Test for Abelian Groups

Matt McCarthy

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Definition 1. Let S be a set. Then a *binary operation on* S is a map $* : S \times S \to S$.

Definition 2. Let G be a set and $*$ be a binary operation on G. Then $(G, *)$ is a *group* if and only if all of the following hold.

1. For each $a, b, c \in G$, $(a * b) * c = a * (b * c)$.
2. There exists a $1 \in G$ such that for every $a \in G$ $1 * a = a * 1 = a$.
3. For each $a \in G$ there exists a $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = 1$.

Definition 3. Let $(G, \cdot)$ be a group. Then we say G is *abelian* if and only if for each $a, b \in G$ $ab = ba$.

Theorem 1. *Suppose $(G, \cdot)$ is a group such that there exists an $n \in \mathbb{N}$ where $(ab)^n = a^n b^n$, $(ab)^{n+1} = a^{n+1} b^{n+1}$, and $(ab)^{n+2} = a^{n+2} b^{n+2}$ for each $a, b \in G$. Then G is abelian.*

Proof. Consider $a^{n+1} b^{n+1}$.

$$a^{n+1} b^{n+1} = (ab)^{n+1} = (ab)^n (ab) = a^n b^n ab$$

If we multiply on the right by a^{-n} and the left by b^{-1}, we get

$$ab^n = b^n a.$$

Consider $a^{n+2} b^{n+2}$.

$$a^{n+2} b^{n+2} = (ab)^{n+1} = (ab)^{n+1} (ab) = a^{n+1} b^{n+1} ab$$

Again, multiplying on the right $a^{-(n+1)}$ and on the right by b^{-1} yields,

$$ab^{n+1} = b^{n+1} a.$$

We now do some manipulation.

$$\begin{aligned} ab^{n+1} &= b^{n+1} a \\ &= b(b^n a) \\ &= b(ab^n) \end{aligned}$$

Multiplying on the right by b^{-n} yields,

$$ab = ba$$

and thus G is abelian. $\square$