Euler's Formula and the Complex Unit Circle

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We want to show the following.

Theorem. *Let $x \in \mathbb{R}$ and i be the positive root of $x^2 + 1$. Then $e^{ix} = \cos x + i \sin x$.*

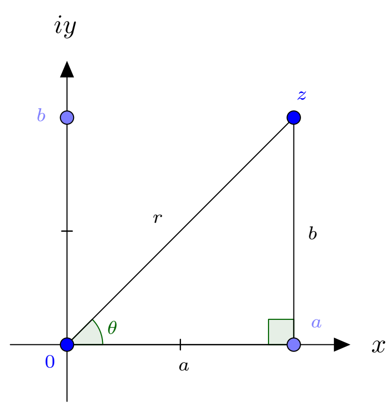
Theorem. *The set $U = \{z \in \mathbb{C} | |z| = 1\}$ forms a group under complex multiplication.*

1 Complex Numbers

We begin by defining the complex numbers.

Definition 1. The set of *complex numbers* is the set $\mathbb{C} = \{a + bi | a, b \in \mathbb{R}\}$ where $i^2 = -1$.

Suppose $z \in \mathbb{C}$, then there exist $a, b \in \mathbb{R}$ such that $z = a + bi$. We can plot this on a plane as follows.



When we plot it, we get a triangle with vertices $0, a, z$. If we consider the line az, we see that it is parallel to the iy axis, which is in turn perpendicular to the x axis. Thus, $\triangle 0az$ is right and $r = \sqrt{a^2 + b^2}$ by Pythagorean theorem. We also define $|z| = \sqrt{a^2 + b^2}$, or the Euclidean distance between z and 0.

Furthermore, consider the angle θ. If we use the change of variables $a = r \cos \theta$ and $b = r \sin \theta$, we get that $\theta = \arctan(b/a)$. Thus, we can write $z = a + bi$ as $r(\cos \theta + i \sin \theta)$, which gives us a polar representation of z. Moreover, we call r the *modulus* of z and θ the *argument* of z; note that both the modulus and argument are real numbers.

2 Euler's Formula

In the 1740's Leonhard Euler noted that

$$e^{ix} = \cos x + i \sin x.$$

We provide a proof of that.

Theorem 1. *Let $x \in \mathbb{R}$ and i be the positive root of x^2+1. Then $e^{ix} = \cos x + i \sin x$.*

Proof. We know e^{ix} is a complex number thus, $e^{ix} = r \cdot (\cos\theta + i\sin\theta)$, where $r = r(x)$ and $\theta = \theta(x)$. Therefore

$$\frac{d}{dx} e^{ix} = \frac{d}{dx}\left(r \cdot (\cos\theta + i\sin\theta)\right)$$

and

$$-r\sin\theta + ir\cos\theta = ie^{ix} = (\cos\theta + i\sin\theta)\frac{dr}{dx} + r\cdot(-\sin\theta + i\cos\theta)\frac{d\theta}{dx}.$$

Thus, when we match real and imaginary parts, we get

$$\cos\theta\frac{dr}{dx} - r\sin\theta\frac{d\theta}{dx} = -r\sin\theta$$

and

$$\sin\theta\frac{dr}{dx} + r\cos\theta\frac{d\theta}{dx} = r\cos\theta.$$

□

3 The Unit Circle