

Trigonometric Sum and Difference Formulae in \mathbb{C}

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June 2016

Problem. Show that for any $z, w \in \mathbb{C}$,

$$\sin(z + w) = \cos z \sin w + \sin z \cos w \text{ and } \cos(z + w) = \cos z \cos w - \sin z \sin w.$$

1 Background

In the world of \mathbb{C} , differentiability is a stronger condition than in \mathbb{R} . To be precise, once a function is differentiable in \mathbb{C} , it is infinitely differentiable. This happens because differentiability has more stringent requirements in \mathbb{C} . In order for a \mathbb{C} -valued function to be differentiable on a domain, it must be *analytic* on that domain.

Definition 1. Let $f : \mathbb{C} \rightarrow \mathbb{C}$, where $f(x + iy) = u(x, y) + iv(x, y)$, be a function such that $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, and $\partial v/\partial y$ exist and are continuous on some disk $D \subseteq \mathbb{C}$ with a nonzero radius. If f satisfies the *Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

then f is said to be *analytic on D* . Furthermore, the largest subset of \mathbb{C} on which f is analytic is called f 's *domain of analyticity*. If the domain of analyticity is \mathbb{C} , then f is said to be *entire*.

Since any analytic function is infinitely differentiable, its Taylor expansion exists as well. Furthermore, the Taylor series converges on any disk in the domain of analyticity.

Theorem 1. Let f be analytic on a disk $D(z_0, r)$, then f 's Taylor series about z_0 converges for all $z \in D$.

2 Solution

Problem. Show that for any $z, w \in \mathbb{C}$,

$$\sin(z + w) = \cos z \sin w + \sin z \cos w \text{ and } \cos(z + w) = \cos z \cos w - \sin z \sin w.$$

To prove these statements, we will find $\sin z$ and $\cos z$ in terms of e^z . From there, we will show the sum formulae. However, before we can get to either of those steps, we need to show that e^z is entire.

2.1 Analyticity of e^z

Proposition 2. e^z is entire.

Proof. Let $z = x + iy$. We want to find \mathbb{R} -valued functions u, v such that $e^{x+iy} = u(x, y) + iv(x, y)$.

$$\begin{aligned} e^z &= e^{x+iy} \\ &= e^x e^{iy} \\ &= e^x \cos y + ie^x \sin y \\ &= u(x, y) + iv(x, y) \end{aligned}$$

Taking partial derivatives yields the following.

$$\begin{aligned}\frac{\partial u}{\partial x} &= e^x \cos y & \frac{\partial v}{\partial y} &= e^x \cos y \\ \frac{\partial u}{\partial y} &= -e^x \sin y & -\frac{\partial v}{\partial x} &= -e^x \sin y\end{aligned}$$

Therefore e^z satisfies the Cauchy-Riemann equations. Furthermore, since the equations hold for all $z \in \mathbb{C}$, e^z is entire. \square

2.2 Complex Trigonometric Functions in Terms of the Complex Exponential

Our next step is to use the Taylor expansion of e^z to get the Taylor expansion of $\cos z$ and $\sin z$.

Proposition 3. *Let $z \in \mathbb{C}$. Then*

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \text{ and } \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Proof. Consider $(e^{iz} + e^{-iz})/2$. Since e^z is entire, its Taylor series converges everywhere in \mathbb{C} .

$$\begin{aligned}\frac{1}{2}(e^{iz} + e^{-iz}) &= \frac{1}{2} \left(\sum_{n=0}^{\infty} \frac{1}{n!} i^n z^n + \frac{(-1)^n}{n!} i^n z^n \right) \\ &= \frac{1}{2} \left(\sum_{n=0}^{\infty} \frac{1 + (-1)^n}{n!} i^n z^n \right) \\ &= \frac{1}{2} \left(\sum_{n=0}^{\infty} \frac{1 + (-1)^{2n}}{(2n)!} i^{2n} z^{2n} + \frac{1 + (-1)^{2n+1}}{(2n+1)!} i^{2n+1} z^{2n+1} \right) \\ &= \frac{1}{2} \left(\sum_{n=0}^{\infty} \frac{2}{(2n)!} (i^2)^n z^{2n} \right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} \\ &= \cos z\end{aligned}$$

Consider $(e^{iz} - e^{-iz})/(2i)$.

$$\begin{aligned}\frac{1}{2i}(e^{iz} - e^{-iz}) &= \frac{1}{2i} \left(\sum_{n=0}^{\infty} \frac{1}{n!} i^n z^n - \frac{(-1)^n}{n!} i^n z^n \right) \\ &= \frac{1}{2i} \left(\sum_{n=0}^{\infty} \frac{1 - (-1)^n}{n!} i^n z^n \right) \\ &= \frac{1}{2i} \left(\sum_{n=0}^{\infty} \frac{1 - (-1)^{2n}}{(2n)!} i^{2n} z^{2n} + \frac{1 - (-1)^{2n+1}}{(2n+1)!} i^{2n+1} z^{2n+1} \right) \\ &= \frac{1}{2i} \left(\sum_{n=0}^{\infty} \frac{2i}{(2n+1)!} (i^2)^n z^{2n+1} \right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \\ &= \sin z\end{aligned}$$

\square

2.3 Trigonometric Sum Identities

We will now use our new identities to show the sum formulae. Afterwards, we will get the difference formulae as corollaries.

Theorem 4. *Let $z, w \in \mathbb{C}$. Then $\sin(z + w) = \cos z \sin w + \sin z \cos w$.*

Proof. Consider $\cos z \sin w + \sin z \cos w$.

$$\begin{aligned}
 \cos z \sin w + \sin z \cos w &= \left(\frac{e^{iz} + e^{-iz}}{2} \right) \left(\frac{e^{iw} - e^{-iw}}{2i} \right) + \left(\frac{e^{iz} - e^{-iz}}{2i} \right) \left(\frac{e^{iw} + e^{-iw}}{2} \right) \\
 &= \frac{e^{i(z+w)} + e^{i(w-z)} - e^{i(z-w)} - e^{-i(z+w)}}{4i} + \frac{e^{i(z+w)} - e^{i(w-z)} + e^{i(z-w)} - e^{-i(z+w)}}{4i} \\
 &= \frac{2e^{i(z+w)} - 2e^{-i(z+w)}}{4i} \\
 &= \frac{e^{i(z+w)} - e^{-i(z+w)}}{2i} \\
 &= \sin(z + w)
 \end{aligned}$$

□

Theorem 5. *Let $z, w \in \mathbb{C}$. Then $\cos(z + w) = \cos z \cos w - \sin z \sin w$.*

Proof. Consider $\cos z \cos w - \sin z \sin w$.

$$\begin{aligned}
 \cos z \cos w - \sin z \sin w &= \left(\frac{e^{iz} + e^{-iz}}{2} \right) \left(\frac{e^{iw} + e^{-iw}}{2} \right) - \left(\frac{e^{iz} - e^{-iz}}{2i} \right) \left(\frac{e^{iw} - e^{-iw}}{2i} \right) \\
 &= \frac{e^{i(z+w)} + e^{i(w-z)} + e^{i(z-w)} + e^{-i(z+w)}}{4} + \frac{e^{i(z+w)} - e^{i(w-z)} - e^{i(z-w)} + e^{-i(z+w)}}{4} \\
 &= \frac{e^{i(z+w)} + e^{-i(z+w)}}{2} \\
 &= \cos(z + w)
 \end{aligned}$$

□

Corollary 6. *Let $z, w \in \mathbb{C}$. Then*

$$\sin(z - w) = \sin z \cos w - \cos z \sin w \text{ and } \cos(z - w) = \cos z \cos w + \sin z \sin w.$$

Proof. Since $\sin w$ is an odd function, $\sin(-w) = -\sin w$. Since $\cos w$ is an even function, $\cos(-w) = \cos w$. These facts combined with the sum formulae for $\cos(z + (-w))$ and $\sin(z + (-w))$ yield the difference formulae. □

References

- [1] John M. Howie. *Complex Analysis*. Springer. ISBN: 978-1-85233-733-9.