

# Box Counting Dimension of the Middle- $\lambda$ Cantor Set

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**Theorem.** *Let  $C$  denote the middle- $\lambda$  Cantor set with  $0 < \lambda < 1$ . Then the box counting dimension of  $C$  is*

$$\dim_B C = \frac{\ln 2}{\ln \left( \frac{2}{1-\lambda} \right)}.$$

## 1 Background

### 1.1 Fractal Analysis

We begin by defining a  $\delta$ -cover of a set.

**Definition 1.** Let  $X \subset \mathbb{R}^n$  and  $\delta > 0$ . Then a  $\delta$ -cover of  $X$  is a set  $D = \{D_i\}_{i=1}^{n_D}$  such that  $X \subseteq \cup_{D_i \in D} D_i$  and  $\text{diam } D_i \leq \delta$  for all  $D_i \in D$ . We denote the collection of all such covers as  $\mathcal{D}_\delta(X)$ .

Now that we have  $\delta$ -covers, we can define the *box-counting dimension* of a set.

**Definition 2.** Let  $X$  be a subset of  $\mathbb{R}^n$ . Then the *box-counting dimension* of  $X$ , denoted  $\dim_B(X)$ , is defined as

$$\lim_{\delta \rightarrow 0} \frac{\ln N_\delta(X)}{-\ln \delta}$$

where

$$N_\delta(X) = \min_{D \in \mathcal{D}_\delta(X)} \text{card } D.$$

Note that the box-counting dimension of a set does not necessarily exist, however when it does exist it is usually easier to find.

**Theorem 1** (Squeeze Theorem). *Let  $(a_n), (b_n), (c_n)$  be real valued sequences such that  $\lim a_n = \lim c_n = x$  and  $a_n \leq b_n \leq c_n$  for all  $n$  greater than some  $n \in \mathbb{N}$ . Then  $b_n$  converges to  $x$ .*

### 1.2 The Cantor Set

## 2 Solution

**Theorem 2.** *Let  $C$  denote the middle- $\lambda$  Cantor set with  $0 < \lambda < 1$ . Then the box counting dimension of  $C$  is*

$$\dim_B C = \frac{\ln 2}{\ln \left( \frac{2}{1-\lambda} \right)}.$$

*Proof.* Consider removing an interval of length  $a\lambda$  from the middle of the interval  $[0, a]$  for some  $a > 0$ . Since we have removed  $a\lambda$  from the interval, the length of this new set is exactly  $a - a\lambda = a(1 - \lambda)$ . Moreover, since we removed it from the middle of the interval, this length is equally distributed among the two resultant

subintervals. Thus the subintervals must have length  $a((1 - \lambda)/2)$ . Furthermore, the leftmost interval must be  $[0, a((1 - \lambda)/2)]$ .

We now need to find the length of any subinterval of  $C_n$ . Since  $C_0 = [0, 1]$ ,  $l_0 = 1$ . This tells us that  $l_1 = (1 - \lambda)/2$  and that the leftmost interval in  $C_1$  is  $[0, (1 - \lambda)/2]$ . We know from the above logic that  $l_{n+1} = l_n(1 - \lambda)/2$  and its leftmost interval will be  $[0, l_n(1 - \lambda)/2]$ . Solving this recursion yields that

$$l_n = \left(\frac{1 - \lambda}{2}\right)^n$$

and the leftmost interval of  $C_n$  is  $[0, ((1 - \lambda)/2)^n]$ . Thus if we were to cover  $C_n$  we would need  $2^n$  sets of diameter  $((1 - \lambda)/2)^n$ .

Suppose  $\delta > 0$ . Thus, we can find an  $n$  such that  $l_{n+1} \leq \delta < l_n$ . Since  $\delta < l_n$ , we need no fewer than  $2^n$  sets of diameter  $\delta$  to cover  $C_n \supset C$ . Since  $\delta \geq l_{n+1}$ , we need no more than  $2^{n+1}$  sets of diameter  $\delta$  to cover  $C_{n+1} \supset C$ . Thus, we can glean the following inequality.

$$2^n \leq N_\delta(C) \leq 2^{n+1}.$$

Since  $\ln$  is a monotone increasing function, we get

$$n \ln 2 \leq \ln N_\delta(C) \leq (n + 1) \ln 2. \quad (1)$$

Furthermore, since  $l_{n+1} \leq \delta < l_n$ ,

$$\frac{1}{-\ln l_{n+1}} \leq \frac{1}{-\ln \delta} \leq \frac{1}{-\ln l_n}. \quad (2)$$

Take Equation 1 and multiply it by  $1/(-\ln \delta)$ .

$$\frac{n \ln 2}{-\ln \delta} \leq \frac{\ln N_\delta(C)}{-\ln \delta} \leq \frac{(n + 1) \ln 2}{-\ln \delta}$$

By Equation 2, we have

$$\frac{n \ln 2}{-\ln l_{n+1}} \leq \frac{\ln N_\delta(C)}{-\ln \delta} \leq \frac{(n + 1) \ln 2}{-\ln l_n}.$$

Consider  $\ln l_n$ .

$$\ln l_n = \ln \left( \left( \frac{1 - \lambda}{2} \right)^n \right) = -n \ln \left( \frac{2}{1 - \lambda} \right)$$

Thus, our inequality becomes,

$$\frac{n \ln 2}{(n + 1) \ln \left( \frac{2}{1 - \lambda} \right)} \leq \frac{\ln N_\delta(C)}{-\ln \delta} \leq \frac{(n + 1) \ln 2}{n \ln \left( \frac{2}{1 - \lambda} \right)}.$$

If we want to find  $\lim_{\delta \rightarrow 0} \ln N_\delta(C)/(-\ln \delta)$ , we need to find the limits of the far left and far right as  $n \rightarrow \infty$ .

On the lefthand side we get

$$\frac{n \ln 2}{(n + 1) \ln \left( \frac{2}{1 - \lambda} \right)} = \frac{n \ln 2}{n \ln \left( \frac{2}{1 - \lambda} \right) + \ln \left( \frac{2}{1 - \lambda} \right)} \rightarrow \frac{\ln 2}{\ln \left( \frac{2}{1 - \lambda} \right)}$$

by l'Hôpital's rule. On the righthand side we get

$$\frac{(n + 1) \ln 2}{n \ln \left( \frac{2}{1 - \lambda} \right)} = \frac{n \ln 2 + \ln 2}{n \ln \left( \frac{2}{1 - \lambda} \right)} \rightarrow \frac{\ln 2}{\ln \left( \frac{2}{1 - \lambda} \right)}$$

again by l'Hôpital's rule. Thus by squeeze theorem,  $\dim_B(C) = \frac{\ln 2}{\ln \left( \frac{2}{1 - \lambda} \right)}$ . □