Exploring the formation and robustness of partially relaxed MHD states

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Part I

Introduction

Preface

- Fusion plasmas are **not ideal**
  - Magnetic flux is not frozen-into the plasma
  - Magnetic flux can form islands in non-ideal plasma
  - and much more ...
- Ideal MHD is thus not an accurate description of fusion plasma

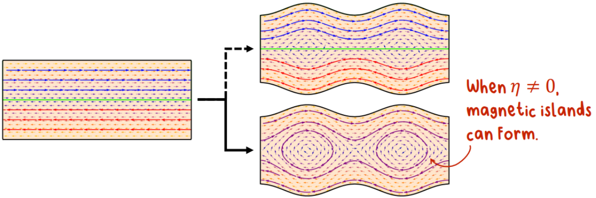


Figure: Credit: "Introduction to MHD" SULI Intro Course 2021. [6]

Multiregion Relaxed MHD (MRxMHD)

- A **Multiregion Relaxed (MRx)** model is used instead
  - Taylor relaxation occurs only within subregions of plasma [2]
- **Taylor Relaxation**
  - Process of plasma transitioning into lower energy state
- **Ideal Interfaces**
  - Thin layers of ideal plasma
  - Forms a current sheet structure
  - In analysis, layers are assumed to have **zero** thickness
- **Applications**
  - Critical tool for optimizing stellarator design
  - Model magnetic field perturbations in tokamaks

Project Overview

Our project aims to test the **robustness** of ideal interfaces in MRxMHD.

- Testing the presence of interfaces in resistive plasma
- Typically expect current sheet structures to break down in resistive plasmas

This presentation will go over the following parts of the project:

- Mathematical Methods — Boundary Layer Theory, Perturbation Methods
- Literature Analysis and Derivations

Part II

Mathematical Methods

Boundary Layer Theory [1]

- **Boundary Layer** — narrow region where DE solution "changes rapidly"
    - perturbing parameter ε (generally small)
    - characteristic size of narrow region δ
- **Inner** and **Outer** solutions . . .
    - approximate the actual solution
    - **inner** — rapidly changing soln.
    - **outer** — slowly changing soln.
    - solutions may be "stitched" together with asymptotic matching
- Motivation
    - interest in MHD equilibrium at small resistivity η (i.e. force balance in a plasma)

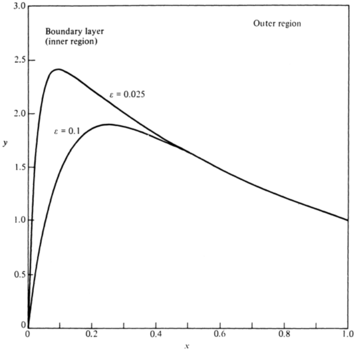
Boundary Layer Theory — Example [1]

Consider the following DE solution with $y(0) = 0,\ y'(1) = 1$

$$\varepsilon y'' + (1 + \varepsilon)\, y' + y = 0 \qquad (1)$$

$$y(x) = \frac{e^{-x} - e^{-x/\varepsilon}}{e^{-1} - e^{-1/\varepsilon}} \qquad (2)$$

- outer solution changes slowly at $1 < \varepsilon$
- inner solution changes rapidly at $0 < \varepsilon < 1$



Perturbation Methods [1]

Approximate a solution from an exact solution from a simpler problem

- Involved in linearizing PDEs
- Express a solution in terms of a series expansion with small "perturbing" parameters

$$\psi = \sum_{n=0}^{\infty} \varepsilon^n \psi_n = \psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + \dots \tag{3}$$

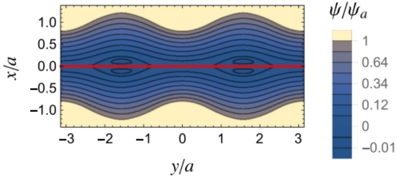
- Can substitute variables within PDEs with series to get a system of equations, sorted by order of ε
- Assumptions made to eliminate certain terms

Part III

Literature Analysis and Derivations

Dewar et al. (2017) [2]

- Description of magnetic flux under rippled boundary conditions
- “HKT-Beltrami” Slab Model
  - Thin, annular section of a toroidal cross-section transformed into Cartesian coords.
  - x-axis — radial direction of toroidal cross-section
  - y-axis — poloidal direction. Periodic. $y \in [-\pi, \pi]$
- Slab boundaries are perturbed with ripples of amplitude α



Dewar et al. (2017) [2]

The ansatz of the magnetic flux function is written as

$$\begin{aligned}\hat{\psi}_{+}(x,y) = {} & c_0 \cos \mu x + \sum_{l=1}^{\infty} c_{lm} \cos \frac{lmy}{a} \cosh \kappa_{lm} x \\ & + d_0 \sin \mu x + \sum_{l=1}^{\infty} d_{lm} \cos \frac{lmy}{a} \sinh \kappa_{lm} x\end{aligned} \tag{4}$$

The solution fits criterion such as:

- Helmholtz differential equation, $\left(\nabla^2 + \mu^2\right) \hat{\psi} = 0$
- Slab model geometry

Dewar et al. (2017) [2]

There are two different boundary conditions we explored in the project

1. **Bdy-1**

$$\psi\left(\pm a, y\right) - \langle\psi\left(\pm a, y\right)\rangle = 2\alpha\psi_a \cos k_y y \tag{5}$$

- boundary is described implicitly, generally easier to solve for
- accurate for small values of α

2. **Bdy-2**

$$x_{\text{bdy}}\left(y\right) = a\left(1 - \alpha \cos k_y y\right) \tag{6}$$

- boundary is described explicitly as a function of position, difficult to solve for
- accurate for greater range of α

Dewar et al. (2017) [2]

- **Bdy-1**

$$\psi(\pm a, y) - \langle \psi(\pm a, y) \rangle = 2\alpha\psi_a \cos k_y y$$

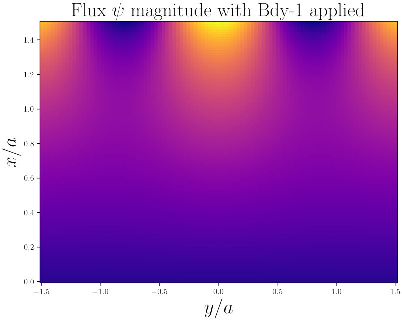


Figure: Plot of flux magnitude using Bdy-1

B. Sinusoidal rippled boundary condition

In Fig. 15, we compare boundaries generated by method Bdy-1, described in Sec. IV A, with the corresponding sinusoidal boundaries defined by Eq. (49). For case (a), small amplitude ripple, method Bdy-1 produces a boundary indistinguishable from the target sinusoid, but for larger amplitude, case (b), strong second harmonic error is clear to the eye.

In Fig. 16, we plot the difference between the pure sinusoid defined by Eq. (49) and boundaries generated by the Bdy-2 method, (a) for small-amplitude ripple, $\alpha = 0.03$ and

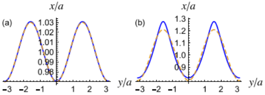


FIG. 15. Panels (a) and (b) show the boundaries generated by method Bdy-1 (blue online) for ripple amplitudes $\alpha = 0.03$ and 0.2, respectively. For comparison, corresponding sinusoidal boundaries (dashed, orange online), as used in method Bdy-2, are also shown. In both panels, $m = 2$, $a\mu_0 = 0.2$.

Figure: At larger values of α Bdy-1 ripples diverge from the actual.

Dewar et al. (2017) [2]

Applying equation (6) for Bdy-2 involves substituting x with

$$x_{\mathrm{bdy}}(y) = a\left(1 - \alpha \cos k_y y\right)$$

- The solution involves compositions of sinusoidal functions, ex. $\cos(\mu a(1 - \alpha \cos k_y y))$
- Two approaches:
  - Expansion of the nested sinusoidals into a series of Bessel functions
  - Taylor Expansion with respect to α, truncating terms $\sim O(\alpha^2)$ and higher

Wang, Bhattacharjee (1992) [5]

- Provides insight on timescales
  - Phases A-D describe temporal evolution of a boundary-deformed magnetic field
  - Exponential weighting of Alfven τ_A and Resistive τ_R timescales (generally $\tau_A < \tau_R$)
- **Insight** — resistive effects and timescales become more significant with time

	τ_A (ideal)	τ_R (non-ideal)
Phase A	2/3	1/3
Phase B	2/5	3/5
Phase C	1/2	1/2
Phase D	1/4	3/4

$$\tau = \tau_A^{(1-s)} \tau_R^{s} \tag{7}$$

Hahm, Kulsrud (1985) [4]

The following equation is a combination of the induction (with resistivity) and the fluid momentum conservation equations:

$$\frac{\partial \psi}{\partial t} + \vec{V} \cdot \vec{\nabla} \psi = \frac{\eta}{4\pi} \nabla^2 \psi \tag{8}$$

where

- $\vec{V} = \hat{z} \times \vec{\nabla} \phi$
- ψ — magnetic flux where $\vec{B} = B_T \hat{z} + \hat{z} \times \nabla \psi$ (B_T is the toroidal magnetic field)

Hahm, Kulsrud (1985) [4]

Expanding (8) with a perturbation series and sorting terms by order of ε,

$$\varepsilon^0 : \frac{\partial \psi_0}{\partial t} + \vec{V}_0 \cdot \vec{\nabla} \psi_0 = \frac{\eta}{4\pi} \nabla^2 \psi_0 \tag{9}$$

$$\varepsilon^1 : \frac{\partial \psi_1}{\partial t} + \vec{V}_0 \cdot \vec{\nabla} \psi_1 + \vec{V}_1 \cdot \vec{\nabla} \psi_0 = \frac{\eta}{4\pi} \nabla^2 \psi_1 \tag{10}$$

$$\varepsilon^2 : \frac{\partial \psi_2}{\partial t} + \vec{V}_0 \cdot \vec{\nabla} \psi_2 + \vec{V}_1 \cdot \vec{\nabla} \psi_1 + \vec{V}_2 \cdot \vec{\nabla} \psi_0 = \frac{\eta}{4\pi} \nabla^2 \psi_2 \tag{11}$$

Hahm, Kulsrud (1985) [4]

Assumptions made with the equations:

- Let $\vec{V}_0 = 0$, making ϕ_0 an arbitrary constant
- $\eta/4\pi \sim O\left(\varepsilon^2\right)$. This allows equations (9) and (10) to be equal to 0

Which results in this equation,

$$\frac{\partial \psi_1}{\partial t} - kB_0 \frac{x}{a} \phi_1(x) = 0 \tag{12}$$

- k — wave number
- B_0 — the maximum magnetic field along the y-component
- a — characteristic spatial size of the system
- The partial derivative of phi is evaluated near $x = 0$

Part IV

Summary and Goals

Summary of Findings

- In the Dewar et al. paper; the solution found in this project for Bdy-2 via Taylor series approximation has modes $l = 1, 2$, as opposed to just $l = 1$ as stated in the paper
- In the Wang, Bhattacharjee paper; resistive timescales become more dominant in determining the overall char. timescale as the plasma evolves over time

Future Work

- Continue work on solutions in the Dewar et al. paper under Bdy-2
  - also comparing results between Bdy-1 and Bdy-2 under more scrutiny
- Test flux functions on the linearized PDEs presented in Hahm, Kulsrud paper
- Begin work on nondimensionalization of DEs, giving more insight on when resistive timescale is non-negligible

Part V

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Part VI

Supplements

Fourier Analysis [1]

Magnetic flux solutions can have the form similar to

$$\sum_{n=1}^{\infty} C_n \sin\left(n\pi y/a\right) = V_0(y) \tag{13}$$

To determine coefficient C_n, multiply equation by $\sin\left(n'\pi y/a\right)$ and integrate from 0 to a [3],

$$\sum_{n=1}^{\infty} C_n \int_0^a \sin\left(n\pi y/a\right) \sin\left(n'\pi y/a\right)\, dy = \int_0^a V_0(y) \sin\left(n'\pi y/a\right)\, dy \tag{14}$$

Fourier Analysis

The LH integral contains mutually orthogonal functions, which evalutates to

$$\int_0^a \sin\left(n\pi y/a\right) \sin\left(n'\pi y/a\right)\, dy = \frac{a}{2}\delta_{nn'} \tag{15}$$

The coefficient is thus

$$C_n = \frac{2}{a}\int_0^a V_0(y) \sin\left(n\pi y/a\right)\, dy \tag{16}$$

Dewar et al. Solutions

Bdy-1 applied:

$$\begin{aligned}\hat{\psi}_{+}(x,y) &= -\overline{\psi}\cos\mu x \\ &\quad + \frac{2\alpha\psi_a}{\sinh\kappa_m a}\sinh\kappa_m x\cos\left(\frac{my}{a}\right) \\ &\quad + \frac{2\alpha\psi_a\gamma_s\kappa_m}{\mu\sinh\left(\kappa_m a\right)}\sin\left(\mu x\right)\end{aligned} \tag{17}$$

Dewar et al. Solutions

Bdy-2 applied:

$$
\begin{aligned}
\hat{\psi}_+(x,y) = &-\overline{\psi}\cos\mu x \\
&+ \frac{\sin\mu x}{\sin\mu a}\left[\psi_a - \overline{\psi} - \frac{\overline{F}}{\mu}(1+\cos\mu a) + \overline{\psi}\cos\mu a\right] \\
&+ \overline{F}\alpha a \sin\mu a \sum_{l=1}^{2} \cos\left(\frac{lmy}{a}\right)\frac{\sinh\kappa_{lm}x}{\sinh\kappa_{lm}a}
\end{aligned}
\tag{18}
$$

Hahm, Kulsrud (1985) [4]

- Description of inner and outer magnetic flux solution
- Outer Solution

$$\psi_1(x) = B_0\delta\left(\frac{\sinh kx}{\sinh ka}\right) \tag{19}$$

- Inner Solution

$$\xi = \frac{2}{\pi}\frac{B_0 k\delta}{\sinh ka}\int_0^{kxt/\tau_A}\frac{\sin u}{u}\,du \tag{20}$$

where $\xi = \frac{\psi_1}{x}$