Trigonometric Sum and Difference Formulas in $\mathbb{C}$

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1 Background

Definition 1. Let $f : \mathbb{C} \to \mathbb{C}$, where $f(x+iy) = u(x,y) + iv(x,y)$, be a function such that $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, and $\partial v/\partial y$ exist and are continuous on some disk $D \subseteq \mathbb{C}$ with a nonzero radius. If f satisfies the *Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

then f is said to be *analytic on* D. Furthermore, the largest subset of $\mathbb{C}$ on which f is analytic is called f's *domain of analyticity*. If the domain of analyticity is $\mathbb{C}$, then f is said to be *entire*.

Theorem 1. *Let f be analytic on a disk D, then the Taylor series for f converges for all $z \in D$.*

2 Solution

2.1 Analyticity of e^z

Proposition 2. *e^z is entire.*

Proof. Let $z = x + iy$. We want to find $\mathbb{R}$-valued functions u, v such that $e^{x+iy} = u(x,y) + iv(x,y)$.

$$\begin{aligned} e^z &= e^{x+iy} \\ &= e^x e^{iy} \\ &= e^x \cos y + ie^x \sin y \\ &= u(x,y) + iv(x,y) \end{aligned}$$

Taking partial derivatives yields the following.

$$\begin{array}{rlrl} \partial u/\partial x = & e^x \cos y & \qquad \partial v/\partial y = & e^x \cos y \\ \partial u/\partial y = & -e^x \sin y & \qquad -\partial v/\partial x = & -e^x \sin y \end{array}$$

Therefore e^z satisfies the Cauchy-Riemann equations. Furthermore, since the equations hold for all $z \in \mathbb{C}$, e^z is entire. $\square$

Proposition 3. *Let $z \in \mathbb{C}$. Then*

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \textit{ and } \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Proof. Since e^z is entire, its Taylor series converges everywhere in $\mathbb{C}$. Take each right hand side, and replace e^{iz} and e^{-iz} with their respective Taylor expansions. Doing so will yield the Taylor expansion of $\cos z$ and $\sin z$ respectively. The details are left as an exercise. $\square$

2.2 Trigonometric Sum Identities

Theorem 4. *Let* $z, w \in \mathbb{C}$. *Then* $\sin(z+w) = \cos z \sin w + \sin z \cos w$.

Proof. Consider $\cos z \sin w + \sin z \cos w$.

$$\begin{aligned}
\cos z \sin w + \sin z \cos w &= \left(\frac{e^{iz}+e^{-iz}}{2}\right)\left(\frac{e^{iw}-e^{-iw}}{2i}\right) + \left(\frac{e^{iz}-e^{-iz}}{2i}\right)\left(\frac{e^{iw}+e^{-iw}}{2}\right) \\
&= \frac{e^{i(z+w)}+e^{i(w-z)}-e^{i(z-w)}-e^{-i(z+w)}}{4i} + \frac{e^{i(z+w)}-e^{i(w-z)}+e^{i(z-w)}-e^{-i(z+w)}}{4i} \\
&= \frac{2e^{i(z+w)}-2e^{-i(z+w)}}{4i} \\
&= \frac{e^{i(z+w)}-e^{-i(z+w)}}{2i} \\
&= \sin(z+w)
\end{aligned}$$

□

Theorem 5. *Let* $z, w \in \mathbb{C}$. *Then* $\cos(z+w) = \cos z \cos w - \sin z \sin w$.

Proof. Consider $\cos z \cos w - \sin z \sin w$.

$$\begin{aligned}
\cos z \cos w - \sin z \sin w &= \left(\frac{e^{iz}+e^{-iz}}{2}\right)\left(\frac{e^{iw}+e^{-iw}}{2}\right) - \left(\frac{e^{iz}-e^{-iz}}{2i}\right)\left(\frac{e^{iw}-e^{-iw}}{2i}\right) \\
&= \frac{e^{i(z+w)}+e^{i(w-z)}+e^{i(z-w)}+e^{-i(z+w)}}{4} + \frac{e^{i(z+w)}-e^{i(w-z)}-e^{i(z-w)}+e^{-i(z+w)}}{4} \\
&= \frac{e^{i(z+w)}+e^{-i(z+w)}}{2} \\
&= \cos(z+w)
\end{aligned}$$

□

References

[1] John M. Howie. *Complex Analysis*. Springer. ISBN: 978-1-85233-733-9.