

MCG 5141, Assignment I

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Question 1

a)

$$\begin{aligned} 0 &= \vec{\nabla} \cdot (\vec{\nabla} \times \vec{v}) \\ 0 &= \vec{\nabla} \cdot (\epsilon_{ijk} \frac{\partial v_k}{\partial x_j}) \\ 0 &= \frac{\partial}{\partial x_i} \cdot (\epsilon_{ijk} \frac{\partial v_k}{\partial x_j}) \\ 0 &= \epsilon_{ijk} \frac{\partial^2 v_k}{\partial x_i \partial x_j} \\ 0 &= \epsilon_{123} \frac{\partial^2 v_3}{\partial x_1 \partial x_2} + \epsilon_{213} \frac{\partial^2 v_3}{\partial x_2 \partial x_1} + \epsilon_{312} \frac{\partial^2 v_2}{\partial x_3 \partial x_1} \\ &\quad + \epsilon_{132} \frac{\partial^2 v_2}{\partial x_1 \partial x_3} + \epsilon_{231} \frac{\partial^2 v_1}{\partial x_2 \partial x_3} + \epsilon_{321} \frac{\partial^2 v_1}{\partial x_3 \partial x_2} \\ 0 &= \frac{\partial^2 v_3}{\partial x_1 \partial x_2} - \frac{\partial^2 v_3}{\partial x_2 \partial x_1} + \frac{\partial^2 v_2}{\partial x_3 \partial x_1} \\ &\quad - \frac{\partial^2 v_2}{\partial x_1 \partial x_3} + \frac{\partial^2 v_1}{\partial x_2 \partial x_3} - \frac{\partial^2 v_1}{\partial x_3 \partial x_2} \\ 0 &= 0 \end{aligned}$$

b)

$$\begin{aligned}\vec{\nabla} \times (\vec{\nabla} \times \vec{v}) &= \vec{\nabla}(\vec{\nabla} \cdot \vec{v} - \nabla^2 \vec{v}) \\ \epsilon_{ijk} \frac{\partial}{\partial x_j} \epsilon_{klm} \frac{\partial v_m}{\partial x_l} &= \vec{\nabla}(\vec{\nabla} \cdot \vec{v} - \nabla^2 \vec{v}) \\ \epsilon_{ijk} \epsilon_{klm} \frac{\partial}{\partial x_j} \frac{\partial v_m}{\partial x_l} &= \vec{\nabla}(\vec{\nabla} \cdot \vec{v} - \nabla^2 \vec{v})\end{aligned}$$

iterate over i, j, k, l, m

the solution for $i = 1$

$$\begin{aligned}\epsilon_{123}\epsilon_{321} \frac{\partial}{\partial x_2} \frac{\partial v_1}{\partial x_2} + \epsilon_{123}\epsilon_{312} \frac{\partial}{\partial x_2} \frac{\partial v_2}{\partial x_1} + \epsilon_{132}\epsilon_{231} \frac{\partial}{\partial x_3} \frac{\partial v_1}{\partial x_3} + \epsilon_{132}\epsilon_{213} \frac{\partial}{\partial x_3} \frac{\partial v_3}{\partial x_1} &= [\vec{\nabla}(\vec{\nabla} \cdot \vec{v} - \nabla^2 \vec{v})]_i \\ (+1)(-1) \frac{\partial}{\partial x_2} \frac{\partial v_1}{\partial x_2} + (+1)(+1) \frac{\partial}{\partial x_2} \frac{\partial v_2}{\partial x_1} + (-1)(+1) \frac{\partial}{\partial x_3} \frac{\partial v_1}{\partial x_3} + (-1)(-1) \frac{\partial}{\partial x_3} \frac{\partial v_3}{\partial x_1} &= [\vec{\nabla}(\vec{\nabla} \cdot \vec{v} - \nabla^2 \vec{v})]_i \\ -\frac{\partial}{\partial x_2} \frac{\partial v_1}{\partial x_2} + \frac{\partial}{\partial x_2} \frac{\partial v_2}{\partial x_1} - \frac{\partial}{\partial x_3} \frac{\partial v_1}{\partial x_3} + \frac{\partial}{\partial x_3} \frac{\partial v_3}{\partial x_1} &= [\vec{\nabla}(\vec{\nabla} \cdot \vec{v} - \nabla^2 \vec{v})]_i \\ \frac{\partial}{\partial x_2} \frac{\partial v_2}{\partial x_1} + \frac{\partial}{\partial x_3} \frac{\partial v_3}{\partial x_1} - \frac{\partial^2 v_1}{\partial x_2^2} - \frac{\partial^2 v_1}{\partial x_3^2} &= [\vec{\nabla}(\vec{\nabla} \cdot \vec{v} - \nabla^2 \vec{v})]_i\end{aligned}$$

The same procedure can be done for j and k

$$\begin{aligned}\frac{\partial}{\partial x_1} \frac{\partial v_2}{\partial x_2} + \frac{\partial}{\partial x_3} \frac{\partial v_3}{\partial x_2} - \frac{\partial^2 v_2}{\partial x_1^2} - \frac{\partial^2 v_2}{\partial x_3^2} &= [\vec{\nabla}(\vec{\nabla} \cdot \vec{v} - \nabla^2 \vec{v})]_j \\ \frac{\partial}{\partial x_1} \frac{\partial v_1}{\partial x_3} + \frac{\partial}{\partial x_1} \frac{\partial v_3}{\partial x_2} - \frac{\partial^2 v_3}{\partial x_2^2} - \frac{\partial^2 v_3}{\partial x_1^2} &= [\vec{\nabla}(\vec{\nabla} \cdot \vec{v} - \nabla^2 \vec{v})]_k\end{aligned}$$

Question 2

a)

$$\begin{aligned} &= \frac{\partial}{\partial v_i}(v_i) \\ &= \frac{\partial}{\partial v_x}(v_x) + \frac{\partial}{\partial v_y}(v_y) + \frac{\partial}{\partial v_z}(v_z) \\ &= 1 + 1 + 1 \\ &= 3 \end{aligned}$$

b)

$$\begin{aligned} &= \frac{\partial}{\partial v_i}(v_j) \\ &= \frac{\partial}{\partial v_x}(v_y) + \frac{\partial}{\partial v_x}(v_z) + \frac{\partial}{\partial v_y}(v_x) + \frac{\partial}{\partial v_y}(v_z) + \frac{\partial}{\partial v_z}(v_x) + \frac{\partial}{\partial v_z}(v_y) \\ &\quad + \frac{\partial}{\partial v_x}(v_x) + \frac{\partial}{\partial v_y}(v_y) + \frac{\partial}{\partial v_z}(v_z) \\ &= \delta_{ij} \end{aligned}$$

c)

$$\begin{aligned}
&= \frac{\partial}{\partial v_i} (v_i v_j) \\
&= \frac{\partial}{\partial v_i} (v_{ij}) \\
&= \frac{\partial}{\partial v_x} (v_{xx}) + \frac{\partial}{\partial v_x} (v_{xy}) + \frac{\partial}{\partial v_x} (v_{xz}) \\
&\quad + \frac{\partial}{\partial v_y} (v_{yx}) + \frac{\partial}{\partial v_y} (v_{yy}) + \frac{\partial}{\partial v_y} (v_{yz}) \\
&\quad + \frac{\partial}{\partial v_z} (v_{zx}) + \frac{\partial}{\partial v_z} (v_{zy}) + \frac{\partial}{\partial v_z} (v_{zz}) \\
&= 2v_x + v_y + v_z \\
&\quad + v_x + 2v_y + v_z \\
&\quad + v_x + v_y + 2v_z \\
&= 4v_i
\end{aligned}$$

d)

$$\begin{aligned}
&= \frac{\partial}{\partial v_i} \frac{\partial}{\partial v_i} (v_k v_k) \\
&= \frac{\partial v_x^2}{\partial^2 v_x} v_x^2 + \frac{\partial v_x^2}{\partial^2 v_x} v_y^2 + \frac{\partial v_x^2}{\partial^2 v_x} v_z^2 \\
&\quad + \frac{\partial v_y^2}{\partial^2 v_x} v_x^2 + \frac{\partial v_y^2}{\partial^2 v_x} v_y^2 + \frac{\partial v_y^2}{\partial^2 v_x} v_z^2 \\
&\quad + \frac{\partial v_z^2}{\partial^2 v_x} v_x^2 + \frac{\partial v_z^2}{\partial^2 v_x} v_y^2 + \frac{\partial v_z^2}{\partial^2 v_x} v_z^2 \\
&= 2 + 0 + 0 \\
&\quad + 0 + 2 + 0 \\
&\quad + 0 + 0 + 2
\end{aligned}$$

$$= \frac{\partial}{\partial v_i} (v_j v_k^2)$$

for $i = x$

$$\begin{aligned}
&= \frac{\partial}{\partial v_x}(v_x v_x^2) + \frac{\partial}{\partial v_x}(v_x v_y^2) + \frac{\partial}{\partial v_x}(v_x v_z^2) \\
&+ \frac{\partial}{\partial v_x}(v_y v_x^2) + \frac{\partial}{\partial v_x}(v_y v_y^2) + \frac{\partial}{\partial v_x}(v_y v_z^2) \\
&= \frac{\partial}{\partial v_x}(v_z v_x^2) + \frac{\partial}{\partial v_x}(v_z v_y^2) + \frac{\partial}{\partial v_x}(v_z v_z^2) \\
&= 3v_x^2 + v_y^2 + v_z^2 \\
&+ 2v_y v_x + 0 + 0 \\
&+ 2v_z v_x + 0 + 0
\end{aligned}$$

for $i = y$

$$\begin{aligned}
&= \frac{\partial}{\partial v_y}(v_x v_x^2) + \frac{\partial}{\partial v_y}(v_x v_y^2) + \frac{\partial}{\partial v_y}(v_x v_z^2) \\
&+ \frac{\partial}{\partial v_y}(v_y v_x^2) + \frac{\partial}{\partial v_y}(v_y v_y^2) + \frac{\partial}{\partial v_y}(v_y v_z^2) \\
&= \frac{\partial}{\partial v_y}(v_z v_x^2) + \frac{\partial}{\partial v_y}(v_z v_y^2) + \frac{\partial}{\partial v_y}(v_z v_z^2) \\
&= 0 + 2v_x v_y + 0 \\
&+ v_x^2 + 3v_y^2 + v_z^2 \\
&+ 0 + 2v_z v_y + 0
\end{aligned}$$

for $i = z$

$$\begin{aligned}
&= \frac{\partial}{\partial v_z}(v_x v_x^2) + \frac{\partial}{\partial v_z}(v_x v_y^2) + \frac{\partial}{\partial v_z}(v_x v_z^2) \\
&+ \frac{\partial}{\partial v_z}(v_y v_x^2) + \frac{\partial}{\partial v_z}(v_y v_y^2) + \frac{\partial}{\partial v_z}(v_y v_z^2) \\
&= \frac{\partial}{\partial v_z}(v_z v_x^2) + \frac{\partial}{\partial v_z}(v_z v_y^2) + \frac{\partial}{\partial v_z}(v_z v_z^2) \\
&= 0 + 0 + 2v_x v_z \\
&+ 0 + 0 + 2v_y v_z \\
&+ v_x^2 + v_y^2 + 3v_z^2
\end{aligned}$$

$$\begin{aligned}
&= \begin{bmatrix} 5v_{xx} & 0 & 2v_{xz} \\ 0 & 5v_{yy} & 0 \\ 0 & 0 & 5v_{zz} \end{bmatrix} \\
&= 5\delta_{ii} + \dots
\end{aligned}$$

Question 3

$$\frac{\partial E_i}{\partial x_i} = \frac{\rho}{\epsilon_0}$$

$$\frac{\partial B_i}{\partial x_i} = 0$$

$$\epsilon_{ijk} \frac{\partial E_k}{\partial x_j} = \frac{\partial B_i}{\partial t}$$

$$\epsilon_{ijk} \frac{\partial B_k}{\partial x_j} = \mu_0 \left(J_i + \epsilon_0 \frac{\partial E_i}{\partial t} \right)$$

Question 4

a)