Matt's Linear Algebra Notes

December 14, 2015

Chapter 1

Material

1.1 Vector Spaces

1.1.1 Introduction to Vector Spaces

Definition 1.1.1 (Vector Space)**.** A *vector space* V over a field $\mathbb{F}$ is a set with two binary operations, $+ : V \times V \to V$ and $\cdot : V \times \mathbb{F} \to V$ such that all of the following hold.

1. For all $x, y \in V$, $x + y = y + x$. (Additive Commutativity)
2. For all $x, y, z \in V$, $x + (y + z) = (x + y) + z$. (Additive Associativity)
3. There exists an element, denoted 0, in V such that for all $x \in V$, $x + 0 = x$.
4. For each $x \in V$ there exists a $y \in V$, denoted $-x$, such that $x + y = 0$.
5. For all $x \in V$, $1x = x$.
6. For all $a, b \in \mathbb{F}$ and $x \in V$, $a(bx) = (ab)x$.
7. For all $a \in \mathbb{F}$ and $x, y \in V$, $a(x + y) = ax + ay$.
8. For all $a, b \in \mathbb{F}$ and $x \in V$, $(a + b)x = ax + bx$.

Furthermore, $x + y$ is called the *sum of x and y* while ax is called the *product of x and a*. Moreover, each $x \in V$ is called a *vector* and each $a \in \mathbb{F}$ is called a *scalar*.

Definition 1.1.2 (n-tuple)**.** An object of the form $(a_1, a_2, \dots, a_n)$ where $a_j \in \mathbb{F}$ for all $1 \leq j \leq n$, is called an *n-tuple*.

Example 1.1.1. *Let $\mathbb{F}$ be a field and $n \in \mathbb{N}$, then $\mathbb{F}^n = \{(a_1, a_2, \dots, a_n) | a_j \in \mathbb{F} \forall 1 \leq j \leq n\}$ forms a vector space under component-wise addition and multiplication as defined below for $(a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_n) \in \mathbb{F}^n$ and $k \in \mathbb{F}$.*

$$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$$
$$k(a_1, a_2, \dots, a_n) = (ka_1, ka_2, \dots, ka_n)$$

Furthermore, it said that

$$(a_1, a_2, \dots, a_n) = (b_1, b_2, \dots, b_n)$$

if and only if $a_j = b_j$ for all $1 \leq j \leq n$.

Proof. $\mathbb{F}^n$ is a vector space trivially from the fact that $\mathbb{F}$ is a field. □

Definition 1.1.3 (Matrix)**.** Let $\mathbb{F}$ be a field and $m, n \in \mathbb{N}$, then an $m \times n$ *matrix* with entries from $\mathbb{F}$ is a rectangular array of the form

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{pmatrix}$$

where $a_{i,j} \in \mathbb{F}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. The entries $(a_{i,1}, a_{i,2}, \dots, a_{i,n})$ is called the *ith row* of the matrix and is a row vector in $\mathbb{F}^n$. The entries $(a_{1,j}, a_{2,j}, \dots, a_{m,j})$ is called the *jth column* of the matrix and is a column vector in $\mathbb{F}^n$. We denote the entry on the ith row and jth column as $A_{i,j}$. Furthermore, two $m \times n$ matrices, A and B, are equal if and only if $A_{i,j} = B_{i,j}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$; we denote this by $A = B$. Moreover, if $n = m$ we say that A is a *square matrix*. Lastly, we denote the set of $m \times n$ matrices over $\mathbb{F}$ as $M_{m \times n}(\mathbb{F})$.

Example 1.1.2. *Let $\mathbb{F}$ be a field and $m, n \in \mathbb{N}$, then $M_{m\times n}(\mathbb{F})$ is a vector space over $\mathbb{F}$ under the following operations for $A, B \in M_{m\times n}(\mathbb{F})$ and $k \in \mathbb{F}$.*

$$(A+B)_{i,j} = A_{i,j} + B_{i,j}$$
$$(kA)_{i,j} = kA_{i,j}$$

Proof. The proof is trivial from the fact that we operating on multiple copies of a field. □

Example 1.1.3. *Let S be a nonempty set and let $\mathbb{F}$ be a field and let $\mathscr{F}(S,\mathbb{F})$ denote the set of all functions from S into $\mathbb{F}$. Two elements $f, g \in \mathscr{F}(S,\mathbb{F})$ are equal if and only if $f(s) = g(s)$ for all $s \in S$. Then $\mathscr{F}(S,\mathbb{F})$ is a vector space under the following operations for $f, g \in \mathscr{F}(S,\mathbb{F})$ and $k \in \mathbb{F}$.*

$$(f+g)(s) = f(s) + g(s)$$
$$(kf)(s) = k\,[f(s)]$$

Proof. The proof is trivial because all operations are done inside the field, and thus the space inherits the structure from $\mathbb{F}$. □

Definition 1.1.4 (Polynomial Ring)**.** Let $\mathbb{F}$ be a field. Then the *ring of polynomials in an indeterminate x over $\mathbb{F}$*, denoted $\mathbb{F}[x]$ is defined as

$$\mathbb{F}[x] := \{\sum_{i=0}^{n} a_i x^i | n \in \mathbb{N}, (a_0, a_1, \ldots, a_n) \in \mathbb{F}^n, a_n \neq 0\}.$$

Additionally, we define $x^0 = 1$. Moreover, for each $p = \sum_{i=0}^{n} p_i x^i \in \mathbb{F}[x]$, the *degree of p*, denoted $\deg p$, is n. Furthermore, if $p = 0$, that is $p_n = p_{n-1} = \ldots = p_0 = 0$, then p is called the *zero polynomial* and $\deg p = -1$ or $\deg p = -\infty$ depending on convention. If $\deg p = 0$, then we say p is a *constant polynomial.* Lastly, $\mathbb{F}[x]$ forms a ring under the following operations where $p, q \in \mathbb{F}[x]$ and without loss of generality assume, $\deg p \geq \deg q$.

$$p + q = \sum_{i=0}^{\deg q} (p_i + q_i) x^i + \sum_{i=\deg q+1}^{\deg p} p_i x^i$$
$$pq = \sum_{k=0}^{\deg p \cdot \deg q} \left(\sum_{i+j=k} p_i q_j \right) x^k$$

Example 1.1.4. *Let $\mathbb{F}$ be a field, then $\mathbb{F}[x]$ is a vector space over $\mathbb{F}$ under polynomial addition and scalar multiplication by constant polynomials.*

Proof. Since $\mathbb{F}[x]$ is a ring, it is closed under addition. Since constant polynomials are polynomials, it is closed under scalar multiplication. Since $\mathbb{F}[x]$ is a ring, addition is commutative and associative. Moreover the zero polynomial is the additive identity and likewise serves as the zero vector. Since $\mathbb{F}[x]$ is a ring, each $p \in \mathbb{F}[x]$ has a unique $-p \in \mathbb{F}[x]$ such that $p + (-p) = 0$. The constant polynomial 1 is the multiplicative identity in the ring and serves as the identity in the vector space. Furthermore, multiplication is associative and distributes over addition because $\mathbb{F}[x]$ is, indeed, a ring. □

Proposition 1.1.1. *If u, v, w are elements of a vector space V such that $x + z = y + z$ then, $x = y$.*

Proof. Since $z \in V$, then there exists a $-z \in V$ such that $z + (-z) = 0$. Thus, $x + z = y + z$ implies

$$x + z - z = y + z - z$$

and ergo $x = y$. □

Proposition 1.1.2. *The zero vector in any vector space V is unique.*

Proof. Assume there exists two zero vectors in V denoted 0_1 and 0_2. Then $v + 0_1 = v$ and $v + 0_2 = v$ for all $v \in V$. Therefore it is true that

$$0_2 = 0_2 + 0_1 = 0_1 + 0_2 = 0_1$$

and thus these identities are in fact the same. $\square$

Proposition 1.1.3. *Let V be a vector space and let $v \in V$, then there exists a unique $u \in V$ such that $v + u = 0$.*

Proof. Assume v has two inverses, namely u_1 and u_2. Then $v + u_1 = 0$ and $v + u_2 = 0$. Therefore,

$$v + u_1 + u_2 = 0 + u_2 = u_2$$

and

$$v + u_1 + u_2 = v + u_2 + u_1 = 0 + u_1 = u_1.$$

Ergo, $u_1 = u_2$ and the inverse of v is unique. $\square$

Proposition 1.1.4. *Let V be a vector space over a field $\mathbb{F}$, then:*

1. $0x = 0$ *for all* $x \in V$*;*
2. $a0 = 0$ *for all* $a \in \mathbb{F}$*;*
3. $(-a)x = -(ax) = a(-x)$ *for all* $x \in V$ *and* $a \in \mathbb{F}$.

Proof (1): Consider $0x + 0x$. Then,

$$0x + 0x = (0 + 0)x = 0x = 0 + 0x.$$

Since $0x + 0x = 0 + 0x$, by cancellation, we have $0x = 0$. $\square$

Proof (2). Consider $a0 + a0$.

$$a0 + a0 = a(0 + 0) = a0 = a0 + 0$$

Thus, by cancellation, $a0 = 0$. $\square$

Proof (3). Consider $-(ax)$. We know that $-(ax)$ is the unique additive inverse of ax, thus it is enough to show that $(-a)x$ and $a(-x)$ are inverses of ax.

$$\begin{aligned} ax + (-a)x &= (a - a)x \\ &= 0x \\ &= 0 \\ ax + a(-x) &= a(x - x) \\ &= a0 \\ &= 0 \end{aligned}$$

Thus $a(-x)$ and $(-a)x$ are inverses of ax and $(-a)x = -(ax) = a(-x)$. $\square$

1.1.2 Subspaces

Definition 1.1.5 (Subspace)**.** A *subspace*, W, of a vector space, V, over a field, $\mathbb{F}$, is a subset of V that is also a vector space over $\mathbb{F}$.

Example 1.1.5. *For any vector space V, V and $\{0\}$ are subspaces of V. The latter is called the zero subspace.*

Theorem 1.1.5. *Let V be a vector space over a field $\mathbb{F}$ and let $W \subseteq V$. Then W is a subspace of V if and only if all of the following are satisfied.*

- $0 \in W$.
- *For all $x, y \in W$, $x + y \in W$.*
- *For all $a \in \mathbb{F}$ and $x \in W$, $ax \in W$.*

Proof. $\Rightarrow$ Since W is a subspace of V, $0 \in W$ and W is closed under V's vector addition and scalar multiplication.

$\Leftarrow$ Since V is a vector space W inherits associativity, commutativity, and distributivity from V as well as V's behavior with respect to the identities. Furthermore, $0 \in W$ by assumption. All that is left to show is that W contains additive inverses. Suppose $x \in W$, then by assumption $-x = (-1)x \in W$. Thus W is a subspace of V. $\square$

Definition 1.1.6 (Matrix Transpose)**.** Let M be an $m \times n$ matrix, then the *transpose of* M, denoted M^T, is the $n \times m$ matrix defined by $(M^T)_{i,j} = M_{j,i}$, that is

$$M^T = \begin{pmatrix} M_{1,1} & M_{2,1} & \dots & M_{m,1} \\ M_{1,2} & M_{2,2} & \dots & M_{m,2} \\ \vdots & \vdots & \ddots & \vdots \\ M_{1,n} & M_{2,n} & \dots & M_{m,n} \end{pmatrix}.$$

Definition 1.1.7 (Symmetric Matrix)**.** Let M be a matrix, then if $M = M^T$, we say M is *symmetric.*

Example 1.1.6. *The set of symmetric $n \times n$ matrices over a field $\mathbb{F}$, denoted $W_{n\times n}(\mathbb{F})$, is a subspace of $M_{n\times n}(\mathbb{F})$.*

Proof. Consider the zero matrix. Since the zero matrix is an $n \times n$ matrix with all entries equal to zero, the transpose of the zero matrix is also an $n \times n$ matrix with all entries equal to zero. Thus, $0 = 0^T$ and the zero matrix is symmetric.

Let $A, B \in W_{n\times n}(\mathbb{F})$. Then $A = A^T$ and $B = B^T$. By definition of symmetry and matrix transpose we have

$$A_{i,j} = (A^T)_{i,j} = A_{j,i} \tag{1.1}$$

and

$$B_{i,j} = (B^T)_{i,j} = B_{j,i} \tag{1.2}$$

for all $1 \leq i, j \leq n$.

Consider $(A + B)$. By definition we have

$$(A + B)_{i,j} = A_{i,j} + B_{i,j}.$$

By Equation 1.1 and Equation 1.2 we have

$$(A + B)_{i,j} = A_{i,j} + B_{i,j} = A_{j,i} + B_{j,i}.$$

The definition of matrix transpose implies that

$$(A+B)_{i,j} = A_{i,j} + B_{i,j} = A_{j,i} + B_{j,i} = (A^T)_{i,j} + (B^T)_{i,j}$$

and thus by definition of matrix addition,

$$(A+B)_{i,j} = A_{i,j} + B_{i,j} = A_{j,i} + B_{j,i} = (A^T)_{i,j} + (B^T)_{i,j} = (A^T + B^T)_{i,j}.$$

Ergo, $A+B$ is symmetric and $W(\mathbb{F})$ is closed under matrix addition.

Let $k \in \mathbb{F}$ and consider kA. We know by definition that

$$(kA)_{i,j} = k \cdot A_{i,j}.$$

We invoke Equation 1.1 again to get that

$$(kA)_{i,j} = k \cdot A_{i,j} = k \cdot A_{j,i}.$$

Applying the definition of matrix transpose yields,

$$(kA)_{i,j} = k \cdot A_{i,j} = k \cdot A_{j,i} = k \cdot (A^T)_{i,j}.$$

Lastly, by definition of scalar multiplication we have,

$$(kA)_{i,j} = k \cdot A_{i,j} = k \cdot A_{j,i} = k \cdot (A^T)_{i,j} = (kA^T)_{i,j}.$$

Thus kA is symmetric and $W(\mathbb{F})$ is closed under scalar multiplication. □

Definition 1.1.8 (Main Diagonal of a Matrix)**.** Let $\mathbb{F}$ be a field and let $M \in M_{n\times n}(\mathbb{F})$, then the *main diagonal of* M is the set $\{M_{i,i}\}_{i=1}^n$.

Definition 1.1.9 (Diagonal Matrix)**.** Let $\mathbb{F}$ be a field and let $A \in M_{n\times n}(\mathbb{F})$, then A is called a *diagonal matrix* if and only if whenever $i \neq j$, $A_{i,j} = 0$.

Example 1.1.7. *Let $\mathbb{F}$ be a field and let $D_n(\mathbb{F})$ be the set of all diagonal matrices in $M_{n\times n}(\mathbb{F})$, then $D_n(\mathbb{F})$ is a subspace on $M_{n\times n}(\mathbb{F})$.*

Proof. We know $0 \in D_n(\mathbb{F})$ since for all i, j, $0_{i,j} = 0$. Let $A, B \in D_n(\mathbb{F})$. Then for all $i \neq j$, $A_{i,j} = B_{i,j} = 0$. Thus, $(A+B)_{i,j} = A_{i,j} + B_{i,j} = 0 + 0 = 0$ whenever $i \neq j$ and $A+B$ is diagonal. Let $k \in \mathbb{F}$. Then $(kA)_{i,j} = k \cdot A_{i,j} = k \cdot 0 = 0$ and kA is diagonal. Therefore $D_n(\mathbb{F})$ forms a subspace of $M_{n\times n}(\mathbb{F})$. □

Definition 1.1.10 (Trace of a Matrix)**.** Let $\mathbb{K}$ be a field and let $M \in M_{n\times n}(\mathbb{K})$, then the *trace of* M denoted $\operatorname{tr} M$ is defined as

$$\operatorname{tr} M = \sum_{i=1}^{n} M_{i,i}$$

or the sum of the elements on the main diagonal.

Example 1.1.8. *Let $\mathbb{K}$ be a field and let $T_n(\mathbb{K})$ be the set of matrices in $M_{n\times n}(\mathbb{K})$ with trace equal to zero, then $T_n(\mathbb{K})$ is a subspace of $M_{n\times n}(\mathbb{K})$.*

Proof. Obviously, the zero matrix has a trace of zero and thus $0 \in T_n(\mathbb{K})$. Let $A, B \in T_n(\mathbb{K})$ then $\operatorname{tr} A = 0$ and $\operatorname{tr} B = 0$. Consider $\operatorname{tr}(A+B)$.

$$\operatorname{tr}(A+B) = \sum_{i=1}^{n}(A+B)_{i,i} = \sum_{i=1^n}(A_{i,i} + B_{i,i}) = \left(\sum_{i=1}^{n} A_{i,i}\right) + \left(\sum_{i=1}^{n} B_{i,i}\right) = \operatorname{tr} A + \operatorname{tr} B = 0$$

Thus, $A+B$ has trace 0 and $A+B \in T_n(\mathbb{K})$. Let $k \in \mathbb{K}$. Consider $\operatorname{tr}(kA)$.

$$\operatorname{tr}(kA) = \sum_{i=1}^{n}(kA)_{i,i} = \sum_{i=1}^{n} k \cdot A_{i,i} = k\sum_{i=1}^{n} A_{i,i} = k \operatorname{tr} A = 0$$

And thus, kA has trace 0 and $kA \in T_n(\mathbb{K})$. Therefore $T_n(\mathbb{K})$ is a subspace of $M_{n\times n}(\mathbb{K})$. □

Theorem 1.1.6. *Let V be a vector space over a field $\mathbb{F}$ and let $\mathcal{W}$ be a countable collection of subspaces of V. Then*

$$W_i = \bigcap_{W \in \mathcal{W}} W$$

is a subspace of V.

Proof. Since $0 \in W$ for all $W \in \mathcal{W}$, $0 \in W_i$. Let $x, y \in W_i$, then $x, y \in W$ for all $W \in \mathcal{W}$ and thus $x + y \in W$ for all $W \in \mathcal{W}$. Therefore, $x + y \in W_i$. Let $a \in \mathbb{F}$. Since $x \in W$ for all $W \in \mathcal{W}$, $ax \in W$ for all $W \in \mathcal{W}$. Ergo, $ax \in W_i$ and W_i is a subspace of V. $\square$

Chapter 2

Definitions

2.1 Vector Spaces

2.1.1 Introduction to Vector Spaces

Definition 2.1.1 (Vector Space)**.** A *vector space* V over a field $\mathbb{F}$ is a set with two binary operations, $+ : V \times V \to V$ and $\cdot : V \times \mathbb{F} \to V$ such that all of the following hold.

1. For all $x, y \in V$, $x + y = y + x$. (Additive Commutativity)
2. For all $x, y, z \in V$, $x + (y + z) = (x + y) + z$. (Additive Associativity)
3. There exists an element, denoted 0, in V such that for all $x \in V$, $x + 0 = x$.
4. For each $x \in V$ there exists a $y \in V$, denoted $-x$, such that $x + y = 0$.
5. For all $x \in V$, $1x = x$.
6. For all $a, b \in \mathbb{F}$ and $x \in V$, $a(bx) = (ab)x$.
7. For all $a \in \mathbb{F}$ and $x, y \in V$, $a(x + y) = ax + ay$.
8. For all $a, b \in \mathbb{F}$ and $x \in V$, $(a + b)x = ax + bx$.

Furthermore, $x + y$ is called the *sum of x and y* while ax is called the *product of x and a*. Moreover, each $x \in V$ is called a *vector* and each $a \in \mathbb{F}$ is called a *scalar*.

Definition 2.1.2 (n-tuple)**.** An object of the form $(a_1, a_2, \ldots, a_n)$ where $a_j \in \mathbb{F}$ for all $1 \leq j \leq n$, is called an *n-tuple*.

Definition 2.1.3. Let $\mathbb{F}$ be a field and $m, n \in \mathbb{N}$, then an $m \times n$ *matrix* with entries from $\mathbb{F}$ is a rectangular array of the form

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,n} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \ldots & a_{m,n} \end{pmatrix}$$

where $a_{i,j} \in \mathbb{F}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. The entries $(a_{i,1}, a_{i,2}, \ldots, a_{i,n})$ is called the *ith row* of the matrix and is a row vector in $\mathbb{F}^n$. The entries $(a_{1,j}, a_{2,j}, \ldots, a_{m,j})$ is called the *jth column* of the matrix and is a column vector in $\mathbb{F}^n$. We denote the entry on the ith row and jth column as $A_{i,j}$. Furthermore, two $m \times n$ matrices, A and B, are equal if and only if $A_{i,j} = B_{i,j}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$; we denote this by $A = B$. Moreover, if $n = m$ we say that A is a *square matrix*. Lastly, we denote the set of $m \times n$ matrices over $\mathbb{F}$ as $M_{m \times n}(\mathbb{F})$.

Definition 2.1.4 (Polynomial Ring)**.** Let $\mathbb{F}$ be a field. Then the *ring of polynomials in an indeterminate x over* $\mathbb{F}$, denoted $\mathbb{F}[x]$ is defined as

$$\mathbb{F}[x] := \{\sum_{i=0}^{n} a_i x^i | n \in \mathbb{N}, (a_0, a_1, \ldots, a_n) \in \mathbb{F}^n, a_n \neq 0\}.$$

Additionally, we define $x^0 = 1$. Moreover, for each $p = \sum_{i=0}^{n} p_i x^i \in \mathbb{F}[x]$, the *degree of p*, denoted $\deg p$, is n. Furthermore, if $p = 0$, that is $p_n = p_{n-1} = \ldots = p_0 = 0$, then p is called the *zero polynomial* and $\deg p = -1$ or $\deg p = -\infty$ depending on convention. If $\deg p = 0$, then we say p is a *constant polynomial.*

Lastly, $\mathbb{F}[x]$ forms a ring under the following operations where $p, q \in \mathbb{F}[x]$ and without loss of generality assume, $\deg p \geq \deg q$.

$$p + q = \sum_{i=0}^{\deg q} (p_i + q_i)x^i + \sum_{i=\deg q+1}^{\deg p} p_i x^i$$

$$pq = \sum_{k=0}^{\deg p \cdot \deg q} \left(\sum_{i+j=k} p_i q_j \right) x^k$$

2.1.2 Subspaces

Definition 2.1.5 (Subspace)**.** A *subspace*, W, of a vector space, V, over a field, $\mathbb{F}$, is a subset of V that is also a vector space over $\mathbb{F}$.

Definition 2.1.6 (Matrix Transpose)**.** Let M be an $m \times n$ matrix, then the *transpose of* M, denoted M^T, is the $n \times m$ matrix defined by $(M^T)_{i,j} = M_{j,i}$, that is

$$M^T = \begin{pmatrix} M_{1,1} & M_{2,1} & \dots & M_{m,1} \\ M_{1,2} & M_{2,2} & \dots & M_{m,2} \\ \vdots & \vdots & \ddots & \vdots \\ M_{1,n} & M_{2,n} & \dots & M_{m,n} \end{pmatrix}.$$

Definition 2.1.7 (Symmetric Matrix)**.** Let M be a matrix, then if $M = M^T$, we say M is *symmetric*.

Definition 2.1.8 (Main Diagonal of a Matrix)**.** Let $\mathbb{F}$ be a field and let $M \in M_{n\times n}(\mathbb{F})$, then the *main diagonal of* M is the set $\{M_{i,i}\}_{i=1}^n$.

Definition 2.1.9 (Diagonal Matrix)**.** Let $\mathbb{F}$ be a field and let $A \in M_{n\times n}(\mathbb{F})$, then A is called a *diagonal matrix* if and only if whenever $i \neq j$, $A_{i,j} = 0$.

Definition 2.1.10 (Trace of a Matrix)**.** Let $\mathbb{K}$ be a field and let $M \in M_{n\times n}(\mathbb{K})$, then the *trace of* M denoted $\operatorname{tr} M$ is defined as

$$\operatorname{tr} M = \sum_{i=1}^{n} M_{i,i}$$

or the sum of the elements on the main diagonal.

Chapter 3

Theorems

3.1 Vector Spaces

3.1.1 Introduction to Vector Spaces

Proposition 3.1.1. *If u, v, w are elements of a vector space V such that $x + z = y + z$ then, $x = y$.*

Proposition 3.1.2. *The zero vector in any vector space V is unique.*

Proposition 3.1.3. *Let V be a vector space and let $v \in V$, then there exists a unique $u \in V$ such that $v + u = 0$.*

Proposition 3.1.4. *Let V be a vector space over a field $\mathbb{F}$, then:*

1. $0x = 0$ *for all* $x \in V$*;*
2. $a0 = 0$ *for all* $a \in \mathbb{F}$*;*
3. $(-a)x = -(ax) = a(-x)$ *for all* $x \in V$ *and* $a \in \mathbb{F}$*.*

3.1.2 Subspaces

Theorem 3.1.5. *Let V be a vector space over a field $\mathbb{F}$ and let $W \subseteq V$. Then W is a subspace of V if and only if all of the following are satisfied.*

- $0 \in W$.
- *For all $x, y \in W$, $x + y \in W$.*
- *For all $a \in \mathbb{F}$ and $x \in W$, $ax \in W$.*

Theorem 3.1.6. *Let V be a vector space over a field $\mathbb{F}$ and let $\mathcal{W}$ be a countable collection of subspaces of V. Then*

$$W_i = \bigcap_{W \in \mathcal{W}} W$$

is a subspace of V.