Matt's Linear Algebra Notes

December 16, 2015

Chapter 1

Material

1.1 Vector Spaces

1.1.1 Introduction to Vector Spaces

Definition 1.1.1 (Vector Space)**.** A *vector space* V over a field $\mathbb{F}$ is a set with two binary operations, $+ : V \times V \to V$ and $\cdot : V \times \mathbb{F} \to V$ such that all of the following hold.

1. For all $x, y \in V$, $x + y = y + x$. (Additive Commutativity)
2. For all $x, y, z \in V$, $x + (y + z) = (x + y) + z$. (Additive Associativity)
3. There exists an element, denoted 0, in V such that for all $x \in V$, $x + 0 = x$.
4. For each $x \in V$ there exists a $y \in V$, denoted $-x$, such that $x + y = 0$.
5. For all $x \in V$, $1x = x$.
6. For all $a, b \in \mathbb{F}$ and $x \in V$, $a(bx) = (ab)x$.
7. For all $a \in \mathbb{F}$ and $x, y \in V$, $a(x + y) = ax + ay$.
8. For all $a, b \in \mathbb{F}$ and $x \in V$, $(a + b)x = ax + bx$.

Furthermore, $x + y$ is called the *sum of* x *and* y while ax is called the *product of* x *and* a. Moreover, each $x \in V$ is called a *vector* and each $a \in \mathbb{F}$ is called a *scalar*.

Definition 1.1.2 (n-tuple)**.** An object of the form $(a_1, a_2, \ldots, a_n)$ where $a_j \in \mathbb{F}$ for all $1 \leq j \leq n$, is called an *n-tuple*.

Example 1.1.1. *Let $\mathbb{F}$ be a field and $n \in \mathbb{N}$, then $\mathbb{F}^n = \{(a_1, a_2, \ldots, a_n) | a_j \in \mathbb{F} \forall 1 \leq j \leq n\}$ forms a vector space under component-wise addition and multiplication as defined below for $(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n) \in \mathbb{F}^n$ and $k \in \mathbb{F}$.*

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$
$$k(a_1, a_2, \ldots, a_n) = (ka_1, ka_2, \ldots, ka_n)$$

Furthermore, it said that

$$(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$$

if and only if $a_j = b_j$ for all $1 \leq j \leq n$.

Proof. $\mathbb{F}^n$ is a vector space trivially from the fact that $\mathbb{F}$ is a field. □

Definition 1.1.3 (Matrix)**.** Let $\mathbb{F}$ be a field and $m, n \in \mathbb{N}$, then an $m \times n$ *matrix* with entries from $\mathbb{F}$ is a rectangular array of the form

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

where $a_{i,j} \in \mathbb{F}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. The entries $(a_{i,1}, a_{i,2}, \ldots, a_{i,n})$ is called the *ith row* of the matrix and is a row vector in $\mathbb{F}^n$. The entries $(a_{1,j}, a_{2,j}, \ldots, a_{m,j})$ is called the *jth column* of the matrix and is a column vector in $\mathbb{F}^n$. We denote the entry on the ith row and jth column as $A_{i,j}$. Furthermore, two $m \times n$ matrices, A and B, are equal if and only if $A_{i,j} = B_{i,j}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$; we denote this by $A = B$. Moreover, if $n = m$ we say that A is a *square matrix*. Lastly, we denote the set of $m \times n$ matrices over $\mathbb{F}$ as $M_{m \times n}(\mathbb{F})$.

Example 1.1.2. *Let $\mathbb{F}$ be a field and $m, n \in \mathbb{N}$, then $M_{m\times n}(\mathbb{F})$ is a vector space over $\mathbb{F}$ under the following operations for $A, B \in M_{m\times n}(\mathbb{F})$ and $k \in \mathbb{F}$.*

$$(A+B)_{i,j} = A_{i,j} + B_{i,j}$$
$$(kA)_{i,j} = kA_{i,j}$$

Proof. The proof is trivial from the fact that we operating on multiple copies of a field. $\square$

Example 1.1.3. *Let S be a nonempty set and let $\mathbb{F}$ be a field and let $\mathscr{F}(S,\mathbb{F})$ denote the set of all functions from S into $\mathbb{F}$. Two elements $f, g \in \mathscr{F}(S,\mathbb{F})$ are equal if and only if $f(s) = g(s)$ for all $s \in S$. Then $\mathscr{F}(S,\mathbb{F})$ is a vector space under the following operations for $f, g \in \mathscr{F}(S,\mathbb{F})$ and $k \in \mathbb{F}$.*

$$(f+g)(s) = f(s) + g(s)$$
$$(kf)(s) = k\,[f(s)]$$

Proof. The proof is trivial because all operations are done inside the field, and thus the space inherits the structure from $\mathbb{F}$. $\square$

Definition 1.1.4 (Polynomial Ring)**.** Let $\mathbb{F}$ be a field. Then the *ring of polynomials in an indeterminate x over $\mathbb{F}$*, denoted $\mathbb{F}[x]$ is defined as

$$\mathbb{F}[x] := \{\sum_{i=0}^{n} a_i x^i | n \in \mathbb{N}, (a_0, a_1, \dots, a_n) \in \mathbb{F}^n, a_n \neq 0\}.$$

Additionally, we define $x^0 = 1$. Moreover, for each $p = \sum_{i=0}^{n} p_i x^i \in \mathbb{F}[x]$, the *degree of p*, denoted $\deg p$, is n. Furthermore, if $p = 0$, that is $p_n = p_{n-1} = \dots = p_0 = 0$, then p is called the *zero polynomial* and $\deg p = -1$ or $\deg p = -\infty$ depending on convention. If $\deg p = 0$, then we say p is a *constant polynomial.* Lastly, $\mathbb{F}[x]$ forms a ring under the following operations where $p, q \in \mathbb{F}[x]$ and without loss of generality assume, $\deg p \geq \deg q$.

$$p + q = \sum_{i=0}^{\deg q} (p_i + q_i)x^i + \sum_{i=\deg q+1}^{\deg p} p_i x^i$$
$$pq = \sum_{k=0}^{\deg p \cdot \deg q} \left(\sum_{i+j=k} p_i q_j \right) x^k$$

Example 1.1.4. *Let $\mathbb{F}$ be a field, then $\mathbb{F}[x]$ is a vector space over $\mathbb{F}$ under polynomial addition and scalar multiplication by constant polynomials.*

Proof. Since $\mathbb{F}[x]$ is a ring, it is closed under addition. Since constant polynomials are polynomials, it is closed under scalar multiplication. Since $\mathbb{F}[x]$ is a ring, addition is commutative and associative. Moreover the zero polynomial is the additive identity and likewise serves as the zero vector. Since $\mathbb{F}[x]$ is a ring, each $p \in \mathbb{F}[x]$ has a unique $-p \in \mathbb{F}[x]$ such that $p + (-p) = 0$. The constant polynomial 1 is the multiplicative identity in the ring and serves as the identity in the vector space. Furthermore, multiplication is associative and distributes over addition because $\mathbb{F}[x]$ is, indeed, a ring. $\square$

Proposition 1.1.1. *If u, v, w are elements of a vector space V such that $x + z = y + z$ then, $x = y$.*

Proof. Since $z \in V$, then there exists a $-z \in V$ such that $z + (-z) = 0$. Thus, $x + z = y + z$ implies

$$x + z - z = y + z - z$$

and ergo $x = y$. $\square$

Proposition 1.1.2. *The zero vector in any vector space V is unique.*

Proof. Assume there exists two zero vectors in V denoted 0_1 and 0_2. Then $v + 0_1 = v$ and $v + 0_2 = v$ for all $v \in V$. Therefore it is true that

$$0_2 = 0_2 + 0_1 = 0_1 + 0_2 = 0_1$$

and thus these identities are in fact the same. $\square$

Proposition 1.1.3. *Let V be a vector space and let $v \in V$, then there exists a unique $u \in V$ such that $v + u = 0$.*

Proof. Assume v has two inverses, namely u_1 and u_2. Then $v + u_1 = 0$ and $v + u_2 = 0$. Therefore,

$$v + u_1 + u_2 = 0 + u_2 = u_2$$

and

$$v + u_1 + u_2 = v + u_2 + u_1 = 0 + u_1 = u_1.$$

Ergo, $u_1 = u_2$ and the inverse of v is unique. $\square$

Proposition 1.1.4. *Let V be a vector space over a field $\mathbb{F}$, then:*

1. $0x = 0$ *for all* $x \in V$;
2. $a0 = 0$ *for all* $a \in \mathbb{F}$;
3. $(-a)x = -(ax) = a(-x)$ *for all* $x \in V$ *and* $a \in \mathbb{F}$.

Proof (1): Consider $0x + 0x$. Then,

$$0x + 0x = (0 + 0)x = 0x = 0 + 0x.$$

Since $0x + 0x = 0 + 0x$, by cancellation, we have $0x = 0$. $\square$

Proof (2). Consider $a0 + a0$.

$$a0 + a0 = a(0 + 0) = a0 = a0 + 0$$

Thus, by cancellation, $a0 = 0$. $\square$

Proof (3). Consider $-(ax)$. We know that $-(ax)$ is the unique additive inverse of ax, thus it is enough to show that $(-a)x$ and $a(-x)$ are inverses of ax.

$$\begin{aligned}
ax + (-a)x &= (a - a)x \\
&= 0x \\
&= 0 \\
ax + a(-x) &= a(x - x) \\
&= a0 \\
&= 0
\end{aligned}$$

Thus $a(-x)$ and $(-a)x$ are inverses of ax and $(-a)x = -(ax) = a(-x)$. $\square$

1.1.2 Subspaces

Definition 1.1.5 (Subspace)**.** A *subspace*, W, of a vector space, V, over a field, $\mathbb{F}$, is a subset of V that is also a vector space over $\mathbb{F}$.

Example 1.1.5. *For any vector space V, V and $\{0\}$ are subspaces of V. The latter is called the zero subspace.*

Theorem 1.1.5. *Let V be a vector space over a field $\mathbb{F}$ and let $W \subseteq V$. Then W is a subspace of V if and only if all of the following are satisfied.*

- $0 \in W$.
- *For all $x, y \in W$, $x + y \in W$.*
- *For all $a \in \mathbb{F}$ and $x \in W$, $ax \in W$.*

Proof. $\Rightarrow$ Since W is a subspace of V, $0 \in W$ and W is closed under V's vector addition and scalar multiplication.

$\Leftarrow$ Since V is a vector space W inherits associativity, commutativity, and distributivity from V as well as V's behavior with respect to the identities. Furthermore, $0 \in W$ by assumption. All that is left to show is that W contains additive inverses. Suppose $x \in W$, then by assumption $-x = (-1)x \in W$. Thus W is a subspace of V. $\square$

Definition 1.1.6 (Matrix Transpose)**.** Let M be an $m \times n$ matrix, then the *transpose of M*, denoted M^T, is the $n \times m$ matrix defined by $(M^T)_{i,j} = M_{j,i}$, that is

$$M^T = \begin{pmatrix} M_{1,1} & M_{2,1} & \dots & M_{m,1} \\ M_{1,2} & M_{2,2} & \dots & M_{m,2} \\ \vdots & \vdots & \ddots & \vdots \\ M_{1,n} & M_{2,n} & \dots & M_{m,n} \end{pmatrix}.$$

Definition 1.1.7 (Symmetric Matrix)**.** Let M be a matrix, then if $M = M^T$, we say M is *symmetric.*

Example 1.1.6. *The set of symmetric $n \times n$ matrices over a field $\mathbb{F}$, denoted $W_{n\times n}(\mathbb{F})$, is a subspace of $M_{n\times n}(\mathbb{F})$.*

Proof. Consider the zero matrix. Since the zero matrix is an $n \times n$ matrix with all entries equal to zero, the transpose of the zero matrix is also an $n \times n$ matrix with all entries equal to zero. Thus, $0 = 0^T$ and the zero matrix is symmetric.

Let $A, B \in W_{n\times n}(\mathbb{F})$. Then $A = A^T$ and $B = B^T$. By definition of symmetry and matrix transpose we have

$$A_{i,j} = (A^T)_{i,j} = A_{j,i} \tag{1.1}$$

and

$$B_{i,j} = (B^T)_{i,j} = B_{j,i} \tag{1.2}$$

for all $1 \le i, j \le n$.

Consider $(A + B)$. By definition we have

$$(A + B)_{i,j} = A_{i,j} + B_{i,j}.$$

By Equation 1.1 and Equation 1.2 we have

$$(A + B)_{i,j} = A_{i,j} + B_{i,j} = A_{j,i} + B_{j,i}.$$

The definition of matrix transpose implies that

$$(A+B)_{i,j} = A_{i,j} + B_{i,j} = A_{j,i} + B_{j,i} = (A^T)_{i,j} + (B^T)_{i,j}$$

and thus by definition of matrix addition,

$$(A+B)_{i,j} = A_{i,j} + B_{i,j} = A_{j,i} + B_{j,i} = (A^T)_{i,j} + (B^T)_{i,j} = (A^T + B^T)_{i,j}.$$

Ergo, $A+B$ is symmetric and $W(\mathbb{F})$ is closed under matrix addition.

Let $k \in \mathbb{F}$ and consider kA. We know by definition that

$$(kA)_{i,j} = k \cdot A_{i,j}.$$

We invoke Equation 1.1 again to get that

$$(kA)_{i,j} = k \cdot A_{i,j} = k \cdot A_{j,i}.$$

Applying the definition of matrix transpose yields,

$$(kA)_{i,j} = k \cdot A_{i,j} = k \cdot A_{j,i} = k \cdot (A^T)_{i,j}.$$

Lastly, by definition of scalar multiplication we have,

$$(kA)_{i,j} = k \cdot A_{i,j} = k \cdot A_{j,i} = k \cdot (A^T)_{i,j} = (kA^T)_{i,j}.$$

Thus kA is symmetric and $W(\mathbb{F})$ is closed under scalar multiplication. □

Definition 1.1.8 (Main Diagonal of a Matrix)**.** Let $\mathbb{F}$ be a field and let $M \in M_{n\times n}(\mathbb{F})$, then the *main diagonal of* M is the set $\{M_{i,i}\}_{i=1}^{n}$.

Definition 1.1.9 (Diagonal Matrix)**.** Let $\mathbb{F}$ be a field and let $A \in M_{n\times n}(\mathbb{F})$, then A is called a *diagonal matrix* if and only if whenever $i \neq j$, $A_{i,j} = 0$.

Example 1.1.7. *Let $\mathbb{F}$ be a field and let $D_n(\mathbb{F})$ be the set of all diagonal matrices in $M_{n\times n}(\mathbb{F})$, then $D_n(\mathbb{F})$ is a subspace on $M_{n\times n}(\mathbb{F})$.*

Proof. We know $0 \in D_n(\mathbb{F})$ since for all i, j, $0_{i,j} = 0$. Let $A, B \in D_n(\mathbb{F})$. Then for all $i \neq j$, $A_{i,j} = B_{i,j} = 0$. Thus, $(A+B)_{i,j} = A_{i,j} + B_{i,j} = 0 + 0 = 0$ whenever $i \neq j$ and $A+B$ is diagonal. Let $k \in \mathbb{F}$. Then $(kA)_{i,j} = k \cdot A_{i,j} = k \cdot 0 = 0$ and kA is diagonal. Therefore $D_n(\mathbb{F})$ forms a subspace of $M_{n\times n}(\mathbb{F})$. □

Definition 1.1.10 (Trace of a Matrix)**.** Let $\mathbb{K}$ be a field and let $M \in M_{n\times n}(\mathbb{K})$, then the *trace of* M denoted $\operatorname{tr} M$ is defined as

$$\operatorname{tr} M = \sum_{i=1}^{n} M_{i,i}$$

or the sum of the elements on the main diagonal.

Example 1.1.8. *Let $\mathbb{K}$ be a field and let $T_n(\mathbb{K})$ be the set of matrices in $M_{n\times n}(\mathbb{K})$ with trace equal to zero, then $T_n(\mathbb{K})$ is a subspace of $M_{n\times n}(\mathbb{K})$.*

Proof. Obviously, the zero matrix has a trace of zero and thus $0 \in T_n(\mathbb{K})$. Let $A, B \in T_n(\mathbb{K})$ then $\operatorname{tr} A = 0$ and $\operatorname{tr} B = 0$. Consider $\operatorname{tr}(A+B)$.

$$\operatorname{tr}(A+B) = \sum_{i=1}^{n} (A+B)_{i,i} = \sum_{i=1^n} (A_{i,i} + B_{i,i}) = \left(\sum_{i=1}^{n} A_{i,i}\right) + \left(\sum_{i=1}^{n} B_{i,i}\right) = \operatorname{tr} A + \operatorname{tr} B = 0$$

Thus, $A+B$ has trace 0 and $A+B \in T_n(\mathbb{K})$. Let $k \in \mathbb{K}$. Consider $\operatorname{tr}(kA)$.

$$\operatorname{tr}(kA) = \sum_{i=1}^{n} (kA)_{i,i} = \sum_{i=1}^{n} k \cdot A_{i,i} = k \sum_{i=1}^{n} A_{i,i} = k \operatorname{tr} A = 0$$

And thus, kA has trace 0 and $kA \in T_n(\mathbb{K})$. Therefore $T_n(\mathbb{K})$ is a subspace of $M_{n\times n}(\mathbb{K})$. □

Theorem 1.1.6. *Let V be a vector space over a field $\mathbb{F}$ and let $\mathcal{W}$ be a countable collection of subspaces of V. Then*

$$W_i = \bigcap_{W \in \mathcal{W}} W$$

is a subspace of V.

Proof. Since $0 \in W$ for all $W \in \mathcal{W}$, $0 \in W_i$. Let $x, y \in W_i$, then $x, y \in W$ for all $W \in \mathcal{W}$ and thus $x + y \in W$ for all $W \in \mathcal{W}$. Therefore, $x + y \in W_i$. Let $a \in \mathbb{F}$. Since $x \in W$ for all $W \in \mathcal{W}$, $ax \in W$ for all $W \in \mathcal{W}$. Ergo, $ax \in W_i$ and W_i is a subspace of V. □

Proposition 1.1.7. *For any matrix A, $[(A^T)^T] = A$.*

Proof. Apply the definition of matrix transposition twice.

$$[(A^T)^T]_{i,j} = (A^T)_{j,i} = A_{i,j}$$

□

Proposition 1.1.8. *For any matrix A, $A + A^T$ is symmetric.*

Proof. Consider $(A + A^T)_{i,j}$.

$$(A + A^T)_{i,j} = A_{i,j} + (A^T)_{i,j} = A_{i,j} + A_{j,i} = A_{j,i} + A_{i,j} = A_{j,i} + (A^T)_{j,i} = (A + A^T)_{j,i} = [(A + A^T)^T]_{i,j}$$

And thus, $(A + A^T)$ is symmetric. □

Proposition 1.1.9. *Let $\mathbb{K}$ be a field and let $A, B \in M_{n \times n}(\mathbb{K})$ and $a, b \in \mathbb{K}$, then $\operatorname{tr}(aA + bB) = a \operatorname{tr} A + b \operatorname{tr} B$.*

Proof.

$$\begin{aligned}
\operatorname{tr}(aA + bB) &= \sum_{i=1}^{n} (aA + bB)_{i,i} \\
&= \sum_{i=1}^{n} [(aA)_{i,i} + (bB)_{i,i}] \\
&= \left(\sum_{i=1}^{n} a \cdot A_{i,i} \right) + \left(\sum_{i=1}^{n} b \cdot B_{i,i} \right) \\
&= a \left(\sum_{i=1}^{n} A_{i,i} \right) + b \left(\sum_{i=1}^{n} B_{i,i} \right) \\
&= a \operatorname{tr} A + b \operatorname{tr} B
\end{aligned}$$

□

Definition 1.1.11 (Sum of Subsets of a Vector Space)**.** Let S, R be nonempty subsets of a vector space V, then the *sum of S and R*, denoted $S + R$ is defined as $S + R = \{s + r | s \in S, r \in R\}$.

Proposition 1.1.10. *Let U, W be subspaces of a vector space V over a field $\mathbb{F}$. Then $U + W$ is a subspace of V and is the smallest subspace containing both U and W.*

Proof. Since U and W are subspaces, $0 \in U$ and $0 \in W$ therefore, $0 = 0 + 0 \in U + W$. Let $x, y \in U + W$ then there exist $u_x, u_y \in U$ and $w_x, w_y \in W$ such that $x = u_x + w_x$ and $y = u_y + w_y$. Thus,

$$x + y = (u_x + w_x) + (u_y + w_y) = (u_x + u_y) + (w_x + w_y).$$

Since U and W are subspaces, $u_x + u_y \in U$ and $w_x + w_y \in W$. Ergo, $x + y = (u_x + u_y) + (w_x + w_y) \in U + W$.

Let $a \in \mathbb{F}$. Then,

$$ax = a(u_x + w_x) = au_x + aw_x.$$

Since U and W are subspaces, $au_x \in U$ and $aw_x \in W$. Ergo, $ax = au_x + aw_x \in U + W$ and $U + W$ is a subspace of V.

We know that, set-wise, $U = \{u + 0\}_{u \in U}$ and $W = \{0 + w\}_{w \in W}$, and thus $U, W \subseteq U + W$. Let X be a subspace of V such that $U, W \subseteq X$. Let $x \in U + W$, then there exists some $u \in U \subseteq X$ and $w \in W \subseteq X$ such that $x = u + w$. Therefore, $x = u + w \in X$ and $U + W \subseteq X$ for all subspaces X containing U and W. Ergo, $U + W$ is the smallest subspace of V containing U and W. □

Definition 1.1.12 (Direct Sum of Vector Spaces)**.** A vector space V is called the *direct sum of* U *and* W, denoted $V = U \oplus W$ if and only if U and W are subspaces of V such that $U \cap W = \emptyset$ and $U + W = V$.

Example 1.1.9. *Let* $\mathbb{K}$ *be a field and let* $U = \{(a_1, a_2, \dots, a_n) \in \mathbb{K}^n | a_n = 0\}$ *and* $V = \{(a_1, a_2 \dots, a_n) \in \mathbb{K}^n | a_1 = a_2 = \dots = a_{n-1} = 0\}$*. Then* $\mathbb{K}^n = U \oplus V$*.*

Proof. The details are obvious and left as an exercise. □

Definition 1.1.13 (Cosets of a Vector Space)**.** Let U be a subspace of a vector space V over a field $\mathbb{K}$. Then for each $v \in V$ the set $\{v\} + W = \{v + w\}_{w \in W}$ is called the *coset of* W *containing* v, denoted $v + W$.

Proposition 1.1.11. *Let* $\mathbb{K}$ *be a field and* W *be a subspace of a vector space* V *over* $\mathbb{F}$ *and let* $v \in V$*. Then* $v + W$ *is a subspace if and only if* $v \in W$*.*

Proof. $\Leftarrow$ Suppose $v \in W$. Then by closure, $v + W = \{v + w\}_{w \in W} = W$.

$\Rightarrow$ Suppose $v + W$ is a subspace of V. Then $0 \in v + W$ and therefore, there exists a $w \in W$ such that $0 = v + w$. This w can only be $-v$ by uniqueness of inverses. Since $-v \in W$, $v \in W$ since W is a subspace. □

Proposition 1.1.12. *Let* $\mathbb{K}$ *be a field and* W *be a subspace of a vector space* V *over* $\mathbb{F}$ *and let* $v \in V$*. Then* $v \in v + W$*.*

Proof. Since W is a subspace, $0 \in W$ and thus $v = v + 0 \in v + W$. □

Proposition 1.1.13. *Let* $\mathbb{K}$ *be a field and* W *be a subspace of a vector space* V *over* $\mathbb{F}$ *and let* $u, v \in V$*. Then* $v + W \cap u + W = \emptyset$ *if and only if* $v + W \neq u + W$*.*

Proof. $\Rightarrow$ Suppose $v + W \cap u + W = \emptyset$. Then since both $v + W$ and $u + W$ are non-empty, $v + W \neq u + W$.

$\Leftarrow$ Suppose $v + W \neq u + W$ with $v + W \cap u + W \neq \emptyset$. Then there exists an $x \in v + W \cap u + W$. Ergo, $x \in v + W$ and $x \in u + W$. Thus, there exists $w_1, w_2 \in W$ such that $x = v + w_1$ and $x = u + w_2$ respectively. Therefore, $v + w_1 = u + w_2$ and $v = u + w_2 - w_1$. Ergo,

$$v + W = \{v + w\}_{w \in W} = \{u + (w_2 - w_1 + w)\}_{w \in W} = u + W.$$

Thus, creating a contradiction. Therefore if $v + W \neq u + W$ then $v + W \cap u + W = \emptyset$. □

Proposition 1.1.14. *Let* $\mathbb{K}$ *be a field and* W *be a subspace of a vector space* V *over* $\mathbb{F}$ *and let* $u, v \in V$*. Then* $v + W = u + W$ *if and only if* $v - u \in W$*.*

Proof. $\Rightarrow$ Assume $v + W = u + W$. Then $v \in v + W$ and thus $v \in u + W$. Ergo, there exists a $w \in W$ such that $v = u + w$. Solving for w yields $w = v - u \in W$.

$\Leftarrow$ Assume $v - u \in W$. Therefore, $u + v - u = v \in u + W$. We know $v \in v + W$ thus, $u + W \cap v + W \neq \emptyset$. This occurs if and only if $u + W = v + W$. □

Definition 1.1.14 (Quotient Space)**.** Let $\mathbb{K}$ be a field and W be a subspace of a vector space V over $\mathbb{F}$. The the *quotient space of* V *modulo* W, denoted V/W is the set of all cosets of W,

$$V/W := \{v + W\}_{v \in V}.$$

Furthermore V/W is a vector space under the following operations.

$$\begin{aligned} (u + W) + (v + W) &= (u + v) + W \\ a(u + W) &= (au) + W \end{aligned}$$

Proof. A bunch of tedious symbol pushing that I refuse to do. $\square$

1.1.3 Linear Combinations

Definition 1.1.15 (Linear Combination)**.** Let V be a vector space over a field $\mathbb{F}$ and let S be a nonempty subset of V. An $x \in V$ is said to be a *linear combination of elements of* S if and only if there exists a $\{s_j\}_{j=1}^n \subseteq S$ and scalars $\{a_j\}_{j=1}^n \subseteq \mathbb{F}$ where $n < \infty$ such that

$$x = \sum_{j=1}^{n} a_j y_j.$$

When this happens, we say x is a *linear combination of* $y_1, y_2, \ldots, y_n$.

Definition 1.1.16 (Spanning Set)**.** Let V be a vector space over a field $\mathbb{F}$ and let S be a nonempty subset of V. Then, the *span of* S, denoted $\operatorname{span} S$, is the set

$$\operatorname{span} S = \{\sum_{j=1}^{n} a_j s_j | \{a_j\}_{j=1}^n \subseteq \mathbb{F}, \{s_j\}_{j=1}^n \subseteq S, n < \infty\}$$

or the set of linear combinations of elements of S. We define $\operatorname{span} \emptyset = \{0\}$.

Theorem 1.1.15. *Let V be a vector space over a field $\mathbb{F}$ and let S be a nonempty subset of V. Then* $\operatorname{span} S$ *is a subspace of V and is the smallest subspace of V containing S.*

Proof. Let $\{s_j\}_{j=1}^n \subseteq S$ where $n < \infty$. Then $0 = \sum_{j=1}^n 0 s_j \in \operatorname{span} S$. Let $x, y \in \operatorname{span} S$. Then there exist $\{s_j\}_{j=1}^n, \{r_j\}_{j=1}^m \subseteq S$ and $\{a_j\}_{j=1}^n, \{b_j\}_{j=1}^m \subseteq \mathbb{F}$ with $m, n < \infty$ such that $x = \sum_{j=1}^n a_j s_j$ and $y = \sum_{j=1}^m b_j r_j$. Define $\{t_j\}_{j=1}^{n+m}$ and $\{c_j\}_{j=1}^{n+m}$ by

$$t_j = \begin{cases} s_j & j \leq n \\ r_j & j > n \end{cases} \qquad\qquad c_j = \begin{cases} a_j & j \leq n \\ b_j & j > n \end{cases}.$$

We can see that $\{t_j\}$ is a finite subset of S and $\{c_j\}$ is a finite subset of $\mathbb{F}$, thus any element made out of scalar multiples of t vectors is in $\operatorname{span} S$. Consider $x + y$.

$$x + y = \sum_{j=1}^{n} a_j s_j + \sum_{j=1}^{m} b_j r_j = \sum_{j=1}^{n} c_j t_j + \sum_{j=n+1}^{n+m} b_{j-n} r_{j-n} = \sum_{j=1}^{n} c_j t_j + \sum_{j=n+1}^{n+m} c_j t_j$$

Therefore, $x+y$ is a linear combination of elements of S and thus $x+y \in \operatorname{span} S$. Consider kx for any $k \in \mathbb{F}$.

$$kx = k \sum_{j=1}^{n} a_j s_j = \sum_{j=1}^{n} (k a_j) s_j$$

Ergo, $kx \in \operatorname{span} S$ and $\operatorname{span} S$ is a subspace of V.

Let W be a subspace of V such that $S \subseteq W$. Then for all $s \in S$ and $a \in \mathbb{F}$, $as \in W$ since W is a subspace. Ergo, for any $\{s_j\}_{j=1}^n \subseteq S$ and $\{a_j\}_{j=1}^n \subseteq \mathbb{F}$ with $n < \infty$

$$\sum_{j=1}^{n} a_j s_j \in W$$

since W is a subspace and any finite sum of vectors in W is in W. Ergo $\operatorname{span} S \subseteq W$ and is the smallest subspace of V containing S. ☐

Definition 1.1.17 (Span)**.** A subset S of a vector space V *spans* V if and only if $\operatorname{span} S = V$.

Proposition 1.1.16. *Let W be a nonempty subset of a vector space V over a field $\mathbb{F}$. Then W is a subspace of V if and only if $W = \operatorname{span} W$.*

Proof. $\Rightarrow$ Suppose W is a subspace. We know $W \subseteq \operatorname{span} W$, by definition. Furthermore, for all $w \in W$ and $a \in \mathbb{F}$, $aw \in W$ since W is a subspace. Thus, for any finite $\{w_j\}_{j=1}^n \subseteq W$ and $\{a_j\}_{j=1}^n \subseteq \mathbb{F}$, $\sum a_j w_j \in W$ by properties of vector spaces. Ergo, $\operatorname{span} W \subseteq W$ and $W = \operatorname{span} W$.

$\Leftarrow$ Suppose $W = \operatorname{span} W$. Since $\operatorname{span} W$ is a subspace and $W = \operatorname{span} W$, W is trivially a subspace. $\square$

Proposition 1.1.17. *Let S, R be nonempty subsets of V such that $S \subseteq R$. Then* $\operatorname{span} S \subseteq \operatorname{span} R$ *and if* $\operatorname{span} S = V$, *then* $\operatorname{span} R = V$.

Proof. I provide a sketch and leave the details to the reader.

All vectors in S are also in R ergo all sums of scalar multiples of vectors in S (read: $\operatorname{span} S$) are in $\operatorname{span} R$.

Furthermore, we know $\operatorname{span} R$ is a subspace of V, and thus $\operatorname{span} R \subseteq V$. If $V = \operatorname{span} S \subseteq \operatorname{span} R$, then $V \subseteq \operatorname{span} R$ and thus $\operatorname{span} R = V$. $\square$

1.1.4 Linear Independence

Definition 1.1.18 (Linear Independence)**.** A subset S of a vector space V over a field $\mathbb{F}$ is *linearly independent* if and only if for any $\{x_j\}_{j=1}^{n} \subseteq V$ where $n < \infty$ the statement

$$\sum_{j=1}^{n} a_j x_j = 0$$

implies that $\{a_j\} = \{0\}$, where $\{a_j\} \subseteq \mathbb{F}$. Furthermore, if S is not linearly independent, we say that S is *linearly dependent.*

Example 1.1.10. *Let $\mathbb{F}$ be a field and $n \in \mathbb{N}$. Define, $e_j := (a_1, a_2, \ldots, a_n) \in \mathbb{F}^n$ where $a_j = 1$ and $a_i = 0$ for all $i \neq j$. Then $\{e_1, e_2, \ldots, e_n\}$ is linearly independent.*

Proof. Let $\{c_j\}_{j=1}^{n} \subseteq \mathbb{F}$ such that $\sum c_j e_j = 0$. Consider $c_j e_j$. On the jth entry of this vector, we will have c_j and all other entries are zero. Therefore,

$$\sum c_j e_j = (c_1, c_2, \ldots, c_n) = (0, 0, \ldots, 0).$$

By our definition of vector equality in $\mathbb{F}^n$ we have $c_j = 0$ for all $1 \leq j \leq n$. Thus, $\{e_j\}_{j=1}^{n}$ is linearly independent. $\square$

Theorem 1.1.18. *A subset S of a vector space V over a field $\mathbb{F}$ is linearly dependent if and only if $x_1 = 0$ or there exists a $k < n$ such that $x_{k+1} \in \operatorname{span}\{x_1, x_2, \ldots, x_k\}$.*

Proof. $\Rightarrow$ Assume S is linearly dependent. Therefore, there exists a $\{a_j\}_{j=1}^{n} \subseteq \mathbb{F}$ with $\{a_j\}_{j=1}^{n} \neq \{0\}$ such that $\sum a_j x_j = 0$. Define $k = \max\{j | a_j \neq 0\}$. If $1 < k \leq n$, then

$$\sum_{j=1}^{n} a_j x_j = \sum_{j=1}^{k} a_j x_j = 0$$

and

$$x_k = \sum_{j=1}^{n} (-a_j a_k^{-1}) \in \operatorname{span}\{x_1, x_2, \ldots, x_{k-1}\}.$$

If $k = 1$, then

$$\sum_{j=1}^{n} a_j x_j = a_1 x_1 = 0$$

with $a_1 \neq 0$. Ergo, $x_1 = 0$.

$\Leftarrow$ Assume $x_1 = 0$. Then $a x_1 = 0$ for all $a \in \mathbb{F}$ and S is linearly dependent. Assume there exists a $k < n$ such that $x_{k+1} \in \operatorname{span}\{x_1, x_2, \ldots, x_k\}$. Then there exists $\{a_j\} \subseteq \mathbb{F}$ such that $x_{k+1} = \sum_{j=1}^{k} a_j x_j$. We know $\sum_{j=1}^{k} a_j x_j - x_{k+1}$ is a linear combination of vectors in $\mathbb{F}$ and

$$\sum_{j=1}^{k} a_j x_j - x_{k+1} = 0$$

thus, S is linearly dependent. $\square$

1.1.5 Bases and Dimension

Definition 1.1.19 (Basis of a Vector Space)**.** A *basis* B for a vector space V over a field $\mathbb{F}$ is a a linearly independent subset of V that spans V.

Chapter 2

Definitions

2.1 Vector Spaces

2.1.1 Introduction to Vector Spaces

Definition 2.1.1 (Vector Space)**.** A *vector space* V over a field $\mathbb{F}$ is a set with two binary operations, $+ : V \times V \to V$ and $\cdot : V \times \mathbb{F} \to V$ such that all of the following hold.

1. For all $x, y \in V$, $x + y = y + x$. (Additive Commutativity)
2. For all $x, y, z \in V$, $x + (y + z) = (x + y) + z$. (Additive Associativity)
3. There exists an element, denoted 0, in V such that for all $x \in V$, $x + 0 = x$.
4. For each $x \in V$ there exists a $y \in V$, denoted $-x$, such that $x + y = 0$.
5. For all $x \in V$, $1x = x$.
6. For all $a, b \in \mathbb{F}$ and $x \in V$, $a(bx) = (ab)x$.
7. For all $a \in \mathbb{F}$ and $x, y \in V$, $a(x + y) = ax + ay$.
8. For all $a, b \in \mathbb{F}$ and $x \in V$, $(a + b)x = ax + bx$.

Furthermore, $x + y$ is called the *sum of* x *and* y while ax is called the *product of* x *and* a. Moreover, each $x \in V$ is called a *vector* and each $a \in \mathbb{F}$ is called a *scalar*.

Definition 2.1.2 (n-tuple)**.** An object of the form $(a_1, a_2, \ldots, a_n)$ where $a_j \in \mathbb{F}$ for all $1 \leq j \leq n$, is called an *n-tuple*.

Definition 2.1.3. Let $\mathbb{F}$ be a field and $m, n \in \mathbb{N}$, then an $m \times n$ *matrix* with entries from $\mathbb{F}$ is a rectangular array of the form

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,n} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \ldots & a_{m,n} \end{pmatrix}$$

where $a_{i,j} \in \mathbb{F}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. The entries $(a_{i,1}, a_{i,2}, \ldots, a_{i,n})$ is called the *ith row* of the matrix and is a row vector in $\mathbb{F}^n$. The entries $(a_{1,j}, a_{2,j}, \ldots, a_{m,j})$ is called the *jth column* of the matrix and is a column vector in $\mathbb{F}^n$. We denote the entry on the ith row and jth column as $A_{i,j}$. Furthermore, two $m \times n$ matrices, A and B, are equal if and only if $A_{i,j} = B_{i,j}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$; we denote this by $A = B$. Moreover, if $n = m$ we say that A is a *square matrix*. Lastly, we denote the set of $m \times n$ matrices over $\mathbb{F}$ as $M_{m \times n}(\mathbb{F})$.

Definition 2.1.4 (Polynomial Ring)**.** Let $\mathbb{F}$ be a field. Then the *ring of polynomials in an indeterminate* x *over* $\mathbb{F}$, denoted $\mathbb{F}[x]$ is defined as

$$\mathbb{F}[x] := \{\sum_{i=0}^{n} a_i x^i | n \in \mathbb{N}, (a_0, a_1, \ldots, a_n) \in \mathbb{F}^n, a_n \neq 0\}.$$

Additionally, we define $x^0 = 1$. Moreover, for each $p = \sum_{i=0}^{n} p_i x^i \in \mathbb{F}[x]$, the *degree of* p, denoted $\deg p$, is n. Furthermore, if $p = 0$, that is $p_n = p_{n-1} = \ldots = p_0 = 0$, then p is called the *zero polynomial* and $\deg p = -1$ or $\deg p = -\infty$ depending on convention. If $\deg p = 0$, then we say p is a *constant polynomial.*

Lastly, $\mathbb{F}[x]$ forms a ring under the following operations where $p, q \in \mathbb{F}[x]$ and without loss of generality assume, $\deg p \geq \deg q$.

$$p + q = \sum_{i=0}^{\deg q} (p_i + q_i) x^i + \sum_{i=\deg q+1}^{\deg p} p_i x^i$$

$$pq = \sum_{k=0}^{\deg p \cdot \deg q} \left(\sum_{i+j=k} p_i q_j \right) x^k$$

2.1.2 Subspaces

Definition 2.1.5 (Subspace)**.** A *subspace*, W, of a vector space, V, over a field, $\mathbb{F}$, is a subset of V that is also a vector space over $\mathbb{F}$.

Definition 2.1.6 (Matrix Transpose)**.** Let M be an $m \times n$ matrix, then the *transpose of* M, denoted M^T, is the $n \times m$ matrix defined by $(M^T)_{i,j} = M_{j,i}$, that is

$$M^T = \begin{pmatrix} M_{1,1} & M_{2,1} & \dots & M_{m,1} \\ M_{1,2} & M_{2,2} & \dots & M_{m,2} \\ \vdots & \vdots & \ddots & \vdots \\ M_{1,n} & M_{2,n} & \dots & M_{m,n} \end{pmatrix}.$$

Definition 2.1.7 (Symmetric Matrix)**.** Let M be a matrix, then if $M = M^T$, we say M is *symmetric*.

Definition 2.1.8 (Main Diagonal of a Matrix)**.** Let $\mathbb{F}$ be a field and let $M \in M_{n\times n}(\mathbb{F})$, then the *main diagonal of* M is the set $\{M_{i,i}\}_{i=1}^n$.

Definition 2.1.9 (Diagonal Matrix)**.** Let $\mathbb{F}$ be a field and let $A \in M_{n\times n}(\mathbb{F})$, then A is called a *diagonal matrix* if and only if whenever $i \neq j$, $A_{i,j} = 0$.

Definition 2.1.10 (Trace of a Matrix)**.** Let $\mathbb{K}$ be a field and let $M \in M_{n\times n}(\mathbb{K})$, then the *trace of* M denoted $\operatorname{tr} M$ is defined as

$$\operatorname{tr} M = \sum_{i=1}^{n} M_{i,i}$$

or the sum of the elements on the main diagonal.

Definition 2.1.11 (Sum of Subsets of a Vector Space)**.** Let S, R be nonempty subsets of a vector space V, then the *sum of* S *and* R, denoted $S + R$ is defined as $S + R = \{s + r | s \in S, r \in R\}$.

Definition 2.1.12 (Direct Sum of Vector Spaces)**.** A vector space V is called the *direct sum of* U *and* W, denoted $V = U \oplus W$ if and only if U and W are subspaces of V such that $U \cap W = \emptyset$ and $U + W = V$.

Definition 2.1.13 (Cosets of a Vector Space)**.** Let U be a subspace of a vector space V over a field $\mathbb{K}$. Then for each $v \in V$ the set $\{v\} + W = \{v + w\}_{w\in W}$ is called the *coset of* W *containing* v, denoted $v + W$.

Definition 2.1.14 (Quotient Space)**.** Let $\mathbb{K}$ be a field and W be a subspace of a vector space V over $\mathbb{F}$. The the *quotient space of* V *modulo* W, denoted V/W is the set of all cosets of W,

$$V/W := \{v + W\}_{v\in V}.$$

Furthermore V/W is a vector space under the following operations.

$$\begin{aligned} (u + W) + (v + W) &= (u + v) + W \\ a(u + W) &= (au) + W \end{aligned}$$

2.1.3 Linear Combinations

Definition 2.1.15 (Linear Combination)**.** Let V be a vector space over a field $\mathbb{F}$ and let S be a nonempty subset of V. An $x \in V$ is said to be a *linear combination of elements of* S if and only if there exists a $\{s_j\}_{j=1}^n \subseteq S$ and scalars $\{a_j\}_{j=1}^n \subseteq \mathbb{F}$ where $n < \infty$ such that

$$x = \sum_{j=1}^{n} a_j y_j.$$

When this happens, we say x is a *linear combination of* $y_1, y_2, \ldots, y_n$.

Definition 2.1.16 (Spanning Set)**.** Let V be a vector space over a field $\mathbb{F}$ and let S be a nonempty subset of V. Then, the *span of* S, denoted $\operatorname{span} S$, is the set

$$\operatorname{span} S = \{\sum_{j=1}^{n} a_j s_j | \{a_j\}_{j=1}^n \subseteq \mathbb{F}, \{s_j\}_{j=1}^n \subseteq S, n < \infty\}$$

or the set of linear combinations of elements of S. We define $\operatorname{span} \emptyset = \{0\}$.

Definition 2.1.17 (Span)**.** A subset S of a vector space V *spans* V if and only if $\operatorname{span} S = V$.

2.1.4 Linear Independence

Definition 2.1.18 (Linear Independence)**.** A subset S of a vector space V over a field $\mathbb{F}$ is *linearly independent* if and only if for any $\{x_j\}_{j=1}^{n} \subseteq V$ where $n < \infty$ the statement

$$\sum_{j=1}^{n} a_j x_j = 0$$

implies that $\{a_j\} = \{0\}$, where $\{a_j\} \subseteq \mathbb{F}$. Furthermore, if S is not linearly independent, we say that S is *linearly dependent*.

Chapter 3

Theorems

3.1 Vector Spaces

3.1.1 Introduction to Vector Spaces

Proposition 3.1.1. *If u, v, w are elements of a vector space V such that $x + z = y + z$ then, $x = y$.*

Proposition 3.1.2. *The zero vector in any vector space V is unique.*

Proposition 3.1.3. *Let V be a vector space and let $v \in V$, then there exists a unique $u \in V$ such that $v + u = 0$.*

Proposition 3.1.4. *Let V be a vector space over a field $\mathbb{F}$, then:*

1. $0x = 0$ *for all* $x \in V$*;*
2. $a0 = 0$ *for all* $a \in \mathbb{F}$*;*
3. $(-a)x = -(ax) = a(-x)$ *for all* $x \in V$ *and* $a \in \mathbb{F}$*.*

3.1.2 Subspaces

Theorem 3.1.5. *Let V be a vector space over a field $\mathbb{F}$ and let $W \subseteq V$. Then W is a subspace of V if and only if all of the following are satisfied.*

- $0 \in W$.
- *For all $x, y \in W$, $x + y \in W$.*
- *For all $a \in \mathbb{F}$ and $x \in W$, $ax \in W$.*

Theorem 3.1.6. *Let V be a vector space over a field $\mathbb{F}$ and let $\mathcal{W}$ be a countable collection of subspaces of V. Then*

$$W_i = \bigcap_{W \in \mathcal{W}} W$$

is a subspace of V.

Proposition 3.1.7. *For any matrix A, $[(A^T)^T] = A$.*

Proposition 3.1.8. *For any matrix A, $A + A^T$ is symmetric.*

Proposition 3.1.9. *Let $\mathbb{K}$ be a field and let $A, B \in M_{n\times n}(\mathbb{K})$ and $a, b \in \mathbb{K}$, then $\operatorname{tr}(aA+bB) = a \operatorname{tr} A + b \operatorname{tr} B$.*

Proposition 3.1.10. *Let U, W be subspaces of a vector space V over a field $\mathbb{F}$. Then $U + W$ is a subspace of V and is the smallest subspace containing both U and W.*

Proposition 3.1.11. *Let $\mathbb{K}$ be a field and W be a subspace of a vector space V over $\mathbb{F}$ and let $v \in V$. Then $v + W$ is a subspace if and only if $v \in W$.*

Proposition 3.1.12. *Let $\mathbb{K}$ be a field and W be a subspace of a vector space V over $\mathbb{F}$ and let $v \in V$. Then $v \in v + W$.*

Proposition 3.1.13. *Let $\mathbb{K}$ be a field and W be a subspace of a vector space V over $\mathbb{F}$ and let $u, v \in V$. Then $v + W \cap u + W = \emptyset$ if and only if $v + W \neq u + W$.*

Proposition 3.1.14. *Let $\mathbb{K}$ be a field and W be a subspace of a vector space V over $\mathbb{F}$ and let $u, v \in V$. Then $v + W = u + W$ if and only if $v - u \in W$.*

3.1.3 Linear Combinations

Theorem 3.1.15. *Let V be a vector space over a field $\mathbb{F}$ and let S be a nonempty subset of V. Then* $\operatorname{span} S$ *is a subspace of V and is the smallest subspace of V containing S.*

Proposition 3.1.16. *Let W be a nonempty subset of a vector space V over a field $\mathbb{F}$. Then W is a subspace of V if and only if $W = \operatorname{span} W$.*

Proposition 3.1.17. *Let S, R be nonempty subsets of V such that $S \subseteq R$. Then* $\operatorname{span} S \subseteq \operatorname{span} R$ *and if* $\operatorname{span} S = V$, *then* $\operatorname{span} R = V$.

Theorem 3.1.18. *A subset S of a vector space V over a field $\mathbb{F}$ is linearly dependent if and only if $x_1 = 0$ or there exists a $k < n$ such that $x_{k+1} \in \operatorname{span}\{x_1, x_2, \ldots, x_k\}$.*