2014.10.19

No. 1

等積中心アフィンの平面曲線（⇝ KdV）が自然に出てくることを説明したい．

$$\det\left(\frac{\overline{r}-r}{\varepsilon}, r\right) = 1$$

$$\Longleftrightarrow \det(\overline{r}, r) = \varepsilon$$

$$\therefore\ 0 = \det\left(\frac{\overline{r}-r}{\varepsilon}, r\right) - \det\left(\frac{r-\underline{r}}{\varepsilon}, \underline{r}\right)$$

$$= \det(\Delta r, r) - \det(\underline{\Delta r}, r - \cancel{\varepsilon\underline{\Delta r}})$$

$$= \det(\Delta r - \underline{\Delta r}, r)$$

$$\Longleftrightarrow \det(\Delta^2 r, r) = 0, \quad \Delta^2 r = \frac{\Delta r - \underline{\Delta r}}{\varepsilon}$$

$$\therefore\ \Delta^2 r = -\kappa r$$

<δ>

$$\frac{\tilde{r}-r}{\delta} = a\Delta r + b r$$

$$\Longleftrightarrow \tilde{r} = r + \delta a\Delta r + \delta b r$$

$$= (1+\delta b) r + \delta a \Delta r$$

$$\therefore\ \overline{\tilde{r}} = \overline{r} + \delta\overline{a}\,\overline{\Delta r} + \delta\overline{b}\,\overline{r}$$

$$= (1+\delta\overline{b})\{r + \varepsilon\Delta r\} + \delta\overline{a}\{-\varepsilon\overline{\kappa} r + (1-\varepsilon^2\overline{\kappa})\Delta r\}$$

$$= (1+\delta\overline{b} - \varepsilon\delta\overline{a}\overline{\kappa}) r$$

$$\cancel{\therefore\ \overline{\tilde{r}} - \tilde{r}}$$

$$+ \left(\varepsilon + \varepsilon\delta\overline{b} + \delta\overline{a}(1-\varepsilon^2\overline{\kappa})\right)\Delta r$$

$$\therefore\ \overline{\tilde{r}} - \tilde{r} = \left(\delta(\overline{b}-b) - \varepsilon\delta\overline{a}\overline{\kappa}\right) r + \left(\varepsilon + \varepsilon\delta\overline{b} + \delta(\overline{a}-a) - \varepsilon^2\delta\overline{a}\overline{\kappa}\right)\Delta r$$

$$\therefore\ \widetilde{\Delta r} = \left(\delta\Delta b - \delta\overline{a}\overline{\kappa}\right) r + \left(1 + \delta\overline{b} + \delta\Delta a - \varepsilon\delta\overline{a}\overline{\kappa}\right)\Delta r$$

$$\frac{\overline{r}-r}{\varepsilon} = \Delta r$$

$$\frac{r-\underline{r}}{\varepsilon} = \underline{\Delta r}$$

$$\underline{r} = r - \varepsilon\underline{\Delta r}$$

$$\Delta r - \underline{\Delta r} = \varepsilon\Delta^2 r = -\varepsilon\kappa r$$

$$\therefore\ \overline{\Delta r} - \Delta r = -\varepsilon\overline{\kappa}\,\overline{r} = -\varepsilon\overline{\kappa}\{r + \varepsilon\Delta r\}$$

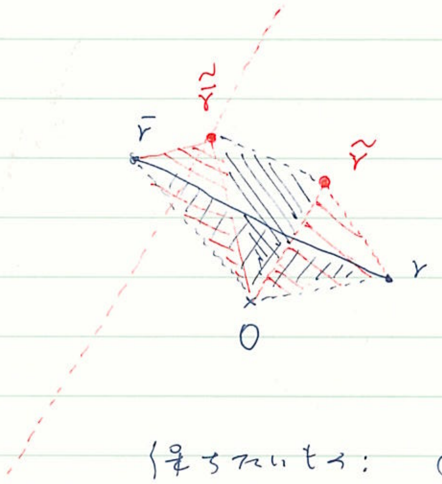
$$\therefore\ \overline{\Delta r} = -\varepsilon\overline{\kappa} r + (1-\varepsilon^2\overline{\kappa})\Delta r$$

$$\phi = (\Delta r, r)$$

$$\bar{\phi} = (\overline{\Delta r}, \bar{r}) = \phi \begin{bmatrix} 1-\varepsilon^2\bar{\kappa} & \varepsilon \\ -\varepsilon\bar{\kappa} & 1 \end{bmatrix}$$

$$\tilde{\phi} = (\widetilde{\Delta r}, \tilde{r}) = \phi \begin{bmatrix} 1+\delta\bar{b}+\delta\Delta a-\varepsilon\delta\bar{a}\bar{\kappa} & \delta a \\ \delta\Delta b-\delta\bar{a}\bar{\kappa} & 1+\delta b \end{bmatrix}$$

—<○>—



保ちたいもの: ⊙ $\det(\bar{r}, r)$

⊙ $\det(\tilde{r}, r)$

$$\begin{aligned} \text{const.} &= \det(\bar{r}, r) \\ &= \det(\bar{r}-r, r) \\ &= \varepsilon \det(\Delta r, r) \\ &= \varepsilon. \end{aligned}$$

$$\begin{aligned} \text{const.} &= \det(\tilde{r}, r) \\ &= \det(\delta a \Delta r, r) \\ &= \delta a \end{aligned}$$

δとする

$\therefore \boxed{a = 1}$

対称性を考えて。

—<○>—

$$\therefore \begin{cases} L = \begin{bmatrix} 1-\varepsilon^2\bar{\kappa} & \varepsilon \\ -\varepsilon\bar{\kappa} & 1 \end{bmatrix} \\ M = \begin{bmatrix} 1+\delta\bar{b}-\varepsilon\delta\bar{\kappa} & \delta \\ \delta\Delta b-\delta\bar{\kappa} & 1+\delta b \end{bmatrix} \end{cases}$$

$$1 = \det M$$
$$= (1 + s\bar{\varphi} - \varepsilon s\bar{u})(1 + s\varphi) - s^2(\Delta\varphi - \bar{u})$$
$$= 1 + s\varphi + s\bar{\varphi} - \varepsilon s\bar{u} + s^2\varphi(\bar{\varphi} - \varepsilon\bar{u}) - \frac{s^2}{\varepsilon}(\bar{\varphi} - \varphi - \varepsilon\bar{u})$$

$$\Longleftrightarrow\quad 0 = \varepsilon\varphi + \varepsilon\bar{\varphi} - \varepsilon^2\bar{u} + \varepsilon s\varphi(\bar{\varphi} - \varepsilon\bar{u}) - s(\bar{\varphi} - \varphi - \varepsilon\bar{u})$$
$$= (\varepsilon - s + \varepsilon s\varphi)\bar{\varphi} + (\varepsilon + s - \varepsilon^2 s\bar{u})\varphi + \varepsilon(s - \varepsilon)\bar{u}$$

$$\Longleftrightarrow\quad \bar{\varphi} = \frac{(s + \varepsilon - s\varepsilon^2\bar{u})\varphi + \varepsilon(s - \varepsilon)\bar{u}}{s - \varepsilon - s\varepsilon\varphi}$$

$$\Longleftrightarrow\quad 0 = (\varepsilon - s + \varepsilon s\varphi)\bar{\varphi} + (s + \varepsilon)\varphi + \varepsilon(s - \varepsilon - \varepsilon s\varphi)\bar{u}$$

$$\Longleftrightarrow\quad \bar{u} = \frac{(\varepsilon - s + \varepsilon s\varphi)\bar{\varphi} + (\varepsilon + s)\varphi}{\varepsilon(\varepsilon - s + \varepsilon s\varphi)}$$
$$= \frac{1}{\varepsilon}\bar{\varphi} + \frac{(\varepsilon + s)\varphi}{\varepsilon(\varepsilon - s + \varepsilon s\varphi)}$$

$$\Longleftrightarrow\quad \bar{\varphi} - \varepsilon\bar{u} = \frac{(s + \varepsilon)\varphi}{s - \varepsilon - s\varepsilon\varphi}$$

（⟸ したがって特に 行列 M から シフトが消える. これはうれしい.（∵ 両立条件 LM = MĨ を 4 頂点の関係式にしているので））

このとき

$$M = \begin{bmatrix} 1 + \dfrac{s(s + \varepsilon)\varphi}{s - \varepsilon - s\varepsilon\varphi} & s \\ -\dfrac{s}{\varepsilon}\varphi + \dfrac{s}{\varepsilon}\cdot\dfrac{(s + \varepsilon)\varphi}{s - \varepsilon - s\varepsilon\varphi} & 1 + s\varphi \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{s - \varepsilon + s^2\varphi}{s - \varepsilon - s\varepsilon\varphi} & s \\ \dfrac{s}{\varepsilon}\varphi\,\dfrac{\varepsilon + s\varepsilon\varphi + \varepsilon}{s - \varepsilon - s\varepsilon\varphi} & 1 + s\varphi \end{bmatrix}$$

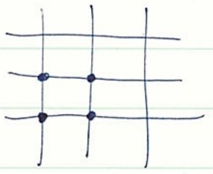
$$= \begin{bmatrix} \dfrac{s - \varepsilon + s^2\varphi}{s - \varepsilon - s\varepsilon\varphi} & s \\ \dfrac{s(2 + s\varphi)\varphi}{s - \varepsilon - s\varepsilon\varphi} & 1 + s\varphi \end{bmatrix}$$

$$L=\begin{bmatrix} 1-\varepsilon^2\left(\frac{1}{\varepsilon}\bar{\delta}+\frac{1}{\varepsilon}\frac{(\varepsilon+s)\delta}{\varepsilon-s+\varepsilon s\delta}\right) & \varepsilon \\ -\varepsilon\left(\frac{1}{\varepsilon}\bar{\delta}+\frac{1}{\varepsilon}\frac{(\varepsilon+s)\delta}{\varepsilon-s+\varepsilon s\delta}\right) & 1 \end{bmatrix}$$

$$=\begin{bmatrix} \frac{\varepsilon-s+\cancel{\varepsilon s}\delta-\varepsilon(\varepsilon+\cancel{s})\delta}{\varepsilon-s+\varepsilon s\delta}-\varepsilon\bar{\delta} & \varepsilon \\ -\bar{\delta}-\frac{(\varepsilon+s)\delta}{\varepsilon-s+\varepsilon s\delta} & 1 \end{bmatrix}$$

$$=\begin{bmatrix} -\varepsilon\bar{\delta}+\frac{\varepsilon-s-\varepsilon^2\delta}{\varepsilon-s+\varepsilon s\delta} & \varepsilon \\ -\bar{\delta}-\frac{(\varepsilon+s)\delta}{\varepsilon-s+\varepsilon s\delta} & 1 \end{bmatrix}$$

$$M=\begin{bmatrix} \frac{\varepsilon-s-s^2\delta}{\varepsilon-s+s\varepsilon\delta} & s \\ \frac{-s(2+s\delta)\delta}{\varepsilon-s+s\varepsilon\delta} & 1+s\delta \end{bmatrix}$$



$\delta=\left(\frac{1}{s}-\frac{1}{\varepsilon}\right)\left(\frac{1}{\nu}-1\right)$ とおく.

In[4]:= `L = {{-ε*b1 + (ε - δ - ε*ε*b)/(ε - δ + ε*δ*b), ε}, {-b1 - ((ε + δ)*b)/(ε - δ + ε*δ*b), 1}};`

In[5]:= `L // MatrixForm`

Out[5]//MatrixForm=

$$\begin{pmatrix} -b1\,\epsilon + \frac{-\delta+\epsilon-b\,\epsilon^2}{-\delta+\epsilon+b\,\delta\,\epsilon} & \epsilon \\ -b1 - \frac{b\,(\delta+\epsilon)}{-\delta+\epsilon+b\,\delta\,\epsilon} & 1 \end{pmatrix}$$

In[6]:= `M = {{(ε - δ - δ*δ*b)/(ε - δ + ε*δ*b), δ}, {(-δ*(2 + δ*b)*b)/(ε - δ + ε*δ*b), 1 + δ*b}};`

In[7]:= `M // MatrixForm`

Out[7]//MatrixForm=

$$\begin{pmatrix} \frac{-\delta-b\,\delta^2+\epsilon}{-\delta+\epsilon+b\,\delta\,\epsilon} & \delta \\ -\frac{b\,\delta\,(2+b\,\delta)}{-\delta+\epsilon+b\,\delta\,\epsilon} & 1+b\,\delta \end{pmatrix}$$

In[13]:= `Z = L.(M /. {b → b1}) - M.(L /. {b → b2, b1 → b12});`

In[14]:= `Simplify[Z]`

Out[14]=

$$\Big\{\Big\{\Big((\delta^2-\epsilon^2)\big(b\,(\epsilon+\delta\,(-1+b1\,\epsilon))\,(\epsilon+\delta\,(-1+b2\,\epsilon)) + (\delta-\epsilon)\,(b2\,(\delta+\epsilon)+b12\,(-\delta+\epsilon+b2\,\delta\,\epsilon)) - b1\,(-\epsilon^2+b12\,\delta\,\epsilon^2+\delta^2\,(1+b12\,\epsilon\,(-1+b2\,\epsilon)))\big)\Big)\Big/ \big((\epsilon+\delta\,(-1+b\,\epsilon))\,(\epsilon+\delta\,(-1+b1\,\epsilon))\,(\epsilon+\delta\,(-1+b2\,\epsilon))\big),\ 0\Big\},$$

$$\Big\{-\frac{b1\,\delta\,(2+b1\,\delta)}{\epsilon+\delta\,(-1+b1\,\epsilon)} + \frac{b\,\delta\,(2+b\,\delta)\left(-b12\,\epsilon+\frac{\delta-\epsilon+b2\,\epsilon^2}{\delta-\epsilon-b2\,\delta\,\epsilon}\right)}{\epsilon+\delta\,(-1+b\,\epsilon)} + \frac{(-\delta-b1\,\delta^2+\epsilon)\left(-b1-\frac{b\,(\delta+\epsilon)}{\epsilon+\delta\,(-1+b\,\epsilon)}\right)}{\epsilon+\delta\,(-1+b1\,\epsilon)} - (1+b\,\delta)\left(-b12-\frac{b2\,(\delta+\epsilon)}{\epsilon+\delta\,(-1+b2\,\epsilon)}\right),\ 0\Big\}\Big\}$$

In[18]:= `sol = Solve[Z[[1, 1]] == 0, b12]`

Out[18]=

$$\left\{\left\{b12 \to \frac{\frac{b1\,\delta\,(2+b1\,\delta)\,\epsilon}{-\delta+\epsilon+b1\,\delta\,\epsilon} - \frac{b2\,\delta\,(\delta+\epsilon)}{-\delta+\epsilon+b2\,\delta\,\epsilon} + \frac{(-\delta-b\,\delta^2+\epsilon)\,(-\delta+\epsilon-b2\,\epsilon^2)}{(-\delta+\epsilon+b\,\delta\,\epsilon)\,(-\delta+\epsilon+b2\,\delta\,\epsilon)} - \frac{(-\delta-b1\,\delta^2+\epsilon)\left(-b1\,\epsilon+\frac{-\delta+\epsilon-b\,\epsilon^2}{-\delta+\epsilon+b\,\delta\,\epsilon}\right)}{-\delta+\epsilon+b1\,\delta\,\epsilon}}{\delta+\frac{\epsilon\,(-\delta-b\,\delta^2+\epsilon)}{-\delta+\epsilon+b\,\delta\,\epsilon}}\right\}\right\}$$

In[22]:= `Simplify[Z /. sol[[1, 1]]]`

Out[22]= {{0, 0}, {0, 0}}

In[51]:= `b12 - b == Simplify[(b12 /. sol[[1, 1]]) - b]`

Out[51]=

$$\boxed{-b+b12 == -\frac{(b1-b2)\,(\delta^2-\epsilon^2)}{(\epsilon+\delta\,(-1+b1\,\epsilon))\,(\epsilon+\delta\,(-1+b2\,\epsilon))}}$$

← これが両立条件。

In[55]:= `mykdv = -b + b12 == ((b1 - b2) * (1/δ/δ - 1/ε/ε))/((1/δ - 1/ε + b1) * (1/δ - 1/ε + b2))`

Out[55]=

$$\boxed{-b+b12 == \frac{(b1-b2)\left(\frac{1}{\delta^2}-\frac{1}{\epsilon^2}\right)}{\boxed{b1+\frac{1}{\delta}-\frac{1}{\epsilon}}\left(b2+\frac{1}{\delta}-\frac{1}{\epsilon}\right)}}$$

← 両立条件を書き直しただけ。

ここを新しい変数とおいてみよう。($= \frac{1}{v}$ とする)

```
In[66]:= mysubs = Solve[1 / V == b + 1 / δ - 1 / ϵ, b]
```

Out[66]= $\left\{\left\{b \to \frac{1}{V} - \frac{1}{\delta} + \frac{1}{\epsilon}\right\}\right\}$

```
In[67]:= mysubs2 = {mysubs[[1, 1]], mysubs[[1, 1]] /. {b → b1, V → V1},
           mysubs[[1, 1]] /. {b → b2, V → V2}, mysubs[[1, 1]] /. {b → b12, V → V12}}
```

Out[67]= $\left\{b \to \frac{1}{V} - \frac{1}{\delta} + \frac{1}{\epsilon},\ b1 \to \frac{1}{V1} - \frac{1}{\delta} + \frac{1}{\epsilon},\ b2 \to \frac{1}{V2} - \frac{1}{\delta} + \frac{1}{\epsilon},\ b12 \to \frac{1}{V12} - \frac{1}{\delta} + \frac{1}{\epsilon}\right\}$

```
In[68]:= solmykdv = Simplify[Solve[mykdv /. mysubs2, V12]]
```

Out[68]= $\left\{\left\{V12 \to \frac{V\,\delta^2\,\epsilon^2}{\delta^2\,\epsilon^2 + V\,(V1 - V2)\,\left(\delta^2 - \epsilon^2\right)}\right\}\right\}$

```
In[71]:= 1 / V12 - 1 / V == Simplify[(1 / V12 /. solmykdv[[1, 1]]) - 1 / V]
```

Out[71]= $\boxed{-\frac{1}{V} + \frac{1}{V12} == \frac{(V1 - V2)\,\left(\delta^2 - \epsilon^2\right)}{\delta^2\,\epsilon^2}}$ ← 新しい変数 V で見ると
両立条件はこのように書ける。
これは discrete KdV。

```
In[26]:= kdv = (1 / δ - 1 / ϵ) * (1 / v12 - 1 / v) == (1 / δ + 1 / ϵ) * (v2 - v1)
```

Out[26]= $\left(-\frac{1}{v} + \frac{1}{v12}\right)\left(\frac{1}{\delta} - \frac{1}{\epsilon}\right) == (-v1 + v2)\left(\frac{1}{\delta} + \frac{1}{\epsilon}\right)$