University of Ottawa

MCG 5141, Assignment I

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Question 1

a)

$$
\begin{aligned}
0 &= \vec{\nabla} \cdot (\vec{\nabla} \times \vec{v}) \\
0 &= \vec{\nabla} \cdot (\epsilon_{ijk} \frac{\partial v_k}{\partial x_j}) \\
0 &= \frac{\partial}{\partial x_i} \cdot (\epsilon_{ijk} \frac{\partial v_k}{\partial x_j}) \\
0 &= \epsilon_{ijk} \frac{\partial^2 v_k}{\partial x_i \partial x_j} \\
0 &= \epsilon_{123} \frac{\partial^2 v_3}{\partial x_1 \partial x_2} + \epsilon_{213} \frac{\partial^2 v_3}{\partial x_2 \partial x_1} + \epsilon_{312} \frac{\partial^2 v_2}{\partial x_3 \partial x_1} \\
&\quad + \epsilon_{132} \frac{\partial^2 v_2}{\partial x_1 \partial x_3} + \epsilon_{231} \frac{\partial^2 v_1}{\partial x_2 \partial x_3} + \epsilon_{321} \frac{\partial^2 v_1}{\partial x_3 \partial x_2} \\
0 &= \frac{\partial^2 v_3}{\partial x_1 \partial x_2} - \frac{\partial^2 v_3}{\partial x_2 \partial x_1} + \frac{\partial^2 v_2}{\partial x_3 \partial x_1} \\
&\quad - \frac{\partial^2 v_2}{\partial x_1 \partial x_3} + \frac{\partial^2 v_1}{\partial x_2 \partial x_3} - \frac{\partial^2 v_1}{\partial x_3 \partial x_2} \\
0 &= 0
\end{aligned}
$$

b)

$$\begin{aligned}
\vec{\nabla} \times (\vec{\nabla} \times \vec{v}) &= \vec{\nabla}(\vec{\nabla} \cdot \vec{v} - \nabla^2 \vec{v}) \\
\epsilon_{ijk} \frac{\partial}{\partial x_j} \epsilon_{klm} \frac{\partial v_m}{\partial x_l} &= \vec{\nabla}(\vec{\nabla} \cdot \vec{v} - \nabla^2 \vec{v} \\
\epsilon_{ijk}\epsilon_{klm} \frac{\partial}{\partial x_j} \frac{\partial v_m}{\partial x_l} &= \vec{\nabla}(\vec{\nabla} \cdot \vec{v} - \nabla^2 \vec{v}
\end{aligned}$$

iterate over i, j, k, l, m
the solution for $i = 1$

$$\begin{aligned}
\epsilon_{123}\epsilon_{321} \frac{\partial}{\partial x_2} \frac{\partial v_1}{\partial x_2} + \epsilon_{123}\epsilon_{312} \frac{\partial}{\partial x_2} \frac{\partial v_2}{\partial x_1} + \epsilon_{132}\epsilon_{231} \frac{\partial}{\partial x_3} \frac{\partial v_1}{\partial x_3} + \epsilon_{132}\epsilon_{213} \frac{\partial}{\partial x_3} \frac{\partial v_3}{\partial x_1} &= \left[\vec{\nabla}(\vec{\nabla} \cdot \vec{v} - \nabla^2 \vec{v})\right]_i \\
(+1)(-1) \frac{\partial}{\partial x_2} \frac{\partial v_1}{\partial x_2} + (+1)(+1) \frac{\partial}{\partial x_2} \frac{\partial v_2}{\partial x_1} + (-1)(+1) \frac{\partial}{\partial x_3} \frac{\partial v_1}{\partial x_3} + (-1)(-1) \frac{\partial}{\partial x_3} \frac{\partial v_3}{\partial x_1} &= \left[\vec{\nabla}(\vec{\nabla} \cdot \vec{v} - \nabla^2 \vec{v})\right]_i \\
-\frac{\partial}{\partial x_2} \frac{\partial v_1}{\partial x_2} + \frac{\partial}{\partial x_2} \frac{\partial v_2}{\partial x_1} - \frac{\partial}{\partial x_3} \frac{\partial v_1}{\partial x_3} + \frac{\partial}{\partial x_3} \frac{\partial v_3}{\partial x_1} &= \left[\vec{\nabla}(\vec{\nabla} \cdot \vec{v} - \nabla^2 \vec{v})\right]_i \\
\frac{\partial}{\partial x_2} \frac{\partial v_2}{\partial x_1} + \frac{\partial}{\partial x_3} \frac{\partial v_3}{\partial x_1} - \frac{\partial^2 v_1}{\partial x_2^2} - \frac{\partial^2 v_1}{\partial x_3^2} &= \left[\vec{\nabla}(\vec{\nabla} \cdot \vec{v} - \nabla^2 \vec{v})\right]_i
\end{aligned}$$

The same procedure can be done for j and k

$$\begin{aligned}
\frac{\partial}{\partial x_1} \frac{\partial v_2}{\partial x_2} + \frac{\partial}{\partial x_3} \frac{\partial v_3}{\partial x_2} - \frac{\partial^2 v_2}{\partial x_1^2} - \frac{\partial^2 v_2}{\partial x_3^2} &= \left[\vec{\nabla}(\vec{\nabla} \cdot \vec{v} - \nabla^2 \vec{v})\right]_j \\
\frac{\partial}{\partial x_1} \frac{\partial v_1}{\partial x_3} + \frac{\partial}{\partial x_1} \frac{\partial v_3}{\partial x_2} - \frac{\partial^2 v_3}{\partial x_2^2} - \frac{\partial^2 v_3}{\partial x_1^2} &= \left[\vec{\nabla}(\vec{\nabla} \cdot \vec{v} - \nabla^2 \vec{v})\right]_k
\end{aligned}$$

By observing each component of the solution we see a pattern where the i component $-\left(\frac{\partial^2 v_1}{\partial x_2^2} + \frac{\partial^2 v_1}{\partial x_3^2}\right)$ is the i term in $-\nabla^2 \vec{v}$. Furthermore the inital term $\vec{\nabla}\vec{\nabla} \cdot \vec{v}$

Question 2

a)

$$
\begin{aligned}
&= \frac{\partial}{\partial v_i}(v_i) \\
&= \frac{\partial}{\partial v_x}(v_x) + \frac{\partial}{\partial v_y}(v_y) + \frac{\partial}{\partial v_z}(v_z) \\
&= 1 + 1 + 1 \\
&= 3
\end{aligned}
$$

b)

$$
\begin{aligned}
&= \frac{\partial}{\partial v_i}(v_j) \\
&= \frac{\partial}{\partial v_x}(v_y) + \frac{\partial}{\partial v_x}(v_z) + \frac{\partial}{\partial v_y}(v_x) + \frac{\partial}{\partial v_y}(v_z) + \frac{\partial}{\partial v_z}(v_x) + \frac{\partial}{\partial v_z}(v_y) \\
&+ \frac{\partial}{\partial v_x}(v_x) + \frac{\partial}{\partial v_y}(v_y) + \frac{\partial}{\partial v_z}(v_z) \\
&= \delta_{ij}
\end{aligned}
$$

c)

$$
\begin{aligned}
&= \frac{\partial}{\partial v_i}(v_i v_j) \\
&= \frac{\partial}{\partial v_i}(v_{ij}) \\
&= \frac{\partial}{\partial v_x}(v_{xx}) + \frac{\partial}{\partial v_x}(v_{xy}) + \frac{\partial}{\partial v_x}(v_{xz}) \\
&+ \frac{\partial}{\partial v_y}(v_{yx}) + \frac{\partial}{\partial v_y}(v_{yy}) + \frac{\partial}{\partial v_y}(v_{yz}) \\
&+ \frac{\partial}{\partial v_z}(v_{zx}) + \frac{\partial}{\partial v_z}(v_{zy}) + \frac{\partial}{\partial v_z}(v_{zz}) \\
&= 2v_x + v_y + v_z \\
&+ v_x + 2v_y + v_z \\
&+ v_x + v_y + 2v_z \\
&= 4v_i
\end{aligned}
$$

d)

$$
\begin{aligned}
&= \frac{\partial}{\partial v_i}\frac{\partial}{\partial v_i}(v_k v_k) \\
&= \frac{\partial v_x^2}{\partial^2 v_x} v_x^2 + \frac{\partial v_x^2}{\partial^2 v_x} v_y^2 + \frac{\partial v_x^2}{\partial^2 v_x} v_z^2 \\
&+ \frac{\partial v_y^2}{\partial^2 v_x} v_x^2 + \frac{\partial v_y^2}{\partial^2 v_x} v_y^2 + \frac{\partial v_y^2}{\partial^2 v_x} v_z^2 \\
&+ \frac{\partial v_z^2}{\partial^2 v_x} v_x^2 + \frac{\partial v_z^2}{\partial^2 v_x} v_y^2 + \frac{\partial v_z^2}{\partial^2 v_x} v_z^2 \\
&= 2 + 0 + 0 \\
&+ 0 + 2 + 0 \\
&+ 0 + 0 + 2 \\
&= 6
\end{aligned}
$$

e)

$$=\frac{\partial}{\partial v_i}(v_j v_k^2)$$

for $i = x$

$$\begin{aligned}
=&\frac{\partial}{\partial v_x}(v_x v_x^2) + \frac{\partial}{\partial v_x}(v_x v_y^2) + \frac{\partial}{\partial v_x}(v_x v_z^2) \\
+&\frac{\partial}{\partial v_x}(v_y v_x^2) + \frac{\partial}{\partial v_x}(v_y v_y^2) + \frac{\partial}{\partial v_x}(v_y v_z^2) \\
=&\frac{\partial}{\partial v_x}(v_z v_x^2) + \frac{\partial}{\partial v_x}(v_z v_y^2) + \frac{\partial}{\partial v_x}(v_z v_z^2) \\
=&3v_x^2 + v_y^2 + v_z^2 \\
+&2v_y v_x + 0 + 0 \\
+&2v_z v_x + 0 + 0
\end{aligned}$$

for $i = y$

$$\begin{aligned}
=&\frac{\partial}{\partial v_y}(v_x v_x^2) + \frac{\partial}{\partial v_y}(v_x v_y^2) + \frac{\partial}{\partial v_y}(v_x v_z^2) \\
+&\frac{\partial}{\partial v_y}(v_y v_x^2) + \frac{\partial}{\partial v_y}(v_y v_y^2) + \frac{\partial}{\partial v_y}(v_y v_z^2) \\
=&\frac{\partial}{\partial v_y}(v_z v_x^2) + \frac{\partial}{\partial v_y}(v_z v_y^2) + \frac{\partial}{\partial v_y}(v_z v_z^2) \\
=&0 + 2v_x v_y + 0 \\
+&v_x^2 + 3v_y^2 + v_z \\
+&0 + 2v_z v_y + 0
\end{aligned}$$

for $i = z$

$$\begin{aligned}
=&\frac{\partial}{\partial v_z}(v_x v_x^2) + \frac{\partial}{\partial v_z}(v_x v_y^2) + \frac{\partial}{\partial v_z}(v_x v_z^2) \\
+&\frac{\partial}{\partial v_z}(v_y v_x^2) + \frac{\partial}{\partial v_z}(v_y v_y^2) + \frac{\partial}{\partial v_z}(v_y v_z^2) \\
=&\frac{\partial}{\partial v_z}(v_z v_x^2) + \frac{\partial}{\partial v_z}(v_z v_y^2) + \frac{\partial}{\partial v_z}(v_z v_z^2) \\
=&0 + 0 + 2v_x v_z \\
+&0 + 0 + 2v_y v_z \\
+&v_x^2 + v_y^2 + 3v_z^2
\end{aligned}$$

$$= 5\delta_{ij} + v_{ij}$$

Question 3

$$\frac{\partial E_i}{\partial x_i} = \frac{\rho}{\epsilon_0}$$

$$\frac{\partial B_i}{\partial x_i} = 0$$

$$\epsilon_{ijk}\frac{\partial E_k}{\partial x_j} = \frac{\partial B_i}{\partial t}$$

$$\epsilon_{ijk}\frac{\partial B_k}{\partial x_j} = \mu_0(J_i + \epsilon_0\frac{\partial E_i}{\partial t})$$