Magic Method of Factoring a Quadratic Trinomial

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Introduction

There are several methods on the web touted as “Magic Method” other than the one described in this paper. The one discussed in this paper has previously been referred to as the Bottoms-Up Method, Slide Method, Slip-and-Slide Method, and Slide-and-Divide Method.

Preliminaries

$ax^2 + bx + c$ is known as a quadratic form due to the x^2 term. Since it is a polynomial with 3 terms, it is a trinomial. Hence it is known as the *quadratic trinomial*.

When $a = 1$, it is well known that $x^2 + bx + c$ can be factored if there are factors of c that add up to b. Specifically, if a_1 and c_1 are such factors, then $ax^2 + bx + c = (x + a_1)(x + c_1)$.

Factoring a Quadratic Trinomial by the Magic Method

We wish to factor $ax^2 + bx + c$, where a, b, and c are relatively prime integers. There are at least 3 different sets of steps that all describe the same method. All start with the same first step. All involve reducing the pairs (a, a_1) and (a, c_1) as much as possible. Finally, all conclude with the same factoring formula.

Magic Method 1.

Step 1. Factor $x^2 + bx + ac$

a. Find factors, a_1 and c_1, of ac that add up to b
b. Then $x^2 + bx + ac = (x + a_1)(x + c_1)$

Step 2. Divide the constants a_1 and c_1 by a and reduce as much as possible:

$$\left(x+\frac{\overset{a_2}{\cancel{a}_1}}{\underset{s}{\cancel{a}}}\right)\left(x+\frac{\overset{c_2}{\cancel{c}_1}}{\underset{r}{\cancel{a}}}\right)$$

Step 3. Multiply by the respective denominators, s and r:

$$ax^2 + bx + c = (sx + a_2)(rx + c_2)$$

Magic Method 2.

Step 1': Find factors, a_1 and c_1, of ac that add up to b, and write $(ax+a_1)(ax+c_1)$

Step 2': Reduce each factor as much as possible:

$$r(sx+a_2)\,s(rx+c_2) = rs(sx+a_2)(rx+c_2).$$

Step 3': Divide by a. It turns out that $a = rs$, so we get

$$ax^2 + bx + c = (sx+a_2)(rx+c_2)$$

Magic Method 3.

Step 1": Find factors, a_1 and c_1, of ac that add up to b, and write $(ax+a_1)(ax+c_1)$

Step 2": Divide each factor by as much as possible: $\left(\dfrac{ax+a_1}{r}\right)\left(\dfrac{ax+c_1}{s}\right)$.

Step 3": Cancel:

$$ax^2 + bx + c = \left(\frac{\overset{rs}{\cancel{a}}\,x + \overset{ra_2}{\cancel{a}_1}}{r}\right)\left(\frac{\overset{rs}{\cancel{a}}\,x + \overset{sc_2}{\cancel{c}_1}}{s}\right) = (sx+a_2)(rx+c_2)$$

Note. The Magic Method does not work if a, b, and c are not relatively prime. However, this is not a problem since the First Rule of Factoring is to factor out the Greatest Common Factor (GCF), leaving a, b, and c relatively prime.

Examples of Magic Method 1

Example 1. Factor $12x^2 - 17x + 6$

Step 1. $ac = (12)(6) = 72$. Factors of 72 that add up to -17 are -8 and -9. So,

$$(x-8)(x-9)$$

Step 2. $$\left(x-\frac{\overset{2}{\cancel{8}}}{\underset{3}{\cancel{12}}}\right)\left(x-\frac{\overset{3}{\cancel{9}}}{\underset{4}{\cancel{12}}}\right)$$

Step 3. $12x^2 - 17x + 6 = (3x - 2)(4x - 3)$

Example 2. Factor $2x^2 + 7x + 6$

Step 1. $ac = (2)(6) = 12$. Factors of 12 that add up to 7 are 3 and 4. So,

$$(x + 3)(x + 4)$$

Step 2. $$\left(x+\frac{3}{2}\right)\left(x+\frac{\overset{2}{\cancel{4}}}{\underset{1}{\cancel{2}}}\right)$$

Step 3. $2x^2 + 7x + 6 = (2x + 3)(x + 2)$

Proof of Magic Method:

We are given $ax^2 + bx + c$ where a, b, and c are relatively prime integers. We also *assume* that there are factors a_1 and c_1 of ac that add up to b because, otherwise, we know from the ac method that $ax^2 + bx + c$ is prime (i.e., it cannot be factored). Thus,

(1) $a_1c_1 = ac$

and

(2) $a_1 + c_1 = b$.

Therefore

(1') $$\left(\frac{a_1}{a}\right)\left(\frac{c_1}{a}\right) = \frac{a_1c_1}{a^2} \overset{(1)}{=} \frac{ac}{a^2} = \frac{c}{a}$$

and

(2') $$\frac{a_1}{a}+\frac{c_1}{a} = \frac{a_1+c_1}{a} \overset{(2)}{=} \frac{b}{a}\ .$$

That is,

$$\frac{a_1}{a} \text{ and } \frac{c_1}{a} \text{ are factors of } \frac{c}{a} \text{ that add up to } \frac{b}{a}.$$

Therefore, these are the factors of a quadratic trinomial with a = 1; namely

$$(3)\quad x^2 + \frac{b}{a}x + \frac{c}{a} = \left(x + \frac{a_1}{a}\right)\left(x + \frac{c_1}{a}\right).$$

Hence

$$(4)\quad ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \overset{(3)}{=} a\left(x + \frac{a_1}{a}\right)\left(x + \frac{c_1}{a}\right).$$

Define

$$(5)\quad r \equiv \text{GCF}(a, a_1)$$

and

$$(6)\quad s \equiv \frac{a}{r}.$$

Since r is a factor of a, we have that s is an integer, and

$$(6')\quad rs = a.$$

Since r is also a factor of a_1, define an integer

$$(7)\quad a_2 \equiv \frac{a_1}{r},$$

which means that

$$(7')\quad a_1 = a_2 r.$$

Also, define a rational number (i.e., not necessarily an integer)

$$(8)\quad c_2 \equiv \frac{c_1}{s},$$

which means that

(8') $c_1 = c_2 s$.

Substituting *rs* for *a* into (4) yields

$$ax^2+bx+c \overset{(4)}{=} \frac{a^2}{a}\left(x+\frac{a_1}{a}\right)\left(x+\frac{c_1}{a}\right) \overset{(6')}{=} \frac{a^2}{rs}\left(x+\frac{a_1}{a}\right)\left(x+\frac{c_1}{a}\right),$$

$$= \frac{a}{r}\left(x+\frac{a_1}{a}\right)\frac{a}{s}\left(x+\frac{c_1}{a}\right) = \left(\frac{a}{r}x+\frac{a_1}{r}\right)\left(\frac{a}{s}x+\frac{c_1}{s}\right)$$

$$(15)\quad ax^2+bx+c \overset{(6),\,(7),\,(8)}{=} \left(sx+a_2\right)\left(rx+c_2\right).$$

which is the Magic Formula.

It might seem that the proof ends here. Indeed, we have shown how the “magic” steps lead to the correct answer. However, there remains 2 details that need proving.

I. Factorization requires integers. We have shown that s, a_2, and r are integers. We must show that c_2 is an integer.

II. In Step 2 we were instructed to reduce the fraction $\frac{c_1}{a}$ as much as possible to get $\frac{c_2}{r}$. But neither r nor c_2 were defined in terms of reducing the fraction as much as possible. We must show that $\frac{c_2}{r}$ cannot be further reduced.

To better understand why we must prove detail II, it is instructive to check that equation (15) is correct by multiplying out the terms in parentheses on the right-hand side:

$$(16)\quad \left(sx+a_2\right)\left(rx+c_2\right) = rsx^2+(a_2r+c_2s)x+a_2c_2 \overset{(6'),\,(7),\,(8)}{=} ax^2+\left(\frac{a_1}{r}r+\frac{c_1}{s}s\right)x+\frac{a_1}{r}\frac{c_1}{s}$$

$$\overset{(1),\,(6')}{=} ax^2+\left(a_1+c_1\right)x+\frac{ac}{a} \overset{(2)}{=} ax^2+bx+c \;\checkmark$$

As we see, we used the fact that $rs = a$ in (16). That is why, in Step 2, when we reduce $\frac{c_1}{a}$ as much as possible we need to end up with r in the denominator. We must prove that we cannot reduce further.

We prove the 2 remaining details now.

Detail I: c_2 is an integer

By definition (8), this is equivalent to showing

(#) s is a factor of c_1.

To prove (#), we first show

(##) s and a_2 have no common factor other than 1.

For suppose that K is a common factor of s and a_2. Then there are integers A and B such that

(9) $s = KA$

and

(10) $a_2 = KB.$

Then

(11) $a = KrA$

because $a \overset{(6')}{=} rs \overset{(9)}{=} KrA$, and

(12) $a_1 = KrB$

because $a_1 \overset{(7')}{=} ra_2 \overset{(10)}{=} KrB$. Thus, Kr is a common factor of a and a_1. But, from (5), r is the greatest common factor of a and a_1. Therefore, $K = 1$. Since K was allowed to be any common factor of s and a_2, this proves (##), and hence

(13) $\mathrm{GCF}(s, a_2) = 1$.

Now, notice that

$$(14)\quad sc = \frac{rsc}{r} \overset{(6')}{=} \frac{ac}{r} \overset{(1)}{=} \frac{a_1 c_1}{r} \overset{(7')}{=} \frac{r a_2 c_1}{r} = a_2 c_1 .$$

This means that s is a factor of $a_2 c_1$. Because of (##), that s and a_2 have no common factor other than 1, all the factors of s must be factors of c_1. That is, s is a factor of c_1, which proves (#) and, hence, (Detail I).

Detail II: Reducing the fraction $\frac{c_1}{a}$ as much as possible results in $\frac{c_2}{r}$.
"As much as possible" means we must reduce c_1 and a by the GCF of c_1 and a. Notice that we get from $\frac{c_1}{a}$ to $\frac{c_2}{r}$ by reducing numerator and denominator by *s:*

$$\frac{c_1}{a} \overset{(6'),\,(8')}{=} \frac{c_2 \cancel{s}}{r \cancel{s}} = \frac{c_2}{r} .$$

This shows that s is a factor of both a and c_1. To finish the proof, we show that $s = \mathrm{GCF}(a, c_1)$. Suppose some multiple of s, call it sK, where K is an integer, is a factor of both a and c_1. We will show that $K = 1$ and that therefore $s = \mathrm{GCF}(a, c_1)$. Both

$$(17)\quad A \equiv \frac{a}{sK}$$

and

$$(18)\quad B \equiv \frac{c_1}{sK}$$

are integers since sK is a factor of both numerators. So, rewriting (17) yields

$$(*)\quad a = K\,(As).$$

Also,

$$(19)\quad r \overset{(6')}{=} \frac{a}{s} \overset{(17)}{=} KA .$$

In addition,

$$(20)\quad a_1 \overset{(7')}{=} r a_2 \overset{(19)}{=} K(A a_2)$$

Thus,

$$(^{**})\quad b \overset{(2)}{=} a_1 + c_1 \overset{(20),\,(18)}{=} K\left(Aa_2\right) + K\left(Bs\right) = K\left(Aa_2 + Bs\right).$$

Finally,

$$(^{***})\quad c \overset{(1)}{=} \frac{a_1 c_1}{a} \overset{(20),\,(18),\,(17)}{=} \frac{K\left(Aa_2\right)K\left(Bs\right)}{K\left(As\right)} = K\left(a_2 B\right)$$

From (*), (**), and (***), K is a factor of *a*, *b*, and *c*. Since *a*, *b*, and *c* are relatively prime, $K = 1$, which proves

(21) $s = \mathrm{GCF}(a, c_1)$

and concludes the proof.

Summary

Ignoring Details I and II, the fundamentals of this proof are only 2 pages long using this very sparse formatting style. This part can be easily understood by a good high school student.

For the mathematically inclined, the key difficulty in this proof lies with *s* and how to incorporate the relative primeness of *a*, *b*, and *c* into the proof. Notice that $s \overset{(6)}{\equiv} \frac{a}{r}$ and also $s \overset{(21)}{=} GCF\left(a, c_1\right)$. The straight-forward approach would have been to **define** *s* as GCF(*a*, c_1), in parallel with the definition of *r* in equation (5). In that case we would have had to prove that $rs = a$, and we would have had to invoke the relative primeness of *a*, *b*, and *c* to do this (not obvious how). Instead, the approach taken in this proof was to define *s* such that $rs = a$ and prove **Detail I**, that *s* is a factor of both *a* and c_1, and then use the relative primeness of *a*, *b*, and *c* to prove **Detail II**, that $s = \mathrm{GCF}(a, c_1)$. This is easy to follow but was somewhat difficult to discover.