The Mystery of 1^π

Matt McCarthy

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Problem. *Find what step is wrong in the following statements.*

$$\begin{aligned} 1 &= 1^\pi \\ &= \left(e^{2i\pi}\right)^\pi \\ &= e^{2i\pi^2} \\ &= \cos 2\pi^2 + i \sin 2\pi^2 \\ &\approx 0.6296 + 0.7768i \end{aligned}$$

1 Background

1.1 Complex Logarithms

In $\mathbb{R}$ we can define logarithms in various ways such as the inverse of the exponential and as

$$\ln x = \int \frac{1}{x} dx.$$

However, in $\mathbb{C}$ the original definition (inverse of the exponential) fails since

$$e^{i\theta} = e^{i\theta + 2ki\pi}$$

where $k \in \mathbb{Z}$. Essentially, the exponential is a bijection between $\mathbb{R}$ and $(0, \infty)$ and is thus invertible. When we move to $\mathbb{C}$, the exponential function loses its injectivity since multiple domain elements get mapped to a single element in the range. However, the defining logarithms as the antiderivative of $1/x$ still works.

Definition 1. Let $z \in \mathbb{C}$. Then $\log z$ is defined as

$$\log z := \ln |z| + i \arg z.$$

If $\arg z \in [0, 2\pi)$ we say it is the *principal logarithm* of z.

As we can see by the definition, the complex logarithm is *multivalued*, thus if we want it to be well-defined we need to restrict the argument of the input to a half-open interval of length 2π.

1.2 Exponentiation with a Complex Base

Because of these problems with the logarithm, exponentiation becomes less straightforward. In the complex world, exponentiation is defined as follows.

Definition 2. Let $z, w \in \mathbb{C}$. Then

$$w^z = e^{z \log w}.$$

However, this definition is equivalent to

$$w^z = e^{z(\ln|w| + i \arg w)}.$$

Thus, exponentiation in the complex plane is now also multivalued, because of the properties of the complex logarithm. As we will see later, this will cause us to lose some properties generally associated with exponentiation.

When exponentiating, the following are true.

1. z^w is single-valued iff $w \in \mathbb{Z}$.
2. $z^{m/n}$ has n distinct values for $m, n \in \mathbb{Q}$.
3. z^x has infinitely many values if x is irrational.

2 Solution

Problem. *Find what step is wrong in the following statements.*

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The problem is

$$1^\pi = 1.$$

By definition of complex exponentiation we have

$$1^\pi = e^{\pi \log 1}.$$

However

$$\log 1 = \ln|1| + i \arg 1 = 0 + 2ki\pi$$

for $k \in \mathbb{Z}$. Thus,

$$1^\pi = e^{2ki\pi^2}.$$

Since π is irrational, there are infinitely many values that $e^{2ki\pi^2}$ can take only one of which will yield 1 as the answer. If we force $k = 0$, then we get the desired result that $1^\pi = 1$, however this excludes infinitely many other values.