Sums Convergent under the p-adic Norm

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Problem. *Show that*

$$\sum_{n=0}^{\infty} 2^n = -1$$

under the 2-adic metric.

1 Background

In order to talk about limits, we first need to understand the concept of a metric.

Definition 1 (Metric Space)**.** Let X be a non-empty set and let $d : X \times X \to \mathbb{R}$ be a function. Then d is a *metric* on X if all of the following hold.

1. For all $x, y \in X$, $d(x, y) \geq 0$.
2. For all $x, y \in X$, $d(x, y) = 0$ iff $x = y$.
3. For all $x, y \in X$, $d(x, y) = d(y, x)$.
4. For all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

If d is a metric on X, then we say (X, d) forms a *metric space*.

After defining the metric space, we can consider whether or not a sequence in that space converges.

Definition 2 (Convergent Sequence)**.** Let (X, d) be a metric space and let $(a_n)_{n\in\mathbb{N}} \subseteq X$ and let $a \in X$. We say a_n is a *converges* to a if for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $d(a_n, a) < \varepsilon$ for all $n \geq N$. If such an a exists, we say a_n is *convergent in* X.

Furthermore, if a sequence is convergent in a metric space, it is also Cauchy in that space.

Definition 3 (Cauchy Sequence)**.** Let (X, d) be a metric space and let $(a_n)_{n\in\mathbb{N}} \subseteq X$. We say a_n is a *Cauchy sequence* if for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $d(a_n, a_m) < \varepsilon$ for all $n, m \geq N$.

While convergent implies Cauchy, the other way does not always hold. For example in $\mathbb{Q}$ under the Euclidean metric ($d(x, y) = |x - y|$), the sequence defined by $(1 + 1/n)^n$ is Cauchy but not convergent. However, if we move to $\mathbb{R}$, $(1 + 1/n)^n$ converges to e. Cauchy sequences are sequences that *should* be convergent in our space. If they are not, then we need to move to what is called the completion of the metric space.

Definition 4 (Complete Metric Space)**.** Let (X, d) be a metric space. We say X is *complete* if all Cauchy sequences in X converge in X.

Theorem 1. *Let (X, d) be a metric space. Then X has a unique completion, $C(X, d)$, up to isometry. Furthermore, this completion is isometric to the space $(B(X), D)$ where $B(X)$ is the set of all bounded functions from X to $\mathbb{R}$, and $D(f, g) = \sup_{x\in X} |f(x) - g(x)|$.*

Now that we have enough background in analysis, lets talk about the p-adic numbers. Before we can define the p-adic's, we need to introduce the p-adic ordinal and p-adic absolute value first.

Definition 5 (p-adic Ordinal)**.** Let p be a prime and let $a \in \mathbb{Z}$ be nonzero. Then the *p-adic ordinal* of a, denoted $\operatorname{ord}_p a$, is defined as

$$\operatorname{ord}_p a = \max\{n : p^n | a\}.$$

Furthermore, for any nonzero $x = b/c \in \mathbb{Q}$,

$$\operatorname{ord}_p x = \operatorname{ord}_p a - \operatorname{ord}_p b.$$

Using the definition of p-adic ordinal, we now provide the definition of the p-adic absolute value.

Definition 6 (p-adic absolute value)**.** Let p be a prime and $x \in \mathbb{Q}$. Then the *p-adic absolute value of x*, denoted $|x|_p$, is defined as

$$|x|_p = \begin{cases} p^{-\operatorname{ord}_p x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

While an absolute value is not a metric in and of itself, it generates a metric. Similarly to how we use $|x - y|$ as a metric, we use $|x - y|_p$ as the p-adic metric. With this, we can define the p-adic numbers.

Definition 7 (p-adic Numbers)**.** The set of *p-adic numbers* is the completion of the metric space $(\mathbb{Q}, |\cdot|_p)$.

Additionally, $\mathbb{Q}_p$ satisfies a stronger version of the triangle inequality.

Theorem 2 (Strong Triangle Inequality)**.** *For all $x, y \in \mathbb{Q}_p$, $|x + y|_p \leq \max\{|x|_p, |y|_p\}$.*

2 Solution

Problem. *Show that*

$$\sum_{n=0}^{\infty} 2^n = -1$$

under the 2-adic norm.

To start our solution, we prove that if a sequence in $\mathbb{Q}_p$ converges to 0 under the p-adic metric, the infinite sum of all of its terms is convergent in $\mathbb{Q}_p$. While this result is not true in $\mathbb{R}$ (e.g. the harmonic series), it works in $\mathbb{Q}_p$ because of the strong triangle inequality.

Theorem 3. *Let $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{Q}_p$ such that $a_n \to 0$. Then $\sum_{n=0}^{\infty} a_n$ converges in $\mathbb{Q}_p$.*

Proof. Let $\varepsilon > 0$ be given. Since $a_n \to 0$, there exists an $N \in \mathbb{N}$ such that $|a_n| < \varepsilon$. Force $n \geq m \geq N$. Consider $|\sum_{k=0}^{n} a_k - \sum_{k=0}^{m}|_p$.

$$\left|\sum_{k=0}^{n} a_k - \sum_{k=0}^{m}\right|_p a_k = \left|\sum_{k=m+1}^{n} a_k\right|_p$$

Thus, by the strong triangle inequality,

$$\left|\sum_{k=0}^{n} a_k - \sum_{k=0}^{m} a_k\right|_p \leq \max_{m+1\leq k\leq n}\{|a_k|_p\}.$$

However, we know that for each $k \geq N$, $|a_k|_p < \varepsilon$. Therefore,

$$\left|\sum_{k=0}^{n} a_k - \sum_{k=0}^{m} a_k\right|_p < \varepsilon$$

and $\left(\sum_{k=0}^{n} a_k\right)_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{Q}_p$. Since $\mathbb{Q}_p$ is defined as a complete metric space, $\left(\sum_{k=0}^{n} a_k\right)_{n\in\mathbb{N}}$ converges in $\mathbb{Q}_p$. □

Now, in order to show that $\sum_{n=0}^{\infty} 2^n$ converges with respect to the p-adic metric, we show that $2^n \to 0$ with respect to the p-adic metric.

Lemma 4. *Under the 2-adic metric,* $2^n \to 0$.

Proof. Let $\varepsilon > 0$ be given. Without loss of generality, assume $\varepsilon < 1$. Thus, $\lg \varepsilon < 0$. Take $N > -\lg \varepsilon$ and let $n \geq N$. Then

$$|2^n|_2 = 2^{-n} \leq 2^{-N} < 2^{\lg \varepsilon} = \varepsilon.$$

Therefore, $2^n \to 0$ under $|\cdot|_2$. □

From here, we do some algebraic manipulation to find the limit point.

Proposition 5. *In* $\mathbb{Q}_2$,

$$\sum_{n=0}^{\infty} 2^n = -1.$$

Proof. Since $2^n \to 0$, $\sum_{n=0}^{\infty} 2^n$ converges to some $S \in \mathbb{Q}_p$. Consider S.

$$S = \sum_{n=0}^{\infty} 2^n = 1 + \sum_{n=1}^{\infty} 2^n = 1 + 2\sum_{n=1}^{\infty} 2^{n-1} = 1 + 2\sum_{n=0} 2^n = 1 + 2S$$

Solving for S yields,

$$S = -1.$$

□

References

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