Roots of Unity

Matt McCarthy

June 2016

Problem. *Factor the polynomial*

$$p = z^5 - 1.$$

1 Background

The complex numbers, denoted as $\mathbb{C}$ are defined as follows.

Definition 1. The set of *complex numbers* is the following two-dimensional vector space over the real numbers.

$$\mathbb{C} := \{a + bi \,:\, a, b \in \mathbb{R},\, i^2 = -1\}$$

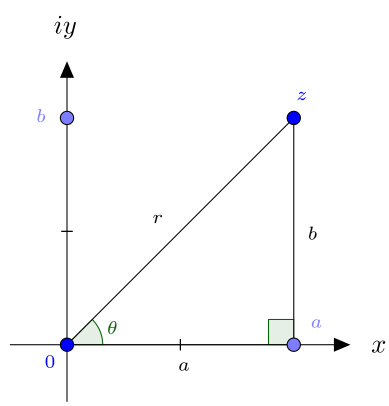

Figure 1: A diagram of the complex plane.

Furthermore, all $z \in \mathbb{C}$ have a *polar form*

$$z = r(\cos\theta + i\sin\theta)$$

where $r, \theta \in \mathbb{R}$ with $r \geq 0$. Moreover, θ is called the *argument* of z. Additionally, we have Euler's formula which allows us to write the polar form of a complex number more concisely.

Theorem 1. *For all* $\theta \in \mathbb{R}$,

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Thus, any $z \in \mathbb{C}$ can be written as

$$z = re^{i\theta}.$$

Moreover, for any $z = a + bi = re^{i\theta} \in \mathbb{C}$, the *modulus* of a z is defined as

$$|z| := \sqrt{a^2 + b^2} = r.$$

Lastly, an *nth root of unity* is a $z \in \mathbb{C}$ such that

$$z^n = 1.$$

2 Solution

Problem. *Factor the polynomial*

$$p = z^5 - 1.$$

In order to factor p, we want to find the roots of the equation

$$z^5 - 1 = 0.$$

Equivalently, we will find the 5th roots of unity, or the solutions to

$$z^5 = 1.$$

To begin, we know that $z = re^{i\theta}$ for some $r \geq 0$ and $\theta \in \mathbb{R}$. Furthermore, we know that $1 = e^{2ki\pi}$ for all $k \in \mathbb{Z}$. Thus,

$$r^5 e^{5i\theta} = e^{2ki\pi}$$

for all $k \in \mathbb{Z}$. Since $|e^{i\phi}| = 1$ for all $\phi \in \mathbb{R}$, we know that $r^5 = 1$. Thus, $r = 1$ because r is a positive real. Thus, we are left with

$$e^{5i\theta} = e^{2ki\pi}.$$

Ergo,

$$\theta = 2ki\pi/5$$

for all $k \in \mathbb{Z}$. However, this is an infinite solution set and there are only 5 *distinct* solutions. To find these distinct solutions, we use the fact that

$$e^{i\theta} = e^{i(\theta+2k\pi)}$$

for any $k \in \mathbb{Z}$. Thus, our distinct values for θ are as follows.

$$\theta \in \{0, 2\pi/5, 4\pi/5, 6\pi/5, 8\pi/5\}$$

When $\theta = 10\pi/5 = 2\pi$, we get the same result as when $\theta = 0$ since $2\pi \equiv 0 \mod 2\pi$. Therefore,

$$z \in S := \{1, e^{2\pi/5}, e^{4\pi/5}, e^{6\pi/5}, e^{8\pi/5}\}.$$

Since each of the elements of the solution set is a root of the polynomial, $z^5 - 1$, we can factor out $z - r$ from p for each $r \in S$. Thus,

$$p = z^5 - 1 = (z-1)(z - e^{2\pi/5})(z - e^{4\pi/5})(z - e^{6\pi/5})(z - e^{8\pi/5}).$$