Pseudo-Marginal Slice Sampling

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THE PROBLEM

We want to sample from some target distribution

$$\pi(\boldsymbol{\theta}) = \frac{f(\boldsymbol{\theta})}{C}$$

but only have a noisy unbiased estimator for $f(\boldsymbol{\theta})$

$$\hat{f}(\boldsymbol{\theta};\boldsymbol{u}) \;:\; \mathbb{E}_{q(\boldsymbol{u})}\left[\hat{f}(\boldsymbol{\theta};\boldsymbol{u})\right] = f(\boldsymbol{\theta})$$

$\boldsymbol{u}$ random variates used in estimator $\sim q(\cdot)$

SPECIAL CASES

DOUBLY-INTRACTABLE DISTRIBUTIONS

Marginal likelihood $p(\boldsymbol{y}\,|\,\boldsymbol{\theta}) = \int p(\boldsymbol{y}\,|\,\boldsymbol{z},\boldsymbol{\theta})\,p(\boldsymbol{z}\,|\,\boldsymbol{\theta})\,\mathrm{d}\boldsymbol{z}$

$$f(\boldsymbol{\theta}) = p(\boldsymbol{y}\,|\,\boldsymbol{\theta})\,p(\boldsymbol{\theta}) \qquad \hat{f}(\boldsymbol{\theta};\boldsymbol{u}) = \frac{p(\boldsymbol{y}\,|\,\mathrm{z}(\boldsymbol{u}),\boldsymbol{\theta})\,p(\mathrm{z}(\boldsymbol{u})\,|\,\boldsymbol{\theta})\,p(\boldsymbol{\theta})}{\mathcal{N}(\mathrm{z}(\boldsymbol{u});\,\boldsymbol{\mu},\,\boldsymbol{\Sigma})} \qquad q(\boldsymbol{u}) = \mathcal{N}(\boldsymbol{u};\,\boldsymbol{0},\,\mathbf{I})$$

z deterministic function $\mathrm{z}(\boldsymbol{u}) = \boldsymbol{\mu} + \mathrm{chol}(\boldsymbol{\Sigma})\,\boldsymbol{u}$

$\boldsymbol{\mu},\boldsymbol{\Sigma}$ Gaussian approximation $\mathcal{N}(\boldsymbol{z};\,\boldsymbol{\mu},\boldsymbol{\Sigma}) \approx p(\boldsymbol{z}\,|\,\boldsymbol{y},\boldsymbol{\theta})$

LATENT VARIABLE MODELS

Likelihood $p(\boldsymbol{y}\,|\,\boldsymbol{\theta}) = \frac{g(\boldsymbol{y};\boldsymbol{\theta})}{Z(\boldsymbol{\theta})}$ with $Z(\boldsymbol{\theta})$ intractable

$$f(\boldsymbol{\theta}) = g(\boldsymbol{y};\boldsymbol{\theta})\,p(\boldsymbol{\theta})\frac{Z(\hat{\boldsymbol{\theta}})}{Z(\boldsymbol{\theta})} \qquad \hat{f}(\boldsymbol{\theta};\boldsymbol{u}) = g(\boldsymbol{y};\boldsymbol{\theta})\,p(\boldsymbol{\theta})\frac{g(\mathrm{y}(\boldsymbol{u});\hat{\boldsymbol{\theta}})}{g(\mathrm{y}(\boldsymbol{u});\boldsymbol{\theta})} \qquad q(\boldsymbol{u}) = \mathcal{U}(\boldsymbol{u})$$

y deterministic function $\boldsymbol{u} \sim \mathcal{U}(\cdot) \Rightarrow \mathrm{y}(\boldsymbol{u}) \sim p(\cdot\,|\,\boldsymbol{\theta})$

$\hat{\boldsymbol{\theta}}$ fixed set of reference parameters

STANDARD APPROACH

Pseudo-Marginal (PM) MCMC, as analysed in Andrieu and Roberts (2009):

Inputs: current parameters $\boldsymbol{\theta}$, previous estimate of unnormalized target probability $\hat{f}$, proposal dist. $r(\boldsymbol{\theta}';\boldsymbol{\theta})$, unbiased estimator : $\mathbb{E}_{\epsilon(\hat{f};\boldsymbol{\theta})}\left[\hat{f}\right] = f(\boldsymbol{\theta})\ \forall\,\boldsymbol{\theta}$,

Output: new state-estimate pair $(\boldsymbol{\theta},\hat{f})$.

1. Propose new state and estimate its probability:

$$\boldsymbol{\theta}' \sim r(\cdot\,;\boldsymbol{\theta})$$
$$\hat{f}' \sim \epsilon(\cdot\,;\boldsymbol{\theta}')$$

2. Metropolis–Hastings style acceptance rule,

with probability $\min\left(1,\ \frac{\hat{f}'}{\hat{f}}\frac{r(\boldsymbol{\theta};\boldsymbol{\theta}')}{r(\boldsymbol{\theta}';\boldsymbol{\theta})}\right)$**:**

Accept: **return** $(\boldsymbol{\theta}',\hat{f}')$

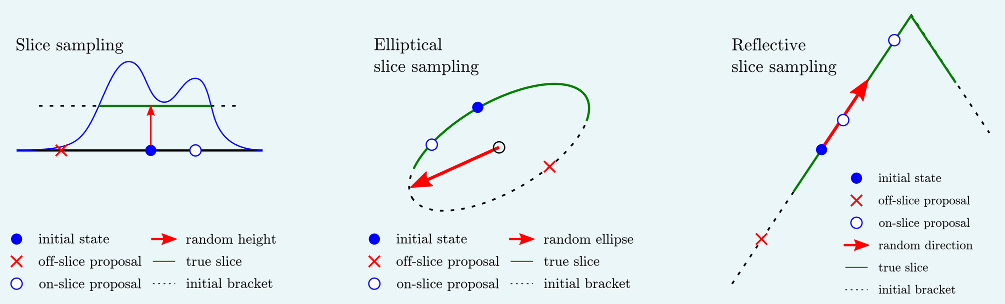
else: Reject: **return** $(\boldsymbol{\theta},\hat{f})$

If estimator distribution is heavy-tailed, moves to large $\hat{f}$ are occasionally accepted which it is hard to move away from.

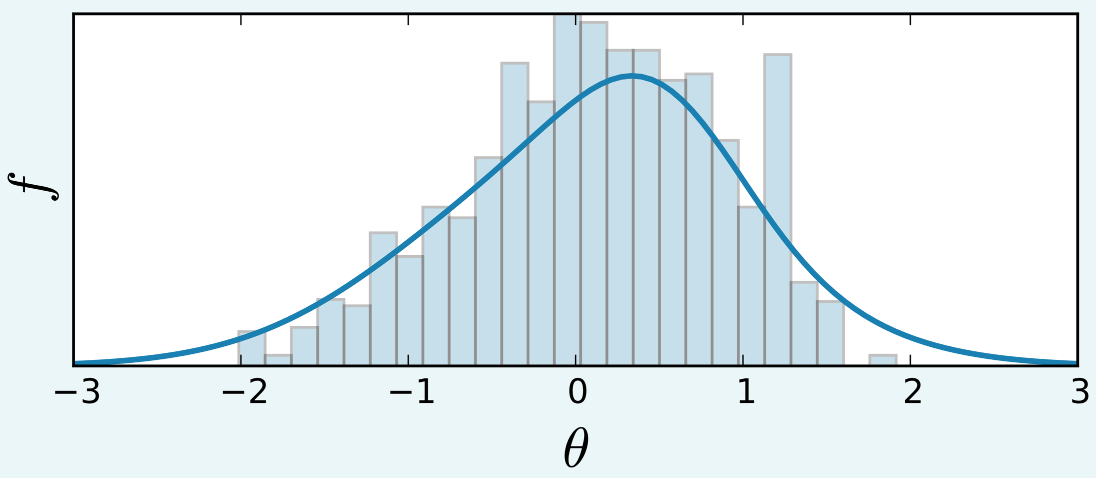
PM methods notorious for this 'sticking' behaviour.

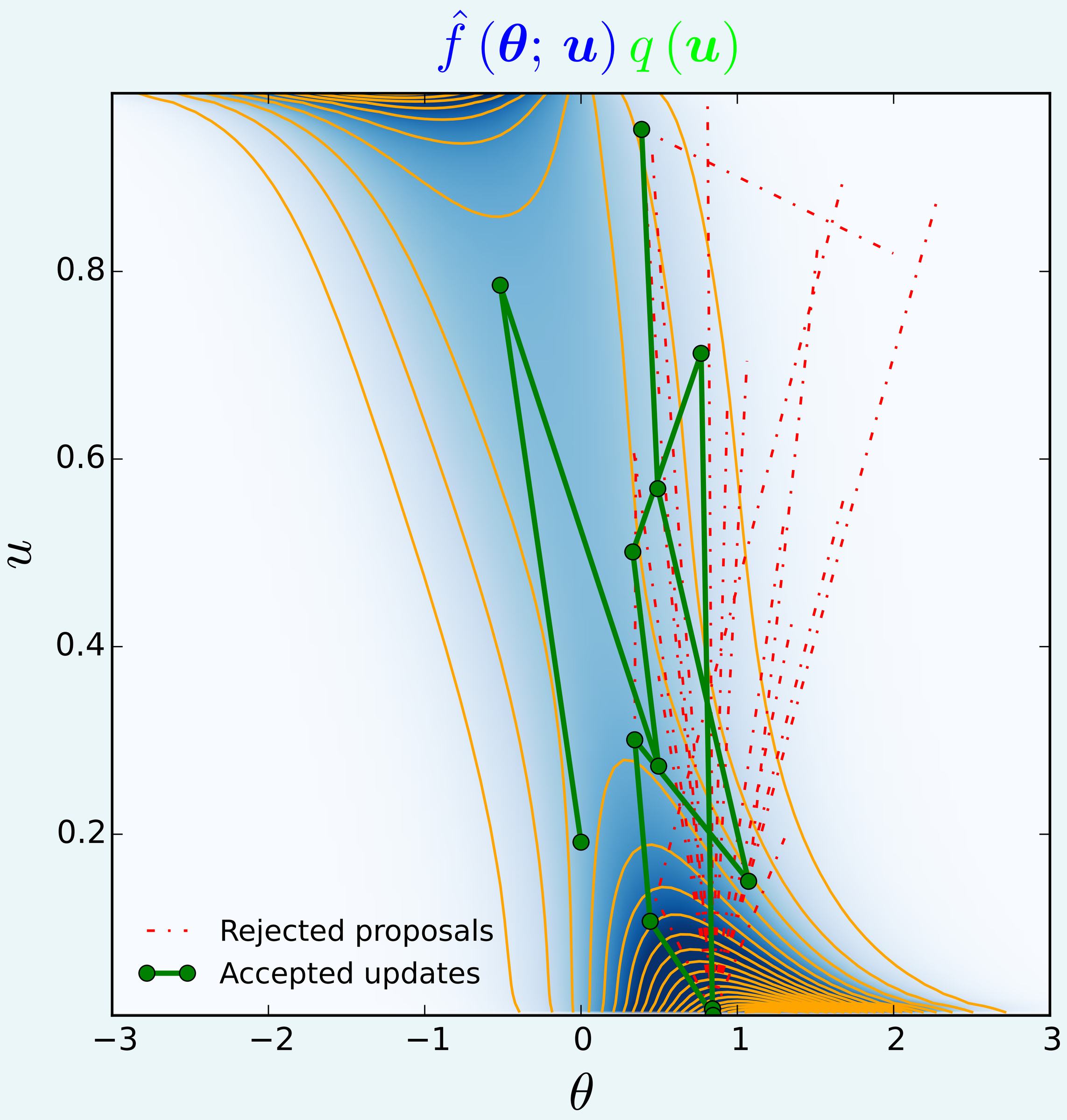
SLICE SAMPLING

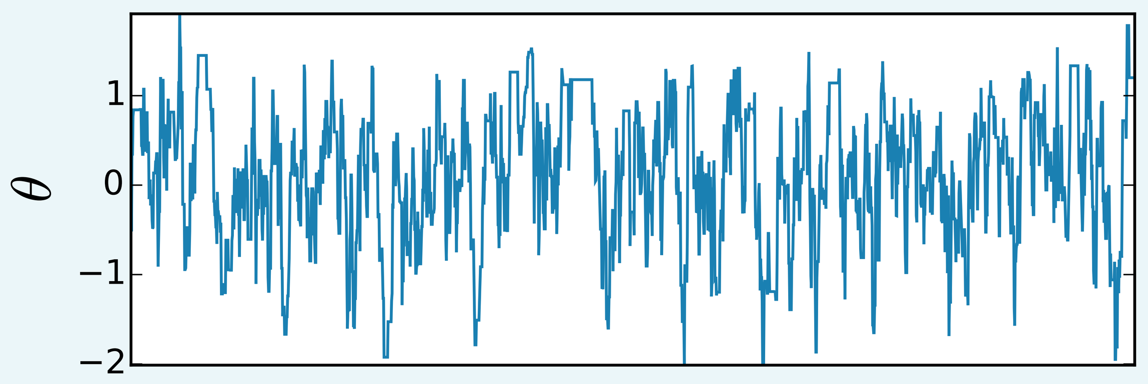
Slice sampling (Neal, 2003) - family of MCMC algorithms with updates that locally adapt to the target density.



Slice sampling algorithms always move the state being updated: can we apply them in the pseudo-marginal setting?







OUR APPROACH

Consider $\boldsymbol{u}$ as auxiliary variables as in Chopin and Singh (2015). This gives a new state $(\boldsymbol{\theta},\boldsymbol{u})$ with joint target density

$$\pi(\boldsymbol{\theta},\,\boldsymbol{u}) = \hat{f}(\boldsymbol{\theta};\,\boldsymbol{u})\,q(\boldsymbol{u})/C$$

$$\int \pi(\boldsymbol{\theta},\,\boldsymbol{u})\,\mathrm{d}\boldsymbol{u} = \int \hat{f}(\boldsymbol{\theta};\,\boldsymbol{u})\,q(\boldsymbol{u})/C\,\mathrm{d}\boldsymbol{u} = f(\boldsymbol{\theta})/C$$

Standard PM MCMC: Metropolis–Hastings update on the joint target with proposal

$$\tilde{r}\left[(\boldsymbol{\theta}',\,\boldsymbol{u}')\,|\,(\boldsymbol{\theta},\,\boldsymbol{u})\right] = r(\boldsymbol{\theta}';\,\boldsymbol{\theta})q(\boldsymbol{u}')$$

Our *Auxiliary Pseudo-Marginal* (APM) framework:

Inputs: current state: parameters $\boldsymbol{\theta}$, randomness $\boldsymbol{u}$; unbiased estimator : $\mathbb{E}_{q(\boldsymbol{u})}\left[\hat{f}(\boldsymbol{\theta};\boldsymbol{u})\right] = f(\boldsymbol{\theta})\ \forall\,\boldsymbol{\theta}$,

Output: new state $(\boldsymbol{\theta},\boldsymbol{u})$.

1. Update $\boldsymbol{u}$ leaving invariant its target conditional:

$$\pi(\boldsymbol{u}\,|\,\boldsymbol{\theta}) \propto \hat{f}(\boldsymbol{\theta};\,\boldsymbol{u})\,q(\boldsymbol{u})$$

2. Update $\boldsymbol{\theta}$ leaving invariant its target conditional:

$$\pi(\boldsymbol{\theta}\,|\,\boldsymbol{u}) \propto \hat{f}(\boldsymbol{\theta};\,\boldsymbol{u})\,q(\boldsymbol{u})$$

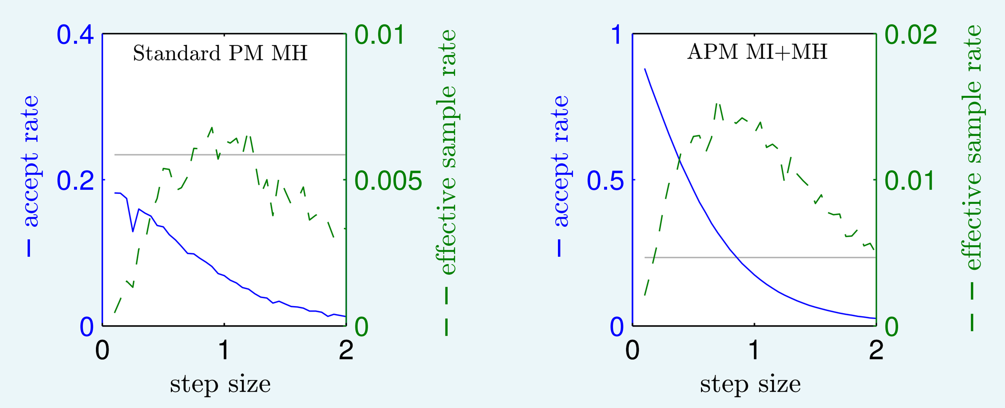
Free to mix and match any standard MCMC operators for the two updates.

With $\boldsymbol{u}$ now clamped during the $\boldsymbol{\theta}$ updates, usual case of a conditional distribution proportional to a deterministic function.

Slice sampling will move $\boldsymbol{\theta}$ if $\hat{f}(\boldsymbol{\theta};\,\boldsymbol{u})$ is continuous almost everywhere.

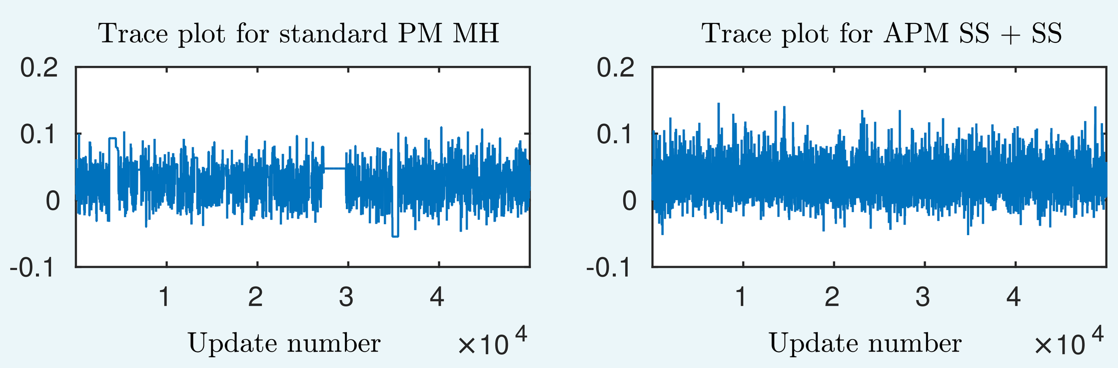
Equally can make perturbative moves to the random variates $\boldsymbol{u}$ with slice sampling or other adaptive methods.

EXPERIMENTS



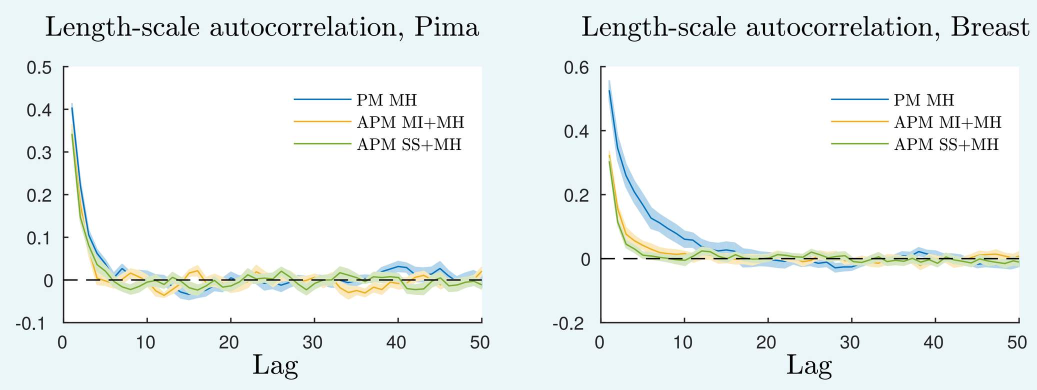
Clearer step-size selection

- Optimal efficiency at step size giving acceptance rate 0.234 for standard Metropolis proposals in high dimensions (Roberts et al. 1997).
- Not applicable to PM approach - in Gaussian test case 0.234 acceptance not even achievable.
- Using APM with independent proposals for $\boldsymbol{u}$ update and Gaussian proposals for $\boldsymbol{\theta}$ update, optimal acceptance rate now close to 0.234.



Parameter inference in Ising model

- Doubly-intractable distribution - 30 × 10 lattice.
- Data generated using exact sampler.
- Using standard PM approach chains stick for long periods.
- Using APM framework with slice sampling updates for $\boldsymbol{u}$ and $\boldsymbol{\theta}$ eliminates sticking and gives much improved cost-scaled autocorrelations.
- With lower-variance estimator using annealed importance sampling, standard PM performs better than APM.



Parameter inference in Gaussian process classifier

- Two UCI datasets, Pima and Breast, modelled following Filippone and Girolami, (2014).
- Estimator for marginal likelihood - importance sampler with using Gaussian latent posterior approximation as importance distribution.
- Adaptive tuning of step size significantly more consistent with APM approaches.
- Cost-normalised effective sample size when using APM updates significantly better than standard PM approach.

CONCLUSION

- General method for clamping and updating random numbers used in a pseudo-marginal method's unbiased estimator.
- Within this framework it is possible to use slice sampling and other adaptive methods.
- Resulting Markov chains are most robust and often mix more quickly.

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