Trigonometric Sum and Difference Formulas in $\mathbb{C}$

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Problem. *Show that for any $z, w \in \mathbb{C}$,*

$$\sin(z+w) = \cos z \sin w + \sin z \cos w \text{ and } \cos(z+w) = \cos z \cos w - \sin z \sin w.$$

1 Background

In the world of $\mathbb{C}$, differentiability is a stronger condition than in $\mathbb{R}$. To be precise, once a function is differentiable in $\mathbb{C}$, it is infinitely differentiable. This happens because differentiability has more stringent requirements in $\mathbb{C}$. In order for a $\mathbb{C}$-valued function to be differentiable on a domain, it must be *analytic* on that domain.

Definition 1. Let $f : \mathbb{C} \to \mathbb{C}$, where $f(x+iy) = u(x,y) + iv(x,y)$, be a function such that $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, and $\partial v/\partial y$ exist and are continuous on some disk $D \subseteq \mathbb{C}$ with a nonzero radius. If f satisfies the *Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

then f is said to be *analytic on D*. Furthermore, the largest subset of $\mathbb{C}$ on which f is analytic is called f's *domain of analyticity*. If the domain of analyticity is $\mathbb{C}$, then f is said to be *entire*.

Since any analytic function is infinitely differentiable, its Taylor expansion exists as well. Furthermore, the Taylor series converges on any disk in the domain of analyticity.

Theorem 1. *Let f be analytic on a disk $D(z_0, r)$, then f's Taylor series about z_0 converges for all $z \in D$.*

2 Solution

Problem. *Show that for any $z, w \in \mathbb{C}$,*

$$\sin(z+w) = \cos z \sin w + \sin z \cos w \text{ and } \cos(z+w) = \cos z \cos w - \sin z \sin w.$$

Our solution will give $\sin z$ and $\cos z$ in terms of e^z. From there, we will show the sum formulas. However, before we can get to either of those steps, we need to show that e^z is entire.

2.1 Analyticity of e^z

Proposition 2. *e^z is entire.*

Proof. Let $z = x + iy$. We want to find $\mathbb{R}$-valued functions u, v such that $e^{x+iy} = u(x,y) + iv(x,y)$.

$$\begin{aligned} e^z &= e^{x+iy} \\ &= e^x e^{iy} \\ &= e^x \cos y + ie^x \sin y \\ &= u(x,y) + iv(x,y) \end{aligned}$$

Taking partial derivatives yields the following.

$$\begin{aligned} \partial u/\partial x &= e^x \cos y & \qquad \partial v/\partial y &= e^x \cos y \\ \partial u/\partial y &= -e^x \sin y & \qquad -\partial v/\partial x &= -e^x \sin y \end{aligned}$$

Therefore e^z satisfies the Cauchy-Riemann equations. Furthermore, since the equations hold for all $z \in \mathbb{C}$, e^z is entire. $\square$

2.2 Complex Trigonometric Functions in Terms of the Complex Exponential

Our next step is to use the Taylor expansion of e^z to get the Taylor expansion of $\cos z$ and $\sin z$.

Proposition 3. *Let $z \in \mathbb{C}$. Then*

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \text{ and } \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Proof. Consider $\left(e^{iz} + e^{-iz}\right)/2$. Since e^z is entire, its Taylor series converges everywhere in $\mathbb{C}$.

$$\begin{aligned} \frac{1}{2}\left(e^{iz} + e^{-iz}\right) &= \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{1}{n!} i^n z^n + \frac{(-1)^n}{n!} i^n z^n\right) \\ &= \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{1+(-1)^n}{n!} i^n z^n\right) \\ &= \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{1+(-1)^{2n}}{(2n)!} i^{2n} z^{2n} + \frac{1+(-1)^{2n+1}}{(2n+1)!} i^{2n+1} z^{2n+1}\right) \\ &= \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{2}{(2n)!} (i^2)^n z^{2n}\right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n!)} z^{2n} \\ &= \cos z \end{aligned}$$

Consider $\left(e^{iz} - e^{-iz}\right)/(2i)$.

$$\begin{aligned} \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) &= \frac{1}{2i}\left(\sum_{n=0}^{\infty} \frac{1}{n!} i^n z^n - \frac{(-1)^n}{n!} i^n z^n\right) \\ &= \frac{1}{2i}\left(\sum_{n=0}^{\infty} \frac{1-(-1)^n}{n!} i^n z^n\right) \\ &= \frac{1}{2i}\left(\sum_{n=0}^{\infty} \frac{1-(-1)^{2n}}{(2n)!} i^{2n} z^{2n} + \frac{1-(-1)^{2n+1}}{(2n+1)!} i^{2n+1} z^{2n+1}\right) \\ &= \frac{1}{2i}\left(\sum_{n=0}^{\infty} \frac{2i}{(2n+1)!} (i^2)^n z^{2n+1}\right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \\ &= \sin z \end{aligned}$$

$\square$

2.3 Trigonometric Sum Identities

We will now use our new identities to show the sum formulas, and get the difference formulas as corollaries.

Theorem 4. *Let* $z, w \in \mathbb{C}$. *Then* $\sin(z+w) = \cos z \sin w + \sin z \cos w$.

Proof. Consider $\cos z \sin w + \sin z \cos w$.

$$\begin{aligned}
\cos z \sin w + \sin z \cos w &= \left(\frac{e^{iz}+e^{-iz}}{2}\right)\left(\frac{e^{iw}-e^{-iw}}{2i}\right) + \left(\frac{e^{iz}-e^{-iz}}{2i}\right)\left(\frac{e^{iw}+e^{-iw}}{2}\right) \\
&= \frac{e^{i(z+w)}+e^{i(w-z)}-e^{i(z-w)}-e^{-i(z+w)}}{4i} + \frac{e^{i(z+w)}-e^{i(w-z)}+e^{i(z-w)}-e^{-i(z+w)}}{4i} \\
&= \frac{2e^{i(z+w)}-2e^{-i(z+w)}}{4i} \\
&= \frac{e^{i(z+w)}-e^{-i(z+w)}}{2i} \\
&= \sin(z+w)
\end{aligned}$$

□

Theorem 5. *Let* $z, w \in \mathbb{C}$. *Then* $\cos(z+w) = \cos z \cos w - \sin z \sin w$.

Proof. Consider $\cos z \cos w - \sin z \sin w$.

$$\begin{aligned}
\cos z \cos w - \sin z \sin w &= \left(\frac{e^{iz}+e^{-iz}}{2}\right)\left(\frac{e^{iw}+e^{-iw}}{2}\right) - \left(\frac{e^{iz}-e^{-iz}}{2i}\right)\left(\frac{e^{iw}-e^{-iw}}{2i}\right) \\
&= \frac{e^{i(z+w)}+e^{i(w-z)}+e^{i(z-w)}+e^{-i(z+w)}}{4} + \frac{e^{i(z+w)}-e^{i(w-z)}-e^{i(z-w)}+e^{-i(z+w)}}{4} \\
&= \frac{e^{i(z+w)}+e^{-i(z+w)}}{2} \\
&= \cos(z+w)
\end{aligned}$$

□

Corollary 6. *Let* $z, w \in \mathbb{C}$. *Then*

$$\sin(z-w) = \sin z \cos w - \cos z \sin w \ \textit{ and } \ \cos(z-w) = \cos z \cos w + \sin z \sin w.$$

Proof. Since $\sin w$ is an odd function, $\sin(-w) = -\sin w$. Since $\cos w$ is an even function, $\cos(-w) = \cos w$. These facts combined with the sum formulas for $\cos(z+(-w))$ and $\sin(z+(-w))$ yield the difference formulae. □

References

[1] John M. Howie. *Complex Analysis*. Springer. ISBN: 978-1-85233-733-9.