Fibonacci Generating Function

Matt McCarthy

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Problem. *Let $(F_n)_{n\in\mathbb{N}}$ denote the sequence of Fibonacci numbers. Find the closed form of F_n.*

1 Background

Lets begin with the definition of the Fibonacci numbers.

Definition 1. The Fibonacci numbers $(F_n)_{n\in\mathbb{N}}$ are defined by $F_1 = F_2 = 1$, and $F_{n+2} = F_{n+1} + F_n$.

From the Fibonacci numbers we get the golden ratio, φ, and its conjugate, $\overline{\varphi}$ where

$$\varphi = \frac{1+\sqrt{5}}{2} \text{ and } \overline{\varphi} = \frac{1-\sqrt{5}}{2}.$$

While the Fibonacci recurrence is simple, its difficult to tell from the definition what the nth Fibonacci number is without computing all of the terms before it. Since the Fibonacci's are defined recursively, we can use *generating functions* to extract a closed form for F_n relatively easily.

Definition 2. Let $(a_n)_{n\in\mathbb{N}}$ be a $\mathbb{R}$-valued sequence. Then the generating function for a_n is the power series

$$G(x) = \sum_{n\in\mathbb{N}} a_n x^n.$$

After we create the generating function, we generally want to find its closed form (that is, a representation without using limits). Then, we use that new representation to find a closed form for the terms of the sequence (in this case, a non-recursive representation).

2 Solution

Problem. *Let $(F_n)_{n\in\mathbb{N}}$ denote the sequence of Fibonacci numbers. Find the closed form of F_n.*

We begin by setting up a generating function, $F(x)$, for F_n.

$$F(x) := \sum_{n=1}^{\infty} F_n x^n.$$

The next step is to use the recurrence, however F_1 and F_2 need to be handled separately since they are not defined recursively. Thus,

$$F(x) = F_1 x + F_2 x^2 + \sum_{n=3}^{\infty} F_n x^n.$$

Now we apply the definition of the Fibonacci numbers yielding,

$$F(x) = x + x^2 + \sum_{n=3}^{\infty} (F_{n-1} + F_{n-2}) x^n.$$

By rearranging we get,

$$F(x) = x + x^2 + \left(\sum_{n=3}^{\infty} F_{n-1} x^n \right) + \left(\sum_{n=3}^{\infty} F_{n-2} x^n \right).$$

Now, we factor out some power of x from each sum, yielding

$$F(x) = x + x^2 + x \left(\sum_{n=3}^{\infty} F_{n-1} x^{n-1} \right) + x^2 \left(\sum_{n=3}^{\infty} F_{n-2} x^{n-2} \right).$$

We now reindex the sums and do some manipulation so that we get the following.

$$F(x) = x + x^2 + x \left(F_1 x - F_1 x + \sum_{n=2}^{\infty} F_n x^n \right) + x^2 \left(\sum_{n=1}^{\infty} F_n x^n \right)$$

Some more manipulation yields,

$$F(x) = x + x^2 - x^2 + x \left(\sum_{n=1}^{\infty} F_n x^n \right) + x^2 \left(\sum_{n=1}^{\infty} F_n x^n \right).$$

Note, that we see the original definition of $F(x)$ in this new form, therefore

$$F(x) = x + xF(x) + x^2 F(x).$$

Solving for $F(x)$ yields,

$$F(x) = \frac{x}{1 - x - x^2} = \frac{x}{-(x^2 + x - 1)}.$$

The next step is to find the closed form of F_n. We know that $-\varphi$ and $-\overline{\varphi}$ are roots of the polynomial in the denominator, therefore

$$F(x) = \frac{x}{-(x + \varphi)(x + \overline{\varphi})}.$$

Using $\varphi\overline{\varphi} = -1$ yields,

$$F(x) = \frac{x}{(1 - \varphi x)(1 - \overline{\varphi} x)}.$$

Next, we perform partial fraction decomposition of $F(x)$, which gives us

$$F(x) = \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \varphi x} - \frac{1}{1 - \overline{\varphi} x} \right).$$

Using the formula for the power series for $(1 - x)^{-1}$ yields,

$$F(x) = \sum_{k=1}^{\infty} \frac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}} x^n = \sum_{k=1}^{\infty} F_n x^n.$$

By uniqueness of power series, we deduce that

$$F_n = \frac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}}.$$

References

[1] Hongwei Chen. *Excursions in Classical Analysis*. Mathematical Association of America. ISBN: 978-0-88385-768-7.