial_d\tau^{eb} + \Gamma^e_{fd}\,\tau^{fb} + \Gamma^b_{fd}\,\tau^{ef}] + \Gamma^b_{ec}\,[\partial_d\tau^{ea} + \Gamma^e_{fd}\,\tau^{fa} + \Gamma^a_{fd}\,\tau^{ef}]$$
$$- \Gamma^e_{dc}\,[\partial_e\tau^{ab} + \Gamma^a_{fe}\,\tau^{fb} + \Gamma^b_{fe}\,\tau^{af}]$$

$$\tau^{ab}{}_{;cd} - \tau^{ab}{}_{;dc} = (\partial_d\Gamma^a_{ec} - \partial_c\Gamma^a_{ed})\tau^{eb} + (\Gamma^a_{ed}\,\Gamma^e_{fc} - \Gamma^a_{ec}\,\Gamma^e_{fd})\,\tau^{fb}$$
$$+ (\partial_d\Gamma^b_{ec} - \partial_c\Gamma^b_{ed})\tau^{ae} + (\Gamma^b_{ed}\,\Gamma^e_{fc} - \Gamma^b_{ec}\,\Gamma^e_{fd})\,\tau^{fa}$$
$$= [\partial_d\Gamma^a_{ec} - \partial_c\Gamma^a_{ed} + \Gamma^a_{fd}\,\Gamma^f_{ec} - \Gamma^a_{fc}\,\Gamma^f_{ed}]\,\tau^{eb}$$
$$+ [\partial_d\Gamma^b_{ec} - \partial_c\Gamma^b_{ed} + \Gamma^b_{fd}\,\Gamma^f_{ec} - \Gamma^b_{fc}\,\Gamma^f_{ed}]\,\tau^{ae}$$
$$= -R^a_{ecd}\,\tau^{eb} - R^b_{ecd}\,\tau^{ae} \qquad \checkmark$$

(c) Guess the form of $\tau^{ab}{}_{c;de} - \tau^{ab}{}_{c;ed}$

First, it is clear that $\tau^{abc}{}_{;de} - \tau^{abc}{}_{;de} = - R^{a}_{fde}\,\tau^{fbc} - R^{b}_{fde}\,\tau^{afc} - R^{c}_{fde}\,\tau^{abf}$.
Next, the mnemonic "co-below and minus" suggests that $\lambda_{a;bc} - \lambda_{a;cb} = R^{d}_{abc}\,\lambda_{d}$.
So a guess is that $\tau^{ab}{}_{c;de} - \tau^{ab}{}_{c;ed} = - R^{a}_{fde}\,\tau^{fb}{}_{c} - R^{b}_{fde}\,\tau^{af}{}_{c} + R^{f}_{cde}\,\tau^{ab}{}_{f}$.

(d) Guess the form of $\tau^{a_1 \ldots a_r}_{b_1 \ldots b_s;cd} - \tau^{a_1 \ldots a_r}_{b_1 \ldots b_s;dc}$

From (c), the general form is clear:

$$\tau^{a_1 \ldots a_r}_{b_1 \ldots b_s;cd} - \tau^{a_1 \ldots a_r}_{b_1 \ldots b_s;dc} = - \sum_{k=1}^{r} R^{a_k}_{ecd}\,\tau^{a_1 \ldots a_{k-1}\,e\,a_{k+1} \ldots a_r}_{b_1 \ldots b_s} + \sum_{k=1}^{s} R^{e}_{b_k cd}\,\tau^{a_1 \ldots a_r}_{b_1 \ldots b_{k-1}\,e\,b_{k+1} \ldots b_s} \qquad \text{(a)}$$

(e) Prove the claim that $R^{d}_{abc} = 0$ is a necessary and sufficient condition for being able to interchange the order of covariant differentiation of fields

If $R^{d}_{abc} = 0$ then then (a) shows that the order of differentiation can be reversed, and if the order can be reversed, then (a) shows that $R^{d}_{abc} = - R^{d}_{abc} \Leftrightarrow R^{d}_{abc} = 0$.

This proves the claim.