Matt's Linear Algebra Notes

December 13, 2015

Chapter 1

Material

1.1 Vector Spaces

Definition 1.1.1 (Vector Space)**.** A *vector space* V over a field $\mathbb{F}$ is a set with two binary operations, $+ : V \times V \to V$ and $\cdot : V \times \mathbb{F} \to V$ such that all of the following hold.

1. For all $x, y \in V$, $x + y = y + x$. (Additive Commutativity)
2. For all $x, y, z \in V$, $x + (y + z) = (x + y) + z$. (Additive Associativity)
3. There exists an element, denoted 0, in V such that for all $x \in V$, $x + 0 = x$.
4. For each $x \in V$ there exists a $y \in V$, denoted $-x$, such that $x + y = 0$.
5. For all $x \in V$, $1x = x$.
6. For all $a, b \in \mathbb{F}$ and $x \in V$, $a(bx) = (ab)x$.
7. For all $a \in \mathbb{F}$ and $x, y \in V$, $a(x + y) = ax + ay$.
8. For all $a, b \in \mathbb{F}$ and $x \in V$, $(a + b)x = ax + bx$.

Furthermore, $x + y$ is called the *sum of* x *and* y while ax is called the *product of* x *and* a. Moreover, each $x \in V$ is called a *vector* and each $a \in \mathbb{F}$ is called a *scalar*.

Definition 1.1.2 (n-tuple)**.** An object of the form $(a_1, a_2, \ldots, a_n)$ where $a_j \in \mathbb{F}$ for all $1 \leq j \leq n$, is called an *n-tuple*.

Example 1.1.1. *Let $\mathbb{F}$ be a field and $n \in \mathbb{N}$, then $\mathbb{F}^n = \{(a_1, a_2, \ldots, a_n) | a_j \in \mathbb{F} \forall 1 \leq j \leq n\}$ forms a vector space under component-wise addition and multiplication as defined below for $(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n) \in \mathbb{F}^n$ and $k \in \mathbb{F}$.*

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$
$$k(a_1, a_2, \ldots, a_n) = (ka_1, ka_2, \ldots, ka_n)$$

Furthermore, it said that

$$(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$$

if and only if $a_j = b_j$ for all $1 \leq j \leq n$.

Proof. $\mathbb{F}^n$ is a vector space trivially from the fact that $\mathbb{F}$ is a field. ☐

Definition 1.1.3 (Matrix)**.** Let $\mathbb{F}$ be a field and $m, n \in \mathbb{N}$, then an $m \times n$ *matrix* with entries from $\mathbb{F}$ is a rectangular array of the form

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,n} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \ldots & a_{m,n} \end{pmatrix}$$

where $a_{i,j} \in \mathbb{F}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. The entries $(a_{i,1}, a_{i,2}, \ldots, a_{i,n})$ is called the *ith row* of the matrix and is a row vector in $\mathbb{F}^n$. The entries $(a_{1,j}, a_{2,j}, \ldots, a_{m,j})$ is called the *jth column* of the matrix and is a column vector in $\mathbb{F}^n$. We denote the entry on the ith row and jth column as $A_{i,j}$. Furthermore, two $m \times n$ matrices, A and B, are equal if and only if $A_{i,j} = B_{i,j}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$; we denote this by $A = B$. Moreover, if $n = m$ we say that A is a *square matrix*. Lastly, we denote the set of $m \times n$ matrices over $\mathbb{F}$ as $M_{m \times n}(\mathbb{F})$.

Example 1.1.2. *Let $\mathbb{F}$ be a field and $m, n \in \mathbb{N}$, then $M_{m \times n}(\mathbb{F})$ is a vector space over $\mathbb{F}$ under the following operations for $A, B \in M_{m \times n}(\mathbb{F})$ and $k \in \mathbb{F}$.*

$$(A + B)_{i,j} = A_{i,j} + B_{i,j}$$
$$(kA)_{i,j} = kA_{i,j}$$

Chapter 2

Definitions

2.1 Vector Spaces

Definition 2.1.1 (Vector Space)**.** A *vector space* V over a field $\mathbb{F}$ is a set with two binary operations, $+ : V \times V \to V$ and $\cdot : V \times \mathbb{F} \to V$ such that all of the following hold.

1. For all $x, y \in V$, $x + y = y + x$. (Additive Commutativity)
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8. For all $a, b \in \mathbb{F}$ and $x \in V$, $(a + b)x = ax + bx$.

Furthermore, $x + y$ is called the *sum of x and y* while ax is called the *product of x and a*. Moreover, each $x \in V$ is called a *vector* and each $a \in \mathbb{F}$ is called a *scalar*.

Definition 2.1.2 (n-tuple)**.** An object of the form $(a_1, a_2, \ldots, a_n)$ where $a_j \in \mathbb{F}$ for all $1 \leq j \leq n$, is called an *n-tuple*.

Definition 2.1.3. Let $\mathbb{F}$ be a field and $m, n \in \mathbb{N}$, then an $m \times n$ *matrix* with entries from $\mathbb{F}$ is a rectangular array of the form

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

where $a_{i,j} \in \mathbb{F}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. The entries $(a_{i,1}, a_{i,2}, \ldots, a_{i,n})$ is called the *ith row* of the matrix and is a row vector in $\mathbb{F}^n$. The entries $(a_{1,j}, a_{2,j}, \ldots, a_{m,j})$ is called the *jth column* of the matrix and is a column vector in $\mathbb{F}^n$. We denote the entry on the ith row and jth column as $A_{i,j}$. Furthermore, two $m \times n$ matrices, A and B, are equal if and only if $A_{i,j} = B_{i,j}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$; we denote this by $A = B$. Moreover, if $n = m$ we say that A is a *square matrix*. Lastly, we denote the set of $m \times n$ matrices over $\mathbb{F}$ as $M_{m \times n}(\mathbb{F})$.

Chapter 3

Theorems