

Sums Convergent under the p -adic Norm

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Problem. Show that

$$\sum_{n=0}^{\infty} 2^n = -1$$

under the 2-adic norm.

1 Background

Definition 1 (Metric Space). Let X be a non-empty set and let $d : X \times X \rightarrow \mathbb{R}$ be a function. Then d is a *metric* on X if all of the following hold.

1. For all $x, y \in X$, $d(x, y) \geq 0$.
2. For all $x, y \in X$, $d(x, y) = 0$ iff $x = y$.
3. For all $x, y \in X$, $d(x, y) = d(y, x)$.
4. For all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

If d is a metric on X , then we say (X, d) forms a *metric space*.

Definition 2 (Cauchy Sequence). Let (X, d) be a metric space and let $(a_n)_{n \in \mathbb{N}} \subseteq X$. We say a_n is a *Cauchy sequence* if for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $d(a_n, a_m) < \varepsilon$ for all $n, m \geq N$.

Definition 3 (Convergent Sequence). Let (X, d) be a metric space and let $(a_n)_{n \in \mathbb{N}} \subseteq X$ and let $a \in X$. We say a_n is a *converges* to a if for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $d(a_n, a) < \varepsilon$ for all $n \geq N$. If such an a exists, we say a_n is *convergent* in X .

Definition 4 (Complete Metric Space). Let (X, d) be a metric space. We say X is *complete* if all Cauchy sequences in X converge in X .

Theorem 1. Let (X, d) be a metric space. Then X has a unique completion, $C(X, d)$, up to isometry.

Definition 5 (p -adic Ordinal). Let p be a prime and let $a \in \mathbb{Z}$ be nonzero. Then the *p -adic ordinal* of a , denoted $\text{ord}_p a$, is defined as

$$\text{ord}_p a = \max\{n \text{ s.t. } p^n | a\}.$$

Furthermore, for any nonzero $x = b/c \in \mathbb{Q}$,

$$\text{ord}_p x = \text{ord}_p a - \text{ord}_p b.$$

Using the definition of p -adic ordinal, we now provide the definition of the p -adic norm.

Definition 6 (p -adic Norm). Let p be a prime and $x \in \mathbb{Q}$. Then the *p -adic norm* of x , denoted $|x|_p$, is defined as

$$|x|_p = \begin{cases} p^{-\text{ord}_p x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Definition 7 (p -adic Numbers). The set of *p -adic numbers* is the completion of the metric space $(\mathbb{Q}, |\cdot|_p)$.

Theorem 2 (Strong Triangle Inequality). For all $x, y \in \mathbb{Q}_p$, $|x - y|_p \leq \max\{|x|_p, |y|_p\}$.

2 Solution

Problem. Show that

$$\sum_{n=0}^{\infty} 2^n = -1$$

under the 2-adic norm.

Theorem 3. Let $(a_n)_{n \in \mathbb{N}} \subseteq \mathbb{Q}_p$ such that $a_n \rightarrow 0$. Then $\sum_{n=0}^{\infty} a_n$ converges in \mathbb{Q}_p .

Proof. Let $\varepsilon > 0$ be given. Since $a_n \rightarrow 0$, there exists an $N \in \mathbb{N}$ such that $|a_n| < \varepsilon$. Force $n \geq m \geq N$. Consider $|\sum_{k=0}^n a_k - \sum_{k=0}^m a_k|_p$.

$$\left| \sum_{k=0}^n a_k - \sum_{k=0}^m a_k \right|_p = \left| \sum_{k=m+1}^n a_k \right|_p$$

Thus, by the strong triangle inequality,

$$\left| \sum_{k=0}^n a_k - \sum_{k=0}^m a_k \right|_p \leq \max_{m+1 \leq k \leq n} \{|a_k|_p\}.$$

However, we know that for each $k \geq N$, $|a_k|_p < \varepsilon$. Therefore,

$$\left| \sum_{k=0}^n a_k - \sum_{k=0}^m a_k \right|_p < \varepsilon$$

and $(\sum_{k=0}^n a_k)_{n \in \mathbb{N}}$ is Cauchy in \mathbb{Q}_p . Since \mathbb{Q}_p is defined as a complete metric space, $(\sum_{k=0}^n a_k)_{n \in \mathbb{N}}$ converges in \mathbb{Q}_p . \square

Lemma 4. Under the 2-adic metric, $2^n \rightarrow 0$.

Proof. Let $\varepsilon > 0$ be given. Without loss of generality, assume $\varepsilon < 1$. Thus, $\lg \varepsilon < 0$. Take $N > -\lg \varepsilon$ and let $n \geq N$. Then

$$|2^n|_2 = 2^{-n} \leq 2^{-N} < 2^{\lg \varepsilon} = \varepsilon.$$

Therefore, $2^n \rightarrow 0$ under $|\cdot|_2$. \square

Proposition 5. In \mathbb{Q}_2 ,

$$\sum_{n=0}^{\infty} 2^n = -1.$$

Proof. Since $2^n \rightarrow 0$, $\sum_{n=0}^{\infty} 2^n$ converges to some $S \in \mathbb{Q}_p$. Consider S .

$$S = \sum_{n=0}^{\infty} 2^n = 1 + \sum_{n=1}^{\infty} 2^n = 1 + 2 \sum_{n=1}^{\infty} 2^{n-1} = 1 + 2 \sum_{n=0}^{\infty} 2^n = 1 + 2S$$

Solving for S yields,

$$S = -1.$$

\square

References

- [1] Neal Koblitz. *p-adic Numbers, p-adic Analysis, and Zeta-Functions*. Springer. ISBN: 9781461270140.