Cauchy-Schwarz Inequality

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Theorem. *Prove that*

$$\frac{(a+b)^2}{x+y} \leq \frac{a^2}{x} + \frac{b^2}{y}$$

for $a, b, x, y > 0$. Use that to the prove the Cauchy-Schwarz inequality,

$$\left(\sum_{i=0}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=0}^{n} a_i^2\right)\left(\sum_{i=0}^{n} b_i^2\right)$$

where $\{a_i\}_{i=0}^n, \{b_i\}_{i=0}^n \subset \mathbb{R}^+$.

To begin, we need to prove the following lemma.

Lemma 1. *Let $a, b, x, y > 0$, then*

$$\frac{(a+b)^2}{x+y} \leq \frac{a^2}{x} + \frac{b^2}{y}.$$

Proof. Proceed via contradiction. Assume that

$$\frac{(a+b)^2}{x+y} > \frac{a^2}{x} + \frac{b^2}{y}.$$

Then,

$$\frac{(a+b)^2}{x+y} - \frac{a^2}{x} - \frac{b^2}{y} > 0.$$

Multiplying out the denominators yields

$$xy(a+b)^2 - y(x+y)a^2 - x(x+y)b^2 > 0$$

and thus

$$xya^2 + 2abxy + xyb^2 - xya^2 - y^2a^2 - xyb^2 - x^2b^2 > 0.$$

Therefore

$$-\left((ya)^2 - 2abxy + (xb)^2\right) > 0$$

and

$$(ya - xb)^2 = (ya)^2 - 2abxy + (xb)^2 < 0.$$

However, $(ya - xb)^2 \geq 0$ since $ya - xb \in \mathbb{R}$. This is a contradiction, and thus

$$\frac{(a+b)^2}{x+y} \leq \frac{a^2}{x} + \frac{b^2}{y}.$$

□

We now generalize the previous lemma.

Lemma 2. *Let $a_i, x_i > 0$ for all i such that $0 \leq i \leq n$. Then,*

$$\frac{\left(\sum_{i=0}^{n} a_i\right)^2}{\sum_{i=0}^{n} x_i} \leq \sum_{i=0}^{n} \frac{a_i^2}{x_i}.$$

Proof. Proceed via induction. Let $H(k)$ represent the hypothesis that the previous statement is true for $n = k$. By Lemma 1, $H(1)$ is true. Assume $H(n)$ is true. We want to show that $H(n+1)$ is true. By $H(1)$, we know that

$$\frac{\left(\sum_{i=0}^{n+1} a_i\right)^2}{\sum_{i=0}^{n+1} x_i} \leq \frac{\left(\sum_{i=0}^{n} a_i\right)^2}{\sum_{i=0}^{n} x_i} + \frac{a_{n+1}^2}{x_{n+1}}.$$

By $H(n)$

$$\frac{\left(\sum_{i=0}^{n+1} a_i\right)^2}{\sum_{i=0}^{n+1} x_i} \leq \frac{\left(\sum_{i=0}^{n} a_i\right)^2}{\sum_{i=0}^{n} x_i} + \frac{a_{n+1}^2}{x_{n+1}} =\leq \sum_{i=0}^{n} \frac{a_i^2}{x_i} + \frac{a_{n+1}^2}{x_{n+1}} = \sum_{i=0}^{n+1} \frac{a_i^2}{x_i}.$$

Thus $H(n+1)$ is true and by induction $H(k)$ is true for all k. $\square$

This leads us to the Cauchy-Schwarz inequality.

Theorem 3. *Let $\{a_i\}_{i=0}^n, \{b_i\}_{i=0}^n \subset \mathbb{R}^+$, then*

$$\left(\sum_{i=0}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=0}^{n} a_i^2\right)\left(\sum_{i=0}^{n} b_i^2\right).$$

Proof. Consider the right hand side.

$$\left(\sum_{i=0}^{n} a_i^2\right)\left(\sum_{j=0}^{n} b_j^2\right) = \sum_{k=0}^{2n} \sum_{i+j=k} a_i^2 b_j^2$$

Since these are all nonnegative, we can drop the odd terms

$$\left(\sum_{i=0}^{n} a_i^2\right)\left(\sum_{j=0}^{n} b_j^2\right) = \sum_{k=0}^{2n} \sum_{i+j=k} a_i^2 b_j^2 \geq \sum_{k=0}^{n} \sum_{i+j=2k} a_i^2 b_j^2.$$

Furthermore, $k + k = 2k$ and thus

$$\left(\sum_{i=0}^{n} a_i^2\right)\left(\sum_{j=0}^{n} b_j^2\right) \geq \sum_{k=0}^{n} \sum_{i+j=2k} a_i^2 b_j^2 \geq \sum_{k=0}^{n} a_k^2 b_k^2.$$

Consider

$$\left(\sum_{i=0}^{n} a_i b_i\right)^2.$$

Then,

$$\left(\sum_{i=0}^{n} a_i b_i\right)^2 = \frac{\left(\sum_{i=0}^{n} a_i b_i\right)^2}{\sum_{i=0}^{n} \frac{1}{n+1}}.$$

If we invoke Lemma 2,

$$\left(\sum_{i=0}^{n} a_i b_i\right)^2 = \frac{\left(\sum_{i=0}^{n} a_i b_i\right)^2}{\sum_{i=0}^{n} \frac{1}{n+1}} \leq \sum_{i=0}^{n} a_i^2 b_i^2 (n+1) \leq \sum_{i=0}^{n} a_i^2 b_i^2 \leq \left(\sum_{i=0}^{n} a_i^2\right)\left(\sum_{i=0}^{n} b_i^2\right).$$

$\square$