Time Series - Homework 2

Matt Isaac

October 02, 2018

Exercise 1

(Brockwell and Davis) 3.1 - Determine which of the following ARMA processes are causal and which of them are invertible. (In each case $\{Z_t\}$ denotes white noise.)

a. $\mathbf{X_t + 0.2X_{t-1} - 0.48X_{t-2} = Z_t}$

$\Phi(z) = 1 + 0.2z - 0.48z^2$

Using R to solve for the roots of $\Phi(z)$:

```
phiz <- polyroot(c(1, 0.2, -0.48))
Mod(phiz)
```

```
## [1] 1.666667 1.250000
```

We see that the polynomial $\Phi(z)$ has roots 1.667 and 1.25, both of which are outside the unit circle. Thus, the process is causal.

$\Theta(z) = 1$

$\Theta(z)$ has no roots, so no roots exist inside the unit circle. Thus, the process is invertible.

The process is causal and invertible

b. $\mathbf{X_t + 1.9X_{t-1} + 0.88X_{t-2} = Z_t + 0.2Z_{t-1} + 0.7Z_{t-2}}$

$\Phi(z) = 1 + 1.9z + 0.88z^2$

Using R to solve for the roots of $\Phi(z)$:

```
phiz <- polyroot(c(1, 1.9, 0.88))
Mod(phiz)
```

```
## [1] 0.9090909 1.2500000
```

The polynomial $\Phi(z)$ has roots 0.9091 and 1.25. One of these is inside the unit circle, so the process in not causal.

$\Theta(z) = 1 + 0.2z + 0.7z^2$

Using R to solve for the roots of $\Theta(z)$:

```
thetaz <- polyroot(c(1, 0.2, 0.7))
Mod(thetaz)
```

```
## [1] 1.195229 1.195229
```

The polynomial $\Theta(z)$ has roots 1.195 and 1.195, both of which are outisde the unit circle. The process is therefore invertible.

The process is not causal but is invertible

c. $\mathbf{X_t + 0.6X_{t-1} = Z_t + 1.2Z_{t-1}}$

$\Phi(z) = 1 + 0.6z$

Using R to solve for the root of $\Phi(z)$:

```r
phiz <- polyroot(c(1, 0.6))
Mod(phiz)
```

```
## [1] 1.666667
```

$\Phi(z)$ has one root, 1.667. This is outside the unit circle, so the process is causal.

$\Theta(z) = 1 + 1.2z$

Using R to solve for the root of $\Theta(z)$:

```r
thetaz <- polyroot(c(1, 1.2))
Mod(thetaz)
```

```
## [1] 0.8333333
```

The root of $\Theta(z)$, 0.833, is not outside the unit circle, so the process is not invertible.

The process is causal but not invertible

d. $\mathbf{X_t + 1.8X_{t-1} + 0.81X_{t-2} = Z_t}$

$\Phi(z) = 1 + 1.8z + 0.81z^2$

Using R to solve for the roots of $\Phi(z)$:

```r
phiz <- polyroot(c(1, 1.8, 0.81))
Mod(phiz)
```

```
## [1] 1.111111 1.111111
```

$\Phi(z)$ has repeated roots with a value of 1.111. Thus both roots are outside the unit circle, so the process is causal.

$\Theta(z) = 1$

$\Theta(z)$ has no roots, and therefore no roots inside the unit circle. The process is invertible.

The process is causal and invertible

e. $\mathbf{X_t + 1.6X_{t-1} = Z_t - 0.4Z_{t-1} + 0.04Z_{t-2}}$

$\Phi(z) = 1 + 1.6z$

Using R to solve for the roots of $\Phi(z)$:

```r
phiz <- polyroot(c(1, 1.6))
Mod(phiz)
```

```
## [1] 0.625
```

$\Phi(z)$ has one root: 0.625. This is inside the unit circle, so the process is not causal.

$\Theta(z) = 1 - 0.4z + 0.04z^2$

Using R to solve for the roots of $\Theta(z)$

```r
thetaz <- polyroot(c(1, -0.4, 0.04))
Mod(thetaz)
```

```
## [1] 5 5
```

$\Theta(z)$ has repeated roots, both with a value of 5. Since both roots are outside the unit circle, the process is invertible.

The process is not causal but is invertible

Exercise 2

For part (a) in the first problem, calculate $\rho(h)$ for $h = 1, 2, ..., 30$, and graph it.

$$X_t + 0.2X_{t-1} - 0.48X_{t-2} = Z_t$$

So, $\phi_1 = -0.2$, and $\phi_2 = 0.48$

We will use the following equations:
For $k = 0$:

$$\gamma(0) - [\phi_1\gamma(1) + ... + \phi_p\gamma(p)] = \sigma^2$$

and for $k \geq 1$:

$$\gamma(k) - [\phi_1\gamma(k-1) + ... + \phi_p\gamma(k-p)] = 0$$

Plugging in our values of ϕ_1 and ϕ_2, we see that

For $k = 0$:

$$\gamma(0) - [-0.2\gamma(1) + 0.48\gamma(2)] = \sigma^2$$

For $k = 1$:

$$\gamma(1) - [-0.2\gamma(0) + 0.48\gamma(1)] = 0$$

For $k = 2$:

$$\gamma(2) - [-0.2\gamma(1) + 0.48\gamma(0)] = 0$$

From the above equations, we obtain the following system of linear equations:

$$\gamma(0) + 0.2\gamma(1) - 0.48\gamma(2) = \sigma^2$$
$$0.2\gamma(0) + 0.52\gamma(1) + 0\gamma(2) = 0$$
$$-0.48\gamma(0) + 0.2\gamma(1) + \gamma(2) = 0$$

Written in matrix form:

$$\begin{bmatrix} 1 & 0.2 & -0.48 \\ 0.2 & 0.52 & 0 \\ -0.48 & 0.2 & 1 \end{bmatrix} \begin{bmatrix} \gamma(0) \\ \gamma(1) \\ \gamma(2) \end{bmatrix} = \begin{bmatrix} \sigma^2 \\ 0 \\ 0 \end{bmatrix}$$

The following R code solves the linear system of equations for the three unknowns, $\gamma(0)$, $\gamma(1)$, and $\gamma(2)$ (letting $\sigma^2 = 1$).

```r
A <- matrix(data = c(1, 0.2, -0.48, 0.2, 0.52, 0, -0.48, 0.2, 1 ), nrow = 3, byrow = TRUE)
B <- matrix(data = c(1, 0, 0), nrow = 3, byrow = TRUE)

g <- solve(a = A, b = B)
g <- as.vector(g)
```

$\gamma(0)$, $\gamma(1)$, and $\gamma(2)$ are 1.525, -0.587, 0.849 respectively.

Using the values obtained by solving the system of linear equations, we recursively solve for $\gamma(h)$ for $h = 3, 4, ..., 30$ by using the equation:

$$\gamma(k) = \phi_1\gamma(k-1) + ... + \phi_p\gamma(k-p)$$

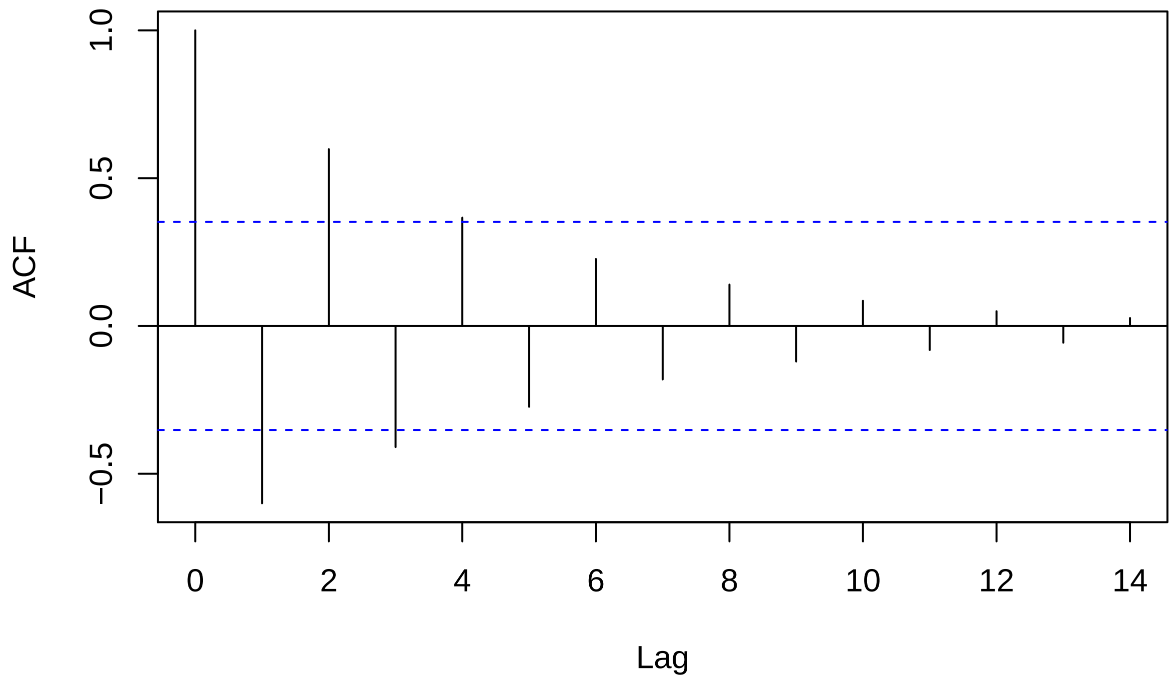
```r
gammas <- c(g, rep(0, 28))

for(k in seq(4,31)){
  gammas[k] <- (-0.2 * gammas[k-1]) + (0.48 * gammas[k-2])
}
```

The values of $\gamma(h)$ for $h = 3, 4, ..., 30$ are stored in the vector `gammas`. The code below obtains the values of $\rho(h) = \frac{\gamma(h)}{\gamma(0)}$ and plots $\rho(h)$.

```
rhos <- gammas/gammas[1]

acf(rhos, main = "")
```



Exercise 3

a. Construct an AR model whose ACF has for form of a damped sin function, with the damping factor equal to 0.8.

In the textbook, it is given that:

$$\xi_1 = re^{i\theta} \text{ and } \xi_2 = re^{-i\theta}$$

Choosing a damping factor of $r^{-1} = 0.8$, we have the following AR(2) process defined by

$$(1 - \xi_1^{-1}B)(1 - \xi_2^{-1}B)X_t = Z_t$$
$$\implies (1 - \xi_2^{-1}B - \xi_1^{-1}B + \xi_1^{-1}\xi_2^{-1}B^2)X_t = Z_t$$
$$\implies X_t - (\xi_2^{-1})X_{t-1} - (\xi_1^{-1})X_{t-1} + (\xi_1^{-1}\xi_2^{-1})X_{t-2} = Z_t$$
$$\implies X_t + (-\xi_2^{-1} - \xi_1^{-1})X_{t-1} + (\xi_1^{-1}\xi_2^{-1})X_{t-2} = Z_t$$
$$\implies X_t - (\xi_2^{-1} + \xi_1^{-1})X_{t-1} - (-\xi_1^{-1}\xi_2^{-1})X_{t-2} = Z_t$$

Recalling the general formula for an AR(p) process:

$$X_t - \phi_1 X_{t-1} - ... - \phi_p X_{t-p} = Z_t$$

Thus,

ϕ_1:

$\phi_1 = \xi_2^{-1} + \xi_1^{-1}$

$\quad \implies \phi_1 = \frac{0.8}{e^{-i}} + \frac{0.8}{e^{i}}$

```
e <- exp(1)
i <- complex(real = 0, imaginary = 1)
phi_1 <- 0.8/(e^-i) + (0.8/(e^i))
phi_1
```

```
## [1] 0.8644837+0i
```

$\quad \implies \phi_1 = 0.8645$

ϕ_2:

$\quad \phi_2 = -\xi_1^{-1}\xi_2^{-1}$

$\quad \implies \phi_2 = -\frac{0.8}{e^{-i}} * \frac{0.8}{e^{i}}$

```
phi_2 <- -0.8/(e^-i) * (0.8/(e^i))
phi_2
```

```
## [1] -0.64+0i
```

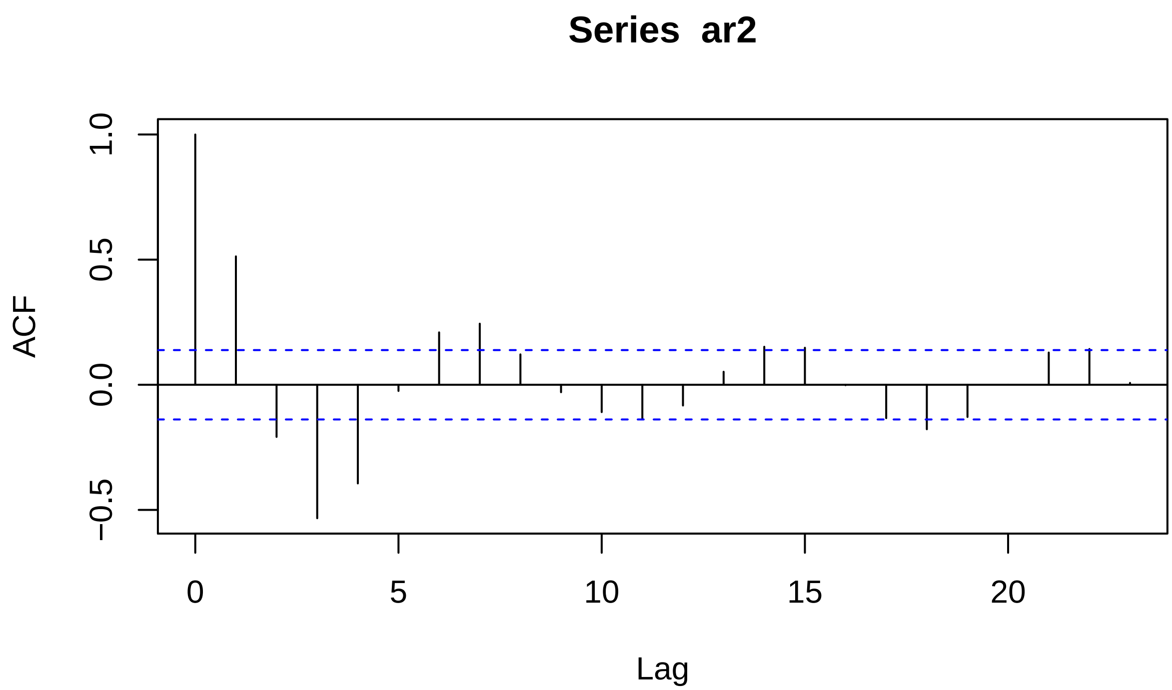
$\quad \implies \phi_2 = -0.64$

Thus, an AR(2) model whose ACF has a form of a damped sin function (damping factor = 0.8) can be represented by

$$\mathbf{X_t - 0.8645X_{t-1} + 0.64X_{t-2} = Z_t}.$$

b. Simulate a realization of the model you constructed, and plots its sample ACF. Does it match your expectation?

```
set.seed(123)
ar2 <- arima.sim(model = list(ar = c(0.8644837, -0.64)), sd = 1, n = 200)
acf(ar2)
```



Yes, we see the decaying sine function, which is what we would expect.