Fractal Analysis

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Chapter 1

Box-counting Dimension

1.1 Introduction

1.2 Covers

Let F be a subset of $\mathbb{R}^n$. We define the following covers:

$$\mathcal{B}_\delta(F) := \{B = \{B(x_j, \delta/2)\}_{j=1}^{n_B} | F \subseteq \cup B\}$$
$$\mathcal{D}_\delta(F) := \{D = \{D_j\}_{j=1}^{n_D} | F \subseteq \cup D \ \wedge \ \text{diameter}(D_j) \leq \delta \ \forall \ D_j \in D\}.$$

We now define the following counts

$$N_\delta^B(F) := \min_{B \in \mathcal{B}_\delta(F)} \text{card}(B)$$
$$N_\delta^D(F) := \min_{D \in \mathcal{D}_\delta(F)} \text{card}(D).$$

Theorem 1.2.1.

$$N_\delta^B(F) = N_\delta^D(F)$$

Proof. Let B be a cover in $\mathcal{B}_\delta(F)$ such that $\text{card}(B) = N_\delta^B(F)$. Likewise let D be a cover in $\mathcal{D}_\delta(F)$ such that $\text{card}(D) = N_\delta^D(F)$. Since each set in B has diameter δ $B \in \mathcal{D}_\delta(F)$, and by minimality, $N_\delta^D(F) \leq N_\delta^B(F)$. Since each set in D has diameter no more than δ, we can encapsulate each set in D with a single ball of radius $\delta/2$, call this new cover $B(D)$. By construction, $D(B) \in \mathcal{B}_\delta(F)$ and $\text{card}(D(B)) = N_\delta^D(F)$. Again, by minimality, $N_\delta^B(F) \leq N_\delta^D(F)$ and $N_\delta^B(F) = N_\delta^D(F)$. $\square$

1.3 Box-counting Dimension

Definition 1.3.1. We say the *lower box-counting dimension* of $F \subset \mathbb{R}^n$ is

$$\underline{\dim}_B(F) := \liminf_{\delta \to 0} \frac{\log N_\delta^B(F)}{-\log \delta}$$

and the *upper box-counting dimension* is

$$\overline{\dim}_B(F) := \limsup_{\delta\to 0} \frac{\log N_\delta^B(F)}{-\log \delta}.$$

If $\underline{\dim}_B(F) = \overline{\dim}_B(F)$, then we say the *box-counting dimension* of F is

$$\dim_B(F) := \lim_{\delta\to 0} \frac{\log N_\delta^B(F)}{-\log \delta}.$$

Theorem 1.3.1. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be Lipschitz on $F \subset \mathbb{R}^n$. Then $\underline{\dim}_B(f(F)) \le \underline{\dim}_B(F)$ and $\overline{\dim}_B(f(F)) \le \overline{\dim}_B(F)$. If f is bi-Lipschitz on F, then $\underline{\dim}_B(f(F)) = \underline{\dim}_B(F)$ and $\overline{\dim}_B(f(F)) = \overline{\dim}_B(F)$.*

Proof. Let $\delta > 0$ be given. Let $\{U_i\}_{i=1}^k$ be a minimal δ-cover of F. Thus, $\{U_i \cap F\}$ is also a minimal δ-cover of F. Note that for all $x, y \in U_i \cap F$, $|x - y| \le \delta$. We also know that $\{f(U_i \cap F)\}$ is a $c\delta$-cover since $|f(x) - f(y)| \le c|x - y| \le c\delta$. However, this cover is not necessarily minimal and thus

$$N_{c\delta}(f(F)) \le N_\delta(F).$$

Ergo,

$$\frac{\ln N_{c\delta}(f(F))}{-\ln(c\delta) + \ln \delta} \le \frac{\ln N_\delta(F)}{-\ln \delta}$$

and $\underline{\dim}_B(f(F)) \le \underline{\dim}_B(F)$ and $\overline{\dim}_B(f(F)) \le \overline{\dim}_B(F)$.

If f is bi-Lipschitz, then f^{-1} is Lipschitz on $f^{-1}(F)$. Thus $\underline{\dim}_B(f^{-1}(f(F))) \le \underline{\dim}_B(f(F))$ and $\overline{\dim}_B(f^{-1}(f(F))) \le \overline{\dim}_B(f(F))$. Therefore, $\underline{\dim}_B(f(F)) = \underline{\dim}_B(F)$ and $\overline{\dim}_B(f(F)) = \overline{\dim}_B(F)$. □

Theorem 1.3.2. *Suppose f satisfies the Hölder condition, that is for some $c, \alpha > 0$*

$$|f(x) - f(y)| \le c|x - y|^\alpha.$$

Then $\underline{\dim}_B(f(F)) \le (1/\alpha)\underline{\dim}_B(F)$ and $\overline{\dim}_B(f(F)) \le (1/\alpha)\overline{\dim}_B(F)$.

Proof. Let $\delta > 0$ be given. Let $\{U_i\}_{i=1}^k$ be a minimal δ-cover of F. Thus, $\{U_i \cap F\}$ is also a minimal δ-cover of F. Note that for all $x, y \in U_i \cap F$, $|x - y| \le \delta$. We also know that $\{f(U_i \cap F)\}$ is a $c\delta^\alpha$-cover since $|f(x) - f(y)| \le c|x - y|^\alpha \le c\delta^\alpha$. However, this cover is not necessarily minimal and thus

$$N_{c\delta^\alpha}(f(F)) \le N_\delta(F).$$

Ergo,

$$\frac{\ln N_{c\delta^\alpha}(f(F))}{-\ln(c\delta^\alpha) + \ln \delta} \le \frac{\ln N_\delta(F)}{-\alpha \ln \delta}$$

and $\underline{\dim}_B(f(F)) \le (1/\alpha)\underline{\dim}_B(F)$ and $\overline{\dim}_B(f(F)) \le (1/\alpha)\overline{\dim}_B(F)$. □

1.4 Examples

Example 1.4.1. *Let C denote the middle-λ Cantor set for $0 < \lambda < 1$.*

$$\dim_B(C) = \frac{\ln 2}{\ln\left(\frac{2}{1-\lambda}\right)}$$

Proof. To begin, we must determine the length of each interval of C_n after we remove the middle $1/\lambda$ from each of them. Assume we are removing the middle λ of the interval $[0, a]$ where a is a positive real number. Since we removed $a\lambda$ from $[0, a]$, we have exactly $a - a\lambda = a(1-\lambda)$ length remaining. Since we removed the middle a/λ, we equally distribute the remaining length into two subintervals of length $a(1-\lambda)/2$.

Let l_n represent the length of each interval in C_n and define $l_0 = 1$. Thus, $l_1 = (1-\lambda)/2$ and $l_{n+1} = l_n(1-\lambda)/2$ by our previous derivation. Solving this recursion yields $l_n = [(1-\lambda)/2]^n$.

Let $\delta > 0$ be given. Choose n such that $l_{n+1} \leq \delta < l_n$. Since $\delta \geq l_{n+1}$, we need no more 2^{n+1} sets of diameter δ to cover C because there are exactly 2^{n+1} intervals of length l_{n+1} in C_{n+1}. Furthermore, since $\delta < l_n$, we need at least 2^n sets of diameter δ to cover C becayse there are exactly 2^n intervals of length l_n is C_n. Thus yielding the inequality

$$2^n \leq N_\delta^D(C) \leq 2^{n+1}.$$

Taking logs and doing some manipulation yields

$$\frac{n \ln 2}{(n+1)\ln\left(\frac{2}{1-\lambda}\right)} \leq \frac{\ln N_\delta^D(C)}{-\ln \delta} \leq \frac{(n+1)\ln 2}{n \ln\left(\frac{2}{1-\lambda}\right)}.$$

Using L'Hôpital's rule yields

$$\dim_B(C) = \frac{\ln 2}{\ln\left(\frac{2}{1-\lambda}\right)}.$$

□

Example 1.4.2. *Let F be the set containing all numbers in $[0, 1]$ that do not have a 5 in their decimal expansion. Then,*

$$\dim_B(F) = \frac{\ln 9}{\ln 10}.$$

Proof. We perform a similar construction to the middle-λ Cantor set. Define $F_n = [0, 10^{-n}]$ where n is a non-negative integer. In our construction, we will spilt F_n into ten intervals of length 10^{n+1} and remove the interval $(5 \cdot 10^{n+1}, 6 \cdot 10^{n+1})$, removing any number with a 5 in the $n + 1$st decimal place from F_n. We now have nine subintervals left, all of which are copies of F_{n+1}. This construction means that we have 9^n copies of F_n for each n.

Let $\delta > 0$ be given. Choose n such that $10^{-(n+1)} \leq \delta < 10^{-n}$. Since $\delta \geq 10^{-(n+1)}$, we need no more than 9^{n+1} sets of diameter δ to cover F. Since $\delta < 10^{-n}$, we need at least 9^n sets of diameter δ to cover F. Thus we have the following inequality

$$9^n \leq N_\delta^D(F) \leq 9^{n+1}.$$

By taking logs and manipulating the inequality we get,

$$\frac{n \ln 9}{(n+1)\ln 10} \leq \frac{\ln N_\delta^D(F)}{-\ln \delta} \leq \frac{(n+1)\ln 9}{n \ln 10}.$$

Since both the far left and far right tend to $\frac{\ln 9}{\ln 10}$, by Squeeze theorem, we have

$$\dim_B(F) = \frac{\ln 9}{\ln 10}.$$

□

Example 1.4.3. *The Cantor Dust has* $\dim_B = 1$.

Proof. To construct the Cantor Dust, we consider the unit square, call this C_0. Then subdivide C_0 into 16 more squares and keep precisely 4 of them, call them copies of C_1. We iterate this process by splitting C_n into 16 squares and keeping 4 of them, while calling each one C_{n+1}. We should note that the side length of all of the 4^n copies of C_n is $(1/4)^n$.

Let $\delta > 0$ be given. We begin by selecting an n such that $(1/4)^{n+1} \leq \delta < (1/4)^n$. Thus we need no more than 4^{n+1} squares of side length δ and no fewer than 4^n squares of side length δ to cover C. However, since this is a cover of squares of side length δ, we have a cover using balls of diameter $\sqrt{2}\delta$, by Pythagorean theorem. Thus yielding the following inequality

$$4^{n+1} \leq N_{\sqrt{2}\delta}^D \leq 4^n.$$

After taking logs and some algebraic manipulation, we have

$$\frac{(n+1)\ln 4}{n \ln 4} \leq \frac{\ln N_{\sqrt{2}\delta}^D}{-\ln \delta} \leq \frac{n \ln 4}{(n+1)\ln 4}.$$

After we take the limit of both sides we get, $\dim_B(C) = 1$. □

However, while the Box-counting dimension is nice for our Cantor-like sets, it has a certain undesirable property, that is countable sets can have non-zero dimension.

Lemma 1.4.4. *Let* $\{x_n\}_{n=1}^\infty$, *be a monotone decreasing sequence in* $[0,1]$ *that tends to 0 and let* $0 < \delta < 1/2$. *Let* $S = \{x_n\}_{n=1}^\infty \cup \{0\}$. *Define*

$$l_S(n) := x_n - x_{n+1}$$
$$r_S(n) := x_{n-1} - x_n.$$

Let $k \in \mathbb{N}$ *such that* $l_S(k) \leq \delta < r_S(k)$, *then*

$$k \leq N_\delta^D(S) \leq 2k.$$

and

$$\frac{\ln k}{-\ln r_S(k)} \leq \frac{\ln N_\delta^C}{-\ln \delta} \leq \frac{\ln(2k)}{-\ln l_S(k)}.$$

Proof. If we cover using sets of diameter $r_S(k)$, then we can fit at most one point from $\{x_j\}_{j=1}^k$ in each set. Thus implying that $k \le N_\delta^D(F)$. If we cover using sets of diameter $l_S(k)$, then we can cover $[0, x_k]$ with at most, $k+1$ sets and $\{x_j\}_{j=1}^{k-1}$ with $k-1$ sets. Thus implying that $k \le N_\delta^D(F) \le 2k$, and the logarithm inequality directly follows. $\square$

Example 1.4.5. *Let $j \in \mathbb{R}^+$, then the box-counting dimension of $S_j = \{n^{-j}\}_{n=1}^{\infty} \cup \{0\}$ is*

$$\dim_B(S_j) = \frac{1}{j+1}.$$

Proof. From the construction of S_j we know that

$$l_{S_j}(k) = \frac{1}{k^j} - \frac{1}{(k+1)^j} = \frac{(k+1)^j - k^j}{k^j(k+1)^j}$$

and

$$r_{S_j}(k) = \frac{1}{(k-1)^j} - \frac{1}{k^j} = \frac{k^j - (k-1)^j}{k^j(k-1)^j}.$$

By Lemma 1.4.4, we know that

$$\frac{\ln k}{-\ln r_{S_j}(k)} \le \frac{\ln N_\delta^C}{-\ln \delta} \le \frac{\ln(2k)}{-\ln l_{S_j}(k)}.$$

Thus, the box-counting dimension is in between the limits of the far left and far right of this inequality. Consider $\ln k/(-\ln r_{S_j}(k))$.

$$\frac{\ln k}{-\ln r_{S_j}(k)} = \frac{\ln k}{-\ln(k^j - (k-1)^j) + j\ln k + j\ln(k+1)}$$

Consider $n^j - (n-1)^j$.

$$n^j - (n-1)^j = n^{j-1}\left(n - (n-1)\left(\frac{n-1}{n}\right)^{j-1}\right) = n^{j-1}\left(n - (n-1)\left(1 - \frac{1}{n}\right)^{j-1}\right)$$

Thus,

$$\frac{\ln k}{-\ln r_{S_j}(k)} = \frac{\ln k}{-\ln(k - (k-1)(1-1/k)^{j-1}) + \ln k + j\ln(k+1)}$$

which tends to $1/(j+1)$. Consider $\ln 2k/(-\ln l_{S_j}(k))$.

$$\frac{\ln 2k}{-\ln l_{S_j}(k)} = \frac{\ln 2k}{-\ln((k+1)^j - k^j) + j\ln k + j\ln(k-1)}$$

Consider $(n+1)^j - n^j$.

$$(n+1)^j - n^j = n^{j-1}\left((n+1)\left(\frac{n+1}{n}\right)^{j-1} - n\right) = n^{j-1}\left((n+1)\left(1 + \frac{1}{n}\right)^{j-1} - n\right)$$

Thus,

$$\frac{\ln 2 + \ln k}{-\ln l_{S_j}(k)} = \frac{\ln 2 + \ln k}{-\ln((k+1)(1+1/k)^{j-1} - k) + \ln k + j\ln(k+1)}$$

which tends to $1/(j+1)$ and by Squeeze theorem, $\dim_B(S_j)$ is $1/(j+1)$. $\square$

Example 1.4.6. *Let* $S = \{(n!)^{-1}\}_{n=1}^{\infty} \cup \{0\}$, *then*

$$\dim_B(S) = 0.$$

Proof. We begin by computing $l_S(n)$ and $r_S(n)$.

$$l_S(n) = \frac{n}{(n+1)!}$$
$$r_S(n) = \frac{n-1}{n!}$$

We then invoke Lemma 1.4.4 to get

$$\frac{\ln k}{-\ln(k-1) + \ln(k!)} \leq \frac{\ln N_\delta^C(S)}{-\ln \delta} \leq \frac{\ln 2 + \ln k}{-\ln k + \ln((k+1)!)}.$$

Since the limit of both the far right and the far left of the previous inequality is zero, $\dim_B(S) = 0$. $\square$

Bibliography

[1] Kenneth Falconer. *Fractal Geometry*. 3rd ed. John Wiley & Sons. ISBN: 9781119942369.