Weil Height and the p-adic Numbers

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1 Introduction

Height functions generally measure the complexity of a number. In particular, the Weil height measure the complexity of a rational number as follows.

Definition 1.1. Let $x = m/n \in \mathbb{Q}$, then the *Weil height of* x, denoted $h(x)$, is defined as

$$h(x) = \max\{|m|, |n|\}.$$

As nice as this height function is, we would like to have a natural extension to other spaces, like the algebraics, where measuring the Euclidean distance of the numerators and denominators makes less sense.

2 Background

Before we extend the Weil height, we introduce the concept of the p-adic ordinal and norm.

Definition 2.1 (p-adic Ordinal [1])**.** Let p be a prime and let $a \in \mathbb{Z}$. Then the *p-adic ordinal* of a, denoted $\operatorname{ord}_p a$, is defined as

$$\operatorname{ord}_p a = \max\{n \text{ s.t. } p^n | a\}.$$

Furthermore, for any $x = b/c \in \mathbb{Q}$,

$$\operatorname{ord}_p x = \operatorname{ord}_p a - \operatorname{ord}_p b.$$

Using the definition of p-adic ordinal, we now provide the definition of the p-adic norm.

Definition 2.2 (p-adic Norm [1])**.** Let p be a prime and $x \in \mathbb{Q}$. Then the *p-adic norm of* x, denoted $|x|_p$, is defined as

$$|x|_p = \begin{cases} p^{-\operatorname{ord}_p x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

3 Problem

We want to show that following redefinition of the Weil height is equivalent to Definition 1.1

Theorem 3.1. *Let* $\alpha = n/d \in \mathbb{Q}$ *with* $n \neq 0$ *and* $\gcd(n, d) = 1$*. Define* $h(\alpha) = \left(\prod_p \max\{1, |\alpha|_p\}\right)(\max\{1, |\alpha|\})$*. Then* $h(\alpha) = \max\{|n|, |d|\}$*.*

Proof. Define $r(\alpha) := \prod_p \max\{1, |\alpha|_p\}$, then $h(\alpha) = r(\alpha) \cdot (\max\{1, |\alpha|\})$. Since $|n|, |d| \in \mathbb{Z}^+$, $|n| = \prod_{i=1}^k p_i^{n_i}$ and $|d| = \prod_{i=1}^j q_i^{m_i}$ where $\{p_i\}$ and $\{q_i\}$ are finite sets of prime numbers and $\{n_i\}, \{m_i\} \subset \mathbb{N}$. Furthermore, $\{p_i\} \cap \{q_i\} = \emptyset$ since $\gcd(n, d) = 1$. Therefore,

$$\begin{aligned} r(\alpha) &= r\left(\left(\prod_{p_i} p_i^{n_i}\right)\left(\prod_{q_i} q_i^{-m_i}\right)\right) \\ &= \left(\prod_{p_i} \max\{1, |\alpha|_{p_i}\}\right)\left(\prod_{q_i} \max\{1, |\alpha|_{q_i}\}\right) \\ &= \left(\prod_{p_i} \max\{1, p_i^{-n_i}\}\right)\left(\prod_{q_i} \max\{1, q_i^{m_i}\}\right) \\ &= 1 \cdot \prod_{q_i} q_i^{m_i} \\ &= |d|. \end{aligned}$$

Thus,

$$h(\alpha) = |d| \cdot \max\{1, |n/d|\}.$$

If $n < d$,

$$h(\alpha) = |d| \cdot 1 = \max\{|n|, |d|\}.$$

If $n = d$,

$$h(\alpha) = |d| = \max\{|n|, |d|\}.$$

Otherwise, $n > d$ and

$$h(\alpha) = |d|\frac{|n|}{|d|} = |n| = \max |n|, |d|.$$

□

Analysis. We begin by splitting up h into the product of p-adic norms and the Euclidean norm. From there, we notice that the product of p-adic norms is equal to $|d|$. To prove this, we use the fact that the prime factorizations of n and d are finite and disjoint, we can break $r(\alpha)$ into the product of $r(n)$ and $r(1/d)$. When we consider $r(n)$ we see that $|n|_p \leq 1$ for all primes p and thus $\max\{1, |n|_p\} = 1$. When we consider $r(1/d)$, we see that $|1/d|_p \geq 1$ for all p and thus, $\max\{1, |1/d|_p\} = |1/d|_p$. For any prime not in d's prime factorization, $|1/d|_p = 1$, for those in d's prime factorization, $|1/d|_p = p^{n_p} = \operatorname{ord}_p d$. Thus, $r(1/d) = |d|$. We then consider cases to get the result.

References

[1] Neal Koblitz. *p-adic Numbers, p-adic Analysis, and Zeta-Functions*. Springer. ISBN: 9781461270140.