Université d'Ottawa
Département de
génie mécanique

uOttawa

University of Ottawa
Department of
mechanical engineering

Assignment 1

Kinetic Theory of Gases, Winter, 2020

---

1) Confirm the following identities using tensor notation

   a) $\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{v}\right)=0$

   b) $\vec{\nabla}\times\left(\vec{\nabla}\times\vec{v}\right)=\vec{\nabla}\left(\vec{\nabla}\cdot\vec{v}\right)-\nabla^2\vec{v}$

2) Evaluate the following derivatives using tensor notation

   a) $\frac{\partial}{\partial v_i}(v_i)$

   b) $\frac{\partial}{\partial v_j}(v_i)$

   c) $\frac{\partial}{\partial v_i}\left(v_i v_j\right)$

   d) $\frac{\partial}{\partial v_i}\frac{\partial}{\partial v_i}(v_k v_k)$

   e) $\frac{\partial}{\partial v_i}\left(v_j v_k v_k\right)$

3) In vector notation, Maxwell's equations of electro-magnetism are

$$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\epsilon_0}, \qquad \vec{\nabla}\cdot\vec{B}=0, \qquad \vec{\nabla}\times\vec{E}=\frac{\partial\vec{B}}{\partial t}, \qquad \vec{\nabla}\times\vec{B}=\mu_0\left(\vec{J}+\epsilon_0\frac{\partial\vec{E}}{\partial t}\right).$$

   Here, $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields, ρ and $\vec{J}$ are the charge density and current density, while μ_0 and ϵ_0 are the magnetic permeability and electric permittivity of a vacuum. Write these four equations in tensor notation.

4) A gas in local equilibrium with bulk velocity, u_i, has the distribution function

$$\mathcal{F}=\frac{\rho}{m}\left(\frac{\rho}{2\pi p}\right)^{\frac{3}{2}}\exp\left[-\frac{\rho}{2p}(v_i-u_i)(v_i-u_i)\right].$$

   Evaluate the following moments of this distribution function

   a) $\langle m v_x \mathcal{F}\rangle$

   b) $\left\langle m v_x^2 \mathcal{F}\right\rangle$

   c) $\left\langle m v_y v_z \mathcal{F}\right\rangle$

   d) $\langle m v_x v_i v_i \mathcal{F}\rangle$

   e) $\left\langle m v_i v_i v_j v_j \mathcal{F}\right\rangle$

   Here, the Einstein summation convention should be applied to general indices (i, j, k, etc.), but not specific Cartesian directions (x, y, z).

5) Consider a gas that is outside local thermodynamic equilibrium with mass density ρ. The particles of this gas have their velocity vectors uniformly distributed in a sphere of velocity space centred on the origin with radius Φ. In other words, no particles have a speed greater than Φ and all velocities with magnitudes less than Φ are equally likely. Evaluate the following moments of this distribution

   a) $\langle m v_x \mathcal{F} \rangle$
   b) $\left\langle m v_x^2 \mathcal{F} \right\rangle$
   c) $\left\langle m v_x v_y \mathcal{F} \right\rangle$
   d) $\langle m v_i v_i \mathcal{F} \rangle$
   e) $\left\langle m v_i v_j v_j \mathcal{F} \right\rangle$
   f) $\left\langle m v_x^2 v_j v_j \mathcal{F} \right\rangle$
   g) $\left\langle m v_i v_i v_j v_j \mathcal{F} \right\rangle$

   Here, the Einstein summation convention should be applied to general indices (i, j, k, etc.), but not specific Cartesian directions (x, y, z).

6) Small particles or droplets, present in multiphase flow, can also be described by a Boltzmann-like kinetic equation,

$$\frac{\partial \mathcal{F}}{\partial t} + v_i \frac{\partial \mathcal{F}}{\partial x_i} + \frac{\partial a_i \mathcal{F}}{\partial v_i} = \frac{\delta \mathcal{F}}{\delta t}.$$

   Assume that the particle concentration is dilute enough that particle collisions can be neglected and $\frac{\delta \mathcal{F}}{\delta t} = 0$. For spherical particles at very low Reynolds number, the acceleration due to drag is given by

$$a_i = \frac{V_i - v_i}{\tau},$$

   where V_i is the velocity of the background fluid and τ is the relaxation time scale for the drag force.

   Assume the distribution of particle velocities is everywhere described by a Maxwell-Boltzmann distribution,

$$\mathcal{F} = \mathcal{M} = \frac{\rho}{m} \left(\frac{\rho}{2\pi p} \right)^{\frac{3}{2}} exp\left[-\frac{\rho}{2p} (v_i - u_i)(v_i - u_i) \right].$$

   Here, ρ, is the local particle-mass density, u_i is their local average velocity, and p can be interpreted as a "pressure" of the particles or simply as a measure of the local spread of velocities.

   Using Maxwell's equation of change with the weights, derive balance laws for the evolution of the particle mass density, momentum density, and energy density. To do this, use the weights: m, $m v_i$, and $\frac{1}{2} m v_i v_i$.

Helpful Integrals

$$\int_0^\infty e^{-\beta x^2}\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{\beta}}$$

$$\int_0^\infty x\,e^{-\beta x^2}\,\mathrm{d}x = \frac{1}{2\beta}$$

$$\int_0^\infty x^2\,e^{-\beta x^2}\,\mathrm{d}x = \frac{1}{4\beta}\sqrt{\frac{\pi}{\beta}}$$

$$\int_0^\infty x^3\,e^{-\beta x^2}\,\mathrm{d}x = \frac{1}{2\beta^2}$$

$$\int_0^\infty x^4\,e^{-\beta x^2}\,\mathrm{d}x = \frac{3}{8\beta^2}\sqrt{\frac{\pi}{\beta}}$$

$$\int_0^\infty x^5\,e^{-\beta x^2}\,\mathrm{d}x = \frac{1}{\beta^3}$$

$$\int_0^\infty x^6\,e^{-\beta x^2}\,\mathrm{d}x = \frac{15}{16\beta^3}\sqrt{\frac{\pi}{\beta}}$$

$$\int_0^\infty x^7\,e^{-\beta x^2}\,\mathrm{d}x = \frac{3}{\beta^4}$$