Fractal Analysis

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Chapter 1

Box-counting Dimension

1.1 Covers

Let F be a subset of $\mathbb{R}^n$. We define the following covers:

$$\mathcal{B}_\delta(F) := \{B = \{B(x_j, \delta/2)\}_{j=1}^{n_B} | F \subseteq \cup B\}$$
$$\mathcal{D}_\delta(F) := \{D = \{D_j\}_{j=1}^{n_D} | F \subseteq \cup D \wedge \operatorname{diam}(D_j) \leq \delta \forall D_j \in D\}.$$

We now define the following counts

$$N_\delta^B(F) := \min_{B \in \mathcal{B}_\delta(F)} \operatorname{card}(B)$$
$$N_\delta^D(F) := \min_{D \in \mathcal{D}_\delta(F)} \operatorname{card}(D).$$

Theorem 1.1.1.

$$N_\delta^B(F) = N_\delta^D(F)$$

Proof. Let B be a cover in $\mathcal{B}_\delta(F)$ such that $\operatorname{card}(B) = N_\delta^B(F)$. Likewise let D be a cover in $\mathcal{D}_\delta(F)$ such that $\operatorname{card}(D) = N_\delta^D(F)$. Since each set in B has diameter δ $B \in \mathcal{D}_\delta(F)$, and by minimality, $N_\delta^D(F) \leq N_\delta^B(F)$. Since each set in D has diameter no more than δ, we can encapsulate each set in D with a single ball of radius $\delta/2$, call this new cover $B(D)$. By construction, $D(B) \in \mathcal{B}_\delta(F)$ and $\operatorname{card}(D(B)) = N_\delta^D(F)$. Again, by minimality, $N_\delta^B(F) \leq N_\delta^D(F)$ and $N_\delta^B(F) = N_\delta^D(F)$. □

1.2 Box-Counting Dimension

Definition 1.2.1. We say the *lower box-counting dimension* of $F \subset \mathbb{R}^n$ is

$$\underline{\dim}_B(F) := \liminf_{\delta \to 0} \frac{\log N_\delta^B(F)}{-\log \delta}$$

and the *upper box-counting dimension* is

$$\overline{\dim}_B(F) := \limsup_{\delta\to 0} \frac{\log N_\delta^B(F)}{-\log \delta}.$$

If $\underline{\dim}_B(F) = \overline{\dim}_B(F)$, then we say the *box-counting dimension* of F is

$$\dim_B(F) := \lim_{\delta\to 0} \frac{\log N_\delta^B(F)}{-\log \delta}.$$

Theorem 1.2.1. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be Lipschitz on $F \subset \mathbb{R}^n$. Then $\underline{\dim}_B(f(F)) \le \underline{\dim}_B(F)$ and $\overline{\dim}_B(f(F)) \le \overline{\dim}_B(F)$. If f is bi-Lipschitz on F, then $\underline{\dim}_B(f(F)) = \underline{\dim}_B(F)$ and $\overline{\dim}_B(f(F)) = \overline{\dim}_B(F)$.*

Proof. Let $\delta > 0$ be given. Let $\{U_i\}_{i=1}^k$ be a minimal δ-cover of F. Thus, $\{U_i \cap F\}$ is also a minimal δ-cover of F. Note that for all $x, y \in U_i \cap F$, $|x - y| \le \delta$. We also know that $\{f(U_i \cap F)\}$ is a $c\delta$-cover since $|f(x) - f(y)| \le c|x - y| \le c\delta$. However, this cover is not necessarily minimal and thus

$$N_{c\delta}(f(F)) \le N_\delta(F).$$

Ergo,

$$\frac{\ln N_{c\delta}(f(F))}{-\ln(c\delta) + \ln\delta} \le \frac{\ln N_\delta(F)}{-\ln\delta}$$

and $\underline{\dim}_B(f(F)) \le \underline{\dim}_B(F)$ and $\overline{\dim}_B(f(F)) \le \overline{\dim}_B(F)$.

If f is bi-Lipschitz, then f^{-1} is Lipschitz on $f^{-1}(F)$. Thus $\underline{\dim}_B(f^{-1}(f(F))) \le \underline{\dim}_B(f(F))$ and $\overline{\dim}_B(f^{-1}(f(F))) \le \overline{\dim}_B(f(F))$. Therefore, $\underline{\dim}_B(f(F)) = \underline{\dim}_B(F)$ and $\overline{\dim}_B(f(F)) = \overline{\dim}_B(F)$. □

Theorem 1.2.2. *Suppose f satisfies the Hölder condition, that is for some $c, \alpha > 0$*

$$|f(x) - f(y)| \le c|x - y|^\alpha.$$

Then $\underline{\dim}_B(f(F)) \le (1/\alpha)\underline{\dim}_B(F)$ and $\overline{\dim}_B(f(F)) \le (1/\alpha)\overline{\dim}_B(F)$.

Proof. Let $\delta > 0$ be given. Let $\{U_i\}_{i=1}^k$ be a minimal δ-cover of F. Thus, $\{U_i \cap F\}$ is also a minimal δ-cover of F. Note that for all $x, y \in U_i \cap F$, $|x - y| \le \delta$. We also know that $\{f(U_i \cap F)\}$ is a $c\delta^\alpha$-cover since $|f(x) - f(y)| \le c|x - y|^\alpha \le c\delta^\alpha$. However, this cover is not necessarily minimal and thus

$$N_{c\delta^\alpha}(f(F)) \le N_\delta(F).$$

Ergo,

$$\frac{\ln N_{c\delta^\alpha}(f(F))}{-\ln(c\delta^\alpha) + \ln\delta} \le \frac{\ln N_\delta(F)}{-\alpha\ln\delta}$$

and $\underline{\dim}_B(f(F)) \le (1/\alpha)\underline{\dim}_B(F)$ and $\overline{\dim}_B(f(F)) \le (1/\alpha)\overline{\dim}_B(F)$. □

1.3 Examples

For our first example, we take the box-counting dimension of one of a canonical fractal, the Cantor Set.

Example 1.3.1. *Let C denote the middle-λ Cantor set for $0 < \lambda < 1$.*

$$\dim_B(C) = \frac{\ln 2}{\ln\left(\frac{2}{1-\lambda}\right)}$$

Proof. To begin, we must determine the length of each interval of C_n after we remove the middle $1/\lambda$ from each of them. Assume we are removing the middle λ of the interval $[0, a]$ where a is a positive real number. Since we removed $a\lambda$ from $[0, a]$, we have exactly $a - a\lambda = a(1-\lambda)$ length remaining. Since we removed the middle a/λ, we equally distribute the remaining length into two subintervals of length $a(1-\lambda)/2$.

Let l_n represent the length of each interval in C_n and define $l_0 = 1$. Thus, $l_1 = (1 - \lambda)/2$ and $l_{n+1} = l_n(1-\lambda)/2$ by our previous derivation. Solving this recursion yields $l_n = [(1-\lambda)/2]^n$.

Let $\delta > 0$ be given. Choose n such that $l_{n+1} \leq \delta < l_n$. Since $\delta \geq l_{n+1}$, we need no more 2^{n+1} sets of diameter δ to cover C because there are exactly 2^{n+1} intervals of length l_{n+1} in C_{n+1}. Furthermore, since $\delta < l_n$, we need at least 2^n sets of diameter δ to cover C because there are exactly 2^n intervals of length l_n is C_n. Thus yielding the inequality

$$2^n \leq N_\delta^D(C) \leq 2^{n+1}.$$

Taking logs and doing some manipulation yields

$$\frac{n \ln 2}{(n+1)\ln\left(\frac{2}{1-\lambda}\right)} \leq \frac{\ln N_\delta^D(C)}{-\ln \delta} \leq \frac{(n+1)\ln 2}{n \ln\left(\frac{2}{1-\lambda}\right)}.$$

Using L'Hôpital's rule yields

$$\dim_B(C) = \frac{\ln 2}{\ln\left(\frac{2}{1-\lambda}\right)}.$$

□

We can expand this logic to more general forms of the Cantor set, and present two examples of it here.

Example 1.3.2. *Let F be the set containing all numbers in $[0, 1]$ that do not have a 5 in their decimal expansion. Then,*

$$\dim_B(F) = \frac{\ln 9}{\ln 10}.$$

Proof. We perform a similar construction to the middle-λ Cantor set. Define $F_n = [0, 10^{-n}]$ where n is a non-negative integer. In our construction, we will spilt F_n into ten intervals of length 10^{n+1} and remove the interval $(5 \cdot 10^{n+1}, 6 \cdot 10^{n+1})$, removing any number with a 5 in the $n+1$st decimal place from F_n. We now have nine subintervals left, all of which are copies of F_{n+1}. This construction means that we have 9^n copies of F_n for each n.

Let $\delta > 0$ be given. Choose n such that $10^{-(n+1)} \leq \delta < 10^{-n}$. Since $\delta \geq 10^{-(n+1)}$, we need no more than 9^{n+1} sets of diameter δ to cover F. Since $\delta < 10^{-n}$, we need at least 9^n sets of diameter δ to cover F. Thus we have the following inequality

$$9^n \leq N_\delta^D(F) \leq 9^{n+1}.$$

By taking logs and manipulating the inequality we get,

$$\frac{n \ln 9}{(n+1)\ln 10} \leq \frac{\ln N_\delta^D(F)}{-\ln \delta} \leq \frac{(n+1)\ln 9}{n \ln 10}.$$

Since both the far left and far right tend to $\frac{\ln 9}{\ln 10}$, by Squeeze theorem, we have

$$\dim_B(F) = \frac{\ln 9}{\ln 10}.$$

□

Example 1.3.3. *The Cantor Dust has* $\dim_B = 1$.

Proof. To construct the Cantor Dust, we consider the unit square, call this C_0. Then subdivide C_0 into 16 more squares and keep precisely 4 of them, call them copies of C_1. We iterate this process by splitting C_n into 16 squares and keeping 4 of them, while calling each one C_{n+1}. We should note that the side length of all of the 4^n copies of C_n is $(1/4)^n$.

Let $\delta > 0$ be given. We begin by selecting an n such that $(1/4)^{n+1} \leq \delta < (1/4)^n$. Thus we need no more than 4^{n+1} squares of side length δ and no fewer than 4^n squares of side length δ to cover C. However, since this is a cover of squares of side length δ, we have a cover using balls of diameter $\sqrt{2}\delta$, by Pythagorean theorem. Thus yielding the following inequality

$$4^{n+1} \leq N_{\sqrt{2}\delta}^D \leq 4^n.$$

After taking logs and some algebraic manipulation, we have

$$\frac{(n+1)\ln 4}{n \ln 4} \leq \frac{\ln N_{\sqrt{2}\delta}^D}{-\ln \delta} \leq \frac{n \ln 4}{(n+1)\ln 4}.$$

After we take the limit of both sides we get, $\dim_B(C) = 1$. □

Furthermore, we can take the box-counting dimension of countable sets as well.

Example 1.3.4. *Let* $S = \{(n!)^{-1}\}_{n=1}^{\infty} \cup \{0\}$, *then*

$$\dim_B(S) = 0.$$

Proof. We begin by computing $l_S(n)$ and $r_S(n)$.

$$l_S(n) = \frac{n}{(n+1)!}$$
$$r_S(n) = \frac{n-1}{n!}$$

We then invoke Lemma 1.4.1 to get

$$\frac{\ln k}{-\ln(k-1)+\ln(k!)} \leq \frac{\ln N_\delta^C(S)}{-\ln\delta} \leq \frac{\ln 2 + \ln k}{-\ln k + \ln((k+1)!)}.$$

Since the limit of both the far right and the far left of the previous inequality is zero, $\dim_B(S) = 0$. □

1.4 Properties of the Box-Counting Dimension

When we speak about dimensions, there are a few properties we want them to have, namely:

1. if $E \subseteq F$, then $\dim E \leq \dim F$;
2. any open set $O \subseteq \mathbb{R}^n$ should have $\dim O = n$;
3. when $F \subseteq \mathbb{R}^n$, $0 \leq \dim F \leq n$;
4. if $\{S_i\}_{i\in I}$ is a countable collection of sets, then $\dim \bigcup_{i\in I} S_i = \sup_{i\in I} \dim S_i$;
5. if S is countable, then $\dim S = 0$.

However, while the Box-counting dimension is nice for our Cantor-like sets and respects our monotonicity, it has a certain undesirable property, that is countable sets can have non-zero dimension.

Lemma 1.4.1. *Let $\{x_n\}_{n=1}^\infty$, be a monotone decreasing sequence in $[0,1]$ that tends to 0 and let $0 < \delta < 1/2$. Let $S = \{x_n\}_{n=1}^\infty \cup \{0\}$. Define*

$$l_S(n) := x_n - x_{n+1}$$
$$r_S(n) := x_{n-1} - x_n.$$

Let $k \in \mathbb{N}$ such that $l_S(k) \leq \delta < r_S(k)$, then

$$k \leq N_\delta^D(S) \leq 2k.$$

and

$$\frac{\ln k}{-\ln r_S(k)} \leq \frac{\ln N_\delta^C}{-\ln\delta} \leq \frac{\ln(2k)}{-\ln l_S(k)}.$$

Proof. If we cover using sets of diameter $r_S(k)$, then we can fit at most one point from $\{x_j\}_{j=1}^k$ in each set. Thus implying that $k \le N_\delta^D(F)$. If we cover using sets of diameter $l_S(k)$, then we can cover $[0, x_k]$ with at most, $k+1$ sets and $\{x_j\}_{j=1}^{k-1}$ with $k-1$ sets. Thus implying that $k \le N_\delta^D(F) \le 2k$, and the logarithm inequality directly follows. $\square$

Example 1.4.2. *Let $j \in \mathbb{R}^+$, then the box-counting dimension of $S_j = \{n^{-j}\}_{n=1}^\infty \cup \{0\}$ is*

$$\dim_B(S_j) = \frac{1}{j+1}.$$

Proof. From the construction of S_j we know that

$$l_{S_j}(k) = \frac{1}{k^j} - \frac{1}{(k+1)^j} = \frac{(k+1)^j - k^j}{k^j(k+1)^j}$$

and

$$r_{S_j}(k) = \frac{1}{(k-1)^j} - \frac{1}{k^j} = \frac{k^j - (k-1)^j}{k^j(k-1)^j}.$$

By Lemma 1.4.1, we know that

$$\frac{\ln k}{-\ln r_{S_j}(k)} \le \frac{\ln N_\delta^C}{-\ln \delta} \le \frac{\ln(2k)}{-\ln l_{S_j}(k)}.$$

Thus, the box-counting dimension is in between the limits of the far left and far right of this inequality. Consider $\ln k/(-\ln r_{S_j}(k))$.

$$\frac{\ln k}{-\ln r_{S_j}(k)} = \frac{\ln k}{-\ln(k^j - (k-1)^j) + j\ln k + j\ln(k+1)}$$

Consider $n^j - (n-1)^j$.

$$n^j - (n-1)^j = n^{j-1}\left(n - (n-1)\left(\frac{n-1}{n}\right)^{j-1}\right) = n^{j-1}\left(n - (n-1)\left(1 - \frac{1}{n}\right)^{j-1}\right)$$

Thus,

$$\frac{\ln k}{-\ln r_{S_j}(k)} = \frac{\ln k}{-\ln(k - (k-1)(1-1/k)^{j-1}) + \ln k + j\ln(k+1)}$$

which tends to $1/(j+1)$. Consider $\ln 2k/(-\ln l_{S_j}(k))$.

$$\frac{\ln 2k}{-\ln l_{S_j}(k)} = \frac{\ln 2k}{-\ln((k+1)^j - k^j) + j\ln k + j\ln(k-1)}$$

Consider $(n+1)^j - n^j$.

$$(n+1)^j - n^j = n^{j-1}\left((n+1)\left(\frac{n+1}{n}\right)^{j-1} - n\right) = n^{j-1}\left((n+1)\left(1 + \frac{1}{n}\right)^{j-1} - n\right)$$

Thus,

$$\frac{\ln 2 + \ln k}{-\ln l_{S_j}(k)} = \frac{\ln 2 + \ln k}{-\ln((k+1)(1+1/k)^{j-1} - k) + \ln k + j\ln(k+1)}$$

which tends to $1/(j+1)$ and by Squeeze theorem, $\dim_B(S_j)$ is $1/(j+1)$. $\square$

Corollary 1.4.3. $\dim_B\{1/n\}_{n=1}^{\infty} = 1/2 \neq 0$

Corollary 1.4.4. *The box-counting dimension is unstable.*

Proof. Proceed via contradiction. Assume that

$$\dim_B \bigcup_{n=1}^{\infty}\{1/n\} = \sup_{n\in\mathbb{N}} \dim_B\{1/n\}. \tag{1.1}$$

We need to find the dimension of a singleton. Since this is a singleton, we can cover it with exactly one set regardless of δ, thus

$$\dim_B\{x\} = \lim_{\delta\to 0}\frac{\ln N_\delta(\{x\})}{-\ln\delta} = \lim_{\delta\to 0}\frac{\ln 1}{-\ln\delta} = 0.$$

Therefore, the righthand side of Equation 1.1 is equal to zero. However, by the Corollary 1.4.3 we know the lefthand side is equal to one half, thus creating a contradiction. $\square$

The intuition behind this result is that $N_\delta(F)$ must count the cardinality of the cover instead of measuring how much it contributes, thus a singleton in the cover contributes the same as an entire interval. In essence, this problem exists because $N_\delta(F)$ is like the counting measure instead of the Lebesgue measure. To solve this problem, we must introduce a way to determine how much each set in the cover contributes to the dimension; that is, we introduce the Hausdorff measure.

Chapter 2

Hausdorff Dimension

After exploring the deficiencies of the box-counting dimension, we need a better way to determine the dimension of the set. In order to do so, we must somehow account for the "size" of each set in our cover. Ideally, we want to encompass the properties of both the counting measure and Lesbesque measure, and thus we introduce the Hausdorff measure.

2.1 The Hausdorff Measure

We begin by defining a δ cover.

Definition 2.1.1. Let F be a subset of a metric space (X, d). Then a δ *cover of* F is $\{U_i\}_{i\in I}$ where I is countable, and $\operatorname{diam} U_i \leq \delta$ for all $i \in I$. We denote the family of all δ-covers of F as $C_\delta(F)$.

We define the s-dimensional Hausdorff-δ thingy as follows.

Definition 2.1.2. Let $F \subseteq \mathbb{R}^n$, and $s, \delta > 0$ then the *s-dimensional Hausdorff-δ thingy of* F is defined as

$$\mathcal{H}^s_\delta = \inf\{\sum_{i\in I} (\operatorname{diam} U_i)^s | \{U_i\}_{i\in I} \text{ a } \delta\text{-cover of } F\}.$$

Note that as δ tends to zero, the collection of covers over which we take the infinimum becomes smaller. This tells us that at the very least, $\mathcal{H}^s_\delta(F)$ is non-increasing and thus will approach a limit as δ tends to zero. In turn, this leads us to the definition of the s-dimensional Hausdorff measure.

Definition 2.1.3. Let $F \subseteq \mathbb{R}^n$ and $s > 0$, then the *s-dimensional Hausdorff measure of* F is defined as

$$\mathcal{H}^s(F) = \lim_{\delta\to 0} \mathcal{H}^s_\delta,$$

Theorem 2.1.1. *For $s > 0$, $\mathcal{H}^s$ is a measure on the Borel σ-algebra of $\mathbb{R}^n$.*

Proof. Let $z \in \mathbb{R}^n$ and $X, Y \subseteq \mathbb{R}^n$. Define $D : \mathbb{R}^n \times \mathcal{P}(\mathbb{R}^n) \to \mathbb{R}^+$ by

$$D(z, X) := \inf_{x \in X} |z - x|$$

and $D : \mathcal{P}(\mathbb{R}^n)^2 \to \mathbb{R}$ by

$$D(X, Y) := \inf_{x \in X,\, y \in Y} |x - y|.$$

To show $\mathcal{H}^s$ is a measure, we need to show that $\mathcal{H}^s(\emptyset) = 0$ and for $A, B \subseteq \mathbb{R}^n$, with $A \cap B = \emptyset$, $\mathcal{H}^s(A \cup B) = \mathcal{H}^s(A) + \mathcal{H}^s(B)$. We begin by showing that $\mathcal{H}^s(\emptyset) = 0$. Note that the minimal δ-cover of $\emptyset$ is an empty cover, thus the sum of its diameters is zero. Ergo $\mathcal{H}^s(\emptyset) = \lim\limits_{\delta \to 0} \mathcal{H}^s_\delta(\emptyset) = \lim\limits_{\delta \to 0} 0 = 0$.

Let $\{F_i\}_{i \in I}$ be a countable collection of subsets of $\mathbb{R}^n$. For each $i \in I$, let $\mathcal{U} \in C_\delta(F_i)$. Then $\mathcal{W} = \cup_{i \in I} \mathcal{U}_\rangle \in C_\delta(\cup_{i \in I} F_i)$ for any combination of covers of the individual sets. Thus,

$$\mathcal{H}^s_\delta = \inf_{\mathcal{W} \in C_\delta(\cup F_i)} \sum_{W \in \mathcal{W}} (\operatorname{diam} W)^s \leq \sum_{i \in I} \inf_{\mathcal{U} \in C_\delta(F_i)} \sum_{U \in \mathcal{U}} (\operatorname{diam} U)^s = \sum_{i \in I} \mathcal{H}^s_\delta(F_i)$$

and by limit laws, $\mathcal{H}^s(\cup F_i) \leq \sum_{i \in I} \mathcal{H}^s(F_i)$. □

It should be noted that if $s = 0$, $\mathcal{H}^s$ is exactly the counting measure, and that if s is a natural number then $\mathcal{H}^s$ is the Lesbesgue measure up to a constant multiple.

Theorem 2.1.2. $\mathcal{H}^0(F) = \operatorname{card} F$

Proof. Suppose F is countable and let $\delta > 0$. Denote $\mathcal{F} = \{\{x\}\}_{x \in F}$, then $\mathcal{F}$ is the minimal δ-cover of F. Thus,

$$\mathcal{H}^0_\delta(F) = \sum_{x \in F} (\operatorname{diam}\{x\})^0 = \operatorname{card} F$$

and $\mathcal{H}(F) = |F|$.

If F is uncountable then, there exists a countably infinite subset of F, which we denote F_c. Since $\mathcal{H}^0$ is a measure, $\mathcal{H}^0(F_c) = \infty \leq \mathcal{H}^0(F)$. Thus, $\mathcal{H}^0(F) = \infty$. □

2.1.1 The Hausdorff and Lesbesgue Measures

To begin, we find the volume of an n-dimensional hyper-sphere of radius r.

The Volume of an n-dimensional Hyper-Sphere of Radius r

Let $V_n(F)$ denote the n-dimensional volume of $F \subseteq \mathbb{R}^n$, a Borel set. Furthermore, $V_n(F) = \mathcal{L}^n(F)$, the n-dimensional Lesbesgue measure of F. Let $B_n(r)$ represent the n-dimensional open ball of radius r. Furthermore, we define $w_n = \int_0^{\pi/2} \cos^n \theta d\theta$ the Wallis integral. Lastly, we have the following recurrence for $V_n(B_n(r))$,

$$V_n(B_n(r)) = 2 \int_0^r V_{n-1}(B_{n-1})(\sqrt{r^2 - \alpha^2}) d\alpha \tag{2.1}$$

Let $r > 0$ be given. Consider $V_1(B_1(r))$.

$$V_1(B_1(r)) = \int_{-r}^{r} dx = 2r = 2r\int_0^{\pi/2} \cos\theta d\theta = 2^1 r^1 w_1$$

Consider $V_2(B_2(r))$.

$$\begin{aligned} V_2(B_2(r)) &= 2\int_0^r V_1(B_1(\sqrt{r^2 - y^2}))dy \\ &= 2\int_0^r 2\sqrt{r^2 - y^2} w_1 dy \\ &= 2^2 w_1 \int_0^r \sqrt{r^2 - y^2} dy \end{aligned}$$

If we let $y = r\sin\theta$ then $dy = r\cos\theta d\theta$, then

$$\begin{aligned} V_2(B_2(r)) &= 2^2 w_1 \int_0^{\pi/2} \sqrt{r^2 - r^2\sin^2\theta} r\cos\theta d\theta \\ &= 2^2 r w_1 \int_0^{\pi/2} r\cos\theta\cos\theta d\theta \\ &= 2^2 r^2 w_1 \int_0^{\pi/2} r\cos^2\theta d\theta \\ &= 2^2 r^2 w_1 w_2. \end{aligned}$$

We can begin to see a pattern unfolding and thus we can conjecture that the following proposition holds.

Proposition 2.1.3. *For $r > 0$, $V_n(B_n(r)) = 2^n r^n \prod_{j=1}^n w_j$.*

Proof. We know that $V_1(B_1(r)) = 2^1 r^1 w_1$. Assume that for all $k \leq n$, $V_k(B_k(r)) = 2^k r^k \prod_{j=1}^k w_j$. Consider $V_{n+1}(B_{n+1}(r))$, then by Equation 2.1

$$\begin{aligned} V_{n+1}(B_{n+1}(r)) &= 2\int_0^r V_n(B_n)(\sqrt{r^2 - \alpha^2})d\alpha \\ &= 2\int_0^r 2^n\left(\sqrt{r^2 - \alpha^2}\right)^n \prod_{j=1}^n w_j d\alpha \\ &= 2^{n+1}\prod_{j=1}^n w_j \int_0^r \left(\sqrt{r^2 - \alpha^2}\right)^n d\alpha. \end{aligned}$$

If we let $\alpha = r\sin\theta$, then $d\alpha = r\cos\theta d\theta$, then

$$\begin{aligned}
V_{n+1}(B_{n+1}(r)) &= 2^{n+1}\prod_{j=1}^{n} w_j \int_0^{\pi/2} \left(\sqrt{r^2 - r^2\sin^2\theta}\right)^n r\cos\theta d\theta \\
&= 2^{n+1} r \prod_{j=1}^{n} w_j \int_0^{\pi/2} r^n \cos^{n+1}\theta d\theta \\
&= 2^{n+1} r^{n+1} \prod_{j=1}^{n+1} w_j.
\end{aligned}$$

□

This form gives us a nicer recurrence relation as well.

Corollary 2.1.4. *For $r > 0$, $V_{n+1}(B_{n+1}(r)) = 2rw_{n+1}V_n(B_n(r))$.*

Corollary 2.1.5. *Define $a_n := V_n(B_n(1))$, then $V_n(B_n(r)) = a_n r^n$.*

Proof. By Proposition 2.1.3, we know that

$$a_n = 2^n \prod_{j=1}^{n} w_j.$$

Thus,

$$V_n(B_n(r)) = \left(2^n \prod_{j=1}^{n} w_j\right) r^n = a_n r^n.$$

□

Furthermore, we may (and by "may" I mean "will") find it useful to know the volume of the ball of diameter one.

Corollary 2.1.6. *If we define $c_n := V_n(B_n(1/2))$ then, $c_n = a_n/2^n$.*

Proof. By the previous corollary

$$c_n = a_n \left(\frac{1}{2}\right)^n = \frac{a_n}{2^n}.$$

□

Moreover, the volume of a ball in terms of its diameter is

$$c_n d^n.$$

Using the form in terms of the Wallis integral, we can find a closed form for the volume of an n-dimensional hyper-sphere of radius $r > 0$.

Theorem 2.1.7.

$$a_n = \begin{cases} \frac{\pi^k}{k!} & n = 2k \\ \frac{2^{2k+1}\pi^k(k!)}{(2k+1)!} & n = 2k+1 \end{cases}.$$

Proof. We know that the Wallis integral equal to

$$w_k = \begin{cases} \frac{(2j)!}{2^{2j}(j!)^2} \cdot \frac{\pi}{2} & k = 2j \\ \frac{2^{2j}(j!)^2}{(2j+1)!} & k = 2j+1 \end{cases}.$$

Consider $n = 2k$. Then by Proposition 2.1.3,

$$\begin{aligned} a_{2k} &= 2^{2k} \prod_{j=1}^{2k} w_j \\ &= 2^{2k} \prod_{j=1}^{k} w_{2j} w_{2j-1} \\ &= 2^{2k} \prod_{j=1}^{k} \frac{\pi}{2} \frac{(2j)!}{2^{2j}(j!)^2} \frac{2^{2j}(j!)^2}{(2j+1)!} \\ &= 2^{2k} \prod_{j=1}^{k} \frac{\pi}{2^2 j} \\ &= \frac{2^{2k}\pi^k}{2^{2k}} \prod_{j=1}^{k} \frac{1}{j} \\ &= \frac{\pi^k}{k!}. \end{aligned}$$

If we use Corollary 2.1.4, then

$$\begin{aligned} a_{2k+1} &= 2w_{2k+1}a_{2k} \\ &= 2\frac{\pi^k}{k!}\frac{2^{2k}(k!)^2}{(2k+1)!} \\ &= \frac{2^{2k+1}\pi^k(k!)}{(2k+1)!}. \end{aligned}$$

□

The Relation

Theorem 2.1.8 (Isodiametric Inequality). *For a convex body F in $\mathbb{R}^n$,*

$$\frac{V_n(F)}{(\operatorname{diam} F)^n} \leq \frac{a_n}{(\operatorname{diam} B_n(1))^n} = \frac{a_n}{2^n} = c_n.$$

Theorem 2.1.9. *For a convex body F in $\mathbb{R}^n$,*

$$\frac{V_n(F)}{c_n} = \mathcal{H}^n(F).$$

Proof. Consider $\mathcal{H}^n(F)$. Then, $\mathcal{H}^n(F) = \lim_{\delta\to 0} \mathcal{H}^n_\delta(F)$. Furthermore, we know that $\mathcal{H}^n_\delta(F)$ is monotone decreasing. Thus, by monotone convergence theorem, $\mathcal{H}^n(F) = \inf_{\delta>0} \mathcal{H}^n_\delta(F)$. Therefore it will suffice to show that $V_n(F)/c_n = \inf_{\delta>0} \mathcal{H}^n_\delta(F)$.

Consider our isodiametric inequality for F.

$$\frac{V_n(F)}{(\operatorname{diam} F)^n} \leq c_n$$

Thus,

$$V_n(F) \leq c_n(\operatorname{diam} F)^n.$$

Furthermore, $V_n(F) = \mathcal{L}^n(F)$. Let $\delta > 0$ and $\mathcal{U} \in C_\delta(F)$. Therefore,

$$\mathcal{L}^n(F) \leq \sum_{U\in\mathcal{U}} \mathcal{L}^n(U) \leq c_n \sum (\operatorname{diam} U)^n$$

Taking the infinimum of both sides yields,

$$\mathcal{L}^n(F) \leq c_n \mathcal{H}^n_\delta(F)$$

for all $\delta > 0$. Therefore, taking the limit as $\delta \to 0$ yields

$$\mathcal{L}^n(F) \leq c_n \mathcal{H}^n(F).$$

Let $\varepsilon > 0$ be given. Therefore, we can find a disjoint cover of cubes with side length at most $l > 0$ $\{C_k\}_{k\in K}$ where K is countable such that $\mathcal{L}^n(\cup C_k) - \mathcal{L}^n(F) \leq \varepsilon$. Moreover $\mathcal{L}^n(C_k) = (\operatorname{diam} C_k/\sqrt{n})^n$. Therefore,

$$\left(\frac{1}{\sqrt{n}}\right)^n \sum (\operatorname{diam} C_k)^n \leq \mathcal{L}^n(F) + \varepsilon$$

and taking the infinimum yields,

$$\left(\frac{1}{\sqrt{n}}\right)^n \sum \mathcal{H}^n_l(F) \leq \mathcal{L}^n(F) + \varepsilon$$

for all $l, \varepsilon > 0$. Thus,

$$\left(\frac{1}{\sqrt{n}}\right)^n \sum \mathcal{H}^n(F) \leq \mathcal{L}^n(F).$$

We know by the Vitali Covering principle that there exists a disjoint family of closed balls, $\{\{B^k_j\}_{j\in J_k} | J_k \subseteq \mathbb{N}\}_{j\in J}$ such that $\mathcal{H}^n(C_k \setminus \cup B^k_j) = 0$ for all $j \in J$ and $\cup B^k_j \subseteq C_k$. Therefore $\mathcal{H}^n(C_k) = \mathcal{H}^n(\cup B^k_j)$, by properties of measures. Thus,

$$\mathcal{L}^(F) + \varepsilon \geq \mathcal{L}^n(\cup C_k) \geq \mathcal{L}^n(\cup_{j,k} B^k_j)$$

and thus

$$\mathcal{L}^(F) + \varepsilon \geq c_n \sum (\operatorname{diam} B_j^k) \geq c_n \mathcal{H}^n(F)$$

since $\{B_j^k\}_{j,k}$ is an l-cover of F for all $\varepsilon > 0$. Ergo,

$$\mathcal{L}^(F) \geq c_n \mathcal{H}^n(F)$$

and

$$\mathcal{L}^(F) = c_n \mathcal{H}^n(F).$$

□

2.1.2 Properties of the Hausdorff Measure

Theorem 2.1.10. *Let $F \subset \mathbb{R}^n$ and let $f : F \to \mathbb{R}^m$ be satisfy the Hölder condition*

$$|f(x) - f(y)| \leq c|x - y|^\alpha.$$

Then for any s,

$$\mathcal{H}^{s/\alpha}(f(F)) \leqq c^{s/\alpha} H^s(F).$$

Proof. Let $\delta > 0$ be given. Let $\mathcal{U} \in C_\delta(F)$. Then by Hölder property,

$$\operatorname{diam} f(F \cap U) \leq c(\operatorname{diam} F \cap U)^\alpha \leq c(\operatorname{diam} U)^\alpha \leq c\delta^\alpha$$

for all $U \in \mathcal{U}$. Thus, $\{f(F \cap U)\}_{U \in \mathcal{U}} \in C_{c\delta^\alpha}(f(F))$. Ergo,

$$\sum_{U \in \mathcal{U}} (\operatorname{diam} f(F \cap U))^{s/\alpha} \leq c^{s/\alpha} \sum_{U \in \mathcal{U}} (\operatorname{diam} U)^s.$$

And thus,

$$\mathcal{H}_\delta^{s/\alpha}(f(F)) \leq c^{s/\alpha} \mathcal{H}_\delta^s F.$$

Taking the limit as $\delta \to 0$, yields $\mathcal{H}^{s/\alpha}(f(F)) \leq c^{s/\alpha} \mathcal{H}^s(F)$. □

Theorem 2.1.11. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a similarity transform with scale constant λ. Then $\mathcal{H}^s(f(F)) = \lambda^s \mathcal{H}^s(F)$.*

Proof. Since f is a similarity transform with scale constant λ,

$$|f(x) - f(y)| = \lambda|x - y|$$

which is the Hölder condition with $\alpha = 1$ and $c = \lambda$. Thus,

$$\mathcal{H}^s(f(F)) \leq \lambda^s \mathcal{H}^s(F).$$

Since f is a similarity, f is invertible and

$$|f^{-1}(x) - f^{-1}(y)| = \lambda^{-1}|x - y|$$

which is the Hölder condition with $\alpha = 1$ and $c = \lambda^{-1}$. Thus,

$$\mathcal{H}^s(F) = \mathcal{H}^s(f^{-1}(f(F))) \leq \lambda^{-s}\mathcal{H}^s(f(F))$$

and

$$\mathcal{H}^s(f(F)) = \lambda^s \mathcal{H}^s(F).$$

□

Proposition 2.1.12. *If we only consider δ-covers consisting of closed sets when evaluating $\mathcal{H}^s$ then the value remains unaltered.*

Proof. Let $F \subseteq \mathbb{R}^n$ and let $\mathcal{U} \in C_\delta(F)$. Let $U \in \mathcal{U}$. We know $\operatorname{diam} U \leq \operatorname{diam} \bar{U}$, the closure of U. Let $\varepsilon > 0$ be given. Let $x, y \in \bar{U}$. Since x, y are in the closure of U, then there exist sequences, $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in U such that $x_n \to x$ and $y_n \to y$. Ergo, by convergence, there exists an $N_x \in \mathbb{N}$ such that for all $n \geq N_x$, $|x_n - x| < \varepsilon/2$. Similarly, we can find an N_y such that $|y_n - y| < \varepsilon/2$ for all $n \geq N_y$. Take $N = \max\{N_x, N_y\}$. Then

$$|x - y| \leq |x - x_N| + |x_N - y| \leq |x - x_N| + |x_N - y_N| + |y_N - y| \leq |x_N - y_N| + \varepsilon$$

and $|x - y| \leq |x_N - y_N|$ since this holds for all $\varepsilon > 0$. Taking supremums yields,

$$\operatorname{diam} \bar{U} = \sup_{x,y\in\bar{U}} |x - y| \leq \sup_{a,b\in U} |a - b| = \operatorname{diam} U$$

and $\operatorname{diam} \bar{U} = \operatorname{diam} U$. Therefore,

$$\sum_{U\in\mathcal{U}} (\operatorname{diam} U)^s = \sum_{U\in\mathcal{U}} (\operatorname{diam} \bar{U})^s$$

for all $\mathcal{U} \in C_\delta F$. Ergo, $\mathcal{H}^s_\delta(F)$ remains unchanged and thus $\mathcal{H}^s(F)$ remains unchanged if we only consider closed covers. □

2.1.3 Examples of the Hausdorff Measure

Example 2.1.13. *Let $F = [0, 1]$, then: if $s < 1$, $\mathcal{H}^s(F) = \infty$; if $s = 1$, $0 < \mathcal{H}^s(F) < \infty$; and if $s > 1$, $\mathcal{H}^s(F) = 0$.*

Proof. Consider the sequence $1/n$. Since $\mathcal{H}^s(F) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(F)$ exists, any subsequence of $\{\mathcal{H}^s_\delta(F)\}_{\delta>0}$ converges to the same limit. Ergo, we will deal with $\{\mathcal{H}^s_{1/n}(F)\}_{n=1}^\infty$ to simplify the proof. We begin by noting that the minimal $1/n$ cover of F is $\{i/n, (i+1)/n\}_{i=0}^{n-1}$. Thus,

$$\mathcal{H}^s_{1/n} = \sum_{i=0}^{n-1} \frac{1}{n^s} = \frac{n}{n^s} = n^{1-s}.$$

If $s < 1$, then $\mathcal{H}^s_{1/n} \to \infty$. If $s > 1$, then $\mathcal{H}^s_{1/n} \to 0$. However, if $s = 1$, $\mathcal{H}^s_{1/n} = n^{1-1} = 1 \to 1$. □

This critical value of $s = 1$ is called the Hausdorff dimension of F.

2.2 The Hausdorff Dimension

Definition 2.2.1. Let $F \subseteq \mathbb{R}^n$, then the *Hausdorff dimension of F* is defined as

$$\dim_H F = \inf\{s \geq 0 | \mathcal{H}^s(F) = 0\} = \sup\{s \geq 0 | \mathcal{H}^s(F) = \infty\}.$$

2.3 Properties

Recall the desired properties of any definition of dimension that we enumerated in chapter one.

Theorem 2.3.1. *The Hausdorff dimension satisfies all of the following properties:*

1. *if $E \subseteq F$, then $\dim_H E \leq \dim_H F$;*
2. *if $\{S_i\}_{i \in I}$ is a countable collection of sets, then $\dim_H \bigcup_{i \in I} S_i = \sup\limits_{i \in I} \dim_H S_i$;*
3. *when $F \subseteq \mathbb{R}^n$, $0 \leq \dim_H F \leq n$;*
4. *any open set $O \subseteq \mathbb{R}^n$ should have $\dim_H O = n$;*
5. *if S is countable, then $\dim_H S = 0$.*

Proof (1). Suppose $E \subseteq F \subseteq \mathbb{R}^n$. Since $\mathcal{H}^s$ is a measure and $E \subseteq F$, then

$$\mathcal{H}^{\dim_H E}(E) \leq \mathcal{H}^{\dim_H E}(F).$$

Since $\mathcal{H}^{\dim_H E}(E)$ is finite and $\mathcal{H}^{\dim_H E}(F)$ may not be finite, $\dim_H E \leq \dim_H F$. □

Proof (2). Let $\{F_i\}_{i \in I}$, be a countable collection of sets. By (1), we know that $\sup\limits_{i \in I} \dim_H F_i \leq \dim_H \cup_{i \in I} F_i$. Let $s > \sup\limits_{i \in I} \dim_H F_i$. Since $\mathcal{H}^s$ is a measure, we know

$$0 \leq \mathcal{H}^s(\cup_{i \in I} F_i) \leq \sum_{i \in I} \mathcal{H}^s(F_i).$$

However, since $s > \sup\limits_{i \in I} \dim_H F_i \geq \dim_H F_i$ for all $i \in I$ we have $\mathcal{H}^s(F_i) = 0$ for all $i \in I$ and $\mathcal{H}^s(\cup_{i \in I} F_i) = 0$. Thus

$$\dim_H \cup_{i \in I} F_i = \inf\{s \geq 0 | \mathcal{H}^s(\cup_{i \in I} F_i) = 0\} = \sup_{i \in I} \dim_H F_i$$

and the Hausdorff dimension is stable. □

Lemma 2.3.2. *In $\mathbb{R}^n$, $\dim_H B(x, r) = n$.*

Proof. □

Proof (3). Let $F \subseteq \mathbb{R}^n$. We know that we can cover F with countably many open balls $B_{i i \in I}$, each of dimension n. Thus $F \subseteq \cup_{i\in I} B_i$ and $\dim_H F \leq \dim_H \cup_{i\in I} B_i$ and by (4) we have

$$\dim_H F \leq \sup_{i\in I} \dim_H B_i = n.$$

Since $\mathcal{H}^s$ is a measure, we have $0 \leq \dim_H F$. □

Proof (4). Let $O \subseteq \mathbb{R}^n$ be open. We know that $\dim_H O \leq n$ by (3). Since O is open for each $x \in O$ there exists an $r > 0$ such that $B(x, r) \subseteq O$. Pick any $x \in O$ and take any r such that $B(x, r) \subseteq O$. Then $n = \dim_H B(x, r) \leq \dim_H O \leq n$ and $\dim_H O = n$. □

Lemma 2.3.3. *Let $x \in \mathbb{R}^n$, then $\dim_H\{x\} = 0$.*

Proof. Note that the minimal δ cover of $\{x\}$ is $\{x\}$. Let $s > 0$, then

$$\mathcal{H}^s_\delta(\{x\}) = \operatorname{diam}\{x\}^s = 0^s = 0$$

and $\mathcal{H}^s(\{x\}) = 0$. Thus,

$$\dim_H\{x\} = \inf s \geq 0 | \mathcal{H}^s(\{x\}) = 0 = 0.$$

□

Proof (5). Let $S \subset \mathbb{R}^n$ be countable. Then for some $I \subseteq \mathbb{N}$, $S = \{s_i\}_{i\in I}$. Thus,

$$\dim_H S = \sup_{i\in I} \dim_H s_i = 0$$

by (4). □

2.3.1 Remark on Stability

Consider the set $S = \{1/n\}_{n\in\mathbb{N}}$. By property (5), we know that $\dim_H S = 0$. However, this raises the question as to what the Hausdorff dimension does differently than the box-counting dimension. Suppose we're using the box-counting dimension with N^D_δ. In the cover of the set we used, when δ was between $1/k^2$ and $1/(k-1)^2$ we had to cover the first k terms with sets of diameter no greater than δ. However, we could cover those k terms, with singletons of the form $\{1/k\}$ or with balls of radius $1/k^2$ without affecting our count.

When we take the Hausdorff dimension of S, we still need to cover those k terms, however we are forced by the infinimum in $\mathcal{H}^s_\delta$ to take minimal cover, or in other words, cover those k terms with singletons. Since singletons have diameter zero, when $s > 0$, they contribute absolutely nothing to our measure. Since these are the majority of the sets in our cover, only a constant number of sets actually contribute any length to our measure and as we make δ approach zero, the contribution of these sets goes to zero. Thus forcing the Hausdorff dimension of S to be zero, since it cannot be positive or negative.

2.3.2 Other Properties

Like the box counting dimension, the Hausdorff dimension also has a nice property involving the Hölder condition.

Theorem 2.3.4. *Let $F \subset \mathbb{R}^n$ and $f : F \to \mathbb{R}^n$ satisfy the Hölder condition*

$$|f(x) - f(y)| \leq c|x - y^{\alpha}.$$

Then $\dim_H f(F) \leq 1/\alpha \dim_H F$*.*

Proof. Suppose $s > \dim_H F$. Then, by Theorem 2.1.10

$$\mathcal{H}^{s/\alpha}(f(F)) \leq c^{s/\alpha}\mathcal{H}^s(F) = 0.$$

Thus,

$$\dim_H f(F) \leq \frac{s}{\alpha}$$

for all $s > \dim_H$ which implies

$$\dim_H f(F) \leq \frac{1}{\alpha} \dim_H F.$$

□

Corollary 2.3.5. *If f is bi-Lipschitz, then* $\dim_H f(F) = \dim_H F$*.*

Proof. Since f is bi-Lipschitz, then $\dim_H f(F) \leq \dim_H F$, f is invertible, and f^{-1} is bi-Lipschitz. Since f^{-1} is bi-Lipschitz, then $\dim_H F \leq \dim_H f(F)$ and $\dim_H f(F) \leq \dim_H F$. □

We also have a theorem that relates the Hausdorff dimension to the Box-counting dimension.

Theorem 2.3.6. *Let $F \subset \mathbb{R}^n$ then*

$$\dim_H F \leq \underline{\dim}_B F.$$

Proof. Suppose that for some $s \geq 0$, $1 < \mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F)$. Let $\mathcal{U} \in C_\delta(F)$, then

$$1 < \sum_{U \in \mathcal{U}} (\operatorname{diam} U)^s \leq \sum_{U \in \mathcal{U}} \delta^s = \delta^s (\operatorname{card} \mathcal{U}).$$

If we take the infinimum of both sides we get,

$$1 < \mathcal{H}^s_\delta(F) = \inf_{\mathcal{U} \in C_\delta(F)} \sum_{U \in \mathcal{U}} (\operatorname{diam} U)^s \leq \delta^s \inf_{\mathcal{U} \in C_\delta(F)} \operatorname{card} \mathcal{U} = \delta^s N^D_\delta(F).$$

Taking logs yields,

$$0 < \ln N^D_\delta(F) + s \ln \delta$$

which is equivalent to

$$s \leq \frac{\ln N^D_\delta(F)}{-\ln \delta}.$$

Thus, $\dim_H F \leq \underline{\dim}_B F$. □

Lemma 2.3.7. *Let $F \subset \mathbb{R}$ with $\mathcal{H}^1(F) = 0$. Then F^c is dense in $\mathbb{R}$*

Proof. We need to show that $\overline{F^c} = \mathbb{R}$ in order to show that F^c is dense in $\mathbb{R}$. Equivalently, we will show that $\mathbb{R} \setminus \overline{F^c} = \emptyset$. Proceed via contradiction. Assume $x \in \mathbb{R} \setminus \overline{F^c}$. Thus, $x \notin \overline{F^c}$. Therefore $x \in \overline{F^c}^c$, an open set. Ergo, there exists an $r > 0$ such that $B(x, r) \subseteq \overline{F^c}$. Thus, $B(x, r) \cap \overline{F^c} = \emptyset$ and $B(x, r) \cap F^c = \emptyset$. Ergo, $B(x, r) \cap F = B(x, r)$ and $B(x, r) \subseteq F$. Therefore, $\mathcal{H}^1(F) \geq \mathcal{H}^1(B(x, r)) = \mathcal{L}^1(F)/\mathcal{L}^1(B(0, 1)) = 2r/2 = r > 0$. This contradicts the assumption that $\mathcal{H}^1(F) = 0$ and thus, $\overline{F^c} = \mathbb{R}$ and F^c is dense in $\mathbb{R}$. $\square$

Theorem 2.3.8. *Let $F \subset \mathbb{R}^n$ with $\dim_H F < 1$, then F is totally disconnected.*

Proof. Let $x, y \in F$ with $x \neq y$. Define $f : \mathbb{R}^n \to \mathbb{R}^+$ by $f(z) = |z - x|$. Thus, by reverse triangle inequality,

$$|f(z) - f(w)| = ||z - x| - |w - x|| \leq |z - x - w + x| = |z - w|$$

and f is Lipschitz. Ergo by Theorem 2.3.4, $\dim_H f(F) \leq \dim_H F < 1$. Ergo, $\mathcal{H}^1(f(F)) = 0$ and by Lemma 2.3.7, $f(F)^C$ is dense in $\mathbb{R}$. Thus, we can choose an $r \notin f(F)$ such that $0 < r < f(y)$. Thus, F can be written as

$$F = \{z \in F | |z - x| < r\} \cup \{z \in F | |z - x| > r\}.$$

Note that both of those sets are disjoint and open. Furthermore, we can find these for any $x, y \in F$, thus F must be totally disconnected. $\square$

2.3.3 Examples of the Hausdorff Dimension

Example 2.3.9. *Let $F \subset \mathbb{R}$ and let $f(x) = x^2$. Show that $\dim_H F = \dim_H f(F)$.*

Proof. Suppose F is bounded, then f is Lipschitz on F. Thus,

$$\dim_H f(F) \leq \dim_H F.$$

Let $F^+ = F \cap \mathbb{R}^+$ and $F^- = F \cap \mathbb{R}^-$. Then $f : F^+ \to f(F^+)$ and $f : F^- \to f(F^-)$ are invertible, and defined as $f_+^{-1}(x) = \sqrt{x}$ and $f_-^{-1}(x) = -\sqrt{x}$ respectively. Choose some $\varepsilon > 0$ such that $F \setminus (-\varepsilon, \varepsilon) \neq \emptyset$. We know that $F^+ \setminus [0, \varepsilon)$ or $F^- \setminus [0, \varepsilon)$ is nonempty. Without loss of generality assume $F^+ \setminus [0, \varepsilon) \neq \emptyset$. Since f_+^{-1} is continuously differentiable on $f(F^+ \setminus [0, \varepsilon))$, f_+^{-1} is Lipschitz. Ergo,

$$\dim_H F^+ \setminus [0, \delta) \leq \dim_H f(F^+) \leq \dim_H f(F)$$

for all $0 < \delta \leq \varepsilon$. Taking the limit as $\delta \to 0$ yields,

$$\dim_H F^+ \leq \dim_H f(F).$$

A similar argument yields

$$\dim_H F^- \leq \dim_H f(F).$$

Thus by countable stability,

$$\dim_H F = \max\{\dim_H F^+, \dim_H F^-\} \leq \dim_H f(F)$$

and $\dim_H f(F) = \dim_H F$ when F is bounded.

Suppose F is unbounded, then F can be covered by countably many open balls of radius $r > 0$ with centers x_i in F. Denote, $F_i = B(x_i, r) \cap F$. Note that $F = \cup_{i\in\mathbb{N}} F_i$ and $f(F) = \cup_{i\in\mathbb{N}} f(F_i)$. Since F_i is bounded, by the previous part, we know that $\dim_H F_i = \dim_H f(F_i)$ for each $i \in \mathbb{N}$. Thus, by countable stability,

$$\dim_H F = \sup_{i\in\mathbb{N}} \dim_H F_i = \sup_{i\in\mathbb{N}} \dim_H f(F_i) = \dim_H f(F).$$

□

Example 2.3.10. *Let $f : [0,1] \to \mathbb{R}$ be Lipschitz. Let* $\operatorname{graph} f = \{(x, f(x)) | x \in [0,1]\}$*, then* $\dim_H \operatorname{graph} f = 1$*.*

Proof. Let $g : \operatorname{graph} f \to [0,1]$ be defined by $g(x, f(x)) = x$. We want to show that g is bi-Lipschitz. Note that $|g(x, f(x)) - g(y, f(y))| = |x - y|$ and that since f is Lipschitz, $|f(x) - f(y)| \leq M|x - y|$. Consider $\sqrt{(x-y)^2 + (f(x) - f(y))^2}$.

$$\sqrt{(x-y)^2 + (f(x) - f(y))^2} \leq \sqrt{(x-y)^2 + M^2(x-y)^2} = \sqrt{M^2+1}|x-y|$$

Thus, $\left(\sqrt{(x-y)^2 + (f(x) - f(y))^2}\right) / \sqrt{M^2+1} \leq |x - y|$. Moreover,

$$|x-y| = \sqrt{(x-y)^2} \leq \sqrt{(x-y)^2 + (f(x) - f(y))^2}.$$

Ergo,

$$\frac{1}{\sqrt{M^2+1}}\sqrt{(x-y)^2 + (f(x) - f(y))^2} \leq |g(x, f(x)) - g(y, f(y))| \leq \sqrt{(x-y)^2 + (f(x) - f(y))^2}$$

and g is bi-Lipschitz. Therefore,

$$\dim_H \operatorname{graph} f = \dim_H g(\operatorname{graph} f) = \dim_H [0,1] = 1.$$

□

Example 2.3.11. *Let F be the subset of $[0,1]$ whose elements have no instance of the digit five in their decimal expansion. Then* $\dim_H F = \ln 9 / \ln 10$*.*

Proof. Define $F_i = [i/10, (i+1)/10] \cap F$. Then, $F = \cup_{i=0}^{9} F_i$, the collection of F_i's is mutually disjoint, $F_5 = \emptyset$, and all other F_i's are similar to F with $\lambda = 1/10$. Thus,

$$\mathcal{H}^s(F) = \sum_{i=0}^{9} \mathcal{H}^s(F_i) = \frac{9}{10^s}\mathcal{H}^s(F).$$

Assuming $0 < \mathcal{H}^s(F) < \infty$ at the critical s, then we have

$$10^s = 9$$

and

$$s = \frac{\ln 9}{\ln 10}.$$

Thus $\dim_H F = \ln 9 / \ln 10$. $\square$

Appendix A: Definitions

Set Theory

Definition. A *relation* is a set of ordered pairs.

Definition. Let A and B be nonempty sets and let f be a relation between them. Then f is a *function* if and only if for $(x, y), (x, z) \in f$ then $y = z$.

Example. *Let A be a nonempty set. Then the identity map on A, $id_A(x) = x$, is a function. On $\mathbb{R}$, $\sin x$, and $\cos x$ are functions.*

Definition. Let $f : A \to B$ be a function. Then:

- f is *injective* if and only if for all $a_1, a_2 \in A$, $f(a_1) = f(a_2)$ implies $a_1 = a_2$, moreover f is called an *injection*;
- f is *surjective* if and only if for all $b \in B$ there exists an $a \in A$ such that $b = f(a)$, moreover f is called a *surjection*;
- f is *bijective* if and only if f is injective and surjective, moreover f is called a *bijection*.

Example. *The map $f : \mathbb{R} \to [-1, 1]$ defined by $f(x) = \sin x$ is surjective, but not injective. The map $g : \mathbb{N} \to \mathbb{R}$ defined by $g(x) = x$ is injective, but not surjective. The map $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = x$ is bijective.*

Definition. A nonempty set A is *finite* if and only if there exists an $n \in \mathbb{N}$ such that there exists a bijection between A and $1, 2, \ldots, n$.

Example. *$\mathbb{Z}_7$ is finite since the map $f : \{1, 2, \ldots, 7\} \to \mathbb{Z}_7$ defined by $f(n) = \overline{n-1}$ is a bijection.*

Definition. A nonempty set A is *infinite* if and only if A is not finite.

Example. *$\mathbb{N}$ is infinite.*

Definition. A nonempty set A is *countably infinite* if and only if there exists a bijection between A and $\mathbb{N}$.

Example. *$\mathbb{Z}$ is countably infinite since the map $f : \mathbb{N} \to \mathbb{Z}$ given by $f(n) = (-1)^n \lfloor n/2 \rfloor$ is a bijection.*

Definition. A nonempty set A is *countable* if and only if A is finite or countably infinite.

Example. *The sets, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{A}$, and $\mathbb{Z}_n$ are countable.*

Definition. A nonempty set A is *uncountable* if and only if $|A| > \aleph_0$.

Example. *The sets $\mathbb{R}, \mathbb{C}, \mathbb{H}$ are uncountable.*

Definition. A *poset* is a nonempty set A with ordering $\preceq$ satisfying the following for all $a, b, c \in A$:

- $a \preceq a$;
- $a \preceq b$ and $b \preceq a$ implies that $a = b$;
- $a \preceq b$ and $b \preceq c$ implies that $a \preceq c$.

Example. *Let A be a nonempty set. Then, $\mathcal{P}(A)$ with the ordering $\subseteq$ is a poset.*

Definition. A poset $(A, \preceq)$ is *totally ordered* if and only if for all $a, b \in A$, $a \preceq b$ or $b \preceq a$.

Example. *$(\mathbb{R}, \leq)$ is a totally ordered set.*

Definition. Let $(S, \leq)$ be a totally ordered set and let $A \subseteq S$. We say that A is *bounded above* if and only if there exists a $s \in S$ such that for all $a \in A$, $a \leq s$; we say that s is an upper bound of A. We say that A is *bounded below* if and only if there exists a $r \in S$ such that for all $a \in A$, $r \leq a$; we say that r is a lower bound of A.

Example. *$\mathbb{N}$ is bounded below by 0. $\{1, 2, \ldots n\}$ is bounded below by 1 and bounded above by n.*

Definition. Let $(S, \leq)$ be a totally ordered set and let $A \subseteq S$ be bounded above. Then the *supremum* of A, denoted $\sup A$, is an upper bound of A, α, with the property that if β is also an upper bound of A then $\alpha \leq \beta$.

Example. *The supremum of $(-1, 1)$ is 1 and the supremum of $(-\pi, \pi]$ is π.*

Definition. Let $(S, \leq)$ be a totally ordered set and let $A \subseteq S$ be bounded below. Then the *infimum* of A, denoted $\inf A$, is a lower bound of A, ζ, with the property that if η is also a lower bound of A then $\eta \leq \zeta$.

Example. *The infimum of $(-1, 1)$ is -1 and the infimum of $[-\pi, \pi)$ is $-\pi$.*

Topological Spaces

Definition. Let X be a nonempty set and let τ be a collection of subsets of X. Then τ is called a *topology* on X if and only if τ satisfies all of the following:

- $X \in \tau$ and $\emptyset \in \tau$;
- if $U, V \in \tau$, then $U \cap V \in \tau$;
- if $\{U_i\}_{i \in I} \subseteq \tau$ then $\cup_{i \in I} U_i \in \tau$.

We say that (X, τ) form a *topological space*. Furthermore, any set in τ is called *open* and for any $x \in X$ a set containing x is called a *neighborhood* of x. Moreover, a set $A \subseteq X$ is said to be *closed* if and only if $(X \setminus A) \in \tau$.

Example. *Let A be a nonempty set, then $(A, \mathcal{P}(A))$ forms a topology.*

Definition. Let (X, τ) be a topological space and let $A \subseteq X$. Then the *interior* of A is defined as

$$A^o = \{x \in A | \exists V \in \tau \text{ s.t. } (x \in V) \wedge (V \subseteq A)\}.$$

Example. *In $\mathbb{R}$, the interior of $\mathbb{Q}$ is the empty set under the usual metric topology.*

Definition. Let (X, τ) be a topological space and let $A \subseteq X$. A point $x \in X$ is called an *accumulation point* of A if and only if for all neighborhoods of x, V, $(V \cap A) \setminus x \neq \emptyset$.

Example. 0 *is an accumulation point of* $(0, 1)$.

Definition. Let (X, τ) be a topological space and let $A \subseteq X$. Then the *closure* of A, denoted $\bar{A}$, is the set A together with all of its accumulation points.

Example. *The closure of* $(0, 1)$ *is* $[0, 1]$.

Definition. Let (X, τ) be a topological space and let $A \subseteq X$. A sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq X$ *coverges* to $x \in X$ if and only if for all neighborhoods of U of x, there exists an $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$.

Example. *The sequence* $\{1/n\}_{n \in \mathbb{N}}$ *converges to* 0 *in* $\mathbb{Q}$.

Definition. Let (X, τ) be a topological space and let $A \subseteq X$. Then A is called *perfect* if and only if all $x \in A$ are accumulation points of A.

Example. *Any open interval is perfect.*

Definition. Let (X, τ_X) and (Y, τ_Y) be topological spaces. A function $f : X \to Y$ is said to be *continuous* at $x \in X$ if and only if for each $V \in \tau_Y$ containing $f(x)$, there exists a $U \in \tau_X$ containing x such that $f(U) \subseteq V$. Furthermore, f is said to be continuous on $A \subseteq X$ if and only if f is continuous at each $x \in A$.

Example. *$f(x) = x^2$ is continuous on $\mathbb{R}$ under the metric topology.*

Definition. Let (X, τ) be a topological space and let $A \subseteq X$ A family $\{A_i\}_{i \in I} \subset \mathcal{P}(X)$ is called a *cover* of A if and only if $A \subseteq \cup_{i \in I} A_i$. If $\{A_j\}_{j \in J \subseteq I}$ is also a cover of A, then, $\{A_j\}_{j \in J}$ is called a *subcover* of A. If a cover of A is formed only by open sets, then it is called an *open cover* of A.

Example. *$[0, 100]$ is a cover of $(10, 11)$ that has a subcover of $[9, 12]$. $\{B(x, 1)\}_{x \in \mathbb{R}}$ is an open cover of $\mathbb{R}$.*

Definition. Let (X, τ) be a topological space and let $A \subseteq X$. We say A is *compact* if and only if each open cover of A admits a finite subcover.

Example. *Any closed interval is compact in $\mathbb{R}$ under the usual metric topology.*

Definition. Let (X, τ) be a topological space and let $A \subseteq X$. We say A is *disconnected* if and only if there exist disjoint sets $U, V \subseteq X$ such that

- $A \subset U \cup V$
- $A \cap U \neq \emptyset \wedge A \cap V \neq \emptyset$
- $A \cap U \cap V = \emptyset$.

If A is not disconnected, we say that A is *connected.*

Example. *$[0, 1]$ is connected and $(0, 1/2) \cup (1/2, 1)$ is disconnected.*

Definition. Let (X, τ) be a topological space and let $A \subseteq X$. Then A is *totally disconnected* if and only if for any $x, y \in A$ there exist disjoint $U, V \in \tau$ such that $x \in U$, $y \in V$ and $A \subset U \cup V$.

Example. *The rational numbers are totally disconnected in $\mathbb{R}$.*

Definition. Let (X, τ) be a topological space and let $A \subseteq X$. We say that A is *dense* in X if and only if $\bar{A} = X$.

Example. *$\mathbb{Q}$ is dense in $\mathbb{R}$.*

Definition. Let (X, τ) be a topological space and let $A \subseteq X$. We say that A is *nowhere dense* in X if and only if $\left(\bar{A}\right)^o = \emptyset$.

Example. *$\mathbb{Z}$ is nowhere dense in $\mathbb{R}$.*

Definition. Let (X, τ_x) and (Y, τ_y) be topological spaces. We say that X and Y are *homeomorphic* if and only if there exists a continuous bijection $f : (X, \tau_x) \to (Y, \tau_y)$ where f^{-1} is also continuous.

Example. *For $a, b \in \mathbb{R}$ with $a < b$, the sets (a, b) and $\mathbb{R}$ are homeomorphic under the standard metric topology.*

Definition. Let X be a nonempty set. Let $\mathcal{B}$ be a collection of subsets of X such that

- for each $x \in X$ there is a $B \in \mathcal{B}$ such that $x \in B$
- and if $x \in B_1 \cap B_2$ where $B_1, B_2 \in \mathcal{B}$, then there exists a $B \in \mathcal{B}$ such that $x \in B \subseteq B_1 \cap B_2$

then $\mathcal{B}$ is called a *basis* for a topology on X. Furthermore, the *topology generated by* $\mathcal{B}$ is given by

$$\tau = \{U \subseteq X | \forall x \in U, \exists B \in \mathcal{B} \text{ s.t. } x \in B \subseteq U\}.$$

Example. *Let* (X, d) *be a metric space then the metric topology induced by* d *is given by*

$$\tau_d = \{U \subseteq X | x \in U \Rightarrow \exists r > 0 \textit{ s.t. } B(x, r) \subseteq U\}.$$

Metric Spaces

Definition. A set X and a metric $d : X^2 \to \mathbb{R}$ form a *metric space* (X, d) if and only if all of the following are satisfied:

1. for all $x, y \in X$, $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$;
2. for all $x, y \in X$, $d(x, y) = d(y, x)$;
3. and for all $x, y, z \in X$ $d(x, y) \leq d(x, z) + d(z, x)$.

Example. *$\mathbb{R}$ and absolute value form a metric space.*

Definition. Let (X, d) be a metric space. Let $x_0 \in X$ and $r \in \mathbb{R}$. An *open ball* in (X, d) around x_0 of radius r is the set $B(x_0, r) = \{x \in X | \, d(x_0, x) < r\}$.

Example. *In $\mathbb{R}$, $B(x_0, r) = (x_0 - r, x_0 + r)$.*

Definition. Let (X, d) be a metric space. A subset of A of (X, d) is called *open* if and only if for each $x \in A$, there exists an $r_x > 0$ such that $B(x, r_x) \subseteq A$.

Example. *The set $(-n, n) \cup (k, k+1)$ is open in $\mathbb{R}$ for $k, n \in \mathbb{R}$.*

Definition. Let (X, d) be a metric space. A subset of A of (X, d) is called *closed* if and only if $X \setminus A$ is open.

Example. *$\mathbb{R}$ is closed in $\mathbb{R}$.*

Definition. Let (X, d) be a metric space. The *closed ball* of radius r about $x \in X$ is $\bar{B}(x, r) = \{y \in X | d(x, y) \leq r\}$.

Example. *In $\mathbb{R}$, $\bar{B}(x_0, r) = [x_0 - r, x_0 + r]$.*

Definition. Let (X, d) be a metric space. The *diameter* of a nonempty subset, A, of X is $\sup\limits_{x,y \in A} d(x, y)$.

Example. *The diameter of $(0, 1) \cup (10, 12)$ in $(\mathbb{R}, |\cdot|)$ is 12.*

Definition. Let (X, d) be a metric space and let $A \subseteq X$. A point $x \in X$ is a *closure point* of A if and only if for all $r > 0$, $B(x, r) \cap A \neq \emptyset$.

Example. *The 0 is the only closure point of the set $\{0\}$. The point 3 is a closure point of $(0, 3)$.*

Definition. Let (X, d) be a metric space and let $A \subseteq X$. A point $x \in X$ is an *accumulation point* of A if and only if for all $r > 0$, $B(x, r) \cap A \setminus \{x\} \neq \emptyset$.

Example. *The set $\{0\}$ has no accumulation points since $B(0, r) \cap \{0\} = \{0\}$. The point 3 is an accumulation point of $(0, 3)$.*

Definition. Let (X, d) be a metric space and let $A \subseteq X$. Let $x \in X$. We say that the *closure* of A, denoted $\bar{A}$, is the set $\bar{A} = \{x \in A | \forall r > 0,\ B(x, r) \cap A \neq \emptyset\}$.

Example. *Under the Euclidean metric* $\mathbb{R}$ *is the closure of* $\mathbb{Q}$ *and* $[a, b]$ *is the closure of* (a, b).

Definition. Let (X, d) be a metric space and let $A \subseteq X$. Let $x \in A$. Then x is an *interior point* of A if and only if there exists an $r > 0$ such that $B(x, r) \subseteq A$. Moreover, the *interior* of A, denoted A^o is the set $A^o = \{x \in A | \exists r > 0,\ B(x, r) \subseteq A\}$.

Example. *An interior point of* $[0, 1]$ *under the Euclidean metric is* $1/2$ *and the interior of* $[a, b]$ *is* (a, b).

Definition. Let (X, d) be a metric space and let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in X. We say that x_n *converges* to x if and only if:

1. the real valued sequence $d(x_n, x) \to 0$;
2. for all $\varepsilon > 0$ there exists a $N_\varepsilon \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n \geq N_\varepsilon$.

Example. *In* $\mathbb{R}$, *the sequence* $\{1/n | n \in \mathbb{N}\}$ *converges to 0.*

Definition. Let (X, d) be a metric space and let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in X. We say that x_n is *Cauchy* in X if and only if for all $\varepsilon > 0$ there exists a $N_\varepsilon \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq N_\varepsilon$.

Example. *In* $\mathbb{Q}$, *the sequence*

$$\{\sum_{k=1}^{n} \frac{1}{n^2}\}_{n \in \mathbb{N}}$$

is Cauchy, but not convergent.

Definition. Let (X, d) be a metric space. We say (X, d) is *complete* if and only if all Cauchy sequences in X converge to some $x \in X$.

Example. $\mathbb{R}$ *is a complete metric space under the Euclidean metric.*

Definition. Let (X, d) be a metric space. We say (X, d) is *incomplete* if and only if there exists some sequence that is Cauchy in X but not convergent in X.

Example. $\mathbb{Q}$ *is incomplete under the Euclidean metric because*

$$\{\sum_{k=1}^{n} \frac{1}{n^2}\}_{n \in \mathbb{N}}$$

is Cauchy, but does not converge in $\mathbb{Q}$.

Definition. Let (X, d) and (Y, ρ) be metric spaces. A function $f : (X, d) \to (Y, \rho)$ is called *continuous* at $x_0 \in X$ if and only if for all $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$ such that $\rho(f(x), f(x_0)) > \varepsilon$ whenever $d(x, x_0) < \delta_\varepsilon$.

Example. *All $\mathbb{C}$-valued polynomials are continuous on $\mathbb{C}$ under the Euclidean metric.*

Definition. Let (X, d) and (Y, ρ) be metric spaces. A function $f : (X, d) \to (Y, \rho)$ is called *uniformly continuous* if and only if for all $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$ such that $\rho(f(x), f(y)) > \varepsilon$ whenever $d(x, y) < \delta_\varepsilon$.

Example. *Any differentiable function with a bounded derivative is uniformly continuous. E.g. $f(x) = ax + b$ where a and b are constants.*

Definition. Let (X, d) and (Y, ρ) be metric spaces. A function $f : (X, d) \to (Y, \rho)$ is called an *isometry* if and only if for all $x, y \in X$, $\rho(f(x), f(y)) = d(x, y)$.

Example. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := x + b$ for any $b \in \mathbb{R}$. Then f is an isometry.*

Definition. We say that metric spaces (X, d) and (Y, ρ) are *isometric* if and only if there exists a surjective isometry between them.

Example. *The map, $f : \mathbb{R}^2 \to \mathbb{C}$ given by $f(x, y) = x + iy$ is a surjective isometry and thus $\mathbb{R}^2$ and $\mathbb{C}$ are isometric.*

Definition. Let (Y, ρ) be a complete metric space and let (X, d) be a metric space. Then, (Y, ρ) is called the *completion* of (X, d) if and only if there exists an isometry $f : (X, d) \to (Y, \rho)$ such that the image $f(X)$ is dense in Y, that is $\overline{f(X)} = Y$.

Example. *Let $f : \mathbb{Q} \to \mathbb{R}$ be defined by $f(x) := x$. Then f is an isometry such that $\overline{f(\mathbb{Q})} = \mathbb{R}$ and thus $(\mathbb{R}, |\cdot|)$ is the completion of $(\mathbb{Q}, |\cdot|)$.*

Measure Theory

Definition. Let S be a nonempty set. A collection $\mathcal{F}(S)$ of subsets of S is called a *σ-algebra* on S if and only if all of the following are satisfied:

- $\emptyset \in \mathcal{F}(S)$;
- $A \in \mathcal{F}(S) \Rightarrow A \cap B \in \mathcal{F}(S)$;
- and $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{F}(S) \Rightarrow \cup_{i\in\mathbb{N}} A_i \in \mathcal{F}(S)$.

Example. *The smallest σ-algebra on any set S is $\{\emptyset, S\}$, while the largest is $\mathcal{P}(S)$.*

Definition. Let (X, τ) be a topological space, then the *σ-algebra generated by τ* is the smallest σ-algebra containing τ.

Example. *In $\mathbb{R}$, the σ-algebra generated by the Euclidean metric topology is called the Borel σ-algebra.*

Definition. Let X be a nonempty set and let $\mathcal{F}(X)$ be a σ-algebra on X. A function $\mu : \mathcal{F}(S) \to [0, \infty)$ is called a *measure* if and only if all of the following are satisfied:

- $\mu(\emptyset) = 0$;
- and if $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{F}(X)$ and $A_i \cap A_j = \emptyset$ if $i \neq j$ then $\mu(A_i \cup A_j) = \mu(A_i) + \mu(A_j)$ and $\mu\left(\cup_{i\in\mathbb{N}} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$.

Furthermore, $(X, \mathcal{F}(X), \mu)$ form a *measure space*. Additionally a function $f : X \to \mathbb{R}$ is called *measurable* if and only if for all $\alpha \in \mathbb{R}$, $A_\alpha = \{x \in X | f(x) > \alpha\} \in \mathcal{F}(\mathcal{X})$.

Example. *Let X be a non-empty set and let $A \subseteq X$. The function $\mu : \mathcal{P}(X) \to [0, \infty]$ given by*

$$\mu(A) = \begin{cases} |A| & A \text{ finite} \\ \infty & A \text{ infinite} \end{cases}$$

is a measure called the counting measure.

Definition. If μ is a measure on a σ-algebra $\mathcal{F}(X)$ on X, then a set $A \subseteq X$ is called a *null set* if and only if $\mu(A) = 0$.

Example. *In the Lesbesque measure, any countable set is a null set.*

Definition. A function $f : X \to \mathbb{R}$ is called a *simple function* if and only if f has only finitely many values.

Example. *Any constant function is simple.*

Definition. The *integral* of a non-negative simple function in standard form $\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$ is

$$\int \phi d\mu = \sum_{i=1}^{n} a_i \mu(A_i).$$

The integral of a non-negative measurable function is

$$\int f d\mu = \sup_{\phi \text{ simple, non-negative, } 0 \leq \phi \leq f} \int \phi d\mu.$$

The integral of a measurable function f is given by

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu$$

where $f^+(x) = \{\sup\} f(x), 0$ and $f^-(x) = \sup\{0, -f(x)\}$.

Bibliography

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