Box Counting Dimension of the Middle-λ Cantor Set

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April 2016

Theorem. *Let C denote the middle-λ Cantor set with $0 < \lambda < 1$. Then the box counting dimension of C is*

$$\dim_B C = \frac{\ln 2}{\ln\left(\frac{2}{1-\lambda}\right)}.$$

1 Background

1.1 Fractal Analysis

We begin by defining a δ-cover of a set.

Definition 1. Let $X \subset \mathbb{R}^n$ and $\delta > 0$. Then a *δ-cover* of X is a set $D = \{D_i\}_{i=1}^{n_D}$ such that $X \subseteq \cup_{D_i \in D} D_i$ and $\operatorname{diam} D_i \leq \delta$ for all $D_i \in D$. We denote the collection of all such covers as $\mathcal{D}_\delta(X)$.

Now that we have δ-covers, we can define the *box-counting dimension* of a set.

Definition 2. Let X be a subset of $\mathbb{R}^n$. Then the *box-counting dimension* of X, denoted $\dim_B(X)$, is defined as

$$\lim_{\delta \to 0} \frac{\ln N_\delta(X)}{-\ln \delta}$$

where

$$N_\delta(X) = \min_{D \in \mathcal{D}_\delta(X)} \operatorname{card} D.$$

Note that the box-counting dimension of a set does not necessarily exist, however when it does exist it is usually easier to find.

Theorem 1 (Squeeze Theorem). *Let $(a_n), (b_n), (c_n)$ be real valued sequences such that $\lim a_n = \lim c_n = x$ and $a_n \leq b_n \leq c_n$ for all n greater than some $n \in \mathbb{N}$. Then b_n converges to x.*

1.2 The Cantor Set

We will now talk about the middle third Cantor set. Begin by defining $C_0 = [0,1]$. We now remove the middle third from C_0 yielding, $C_1 = [0,1/3] \cup [2/3,1]$. We then remove the middle third from *each* subinterval of C_1, yielding $C_2 = [0,1/9] \cup [2/9,3/9] \cup [6/9,7/9] \cup [8/9,1]$. We iterate this process by removing the middle-third from each subinterval of C_n and labeling the remaining set C_{n+1}. Finally, we define the Cantor set as $C = \cap_{n \in \mathbb{N}} C_n$. We can similarly construct the middle-λ Cantor set by performing the same construction but removing λ instead of one third in each step. Furthermore, C is an uncountable set that has no length left to it.

2 Solution

Theorem 2. *Let C denote the middle-λ Cantor set with $0 < \lambda < 1$. Then the box counting dimension of C is*

$$\dim_B C = \frac{\ln 2}{\ln\left(\frac{2}{1-\lambda}\right)}.$$

Proof. Consider removing an interval of length $a\lambda$ from the middle of the interval $[0, a]$ for some $a > 0$. Since we have removed $a\lambda$ from the interval, the length of this new set is exactly $a - a\lambda = a(1-\lambda)$. Moreover, since we removed it from the middle of the interval, this length is equally distributed among the two resultant subintervals. Thus the subintervals must have length $a((1-\lambda)/2)$. Furthermore, the leftmost interval must be $[0, a((1-\lambda)/2)]$.

We now need to find the length of any subinterval of C_n. Since $C_0 = [0, 1]$, $l_0 = 1$. This tells us that $l_1 = (1-\lambda)/2$ and that the leftmost interval in C_1 is $[0, (1-\lambda)/2]$. We know from the above logic that $l_{n+1} = l_n(1-\lambda)/2$ and its leftmost interval will be $[0, l_n(1-\lambda)/2]$. Solving this recursion yields that

$$l_n = \left(\frac{1-\lambda}{2}\right)^n$$

and the leftmost interval of C_n is $[0, ((1-\lambda)/2)^n]$. Thus if we were to cover C_n we would need 2^n sets of diameter $((1-\lambda)/2)^n$.

Suppose $\delta > 0$. Thus, we can find an n such that $l_{n+1} \leq \delta < l_n$. Since $\delta < l_n$, we need no fewer than 2^n sets of diameter δ to cover $C_n \supset C$. Since $\delta \geq l_{n+1}$, we need no more than 2^{n+1} sets of diameter δ to cover $C_{n+1} \supset C$. Thus, we can glean the following inequality.

$$2^n \leq N_\delta(C) \leq 2^{n+1}.$$

Since ln is a monotone increasing function, we get

$$n \ln 2 \leq \ln N_\delta(C) \leq (n+1)\ln 2. \tag{1}$$

Furthermore, since $l_{n+1} \leq \delta < l_n$,

$$\frac{1}{-\ln l_{n+1}} \leq \frac{1}{-\ln \delta} \leq \frac{1}{-\ln l_n}. \tag{2}$$

Take Equation 1 and multiply it by $1/(-\ln\delta)$.

$$\frac{n\ln 2}{-\ln\delta} \leq \frac{\ln N_\delta(C)}{-\ln\delta} \leq \frac{(n+1)\ln 2}{-\ln\delta}$$

By Equation 2, we have

$$\frac{n\ln 2}{-\ln l_{n+1}} \leq \frac{\ln N_\delta(C)}{-\ln\delta} \leq \frac{(n+1)\ln 2}{-\ln l_n}.$$

Consider $\ln l_n$.

$$\ln l_n = \ln\left(\left(\frac{1-\lambda}{2}\right)^n\right) = -n\ln\left(\frac{2}{1-\lambda}\right)$$

Thus, our inequality becomes,

$$\frac{n\ln 2}{(n+1)\ln\left(\frac{2}{1-\lambda}\right)} \leq \frac{\ln N_\delta(C)}{-\ln\delta} \leq \frac{(n+1)\ln 2}{n\ln\left(\frac{2}{1-\lambda}\right)}.$$

If we want to find $\lim\limits_{\delta\to 0} \ln N_\delta(C)/(-\ln\delta)$, we need to find the limits of the far left and far right as $n \to \infty$. On the lefthand side we get

$$\frac{n\ln 2}{(n+1)\ln\left(\frac{2}{1-\lambda}\right)} = \frac{n\ln 2}{n\ln\left(\frac{2}{1-\lambda}\right) + \ln\left(\frac{2}{1-\lambda}\right)} \to \frac{\ln 2}{\ln\left(\frac{2}{1-\lambda}\right)}$$

by l'Hôpital's rule. On the righthand side we get

$$\frac{(n+1)\ln 2}{n\ln\left(\frac{2}{1-\lambda}\right)} = \frac{n\ln 2 + \ln 2}{n\ln\left(\frac{2}{1-\lambda}\right)} \to \frac{\ln 2}{\ln\left(\frac{2}{1-\lambda}\right)}$$

again by l'Hôpital's rule. Thus by squeeze theorem, $\dim_B(C) = \frac{\ln 2}{\ln\left(\frac{2}{1-\lambda}\right)}$. $\square$