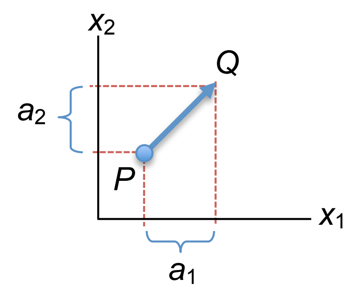
Differential Geometry

by William Graustein 1935, 1966

1 INTRODUCTION



Notation

Points: $\boldsymbol{P} = (x_1, x_2, x_3)$, $\boldsymbol{Q} = (y_1, y_2, y_3)$

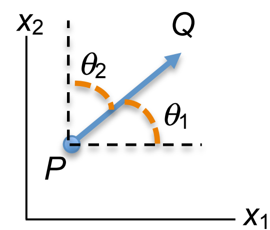
Free Vector:

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad \boldsymbol{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Directed Line Segment = Fixed Vector = Vector Localized at P = Vector at *P*:

$\overrightarrow{\boldsymbol{PQ}} = (x_1, x_2, x_3, a_1, a_2, a_3)$ where a_i is length of projection of $\overrightarrow{\boldsymbol{PQ}}$ onto x_i-axis

Length of $\overrightarrow{\boldsymbol{PQ}}$ = Length of *a*: $a = \sqrt{a_1^2 + a_2^2 + a_3^2}$ (1.2)



Proper Vector: $\boldsymbol{a} \neq 0$

Angles $\overrightarrow{\boldsymbol{PQ}}$ makes with axes: $\theta_1, \theta_2, \theta_3$

$a_i = a\cos\theta_i, \; i = 1,2,3$ (1.3)

Direction cosines: $\cos\theta_1, \cos\theta_2, \cos\theta_3$

Note: The components of a unit vector are its direction cosines

Convention: Null vector is both parallel and perpendicular to every vector.

Definitions The **inner product** of 2 vectors is

$$\langle \boldsymbol{a} | \boldsymbol{b} \rangle \equiv \sum_{i=1}^{3} a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 . \tag{1.11}$$

The **norm** of a vector is $\| \boldsymbol{a} \| = \sqrt{\langle \boldsymbol{a} | \boldsymbol{a} \rangle}$. (1.11b)

Note that $\| \boldsymbol{a} \| = \sqrt{a_1^2 + a_2^2 + a_3^2} \overset{(1.2)}{=} a$. (1.11c)

Also, $\| \boldsymbol{a} \| = 0$ iff $\boldsymbol{a} = \boldsymbol{0}$

The **outer product** of 2 vectors is the vector

$$\boldsymbol{a} \times \boldsymbol{b} \equiv \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} . \tag{1.12}$$

Theorem 3.1 $\boldsymbol{b} \parallel \boldsymbol{a} \Leftrightarrow \exists k \in \mathbb{R} \ni b_i = k a_i$ for $i = 1, 2, 3$.

Proof. To be parallel, the components of ***b*** must be a multiple of those of ***a***. ■

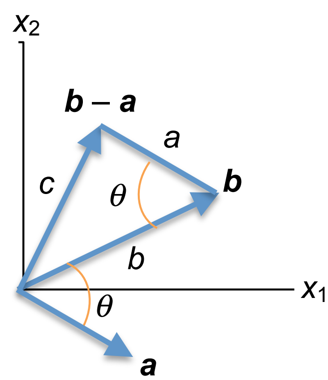
Note that k = 0 iff ***b*** = 0.

Theorem 3.2 $\boxed{\boldsymbol{b} \parallel \boldsymbol{a} \quad \Leftrightarrow \quad \boldsymbol{a} \times \boldsymbol{b} = 0}$.

Proof: $\boldsymbol{b} \parallel \boldsymbol{a} \quad \Leftrightarrow \quad \exists \text{ scalar } k \ni \boldsymbol{b} = k\boldsymbol{a} \quad \Leftrightarrow \quad \boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} a_2(ka_3) - a_3(ka_2) \\ a_3(ka_1) - a_1(ka_3) \\ a_1(ka_2) - a_2(ka_1) \end{pmatrix} = \boldsymbol{0}$ ■

Theorem Let ***a*** and ***b*** be proper vectors and θ the angle between them, $0 \le \theta \le \pi$. Then

$$\cos\theta = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}} \qquad (1.6)$$



Proof. This follows from the Law of Cosines:
$c^2 = a^2 + b^2 - 2ab\cos\theta$. ■

Corollary $\boxed{\langle \boldsymbol{a} | \boldsymbol{b} \rangle = ab\cos\theta}$. (1.6b)

Theorem 3.3 $\boxed{\boldsymbol{b} \perp \boldsymbol{a} \Leftrightarrow \langle \boldsymbol{a} | \boldsymbol{b} \rangle = 0} \Leftrightarrow a_1 b_1 + a_2 b_2 + a_3 b_3 = 0$.

Proof: Follows from corollary, above. ■

Theorem 3.4 If vectors ***a*** and ***b*** are not parallel, then $\boldsymbol{a} \times \boldsymbol{b}$ is a proper vector that is perpendicular to each of them.

Proof. Denote $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$. Then
$\langle \boldsymbol{a} | \boldsymbol{c} \rangle = a_1(a_2 b_3 - a_3 b_2) + a_2(a_3 b_1 - a_1 b_3) + a_3(a_1 b_2 - a_2 b_1) = 0$. ■

Theorem Lagrange Identity [Also, see (1.20)]

$$\begin{aligned}&(a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2 \\ &= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2\end{aligned} \qquad (1.9)$$

Proof. Easy to just multiply out. ■

Definition A **trihedral** is an oriented surface composed of 3 sides of a tetrahedron. A trihedral is generated by 3 vectors called the **edges of the trihedral.** Let ***a***, ***b***, and ***c*** be mutually orthogonal unit vectors. When they are localized at the origin, if they can be oriented by a rigid transformation of the trihedral ***abc*** to point along the positive *x*-, *y*-, and *z*-axes, we say ***abc* has the same disposition as the coordinate axes.**

Theorem 5.1 Let ***a***, ***b***, and ***c*** be mutually orthogonal unit vectors. If ***abc*** has the same disposition as the coordinate axes, then $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$. Moreover, $\boldsymbol{a} = \boldsymbol{b} \times \boldsymbol{c}$ and $\boldsymbol{b} = \boldsymbol{c} \times \boldsymbol{a}$.

Proof. There are only two unit vectors perpendicular to ***a*** and ***b***: ***c*** and $-\boldsymbol{c}$. So, either ***c*** or -***c*** yields the correct disposition. When ***a*** = (1,0,0) and ***b*** = (0,1,0), the formula for $\boldsymbol{a} \times \boldsymbol{b}$ is ***c*** = (0,0,1). Thus, ***abc*** has the desired disposition. Similarly for the other two cases. ■

Notation (***a b c***) represents the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$. (1.13)

Theorem $(\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}) = \langle \boldsymbol{a} \times \boldsymbol{b} | \boldsymbol{c} \rangle = \langle \boldsymbol{b} \times \boldsymbol{c} | \boldsymbol{a} \rangle = \langle \boldsymbol{c} \times \boldsymbol{a} | \boldsymbol{b} \rangle$. (1.14)

Proof. $\langle \boldsymbol{a} \times \boldsymbol{b} | \boldsymbol{c} \rangle = \langle\, (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)\ | (c_1, c_2, c_3) \rangle$

$$= c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & b_3 \\ a_3 & b_3 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

which is (***a b c***) expanded by minors. Similarly for the other two cases. ■

Theorem 5.2 Let ***a***, ***b***, and ***c*** be mutually orthogonal unit vectors. ***abc*** has the same disposition as the coordinate axes iff $(\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}) = 1$.

Proof. $(\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}) = \langle \boldsymbol{a} \times \boldsymbol{b} | \boldsymbol{c} \rangle \overset{\text{(Th 3.4)}}{=} \langle \boldsymbol{c} | \boldsymbol{c} \rangle = 1$.

Theorem 5.3 Three vectors ***a***, ***b***, and ***c*** are parallel to a plane iff $(\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}) = 0$.

Proof. If ***a***, ***b***, and ***c*** are parallel to a plane, $(\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}) = \langle \boldsymbol{a} \times \boldsymbol{b} | \boldsymbol{c} \rangle$. Since ***a*** and ***b*** are parallel to the plane, $\boldsymbol{a} \times \boldsymbol{b}$ is perpendicular to the plane and, hence, perpendicular to ***c***. Therefore, $\langle \boldsymbol{a} \times \boldsymbol{b} | \boldsymbol{c} \rangle = 0$. ✓

Conversely, suppose $(\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}) = \langle \boldsymbol{a} \times \boldsymbol{b} | \boldsymbol{c} \rangle = 0$. If $\boldsymbol{a} \times \boldsymbol{b} \neq 0$, then $\boldsymbol{a} \times \boldsymbol{b}$ is a proper vector perpendicular to ***c*** as well as to the plane containing ***a*** and ***b***. So ***c*** is parallel to the plane containing ***a*** and ***b***. If $\boldsymbol{a} \times \boldsymbol{b} = 0$ then $\boldsymbol{a} \parallel \boldsymbol{b}$, so a plane containing ***c*** and ***a*** is parallel to ***b***. ✓ ■

Theorem 5.4 Vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are parallel to a plane iff scalars k, l, and m, not all zero, exist so that $k\boldsymbol{a}+l\boldsymbol{b}+m\boldsymbol{c}=0$.

Proof. $\begin{cases} ka_1+lb_1+mc_1=0 \\ ka_2+lb_2+mc_2=0 \\ ka_3+lb_3+mc_3=0 \end{cases}$ has a non-zero solution for k, l, and m

iff $(\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c})=0$ (by Cramer's Rule)

iff $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are parallel to a plane (by Theorem 5.3) ■

Theorem 5.5 If vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are not parallel to a plane then they constitute a basis for the vectors.

Proof. It suffices to show that a vector $\boldsymbol{d}$ can be expressed as a linear combination of $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$. For i = 1, 2, and 3 the following determinants are zero:

$$\begin{vmatrix} a_i & b_i & c_i & d_i \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}.$$

Expanding the first row by minors yields

$$(\boldsymbol{b}\ \boldsymbol{c}\ \boldsymbol{d})a_i-(\boldsymbol{a}\ \boldsymbol{c}\ \boldsymbol{d})b_i+(\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{d})c_i-(\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c})d_i=0.$$

By Theorem 5.3, $(\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c})\neq 0$. Thus

$$d_i=\frac{1}{(\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c})}\left[(\boldsymbol{b}\ \boldsymbol{c}\ \boldsymbol{d})a_i-(\boldsymbol{a}\ \boldsymbol{c}\ \boldsymbol{d})b_i+(\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{d})c_i\right]. \quad ■$$

Theorem 5.5 is basically a proof of Cramer's Rule, which says the there is a unique solution for the 3 equations in 3 unknowns ($\boldsymbol{a}$ $\boldsymbol{b}$ $\boldsymbol{c}$) = 0.

Theorem Identities

1. $\langle k\boldsymbol{a}|\boldsymbol{b}\rangle=k\langle\boldsymbol{a}|\boldsymbol{b}\rangle$ (1.18b)
2. $\langle\boldsymbol{a}+\boldsymbol{b}|\boldsymbol{c}\rangle=\langle\boldsymbol{a}|\boldsymbol{b}\rangle+\langle\boldsymbol{a}|\boldsymbol{c}\rangle$ (1.18c)
3. $(k\boldsymbol{a})\times\boldsymbol{b}=k(\boldsymbol{a}\times\boldsymbol{b})$ (1.18d)
4. $(\boldsymbol{a}+\boldsymbol{b})\times\boldsymbol{c}=\boldsymbol{a}\times\boldsymbol{c}+\boldsymbol{b}\times\boldsymbol{c}$ (1.18e)
5. $(k\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c})=k(\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c})$ (1.18f)
6. $(\boldsymbol{a}+\boldsymbol{b}\ \boldsymbol{c}\ \boldsymbol{d})=(\boldsymbol{a}\ \boldsymbol{c}\ \boldsymbol{d})+(\boldsymbol{b}\ \boldsymbol{c}\ \boldsymbol{d})$ (1.18g)

7. $\langle \mathbf{a}\times\mathbf{b}|\mathbf{c}\times\mathbf{d}\rangle = \langle \mathbf{a}|\mathbf{c}\rangle\langle \mathbf{b}|\mathbf{d}\rangle - \langle \mathbf{a}|\mathbf{d}\rangle\langle \mathbf{b}|\mathbf{c}\rangle$ (Generalized Lagrange Identity) (1.19)

8. $\langle \mathbf{a}\times\mathbf{b}|\mathbf{a}\times\mathbf{b}\rangle = \langle \mathbf{a}|\mathbf{a}\rangle\langle \mathbf{b}|\mathbf{b}\rangle - \langle \mathbf{a}|\mathbf{b}\rangle^2$ (Lagrange Identity) (1.20)

9. $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \langle \mathbf{a}|\mathbf{c}\rangle\mathbf{b} - \langle \mathbf{b}|\mathbf{c}\rangle\mathbf{a}$ (1.22)

10. $(\mathbf{b}\,\mathbf{c}\,\mathbf{d})\mathbf{a} - (\mathbf{a}\,\mathbf{c}\,\mathbf{d})\mathbf{b} + (\mathbf{a}\,\mathbf{b}\,\mathbf{d})\mathbf{c} - (\mathbf{a}\,\mathbf{b}\,\mathbf{c})\mathbf{d} = 0$ (1.23)

11. $(\mathbf{b}\,\mathbf{c}\,\mathbf{d})\langle \mathbf{a}|\mathbf{e}\rangle - (\mathbf{a}\,\mathbf{c}\,\mathbf{d})\langle \mathbf{b}|\mathbf{e}\rangle + (\mathbf{a}\,\mathbf{b}\,\mathbf{d})\langle \mathbf{c}|\mathbf{e}\rangle - (\mathbf{a}\,\mathbf{b}\,\mathbf{c})\langle \mathbf{d}|\mathbf{e}\rangle = 0$ (1.24)

Theorem $\boxed{\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\,\|\mathbf{b}\|\sin\theta}$ (1.24b)

Proof. Let $a = \|\mathbf{a}\|$ and $b = \|\mathbf{b}\|$.

$$\cos\theta \overset{(1.6b)}{=} \frac{\langle \mathbf{a}|\mathbf{b}\rangle}{ab} \quad\Rightarrow\quad \sin^2\theta = 1 - \frac{\langle \mathbf{a}|\mathbf{b}\rangle^2}{a^2b^2} = \frac{a^2b^2 - \langle \mathbf{a}|\mathbf{b}\rangle^2}{a^2b^2} \overset{(1.20)}{=} \frac{\langle \mathbf{a}\times\mathbf{b}|\mathbf{a}\times\mathbf{b}\rangle}{a^2b^2} = \frac{\|\mathbf{a}\times\mathbf{b}\|^2}{a^2b^2} \quad \blacksquare$$

Theorem Let $\mathbf{x}$ be a fixed point in a plane Π, and let $\mathbf{a} \neq 0$ be a vector perpendicular to Π. The equation of the plane is the set of points $\mathbf{z}$ that satisfy

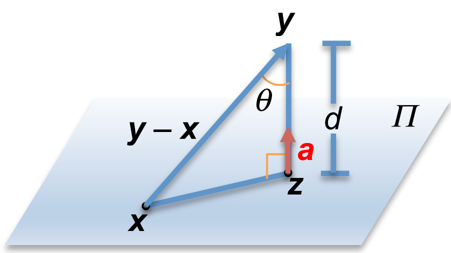
$$\boxed{\langle \mathbf{z}-\mathbf{x}|\mathbf{a}\rangle = 0} \qquad (1.26a)$$

Proof. $\langle \mathbf{z}-\mathbf{x}|\mathbf{a}\rangle = 0 \;\Leftrightarrow\; \mathbf{z}-\mathbf{x}\perp\mathbf{a} \;\overset{\mathbf{a}\neq 0}{\Leftrightarrow}\;$ vector $\mathbf{z}-\mathbf{x}$ is parallel to Π

$\Leftrightarrow$ $\mathbf{z}$ is in Π. $\blacksquare$

Theorem Let $\mathbf{x}$ be a fixed point in a plane Π, and let $\mathbf{a} \neq 0$ be a vector perpendicular to Π. The distance to the plane from a point $\mathbf{y}$ is

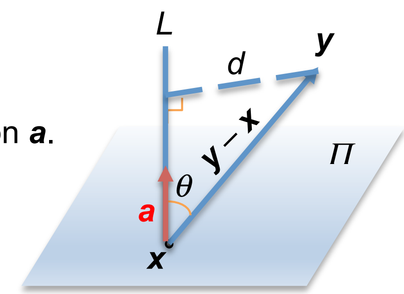
$$\boxed{d = \pm\frac{\langle \mathbf{y}-\mathbf{x}|\mathbf{a}\rangle}{a}}, \text{ where } a = \|\mathbf{a}\|. \qquad (1.27)$$



Proof. Let θ be the angle between $\mathbf{y}-\mathbf{x}$ and $\mathbf{a}$. Let the perpendicular line from $\mathbf{y}$ meet the plane Π at the point $\mathbf{z}$. $d = \pm\|\mathbf{y}-\mathbf{x}\|\cos\theta$, depending on whether or not $\theta > \pi/2$. $\langle \mathbf{y}-\mathbf{x}|\mathbf{a}\rangle = \|\mathbf{y}-\mathbf{x}\|\,\|\mathbf{a}\|\cos\theta = \|\mathbf{y}-\mathbf{x}\|\,a\cos\theta$.

So, $d = \pm\dfrac{\langle \mathbf{y}-\mathbf{x}|\mathbf{a}\rangle}{a}$ $\blacksquare$

Theorem Let L be the line through $\mathbf{x}$ with direction $\mathbf{a}$. The distance from a point $\mathbf{y}$ to the line L is



$$\boxed{d = \frac{\|(\mathbf{y}-\mathbf{x})\times\mathbf{a}\|}{a}} \text{ where } a = \|\mathbf{a}\| \qquad (1.28)$$

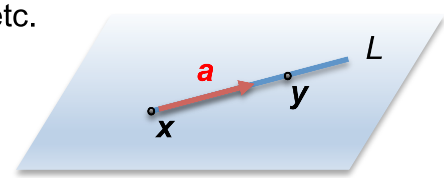
Proof.

$$\left\|(\boldsymbol{y}-\boldsymbol{x})\times\boldsymbol{a}\right\| \overset{(1.24b)}{=} \left\|(\boldsymbol{y}-\boldsymbol{x})\right\|\left\|\boldsymbol{a}\right\|\sin\theta = \cancel{\left\|\boldsymbol{y}-\boldsymbol{x}\right\|}\left\|\boldsymbol{a}\right\|\frac{d}{\cancel{\left\|\boldsymbol{y}-\boldsymbol{x}\right\|}} = d\left\|\boldsymbol{a}\right\| \quad ■$$

2 SPACE CURVES

Definition A **space curve** is the 3-dimensional graph of an equation $\boldsymbol{y}=\boldsymbol{x}(t)$.

Convention Unless otherwise specified, curves start at $\boldsymbol{x}=\boldsymbol{x}(0)$. When we refer to the point $\boldsymbol{x}$ we mean $\boldsymbol{x}(0)$. Moreover, we will use $\boldsymbol{x}'$, $\boldsymbol{x}''$, etc. to mean $\boldsymbol{x}'(0)$, $\boldsymbol{x}''(0)$, etc. To refer to other point, say $\boldsymbol{y}$, we will use $\boldsymbol{x}(t), \boldsymbol{x}'(t), \boldsymbol{x}''(t)$, etc.



Example 8.1 Parametric (or vector) equation of a line:
$\boldsymbol{y}=\boldsymbol{x}+\boldsymbol{a}t$, where

$$\boldsymbol{x}=\boldsymbol{x}(0)=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix},\ \boldsymbol{y}=\boldsymbol{x}(t)=\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix},\text{ and direction }\boldsymbol{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}.$$

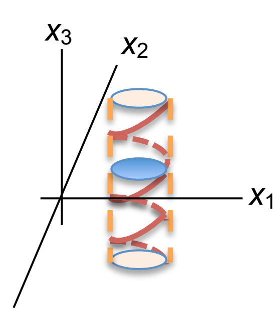
This can also be written as $\begin{cases}y_1=x_1+a_1t\\y_2=x_2+a_2t\\y_3=x_3+a_3t\end{cases}$. ■

Example 8.2 Circle in x_1x_2-plane with center (x_1, x_2) and radius $r > 0$:

$$\begin{cases}y_1=r\cos t\\y_2=r\sin t\\y_3=0\end{cases},\text{ or } \boldsymbol{y}=\boldsymbol{x}(t). \text{ The curve begins at } \boldsymbol{x}=\begin{pmatrix}r\\0\\0\end{pmatrix}. \quad ■$$

Example 8.3 Circular helix, r > 0, $k \neq 0$:

$$\begin{cases}y_1=r\cos t\\y_2=r\sin t\\y_3=kt\end{cases},\text{ or } \boldsymbol{y}=\boldsymbol{x}(t). \text{ The curve begins at } \boldsymbol{x}=\begin{pmatrix}r\\0\\0\end{pmatrix}.$$



This corkscrew around a cylinder is an example of a twisted space curve, one that does not lie in a plane. ■

General parametric formula of a space curve:

$$\begin{cases} x_1 = x_1(t) \\ x_2 = x_2(t) \\ x_3 = x_3(t) \end{cases} \quad \text{or} \quad \boldsymbol{y} = \boldsymbol{x}(t)$$

If $x_1 = x_2 = x_3$ are constant functions, then we get a point (i.e., degenerate curve).

If $x_i(t)$ is analytic about a value t for i = 1, 2, 3, then the curve can be represented by a Taylor series. Let t be a value within the radius of convergence about 0, and let $\boldsymbol{y} = \boldsymbol{x}(t)$. The familiar Taylor series expansion in a single dimension is

$$y = f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots.$$

To get the Taylor series expansion for the vector $\boldsymbol{y}$, replace $f(x)$ by $\boldsymbol{y}$, $f(a)$ by $\boldsymbol{x}$, and $x-a$ by t. Then $f'(a) = \boldsymbol{x}'$, $f''(a) = \boldsymbol{x}''$, … and

$$\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{x}'t + \frac{1}{2!}\boldsymbol{x}''t^2 + \frac{1}{3!}\boldsymbol{x}'''t^3 + \cdots \tag{2.7}$$

or more specifically,

$$\boldsymbol{y} = \boldsymbol{x}(0) + \boldsymbol{x}'(0)\,t + \frac{1}{2!}\boldsymbol{x}''(0)\,t^2 + \frac{1}{3!}\boldsymbol{x}'''(0)\,t^3 + \cdots$$

Definition If a space curve has a representation $\boldsymbol{y} = \boldsymbol{x}(t)$ where $\boldsymbol{x}(t)$ is analytic, we say the **space curve is analytic**. If, in addition, x_1', x_2', x_3' never vanish simultaneously, we say the **space curve is regular analytic** and that ***t* is a regular parameter.**

Example 9.1 For the curve

$$\begin{cases} x_1 = t^3 \\ x_2 = t^6 \\ x_3 = t^9 \end{cases},$$

$\boldsymbol{x}'(0) = 0$, so t cannot be a regular parameter. But this does not mean that the curve is not a regular analytic curve! In fact, this curve can also be written

$$\begin{cases} x_1 = s \\ x_2 = s^2 \\ x_3 = s^3 \end{cases},$$

which shows that it is a regular analytic curve at $t = s = 0$ and that s is a regular parameter. ■

Convention Henceforth, unless otherwise specified, all space curves are assumed to be regular analytic curves represented by regular parameters.

Theorem Suppose $\boldsymbol{y} = \boldsymbol{u}(t)$ and $\boldsymbol{y} = \boldsymbol{v}(t)$ are analytic curves. Then

$$\langle \boldsymbol{u} | \boldsymbol{v} \rangle' = \langle \boldsymbol{u} | \boldsymbol{v}' \rangle + \langle \boldsymbol{u}' | \boldsymbol{v} \rangle \tag{2.8a}$$

$$(\boldsymbol{u} \times \boldsymbol{v})' = (\boldsymbol{u} \times \boldsymbol{v}') = (\boldsymbol{u}' \times \boldsymbol{v}) \tag{2.8b}$$

If, in addition, $\boldsymbol{u}$ is regular, then

$$\| \boldsymbol{u} \|' = \frac{\langle \boldsymbol{u} | \boldsymbol{u}' \rangle}{\| \boldsymbol{u} \|} \tag{2.8c}$$

Proof. Let $\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$. Then

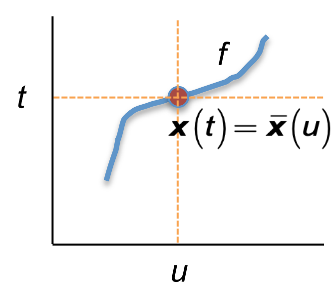
$$\langle \boldsymbol{u} | \boldsymbol{v} \rangle' = \frac{d}{dt}\left(\sum_{i=1}^{3} u_i v_i \right) = \sum (u_i v_i' + u_i' v_i) = \sum u_i v_i' + \sum u_i' v_i = \langle \boldsymbol{u} | \boldsymbol{v}' \rangle + \langle \boldsymbol{u}' | \boldsymbol{v} \rangle \quad ✓$$

$$(\boldsymbol{u} \times \boldsymbol{v}) = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_3 v_1 \end{pmatrix} \quad \Rightarrow \quad (\boldsymbol{u} \times \boldsymbol{v})' = \begin{pmatrix} u_2 v_3' + u_2' v_3 - u_3 v_2' - u_3' v_2 \\ u_3 v_1' + u_3' v_1 - u_1 v_3' - u_1' v_3 \\ u_1 v_2' + u_1' v_2 - u_2 v_1' - u_2' v_1 \end{pmatrix}$$

$$= \begin{pmatrix} u_2 v_3' - u_3 v_2' \\ u_3 v_1' - u_1 v_3' \\ u_1 v_2' - u_2 v_1' \end{pmatrix} + \begin{pmatrix} u_2' v_3 - u_3' v_2 \\ u_3' v_1 - u_1' v_3 \\ u_1' v_2 - u_2' v_1 \end{pmatrix} = (\boldsymbol{u} \times \boldsymbol{v}') + (\boldsymbol{u}' \times \boldsymbol{v}) \quad ✓$$

$$\| \boldsymbol{u} \|' = \left(\sqrt{\langle \boldsymbol{u} | \boldsymbol{u} \rangle} \right)' \overset{(2.8a)}{=} \frac{1}{2} \left(\langle \boldsymbol{u} | \boldsymbol{u} \rangle \right)^{-\frac{1}{2}} 2 \langle \boldsymbol{u} | \boldsymbol{u}' \rangle = \frac{\langle \boldsymbol{u} | \boldsymbol{u}' \rangle}{\sqrt{\langle \boldsymbol{u} | \boldsymbol{u} \rangle}} = \frac{\langle \boldsymbol{u} | \boldsymbol{u}' \rangle}{\| \boldsymbol{u} \|} \quad ✓ \quad ■$$

Notation **Change of Parameter** Let a curve have two parametric representations: $\bar{\boldsymbol{x}}(u) = \boldsymbol{x}(t)$, where $t = f(u)$, $\boldsymbol{y} = \boldsymbol{x}(t)$, and $\boldsymbol{y} = \bar{\boldsymbol{x}}(u)$. Set $\boldsymbol{x}' = \dfrac{d\boldsymbol{x}}{dt}$ and $\bar{\boldsymbol{x}}' = \dfrac{d\bar{\boldsymbol{x}}}{du}$.



Theorem 9.1 Let t be a regular parameter. Then u is a regular parameter $\Leftrightarrow$ $\frac{dt}{du} = f'(u)$ never vanishes $\Leftrightarrow$ $\frac{du}{dt}$ never vanishes.

Proof. t is regular $\Rightarrow$ $\boldsymbol{x}'(t) \neq 0\ \forall t$. (*)

$$\bar{\boldsymbol{x}}'(u) = \frac{d\bar{\boldsymbol{x}}(u)}{du} = \frac{d\boldsymbol{x}(t)}{du} = \frac{d\boldsymbol{x}}{dt}\frac{dt}{du} = \boldsymbol{x}'(t)\, f'(u). \qquad (**)$$

So, u is regular $\overset{\text{def}}{\Leftrightarrow}$ $\bar{\boldsymbol{x}}'(u) \neq 0\ \ \forall u$ $\overset{(*,**)}{\Leftrightarrow}$ $f'(u) \neq 0\ \ \forall u$. ✓

Since $\frac{dt}{du} \neq 0$, $\frac{du}{dt} = \frac{1}{\frac{dt}{du}}$ exists, and it clearly never vanishes ■

Note: If $f' \neq 0$, then the slope is never zero. So, f is either increasing or decreasing. Thus, f^{-1} is single-valued and analytic. Since each t corresponds to a point of the space curve, then similarly each $u = f^{-1}(t)$ corresponds to a point on the curve.

Corollary A curve $\boldsymbol{y} = \boldsymbol{x}(t)$ is regular in a neighborhood of a point $P = \boldsymbol{x}$ iff at least one coordinate of $\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$ is a regular parameter.

Proof. $\mathbf{x}'(t) \neq 0$. WLOG $x_1'(t) \neq 0$. Since x_1' is continuous, there is a neighborhood U of P where x_1' does not vanish. Set $u = x_1$. Since $\frac{du}{dt} = x_1'(t)$ never vanishes in U, by Theorem 9.1, $x_1 = u$ is regular.

Conversely, suppose x_1 is a regular parameter. Then $x_1' \neq 0$ in a neighborhood of U. Therefore,

$$\boldsymbol{x}'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} \neq 0 \ \text{ for all } t \text{ in } U.$$

So, $\boldsymbol{y} = \boldsymbol{x}(t)$ is regular with regular parameter t. ■

The parameter t affects the speed of traversing the curve $\boldsymbol{y} = \boldsymbol{x}(t)$. The curve from $\boldsymbol{x}(t)$ to $\boldsymbol{x}(t+1)$ may trace more than one unit or less than one unit of arc, and it may vary from point to point as t changes. This causes $\boldsymbol{x}'(t)$ to have spurious magnitudes that make it difficult to discover trends. (See Theorem 10.1, below.) However, arc length, s, is itself a parameter and it traces the curve at a constant speed. That is, for one unit of

s, one unit of arc length is traversed. Thus, s is the most useful parameter when it comes to developing theory.

Notation $s = \widehat{\boldsymbol{PQ}}$ is the **arc length** along a curve from a point P to a point Q.

From calculus we know that

$$s = \int_{t_0}^{t_1} \sqrt{(x_1')^2 + (x_2')^2 + (x_3')^2}\, dt = \int_{t_0}^{t_1} \sqrt{\langle \mathbf{x}'(t) | \mathbf{x}'(t) \rangle}\, dt = \int_{t_0}^{t_1} ds. \tag{2.10}$$

Thus,

$$ds = \sqrt{\langle \mathbf{x}'(t) | \mathbf{x}'(t) \rangle}\, dt = \| \mathbf{x}'(t) \| dt, \tag{2.11}$$

$$ds^2 = \langle \mathbf{x}'(t) | \mathbf{x}'(t) \rangle dt^2 = \langle \mathbf{x}'(t)\, dt | \mathbf{x}'(t)\, dt \rangle = \langle d\mathbf{x} | d\mathbf{x} \rangle = dx_1^{\,2} + dx_2^{\,2} + dx_3^{\,2}. \tag{2.12}$$

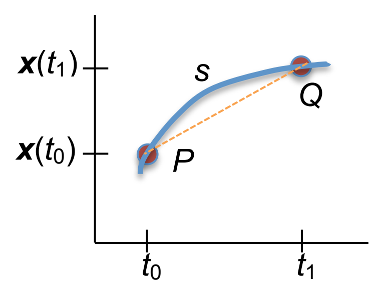
Since $\mathbf{y} = \mathbf{x}(t)$ is regular analytic, $\langle \mathbf{x}'(t) | \mathbf{x}'(t) \rangle$ is never zero, so $\dfrac{ds}{dt}$ never vanishes by (2.11). By Theorem 9.1, s is a regular parameter. Also note that $\| \mathbf{x}'(t) \| \overset{(2.11)}{=} \dfrac{ds}{dt}$.

Notation For points P and Q on a curve, $\overline{\boldsymbol{PQ}}$ denotes the **length of the chord PQ**.

Theorem 10.1 Let $\mathbf{y} = \mathbf{x}(t)$ be a curve. $\| \mathbf{x}'(t) \| = 1$ iff $t = s$ (arc length).

Proof. $\| \mathbf{x}'(t) \| \overset{(2.11)}{=} \dfrac{ds}{dt}$. So

$\| \mathbf{x}'(t) \| = 1$ iff $\dfrac{ds}{dt} = 1$ iff $s = t$. ■



Theorem 10.2 The ratio of chord length $\overline{\boldsymbol{PQ}}$ to arc length $\widehat{\boldsymbol{PQ}}$ approaches unity as Q approaches P. (See figure.)

Proof. $$\lim_{\Delta t \to 0} \frac{\text{chord}}{\text{arc}} = \lim_{\Delta t \to 0} \frac{\sqrt{\Delta x_1^{\,2} + \Delta x_2^{\,2} + \Delta x_3^{\,2}}}{|\Delta s|} = \lim_{\Delta t \to 0} \frac{\sqrt{\left(\dfrac{\Delta x_1}{\Delta t}\right)^2 + \left(\dfrac{\Delta x_2}{\Delta t}\right)^2 + \left(\dfrac{\Delta x_3}{\Delta t}\right)^2}}{\dfrac{\Delta s}{\Delta t}}$$

$$= \frac{\sqrt{\left(\dfrac{dx_1}{dt}\right)^2 + \left(\dfrac{dx_2}{dt}\right)^2 + \left(\dfrac{dx_3}{dt}\right)^2}}{\dfrac{ds}{dt}} = \frac{\sqrt{\left\langle \dfrac{d\mathbf{x}}{dt} \middle| \dfrac{d\mathbf{x}}{dt} \right\rangle}}{\dfrac{ds}{dt}} \overset{(2.11)}{=} \frac{\dfrac{ds}{dt}}{\dfrac{ds}{dt}} = 1. \quad ■$$

Example 10.1 Circular Helix, $r > 0$, $k \neq 0$

$$\boldsymbol{x}(t) = \begin{pmatrix} r\cos t \\ r\sin t \\ kt \end{pmatrix}, \qquad \boldsymbol{x}'(t) = \begin{pmatrix} -r\sin t \\ r\cos t \\ k \end{pmatrix} \qquad \left\langle \boldsymbol{x}'(t) \middle| \boldsymbol{x}'(t) \right\rangle = r^2 + k^2.$$

$$s \overset{(2.10)}{=} \int_0^t \sqrt{\left\langle \boldsymbol{x}'(\tau) \middle| \boldsymbol{x}'(\tau) \right\rangle}\, d\tau = \int_0^t \sqrt{r^2+k^2}\, d\tau = \sqrt{r^2+k^2}\, t \;\Rightarrow\; t = \frac{s}{\sqrt{r^2+k^2}}.$$

Therefore, the parametric equation of the helix in terms of arc length is

$$\boldsymbol{y} = \boldsymbol{x}(s) = \begin{pmatrix} r\cos\dfrac{s}{\sqrt{r^2+k^2}} \\ r\sin\dfrac{s}{\sqrt{r^2+k^2}} \\ \dfrac{ks}{\sqrt{r^2+k^2}} \end{pmatrix}. \quad \blacksquare$$

Example 10.2 Twisted Cubic

$$\boldsymbol{x}(t) = \begin{pmatrix} 6t \\ 3t^2 \\ t^3 \end{pmatrix}, \qquad \boldsymbol{x}'(t) = \begin{pmatrix} 6 \\ 6t \\ 3t^2 \end{pmatrix}, \qquad \left\langle \boldsymbol{x}'(t) \middle| \boldsymbol{x}'(t) \right\rangle = 36 + 36t^2 + 9t^4 = 9\left(4 + 4t^2 + t^4\right).$$

$$s = \int_0^t 3\sqrt{4 + 4\tau^2 + \tau^4}\, d\tau = \cdots = t^3 + 6t$$

$$\Rightarrow \qquad t = \cdots = \sqrt[3]{\frac{s}{2} + \sqrt{\frac{s^4}{4} + 8}} + \sqrt[3]{\frac{s}{2} - \sqrt{\frac{s^4}{4} + 8}}\,. \quad \blacksquare$$

So, this twisted cubic can be written in terms of *s*, but it is not very useful. Even worse, for the general twisted cubic, the arc length integral is elliptical and there is no simply-expressed solution.

The point of these examples is that while arc length is very useful for theory, it is not usually helpful in doing calculations.

Conventions Henceforth we will use "prime", like $\boldsymbol{x}'$, $\boldsymbol{x}'(t)$, etc. to mean derivative with respect to t. Derivatives with respect to s will be written

$$\boldsymbol{x}_s' \equiv \frac{d\boldsymbol{x}}{ds}, \;\; \boldsymbol{x}_s'' \equiv \frac{d^2\boldsymbol{x}}{ds^2}, \;\; \cdots.$$

Moreover, while it remains convenient for $\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{x}''$ to refer to $\mathbf{x}(0)$, it will be convenient for $\boldsymbol{x}_s', \boldsymbol{x}_s'', \ldots$ to refer to a general point $\boldsymbol{x}(t)$. For example, $\boldsymbol{x}_s'$ will mean $\boldsymbol{x}_s'(t)$.

The following facts about arc length derivatives will be useful.

$\boldsymbol{x}_s'$ is a unit vector: $\left\|\boldsymbol{x}_s'\right\|=1$ (from Theorem 10.1) (2.16)

$\boldsymbol{x}_s'' \perp \boldsymbol{x}_s'$, or $\left\langle \boldsymbol{x}_s' \middle| \boldsymbol{x}_s'' \right\rangle = 0$ (2.17)

$\left\langle \boldsymbol{x}_s' \times \boldsymbol{x}_s'' \middle| \boldsymbol{x}_s' \times \boldsymbol{x}_s'' \right\rangle = \left\langle \boldsymbol{x}_s'' \middle| \boldsymbol{x}_s'' \right\rangle$: (2.18)

$$\left\langle \boldsymbol{x}_s' \times \boldsymbol{x}_s'' \middle| \boldsymbol{x}_s' \times \boldsymbol{x}_s'' \right\rangle \overset{(1.20)}{=} \cancelto{1}{\left\langle \boldsymbol{x}_s' \middle| \boldsymbol{x}_s' \right\rangle} \left\langle \boldsymbol{x}_s'' \middle| \boldsymbol{x}_s'' \right\rangle - \cancelto{0}{\left\langle \boldsymbol{x}_s' \middle| \boldsymbol{x}_s'' \right\rangle}^2$$

$$\overset{(2.16,\ 2.17)}{=} \left\langle \boldsymbol{x}_s'' \middle| \boldsymbol{x}_s'' \right\rangle \qquad \checkmark$$

$\boldsymbol{x}_s' = \dfrac{\boldsymbol{x}'(t)}{\left\|\boldsymbol{x}'(t)\right\|}$: (2.18a)

$$\boldsymbol{x}_s' = \frac{d\boldsymbol{x}/dt}{ds/dt} \overset{(2.11)}{=} \frac{\boldsymbol{x}'(t)}{\left\|\boldsymbol{x}'(t)\right\|} \quad \checkmark$$

$\boldsymbol{x}_s'' = \dfrac{\boldsymbol{x}''(t)}{\left\|\boldsymbol{x}'(t)\right\|^2}$: (2.18b)

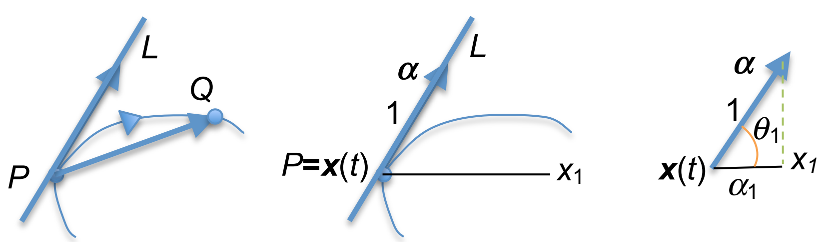
$$\boldsymbol{x}_s'' = \frac{d}{ds}\frac{d\boldsymbol{x}}{ds} = \frac{d}{ds}\left(\frac{d\boldsymbol{x}(t)}{dt}\frac{dt}{ds}\right) = \frac{d}{ds}\left(\frac{\boldsymbol{x}'(t)}{ds/dt}\right) \overset{(2.11)}{=} \frac{d}{ds}\frac{\boldsymbol{x}'(t)}{\left\|\boldsymbol{x}'(t)\right\|} = \frac{1}{\left\|\boldsymbol{x}'(t)\right\|}\frac{d\boldsymbol{x}'(t)}{ds}$$

$$= \frac{1}{\left\|\boldsymbol{x}'(t)\right\|}\frac{d\boldsymbol{x}'(t)/dt}{ds/dt} \overset{(2.11)}{=} \frac{\boldsymbol{x}''(t)}{\left\|\boldsymbol{x}'(t)\right\|^2}$$

$\boldsymbol{x}_s''' = \dfrac{\boldsymbol{x}'''(t)}{\left\|\boldsymbol{x}'(t)\right\|^3}$: (2.18c)

$$\boldsymbol{x}_s''' = \frac{d\boldsymbol{x}_s''}{ds} = \frac{d\boldsymbol{x}_s''}{dt}\frac{dt}{ds} \overset{(2.18c)}{=} \frac{1}{\left\|\boldsymbol{x}'(t)\right\|^2}\frac{d\boldsymbol{x}''(t)}{dt}\frac{dt}{ds} \overset{(2.11)}{=} \frac{\boldsymbol{x}'''(t)}{\left\|\boldsymbol{x}'(t)\right\|^2}\frac{1}{\left\|\boldsymbol{x}'(t)\right\|}$$

Convention We assign the positive direction of the tangent line L at P and the secant line PQ to match the positive direction on the curve from P.



Definition The **tangent vector at a point $\boldsymbol{P} = \boldsymbol{x}(\boldsymbol{t})$** is the positively-oriented unit vector

$$\boldsymbol{\alpha} = \boldsymbol{x_s}'\,. \tag{2.19}$$

Like $\boldsymbol{x}_s\,'$, $\boldsymbol{\alpha}$ will be used to refer to a general point $\boldsymbol{x}(t)$ and not specifically to $\boldsymbol{x}(0)$.

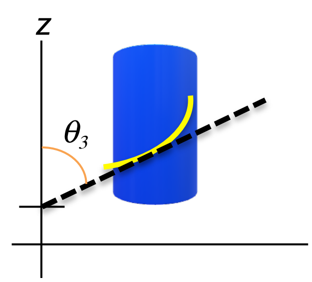
The components $\alpha_i = \dfrac{dx_i}{ds}$ are the **direction cosines of the tangent vector.**

In terms of t, $$\boldsymbol{\alpha} \overset{(2.18b)}{=} \frac{\boldsymbol{x}'(t)}{\left\| \boldsymbol{x}'(t) \right\|}. \tag{2.20}$$

Example Circular Helix, $r > 0$, $k \neq 0$

$$\boldsymbol{x}(t) = \begin{pmatrix} r\cos t \\ r\sin t \\ k\,t \end{pmatrix}, \qquad \boldsymbol{x}'(t) = \begin{pmatrix} -r\sin t \\ r\cos t \\ k \end{pmatrix} \qquad \left\langle \boldsymbol{x}'(t) \middle| \boldsymbol{x}'(t) \right\rangle = r^2 + k^2$$

$$\boldsymbol{\alpha} \overset{(2.20)}{=} \frac{\boldsymbol{x}'(t)}{\sqrt{\left\langle \boldsymbol{x}'(t) \middle| \boldsymbol{x}'(t) \right\rangle}} = \begin{pmatrix} -\dfrac{r}{\sqrt{r^2+k^2}}\sin t \\ +\dfrac{r}{\sqrt{r^2+k^2}}\cos t \\ \dfrac{k}{\sqrt{r^2+k^2}} \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$



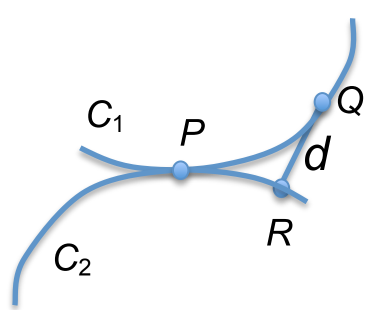
In particular, $\alpha_3 = \cos\theta_3$ is a constant. The helix cuts the rulings of the cylinder at a constant angle.

$$\cos\theta_3 = \boldsymbol{\alpha}_3 = \frac{k}{\sqrt{r^2+k^2}} \quad \Rightarrow \quad k^2 = \left(r^2+k^2\right)\cos^2\theta_3 \quad \Rightarrow \quad k^2 = r^2\,\frac{\cos^2\theta_3}{\sin^2\theta_3}$$

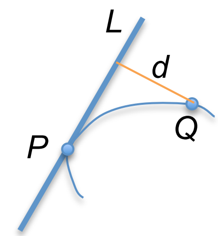
$$\Rightarrow \quad k = r\cot\theta_3 \tag{2.20a}$$

$$\Rightarrow \quad \sqrt{r^2+k^2} = \frac{r}{\sin\theta_3} \quad \Rightarrow \quad \boldsymbol{\alpha} = \begin{pmatrix} -\sin\theta_3\,\sin t \\ \sin\theta_3\,\cos t \\ \cos\theta_3 \end{pmatrix} \qquad \blacksquare$$

We now begin development of the topic of curvature.



Definition Suppose curves C_1 and C_2 are tangential at a point P. Let Q be a point on the first curve within the radius of convergence of the Taylor series for P. Let R be the point on C_2 that is closest to Q. As $Q \to P$, so does R, so we can assume that R, too, is within the radius of convergence of the Taylor's series for P. The perpendicular distance $d = \overline{QR}$ can then be expressed as an infinite series $d = d_0 + d_1 t + d_2 t^2 + \cdots$. Let d_k be the first non-zero coefficient. We say the **two curves have contact of order k–1 at P.**



Consider the case of the tangent line to a curve at $P = \boldsymbol{x} = \boldsymbol{x}(0)$. The tangent line has direction $\boldsymbol{x}' = \boldsymbol{x}'(0)$. We can find d^2 by expanding the formula for the distance from $Q = \boldsymbol{y} = \boldsymbol{x}(t)$ to the tangent line.

$$d^2 \overset{(1.28)}{=} \frac{\left\langle \boldsymbol{x}' \times (\boldsymbol{y} - \boldsymbol{x}) \,\middle|\, \boldsymbol{x}' \times (\boldsymbol{y} - \boldsymbol{x}) \right\rangle}{\left\| \boldsymbol{x}' \right\|^2} \tag{*}$$

$$\boldsymbol{y} - \boldsymbol{x} \overset{(2.7)}{=} \boldsymbol{x}' t + \frac{1}{2!} \boldsymbol{x}'' t^2 + \frac{1}{3!} \boldsymbol{x}''' t^3 + \cdots + \frac{1}{n!} \boldsymbol{x}^{(n)} t^n + \cdots$$

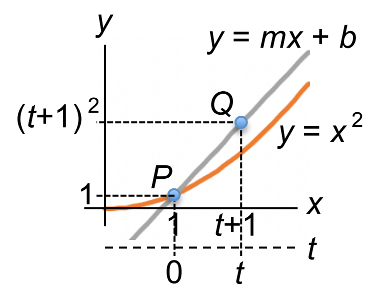
$$\begin{aligned} \boldsymbol{x}' \times (\boldsymbol{y} - \boldsymbol{x}) &\overset{(1.18e)}{=} \cancel{(\boldsymbol{x}' \times \boldsymbol{x}')}^{0}\, t + \frac{1}{2!} (\boldsymbol{x}' \times \boldsymbol{x}'') t^2 + \frac{1}{3!} (\boldsymbol{x}' \times \boldsymbol{x}''') t^3 + \cdots + \frac{1}{n!} \left(\boldsymbol{x}' \times \boldsymbol{x}^{(n)}\right) t^n + \cdots \\ &\overset{(\text{Th } 3.2)}{=} \frac{1}{2!} (\boldsymbol{x}' \times \boldsymbol{x}'') t^2 + \frac{1}{3!} (\boldsymbol{x}' \times \boldsymbol{x}''') t^3 + \cdots + \frac{1}{n!} \left(\boldsymbol{x}' \times \boldsymbol{x}^{(n)}\right) t^n + \cdots \end{aligned}$$

$$\begin{aligned} \left\| \boldsymbol{x}' \right\|^2 d^2 &\overset{(*)}{=} \left\langle \boldsymbol{x}' \times (\boldsymbol{y} - \boldsymbol{x}) \,\middle|\, \boldsymbol{x}' \times (\boldsymbol{y} - \boldsymbol{x}) \right\rangle \\ &\overset{(1.22)}{=} \frac{\left\| \boldsymbol{x}' \times \boldsymbol{x}'' \right\|^2}{(2!)^2} t^4 + \frac{\left\langle \boldsymbol{x}' \times \boldsymbol{x}'' \,\middle|\, \boldsymbol{x}' \times \boldsymbol{x}''' \right\rangle}{2!\,3!} t^5 \\ &\quad + \frac{\left\| \boldsymbol{x}' \times \boldsymbol{x}''' \right\|^2}{(3!)^2} t^6 + \frac{\left\langle \boldsymbol{x}' \times \boldsymbol{x}''' \,\middle|\, \boldsymbol{x}' \times \boldsymbol{x}^{(4)} \right\rangle}{3!\,4!} t^7 + \cdots \\ &\quad + \frac{\left\| \boldsymbol{x}' \times \boldsymbol{x}^{(n)} \right\|^2}{(n!)^2} t^{2n} + \frac{\left\langle \boldsymbol{x}' \times \boldsymbol{x}^{(n)} \,\middle|\, \boldsymbol{x}' \times \boldsymbol{x}^{(n+1)} \right\rangle}{n!\,(n+1)!} t^{2n+1} + \cdots \end{aligned} \tag{2.21}$$

In the infinite series (2.21), d^2 has order ≥ 4 with respect to t. A series for d would have order ≥ 2, and we say that **the tangent line at P has contact order ≥ 1**. Since $\frac{ds}{dt} \overset{(2.11)}{=} \left\| \boldsymbol{x}'(t) \right\| \neq 0$, the arc length s has the same contact order as t, and we say that the **arc $\overset{\frown}{PQ}$ has the same contact order as the tangent line at P**.

Caution In equation (2.21), d and $\left\| \boldsymbol{x}' \right\|$ are just numbers, multiplied together. As Q approaches P there may be a special point $Q = \boldsymbol{x}(t)$ where $\left\| \boldsymbol{x}'(t) \right\| = d$, and so in (2.21) it may appear for that special point that d^4, not d^2, has minimal contact order 2. But since we are considering a limit where d shrinks while the term $\left\| \boldsymbol{x}' \right\|$ remains constant, it is not proper to try to resolve the value of $\left\| \boldsymbol{x}' \right\|$ in terms of d.

Why are we interested in contact order? If d is of order 3, then near P the distance $\in^3$ between the tangent and curve is smaller than $\in^2$ were d only of order 2. This means that the higher the contact order, the better the fit of the tangent line at P. We illustrate this in the following example.



Example The tangent line L to the curve $y = x^2$ at $P = (1,1)$ has equation $y = 2x - 1$. Let's show that the tangent line is the line through P with the highest contact order.

Let $y = mx + b$ be any line through (1,1). We find $b = 1 - m$, so we have $y = mx + 1 - m$. Let's put this into parametric form $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{a}t$. This formula holds only when $\boldsymbol{x} = \boldsymbol{x}(0)$.

$$P = \boldsymbol{x} = \boldsymbol{x}(0) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \; \boldsymbol{a} = \begin{pmatrix} 1 \\ m \\ 0 \end{pmatrix}.$$

Since $P = \boldsymbol{x}(0)$ is located at $x = 1$, we need a new parameter $t = x - 1$ so that when $x = 1$ we have $t = 0$. Let

$$Q = x(t) = \begin{pmatrix} t \\ (t+1)^2 \\ 0 \end{pmatrix}. \quad (\boldsymbol{y} - \boldsymbol{x}) \times \boldsymbol{a} = \begin{pmatrix} t \\ (t+1)^2 - 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ m \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ mt - (t+1)^2 - 1 \end{pmatrix}.$$

$$d = \frac{-t^2 + (m-2)t}{\sqrt{1+m^2}}.$$

We see that d has order 2 iff $m = 2$. That is, $m = 2$ maximizes the order of d. Plugging in, we get the tangent line $y = mx + 1 - m = 2x + 1 - 2 = 2x - 1$. ■

From equation (2.21), derived with $P = \boldsymbol{x}$, we immediately get the following theorem. Remember that $\boldsymbol{x}' = \boldsymbol{x}'(0)$ and $\boldsymbol{x}'' = \boldsymbol{x}''(0)$.

Theorem d has order ≥ 3 iff $\boldsymbol{x}' \times \boldsymbol{x}'' = 0$.

The next theorem is closely related to this.

Theorem 13.1 A curve $\boldsymbol{y} = \boldsymbol{x}(t)$ is a straight line iff $\boldsymbol{x}'(t) \times \boldsymbol{x}''(t) = 0$ at every point on the curve.

Proof. Suppose $\boldsymbol{x}'(t) \times \boldsymbol{x}''(t) = 0$ for all t. Then at every point, $\boldsymbol{x}''(t) \parallel \boldsymbol{x}'(t)$. By convention, $\boldsymbol{x}'(t) \neq 0$. So, there is a scalar function $f(t)$ such that $\boldsymbol{x}''(t) = f(t)\, \boldsymbol{x}'(t)$. So,

$$\frac{d\boldsymbol{x}'(t)}{dt} = \boldsymbol{x}''(t) = f(t)\, \boldsymbol{x}'(t) \quad \Rightarrow \quad \frac{d\boldsymbol{x}'(t)}{\boldsymbol{x}'(t)} = f(t)\, dt \text{, a triple of equations.}$$

$$\Rightarrow \quad \exists \text{ constant } \boldsymbol{a} \ni \ \ln \boldsymbol{x}'(t) - \ln \boldsymbol{a} = \int_a^x \frac{d\boldsymbol{u}'}{\boldsymbol{u}'} = \int f(t)\, dt \quad \Rightarrow \quad \ln \boldsymbol{x}'(t) = \ln \boldsymbol{a} + \int f(t) dt$$

$$\Rightarrow \quad \boldsymbol{x}'(t) = \boldsymbol{a}\, e^{f(t)\,dt} \quad \Rightarrow \quad \boldsymbol{x}(t) = \boldsymbol{a} \int e^{\int f(t)dt}\, dt + \boldsymbol{r} \quad \text{for some constant } \boldsymbol{r}.$$

Set $u = \int e^{\int f(t)dt}\, dt$, a scalar function of t. Then $\boldsymbol{y} = \boldsymbol{x}(t) = u\, \boldsymbol{a} + \boldsymbol{r}$, a straight line.

Conversely, if $\boldsymbol{y} = \boldsymbol{x}(t)$ is a straight line, it has a parametric representation of the form $\boldsymbol{x}(t) = u(t)\, \boldsymbol{a} + \boldsymbol{r}$. So,

$$\boldsymbol{x}'(t) = u'(t)\, \boldsymbol{a}, \quad \boldsymbol{x}''(t) = u''(t)\, \boldsymbol{a}, \text{ and } \boldsymbol{x}'(t) \times \boldsymbol{x}''(t) = u'(t)\, u''(t)\, \boldsymbol{a} \times \boldsymbol{a} = 0. \quad \blacksquare$$

Definition Let $P = \boldsymbol{x}(t)$ be a point of the curve $\boldsymbol{y} = \boldsymbol{x}(t)$. We say ***P* is regular** if $\boldsymbol{x}'(t) \times \boldsymbol{x}''(t) \neq 0$. If $\boldsymbol{x}(t)' \times \boldsymbol{x}(t)'' = 0$ we say that ***P* is singular**.

The term "singular" is used because the curvature (yet to be defined) is zero. Every point on a straight line is singular and the line has curvature zero.

Theorem 13.2 Vectors $\boldsymbol{a}(t)$ all have the same direction iff $\boldsymbol{a}(t) \times \boldsymbol{a}'(t) = 0 \ \ \forall t$.

Proof. Let $\boldsymbol{a} = \boldsymbol{a}(t)$ and $\boldsymbol{a} + \Delta\boldsymbol{a} = \boldsymbol{a}(t + \Delta t)$. If the vectors all have the same direction, then $\boldsymbol{a}$ and $\boldsymbol{a} + \Delta\boldsymbol{a}$ are parallel: $\boldsymbol{a} \times (\boldsymbol{a} + \Delta\boldsymbol{a}) = 0$. So $\boldsymbol{a} \times \boldsymbol{a} \ + \ \boldsymbol{a} \times \Delta\boldsymbol{a} = 0$

$$\Rightarrow \quad \boldsymbol{a} \times \Delta\boldsymbol{a} = 0 \quad \Rightarrow \quad \boldsymbol{a} \times \frac{\Delta\boldsymbol{a}}{\Delta t} = 0 \quad \Rightarrow \quad \boldsymbol{a} \times \boldsymbol{a}' = 0. \quad \checkmark$$

Conversely, if $\boldsymbol{a} \times \boldsymbol{a}' = 0 \ \ \forall t$, they are parallel. So, there is a scalar function $f(t)$ such that $\boldsymbol{a}' = f(t)\, \boldsymbol{a}$. There is a constant $\boldsymbol{b}$ such that

$$\ln \boldsymbol{a} + \boldsymbol{b} = \int \frac{d\boldsymbol{a}}{\boldsymbol{a}} = \int \frac{\boldsymbol{a}'}{\boldsymbol{a}} dt = \int f(t) dt \quad \Rightarrow \quad \boldsymbol{a} = e^{\boldsymbol{b}}\, e^{\int f(tdt}.$$

Set $\boldsymbol{c} = e^{\boldsymbol{b}}$, a constant, and $\phi(t) = e^{\int f(t)dt}$. Then $\boldsymbol{a} = \phi(t)\, \boldsymbol{c}$. Thus, all $\boldsymbol{a}(t)$ have the same direction. $\blacksquare$

In light of Theorem 13.2, we can rephrase Theorem 13.1.

Theorem 13.1 A curve is a straight line iff all tangent vectors $\boldsymbol{x}'(t)$ have the same direction.

Exercise 13 The tangent line L at $P = \boldsymbol{x}(0)$ has contact of order m with the curve iff the vectors $\boldsymbol{x}'$, $\boldsymbol{x}''$, …, $\boldsymbol{x}^{(m)}$ are colinear but $\boldsymbol{x}^{(m+1)}$ does not lie on the common line.

Solution. L has contact order m $\overset{\text{Defn}}{\Leftrightarrow}$ d has order $(m+1)$ $\Leftrightarrow$ d^2 has order $2(m+1)$ $\overset{(2.21)}{\Leftrightarrow}$ $\boldsymbol{x}'\times\boldsymbol{x}^{(k)}=0$ for $2\le k\le m$ and $\boldsymbol{x}'\times\boldsymbol{x}^{(m+1)}\ne 0$ $\overset{\text{Th 13.2}}{\Leftrightarrow}$ $\boldsymbol{x}',\boldsymbol{x}'',\cdots,\boldsymbol{x}^{(m)}$ are parallel but not $\boldsymbol{x}^{(m+1)}$. ■

Contact order for a plane is defined analogously to contact order for curves.

Definition Let L be the tangent line at a regular point $P = \boldsymbol{x}(0)$ on a curve $\boldsymbol{y} = \boldsymbol{x}(t)$, Π a plane containing L, $Q = \boldsymbol{x}(t)$ a point on the curve within the radius of convergence of the Taylor series for $\boldsymbol{x}$, and $d=\operatorname{dist}(Q,\Pi)$. The **contact order of the plane Π** is $(k-1)$, where k is the order of the infinite series $d = d(t)$.

We will show there is a plane Π of highest contact order. It will be called it the osculating plane. We begin construction of the osculating plane by recalling from (1.27) the formula for distance d from a point $\boldsymbol{y}=\boldsymbol{x}+\Delta\boldsymbol{x}$ of a curve to a plane Π through $\boldsymbol{x}$ having normal $\boldsymbol{a}$:

$$d=\pm\frac{\langle \boldsymbol{y}-\boldsymbol{x}\,|\,\boldsymbol{a}\rangle}{\|\boldsymbol{a}\|}.$$

Since

$$\boldsymbol{y}-\boldsymbol{x}\overset{(2.7)}{=}\boldsymbol{x}'t+\frac{1}{2!}\boldsymbol{x}''t^2+\frac{1}{3!}\boldsymbol{x}'''t^3+\cdots+\frac{1}{n!}\boldsymbol{x}^{(n)}t^n+\cdots.$$

$$\pm\|\boldsymbol{a}\|d=\langle \boldsymbol{x}'|\boldsymbol{a}\rangle t+\frac{1}{2!}\langle \boldsymbol{x}''|\boldsymbol{a}\rangle t^2+\frac{1}{3!}\langle \boldsymbol{x}'''|\boldsymbol{a}\rangle t^3+\cdots+\frac{1}{n!}\langle \boldsymbol{x}^{(n)}|\boldsymbol{a}\rangle t^n+\cdots. \qquad (2.22b)$$

d has order ≥ 1 with respect to t.

(I) d has order ≥ 2 iff $\langle \boldsymbol{x}'|\boldsymbol{a}\rangle=0$,

and

(II) d has order ≥ 3 iff both $\langle \boldsymbol{x}'|\boldsymbol{a}\rangle=0$ and $\langle \boldsymbol{x}''|\boldsymbol{a}\rangle=0$:

(I) To satisfy $\langle \boldsymbol{x}'|\boldsymbol{a}\rangle=0$, $\boldsymbol{x}'\perp\boldsymbol{a}$ $\Rightarrow$ $\boldsymbol{x}'(\tau)\in\Pi$. That is, let L be the tangent line at P. Then $L\subset\Pi$. Thus, the planes containing the tangent line have contact order ≥ 1, and no other plane has this property since $\langle \boldsymbol{x}'|\boldsymbol{a}\rangle$ must be zero.

(II) To satisfy both $\langle \boldsymbol{x}' | \boldsymbol{a} \rangle = 0$ and $\langle \boldsymbol{x}'' | \boldsymbol{a} \rangle = 0$, $\boldsymbol{a} \perp \boldsymbol{x}'$ and $\boldsymbol{a} \perp \boldsymbol{x}''$. Since $\boldsymbol{x}' \times \boldsymbol{x}''$ is perpendicular to both $\boldsymbol{x}'$ and $\boldsymbol{x}''$, $\boldsymbol{a}$ is parallel to $\boldsymbol{x}' \times \boldsymbol{x}''$. But, $\boldsymbol{a}$ is perpendicular to Π. Thus, $\boldsymbol{x}' \times \boldsymbol{x}'' \perp \Pi$. There is only one plane through P that is perpendicular to $\boldsymbol{x}' \times \boldsymbol{x}''$, so Π is the unique plane with contact order ≥ 2.

Theorem The unique plane Π having contact order ≥ 2 is the plane of highest contact order.

Proof. If a plane Ψ had higher contact order than Π, it would still be true from (II) that $\langle \boldsymbol{x}' | \boldsymbol{a} \rangle = 0$ and $\langle \boldsymbol{x}'' | \boldsymbol{a} \rangle = 0$. So, $\Psi \subseteq \Pi$, which means $\Psi = \Pi$. ■

Definition Let $P = \boldsymbol{x}(t)$ be a regular point on a non-linear curve. The unique plane Π through P that is perpendicular to $\boldsymbol{x}'(t) \times \boldsymbol{x}''(t)$ is called the **osculating plane of the curve at P.**

From (1.26a), the equation of Π is $\langle\, \boldsymbol{z} - \boldsymbol{x}(t) \,|\, \boldsymbol{x}'(t) \times \boldsymbol{x}''(t) \,\rangle = 0.$

Theorem A non-linear curve $\boldsymbol{y} = \boldsymbol{x}(t)$ in a plane Π has Π as it osculating plane for every regular point on the curve.

Proof. First, suppose Π is the xy-plane. Let P be a regular point on the curve. WLOG $P = \boldsymbol{x}(0)$. Since P is regular,

$$\boldsymbol{x}' \times \boldsymbol{x}'' \neq 0 \text{ and } \boldsymbol{x}' \perp \boldsymbol{x}'' \quad \Rightarrow \quad \boldsymbol{x}' \times \boldsymbol{x}'' \perp xy\text{-plane}$$

$$\Rightarrow \quad \boldsymbol{x}' \times \boldsymbol{x}'' = \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix} \text{ for some constant } b \neq 0.$$

The equation of Π at P is $\langle \boldsymbol{z} - \boldsymbol{x} | \boldsymbol{x}' \times \boldsymbol{x}'' \rangle = 0$. $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$, $\boldsymbol{z} - \boldsymbol{x} = \begin{pmatrix} z_1 - x_1 \\ z_2 - x_2 \\ z_3 \end{pmatrix}$, and

$0 = \langle \boldsymbol{z} - \boldsymbol{x} | \boldsymbol{x}' \times \boldsymbol{x}'' \rangle = (z_1 - x_1)(0) + (z_2 - x_2)(0) + z_3\, b$. Since $b \neq 0$, $z_3 = 0$. So,

$\Pi = \left\{ \begin{pmatrix} z_1 \\ z_2 \\ 0 \end{pmatrix} \right\} = xy$-plane. For an arbitrary plane, rotate and/or translate the plane to the xy-axis, apply the result, and rotate-translate back. ■

Example Circular helix, $r > 0$, $k \neq 0$

$$\boldsymbol{y} = \boldsymbol{x}(t) = \begin{pmatrix} r\cos t \\ r\sin t \\ k t \end{pmatrix}, \qquad \boldsymbol{x}'(t) = \begin{pmatrix} -r\sin t \\ r\cos t \\ k \end{pmatrix} \qquad \boldsymbol{x}''(t) = \begin{pmatrix} -r\cos t \\ -r\sin t \\ 0 \end{pmatrix}$$

$$\boldsymbol{x}'(t)\times\boldsymbol{x}''(t) = \begin{pmatrix} r\,k\sin t \\ -r\,k\cos t \\ r^2 \end{pmatrix}. \qquad \text{Let } \boldsymbol{z} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in \Pi. \qquad \boldsymbol{z} - \boldsymbol{x}(t) = \begin{pmatrix} z_1 - r\cos t \\ z_2 - r\sin t \\ z_3 - k\,t \end{pmatrix}$$

$$0 = \left\langle \boldsymbol{z} - \boldsymbol{x}(t) \,\middle|\, \boldsymbol{x}'(t)\times\boldsymbol{x}''(t) \right\rangle$$

$$= r\,k\left(z_1\sin t - \cancel{r\sin t\cos t}\right) - r\,k\left(z_2\cos t - \cancel{r\sin t\cos t}\right) + r^2 z_3 - r^2 k\,t$$

$$0 = k\,z_1\sin t - k\,z_2\cos t + r\,z_3 - r\,k\,t$$

This is the equation of the osculating plane Π at a point $P = \boldsymbol{x}(t)$. That is, fix t and set constants $A = k\sin t$, $B = -k\cos t$, $C = -r\,k\,t$. Then

$A\,z_1 + B\,z_2 + r\,z_3 + C = 0$ is the equation of the plane.

At the point $P = \boldsymbol{x}(0)$, the equation is

$k\,z_2 + r\,z_3 = 0$. ■

Definition Let $P = \boldsymbol{x}(t)$ be a point on a curve $\boldsymbol{y} = \boldsymbol{x}(t)$. The tangent vector $\boldsymbol{\alpha}$ lies in the osculating plane Π. The positively-oriented unit vector in the plane Π that is perpendicular to $\boldsymbol{\alpha}$ is called the **principal normal vector at *P*, $\boldsymbol{\beta}$.** The positively-oriented unit vector that is perpendicular to the plane Π is called the **binormal vector, $\boldsymbol{\gamma}$.** The **trihedral at *P*** is the oriented surface $\boldsymbol{\alpha\beta\gamma}$ (see trihedral definition before Theorem 5.1).

It is useful to derive $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\gamma}$ in terms of arc length, *s*.

- $\boldsymbol{\alpha} \overset{(2.19)}{=} \boldsymbol{x}_s'$
- By (2.17), $\boldsymbol{x}_s'' \perp \boldsymbol{x}_s'$ and $\boldsymbol{x}_s'' \in \Pi \;\Rightarrow\; \boldsymbol{\beta} = \dfrac{x_s''}{\lVert x_s'' \rVert}$
- $\boldsymbol{x}_s' \times \boldsymbol{x}_s'' \perp \Pi. \quad \therefore\ \boldsymbol{\gamma} = \dfrac{\boldsymbol{x}_s'\times\boldsymbol{x}_s''}{\lVert \boldsymbol{x}_s'\times\boldsymbol{x}_s'' \rVert} \overset{(2.18)}{=} \dfrac{\boldsymbol{x}_s'\times\boldsymbol{x}_s''}{\lVert \boldsymbol{x}_s'' \rVert}$

$$\boxed{\boldsymbol{\alpha} = \boldsymbol{x}_s' \qquad \boldsymbol{\beta} = \frac{\boldsymbol{x}_s''}{\lVert \boldsymbol{x}_s'' \rVert} \qquad \boldsymbol{\gamma} = \frac{\boldsymbol{x}_s'\times\boldsymbol{x}_s''}{\lVert \boldsymbol{x}_s'' \rVert}} \tag{2.25}$$

By (1.3), the components of $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\gamma}$ are their direction cosines.

By (1.26a), the equations of the planes determined by the edges of the trihedral are

$$\left\langle \boldsymbol{z}-\boldsymbol{x}(t) \middle| \alpha \right\rangle \qquad \left\langle \boldsymbol{z}-\boldsymbol{x}(t) \middle| \beta \right\rangle \qquad \left\langle \boldsymbol{z}-\boldsymbol{x}(t) \middle| \gamma \right\rangle \tag{2.26}$$

- The first of these, the plane containing $\beta \times \gamma$ (normal x binormal), is the **normal plane at *P* = *x*(*t*)**.
- The second, containing $\gamma \times \alpha$ (binormal x tangent), is called the **rectifying plane**.
- The third, containing $\alpha \times \beta$ (tangent x normal), is the osculating plane.

Example Circular Helix, $r > 0$, with $k = r \cot \theta$, $\cot \theta \neq 0$

$$\boldsymbol{x}(t)=\begin{pmatrix} r\cos t \\ r\sin t \\ kt \end{pmatrix}, \qquad \boldsymbol{x}'(t)=\begin{pmatrix} -r\sin t \\ r\cos t \\ k \end{pmatrix} \qquad \boldsymbol{x}''(t)=\begin{pmatrix} -r\cos t \\ -r\sin t \\ 0 \end{pmatrix} \qquad \boldsymbol{x}'''(t)=\begin{pmatrix} r\sin t \\ -r\cos t \\ 0 \end{pmatrix}$$

$$\boldsymbol{x}'(t)\times\boldsymbol{x}''(t)=\begin{pmatrix} r^2\cot\theta\sin t \\ -r^2\cot\theta\cos t \\ r^2 \end{pmatrix}. \qquad \left\langle \boldsymbol{x}'(t) \middle| \boldsymbol{x}'(t) \right\rangle = r^2\left(1+\cot^2\theta\right)=r^2\csc^2\theta$$

$$\left\| \boldsymbol{x}'(t) \right\| = r\csc\theta \qquad \left\langle \boldsymbol{x}''(t) \middle| \boldsymbol{x}''(t) \right\rangle = r^2 \qquad \left\| \boldsymbol{x}''(t) \right\| = r$$

$$\left\langle \boldsymbol{x}'(t)\times\boldsymbol{x}''(t) \middle| \boldsymbol{x}'(t)\times\boldsymbol{x}''(t) \right\rangle = r^4\left(r+\cot^2\theta\right)=r^4\csc^2\theta \qquad \left\| \boldsymbol{x}'(t)\times\boldsymbol{x}''(t) \right\| = r^2\csc\theta$$

$$\alpha \overset{(2.20)}{=} \frac{\boldsymbol{x}'(t)}{\left\|\boldsymbol{x}'(t)\right\|}=\begin{pmatrix} -\sin\theta\sin t \\ \sin\theta\cos t \\ \cos\theta \end{pmatrix} \qquad \beta \overset{(2.25)}{=} \frac{\boldsymbol{x}''(t)}{\left\|\boldsymbol{x}''(t)\right\|}=\begin{pmatrix} -\cos t \\ -\sin t \\ 0 \end{pmatrix}$$

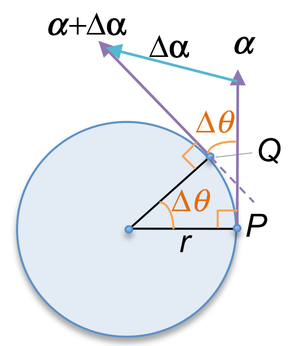
$$\gamma = \frac{\boldsymbol{x}'(t)\times\boldsymbol{x}''(t)}{\left\|\boldsymbol{x}'(t)\times\boldsymbol{x}''(t)\right\|}=\begin{pmatrix} \cos\theta\sin t \\ -\cos\theta\cos t \\ \sin\theta \end{pmatrix}$$

For later,

$$\left(\boldsymbol{x}'(t)\;\; \boldsymbol{x}''(t)\;\; \boldsymbol{x}'''(t)\right)=\begin{vmatrix} -r\sin t & -r\cos t & r\sin t \\ r\cos t & -r\sin t & -r\cos t \\ r\cot\theta & 0 & 0 \end{vmatrix} = r^3\cot\theta\left[\cos^2 t+\sin^2 t\right]=r^3\cot\theta \quad \blacksquare$$

We are now in a position to introduce curvature. On large circles, like the equator on Earth, curvature is very small, almost zero. At the other extreme, small circles have large curvature. So it makes sense to define curvature for a circle as 1/*r*.

For a space curve that is not a circle, we will define curvature in terms of circles that are tangent to the curve. However, there are an infinite number of circles having different radii that are tangent to any regular point of the curve. So, we need a definition of curvature that knows how to select the "correct" circle.



For a circle in the *xy*-plane, the figure at the right shows that the central angle $\Delta\theta$ is the same as the angle between the tangent lines at *P* and *Q*. So, we can define curvature for a circle as

$$\kappa = \frac{1}{r} = \frac{\Delta\theta}{r\,\Delta\theta} = \frac{\Delta\theta}{\Delta s} \text{ where } \Delta\theta \text{ is the angle between the}$$

tangent lines and where Δs is the arc length $\overset{\frown}{PQ}$.

At the opposite extreme from a circle, for a line

$$\kappa = \frac{\Delta\theta}{\Delta s} = 0.$$

Thus, we see that the ratio $\Delta\theta/\Delta s$ quantifies the tendency to curve away from the tangent line. This is the motivation for following definition.

Definition Let *P* be a point on a curve $\boldsymbol{y} = \boldsymbol{x}(t)$, *Q* a nearby point, $\Delta s = \overset{\frown}{PQ}$, $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha} + \Delta\boldsymbol{\alpha}$ the directed tangents at *P* and *Q*, respectively, and $\Delta\theta$ the angle between $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha} + \Delta\boldsymbol{\alpha}$. **Curvature at *P*** is defined as

$$\kappa = \frac{1}{R} = \lim_{\Delta s \to 0} \left| \frac{\Delta\theta}{\Delta s} \right| = \frac{d\theta}{ds}. \tag{2.28}$$

R is the **radius of curvature**. Of course, $R \equiv 0$ if $\kappa = \infty$ and $R = \infty$ if $\kappa = 0$.

Theorem $\boxed{\kappa = \frac{1}{R} = \left\| \boldsymbol{\alpha}_s' \right\|}$ (2.29)

Proof. Since $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha} + \Delta\boldsymbol{\alpha}$ are unit vectors,

$$\cos\Delta\theta = \langle \boldsymbol{\alpha} | \boldsymbol{\alpha} + \Delta\boldsymbol{\alpha} \rangle = \langle \boldsymbol{\alpha} | \boldsymbol{\alpha} \rangle + \langle \boldsymbol{\alpha} | \Delta\boldsymbol{\alpha} \rangle = 1 + \langle \boldsymbol{\alpha} | \Delta\boldsymbol{\alpha} \rangle$$

$$1 = \langle \boldsymbol{\alpha} + \Delta\boldsymbol{\alpha} | \boldsymbol{\alpha} + \Delta\boldsymbol{\alpha} \rangle = 1 + 2\langle \boldsymbol{\alpha} | \Delta\boldsymbol{\alpha} \rangle + \langle \Delta\boldsymbol{\alpha} | \Delta\boldsymbol{\alpha} \rangle$$

$$\Rightarrow \quad \langle \boldsymbol{\alpha} | \Delta\boldsymbol{\alpha} \rangle = -\frac{1}{2} \langle \Delta\boldsymbol{\alpha} | \Delta\boldsymbol{\alpha} \rangle$$

$$\Rightarrow \quad \cos\Delta\theta = 1 - \frac{1}{2} \langle \Delta\boldsymbol{\alpha} | \Delta\boldsymbol{\alpha} \rangle, \text{ or } \langle \Delta\boldsymbol{\alpha} | \Delta\boldsymbol{\alpha} \rangle = 2(1 - \cos\Delta\theta)$$

$$\Rightarrow \quad \left\langle \frac{\Delta\boldsymbol{\alpha}}{\Delta s} \middle| \frac{\Delta\boldsymbol{\alpha}}{\Delta s} \right\rangle = \frac{1}{(\Delta s)^2} \langle \Delta\boldsymbol{\alpha} | \Delta\boldsymbol{\alpha} \rangle = \frac{2(1 - \cos\Delta\theta)}{(\Delta s)^2} = \frac{2(1 - \cos\Delta\theta)}{(\Delta\theta)^2} \frac{(\Delta\theta)^2}{(\Delta s)^2}$$

By L'Hospital's Rule, $\lim\limits_{\Delta\theta\to 0}\dfrac{2(1-\cos\Delta\theta)}{(\Delta\theta)^2} = \lim\limits_{\Delta\theta\to 0}\dfrac{\sin\Delta\theta}{\Delta\theta} = 1$. So,

$$\left\|\frac{d\boldsymbol{\alpha}}{ds}\right\|^2 = \left\langle \frac{d\boldsymbol{\alpha}}{ds} \middle| \frac{d\boldsymbol{\alpha}}{ds} \right\rangle = \lim_{\Delta\theta\to 0}\left\langle \frac{\Delta\boldsymbol{\alpha}}{\Delta s} \middle| \frac{\Delta\boldsymbol{\alpha}}{\Delta s} \right\rangle = \lim_{\Delta\theta\to 0} \cancelto{1}{\frac{2(1-\cos\Delta\theta)}{(\Delta\theta)^2}} \lim_{\Delta s\to 0}\frac{(\Delta\theta)^2}{(\Delta s)^2} \overset{(2.28)}{=} \kappa^2$$

Thus, $\kappa = \left\|\dfrac{d\boldsymbol{\alpha}}{ds}\right\| = \|\boldsymbol{\alpha}_s{}'\|$ ■

Equation (2.28) specifies that $\kappa = \left|\dfrac{d\theta}{ds}\right|$. Equation (2.29) shows that $\kappa = \left\|\dfrac{d\boldsymbol{\alpha}}{ds}\right\|$. Thus, $\|d\boldsymbol{\alpha}\| = |d\theta|$, which can be inferred from the figure above since $\|\Delta\boldsymbol{\alpha}\| \approx 1|\Delta\theta|$. More geometrically, the rate of change of the normal $\boldsymbol{\alpha}$ is another representation of curvature, the tendency to curve away from the tangent line.

Theorem Let $\boldsymbol{y} = \boldsymbol{x}(t)$ be a curve. Then the curvature at a point $P = \boldsymbol{x}(t)$ is zero iff P is singular.

Proof. $\boldsymbol{x}(t)$ is singular

$$\Leftrightarrow \quad \boldsymbol{x}'(t)\times\boldsymbol{x}''(t) = 0 \quad \Leftrightarrow \quad \frac{\boldsymbol{x}'(t)}{\|\boldsymbol{x}'(t)\|}\times\frac{\boldsymbol{x}''(t)}{\|\boldsymbol{x}'(t)\|^2} = 0 \quad \overset{(2.18a,\ 2.18b)}{(\Leftrightarrow)} \quad \boldsymbol{x}_s{}'\times\boldsymbol{x}_s{}'' = 0$$

$$\Leftrightarrow \quad \langle \boldsymbol{x}_s{}'\times\boldsymbol{x}_s{}'' | \boldsymbol{x}_s{}'\times\boldsymbol{x}_s{}'' \rangle = 0 \quad \overset{(2.18)}{\Leftrightarrow} \quad \langle \boldsymbol{x}_s{}'' | \boldsymbol{x}_s{}'' \rangle = 0$$

$$\Leftrightarrow \quad \kappa \overset{(2.29)}{=} \|\boldsymbol{\alpha}_s{}'\| = \sqrt{\langle \boldsymbol{\alpha}_s{}' | \boldsymbol{\alpha}_s{}' \rangle} \overset{(2.25)}{=} \sqrt{\langle \boldsymbol{x}_s{}'' | \boldsymbol{x}_s{}'' \rangle} = 0 \qquad ■$$

Theorem 16.1 A curve has curvature zero everywhere iff it is a straight line

Proof. From Theorem 13.1, a curve is a straight line iff $\boldsymbol{x}'(t)\times\boldsymbol{x}''(t) = 0$ for all t. That means that every point is singular. By the prior theorem, the curvature at each point is zero. ■

Theorem Let $\boldsymbol{y} = \boldsymbol{x}(t)$ be a nonlinear curve. At a regular point of the curve,

$$\frac{d\boldsymbol{\alpha}}{ds} = \frac{\boldsymbol{\beta}}{R}. \tag{2.30}$$

Proof. $\boldsymbol{\beta} = \dfrac{\boldsymbol{x}_s{}''}{\|\boldsymbol{x}_s{}''\|} \overset{(2.25)}{=} \dfrac{\boldsymbol{\alpha}_s{}'}{\|\boldsymbol{\alpha}_s{}'\|} \overset{(2.29)}{=} \dfrac{\boldsymbol{\alpha}_s{}'}{\frac{1}{R}} \Leftrightarrow \dfrac{\boldsymbol{\beta}}{R} = \boldsymbol{\alpha}_s{}'$ ■

Corollary At a regular point of a curve, $\kappa = \dfrac{\|\beta\|}{R}$.

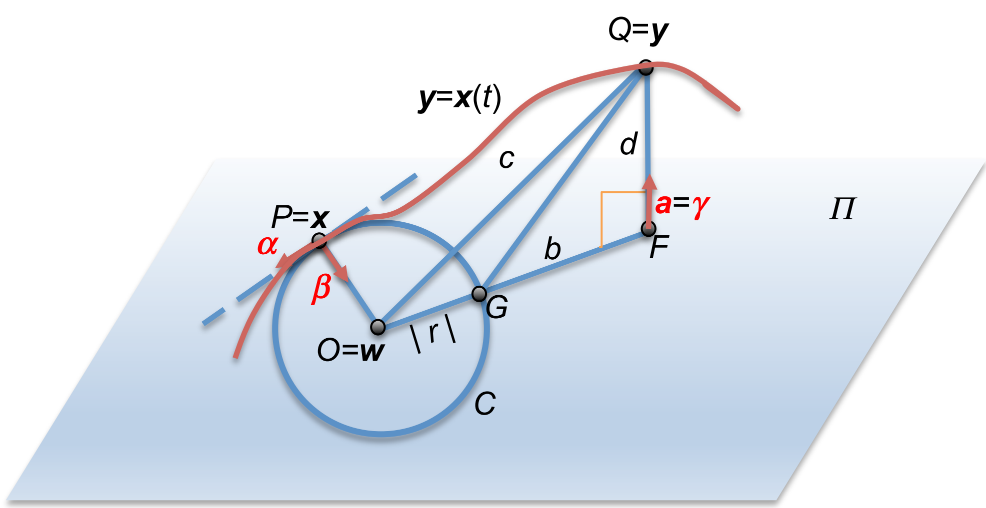
Proof. From (2.29). $\kappa = \|\alpha_s'\|$ ■

We are now in a position to identify the circle that has maximum contact at a regular point P = ***x*** of a curve. Any circle that has ***x***' as a tangent has its center on a line that is normal to ***x***'. There are infinitely many such lines, and infinitely many points on each line capable of being the center of the circle. Even worse, each circle can be spun from 0° to 360° about the normal line with each rotated circle being yet another candidate. There are many, many possibilities for the one circle.

How do we select just one circle? Intuitively, it is easy. The circle should be in the osculating plane. Since the principal normal lies in the osculating plane, the circle should have its center on that line. Finally, the center should at a distance R from P on the side of the curve of positive curvature, where R is the radius of curvature of the curve at point P.

In order to prove this, we need to find the circle that maximizes contact order at P with the curve ***y*** = ***x***(t).

That is, we need to develop a formula for the perpendicular distance to a circle from a nearby point Q on the curve, and then maximize its order with respect to t.



We initially make no assumptions as to the size of the radius of the circle or even what plane the circle lies in, only that it is tangent to the curve at P = ***x*** = ***x***(0). As shown in the figure, let Q = ***y*** = ***x***(t) be a point on the curve that lies within the radius of convergence of the Taylor series at P, O the center of the circle, F the closest point in the plane Π to Q, d the distance from Q to Π, G the nearer of the 2 points of intersection of the line through *OF* and the circle *C*, $\boldsymbol{a}$ a unit vector perpendicular to Π, *b* the distance from *F* to *G*, *c* the distance from *Q* to *O*, and $\boldsymbol{\alpha}$ the unit tangent vector at *P*.

Observe that *G* is the point on circle *c* that is closest to *Q*. That is because the plane containing *QFO* cuts *C* at *G*. Our problem is reduced to maximizing the order of $\overline{QG}$. We begin by maximizing the order of *d.*

Equations (2.7) and (2.22b) were written in terms of a generic parameter *t*, so they hold for the specific parameter *s*, arc length:

$$\boldsymbol{y} \overset{(2.7)}{=} \boldsymbol{x} + \boldsymbol{x}_s{}'s + \frac{1}{2!}\boldsymbol{x}_s{}''s^2 + \frac{1}{3!}\boldsymbol{x}_s{}'''s^3 + o\left(s^4\right) \tag{A}$$

$$\pm d \overset{(2.22b)}{=} \left\langle \boldsymbol{x}_s{}' \middle| \boldsymbol{a} \right\rangle s + \frac{1}{2!}\left\langle \boldsymbol{x}_s{}'' \middle| \boldsymbol{a} \right\rangle s^2 + \frac{1}{3!}\left\langle \boldsymbol{x}_s{}''' \middle| \boldsymbol{a} \right\rangle s^3 + \frac{1}{4!}\left\langle \boldsymbol{x}_s{}^{(4)} \middle| \boldsymbol{a} \right\rangle s^4 + o\left(s^5\right) \tag{B}$$

By definition, the order of *d* is maximized to be ≥ 3 if Π is the osculating plane:

$$\pm d \overset{(A)}{=} \frac{1}{3!}\left\langle \boldsymbol{x}_s{}''' \middle| \boldsymbol{a} \right\rangle s^3 + \frac{1}{4!}\left\langle \boldsymbol{x}_s{}^{(4)} \middle| \boldsymbol{a} \right\rangle s^4 + o\left(s^5\right) \tag{C}$$

In the osculating plane, the unit normal $\boldsymbol{a}$ becomes the binormal $\boldsymbol{\gamma}$, and the principal normal $\boldsymbol{\beta}$ lies along the radius *OP* as shown in the figure. We can write

$$\boldsymbol{w} = \boldsymbol{x} + r\beta \tag{D}$$

where | *r* | is the radius of *C*. At this point we don't know if *r* is positive or negative, but we have at least narrowed the center $\boldsymbol{w}$ down to 2 locations, and we have identified the plane in which the circle lies.

We briefly digress to make an observation. If any 2 sides of a right triangle are of order *m* with respect to *s* then then other side is also. For if the sides have lengths *a*, *b*, and *c*, and *a* and *b* are expressed as series in *s* of order *m*, then a^2 and b^2 are series of order m^2. By combining the series for a^2 and b^2, we find that $c^2 = a^2 + b^2$ is a series of order m^2, and thus *c* has order *m*.

Thus, if we can find a unique value for *r* such that *b* has order ≥ 3, then the diagonal $\overline{QG}$ of the right triangle *QFG* would also have order ≥ 3, and that would complete our derivation.

In the right triangle *QGO*, $\left(b + |r|\right)^2 = c^2 - d^2$. This can be rearranged into

$$\left(2|r| + b\right)b = c^2 - r^2 - d^2. \tag{2.32}$$

Then

$$\boldsymbol{y} - \boldsymbol{w} \overset{(A,\ D)}{=} -r\beta + \boldsymbol{x}_s{}'s + \frac{1}{2!}\boldsymbol{x}_s{}''s^2 + \frac{1}{3!}\boldsymbol{x}_s{}'''s^3 + o\left(s^4\right).$$

By (2.25)

$$
\begin{aligned}
c^2 &= \langle \boldsymbol{y}-\boldsymbol{w} | \boldsymbol{y}-\boldsymbol{w} \rangle \\
&= r^2 \cancel{\langle \beta | \beta \rangle}^{1} - 2r \cancel{\langle \beta | \boldsymbol{x}_s' \rangle}^{0} s + \cancel{\langle \boldsymbol{x}_s' | \boldsymbol{x}_s' \rangle}^{1} s^2 - r \langle \beta | \boldsymbol{x}_s'' \rangle s^2 \\
&\quad + \cancel{\langle \boldsymbol{x}_s' | \boldsymbol{x}_s'' \rangle}^{0} s^3 - \frac{1}{3} r \langle \beta | \boldsymbol{x}_s''' \rangle s^3 + o(s^4) \\
&= r^2 + \left(1 - r\langle \beta | \boldsymbol{x}_s'' \rangle\right) s^2 - \frac{1}{3} r \langle \beta | \boldsymbol{x}_s''' \rangle s^3 + o(s^4)
\end{aligned}
$$

Since $\boldsymbol{x}_s'' \overset{(2.25)}{=} \frac{\beta}{R}$, , we get that $\langle \beta | \boldsymbol{x}_s'' \rangle = \frac{1}{R} \langle \beta | \beta \rangle = \frac{1}{R}$, where R is the radius of curvature at P. So,

$$
c^2 = r^2 + \left(1 - \frac{r}{R}\right) s^2 - \frac{1}{3} r \langle \beta | \boldsymbol{x}_s''' \rangle s^3 + o(s^4)
$$

$$
\left(2|r| + b\right) b \overset{(2.32)}{=} \left(1 - \frac{r}{R}\right) s^2 - \frac{1}{3} r \langle \beta | \boldsymbol{x}_s''' \rangle s^3 + o(s^4) - d^2
$$

From (C), d^2 has order ≥ 6 with respect to s, so

$$
\left(2|r| + b\right) b = \left(1 - \frac{r}{R}\right) s^2 - \frac{1}{3} r \langle \beta | \boldsymbol{x}_s''' \rangle s^3 + o(s^4).
$$

As $Q \to P$, $\left(2|r| + b\right) b \to 2|r| b$. So, we have

$$
2|r| b \approx \left(1 - \frac{r}{R}\right) s^2 - \frac{1}{3} r \langle \beta | \boldsymbol{x}_s''' \rangle s^3 + o(s^4).
$$

Observe that b is of order ≥ 3 iff $r = R$. We have found the unique value of r such that b has order ≥ 3, and that completes our derivation. ■

We summarize this in the following theorem.

Theorem There is a unique circle that has contact of at least the 2nd order with a given curve at a regular point $\boldsymbol{x}$. It is the circle in the osculating plane whose radius is R and whose center is on the positive half of the principal normal at the point $\boldsymbol{x} + R\boldsymbol{\beta}$.

Definition The unique circle above is called the **osculating circle**, or the **circle of curvature**, at the point $\boldsymbol{x}$. Its center is called the **center of curvature** and its radius is called the **radius of curvature** for the point.

Example The osculating circle at any point on a circle is the circle itself.
The equation of a circle in the xy-plane is

$$\boldsymbol{y} = \boldsymbol{x}(\theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 0 \end{pmatrix} \qquad \boldsymbol{x}'(\theta) = r\begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} \qquad \frac{ds}{d\theta} \overset{(2.11)}{=} \left\| \boldsymbol{x}'(\theta) \right\| = r\,.$$

$$\boldsymbol{\alpha} \overset{(2.19)}{=} \boldsymbol{x}_s{}' \overset{(2.20)}{=} \frac{x'(\theta)}{\left\| x'(\theta) \right\|} = \frac{r}{r}\begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}, \qquad \boldsymbol{\alpha}_s{}' = \frac{d\boldsymbol{\alpha}/d\theta}{ds/d\theta} = -\frac{1}{r}\begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$$

$\kappa \overset{(2.29)}{=} \left\| \boldsymbol{\alpha}_s{}' \right\| = \dfrac{1}{r}$ and $R \overset{(2.25)}{=} \dfrac{1}{\kappa} = r$ at any point on the circle. ■

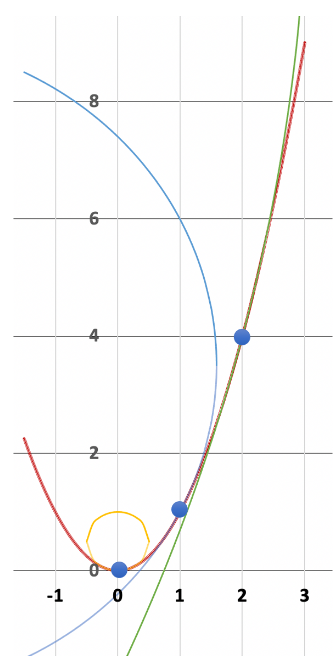
Example Circular Helix, $r > 0$, $\cot\theta_3 \neq 0$

$$\boldsymbol{y} = \boldsymbol{x}(t) \overset{(2.20a)}{=} r\begin{pmatrix} \cos t \\ \sin t \\ t\cot\theta_3 \end{pmatrix}, \qquad \boldsymbol{x}'(t) = r\begin{pmatrix} -\sin t \\ \cos t \\ \cot\theta_3 \end{pmatrix}, \qquad \frac{ds}{dt} \overset{(2.11)}{=} \left\| \boldsymbol{x}'(t) \right\| = r\csc\theta_3$$

$$\boldsymbol{\alpha} \overset{(2.20)}{=} \frac{\boldsymbol{x}'(t)}{\left\| \boldsymbol{x}'(t) \right\|} = \sin\theta_3\begin{pmatrix} -\sin t \\ \cos t \\ \cot\theta_3 \end{pmatrix} \qquad \boldsymbol{\alpha}_s{}' = \frac{d\boldsymbol{\alpha}/dt}{ds/dt} = -\frac{\sin^2\theta_3}{r}\begin{pmatrix} -\cos t \\ -\sin t \\ 0 \end{pmatrix}$$

$$\kappa \overset{(2.29)}{=} \left\| \alpha_s{}' \right\| = \frac{\sin^2\theta_3}{r} \quad \Rightarrow \quad R = \frac{r}{\sin^2\theta_3}$$

This has a nice geometric interpretation. There are 2 effects occurring. First, observe that $\boldsymbol{x}''(t) = -r\begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix}$ is in the *xy*-plane but $\boldsymbol{x}'(t)$ is not. Thus, the osculating circles are slanted rather than parallel to the *xy*-plane in order to match the slant of the upward spiral. Second, the radius of curvature, R, is constant for all t but it is larger than R in order to match the stretch due to the upwards spiral. ■



Example Parabola $y = x^2$ in *xy*-plane. Find the osculating circles at $x = 0$, 1, and 2. See figure at right.

Solution $\boldsymbol{y} = \boldsymbol{x}(t) = \begin{pmatrix} t \\ t^2 \\ 0 \end{pmatrix} \quad \boldsymbol{x}'(t) = \begin{pmatrix} 1 \\ 2t \\ 0 \end{pmatrix},$

$$\frac{ds}{dt} \overset{(2.11)}{=} \left\| \boldsymbol{x}'(t) \right\| = \sqrt{1+4t^2}$$

$$\alpha_s \overset{(2.20)}{=} \frac{\mathbf{x}'(t)}{\left\| \mathbf{x}'(t) \right\|} = \frac{1}{\sqrt{1+4t^2}} \begin{pmatrix} 1 \\ 2t \\ 0 \end{pmatrix} \qquad \alpha_s' = \frac{d\alpha_s}{ds} = \frac{d\alpha_s}{dt}\frac{dt}{ds} = \frac{1}{\sqrt{1+4t^2}}\frac{d\alpha_s}{dt}$$

$$\frac{d}{dt}\frac{1}{\sqrt{1+4t^2}} = -\frac{4t}{\left(1+4t^2\right)^{3/2}}, \quad \frac{d}{dt}\frac{2t}{\sqrt{1+4t^2}} = -\frac{8t^2}{\left(1+4t^2\right)^{3/2}} + \frac{2}{\sqrt{1+4t^2}} = \frac{2}{\left(1+4t^2\right)^{3/2}}$$

$$\alpha_s' = \frac{1}{\left(1+4t^2\right)^2} \begin{pmatrix} -4t \\ 2 \\ 0 \end{pmatrix}, \quad \kappa = \left\| \alpha_s' \right\| = \frac{2\sqrt{1+4t^2}}{\left(1+4t^2\right)^2} = \frac{2}{\left(1+4t^2\right)^{3/2}}$$

At $x = 0 = t$, $\kappa = 2$, $R = .5$ (yellow circle), center located on y-axis at (0,.5).
At $x = 1 = t$, $\kappa \approx .178$, $R \approx 5.6$ (blue circle), center on slope -½ line through (1,1).
At $x = 2 = t$, $\kappa \approx .029$, $R \approx 35$ (green circle), center on slope -¼ line through (2,4). ■

Definition Let P be a regular point of a curve. The **torsion of the curve at P** is

$$\tau = \frac{1}{T} = \pm \lim_{\Delta s \to 0} \frac{\Delta\theta}{\Delta s} = \pm \frac{d\theta}{ds}$$

where $\Delta\theta$ is the angle between the osculating planes at P and a nearby point Q in the curve. A specific choice between plus and minus will be made shortly.

Just as κ measures the rate at which a curve turns away from the tangent line, τ measures the rate at which the curve twists out of the osculating plane. Note that $\Delta\theta$ is also the angle between the directed binormals at P and Q; i.e., the angle between γ and $\gamma + \Delta\gamma$. Thus, intuitively, $\lim_{\Delta s \to 0} \frac{\Delta\theta}{\Delta s}$ is the rate of twisting away from the osculating plane.

Theorem $\tau^2 = \frac{1}{T^2} = \left\| \gamma_s' \right\|^2$. (2.33)

Proof. As in the proof of (2.29), since γ and $\gamma+\Delta\gamma$ are unit vectors,

$$\cos\Delta\theta = \left\langle \gamma \middle| \gamma + \Delta\gamma \right\rangle \quad \Rightarrow \quad \left\langle \Delta\gamma \middle| \Delta\gamma \right\rangle = 2\left(1 - \cos\Delta\theta\right)$$

$$\left\| \frac{d\gamma}{ds} \right\|^2 = \lim_{\Delta s \to 0} \left\langle \frac{\Delta\gamma}{\Delta s} \middle| \frac{\Delta\gamma}{\Delta s} \right\rangle = \lim_{\Delta s \to 0} \frac{2\left(1-\cos\Delta\theta\right)}{\left(\Delta\theta\right)^2} \frac{\left(\Delta\theta\right)^2}{\left(\Delta s\right)^2} = \lim_{\Delta s \to 0} \left(\frac{\Delta\theta}{\Delta s}\right)^2 = \left(\frac{d\theta}{ds}\right)^2 \overset{\text{(defn)}}{=} \frac{1}{T^2}. \ ■$$

Since $\gamma = \dfrac{\mathbf{x}_s' \times \mathbf{x}_s''}{\left\| \mathbf{x}_s'' \right\|}$,

$$\gamma_s' \overset{(2.8b,\ 2.8c)}{=} \frac{\left\| \mathbf{x}_s'' \right\| \left(\mathbf{x}_s' \times \mathbf{x}_s''' + \cancelto{0}{\mathbf{x}_s'' \times \mathbf{x}_s''} \right) - \left(\mathbf{x}_s' \times \mathbf{x}_s'' \right) \dfrac{\left\langle \mathbf{x}_s'' \middle| \mathbf{x}_s''' \right\rangle}{\left\| \mathbf{x}_s'' \right\|}}{\left\| \mathbf{x}_s'' \right\|^2}$$

$$\gamma_s' = \frac{\| \boldsymbol{x}_s'' \|^2 (\boldsymbol{x}_s' \times \boldsymbol{x}_s''') - (\boldsymbol{x}_s' \times \boldsymbol{x}_s'') \langle \boldsymbol{x}_s'' | \boldsymbol{x}_s''' \rangle}{\| \boldsymbol{x}_s'' \|^3}. \tag{2.34}$$

From (1.24), setting $a = \boldsymbol{x}_s', b = \boldsymbol{x}_s'', c = \boldsymbol{x}_s''', d = \omega$, and $e = \boldsymbol{x}_s''$, we get the identity

$$(\boldsymbol{x}_s' \, \boldsymbol{x}_s'' \, \boldsymbol{x}_s''') \langle \boldsymbol{x}_s'' | \omega \rangle = -\langle \boldsymbol{x}_s'' | \boldsymbol{x}_s'' \rangle (\boldsymbol{x}_s' \, \boldsymbol{x}_s''' \, \omega) + \langle \boldsymbol{x}_s'' | \boldsymbol{x}_s''' \rangle (\boldsymbol{x}_s' \, \boldsymbol{x}_s'' \, \omega)$$

that holds for any vector ω. Applying (1.14) to RHS yields

$$(\boldsymbol{x}_s' \, \boldsymbol{x}_s'' \, \boldsymbol{x}_s''') \langle \boldsymbol{x}_s'' | \omega \rangle = -\| \boldsymbol{x}_s'' \|^2 \langle \boldsymbol{x}_s' \times \boldsymbol{x}_s''' | \omega \rangle + \langle \boldsymbol{x}_s'' | \boldsymbol{x}_s''' \rangle \langle \boldsymbol{x}_s' \times \boldsymbol{x}_s'' | \omega \rangle$$

$$\Rightarrow \quad \| \boldsymbol{x}_s'' \|^2 \langle \boldsymbol{x}_s' \times \boldsymbol{x}_s''' | \omega \rangle - \langle \boldsymbol{x}_s'' | \boldsymbol{x}_s''' \rangle \langle \boldsymbol{x}_s' \times \boldsymbol{x}_s'' | \omega \rangle = -(\boldsymbol{x}_s' \, \boldsymbol{x}_s'' \, \boldsymbol{x}_s''') \langle \boldsymbol{x}_s'' | \omega \rangle \; .$$

Since this holds for all ω, we can drop ω to get

$$\Rightarrow \quad \| \boldsymbol{x}_s'' \|^2 (\boldsymbol{x}_s' \times \boldsymbol{x}_s''') - \langle \boldsymbol{x}_s'' | \boldsymbol{x}_s''' \rangle (\boldsymbol{x}_s' \times \boldsymbol{x}_s'') = -(\boldsymbol{x}_s' \, \boldsymbol{x}_s'' \, \boldsymbol{x}_s''') \, \boldsymbol{x}_s'' .$$

LHS is the numerator of (2.34), so we get

$$\gamma_s' = -\frac{(\boldsymbol{x}_s' \, \boldsymbol{x}_s'' \, \boldsymbol{x}_s''') \, \boldsymbol{x}_s''}{\| \boldsymbol{x}_s'' \|^3}. \tag{2.35}$$

$$\frac{1}{T^2} \overset{(2.33)}{=} \| \gamma_s' \|^2 \overset{(2.35)}{=} \frac{(\boldsymbol{x}_s' \, \boldsymbol{x}_s'' \, \boldsymbol{x}_s''')^2 \| \boldsymbol{x}_s'' \|^2}{\| \boldsymbol{x}_s'' \|^6} = \frac{(\boldsymbol{x}_s' \, \boldsymbol{x}_s'' \, \boldsymbol{x}_s''')^2}{\| \boldsymbol{x}_s'' \|^4}$$

$$\Rightarrow \quad \frac{1}{T} = \pm \frac{(\boldsymbol{x}_s' \, \boldsymbol{x}_s'' \, \boldsymbol{x}_s''')}{\| \boldsymbol{x}_s'' \|^2}.$$

To complete this equation for torsion, we choose the minus sign:

$$\frac{1}{T} = -\frac{(\boldsymbol{x}_s' \, \boldsymbol{x}_s'' \, \boldsymbol{x}_s''')}{\| \boldsymbol{x}_s'' \|^2} \tag{2.36}$$

Since $\beta \overset{(2.25)}{=} \dfrac{\boldsymbol{x}_s''}{\| \boldsymbol{x}_s'' \|}$, then

$$\gamma_s' \overset{(2.35)}{=} -\frac{\boldsymbol{x}_s''}{\| \boldsymbol{x}_s'' \|} \frac{(\boldsymbol{x}_s' \, \boldsymbol{x}_s'' \, \boldsymbol{x}_s''')}{\| \boldsymbol{x}_s'' \|^2} \overset{(1.14)}{=} -\boldsymbol{\beta} \frac{(\boldsymbol{x}_s' \, \boldsymbol{x}_s'' \, \boldsymbol{x}_s''')}{\| \boldsymbol{x}_s'' \|^2} \overset{(2.36)}{=} \frac{\boldsymbol{\beta}}{T}. \tag{2.37}$$

Consequently,

$$\boxed{\tau = \frac{1}{T} = \| \gamma_s' \|}.$$

Compare this to

$$\kappa = \frac{1}{R} = \| \boldsymbol{\alpha}_s' \|.$$

The book provides the following 2nd proof of (2.36) and (2.37).

$$\boldsymbol{\alpha} \perp \boldsymbol{\gamma} \quad \Rightarrow \quad \langle \boldsymbol{\alpha} | \boldsymbol{\gamma} \rangle = 0 \quad \Rightarrow \quad \frac{d}{ds}\langle \boldsymbol{\alpha} | \boldsymbol{\gamma} \rangle \overset{(2.8a)}{=} \langle \boldsymbol{\alpha}' | \boldsymbol{\gamma} \rangle + \langle \boldsymbol{\alpha} | \boldsymbol{\gamma}' \rangle = 0 \quad \Rightarrow \quad \boldsymbol{\gamma}_s' \perp \boldsymbol{\alpha}.$$

Since $\boldsymbol{\alpha}_s' \overset{(2.30)}{=} \dfrac{\boldsymbol{\beta}}{R}$, this becomes

$$\frac{1}{R}\langle \boldsymbol{\beta} | \boldsymbol{\gamma} \rangle + \langle \boldsymbol{\alpha} | \boldsymbol{\gamma}_s' \rangle = 0.$$

Since $\langle \boldsymbol{\gamma} | \boldsymbol{\gamma} \rangle = 1$,

$$0 = \frac{d}{ds}\langle \boldsymbol{\gamma} | \boldsymbol{\gamma} \rangle \overset{(2.8a)}{=} 2\langle \boldsymbol{\gamma} | \boldsymbol{\gamma}_s' \rangle \quad \Rightarrow \quad \boldsymbol{\gamma}_s' \perp \boldsymbol{\gamma}.$$

Since $\boldsymbol{\gamma}_s' \perp \boldsymbol{\alpha}$ and $\boldsymbol{\gamma}_s' \perp \boldsymbol{\gamma}$, then

$$\boldsymbol{\gamma}_s' \parallel \boldsymbol{\beta} \quad \Rightarrow \quad \exists c \ni \boldsymbol{\gamma}_s' = c\,\boldsymbol{\beta} \quad \Rightarrow \quad \| \boldsymbol{\gamma}_s' \|^2 = c^2 \| \boldsymbol{\beta} \|^2 = c^2.$$

Therefore

$$\frac{1}{T^2} \overset{(2.33)}{=} \| \boldsymbol{\gamma}_s' \|^2 = c^2.$$

To complete the equation for $\dfrac{1}{T}$ we choose $\dfrac{1}{T} = +c$. Then

$$\boldsymbol{\gamma}_s' = c\,\boldsymbol{\beta} = \frac{\boldsymbol{\beta}}{T}. \quad ✓ \tag{2.37}$$

So,

$$\langle \boldsymbol{\gamma}_s' | \boldsymbol{\beta} \rangle \overset{(2.37)}{=} \frac{1}{T}\langle \boldsymbol{\beta} | \boldsymbol{\beta} \rangle = \frac{1}{T}.$$

Substituting (2.34) for $\boldsymbol{\gamma}_s'$ and $\dfrac{\boldsymbol{x}_s''}{\| \boldsymbol{x}_s'' \|}$ for $\boldsymbol{\beta}$ in LHS yields

$$\frac{1}{T} = \langle \boldsymbol{\gamma}_s' | \boldsymbol{\beta} \rangle = \frac{\langle \boldsymbol{\gamma}_s' | \boldsymbol{x}_s'' \rangle}{\| \boldsymbol{x}_s'' \|}$$

$$= \frac{1}{\| \boldsymbol{x}_s'' \|^4}\left[\| \boldsymbol{x}_s'' \|^2 \langle \boldsymbol{x}_s' \times \boldsymbol{x}_s''' | \boldsymbol{x}_s'' \rangle - \langle \boldsymbol{x}_s'' | \boldsymbol{x}_s''' \rangle \cancelto{0}{\langle \boldsymbol{x}_s' \times \boldsymbol{x}_s'' | \boldsymbol{x}_s'' \rangle} \right]$$

$$\frac{1}{T} = \frac{\langle \boldsymbol{x}_s' \times \boldsymbol{x}_s'' | \boldsymbol{x}_s'' \rangle}{\| \boldsymbol{x}_s'' \|^2}. \quad ✓ \tag{2.36}$$

Thus far we have

$$\boxed{\frac{d\boldsymbol{\alpha}}{ds} = \frac{\boldsymbol{\beta}}{R}} \text{ and } \boxed{\frac{d\boldsymbol{\gamma}}{ds} = \frac{\boldsymbol{\beta}}{T}}. \tag{2.38}$$

That suggests we investigate $\frac{d\boldsymbol{\beta}}{ds}$. Since $\boldsymbol{\beta}=\boldsymbol{\gamma}\times\boldsymbol{\alpha}$, it results in an equation in terms of R and T, but it does not result in a new "radius" beyond curvature and torsion:

$$\boxed{\frac{d\boldsymbol{\beta}}{ds}=-\frac{\boldsymbol{\alpha}}{R}-\frac{\boldsymbol{\gamma}}{T}}: \tag{2.39}$$

$$\frac{d\boldsymbol{\beta}}{ds}\overset{\text{(2.8b)}}{=}\boldsymbol{\gamma}\times\frac{d\boldsymbol{\alpha}}{ds}+\frac{d\boldsymbol{\gamma}}{ds}\times\boldsymbol{\alpha}=\frac{\boldsymbol{\gamma}\times\boldsymbol{\beta}}{R}+\frac{\boldsymbol{\beta}\times\boldsymbol{\alpha}}{T}=-\frac{\boldsymbol{\alpha}}{R}-\frac{\boldsymbol{\gamma}}{T}\quad ✓$$

To calculate R and T, it is best to express them in terms of t. We use the following:

$$\frac{1}{R}\overset{\text{(2.29)}}{=}\left\|\boldsymbol{\alpha}_s{}'\right\|\overset{\text{(2.25)}}{=}\left\|x_s{}''\right\| \qquad \frac{1}{T}\overset{\text{(2.36)}}{=}-\frac{\left(\boldsymbol{x}_s{}'\,\boldsymbol{x}_s{}''\,\boldsymbol{x}_s{}'''\right)}{\left\|\boldsymbol{x}_s{}''\right\|^2} \tag{2.40}$$

$$\left(\boldsymbol{x}_s{}'\,\boldsymbol{x}_s{}''\,\boldsymbol{x}_s{}'''\right)\overset{\text{(1.18f)}}{=}\frac{\left(\boldsymbol{x}'\,\boldsymbol{x}''\,\boldsymbol{x}'''\right)}{\left\|\boldsymbol{x}_s{}'\right\|^6},$$

$$\boldsymbol{x}_s{}'\overset{\text{(2.18b)}}{=}\frac{\boldsymbol{x}'}{\left\|\boldsymbol{x}'\right\|},\quad \boldsymbol{x}_s{}''\overset{\text{(2.18c)}}{=}\frac{\boldsymbol{x}''}{\left\|\boldsymbol{x}'\right\|^2},\quad \boldsymbol{x}_s{}'''\overset{\text{(2.18d)}}{=}\frac{\boldsymbol{x}'''}{\left\|\boldsymbol{x}'\right\|^3},\quad \left\|\boldsymbol{x}_s{}''\right\|=\frac{\left\|\boldsymbol{x}''\right\|}{\left\|\boldsymbol{x}_s{}'\right\|^2}.$$

$$\boxed{\frac{1}{R}=\left\|\boldsymbol{x}_s{}''\right\|=\frac{\left\|\boldsymbol{x}''\right\|}{\left\|\boldsymbol{x}'\right\|^2}}\quad\text{and}\quad\boxed{\frac{1}{T}=-\frac{\left(\boldsymbol{x}'\,\boldsymbol{x}''\,\boldsymbol{x}'''\right)}{\left\|\boldsymbol{x}'\right\|^6}\frac{\left\|\boldsymbol{x}'\right\|^4}{\left\|\boldsymbol{x}''\right\|^2}=-\frac{\left(\boldsymbol{x}'\,\boldsymbol{x}''\,\boldsymbol{x}'''\right)}{\left\|\boldsymbol{x}'\right\|^2\left\|\boldsymbol{x}''\right\|^2}} \tag{2.41}$$

The book developed the equivalent expressions

$$\boxed{\frac{1}{R}=\frac{\left\|\boldsymbol{x}'\times\boldsymbol{x}''\right\|}{\left\|\boldsymbol{x}'\right\|^3}}=\left(\frac{\left\|\boldsymbol{x}'\right\|\left\|\boldsymbol{x}''\right\|}{\left\|\boldsymbol{x}'\right\|^3}\right)\quad\text{and}\quad\boxed{\frac{1}{T}=-\frac{\left(\boldsymbol{x}'\,\boldsymbol{x}''\,\boldsymbol{x}'''\right)}{\left\|\boldsymbol{x}'\times\boldsymbol{x}''\right\|^2}}=\left(\frac{\left(\boldsymbol{x}'\,\boldsymbol{x}''\,\boldsymbol{x}'''\right)}{\left\|\boldsymbol{x}'\right\|^2\left\|\boldsymbol{x}''\right\|^2}\right). \tag{2.41}$$

Example Circular Helix, $r > 0$, $\cot\theta \neq 0$

$$\boldsymbol{x}(t)=r\begin{pmatrix}\cos t\\ \sin t\\ t\cot\theta\end{pmatrix},\quad \boldsymbol{x}'(t)=r\begin{pmatrix}-\sin t\\ \cos t\\ \cot\theta\end{pmatrix},\quad \boldsymbol{x}''(t)=-r\begin{pmatrix}\cos t\\ \sin t\\ 0\end{pmatrix},\quad \boldsymbol{x}'''(t)=r\begin{pmatrix}\sin t\\ -\cos t\\ 0\end{pmatrix},$$

$$\frac{ds}{dt}=\left\|\boldsymbol{x}'(t)\right\|=\frac{r}{\sin\theta},\quad \left\|\boldsymbol{x}''\right\|=r.$$

$$\left(\boldsymbol{x}'\,\boldsymbol{x}''\,\boldsymbol{x}'''\right)=-r^3\begin{vmatrix}-\sin t & \cos t & \sin t\\ \cos t & \sin t & -\cos t\\ \cot\theta & 0 & 0\end{vmatrix}=-r^3\frac{\cos\theta}{\sin\theta}(-1)=r^3\frac{\cos\theta}{\sin\theta}$$

$$\frac{1}{R} = r\frac{\sin^2\theta}{r^2} = \frac{\sin^2\theta}{r} \qquad \frac{1}{T} = -\frac{r^3\cos\theta}{\sin\theta}\frac{\sin^2\theta}{r^2}\frac{1}{r^2} = -\frac{\sin\theta\cos\theta}{r} \quad \blacksquare \tag{2.42}$$

Definition Equations (2.38) and (2.39) are called the **Frenet-Serret formulas**.

They are the central formulas of space curve theory. An easy induction argument shows that $\boldsymbol{x}_s', \boldsymbol{x}_s'', \boldsymbol{x}_s''', \cdots$ are linear combinations of $\alpha, \beta, and\ \gamma$ where the coefficients are rational functions of $\frac{1}{R}, \frac{1}{T}$, and derivatives (of various orders) of $\frac{1}{R}$ and $\frac{1}{T}$ with respect to s. For example,

$$\boldsymbol{x}_s' = \boldsymbol{\alpha}, \quad \boldsymbol{x}_s'' = \frac{\beta}{R}, \quad \boldsymbol{x}_s''' = \frac{1}{R}\left(-\frac{\alpha}{R} - \frac{\gamma}{T}\right) + \left[\frac{d}{ds}\left(\frac{1}{R}\right)\right]\beta \tag{2.44}$$

Thus, if $P = \boldsymbol{x}$ is a point on a curve, the Taylor series for a nearby point $Q = \boldsymbol{y}$ is

$$\boldsymbol{y} = \boldsymbol{x} + \alpha s + \frac{\beta}{R}\frac{s^2}{2!} + \left\{-\frac{\alpha}{R^2} + \left[\frac{d}{ds}\left(\frac{1}{R}\right)\right]\beta - \frac{\gamma}{RT}\right\}s^3 + \cdots \tag{2.45}$$

If P is the origin and the trihedral $\boldsymbol{\alpha\beta\gamma}$ is the xyz-trihedral, this reduces to

$$\begin{cases} y_1 = s - \frac{1}{6}\frac{1}{R^2}s^3 + \cdots \\ y_2 = \frac{1}{R}s^2 + \left[\frac{1}{6}\frac{d}{ds}\left(\frac{1}{R}\right)\right]s^3 + \cdots \\ y_3 = - \frac{1}{6}\frac{1}{RT}s^3 + \cdots \end{cases} \tag{2.46}$$

These equations constitute the **canonical representation** of the curve in a neighborhood or P. Various properties of the curve in a neighborhood of P can be derived from them.

For example, y_3 is the directed distance from the osculating plane at P to the point Q. This is because in the canonical representation, the curve lies in the xy-plane, and so the distance from Q to the (osculating) xy-plane is y_3. So, y_3 determines the contact order of the osculating plane. If P is a regular point, i.e., $\frac{1}{R} \neq 0$, and $T \neq 0$, then the contact order is precisely 2 (since the s^3 coefficient of y_3 has a non-zero coefficient).

Definition Two space curves are **congruent** if a rigid transformation can map one onto the other.

Theorem Two space curves are congruent iff there is a 1-1 mapping of their points such that arc length, curvature, and torsion are preserved.

Consequently, a curve is uniquely determined, except for its position in space, by curvature and torsion expressed as functions of arc length. If these functions are $f(s)$ and $\phi(s)$, respectively, then the equations

$$\boxed{\frac{1}{R} = f(s)}, \qquad \boxed{\frac{1}{T} = \phi(s)} \tag{2.50}$$

define a curve completely except for its position. Equations (2.50) are called the **intrinsic** or **natural equations of the curve**.

Example A circle (in a plane) has a single equation $\frac{1}{R} = c, \quad c > 0$.

Example A plane curve has a single equation $\frac{1}{R} = f(s)$.

Example A circular helix has natural equations $\frac{1}{R} = c, \quad \frac{1}{T} = k, \quad c > 0,\ k \neq 0$:

From (2.42), define $c \equiv \frac{\sin^2\theta}{r} > 0$, and $k \neq 0$. ■

The upshot of equations (2.50) is that although a curve $\boldsymbol{y} = \boldsymbol{x}(\mathrm{t})$ requires 3 parametric equations (y_1, y_2, and y_3), if we ignore its position in space, it requires only two parametric equations.

This development of curvature and torsion, especially equation (2.50) plays a central role in the rest of this book that I choose not to pursue at this time. Chapter 3 develops the theory of curves on surfaces including a kind of coordinate system (I think) composed of involutes and evolutes. Chapter 4 develops the theory of surfaces, and Chapter 5 develops curvature of curves on surfaces. I suspect that some of these additional concepts such as geodesics on curved surfaces are prerequisites for a course in general relativity.