Theorem. Let $A=(a_{ij})$ be an invertible matrix, c_{ij} the cofactor of element a_{ij}, and adj A = (c_{ji}) the adjugate matrix of A. Then $A^{-1}=\dfrac{adj\ A}{\mathbf{det}\,A}$.

Proof.

The **adjugate** c_{ij} is defined as $c_{ij}=(-1)^{i+j}M_{ij}$ where the minor M_{ij} is the determinant of the matrix obtained from A by removing row *i* and column *j*.

The (*i*, *j*) entry of *A* adj *A*

$$= \begin{pmatrix} & & \vdots & & \\ a_{i1} & \cdots & a_{ik} & \cdots & a_{in} \\ & & \vdots & & \end{pmatrix} \begin{pmatrix} & c_{j1} & \\ & \vdots & \\ \cdots & c_{jk} & \cdots \\ & \vdots & \\ & c_{jn} & \end{pmatrix}$$

$$= a_{i1}c_{j1}+\cdots+a_{ik}c_{jk}+\cdots+a_{in}c_{jn} \tag{1}$$

The (*i*, *i*) entry of *A* adj *A* = $a_{i1}c_{i1}+\cdots+a_{ij}c_{ij}+\cdots+a_{in}c_{in}$ = $\mathbf{det}\,A$ since this is the cofactor expansion of *A* along row *i*.

We will show that the (*i*, *j*) entry of *A* adj *A* = 0 if *i* ≠ *j*.

Start by generating a matrix *B* from *A* by replacing row *j* by row *i*:

$$B=\begin{pmatrix} a_{11} & \cdots & a_{1k-1} & a_{1k} & a_{1k+1} & \cdots & a_{1n} \\ \vdots & & & \vdots & & & \vdots \\ a_{j-11} & & a_{j-1k-1} & a_{j-1k} & a_{j-1k+1} & & a_{j-1n} \\ a_{i1} & \cdots & a_{ik-1} & a_{ik} & a_{ik+1} & \cdots & a_{in} \\ a_{j+11} & & a_{j+1k-1} & a_{j+1k} & a_{j+1k+1} & & a_{j+1n} \\ \vdots & & & \vdots & & & \vdots \\ a_{n1} & \cdots & a_{nk-1} & a_{nk} & a_{nk+1} & \cdots & a_{nn} \end{pmatrix}$$

The cofactor of the (*j*, *k*) element of matrix *B* is $(-1)^{j+k}$ times the determinant of the matrix highlighted in magenta.

Next re-write matrix *A* so as to compare its (*j*, *k*) cofactor to that of *B*.

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1k-1} & a_{1k} & a_{1k+1} & \cdots & a_{1n} \\ \vdots & & & \vdots & & & \vdots \\ a_{j-11} & & a_{j-1k-1} & a_{j-1k} & a_{j-1k+1} & & a_{j-1n} \\ a_{j1} & \cdots & a_{jk-1} & a_{jk} & a_{jk+1} & \cdots & a_{jn} \\ a_{j+11} & & a_{j+1k-1} & a_{j+1k} & a_{j+1k+1} & & a_{j+1n} \\ \vdots & & & \vdots & & & \vdots \\ a_{n1} & \cdots & a_{nk-1} & a_{nk} & a_{nk+1} & \cdots & a_{nn} \end{pmatrix}$$

Observe that the magenta cofactor is the same. That is, c_{jk} is the common cofactor for matrices *A* and *B* as long as attention is restricted to row *j*.

Now generate the determinant of *B* by expanding *B* by cofactors along row *j*. Since *B* has two identical rows, its determinant is zero. From equation (1) we see that

$$\begin{aligned} 0 = \mathbf{det}\, B &= a_{i1} c_{j1} + \cdots + a_{ik} c_{jk} + \cdots + a_{in} c_{jn} \\ &\overset{(1)}{=} (i, j) \text{ entry of } A \text{ adj } A \end{aligned}$$

Therefore

$$A \text{ adj } A = \left(\delta_{ij} \, \mathbf{det}\, A\right) = \begin{pmatrix} \mathbf{det}\, A & & 0 \\ & \ddots & \\ 0 & & \mathbf{det}\, A \end{pmatrix} = \left(\mathbf{det}\, A\right) \boldsymbol{I}$$

Left multiplying both sides by A^{-1} and dividing both sides by $\mathbf{det}\, A$ yields

$$A^{-1} = \frac{\text{adj } A}{\mathbf{det}\, A}.$$