Fractal Analysis

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November 2015

Chapter 1

Box-counting Dimension

1.1 Covers

Let F be a subset of $\mathbb{R}^n$. We define the following covers:

$$\mathcal{B}_\delta(F) := \{B = \{B(x_j, \delta/2)\}_{j=1}^{n_B} | F \subseteq \cup B\}$$
$$\mathcal{D}_\delta(F) := \{D = \{D_j\}_{j=1}^{n_D} | F \subseteq \cup D \ \wedge \ \text{diameter}(D_j) \leq \delta \ \forall \ D_j \in D\}.$$

We now define the following counts

$$N_\delta^B(F) := \min_{B \in \mathcal{B}_\delta(F)} \text{card}(B)$$
$$N_\delta^D(F) := \min_{D \in \mathcal{D}_\delta(F)} \text{card}(D).$$

Theorem 1.1.1.

$$N_\delta^B(F) = N_\delta^D(F)$$

Proof. Let B be a cover in $\mathcal{B}_\delta(F)$ such that $\text{card}(B) = N_\delta^B(F)$. Likewise let D be a cover in $\mathcal{D}_\delta(F)$ such that $\text{card}(D) = N_\delta^D(F)$. Since each set in B has diameter δ $B \in \mathcal{D}_\delta(F)$, and by minimality, $N_\delta^D(F) \leq N_\delta^B(F)$. Since each set in D has diameter no more than δ, we can encapsulate each set in D with a single ball of radius $\delta/2$, call this new cover $B(D)$. By construction, $D(B) \in \mathcal{B}_\delta(F)$ and $\text{card}(D(B)) = N_\delta^D(F)$. Again, by minimality, $N_\delta^B(F) \leq N_\delta^D(F)$ and $N_\delta^B(F) = N_\delta^D(F)$. $\square$

1.2 Box-Counting Dimension

Definition 1.2.1. We say the *lower box-counting dimension* of $F \subset \mathbb{R}^n$ is

$$\underline{\dim}_B(F) := \liminf_{\delta \to 0} \frac{\log N_\delta^B(F)}{-\log \delta}$$

and the *upper box-counting dimension* is

$$\overline{\dim}_B(F) := \limsup_{\delta \to 0} \frac{\log N_\delta^B(F)}{-\log \delta}.$$

If $\underline{\dim}_B(F) = \overline{\dim}_B(F)$, then we say the *box-counting dimension* of F is

$$\dim_B(F) := \lim_{\delta \to 0} \frac{\log N_\delta^B(F)}{-\log \delta}.$$

Theorem 1.2.1. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be Lipschitz on $F \subset \mathbb{R}^n$. Then $\underline{\dim}_B(f(F)) \leq \underline{\dim}_B(F)$ and $\overline{\dim}_B(f(F)) \leq \overline{\dim}_B(F)$. If f is bi-Lipschitz on F, then $\underline{\dim}_B(f(F)) = \underline{\dim}_B(F)$ and $\overline{\dim}_B(f(F)) = \overline{\dim}_B(F)$.*

Proof. Let $\delta > 0$ be given. Let $\{U_i\}_{i=1}^k$ be a minimal δ-cover of F. Thus, $\{U_i \cap F\}$ is also a minimal δ-cover of F. Note that for all $x, y \in U_i \cap F$, $|x - y| \leq \delta$. We also know that $\{f(U_i \cap F)\}$ is a $c\delta$-cover since $|f(x) - f(y)| \leq c|x - y| \leq c\delta$. However, this cover is not necessarily minimal and thus

$$N_{c\delta}(f(F)) \leq N_\delta(F).$$

Ergo,

$$\frac{\ln N_{c\delta}(f(F))}{-\ln(c\delta) + \ln \delta} \leq \frac{\ln N_\delta(F)}{-\ln \delta}$$

and $\underline{\dim}_B(f(F)) \leq \underline{\dim}_B(F)$ and $\overline{\dim}_B(f(F)) \leq \overline{\dim}_B(F)$.

If f is bi-Lipschitz, then f^{-1} is Lipschitz on $f^{-1}(F)$. Thus $\underline{\dim}_B(f^{-1}(f(F))) \leq \underline{\dim}_B(f(F))$ and $\overline{\dim}_B(f^{-1}(f(F))) \leq \overline{\dim}_B(f(F))$. Therefore, $\underline{\dim}_B(f(F)) = \underline{\dim}_B(F)$ and $\overline{\dim}_B(f(F)) = \overline{\dim}_B(F)$. □

Theorem 1.2.2. *Suppose f satisfies the Hölder condition, that is for some $c, \alpha > 0$*

$$|f(x) - f(y)| \leq c|x - y|^\alpha.$$

Then $\underline{\dim}_B(f(F)) \leq (1/\alpha)\underline{\dim}_B(F)$ and $\overline{\dim}_B(f(F)) \leq (1/\alpha)\overline{\dim}_B(F)$.

Proof. Let $\delta > 0$ be given. Let $\{U_i\}_{i=1}^k$ be a minimal δ-cover of F. Thus, $\{U_i \cap F\}$ is also a minimal δ-cover of F. Note that for all $x, y \in U_i \cap F$, $|x - y| \leq \delta$. We also know that $\{f(U_i \cap F)\}$ is a $c\delta^\alpha$-cover since $|f(x) - f(y)| \leq c|x - y|^\alpha \leq c\delta^\alpha$. However, this cover is not necessarily minimal and thus

$$N_{c\delta^\alpha}(f(F)) \leq N_\delta(F).$$

Ergo,

$$\frac{\ln N_{c\delta^\alpha}(f(F))}{-\ln(c\delta^\alpha) + \ln \delta} \leq \frac{\ln N_\delta(F)}{-\alpha \ln \delta}$$

and $\underline{\dim}_B(f(F)) \leq (1/\alpha)\underline{\dim}_B(F)$ and $\overline{\dim}_B(f(F)) \leq (1/\alpha)\overline{\dim}_B(F)$. □

1.3 Examples

For our first example, we take the box-counting dimension of one of a canonical fractal, the Cantor Set.

Example 1.3.1. *Let C denote the middle-λ Cantor set for $0 < \lambda < 1$.*

$$\dim_B(C) = \frac{\ln 2}{\ln\left(\frac{2}{1-\lambda}\right)}$$

Proof. To begin, we must determine the length of each interval of C_n after we remove the middle $1/\lambda$ from each of them. Assume we are removing the middle λ of the interval $[0, a]$ where a is a positive real number. Since we removed $a\lambda$ from $[0, a]$, we have exactly $a - a\lambda = a(1 - \lambda)$ length remaining. Since we removed the middle a/λ, we equally distribute the remaining length into two subintervals of length $a(1 - \lambda)/2$.

Let l_n represent the length of each interval in C_n and define $l_0 = 1$. Thus, $l_1 = (1 - \lambda)/2$ and $l_{n+1} = l_n(1 - \lambda)/2$ by our previous derivation. Solving this recursion yields $l_n = [(1 - \lambda)/2]^n$.

Let $\delta > 0$ be given. Choose n such that $l_{n+1} \leq \delta < l_n$. Since $\delta \geq l_{n+1}$, we need no more 2^{n+1} sets of diameter δ to cover C because there are exactly 2^{n+1} intervals of length l_{n+1} in C_{n+1}. Furthermore, since $\delta < l_n$, we need at least 2^n sets of diameter δ to cover C becayse there are exactly 2^n intervals of length l_n is C_n. Thus yielding the inequality

$$2^n \leq N_\delta^D(C) \leq 2^{n+1}.$$

Taking logs and doing some manipulation yields

$$\frac{n \ln 2}{(n+1)\ln\left(\frac{2}{1-\lambda}\right)} \leq \frac{\ln N_\delta^D(C)}{-\ln \delta} \leq \frac{(n+1)\ln 2}{n \ln\left(\frac{2}{1-\lambda}\right)}.$$

Using L'Hôpital's rule yields

$$\dim_B(C) = \frac{\ln 2}{\ln\left(\frac{2}{1-\lambda}\right)}.$$

□

We can expand this logic to more general forms of the Cantor set, and present two examples of it here.

Example 1.3.2. *Let F be the set containing all numbers in $[0, 1]$ that do not have a 5 in their decimal expansion. Then,*

$$\dim_B(F) = \frac{\ln 9}{\ln 10}.$$

Proof. We perform a similar construction to the middle-λ Cantor set. Define $F_n = [0, 10^{-n}]$ where n is a non-negative integer. In our construction, we will spilt F_n into ten intervals of length 10^{n+1} and remove the interval $(5 \cdot 10^{n+1}, 6 \cdot 10^{n+1})$, removing any number with a 5 in the $n+1$st decimal place from F_n. We now have nine subintervals left, all of which are copies of F_{n+1}. This construction means that we have 9^n copies of F_n for each n.

Let $\delta > 0$ be given. Choose n such that $10^{-(n+1)} \leq \delta < 10^{-n}$. Since $\delta \geq 10^{-(n+1)}$, we need no more than 9^{n+1} sets of diameter δ to cover F. Since $\delta < 10^{-n}$, we need at least 9^n sets of diameter δ to cover F. Thus we have the following inequality

$$9^n \leq N_\delta^D(F) \leq 9^{n+1}.$$

By taking logs and manipulating the inequality we get,

$$\frac{n \ln 9}{(n+1)\ln 10} \leq \frac{\ln N_\delta^D(F)}{-\ln \delta} \leq \frac{(n+1)\ln 9}{n \ln 10}.$$

Since both the far left and far right tend to $\frac{\ln 9}{\ln 10}$, by Squeeze theorem, we have

$$\dim_B(F) = \frac{\ln 9}{\ln 10}.$$

□

Example 1.3.3. *The Cantor Dust has* $\dim_B = 1$*.*

Proof. To construct the Cantor Dust, we consider the unit square, call this C_0. Then subdivide C_0 into 16 more squares and keep precisely 4 of them, call them copies of C_1. We iterate this process by splitting C_n into 16 squares and keeping 4 of them, while calling each one C_{n+1}. We should note that the side length of all of the 4^n copies of C_n is $(1/4)^n$.

Let $\delta > 0$ be given. We begin by selecting an n such that $(1/4)^{n+1} \leq \delta < (1/4)^n$. Thus we need no more than 4^{n+1} squares of side length δ and no fewer than 4^n squares of side length δ to cover C. However, since this is a cover of squares of side length δ, we have a cover using balls of diameter $\sqrt{2}\delta$, by Pythagorean theorem. Thus yielding the following inequality

$$4^{n+1} \leq N_{\sqrt{2}\delta}^D \leq 4^n.$$

After taking logs and some algebraic manipulation, we have

$$\frac{(n+1)\ln 4}{n \ln 4} \leq \frac{\ln N_{\sqrt{2}\delta}^D}{-\ln \delta} \leq \frac{n \ln 4}{(n+1)\ln 4}.$$

After we take the limit of both sides we get, $\dim_B(C) = 1$. □

Furthermore, we can take the box-counting dimension of countable sets as well.

Example 1.3.4. *Let* $S = \{(n!)^{-1}\}_{n=1}^\infty \cup \{0\}$*, then*

$$\dim_B(S) = 0.$$

Proof. We begin by computing $l_S(n)$ and $r_S(n)$.

$$l_S(n) = \frac{n}{(n+1)!}$$
$$r_S(n) = \frac{n-1}{n!}$$

We then invoke Lemma 1.4.1 to get

$$\frac{\ln k}{-\ln(k-1) + \ln(k!)} \leq \frac{\ln N_\delta^C(S)}{-\ln \delta} \leq \frac{\ln 2 + \ln k}{-\ln k + \ln((k+1)!)}.$$

Since the limit of both the far right and the far left of the previous inequality is zero, $\dim_B(S) = 0$. □

1.4 Properties of the Box-Counting Dimension

When we speak about dimensions, there are a few properties we want them to have, namely:

1. if $E \subseteq F$, then $\dim E \leq \dim F$;
2. any open set $O \subseteq \mathbb{R}^n$ should have $\dim O = n$;
3. when $F \subseteq \mathbb{R}^n$, $0 \leq \dim F \leq n$;
4. if $\{S_i\}_{i\in I}$ is a countable collection of sets, then $\dim \bigcup_{i\in I} S_i = \sup\limits_{i \in I} \dim S_i$;
5. if S is countable, then $\dim S = 0$.

However, while the Box-counting dimension is nice for our Cantor-like sets and respects our monotonicity, it has a certain undesirable property, that is countable sets can have non-zero dimension.

Lemma 1.4.1. *Let $\{x_n\}_{n=1}^\infty$, be a monotone decreasing sequence in $[0, 1]$ that tends to 0 and let $0 < \delta < 1/2$. Let $S = \{x_n\}_{n=1}^\infty \cup \{0\}$. Define*

$$l_S(n) := x_n - x_{n+1}$$
$$r_S(n) := x_{n-1} - x_n.$$

Let $k \in \mathbb{N}$ such that $l_S(k) \leq \delta < r_S(k)$, then

$$k \leq N_\delta^D(S) \leq 2k.$$

and

$$\frac{\ln k}{-\ln r_S(k)} \leq \frac{\ln N_\delta^C}{-\ln \delta} \leq \frac{\ln(2k)}{-\ln l_S(k)}.$$

Proof. If we cover using sets of diameter $r_S(k)$, then we can fit at most one point from $\{x_j\}_{j=1}^k$ in each set. Thus implying that $k \leq N_\delta^D(F)$. If we cover using sets of diameter $l_S(k)$, then we can cover $[0, x_k]$ with at most, $k+1$ sets and $\{x_j\}_{j=1}^{k-1}$ with $k-1$ sets. Thus implying that $k \leq N_\delta^D(F) \leq 2k$, and the logarithm inequality directly follows. $\square$

Example 1.4.2. *Let $j \in \mathbb{R}^+$, then the box-counting dimension of $S_j = \{n^{-j}\}_{n=1}^\infty \cup \{0\}$ is*

$$\dim_B(S_j) = \frac{1}{j+1}.$$

Proof. From the construction of S_j we know that

$$l_{S_j}(k) = \frac{1}{k^j} - \frac{1}{(k+1)^j} = \frac{(k+1)^j - k^j}{k^j(k+1)^j}$$

and

$$r_{S_j}(k) = \frac{1}{(k-1)^j} - \frac{1}{k^j} = \frac{k^j - (k-1)^j}{k^j(k-1)^j}.$$

By Lemma 1.4.1, we know that

$$\frac{\ln k}{-\ln r_{S_j}(k)} \leq \frac{\ln N_\delta^C}{-\ln \delta} \leq \frac{\ln(2k)}{-\ln l_{S_j}(k)}.$$

Thus, the box-counting dimension is in between the limits of the far left and far right of this inequality. Consider $\ln k/(-\ln r_{S_j}(k))$.

$$\frac{\ln k}{-\ln r_{S_j}(k)} = \frac{\ln k}{-\ln(k^j - (k-1)^j) + j\ln k + j\ln(k+1)}$$

Consider $n^j - (n-1)^j$.

$$n^j - (n-1)^j = n^{j-1}\left(n - (n-1)\left(\frac{n-1}{n}\right)^{j-1}\right) = n^{j-1}\left(n - (n-1)\left(1 - \frac{1}{n}\right)^{j-1}\right)$$

Thus,

$$\frac{\ln k}{-\ln r_{S_j}(k)} = \frac{\ln k}{-\ln(k - (k-1)(1-1/k)^{j-1}) + \ln k + j\ln(k+1)}$$

which tends to $1/(j+1)$. Consider $\ln 2k/(-\ln l_{S_j}(k))$.

$$\frac{\ln 2k}{-\ln l_{S_j}(k)} = \frac{\ln 2k}{-\ln((k+1)^j - k^j) + j\ln k + j\ln(k-1)}$$

Consider $(n+1)^j - n^j$.

$$(n+1)^j - n^j = n^{j-1}\left((n+1)\left(\frac{n+1}{n}\right)^{j-1} - n\right) = n^{j-1}\left((n+1)\left(1 + \frac{1}{n}\right)^{j-1} - n\right)$$

Thus,

$$\frac{\ln 2+\ln k}{-\ln l_{S_j}(k)} = \frac{\ln 2+\ln k}{-\ln((k+1)(1+1/k)^{j-1}-k)+\ln k+j\ln(k+1)}$$

which tends to $1/(j+1)$ and by Squeeze theorem, $\dim_B(S_j)$ is $1/(j+1)$. □

Corollary 1.4.3. $\dim_B\{1/n\}_{n=1}^{\infty} = 1/2 \neq 0$

Corollary 1.4.4. *The box-counting dimension is unstable.*

Proof. Proceed via contradiction. Assume that

$$\dim_B \bigcup_{n=1}^{\infty}\{1/n\} = \sup_{n\in\mathbb{N}} \dim_B\{1/n\}. \tag{1.1}$$

We need to find the dimension of a singleton. Since this is a singleton, we can cover it with exactly one set regardless of δ, thus

$$\dim_B\{x\} = \lim_{\delta\to 0}\frac{\ln N_\delta(\{x\})}{-\ln\delta} = \lim_{\delta\to 0}\frac{\ln 1}{-\ln\delta} = 0.$$

Therefore, the righthand side of Equation 1.1 is equal to zero. However, by the Corollary 1.4.3 we know the lefthand side is equal to one half, thus creating a contradiction. □

The intuition behind this result is that $N_\delta(F)$ must count the cardinality of the cover instead of measuring how much it contributes, thus a singleton in the cover contributes the same as an entire interval. In essence, this problem exists because $N_\delta(F)$ is like the counting measure instead of the Lebesgue measure. To solve this problem, we must introduce a way to determine how much each set in the cover contributes to the dimension; that is, we introduce the Hausdorff measure.

Chapter 2

Hausdorff Dimension

2.1 Hausdorff Measure

2.2 Definition

2.3 Properties

Appendix A: Definitions

Set Theory

Definition. A *relation* is a set of ordered pairs.

Definition. Let A and B be nonempty sets and let f be a relation between them. Then f is a *function* if and only if for $(x, y), (x, z) \in f$ then $y = z$.

Example. *Let A be a nonempty set. Then the identity map on A, $id_A(x) = x$, is a function. On $\mathbb{R}$, $\sin x$, and $\cos x$ are functions.*

Definition. Let $f : A \to B$ be a function. Then:

- f is *injective* if and only if for all $a_1, a_2 \in A$, $f(a_1) = f(a_2)$ implies $a_1 = a_2$, moreover f is called an *injection*;
- f is *surjective* if and only if for all $b \in B$ there exists an $a \in A$ such that $b = f(a)$, moreover f is called a *surjection*;
- f is *bijective* if and only if f is injective and surjective, moreover f is called a *bijection*.

Example. *The map $f : \mathbb{R} \to [-1, 1]$ defined by $f(x) = \sin x$ is surjective, but not injective. The map $g : \mathbb{N} \to \mathbb{R}$ defined by $g(x) = x$ is injective, but not surjective. The map $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = x$ is bijective.*

Definition. A nonempty set A is *finite* if and only if there exists an $n \in \mathbb{N}$ such that there exists a bijection between A and $1, 2, \ldots, n$.

Example. *$\mathbb{Z}_7$ is finite since the map $f : \{1, 2, \ldots, 7\} \to \mathbb{Z}_7$ defined by $f(n) = \overline{n - 1}$ is a bijection.*

Definition. A nonempty set A is *infinite* if and only if A is not finite.

Example. *$\mathbb{N}$ is infinite.*

Definition. A nonempty set A is *countably infinite* if and only if there exists a bijection between A and $\mathbb{N}$.

Example. *$\mathbb{Z}$ is countably infinite since the map $f : \mathbb{N} \to \mathbb{Z}$ given by $f(n) = (-1)^n \lfloor n/2 \rfloor$ is a bijection.*

Definition. A nonempty set A is *countable* if and only if A is finite or countably infinite.

Example. *The sets, $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{A},$ and $\mathbb{Z}_n$ are countable.*

Definition. A nonempty set A is *uncountable* if and only if $|A| > \aleph_0$.

Example. *The sets $\mathbb{R}, \mathbb{C}, \mathbb{H}$ are uncountable.*

Definition. A *poset* is a nonempty set A with ordering $\preceq$ satisfying the following for all $a, b, c \in A$:

- $a \preceq a$;
- $a \preceq b$ and $b \preceq a$ implies that $a = b$;
- $a \preceq b$ and $b \preceq c$ implies that $a \preceq c$.

Example. *Let A be a nonempty set. Then, $\mathcal{P}(A)$ with the ordering $\subseteq$ is a poset.*

Definition. A poset $(A, \preceq)$ is *totally ordered* if and only if for all $a, b \in A$, $a \preceq b$ or $b \preceq a$.

Example. *$(\mathbb{R}, \leq)$ is a totally ordered set.*

Definition. Let $(S, \leq)$ be a totally ordered set and let $A \subseteq S$. We say that A is *bounded above* if and only if there exists a $s \in S$ such that for all $a \in A$, $a \leq s$; we say that s is an upper bound of A. We say that A is *bounded below* if and only if there exists a $r \in S$ such that for all $a \in A$, $r \leq a$; we say that r is a lower bound of A.

Example. *$\mathbb{N}$ is bounded below by 0. $\{1, 2, \ldots n\}$ is bounded below by 1 and bounded above by n.*

Definition. Let $(S, \leq)$ be a totally ordered set and let $A \subseteq S$ be bounded above. Then the *supremum* of A, denoted $\sup A$, is an upper bound of A, α, with the property that if β is also an upper bound of A then $\alpha \leq \beta$.

Example. *The supremem of $(-1, 1)$ is 1 and the supremum of $(-\pi, \pi]$ is π.*

Definition. Let $(S, \leq)$ be a totally ordered set and let $A \subseteq S$ be bounded below. Then the *infinimum* of A, denoted $\inf A$, is a lower bound of A, ζ, with the property that if η is also a lower bound of A then $\eta \leq \zeta$.

Example. *The infinimum of $(-1, 1)$ is -1 and the infinimum of $[-\pi, \pi)$ is $-\pi$.*

Topological Spaces

Definition. Let X be a nonempty set and let τ be a collection of subsets of X. Then τ is called a *topology* on X if and only if τ satisfies all of the following:

- $X \in \tau$ and $\emptyset \in \tau$;
- if $U, V \in \tau$, then $U \cap V \in \tau$;
- if $\{U_i\}_{i \in I} \subseteq \tau$ then $\cup_{i \in I} U_i \in \tau$.

We say that (X, τ) form a *topological space*. Furthermore, any set in τ is called *open* and for any $x \in X$ a set containing x is called a *neighborhood* of x. Moreover, a set $A \subseteq X$ is said to be *closed* if and only if $(X \setminus A) \in \tau$.

Example. *Let A be a nonempty set, then $(A, \mathcal{P}(A))$ forms a topology.*

Definition. Let (X, τ) be a topological space and let $A \subseteq X$. Then the *interior* of A is defined as

$$A^o = \{x \in A | \exists V \in \tau \text{ s.t. } (x \in V) \wedge (V \subseteq A)\}.$$

Example. *In $\mathbb{R}$, the interior of $\mathbb{Q}$ is the empty set under the usual metric topology.*

Definition. Let (X, τ) be a topological space and let $A \subseteq X$. A point $x \in X$ is called an *accumulation point* of A if and only if for all neighborhoods of x, V, $(V \cap A) \setminus x \neq \emptyset$.

Example. 0 *is an accumulation point of* $(0, 1)$.

Definition. Let (X, τ) be a topological space and let $A \subseteq X$. Then the *closure* of A, denoted $\bar{A}$, is the set A together with all of its accumulation points.

Example. *The closure of* $(0, 1)$ *is* $[0, 1]$.

Definition. Let (X, τ) be a topological space and let $A \subseteq X$. A sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq X$ *coverges* to $x \in X$ if and only if for all neighborhoods of U of x, there exists an $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$.

Example. *The sequence* $\{1/n\}_{n \in \mathbb{N}}$ *converges to* 0 *in* $\mathbb{Q}$.

Definition. Let (X, τ) be a topological space and let $A \subseteq X$. Then A is called *perfect* if and only if all $x \in A$ are accumulation points of A.

Example. *Any open interval is perfect.*

Definition. Let (X, τ_X) and (Y, τ_Y) be topological spaces. A function $f : X \to Y$ is said to be *continuous* at $x \in X$ if and only if for each $V \in \tau_Y$ containing $f(x)$, there exists a $U \in \tau_X$ containing x such that $f(U) \subseteq V$. Furthermore, f is said to be continuous on $A \subseteq X$ if and only if f is continuous at each $x \in A$.

Example. *$f(x) = x^2$ is continuous on $\mathbb{R}$ under the metric topology.*

Definition. Let (X, τ) be a topological space and let $A \subseteq X$ A family $\{A_i\}_{i \in I} \subset \mathcal{P}(X)$ is called a *cover* of A if and only if $A \subseteq \cup_{i \in I} A_i$. If $\{A_j\}_{j \in J \subseteq I}$ is also a cover of A, then, $\{A_j\}_{j \in J}$ is called a *subcover* of A. If a cover of A is formed only by open sets, then it is called an *open cover* of A.

Example. *$[0, 100]$ is a cover of $(10, 11)$ that has a subcover of $[9, 12]$. $\{B(x, 1)\}_{x \in \mathbb{R}}$ is an open cover of $\mathbb{R}$.*

Definition. Let (X, τ) be a topological space and let $A \subseteq X$. We say A is *compact* if and only if each open cover of A admits a finite subcover.

Example. *Any closed interval is compact in $\mathbb{R}$ under the usual metric topology.*

Definition. Let (X, τ) be a topological space and let $A \subseteq X$. We say A is *disconnected* if and only if there exist disjoint sets $U, V \subseteq X$ such that

- $A \subset U \cup V$
- $A \cap U \neq \emptyset \ \wedge \ A \cap V \neq \emptyset$
- $A \cap U \cap V = \emptyset$.

If A is not disconnected, we say that A is *connected*.

Example. *$[0, 1]$ is connected and $(0, 1/2) \cup (1/2, 1)$ is disconnected.*

Definition. Let (X, τ) be a topological space and let $A \subseteq X$. Then A is *totally disconnected* if and only if for any $x, y \in A$ there exist disjoint $U, V \in \tau$ such that $x \in U$, $y \in V$ and $A \subset U \cup V$.

Example. *The rational numbers are totally disconnected in $\mathbb{R}$.*

Definition. Let (X, τ) be a topological space and let $A \subseteq X$. We say that A is *dense* in X if and only if $\bar{A} = X$.

Example. *$\mathbb{Q}$ is dense in $\mathbb{R}$.*

Definition. Let (X, τ) be a topological space and let $A \subseteq X$. We say that A is *nowhere dense* in X if and only if $\left(\bar{A}\right)^o = \emptyset$.

Example. *$\mathbb{Z}$ is nowhere dense in $\mathbb{R}$.*

Definition. Let (X, τ_x) and (Y, τ_y) be topological spaces. We say that X and Y are *homeomorphic* if and only if there exists a continuous bijection $f : (X, \tau_x) \to (Y, \tau_y)$ where f^{-1} is also continuous.

Example. *For $a, b \in \mathbb{R}$ with $a < b$, the sets (a, b) and $\mathbb{R}$ are homeomorphic under the standard metric topology.*

Definition. Let X be a nonempty set. Let $\mathcal{B}$ be a collection of subsets of X such that

- for each $x \in X$ there is a $B \in \mathcal{B}$ such that $x \in B$
- and if $x \in B_1 \cap B_2$ where $B_1, B_2 \in \mathcal{B}$, then there exists a $B \in \mathcal{B}$ such that $x \in B \subseteq B_1 \cap B_2$

then $\mathcal{B}$ is called a *basis* for a topology on X. Furthermore, the *topology generated by* $\mathcal{B}$ is given by

$$\tau = \{U \subseteq X | \forall x \in U, \exists B \in \mathcal{B} \text{ s.t. } x \in B \subseteq U\}.$$

Example. *Let (X, d) be a metric space then the metric topology induced by d is given by*

$$\tau_d = \{U \subseteq X | x \in U \Rightarrow \exists r > 0 \textit{ s.t. } B(x, r) \subseteq U\}.$$

Metric Spaces

Definition. A set X and a metric $d : X^2 \to \mathbb{R}$ form a *metric space* (X, d) if and only if all of the following are satisfied:

1. for all $x, y \in X$, $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$;
2. for all $x, y \in X$, $d(x, y) = d(y, x)$;
3. and for all $x, y, z \in X$ $d(x, y) \leq d(x, z) + d(z, x)$.

Example. $\mathbb{R}$ *and absolute value form a metric space.*

Definition. Let (X, d) be a metric space. Let $x_0 \in X$ and $r \in \mathbb{R}$. An *open ball* in (X, d) around x_0 of radius r is the set $B(x_0, r) = \{x \in X | \, d(x_0, x) < r\}$.

Example. *In* $\mathbb{R}$*,* $B(x_0, r) = (x_0 - r, x_0 + r)$.

Definition. Let (X, d) be a metric space. A subset of A of (X, d) is called *open* if and only if for each $x \in A$, there exists an $r_x > 0$ such that $B(x, r_x) \subseteq A$.

Example. *The set* $(-n, n) \cup (k, k + 1)$ *is open in* $\mathbb{R}$ *for* $k, n \in \mathbb{R}$.

Definition. Let (X, d) be a metric space. A subset of A of (X, d) is called *closed* if and only if $X \setminus A$ is open.

Example. $\mathbb{R}$ *is closed in* $\mathbb{R}$.

Definition. Let (X, d) be a metric space. The *closed ball* of radius r about $x \in X$ is $\bar{B}(x, r) = \{y \in X | d(x, y) \leq r\}$.

Example. *In* $\mathbb{R}$*,* $\bar{B}(x_0, r) = [x_0 - r, x_0 + r]$.

Definition. Let (X, d) be a metric space. The *diameter* of a nonempty subset, A, of X is $\sup\limits_{x,y \in A} d(x, y)$.

Example. *The diameter of* $(0, 1) \cup (10, 12)$ *in* $(\mathbb{R}, |\cdot|)$ *is 12.*

Definition. Let (X, d) be a metric space and let $A \subseteq X$. A point $x \in X$ is a *closure point* of A if and only if for all $r > 0$, $B(x, r) \cap A \neq \emptyset$.

Example. *The* 0 *is the only closure point of the set* $\{0\}$*. The point* 3 *is a closure point of* $(0, 3)$.

Definition. Let (X, d) be a metric space and let $A \subseteq X$. A point $x \in X$ is an *accumulation point* of A if and only if for all $r > 0$, $B(x, r) \cap A \setminus \{x\} \neq \emptyset$.

Example. *The set* $\{0\}$ *has no accumulation points since* $B(0, r) \cap \{0\} = \{0\}$*. The point* 3 *is an accumulation point of* $(0, 3)$.

Definition. Let (X, d) be a metric space and let $A \subseteq X$. Let $x \in X$. We say that the *closure* of A, denoted $\bar{A}$, is the set $\bar{A} = \{x \in A | \, \forall r > 0, \, B(x, r) \cap A \neq \emptyset\}$.

Example. *Under the Euclidean metric* $\mathbb{R}$ *is the closure of* $\mathbb{Q}$ *and* $[a, b]$ *is the closure of* (a, b).

Definition. Let (X, d) be a metric space and let $A \subseteq X$. Let $x \in A$. Then x is an *interior point* of A if and only if there exists an $r > 0$ such that $B(x, r) \subseteq A$. Moreover, the *interior* of A, denoted A^o is the set $A^o = \{x \in A | \, \exists r > 0, \, B(x, r) \subseteq A\}$.

Example. *An interior point of* $[0, 1]$ *under the Euclidean metric is* $1/2$ *and the interior of* $[a, b]$ *is* (a, b).

Definition. Let (X, d) be a metric space and let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in X. We say that x_n *converges* to x if and only if:

1. the real valued sequence $d(x_n, x) \to 0$;
2. for all $\varepsilon > 0$ there exists a $N_\varepsilon \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n \geq N_\varepsilon$.

Example. *In* $\mathbb{R}$*, the sequence* $\{1/n | n \in \mathbb{N}\}$ *converges to 0.*

Definition. Let (X, d) be a metric space and let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in X. We say that x_n is *Cauchy* in X if and only if for all $\varepsilon > 0$ there exists a $N_\varepsilon \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq N_\varepsilon$.

Example. *In* $\mathbb{Q}$*, the sequence*

$$\{\sum_{k=1}^{n} \frac{1}{n^2}\}_{n \in \mathbb{N}}$$

is Cauchy, but not convergent.

Definition. Let (X, d) be a metric space. We say (X, d) is *complete* if and only if all Cauchy sequences in X converge to some $x \in X$.

Example. $\mathbb{R}$ *is a complete metric space under the Euclidean metric.*

Definition. Let (X, d) be a metric space. We say (X, d) is *incomplete* if and only if there exists some sequence that is Cauchy in X but not convergent in X.

Example. $\mathbb{Q}$ *is incomplete under the Euclidean metric because*

$$\{\sum_{k=1}^{n} \frac{1}{n^2}\}_{n \in \mathbb{N}}$$

is Cauchy, but does not converge in $\mathbb{Q}$.

Definition. Let (X, d) and (Y, ρ) be metric spaces. A function $f : (X, d) \to (Y, \rho)$ is called *continuous* at $x_0 \in X$ if and only if for all $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$ such that $\rho(f(x), f(x_0)) > \varepsilon$ whenever $d(x, x_0) < \delta_\varepsilon$.

Example. *All $\mathbb{C}$-valued polynomials are continuous on $\mathbb{C}$ under the Euclidean metric.*

Definition. Let (X, d) and (Y, ρ) be metric spaces. A function $f : (X, d) \to (Y, \rho)$ is called *uniformly continuous* if and only if for all $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$ such that $\rho(f(x), f(y)) > \varepsilon$ whenever $d(x, y) < \delta_\varepsilon$.

Example. *Any differentiable function with a bounded derivative is uniformly continuous. E.g. $f(x) = ax + b$ where a and b are constants.*

Definition. Let (X, d) and (Y, ρ) be metric spaces. A function $f : (X, d) \to (Y, \rho)$ is called an *isometry* if and only if for all $x, y \in X$, $\rho(f(x), f(y)) = d(x, y)$.

Example. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := x + b$ for any $b \in \mathbb{R}$. Then f is an isometry.*

Definition. We say that metric spaces (X, d) and (Y, ρ) are *isometric* if and only if there exists a surjective isometry between them.

Example. *The map, $f : \mathbb{R}^2 \to \mathbb{C}$ given by $f(x, y) = x + iy$ is a surjective isometry and thus $\mathbb{R}^2$ and $\mathbb{C}$ are isometric.*

Definition. Let (Y, ρ) be a complete metric space and let (X, d) be a metric space. Then, (Y, ρ) is called the *completion* of (X, d) if and only if there exists an isometry $f : (X, d) \to (Y, \rho)$ such that the image $f(X)$ is dense in Y, that is $\overline{f(X)} = Y$.

Example. *Let $f : \mathbb{Q} \to \mathbb{R}$ be defined by $f(x) := x$. Then f is an isometry such that $\overline{f(\mathbb{Q})} = \mathbb{R}$ and thus $(\mathbb{R}, |\cdot|)$ is the completion of $(\mathbb{Q}, |\cdot|)$.*

Measure Theory

Definition. Let S be a nonempty set. A collection $\mathcal{F}(S)$ of subsets of S is called a *σ-algebra* on S if and only if all of the following are satisfied:

- $\emptyset \in \mathcal{F}(S)$;
- $A \in \mathcal{F}(S) \Rightarrow A \cap B \in \mathcal{F}(S)$;
- and $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{F}(S) \Rightarrow \cup_{i\in\mathbb{N}} A_i \in \mathcal{F}(S)$.

Example. *The smallest σ-algebra on any set S is $\{\emptyset, S\}$, while the largest is $\mathcal{P}(S)$.*

Definition. Let (X, τ) be a topological space, then the *σ-algebra generated by τ* is the smallest σ-algebra containing τ.

Example. *In $\mathbb{R}$, the σ-algebra generated by the Euclidean metric topology is called the Borel σ-algebra.*

Definition. Let X be a nonempty set and let $\mathcal{F}(X)$ be a σ-algebra on X. A function $\mu : \mathcal{F}(S) \to [0, \infty)$ is called a *measure* if and only if all of the following are satisfied:

- $\mu(\emptyset) = 0$;
- and if $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{F}(X)$ and $A_i \cap A_j = \emptyset$ if $i \neq j$ then $\mu(A_i \cup A_j) = \mu(A_i) + \mu(A_j)$ and $\mu\left(\cup_{i\in\mathbb{N}} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$.

Furthermore, $(X, \mathcal{F}(X), \mu)$ form a *measure space*. Additionally a function $f : X \to \mathbb{R}$ is called *measurable* if and only if for all $\alpha \in \mathbb{R}$, $A_\alpha = \{x \in X | f(x) > \alpha\} \in \mathcal{F}(\mathcal{X})$.

Example. *Let X be a non-empty set and let $A \subseteq X$. The function $\mu : \mathcal{P}(X) \to [0, \infty]$ given by*

$$\mu(A) = \begin{cases} |A| & A \text{ finite} \\ \infty & A \text{ infinite} \end{cases}$$

is a measure called the counting measure.

Definition. If μ is a measure on a σ-algebra $\mathcal{F}(X)$ on X, then a set $A \subseteq X$ is called a *null set* if and only if $\mu(A) = 0$.

Example. *In the Lesbesque measure, any countable set is a null set.*

Definition. A function $f : X \to \mathbb{R}$ is called a *simple function* if and only if f has only finitely many values.

Example. *Any constant function is simple.*

Definition. The *integral* of a non-negative simple function in standard form $\phi = \sum_{i=1}^n a_i \chi_{A_i}$ is

$$\int \phi d\mu = \sum_{i=1}^n a_i \mu(A_i).$$

The integral of a non-negative measurable function is

$$\int f d\mu = \sup_{\phi \text{ simple, non-negative, } 0 \le \phi \le f} \int \phi d\mu.$$

The integral of a measurable function f is given by

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu$$

where $f^+(x) = \{\sup\} f(x), 0$ and $f^-(x) = \sup\{0, -f(x)\}$.

Bibliography

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