

Matt's Linear Algebra Notes

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Chapter 1

Material

1.1 Vector Spaces

1.1.1 Introduction to Vector Spaces

Definition 1.1.1 (Vector Space). A *vector space* V over a field \mathbb{F} is a set with two binary operations, $+: V \times V \rightarrow V$ and $\cdot: V \times \mathbb{F} \rightarrow V$ such that all of the following hold.

1. For all $x, y \in V$, $x + y = y + x$. (Additive Commutativity)
2. For all $x, y, z \in V$, $x + (y + z) = (x + y) + z$. (Additive Associativity)
3. There exists an element, denoted 0 , in V such that for all $x \in V$, $x + 0 = x$.
4. For each $x \in V$ there exists a $y \in V$, denoted $-x$, such that $x + y = 0$.
5. For all $x \in V$, $1x = x$.
6. For all $a, b \in \mathbb{F}$ and $x \in V$, $a(bx) = (ab)x$.
7. For all $a \in \mathbb{F}$ and $x, y \in V$, $a(x + y) = ax + ay$.
8. For all $a, b \in \mathbb{F}$ and $x \in V$, $(a + b)x = ax + bx$.

Furthermore, $x + y$ is called the *sum of x and y* while ax is called the *product of x and a* . Moreover, each $x \in V$ is called a *vector* and each $a \in \mathbb{F}$ is called a *scalar*.

Definition 1.1.2 (n -tuple). An object of the form (a_1, a_2, \dots, a_n) where $a_j \in \mathbb{F}$ for all $1 \leq j \leq n$, is called an *n -tuple*.

Example 1.1.1. Let \mathbb{F} be a field and $n \in \mathbb{N}$, then $\mathbb{F}^n = \{(a_1, a_2, \dots, a_n) | a_j \in \mathbb{F} \forall 1 \leq j \leq n\}$ forms a vector space under component-wise addition and multiplication as defined below for $(a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_n) \in \mathbb{F}^n$ and $k \in \mathbb{F}$.

$$\begin{aligned}(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) &= (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n) \\ k(a_1, a_2, \dots, a_n) &= (ka_1, ka_2, \dots, ka_n)\end{aligned}$$

Furthermore, it said that

$$(a_1, a_2, \dots, a_n) = (b_1, b_2, \dots, b_n)$$

if and only if $a_j = b_j$ for all $1 \leq j \leq n$.

Proof. \mathbb{F}^n is a vector space trivially from the fact that \mathbb{F} is a field. □

Definition 1.1.3 (Matrix). Let \mathbb{F} be a field and $m, n \in \mathbb{N}$, then an $m \times n$ *matrix* with entries from \mathbb{F} is a rectangular array of the form

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{pmatrix}$$

where $a_{i,j} \in \mathbb{F}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. The entries $(a_{i,1}, a_{i,2}, \dots, a_{i,n})$ is called the *i th row* of the matrix and is a row vector in \mathbb{F}^n . The entries $(a_{1,j}, a_{2,j}, \dots, a_{m,j})$ is called the *j th column* of the matrix and is a column vector in \mathbb{F}^m . We denote the entry on the i th row and j th column as $A_{i,j}$. Furthermore, two $m \times n$ matrices, A and B , are equal if and only if $A_{i,j} = B_{i,j}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$; we denote this by $A = B$. Moreover, if $n = m$ we say that A is a *square matrix*. Lastly, we denote the set of $m \times n$ matrices over \mathbb{F} as $M_{m \times n}(\mathbb{F})$.

Example 1.1.2. Let \mathbb{F} be a field and $m, n \in \mathbb{N}$, then $M_{m \times n}(\mathbb{F})$ is a vector space over \mathbb{F} under the following operations for $A, B \in M_{m \times n}(\mathbb{F})$ and $k \in \mathbb{F}$.

$$\begin{aligned}(A + B)_{i,j} &= A_{i,j} + B_{i,j} \\ (kA)_{i,j} &= kA_{i,j}\end{aligned}$$

Proof. The proof is trivial from the fact that we operating on multiple copies of a field. \square

Example 1.1.3. Let S be a nonempty set and let \mathbb{F} be a field and let $\mathcal{F}(S, \mathbb{F})$ denote the set of all functions from S into \mathbb{F} . Two elements $f, g \in \mathcal{F}(S, \mathbb{F})$ are equal if and only if $f(s) = g(s)$ for all $s \in S$. Then $\mathcal{F}(S, \mathbb{F})$ is a vector space under the following operations for $f, g \in \mathcal{F}(S, \mathbb{F})$ and $k \in \mathbb{F}$.

$$\begin{aligned}(f + g)(s) &= f(s) + g(s) \\ (kf)(s) &= k[f(s)]\end{aligned}$$

Proof. The proof is trivial because all operations are done inside the field, and thus the space inherits the structure from \mathbb{F} . \square

Definition 1.1.4 (Polynomial Ring). Let \mathbb{F} be a field. Then the ring of polynomials in an indeterminate x over \mathbb{F} , denoted $\mathbb{F}[x]$ is defined as

$$\mathbb{F}[x] := \left\{ \sum_{i=0}^n a_i x^i \mid n \in \mathbb{N}, (a_0, a_1, \dots, a_n) \in \mathbb{F}^n, a_n \neq 0 \right\}.$$

Additionally, we define $x^0 = 1$. Moreover, for each $p = \sum_{i=0}^n p_i x^i \in \mathbb{F}[x]$, the degree of p , denoted $\deg p$, is n . Furthermore, if $p = 0$, that is $p_n = p_{n-1} = \dots = p_0 = 0$, then p is called the zero polynomial and $\deg p = -1$ or $\deg p = -\infty$ depending on convention. If $\deg p = 0$, then we say p is a constant polynomial. Lastly, $\mathbb{F}[x]$ forms a ring under the following operations where $p, q \in \mathbb{F}[x]$ and without loss of generality assume, $\deg p \geq \deg q$.

$$\begin{aligned}p + q &= \sum_{i=0}^{\deg q} (p_i + q_i) x^i + \sum_{i=\deg q+1}^{\deg p} p_i x^i \\ pq &= \sum_{k=0}^{\deg p + \deg q} \left(\sum_{i+j=k} p_i q_j \right) x^k\end{aligned}$$

Example 1.1.4. Let \mathbb{F} be a field, then $\mathbb{F}[x]$ is a vector space over \mathbb{F} under polynomial addition and scalar multiplication by constant polynomials.

Proof. Since $\mathbb{F}[x]$ is a ring, it is closed under addition. Since constant polynomials are polynomials, it is closed under scalar multiplication. Since $\mathbb{F}[x]$ is a ring, addition is commutative and associative. Moreover the zero polynomial is the additive identity and likewise serves as the zero vector. Since $\mathbb{F}[x]$ is a ring, each $p \in \mathbb{F}[x]$ has a unique $-p \in \mathbb{F}[x]$ such that $p + (-p) = 0$. The constant polynomial 1 is the multiplicative identity in the ring and serves as the identity in the vector space. Furthermore, multiplication is associative and distributes over addition because $\mathbb{F}[x]$ is, indeed, a ring. \square

Proposition 1.1.1. If u, v, w are elements of a vector space V such that $x + z = y + z$ then, $x = y$.

Proof. Since $z \in V$, then there exists a $-z \in V$ such that $z + (-z) = 0$. Thus, $x + z = y + z$ implies

$$x + z - z = y + z - z$$

and ergo $x = y$. \square

Proposition 1.1.2. *The zero vector in any vector space V is unique.*

Proof. Assume there exists two zero vectors in V denoted 0_1 and 0_2 . Then $v + 0_1 = v$ and $v + 0_2 = v$ for all $v \in V$. Therefore it is true that

$$0_2 = 0_2 + 0_1 = 0_1 + 0_2 = 0_1$$

and thus these identities are in fact the same. \square

Proposition 1.1.3. *Let V be a vector space and let $v \in V$, then there exists a unique $u \in V$ such that $v + u = 0$.*

Proof. Assume v has two inverses, namely u_1 and u_2 . Then $v + u_1 = 0$ and $v + u_2 = 0$. Therefore,

$$v + u_1 + u_2 = 0 + u_2 = u_2$$

and

$$v + u_1 + u_2 = v + u_2 + u_1 = 0 + u_1 = u_1.$$

Ergo, $u_1 = u_2$ and the inverse of v is unique. \square

Proposition 1.1.4. *Let V be a vector space over a field \mathbb{F} , then:*

1. $0x = 0$ for all $x \in V$;
2. $a0 = 0$ for all $a \in \mathbb{F}$;
3. $(-a)x = -(ax) = a(-x)$ for all $x \in V$ and $a \in \mathbb{F}$.

Proof (1): Consider $0x + 0x$. Then,

$$0x + 0x = (0 + 0)x = 0x = 0 + 0x.$$

Since $0x + 0x = 0 + 0x$, by cancellation, we have $0x = 0$. \square

Proof (2): Consider $a0 + a0$.

$$a0 + a0 = a(0 + 0) = a0 = a0 + 0$$

Thus, by cancellation, $a0 = 0$. \square

Proof (3): Consider $-(ax)$. We know that $-(ax)$ is the unique additive inverse of ax , thus it is enough to show that $(-a)x$ and $a(-x)$ are inverses of ax .

$$\begin{aligned} ax + (-a)x &= (a - a)x \\ &= 0x \\ &= 0 \\ ax + a(-x) &= a(x - x) \\ &= a0 \\ &= 0 \end{aligned}$$

Thus $a(-x)$ and $(-a)x$ are inverses of ax and $(-a)x = -(ax) = a(-x)$. \square

1.1.2 Subspaces

Definition 1.1.5 (Subspace). A *subspace*, W , of a vector space, V , over a field, \mathbb{F} , is a subset of V that is also a vector space over \mathbb{F} .

Example 1.1.5. For any vector space V , V and $\{0\}$ are subspaces of V . The latter is called the zero subspace.

Theorem 1.1.5. Let V be a vector space over a field \mathbb{F} and let $W \subseteq V$. Then W is a subspace of V if and only if all of the following are satisfied.

- $0 \in W$.
- For all $x, y \in W$, $x + y \in W$.
- For all $a \in \mathbb{F}$ and $x \in W$, $ax \in W$.

Proof. \Rightarrow Since W is a subspace of V , $0 \in W$ and W is closed under V 's vector addition and scalar multiplication.

\Leftarrow Since V is a vector space W inherits associativity, commutativity, and distributivity from V as well as V 's behavior with respect to the identities. Furthermore, $0 \in W$ by assumption. All that is left to show is that W contains additive inverses. Suppose $x \in W$, then by assumption $-x = (-1)x \in W$. Thus W is a subspace of V . \square

Definition 1.1.6 (Matrix Transpose). Let M be an $m \times n$ matrix, then the *transpose* of M , denoted M^T , is the $n \times m$ matrix defined by $(M^T)_{i,j} = M_{j,i}$, that is

$$M^T = \begin{pmatrix} M_{1,1} & M_{2,1} & \dots & M_{m,1} \\ M_{1,2} & M_{2,2} & \dots & M_{m,2} \\ \vdots & \vdots & \ddots & \vdots \\ M_{1,n} & M_{2,n} & \dots & M_{m,n} \end{pmatrix}.$$

Definition 1.1.7 (Symmetric Matrix). Let M be a matrix, then if $M = M^T$, we say M is *symmetric*.

Example 1.1.6. The set of symmetric $n \times n$ matrices over a field \mathbb{F} , denoted $W_{n \times n}(\mathbb{F})$, is a subspace of $M_{n \times n}(\mathbb{F})$.

Proof. Consider the zero matrix. Since the zero matrix is an $n \times n$ matrix with all entries equal to zero, the transpose of the zero matrix is also an $n \times n$ matrix with all entries equal to zero. Thus, $0 = 0^T$ and the zero matrix is symmetric.

Let $A, B \in W_{n \times n}(\mathbb{F})$. Then $A = A^T$ and $B = B^T$. By definition of symmetry and matrix transpose we have

$$A_{i,j} = (A^T)_{i,j} = A_{j,i} \tag{1.1}$$

and

$$B_{i,j} = (B^T)_{i,j} = B_{j,i} \tag{1.2}$$

for all $1 \leq i, j \leq n$.

Consider $(A + B)$. By definition we have

$$(A + B)_{i,j} = A_{i,j} + B_{i,j}.$$

By Equation 1.1 and Equation 1.2 we have

$$(A + B)_{i,j} = A_{i,j} + B_{i,j} = A_{j,i} + B_{j,i}.$$

The definition of matrix transpose implies that

$$(A + B)_{i,j} = A_{i,j} + B_{i,j} = A_{j,i} + B_{j,i} = (A^T)_{i,j} + (B^T)_{i,j}$$

and thus by definition of matrix addition,

$$(A + B)_{i,j} = A_{i,j} + B_{i,j} = A_{j,i} + B_{j,i} = (A^T)_{i,j} + (B^T)_{i,j} = (A^T + B^T)_{i,j}.$$

Ergo, $A + B$ is symmetric and $W(\mathbb{F})$ is closed under matrix addition.

Let $k \in \mathbb{F}$ and consider kA . We know by definition that

$$(kA)_{i,j} = k \cdot A_{i,j}.$$

We invoke Equation 1.1 again to get that

$$(kA)_{i,j} = k \cdot A_{i,j} = k \cdot A_{j,i}.$$

Applying the definition of matrix transpose yields,

$$(kA)_{i,j} = k \cdot A_{i,j} = k \cdot A_{j,i} = k \cdot (A^T)_{i,j}.$$

Lastly, by definition of scalar multiplication we have,

$$(kA)_{i,j} = k \cdot A_{i,j} = k \cdot A_{j,i} = k \cdot (A^T)_{i,j} = (kA^T)_{i,j}.$$

Thus kA is symmetric and $W(\mathbb{F})$ is closed under scalar multiplication. \square

Definition 1.1.8 (Main Diagonal of a Matrix). Let \mathbb{F} be a field and let $M \in M_{n \times n}(\mathbb{F})$, then the *main diagonal* of M is the set $\{M_{i,i}\}_{i=1}^n$.

Definition 1.1.9 (Diagonal Matrix). Let \mathbb{F} be a field and let $A \in M_{n \times n}(\mathbb{F})$, then A is called a *diagonal matrix* if and only if whenever $i \neq j$, $A_{i,j} = 0$.

Example 1.1.7. Let \mathbb{F} be a field and let $D_n(\mathbb{F})$ be the set of all diagonal matrices in $M_{n \times n}(\mathbb{F})$, then $D_n(\mathbb{F})$ is a subspace on $M_{n \times n}(\mathbb{F})$.

Proof. We know $0 \in D_n(\mathbb{F})$ since for all i, j , $0_{i,j} = 0$. Let $A, B \in D_n(\mathbb{F})$. Then for all $i \neq j$, $A_{i,j} = B_{i,j} = 0$. Thus, $(A + B)_{i,j} = A_{i,j} + B_{i,j} = 0 + 0 = 0$ whenever $i \neq j$ and $A + B$ is diagonal. Let $k \in \mathbb{F}$. Then $(kA)_{i,j} = k \cdot A_{i,j} = k \cdot 0 = 0$ and kA is diagonal. Therefore $D_n(\mathbb{F})$ forms a subspace of $M_{n \times n}(\mathbb{F})$. \square

Definition 1.1.10 (Trace of a Matrix). Let \mathbb{K} be a field and let $M \in M_{n \times n}(\mathbb{K})$, then the *trace* of M denoted $\text{tr } M$ is defined as

$$\text{tr } M = \sum_{i=1}^n M_{i,i}$$

or the sum of the elements on the main diagonal.

Example 1.1.8. Let \mathbb{K} be a field and let $T_n(\mathbb{K})$ be the set of matrices in $M_{n \times n}(\mathbb{K})$ with trace equal to zero, then $T_n(\mathbb{K})$ is a subspace of $M_{n \times n}(\mathbb{K})$.

Proof. Obviously, the zero matrix has a trace of zero and thus $0 \in T_n(\mathbb{K})$. Let $A, B \in T_n(\mathbb{K})$ then $\text{tr } A = 0$ and $\text{tr } B = 0$. Consider $\text{tr}(A + B)$.

$$\text{tr}(A + B) = \sum_{i=1}^n (A + B)_{i,i} = \sum_{i=1}^n (A_{i,i} + B_{i,i}) = \left(\sum_{i=1}^n A_{i,i} \right) + \left(\sum_{i=1}^n B_{i,i} \right) = \text{tr } A + \text{tr } B = 0$$

Thus, $A + B$ has trace 0 and $A + B \in T_n(\mathbb{K})$. Let $k \in \mathbb{K}$. Consider $\text{tr}(kA)$.

$$\text{tr}(kA) = \sum_{i=1}^n (kA)_{i,i} = \sum_{i=1}^n k \cdot A_{i,i} = k \sum_{i=1}^n A_{i,i} = k \text{tr } A = 0$$

And thus, kA has trace 0 and $kA \in T_n(\mathbb{K})$. Therefore $T_n(\mathbb{K})$ is a subspace of $M_{n \times n}(\mathbb{K})$. \square

Theorem 1.1.6. Let V be a vector space over a field \mathbb{F} and let \mathcal{W} be a countable collection of subspaces of V . Then

$$W_i = \bigcap_{W \in \mathcal{W}} W$$

is a subspace of V .

Proof. Since $0 \in W$ for all $W \in \mathcal{W}$, $0 \in W_i$. Let $x, y \in W_i$, then $x, y \in W$ for all $W \in \mathcal{W}$ and thus $x + y \in W$ for all $W \in \mathcal{W}$. Therefore, $x + y \in W_i$. Let $a \in \mathbb{F}$. Since $x \in W$ for all $W \in \mathcal{W}$, $ax \in W$ for all $W \in \mathcal{W}$. Ergo, $ax \in W_i$ and W_i is a subspace of V . \square

Proposition 1.1.7. For any matrix A , $[(A^T)^T] = A$.

Proof. Apply the definition of matrix transposition twice.

$$[(A^T)^T]_{i,j} = (A^T)_{j,i} = A_{i,j}$$

\square

Proposition 1.1.8. For any matrix A , $A + A^T$ is symmetric.

Proof. Consider $(A + A^T)_{i,j}$.

$$(A + A^T)_{i,j} = A_{i,j} + (A^T)_{i,j} = A_{i,j} + A_{j,i} = A_{j,i} + A_{i,j} = A_{j,i} + (A^T)_{j,i} = (A + A^T)_{j,i} = [(A + A^T)^T]_{i,j}$$

And thus, $(A + A^T)$ is symmetric. \square

Proposition 1.1.9. Let \mathbb{K} be a field and let $A, B \in M_{n \times n}(\mathbb{K})$ and $a, b \in \mathbb{K}$, then $\text{tr}(aA + bB) = a \text{tr } A + b \text{tr } B$.

Proof.

$$\begin{aligned} \text{tr}(aA + bB) &= \sum_{i=1}^n (aA + bB)_{i,i} \\ &= \sum_{i=1}^n [(aA)_{i,i} + (bB)_{i,i}] \\ &= \left(\sum_{i=1}^n a \cdot A_{i,i} \right) + \left(\sum_{i=1}^n b \cdot B_{i,i} \right) \\ &= a \left(\sum_{i=1}^n A_{i,i} \right) + b \left(\sum_{i=1}^n B_{i,i} \right) \\ &= a \text{tr } A + b \text{tr } B \end{aligned}$$

\square

Definition 1.1.11 (Sum of Subsets of a Vector Space). Let S, R be nonempty subsets of a vector space V , then the *sum* of S and R , denoted $S + R$ is defined as $S + R = \{s + r | s \in S, r \in R\}$.

Proposition 1.1.10. Let U, W be subspaces of a vector space V over a field \mathbb{F} . Then $U + W$ is a subspace of V and is the smallest subspace containing both U and W .

Proof. Since U and W are subspaces, $0 \in U$ and $0 \in W$ therefore, $0 = 0 + 0 \in U + W$. Let $x, y \in U + W$ then there exist $u_x, u_y \in U$ and $w_x, w_y \in W$ such that $x = u_x + w_x$ and $y = u_y + w_y$. Thus,

$$x + y = (u_x + w_x) + (u_y + w_y) = (u_x + u_y) + (w_x + w_y).$$

Since U and W are subspaces, $u_x + u_y \in U$ and $w_x + w_y \in W$. Ergo, $x + y = (u_x + u_y) + (w_x + w_y) \in U + W$.

Let $a \in \mathbb{F}$. Then,

$$ax = a(u_x + w_x) = au_x + aw_x.$$

Since U and W are subspaces, $au_x \in U$ and $aw_x \in W$. Ergo, $ax = au_x + aw_x \in U + W$ and $U + W$ is a subspace of V .

We know that, set-wise, $U = \{u + 0\}_{u \in U}$ and $W = \{0 + w\}_{w \in W}$, and thus $U, W \subseteq U + W$. Let X be a subspace of V such that $U, W \subseteq X$. Let $x \in U + W$, then there exists some $u \in U \subseteq X$ and $w \in W \subseteq X$ such that $x = u + w$. Therefore, $x = u + w \in X$ and $U + W \subseteq X$ for all subspaces X containing U and W . Ergo, $U + W$ is the smallest subspace of V containing U and W . \square

Definition 1.1.12 (Direct Sum of Vector Spaces). A vector space V is called the *direct sum of U and W* , denoted $V = U \oplus W$ if and only if U and W are subspaces of V such that $U \cap W = \emptyset$ and $U + W = V$.

Example 1.1.9. Let \mathbb{K} be a field and let $U = \{(a_1, a_2, \dots, a_n) \in \mathbb{K}^n | a_n = 0\}$ and $V = \{(a_1, a_2, \dots, a_n) \in \mathbb{K}^n | a_1 = a_2 = \dots = a_{n-1} = 0\}$. Then $\mathbb{K}^n = U \oplus V$.

Proof. The details are obvious and left as an exercise. \square

Definition 1.1.13 (Cosets of a Vector Space). Let U be a subspace of a vector space V over a field \mathbb{K} . Then for each $v \in V$ the set $\{v\} + W = \{v + w\}_{w \in W}$ is called the *coset of W containing v* , denoted $v + W$.

Proposition 1.1.11. Let \mathbb{K} be a field and W be a subspace of a vector space V over \mathbb{F} and let $v \in V$. Then $v + W$ is a subspace if and only if $v \in W$.

Proof. \Leftarrow Suppose $v \in W$. Then by closure, $v + W = \{v + w\}_{w \in W} = W$.

\Rightarrow Suppose $v + W$ is a subspace of V . Then $0 \in v + W$ and therefore, there exists a $w \in W$ such that $0 = v + w$. This w can only be $-v$ by uniqueness of inverses. Since $-v \in W$, $v \in W$ since W is a subspace. \square

Proposition 1.1.12. Let \mathbb{K} be a field and W be a subspace of a vector space V over \mathbb{F} and let $v \in V$. Then $v \in v + W$.

Proof. Since W is a subspace, $0 \in W$ and thus $v = v + 0 \in v + W$. \square

Proposition 1.1.13. Let \mathbb{K} be a field and W be a subspace of a vector space V over \mathbb{F} and let $u, v \in V$. Then $v + W \cap u + W = \emptyset$ if and only if $v + W \neq u + W$.

Proof. \Rightarrow Suppose $v + W \cap u + W = \emptyset$. Then since both $v + W$ and $u + W$ are non-empty, $v + W \neq u + W$.

\Leftarrow Suppose $v + W \neq u + W$ with $v + W \cap u + W \neq \emptyset$. Then there exists an $x \in v + W \cap u + W$. Ergo, $x \in v + W$ and $x \in u + W$. Thus, there exists $w_1, w_2 \in W$ such that $x = v + w_1$ and $x = u + w_2$ respectively. Therefore, $v + w_1 = u + w_2$ and $v = u + w_2 - w_1$. Ergo,

$$v + W = \{v + w\}_{w \in W} = \{u + (w_2 - w_1 + w)\}_{w \in W} = u + W.$$

Thus, creating a contradiction. Therefore if $v + W \neq u + W$ then $v + W \cap u + W = \emptyset$. \square

Proposition 1.1.14. Let \mathbb{K} be a field and W be a subspace of a vector space V over \mathbb{F} and let $u, v \in V$. Then $v + W = u + W$ if and only if $v - u \in W$.

Proof. \Rightarrow Assume $v + W = u + W$. Then $v \in v + W$ and thus $v \in u + W$. Ergo, there exists a $w \in W$ such that $v = u + w$. Solving for w yields $w = v - u \in W$.

\Leftarrow Assume $v - u \in W$. Therefore, $u + v - u = v \in u + W$. We know $v \in v + W$ thus, $u + W \cap v + W \neq \emptyset$. This occurs if and only if $u + W = v + W$. \square

Definition 1.1.14 (Quotient Space). Let \mathbb{K} be a field and W be a subspace of a vector space V over \mathbb{F} . The the *quotient space of V modulo W* , denoted V/W is the set of all cosets of W ,

$$V/W := \{v + W\}_{v \in V}.$$

Furthermore V/W is a vector space under the following operations.

$$\begin{aligned}(u + W) + (v + W) &= (u + v) + W \\ a(u + W) &= (au) + W\end{aligned}$$

Proof. A bunch of tedious symbol pushing that I refuse to do. □

1.1.3 Linear Combinations

Definition 1.1.15 (Linear Combination). Let V be a vector space over a field \mathbb{F} and let S be a nonempty subset of V . An $x \in V$ is said to be a *linear combination of elements of S* if and only if there exists a $\{s_j\}_{j=1}^n \subseteq S$ and scalars $\{a_j\}_{j=1}^n \subseteq \mathbb{F}$ where $n < \infty$ such that

$$x = \sum_{j=1}^n a_j y_j.$$

When this happens, we say x is a *linear combination of y_1, y_2, \dots, y_n* .

Definition 1.1.16 (Spanning Set). Let V be a vector space over a field \mathbb{F} and let S be a nonempty subset of V . Then, the *span of S* , denoted $\text{span } S$, is the set

$$\text{span } S = \left\{ \sum_{j=1}^n a_j s_j \mid \{a_j\}_{j=1}^n \subseteq \mathbb{F}, \{s_j\}_{j=1}^n \subseteq S, n < \infty \right\}$$

or the set of linear combinations of elements of S . We define $\text{span } \emptyset = \{0\}$.

Theorem 1.1.15. *Let V be a vector space over a field \mathbb{F} and let S be a nonempty subset of V . Then $\text{span } S$ is a subspace of V and is the smallest subspace of V containing S .*

Proof. Let $\{s_j\}_{j=1}^n \subseteq S$ where $n < \infty$. Then $0 = \sum_{j=1}^n 0s_j \in \text{span } S$. Let $x, y \in \text{span } S$. Then there exist $\{s_j\}_{j=1}^n, \{r_j\}_{j=1}^m \subseteq S$ and $\{a_j\}_{j=1}^n, \{b_j\}_{j=1}^m \subseteq \mathbb{F}$ with $m, n < \infty$ such that $x = \sum_{j=1}^n a_j s_j$ and $y = \sum_{j=1}^m b_j r_j$. Define $\{t_j\}_{j=1}^{n+m}$ and $\{c_j\}_{j=1}^{n+m}$ by

$$t_j = \begin{cases} s_j & j \leq n \\ r_j & j > n \end{cases} \quad c_j = \begin{cases} a_j & j \leq n \\ b_j & j > n \end{cases}.$$

We can see that $\{t_j\}$ is a finite subset of S and $\{c_j\}$ is a finite subset of \mathbb{F} , thus any element made out of scalar multiples of t vectors is in $\text{span } S$. Consider $x + y$.

$$x + y = \sum_{j=1}^n a_j s_j + \sum_{j=1}^m b_j r_j = \sum_{j=1}^n c_j t_j + \sum_{j=n+1}^{n+m} b_{j-n} r_{j-n} = \sum_{j=1}^n c_j t_j + \sum_{j=n+1}^{n+m} c_j t_j$$

Therefore, $x + y$ is a linear combination of elements of S and thus $x + y \in \text{span } S$. Consider kx for any $k \in \mathbb{F}$.

$$kx = k \sum_{j=1}^n a_j s_j = \sum_{j=1}^n (ka_j) s_j$$

Ergo, $kx \in \text{span } S$ and $\text{span } S$ is a subspace of V .

Let W be a subspace of V such that $S \subseteq W$. Then for all $s \in S$ and $a \in \mathbb{F}$, $as \in W$ since W is a subspace. Ergo, for any $\{s_j\}_{j=1}^n \subseteq S$ and $\{a_j\}_{j=1}^n \subseteq \mathbb{F}$ with $n < \infty$

$$\sum_{j=1}^n a_j s_j \in W$$

since W is a subspace and any finite sum of vectors in W is in W . Ergo $\text{span } S \subseteq W$ and is the smallest subspace of V containing S . \square

Definition 1.1.17 (Span). A subset S of a vector space V *spans V* if and only if $\text{span } S = V$.

Proposition 1.1.16. *Let W be a nonempty subset of a vector space V over a field \mathbb{F} . Then W is a subspace of V if and only if $W = \text{span } W$.*

Proof. \Rightarrow Suppose W is a subspace. We know $W \subseteq \text{span } W$, by definition. Furthermore, for all $w \in W$ and $a \in \mathbb{F}$, $aw \in W$ since W is a subspace. Thus, for any finite $\{w_j\}_{j=1}^n \subseteq W$ and $\{a_j\}_{j=1}^n \subseteq \mathbb{F}$, $\sum a_j w_j \in W$ by properties of vector spaces. Ergo, $\text{span } W \subseteq W$ and $W = \text{span } W$.

\Leftarrow Suppose $W = \text{span } W$. Since $\text{span } W$ is a subspace and $W = \text{span } W$, W is trivially a subspace. \square

Proposition 1.1.17. *Let S, R be nonempty subsets of V such that $S \subseteq R$. Then $\text{span } S \subseteq \text{span } R$ and if $\text{span } S = V$, then $\text{span } R = V$.*

Proof. I provide a sketch and leave the details to the reader.

All vectors in S are also in R ergo all sums of scalar multiples of vectors in S (read: $\text{span } S$) are in $\text{span } R$.

Furthermore, we know $\text{span } R$ is a subspace of V , and thus $\text{span } R \subseteq V$. If $V = \text{span } S \subseteq \text{span } R$, then $V \subseteq \text{span } R$ and thus $\text{span } R = V$. \square

Chapter 2

Definitions

2.1 Vector Spaces

2.1.1 Introduction to Vector Spaces

Definition 2.1.1 (Vector Space). A *vector space* V over a field \mathbb{F} is a set with two binary operations, $+: V \times V \rightarrow V$ and $\cdot: V \times \mathbb{F} \rightarrow V$ such that all of the following hold.

1. For all $x, y \in V$, $x + y = y + x$. (Additive Commutativity)
2. For all $x, y, z \in V$, $x + (y + z) = (x + y) + z$. (Additive Associativity)
3. There exists an element, denoted 0 , in V such that for all $x \in V$, $x + 0 = x$.
4. For each $x \in V$ there exists a $y \in V$, denoted $-x$, such that $x + y = 0$.
5. For all $x \in V$, $1x = x$.
6. For all $a, b \in \mathbb{F}$ and $x \in V$, $a(bx) = (ab)x$.
7. For all $a \in \mathbb{F}$ and $x, y \in V$, $a(x + y) = ax + ay$.
8. For all $a, b \in \mathbb{F}$ and $x \in V$, $(a + b)x = ax + bx$.

Furthermore, $x + y$ is called the *sum of x and y* while ax is called the *product of x and a* . Moreover, each $x \in V$ is called a *vector* and each $a \in \mathbb{F}$ is called a *scalar*.

Definition 2.1.2 (n -tuple). An object of the form (a_1, a_2, \dots, a_n) where $a_j \in \mathbb{F}$ for all $1 \leq j \leq n$, is called an *n -tuple*.

Definition 2.1.3. Let \mathbb{F} be a field and $m, n \in \mathbb{N}$, then an $m \times n$ *matrix* with entries from \mathbb{F} is a rectangular array of the form

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{pmatrix}$$

where $a_{i,j} \in \mathbb{F}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. The entries $(a_{i,1}, a_{i,2}, \dots, a_{i,n})$ is called the *i th row* of the matrix and is a row vector in \mathbb{F}^n . The entries $(a_{1,j}, a_{2,j}, \dots, a_{m,j})$ is called the *j th column* of the matrix and is a column vector in \mathbb{F}^n . We denote the entry on the i th row and j th column as $A_{i,j}$. Furthermore, two $m \times n$ matrices, A and B , are equal if and only if $A_{i,j} = B_{i,j}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$; we denote this by $A = B$. Moreover, if $n = m$ we say that A is a *square matrix*. Lastly, we denote the set of $m \times n$ matrices over \mathbb{F} as $M_{m \times n}(\mathbb{F})$.

Definition 2.1.4 (Polynomial Ring). Let \mathbb{F} be a field. Then the *ring of polynomials in an indeterminate x over \mathbb{F}* , denoted $\mathbb{F}[x]$ is defined as

$$\mathbb{F}[x] := \left\{ \sum_{i=0}^n a_i x^i \mid n \in \mathbb{N}, (a_0, a_1, \dots, a_n) \in \mathbb{F}^n, a_n \neq 0 \right\}.$$

Additionally, we define $x^0 = 1$. Moreover, for each $p = \sum_{i=0}^n p_i x^i \in \mathbb{F}[x]$, the *degree of p* , denoted $\deg p$, is n . Furthermore, if $p = 0$, that is $p_n = p_{n-1} = \dots = p_0 = 0$, then p is called the *zero polynomial* and $\deg p = -1$ or $\deg p = -\infty$ depending on convention. If $\deg p = 0$, then we say p is a *constant polynomial*.

Lastly, $\mathbb{F}[x]$ forms a ring under the following operations where $p, q \in \mathbb{F}[x]$ and without loss of generality assume, $\deg p \geq \deg q$.

$$p + q = \sum_{i=0}^{\deg q} (p_i + q_i)x^i + \sum_{i=\deg q+1}^{\deg p} p_i x^i$$

$$pq = \sum_{k=0}^{\deg p + \deg q} \left(\sum_{i+j=k} p_i q_j \right) x^k$$

2.1.2 Subspaces

Definition 2.1.5 (Subspace). A *subspace*, W , of a vector space, V , over a field, \mathbb{F} , is a subset of V that is also a vector space over \mathbb{F} .

Definition 2.1.6 (Matrix Transpose). Let M be an $m \times n$ matrix, then the *transpose of M* , denoted M^T , is the $n \times m$ matrix defined by $(M^T)_{i,j} = M_{j,i}$, that is

$$M^T = \begin{pmatrix} M_{1,1} & M_{2,1} & \cdots & M_{m,1} \\ M_{1,2} & M_{2,2} & \cdots & M_{m,2} \\ \vdots & \vdots & \ddots & \vdots \\ M_{1,n} & M_{2,n} & \cdots & M_{m,n} \end{pmatrix}.$$

Definition 2.1.7 (Symmetric Matrix). Let M be a matrix, then if $M = M^T$, we say M is *symmetric*.

Definition 2.1.8 (Main Diagonal of a Matrix). Let \mathbb{F} be a field and let $M \in M_{n \times n}(\mathbb{F})$, then the *main diagonal of M* is the set $\{M_{i,i}\}_{i=1}^n$.

Definition 2.1.9 (Diagonal Matrix). Let \mathbb{F} be a field and let $A \in M_{n \times n}(\mathbb{F})$, then A is called a *diagonal matrix* if and only if whenever $i \neq j$, $A_{i,j} = 0$.

Definition 2.1.10 (Trace of a Matrix). Let \mathbb{K} be a field and let $M \in M_{n \times n}(\mathbb{K})$, then the *trace of M* denoted $\text{tr } M$ is defined as

$$\text{tr } M = \sum_{i=1}^n M_{i,i}$$

or the sum of the elements on the main diagonal.

Definition 2.1.11 (Sum of Subsets of a Vector Space). Let S, R be nonempty subsets of a vector space V , then the *sum of S and R* , denoted $S + R$ is defined as $S + R = \{s + r | s \in S, r \in R\}$.

Definition 2.1.12 (Direct Sum of Vector Spaces). A vector space V is called the *direct sum of U and W* , denoted $V = U \oplus W$ if and only if U and W are subspaces of V such that $U \cap W = \emptyset$ and $U + W = V$.

Definition 2.1.13 (Cosets of a Vector Space). Let U be a subspace of a vector space V over a field \mathbb{K} . Then for each $v \in V$ the set $\{v\} + W = \{v + w\}_{w \in W}$ is called the *coset of W containing v* , denoted $v + W$.

Definition 2.1.14 (Quotient Space). Let \mathbb{K} be a field and W be a subspace of a vector space V over \mathbb{F} . The *quotient space of V modulo W* , denoted V/W is the set of all cosets of W ,

$$V/W := \{v + W\}_{v \in V}.$$

Furthermore V/W is a vector space under the following operations.

$$\begin{aligned} (u + W) + (v + W) &= (u + v) + W \\ a(u + W) &= (au) + W \end{aligned}$$

2.1.3 Linear Combinations

Definition 2.1.15 (Linear Combination). Let V be a vector space over a field \mathbb{F} and let S be a nonempty subset of V . An $x \in V$ is said to be a *linear combination of elements of S* if and only if there exists a $\{s_j\}_{j=1}^n \subseteq S$ and scalars $\{a_j\}_{j=1}^n \subseteq \mathbb{F}$ where $n < \infty$ such that

$$x = \sum_{j=1}^n a_j y_j.$$

When this happens, we say x is a *linear combination of y_1, y_2, \dots, y_n* .

Definition 2.1.16 (Spanning Set). Let V be a vector space over a field \mathbb{F} and let S be a nonempty subset of V . Then, the *span of S* , denoted $\text{span } S$, is the set

$$\text{span } S = \left\{ \sum_{j=1}^n a_j s_j \mid \{a_j\}_{j=1}^n \subseteq \mathbb{F}, \{s_j\}_{j=1}^n \subseteq S, n < \infty \right\}$$

or the set of linear combinations of elements of S . We define $\text{span } \emptyset = \{0\}$.

Definition 2.1.17 (Span). A subset S of a vector space V *spans V* if and only if $\text{span } S = V$.

Chapter 3

Theorems

3.1 Vector Spaces

3.1.1 Introduction to Vector Spaces

Proposition 3.1.1. *If u, v, w are elements of a vector space V such that $x + z = y + z$ then, $x = y$.*

Proposition 3.1.2. *The zero vector in any vector space V is unique.*

Proposition 3.1.3. *Let V be a vector space and let $v \in V$, then there exists a unique $u \in V$ such that $v + u = 0$.*

Proposition 3.1.4. *Let V be a vector space over a field \mathbb{F} , then:*

1. $0x = 0$ for all $x \in V$;
2. $a0 = 0$ for all $a \in \mathbb{F}$;
3. $(-a)x = -(ax) = a(-x)$ for all $x \in V$ and $a \in \mathbb{F}$.

3.1.2 Subspaces

Theorem 3.1.5. *Let V be a vector space over a field \mathbb{F} and let $W \subseteq V$. Then W is a subspace of V if and only if all of the following are satisfied.*

- $0 \in W$.
- For all $x, y \in W$, $x + y \in W$.
- For all $a \in \mathbb{F}$ and $x \in W$, $ax \in W$.

Theorem 3.1.6. *Let V be a vector space over a field \mathbb{F} and let \mathcal{W} be a countable collection of subspaces of V . Then*

$$W_i = \bigcap_{W \in \mathcal{W}} W$$

is a subspace of V .

Proposition 3.1.7. *For any matrix A , $[(A^T)^T] = A$.*

Proposition 3.1.8. *For any matrix A , $A + A^T$ is symmetric.*

Proposition 3.1.9. *Let \mathbb{K} be a field and let $A, B \in M_{n \times n}(\mathbb{K})$ and $a, b \in \mathbb{K}$, then $\text{tr}(aA + bB) = a \text{tr } A + b \text{tr } B$.*

Proposition 3.1.10. *Let U, W be subspaces of a vector space V over a field \mathbb{F} . Then $U + W$ is a subspace of V and is the smallest subspace containing both U and W .*

Proposition 3.1.11. *Let \mathbb{K} be a field and W be a subspace of a vector space V over \mathbb{F} and let $v \in V$. Then $v + W$ is a subspace if and only if $v \in W$.*

Proposition 3.1.12. *Let \mathbb{K} be a field and W be a subspace of a vector space V over \mathbb{F} and let $v \in V$. Then $v \in v + W$.*

Proposition 3.1.13. *Let \mathbb{K} be a field and W be a subspace of a vector space V over \mathbb{F} and let $u, v \in V$. Then $v + W \cap u + W = \emptyset$ if and only if $v + W \neq u + W$.*

Proposition 3.1.14. *Let \mathbb{K} be a field and W be a subspace of a vector space V over \mathbb{F} and let $u, v \in V$. Then $v + W = u + W$ if and only if $v - u \in W$.*

3.1.3 Linear Combinations

Theorem 3.1.15. *Let V be a vector space over a field \mathbb{F} and let S be a nonempty subset of V . Then $\text{span } S$ is a subspace of V and is the smallest subspace of V containing S .*

Proposition 3.1.16. *Let W be a nonempty subset of a vector space V over a field \mathbb{F} . Then W is a subspace of V if and only if $W = \text{span } W$.*

Proposition 3.1.17. *Let S, R be nonempty subsets of V such that $S \subseteq R$. Then $\text{span } S \subseteq \text{span } R$ and if $\text{span } S = V$, then $\text{span } R = V$.*