Matt's Linear Algebra Notes

December 13, 2015

Chapter 1

Material

1.1 Vector Spaces

Definition 1.1.1 (Vector Space)**.** A *vector space* V over a field $\mathbb{F}$ is a set with two binary operations, $+ : V \times V \to V$ and $\cdot : V \times \mathbb{F} \to V$ such that all of the following hold.

1. For all $x, y \in V$, $x + y = y + x$. (Additive Commutativity)
2. For all $x, y, z \in V$, $x + (y + z) = (x + y) + z$. (Additive Associativity)
3. There exists an element, denoted 0, in V such that for all $x \in V$, $x + 0 = x$.
4. For each $x \in V$ there exists a $y \in V$, denoted $-x$, such that $x + y = 0$.
5. For all $x \in V$, $1x = x$.
6. For all $a, b \in \mathbb{F}$ and $x \in V$, $a(bx) = (ab)x$.
7. For all $a \in \mathbb{F}$ and $x, y \in V$, $a(x + y) = ax + ay$.
8. For all $a, b \in \mathbb{F}$ and $x \in V$, $(a + b)x = ax + bx$.

Furthermore, $x + y$ is called the *sum of x and y* while ax is called the *product of x and a*. Moreover, each $x \in V$ is called a *vector* and each $a \in \mathbb{F}$ is called a *scalar*.

Definition 1.1.2 (n-tuple)**.** An object of the form $(a_1, a_2, \ldots, a_n)$ where $a_j \in \mathbb{F}$ for all $1 \leq j \leq n$, is called an *n-tuple*.

Example 1.1.1. *Let $\mathbb{F}$ be a field and $n \in \mathbb{N}$, then $\mathbb{F}^n = \{(a_1, a_2, \ldots, a_n) | a_j \in \mathbb{F} \forall 1 \leq j \leq n\}$ forms a vector space under component-wise addition and multiplication as defined below for $(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n) \in \mathbb{F}^n$ and $k \in \mathbb{F}$.*

$$\begin{aligned}(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) &= (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) \\ k(a_1, a_2, \ldots, a_n) &= (ka_1, ka_2, \ldots, ka_n)\end{aligned}$$

Furthermore, it said that

$$(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$$

if and only if $a_j = b_j$ for all $1 \leq j \leq n$.

Proof. $\mathbb{F}^n$ is a vector space trivially from the fact that $\mathbb{F}$ is a field. $\square$

Definition 1.1.3 (Matrix)**.** Let $\mathbb{F}$ be a field and $m, n \in \mathbb{N}$, then an $m \times n$ *matrix* with entries from $\mathbb{F}$ is a rectangular array of the form

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

where $a_{i,j} \in \mathbb{F}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. The entries $(a_{i,1}, a_{i,2}, \ldots, a_{i,n})$ is called the *ith row* of the matrix and is a row vector in $\mathbb{F}^n$. The entries $(a_{1,j}, a_{2,j}, \ldots, a_{m,j})$ is called the *jth column* of the matrix and is a column vector in $\mathbb{F}^n$. We denote the entry on the ith row and jth column as $A_{i,j}$. Furthermore, two $m \times n$ matrices, A and B, are equal if and only if $A_{i,j} = B_{i,j}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$; we denote this by $A = B$. Moreover, if $n = m$ we say that A is a *square matrix*. Lastly, we denote the set of $m \times n$ matrices over $\mathbb{F}$ as $M_{m \times n}(\mathbb{F})$.

Example 1.1.2. *Let $\mathbb{F}$ be a field and $m, n \in \mathbb{N}$, then $M_{m\times n}(\mathbb{F})$ is a vector space over $\mathbb{F}$ under the following operations for $A, B \in M_{m\times n}(\mathbb{F})$ and $k \in \mathbb{F}$.*

$$(A+B)_{i,j} = A_{i,j} + B_{i,j}$$
$$(kA)_{i,j} = kA_{i,j}$$

Proof. The proof is trivial from the fact that we operating on multiple copies of a field. □

Example 1.1.3. *Let S be a nonempty set and let $\mathbb{F}$ be a field and let $\mathscr{F}(S,\mathbb{F})$ denote the set of all functions from S into $\mathbb{F}$. Two elements $f, g \in \mathscr{F}(S,\mathbb{F})$ are equal if and only if $f(s) = g(s)$ for all $s \in S$. Then $\mathscr{F}(S,\mathbb{F})$ is a vector space under the following operations for $f, g \in \mathscr{F}(S,\mathbb{F})$ and $k \in \mathbb{F}$.*

$$(f+g)(s) = f(s) + g(s)$$
$$(kf)(s) = k\,[f(s)]$$

Proof. The proof is trivial because all operations are done inside the field, and thus the space inherits the structure from $\mathbb{F}$. □

Definition 1.1.4 (Polynomial Ring)**.** Let $\mathbb{F}$ be a field. Then the *ring of polynomials in an indeterminate x over* $\mathbb{F}$, denoted $\mathbb{F}[x]$ is defined as

$$\mathbb{F}[x] := \{\sum_{i=0}^{n} a_i x^i | n \in \mathbb{N}, (a_0, a_1, \ldots, a_n) \in \mathbb{F}^n, a_n \neq 0\}.$$

Additionally, we define $x^0 = 1$. Moreover, for each $p = \sum_{i=0}^{n} p_i x^i \in \mathbb{F}[x]$, the *degree of* p, denoted $\deg p$, is n. Furthermore, if $p = 0$, that is $p_n = p_{n-1} = \ldots = p_0 = 0$, then p is called the *zero polynomial* and $\deg p = -1$ or $\deg p = -\infty$ depending on convention. If $\deg p = 0$, then we say p is a *constant polynomial.* Lastly, $\mathbb{F}[x]$ forms a ring under the following operations where $p, q \in \mathbb{F}[x]$ and without loss of generality assume, $\deg p \geq \deg q$.

$$p + q = \sum_{i=0}^{\deg q} (p_i + q_i) + \sum_{i=\deg q+1}^{\deg p} p_i$$
$$pq = \sum_{k=0}^{\deg p \cdot \deg q} \sum_{i+j=k} p_i q_j$$

Example 1.1.4. *Let $\mathbb{F}$ be a field, then $\mathbb{F}[x]$ is a vector space over $\mathbb{F}$ under polynomial addition and scalar multiplication by constant polynomials.*

Proof. Since $\mathbb{F}[x]$ is a ring, it is closed under addition. Since constant polynomials are polynomials, it is closed under scalar multiplication. Since $\mathbb{F}[x]$ is a ring, addition is commutative and associative. Moreover the zero polynomial is the additive identity and likewise serves as the zero vector. Since $\mathbb{F}[x]$ is a ring, each $p \in \mathbb{F}[x]$ has a unique $-p \in \mathbb{F}[x]$ such that $p + (-p) = 0$. The constant polynomial 1 is the multiplicative identity in the ring and serves as the identity in the vector space. Furthermore, multiplication is associative and distributes over addition because $\mathbb{F}[x]$ is, indeed, a ring. □

Proposition 1.1.1. *The zero vector in any vector space V is unique.*

Proof. Assume there exists two zero vectors in V denoted 0_1 and 0_2. Then $v + 0_1 = v$ and $v + 0_2 = v$ for all $v \in V$. Therefore it is true that

$$0_2 = 0_2 + 0_1 = 0_1 + 0_2 = 0_1$$

and thus these identities are in fact the same. □

Proposition 1.1.2. *Let V be a vector space and let $v \in V$, then there exists a unique $u \in V$ such that $v + u = 0$.*

Proof. Assume v has two inverses, namely u_1 and u_2. Then $v + u_1 = 0$ and $v + u_2 = 0$. Therefore,

$$v + u_1 + u_2 = 0 + u_2 = u_2$$

and

$$v + u_1 + u_2 = v + u_2 + u_1 = 0 + u_1 = u_1.$$

Ergo, $u_1 = u_2$ and the inverse of v is unique. $\square$

Chapter 2

Definitions

2.1 Vector Spaces

Definition 2.1.1 (Vector Space)**.** A *vector space* V over a field $\mathbb{F}$ is a set with two binary operations, $+ : V \times V \to V$ and $\cdot : V \times \mathbb{F} \to V$ such that all of the following hold.

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where $a_{i,j} \in \mathbb{F}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. The entries $(a_{i,1}, a_{i,2}, \dots, a_{i,n})$ is called the *ith row* of the matrix and is a row vector in $\mathbb{F}^n$. The entries $(a_{1,j}, a_{2,j}, \dots, a_{m,j})$ is called the *jth column* of the matrix and is a column vector in $\mathbb{F}^n$. We denote the entry on the ith row and jth column as $A_{i,j}$. Furthermore, two $m \times n$ matrices, A and B, are equal if and only if $A_{i,j} = B_{i,j}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$; we denote this by $A = B$. Moreover, if $n = m$ we say that A is a *square matrix*. Lastly, we denote the set of $m \times n$ matrices over $\mathbb{F}$ as $M_{m\times n}(\mathbb{F})$.

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Chapter 3

Theorems

3.1 Vector Spaces

Proposition 3.1.1. *The zero vector in any vector space V is unique.*

Proof. Assume there exists two zero vectors in V denoted 0_1 and 0_2. Then $v + 0_1 = v$ and $v + 0_2 = v$ for all $v \in V$. Therefore it is true that

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Proposition 3.1.2. *Let V be a vector space and let $v \in V$, then there exists a unique $u \in V$ such that $v + u = 0$.*

Proof. Assume v has two inverses, namely u_1 and u_2. Then $v + u_1 = 0$ and $v + u_2 = 0$. Therefore,

$$v + u_1 + u_2 = 0 + u_2 = u_2$$

and

$$v + u_1 + u_2 = v + u_2 + u_1 = 0 + u_1 = u_1.$$

Ergo, $u_1 = u_2$ and the inverse of v is unique. □