Eigenvalues and Eigenvectors

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Problem. *Let* $\tau : \mathbb{F}_5^3 \to \mathbb{F}_5^3$ *be the endomorphism defined by*

$$\tau = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}.$$

Find all eigenvalues and eigenvectors of τ.
(Note: $\mathbb{F}_5$ *denotes* $\mathbb{Z}/5\mathbb{Z}$*, the integers modulo 5.)*

1 Background

Lets begin with the definition of eigenvalues and eigenvectors.

Definition 1. Let V be a vector space over a field $\mathbb{F}$ and let τ be an endomorphism on V. A scalar $\lambda \in \mathbb{F}$ is an *eigenvalue* of τ if there exists a nonzero vector v such that

$$\tau v = \lambda v.$$

If such a v exists, it is called an *eigenvector* of τ associated with λ.

Furthermore, in finite-dimensional vector spaces, each endomorphism has a minimal polynomial.

Definition 2. Let V be a finite-dimensional vector space over a field $\mathbb{F}$ and let τ be an endomorphism on V. Then the *minimal polynomial* of τ is the generator of the ideal

$$I_\tau = \{p \in \mathbb{F}[x] | p(\tau) = 0\}.$$

Moreover, the eigenvalues of an endomorphism are the zeros of this minimal polynomial.

Theorem 1. *Let* τ *be an endomorphism with minimal polynomial* p*. Then the set of zeros of* p *and the set of eigenvalues of* τ *are equal.*

2 Solution

Problem. *Let* $\tau : \mathbb{F}_5^3 \to \mathbb{F}_5^3$ *be the endomorphism defined by*

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We begin by finding the minimal polynomial of τ. To do so, we apply τ multiple times to the vector $e_1 = (1, 0, 0)$.

$$\tau^0 e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad \tau^1 e_1 = \begin{pmatrix}1&0&2\\2&0&1\\1&2&1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\2\\1\end{pmatrix} \qquad \tau^2 e_1 = \begin{pmatrix}1&0&2\\2&0&1\\1&2&1\end{pmatrix}\begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}3\\3\\1\end{pmatrix}$$

We now need to check that $B = \{e_1, \tau e_1, \tau^2 e_1\}$ is linearly independent. Therefore, we must solve the equation

$$a(1, 0, 0) + b(1, 2, 1) + c(3, 3, 1) = 0.$$

Indeed, this equation implies that $a = b = c = 0$. Ergo B is linearly independent. Next, we compute one more power of τ.

$$\tau^3 e_1 = \begin{pmatrix}1&0&2\\2&0&1\\1&2&1\end{pmatrix}\begin{pmatrix}3\\3\\1\end{pmatrix} = \begin{pmatrix}0\\2\\0\end{pmatrix}$$

Consider the equation

$$a(1, 0, 0) + b(1, 2, 1) + c(3, 3, 1) = (0, 2, 0).$$

This implies that $a = 1$, $b = 3$, and $c = 2$. Thus,

$$\tau^3 e_1 = e_1 + 3\tau e_1 + 2\tau^2 e_1$$

which yields a minimal polynomial of

$$\mu_{\tau,e_1} = X^3 + 3X^2 + 2X + 4.$$

Recall that the set of eigenvalues of a linear transform is the set of zeros of its minimal polynomial. We notice that $\mu_{\tau,e_1}(1) = 0$, thus $X - 1 | \mu_{\tau,e_1}$. Therefore,

$$\mu_{\tau,e_1} = (X - 1)(X^2 + 4X + 1).$$

When we apply the quadratic formula, we get that

$$X = 3 + 3\sqrt{2}, 3 + 2\sqrt{2}$$

are also zeros of μ_{τ,e_1}. Therefore the eigenvalues of μ_{τ,e_1} are $1, 3 + 3\sqrt{2}, 3 + 2\sqrt{2}$. To find the eigenvectors that correspond to 1, we solve the following equation.

$$\begin{pmatrix}1&0&2\\2&0&1\\1&2&1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = 1 \cdot \begin{pmatrix}a\\b\\c\end{pmatrix}$$

This corresponds to the following system of equations.

$$\begin{aligned} a + 0b + 2c &= a \\ 2a + 0b + c &= b \\ a + 2b + c &= c \end{aligned}$$

This system implies that $c = 0$ and $b = 2a$, thus any vector in $\mathbb{F}_5^3$ of the form $(a, 2a, 0)$ is an eigenvector of 1. Even though $\sqrt{2} \notin \mathbb{F}_5$, we can embed $\mathbb{F}_5$ inside $\mathbb{F}_5[\sqrt{2}]$. By performing a similar process we get that any vector of the form $(a, 3\sqrt{2}a, (1+4\sqrt{2})a)$ is an eigenvector of $3+3\sqrt{2}$ and $(a, 2\sqrt{2}a, (1+\sqrt{2})a)$ is an eigenvector of $3 + 2\sqrt{2}$ where $a \in \mathbb{F}_5[\sqrt{2}]$.

References

[1] Steven Roman. *Advanced Linear Algebra*. Springer. ISBN: 978-0-387-72828-5.

[2] Wikipedia. *Minimal polynomial (linear algebra)*. URL: `https://en.wikipedia.org/wiki/Minimal_polynomial_(linear_algebra)`.