DERIVATION OF GENERALISED NONLINEAR SHIELDED SOLUTIONS

From Eq. (64) of Dewar *et al.*[1], the ansatz for the general solution for $\hat{\psi}(x,y)$ is given by,

$$\hat{\psi}(x,y) = c_0 \cos(\mu x) + \sum_{l=1}^{\infty} c_l \cosh(\kappa_l x) \cos\left(\frac{lmy}{a}\right) + d_0 \sin(\mu x) + \sum_{l=1}^{\infty} d_l \sinh(\kappa_l x) \cos\left(\frac{lmy}{a}\right), \tag{1}$$

where $\kappa_l^2 = (lm/a)^2 - \mu^2$ and $\kappa_l \in \mathbb{R}$ since $(lm/a)^2 > \mu^2$.

On the domain Ω_+, the two boundary conditions are,

$$\hat{\psi}(x=0,y) = -\langle\psi\rangle, \tag{2}$$

$$\hat{\psi}(x_{bdy}(y),y) = \psi_a - \langle\psi\rangle - \langle F_0\rangle \psi_U(x_{bdy}(y);\mu), \tag{3}$$

where,

$$x_{bdy}(y) = a\left[1 - \alpha\cos(k_y y)\right], \tag{4}$$

$$\psi_U(x;\mu) = \frac{1-\cos(\mu_0 x)}{\mu_0}. \tag{5}$$

Using (2), it follows that,

$$c_0 = -\langle\psi\rangle, \tag{6}$$

$$c_l = 0, \quad \text{for all } l \geq 1. \tag{7}$$

Next, we want to solve for d_0 and d_l using (3). Before proceeding, we first note the following identities and relations;

$$\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)\,, \tag{8}$$

$$\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)\,, \tag{9}$$

$$\sinh(\alpha-\beta) = \sinh(\alpha)\cosh(\beta) - \cosh(\alpha)\sinh(\beta)\,, \tag{10}$$

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right], \tag{11}$$

$$\cosh(x) = \cos(ix)\,, \tag{12}$$

$$\sinh(x) = -i\sin(ix)\,, \tag{13}$$

$$\cos(t\cos(x)) = J_0(t) + 2\sum_{k=1}^{\infty}(-1)^k J_{2k}(t)\cos(2kx)\,, \tag{14}$$

$$\sin(t\cos(x)) = 2\sum_{k=0}^{\infty}(-1)^k J_{2k+1}(t)\cos((2k+1)x)\,, \tag{15}$$

$$I_n(x) = i^{-n}J_n(ix)\,, \tag{16}$$

for $\{x,t\}\in\mathbb{C}$.

First, we introduce the change of variables, $z = my/a$. Using (6) and (7), together with (13), (1) becomes,

$$d_0\sin(\mu a\left[1-\alpha\cos(z)\right]) + \sum_{l=1}^{\infty} d_l\sinh(\kappa_l a\left[1-\alpha\cos(z)\right])\cos(z) \tag{17}$$

$$= d_0\sin(\mu a\left[1-\alpha\cos(z)\right]) - i\sum_{l=1}^{\infty} d_l\sin(i\kappa_l a\left[1-\alpha\cos(z)\right])\cos(z)\,. \tag{18}$$

Next, we use (8) to expand the sine terms in (18). Noting that,

$$\sin(\mu a\left[1-\alpha\cos(z)\right]) = \sin(\mu a)\cos(\mu a\alpha\cos(z)) - \cos(\mu a)\sin(\mu a\alpha\cos(z))\,, \tag{19}$$

$$\sin(i\kappa_l a\left[1-\alpha\cos(z)\right]) = \sin(i\kappa_l a)\cos(i\kappa_l a\alpha\cos(z)) - \cos(i\kappa_l a)\sin(i\kappa_l a\alpha\cos(z))\,, \tag{20}$$

then (18) becomes,

$$\begin{aligned} &d_0\left[\sin(\mu a)\cos(\mu a\alpha\cos(z)) - \cos(\mu a)\sin(\mu a\alpha\cos(z))\right] \\ &\quad - i\sum_{l=1}^{\infty} d_l\cos(lz)\left[\sin(i\kappa_l a)\cos(i\kappa_l a\alpha\cos(z)) - \cos(i\kappa_l a)\sin(i\kappa_l a\alpha\cos(z))\right]. \end{aligned} \tag{21}$$

Using (14) and (15), we can expand the nested sine and cosine functions in (21);

$$\cos\left(\mu a\alpha\cos\left(z\right)\right) = J_0\left(\mu a\alpha\right) + 2\sum_{n=1}^{\infty}\left(-1\right)^n J_{2n}\left(\mu a\alpha\right)\cos\left(2nz\right), \tag{22}$$

$$\sin\left(\mu a\alpha\cos\left(z\right)\right) = 2\sum_{n=0}^{\infty}\left(-1\right)^n J_{2n+1}\left(\mu a\alpha\right)\cos\left(\left(2n+1\right)z\right), \tag{23}$$

$$\cos\left(i\kappa_l a\alpha\cos\left(z\right)\right) = J_0\left(i\kappa_l a\alpha\right) + 2\sum_{n=1}^{\infty}\left(-1\right)^n J_{2n}\left(i\kappa_l a\alpha\right)\cos\left(2nz\right), \tag{24}$$

$$\sin\left(i\kappa_l a\alpha\cos\left(z\right)\right) = 2\sum_{n=0}^{\infty}\left(-1\right)^n J_{2n+1}\left(i\kappa_l a\alpha\right)\cos\left(\left(2n+1\right)z\right). \tag{25}$$

Using (16), we can write (24) and (25) as,

$$\cos\left(i\kappa_l a\alpha\cos\left(z\right)\right) = I_0\left(\kappa_l a\alpha\right) + 2\sum_{n=1}^{\infty} I_{2n}\left(\kappa_l a\alpha\right)\cos\left(2nz\right), \tag{26}$$

$$\sin\left(i\kappa_l a\alpha\cos\left(z\right)\right) = 2i\sum_{n=0}^{\infty} I_{2n+1}\left(\kappa_l a\alpha\right)\cos\left(\left(2n+1\right)z\right). \tag{27}$$

Substituting (22), (23), (26) and (27) into (21) yields,

$$\begin{aligned} d_0 \Bigg[&\sin\left(\mu a\right)\left(J_0\left(\mu a\alpha\right) + 2\sum_{n=1}^{\infty}\left(-1\right)^n J_{2n}\left(\mu a\alpha\right)\cos\left(2nz\right)\right) \\ &\qquad -2\cos\left(\mu a\right)\left(\sum_{n=0}^{\infty}\left(-1\right)^n J_{2n+1}\left(\mu a\alpha\right)\cos\left(\left(2n+1\right)z\right)\right)\Bigg] \\ &- i\sum_{l=1}^{\infty} d_l\cos\left(lz\right)\Bigg[\sin\left(i\kappa_l a\right)\left(I_0\left(\kappa_l a\alpha\right) + 2\sum_{n=1}^{\infty} I_{2n}\left(\kappa_l a\alpha\right)\cos\left(2nz\right)\right) \\ &\qquad -2i\cos\left(i\kappa_l a\right)\left(\sum_{n=0}^{\infty} I_{2n+1}\left(\kappa_l a\alpha\right)\cos\left(\left(2n+1\right)z\right)\right)\Bigg], \end{aligned} \tag{28}$$

which can be further simplified to,

$$\begin{aligned} d_0 \Bigg[&\sin\left(\mu a\right)\left(J_0\left(\mu a\alpha\right) + 2\sum_{n=1}^{\infty}\left(-1\right)^n J_{2n}\left(\mu a\alpha\right)\cos\left(2nz\right)\right) \\ &\qquad -2\cos\left(\mu a\right)\left(\sum_{n=0}^{\infty}\left(-1\right)^n J_{2n+1}\left(\mu a\alpha\right)\cos\left(\left(2n+1\right)z\right)\right)\Bigg] \\ &+ \sum_{l=1}^{\infty} d_l\cos\left(lz\right)\Bigg[\sinh\left(\kappa_l a\right)\left(I_0\left(\kappa_l a\alpha\right) + 2\sum_{n=1}^{\infty} I_{2n}\left(\kappa_l a\alpha\right)\cos\left(2nz\right)\right) \\ &\qquad -2\cosh\left(\kappa_l a\right)\left(\sum_{n=0}^{\infty} I_{2n+1}\left(\kappa_l a\alpha\right)\cos\left(\left(2n+1\right)z\right)\right)\Bigg], \end{aligned} \tag{29}$$

using (12) and (13). Using (11), note that,

$$\cos(lz)\cos(2nz) = \frac{1}{2}\left[\cos\left(\left(l-2n\right)z\right) + \cos\left(\left(l+2n\right)z\right)\right], \tag{30}$$

$$\cos(lz)\cos\left(\left(2n+1\right)z\right) = \frac{1}{2}\left[\cos\left(\left(l-2n-1\right)z\right) + \cos\left(\left(l+2n+1\right)z\right)\right]. \tag{31}$$

Substituting (30) and (31) into (29) yields,

$$\begin{aligned} d_0 \Bigg[\sin(\mu a) \left(J_0(\mu a \alpha) + 2\sum_{n=1}^{\infty} (-1)^n J_{2n}(\mu a \alpha)\cos(2nz) \right) \\ -2\cos(\mu a)\left(\sum_{n=0}^{\infty} (-1)^n J_{2n+1}(\mu a\alpha)\cos\left(\left(2n+1\right)z\right)\right)\Bigg] \\ + \sum_{l=1}^{\infty} d_l \sinh(\kappa_l a) I_0(\kappa_l a \alpha)\cos(lz) \\ + \sum_{l=1}^{\infty} d_l \sinh(\kappa_l a)\left(\sum_{n=1}^{\infty} I_{2n}(\kappa_l a\alpha)\left[\cos\left(\left(l-2n\right)z\right)+\cos\left(\left(l+2n\right)z\right)\right]\right) \\ - \sum_{l=1}^{\infty} d_l \cosh(\kappa_l a)\left(\sum_{n=0}^{\infty} I_{2n+1}(\kappa_l a\alpha)\left[\cos\left(\left(l-2n-1\right)z\right)+\cos\left(\left(l+2n+1\right)z\right)\right]\right). \end{aligned} \tag{32}$$

This is the unknown side, which must be equated with the known side in order to solve for the unknown coefficients, d_0 and d_l. The known side is given by the RHS of (3), which can be written as,

$$\psi_a - \langle\psi\rangle - \langle F_0\rangle\,\psi_U\left(x_{bdy}(y)(y);\mu\right) = \psi_a - \langle\psi\rangle - \frac{\langle F_0\rangle}{\mu} - \frac{\langle F_0\rangle}{\mu}\cos\left(\mu a\left[1-\alpha\cos(z)\right]\right). \tag{33}$$

Using (9) together with (22) and (23), this becomes,

$$\begin{aligned} \psi_a - \langle\psi\rangle - \frac{\langle F_0\rangle}{\mu}\left(1-\cos(\mu a)J_0(\mu a\alpha)\right) - \frac{2\langle F_0\rangle}{\mu}\Bigg[\cos(\mu a)\left(\sum_{n=1}^{\infty}(-1)^n J_{2n}(\mu a\alpha)\cos(2nz)\right) \\ + \sin(\mu a)\left(\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(\mu a\alpha)\cos\left(\left(2n+1\right)z\right)\right)\Bigg]. \end{aligned} \tag{34}$$

To solve for the unknown coefficients, d_0 and d_l, we will use the orthogonality of the trigonometric functions. Specifically, we will use the fact that,

$$\int_0^{2\pi} \cos(mz)\cos(nz)\,dz = \pi\delta_m^n. \tag{35}$$

Multiplying the unknown side (32) by $\cos(pz)$ for some $p \geq 0$, integrating from 0 to 2π and using (35) yields,

$$2\pi d_0 \left[\sin(\mu a)\left(J_0(\mu a\alpha)\,\delta_0^p + \sum_{n=1}^{\infty}(-1)^n J_{2n}(\mu a\alpha)\,\delta_{2n}^p\right)\right.$$
$$\left. - \cos(\mu a)\left(\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(\mu a\alpha)\,\delta_{2n+1}^p\right)\right]$$
$$+ \pi\sum_{l=1}^{\infty} d_l \sinh(\kappa_l a)\, I_0(\kappa_l a\alpha)\,\delta_l^p$$
$$+ \pi\sum_{l=1}^{\infty} d_l \sinh(\kappa_l a)\left(\sum_{n=1}^{\infty} I_{2n}(\kappa_l a\alpha)\left[\delta_{l-2n}^p + \delta_{l+2n}^p\right]\right)$$
$$- \pi\sum_{l=1}^{\infty} d_l \cosh(\kappa_l a)\left(\sum_{n=0}^{\infty} I_{2n+1}(\kappa_l a\alpha)\left[\delta_{l-2n-1}^p + \delta_{l+2n+1}^p\right]\right). \quad (36)$$

The known side (34) becomes,

$$2\pi\left[\psi_a - \langle\psi\rangle - \frac{\langle F_0\rangle}{\mu}\left(1 - \cos(\mu a)\, J_0(\mu a\alpha)\right)\right]\delta_0^p$$
$$- \frac{2\pi\langle F_0\rangle}{\mu}\left[\cos(\mu a)\left(\sum_{n=1}^{\infty}(-1)^n J_{2n}(\mu a\alpha)\,\delta_{2n}^p\right) + \sin(\mu a)\left(\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(\mu a\alpha)\,\delta_{2n+1}^p\right)\right].$$

(37)

To evaluate the Kronecker-δ functions, we will sum over n on both the unknown and known sides. For the unknown side, (36) becomes,

$$2\pi d_0\left[\sin(\mu a)\left(J_0(\mu a\alpha)\,\delta_0^p + (-1)^{p/2} J_p(\mu a\alpha)\right) - \cos(\mu a)\,(-1)^{(p-1)/2} J_p(\mu a\alpha)\right]$$
$$+ \pi d_p \sinh(\kappa_p a)\, I_0(\kappa_p a\alpha)$$
$$+\pi\sum_{l=1}^{\infty} d_l \sinh(\kappa_l a)\left[I_{l-p}(\kappa_l a\alpha) + I_{p-l}(\kappa_l a\alpha)\right] - \pi\sum_{l=1}^{\infty} d_l \cosh(\kappa_l a)\left[I_{l-p}(\kappa_l a\alpha) + I_{p-l}(\kappa_l a\alpha)\right].$$

(38)

Using the fact that,

$$I_{-n}(z) = I_n(x)\,, \quad (39)$$

we can write (38) as,

$$2\pi d_0 \left[\sin(\mu a) J_0(\mu a\alpha)\,\delta_0^p + i^p J_p(\mu a\alpha)(\sin(\mu a) + i\cos(\mu a))\right] \\ + \pi d_p \sinh(\kappa_p a) I_0(\kappa_p a\alpha) + 2\pi \sum_{l=1}^{\infty} d_l \left[\sinh(\kappa_l a) - \cosh(\kappa_l a)\right] I_{l-p}(\kappa_l a\alpha). \quad (40)$$

The known side (37) evaluates to,

$$2\pi \left[\psi_a - \langle\psi\rangle - \frac{\langle F_0\rangle}{\mu}(1 - \cos(\mu a) J_0(\mu a\alpha))\right]\delta_0^p \\ - \frac{2\pi i^p \langle F_0\rangle}{\mu}\left[\cos(\mu a) - i\sin(\mu a)\right] J_p(\mu a\alpha). \quad (41)$$

Finally, using the fact that $e^{-ix} = \cos(x) - i\sin(x)$, (40) can be written as,

$$2\pi d_0 \left[\sin(\mu a) J_0(\mu a\alpha)\,\delta_0^p + i^{p+1} e^{-i\mu a} J_p(\mu a\alpha)\right] \\ + \pi d_p \sinh(\kappa_p a) I_0(\kappa_p a\alpha) + 2\pi \sum_{l=1}^{\infty} d_l \left[\sinh(\kappa_l a) - \cosh(\kappa_l a)\right] I_{l-p}(\kappa_l a\alpha), \quad (42)$$

whereas (41) becomes,

$$2\pi \left[\psi_a - \langle\psi\rangle - \frac{\langle F_0\rangle}{\mu}(1 - \cos(\mu a) J_0(\mu a\alpha))\right]\delta_0^p - \frac{2\pi i^p e^{-i\mu a}\langle F_0\rangle}{\mu} J_p(\mu a\alpha). \quad (43)$$

Equating (42) and (43) yields,

$$\text{For } p = 0: \quad d_0 \sin(\mu a) J_0(\mu a\alpha) = \left(\psi_a - \langle\psi\rangle - \frac{\langle F_0\rangle}{\mu}\left[1 - \cos(\mu a) J_0(\mu a\alpha)\right]\right), \quad (44)$$

and

Otherwise:

$$d_0 \left[i^{p+1} e^{-i\mu a} J_p(\mu a\alpha)\right] + \frac{d_p}{2}\sinh(\kappa_p a) I_0(\kappa_p a\alpha) + \sum_{l=1}^{\infty} d_l \left[\sinh(\kappa_l a) - \cosh(\kappa_l a)\right] I_{l-p}(\kappa_l a\alpha) \\ = -\frac{i^p e^{-i\mu a}\langle F_0\rangle}{\mu} J_p(\mu a\alpha). \quad (45)$$

From (44) we find,

$$d_0 = \frac{\mu\left[\psi_a - \langle\psi\rangle\right] - \langle F_0\rangle}{\mu\sin(\mu a) J_0(\mu a\alpha)} + \frac{\langle F_0\rangle \cot(\mu a)}{\mu}. \quad (46)$$

It then follows from (45) that,

$$\frac{d_p}{2}\sinh(\kappa_p a) I_0(\kappa_p a\alpha) + \sum_{l=1}^{\infty} d_l \left[\sinh(\kappa_l a) - \cosh(\kappa_l a)\right] I_{l-p}(\kappa_l a\alpha) \\ = -d_0\left[i^{p+1} e^{-i\mu a} J_p(\mu a\alpha)\right] - \frac{i^p e^{-i\mu a}\langle F_0\rangle}{\mu} J_p(\mu a\alpha), \quad (47)$$

where d_0 is given by (46).

Equation (47) contains two parameters (p and $\max(l)$) and should be interpreted as describing a system of simultaneous equations, the solution of which will give each coefficient d_l up to $\max(l)$, where $\max(l)$ is the maximum number of Fourier modes retained in the ansatz given by Eq. (64) of Dewar *et al.*[1].

REFERENCES

[1] Dewar, R. L., Hudson, S. R., Bhattacharjee, A. & Yoshida, Z. 2017 Multi-region relaxed magnetohydrodynamics in plasmas with slowly changing boundaries—resonant response of a plasma slab. *Physics of Plasmas* **24** (4), 042507, arXiv: https://doi.org/10.1063/1.4979350.