Generating Monoids from Categories

Matt McCarthy

June 2016

Theorem. *Let C be a category and let A be an object in C. Then $C[A, A]$ forms a monoid under arrow composition.*

1 Background

2 Solution

Theorem 1. *Let C be a category and let A be an object in C. Then $C[A, A]$ forms a monoid under arrow composition.*

Proof. Let $f, g \in C[A, A]$. Then

$$A \xrightarrow{f} A \xrightarrow{g} A$$

and thus

$$A \xrightarrow{fg} A.$$

Therefore, arrow composition forms a binary operation on $C[A, A]$.
Next, we claim that id_A is the identity for $C[A, A]$ with respect to arrow composition. Let $f \in C[A, A]$. Then, by definition, we know that $f\, id_A = id_A\, f = f$. Thus, id_A is the identity for $C[A, A]$ with respect to arrow composition.
Lastly, we must show that arrow composition is associative for all arrows in $C[A, A]$. Let $f, g, h \in C[A, A]$. Consider $f(gh)$.

$$A \xrightarrow{f(gh)} A = A \xrightarrow{gh} A \xrightarrow{f} = A \xrightarrow{h} A \xrightarrow{g} A \xrightarrow{f} A$$

□