Euler's Formula and the Complex Unit Circle

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We want to show the following.

Theorem. *Let $x \in \mathbb{R}$ and i be the positive root of x^2+1. Then $e^{ix} = \cos x + i \sin x$.*

Theorem. *The set $U = \{z \in \mathbb{C} | |z| = 1\}$ forms a group under complex multiplication.*

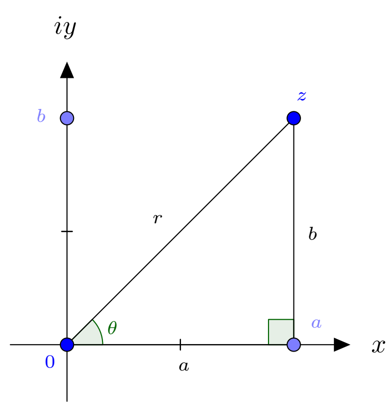
1 Complex Numbers

We begin by defining the complex numbers.

Definition 1. The set of *complex numbers* is the set $\mathbb{C} = \{a+bi|a, b \in \mathbb{R}\}$ where $i^2 = -1$. Complex addition is defined component wise, that is $(a+bi)+(c+di) = (a+c)+(b+d)i$ and complex multiplication is defined as

$$(a+bi)(c+di) = (ac-bd) + (ad+bc)i.$$

Suppose $z \in \mathbb{C}$, then there exist $a, b \in \mathbb{R}$ such that $z = a+bi$. We can plot this on a plane as follows.



When we plot it, we get a triangle with vertices $0, a, z$. If we consider the line az, we see that it is parallel to the iy axis, which is in turn perpendicular to the x axis. Thus, $\triangle 0az$ is right and $r = \sqrt{a^2+b^2}$ by Pythagorean theorem. We also define $|z| = \sqrt{a^2+b^2}$, or the Euclidean distance between z and 0.

Furthermore, consider the angle θ. If we use the change of variables $a = r\cos\theta$ and $b = r\sin\theta$, we get that $\theta = \arctan(b/a)$. Thus, we can write $z = a+bi$ as $r(\cos\theta + i\sin\theta)$, which gives us a polar representation of z. Moreover, we call r the *modulus* of z and θ the *argument* of z; note that both the modulus and argument are real numbers.

2 Euler's Formula

In the 1740's Leonhard Euler noted that

$$e^{ix} = \cos x + i \sin x.$$

We provide a proof of that.

Theorem 1. *Let $x \in \mathbb{R}$ and i be the positive root of $x^2 + 1$. Then $e^{ix} = \cos x + i \sin x$.*

Proof. We know e^{ix} is a complex number thus, $e^{ix} = r \cdot (\cos\theta + i \sin\theta)$, where $r = r(x)$ and $\theta = \theta(x)$. Therefore

$$\frac{d}{dx} e^{ix} = \frac{d}{dx} \left(r \cdot (\cos\theta + i \sin\theta) \right)$$

and

$$-r \sin\theta + ir \cos\theta = ie^{ix} = (\cos\theta + i \sin\theta) \frac{dr}{dx} + r \cdot (-\sin\theta + i \cos\theta) \frac{d\theta}{dx}.$$

Thus, when we match real and imaginary parts, we get the following linear system of equations.

$$\begin{aligned} \cos\theta \frac{dr}{dx} - r \sin\theta \frac{d\theta}{dx} &= -r \sin\theta \\ \sin\theta \frac{dr}{dx} + r \cos\theta \frac{d\theta}{dx} &= r \cos\theta \end{aligned}$$

Solving the system yields the differential equations $dr/dx = 0$ and $d\theta/dx = 1$.

Consider e^{i0}.

$$r(0)(\cos(\theta(0)) + i \sin(\theta(0))) = e^{i0} = 1 = 1(\cos 0 + i \sin 0)$$

Matching terms gives us, $r(0) = 1$ and $\theta(0) = 0$. When combined with our differential equations, this yields the following initial value problems.

$$\begin{cases} \frac{dr}{dx} = 0 \\ r(0) = 1 \end{cases} \qquad \begin{cases} \frac{d\theta}{dx} = 1 \\ \theta(0) = 0 \end{cases}$$

The solutions to these initial value problems are $r(x) = 1$ and $\theta(x) = x$ respectively. Thus,

$$e^{ix} = \cos x + i \sin x.$$

$\square$

We can quickly extend this result to any complex number.

Corollary 2. *For any $z \in \mathbb{C}$, $z = re^{i\theta}$ where $\theta = \arg z$ and $r = |z|$.*

Proof. Since $z \in \mathbb{C}$, $z = r(\cos\theta + i \sin\theta)$ where $\theta = \arg z$ and $r = |z|$. Thus by Euler's formula,

$$z = r(\cos\theta + i \sin\theta) = re^{i\theta}.$$

$\square$

3 The Unit Circle

Consider $e^{i\theta}$ where θ is real. We know that $\left|e^{i\theta}\right| = 1$ by the corollary to Euler's formula. Thus $e^{i\theta}$ lies on the unit circle of the complex plane.

Definition 2. The unit circle of the complex plane is the subset of $\mathbb{C}$ whose elements all have a modulus of one.

$$U = \{z \in \mathbb{C} | |z| = 1\}.$$

Conversely, all points on the unit circle also have the form $e^{i\theta}$ for some $\theta \in [0, 2\pi)$.

Our goal is to show that U is a group under complex multiplication, but before we do so, let us consider $\mathbb{C}$. Most notably, $\mathbb{C}$ is a field, which implies that $\mathbb{C} \setminus \{0\}$ is an abelian group under complex multiplication. Thus, in order to show that $(U, \cdot)$ forms a group, it is sufficient to show that U is a *subgroup* of $\mathbb{C} \setminus \{0\}$. In short, a subset of a group is a subgroup if it is also a group under the group operation in its own right. Additionally, we have a quick test to determine whether or not a subset is a subgroup.

Proposition 3 (One step test)**.** *Let $(G, *)$ be a group and H a non empty subset of G. Then H is a subgroup of G if for all $a, b \in H$,*

$$a * b^{-1} \in H.$$

With the Euler's formula and the one step subgroup test in our toolbox, we are now ready to show that U is a group.

Theorem 4. *The set $U = \{z \in \mathbb{C} | |z| = 1\}$ forms a group under complex multiplication.*

Proof. We know that $|1| = 1$, therefore $1 \in U$ and U is nonempty. Furthermore, $U \subset \mathbb{C} \setminus \{0\}$ by definition of U.

Proceed via one step subgroup test. Let $z, w \in U$. Therefore $|z| = |w| = 1$ and $z = e^{i\theta}$ and $w = e^{i\phi}$ where $\theta, \phi \in \mathbb{R}$ by corollary to Euler's formula.

Consider zw^{-1}.

$$zw^{-1} = e^{i\theta}e^{-i\phi} = e^{i\theta - i\phi} = e^{i(\theta-\phi)}$$

We know that for any $x \in \mathbb{R}$, $|e^{ix}| = 1$ therefore $|zw^{-1}| = |e^{i(\theta-\phi)}| = 1$ and $zw^{-1} \in U$. Thus, U is a subgroup of $\mathbb{C} \setminus \{0\}$ under multiplication by the one step subgroup test. Therefore $(U, \cdot)$ is a group. $\square$

A neat corollary to this is that the roots of unity in the complex plane are closed under multiplication.

Definition 3. Let $n \in \mathbb{N}$. Then the *nth roots of unity* are solutions to the equation

$$z^n = 1.$$