Sum of the Natural Numbers and the Riemann-Zeta Function

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1 Introduction

We begin by introducing a rather interesting result, namely that the sum of the natural numbers is $-1/12$.

Theorem 1.1.

$$\sum_{n=1}^{\infty} n = -\frac{1}{12}$$

We now provide the canonical proof as shown by [2].

Proof (Theorem 1.1). Define $s := 1 + 2 + 3 + 4 + \ldots$, $s_1 := \sum_{n=0}^{\infty}(-1)^n$, and $s_2 := 1 - 2 + 3 - 4 + \ldots$. If we consider s_1, when we stop after an odd number of terms, the partial sum is 1 but when we stop after an even number of terms, the partial sum is zero. Obviously, the series converges to the average of the two, thus $s_1 = 1/2$. Consider $2s_2$.

$$\begin{array}{cccccccc} & 1 & -2 & +3 & -4 & +\ldots & & \\ + & & 1 & -2 & +3 & -4 & +\ldots & \\ \hline & 1 & -1 & +1 & -1 & +\ldots & & \end{array}$$

Since $2s_2 = s_1 = 1/2$, $s_2 = 1/4$. Consider $s - s_2$.

$$\begin{array}{cccccc} & 1 & +2 & +3 & +4 & +\ldots \\ - & 1 & -2 & +3 & -4 & +\ldots \\ \hline & 0 & +4 & +0 & +8 & +\ldots \end{array}$$

Moreover, we can factor out the four to get $s - s_2 = 4[1+2+3+\ldots] = 4s$ Solving for s then yields $s = -1/12$ and completes the proof. $\square$

Analysis. The trick to this proof lies in exploiting the commutativity of addition. By doing so, we can manipulate the subtraction of two infinite series into a single alternating series. After that step, we then use the Taylor series we found in the lemma to complete the proof.

Now that we have "shown" that the natural numbers sum up to $-1/12$ the only things left to do are show that it is wrong and how this incorrect result came to be so prolific.

2 Refutation

We begin refuting this result by stating two of the limit laws and by providing the definition of an infinite series.

Theorem 2.1 (Limit Laws)**.** *Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be real-valued sequences that converge to x and y respectively. Then all of the following hold.*

1. *The sequence* $\{x_n + y_n\}_{n=1}^{\infty}$ *converges to* $x + y$.

2. *Let* $k \in \mathbb{R}$ *then* $\{k \cdot x_n\}_{n=1}^{\infty}$ *converges to* $k \cdot x$.

These laws should seem familiar, considering that we used both in our "proof" of Theorem 1.1 however we may have invoked them without respecting their hypotheses.

Definition 2.1 (Infinite Series)**.** Let $\{x_k\}_{k=1}^{\infty}$ be a sequence of real numbers and let $x \in \mathbb{R}$. An *infinite series* is an expression of the form

$$\sum_{k=1}^{\infty} x_k = x_1 + x_2 + \dots.$$

The corresponding *sequence of partial sums* is defined by

$$s_n := \sum_{k=1}^{n} x_k.$$

The series, $\sum_{k=1}^{\infty} x_k$ *converges* to x if and only if the sequence $\{s_n\}_{n=1}^{\infty}$ converges to x.

This definition tells us that an infinite sum exists if and only if the its corresponding sequence of partial sums converges, which leads us to our biggest error: assuming the sum exists.

To determine whether or not the sum exists, we need to consider the sequence of partial sums,

$$s_n := \sum_{k=1}^{n} k.$$

If we inspect the sequence long enough, we see that the sum only increases and is what we call a *monotone increasing sequence.*

Definition 2.2 (Monotone Increasing Sequence)**.** Let $\{x_n\}_{n=1}^{\infty}$ be a real-valued sequence. We say $\{x_n\}$ is *monotone increasing* if and only if $x_n \leq x_{n+1}$ for each $n \in \mathbb{N}$.

If we consider the $(n+1)$ partial sum, we see

$$s_{n+1} = \sum_{k=1}^{n+1} k = \left(\sum_{k=1}^{n} k\right) + (n+1) \geq \sum_{k=1}^{n} k = s_n.$$

Thus, $\{s_n\}$ is a monotone increasing sequence.

Since s_n is monotone increasing, we have a very nice theorem that turns our limit problem into a boundedness problem.

Definition 2.3 (Upper Bound)**.** Let $\{x_n\}_{n=1}^{\infty}$ be a real valued sequence and let $b \in \mathbb{R}$. We say b is an *upper bound* of $\{x_n\}$ if and only if $x_n \leq b$ for each $n \in \mathbb{N}$. If any such b exists, we say $\{x_n\}$ is *bounded above.*

Theorem 2.2 (Monotone Convergence Theorem)**.** *Let* $\{x_n\}_{n=1}^{\infty}$ *be a real-valued monotone increasing sequence. Then* $\{x_n\}$ *converges if and only if* $\{x_n\}$ *is bounded above.*

With this theorem in our arsenal, we will now show that this infamous sum does not converge.

Proposition 2.3. *The sequence defined by*

$$s_n := \sum_{k=1}^{n} k$$

is unbounded.

Proof. Assume that s_n is bounded above. Ergo there exists a natural number $N > 0$ such that $s_n := \sum_{k=1}^{n} k \leq N$ for every $n \in \mathbb{N}$. We know that

$$s_n := \sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

Consider s_{2N}.

$$s_{2N} = \frac{2N(2N+1)}{2} = N(2N+1) = 2N^2 + N > N$$

The previous statement contradicts our assumption and thus s_n is an unbounded sequence. □

Analysis. We perform a standard contradiction proof. The only tricks we use are the fact that

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

and a careful selection of n. Essentially, the proof boils down to picking a candidate for a bound and showing that we have a term in the sequence that is greater than our bound.

Since s_n is unbounded, by the monotone convergence theorem, s_n does not converge and neither does our sum. Thus, we cannot invoke limit laws in order to compute $-4s$ and $s - 4s$ in our proof of Theorem 1.1. In turn, this invalidates our derivation of $-3s$ which destroys the rest of the proof.

3 The Riemann-Zeta Function

To figure out why people believe this nonsense, we turn to the Riemann-Zeta function. Historically, the Riemann-Zeta function arose as a way to extend the Euler-Zeta function, defined on real numbers greater than 1

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

to the rest of the complex plane [1]. When viewed in this light, one may think that

$$\zeta(-1) = \sum_{n=1}^{\infty} n.$$

However, this is not the case since the Riemann-Zeta function is defined as follows.

Definition 3.1 (Riemann-Zeta Function from [1])**.** The *Riemann-Zeta function* is the map $\zeta : \mathbb{C} \to \mathbb{C}$ defined by the contour integral

$$\zeta(s) := \frac{\Pi(-s)}{2\pi i} \int_{\gamma} \frac{(-x)^s}{(e^x - 1)x} dx$$

where γ is a curve that starts at $+\infty$, moves towards the origin along the positive real axis, circles the origin in a counterclockwise direction, and returns to $+\infty$ along the positive real axis. Furthermore, $\Pi(s)$ is defined as

$$\Pi(s) := \int_{0}^{\infty} e^{-x} x^s dx.$$

As we can see from the definition, this function does not resemble the geometric series in any way. However, if s is real and greater than 1 (note: strictly greater than, not greater than or equal to) the zeta function can be written as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1}$$

Furthermore, since -1 is less than 1,

$$\zeta(-1) \neq \sum_{n=1}^{\infty} n.$$

Another interesting property of the Riemann-Zeta function pops up when we let s be a negative integer.

Proposition 3.1 (From [1])**.** *For any natural number n,*

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}$$

where B_{n+1} is the $n+1$th Bernoulli number.

By using this property we get

$$\zeta(-1) = (-1)\frac{B_2}{2} = (-1)\left(\frac{1}{6}\right)\left(\frac{1}{2}\right) = -\frac{1}{12}. \tag{2}$$

Very likely, the incorrect conclusion that $1+2+3+\ldots = -1/12$ came out of some confusion arising from Equation 1, Equation 2, and the history of the zeta function.

References

[1] H. M. Edwards. *Riemann's Zeta Function*. Dover. ISBN: 9780486417400.

[2] Numberphile. *ASTOUNDING: 1 + 2 + 3 + 4 + 5 + ... = -1/12*. URL: `https://www.youtube.com/watch?v=w-I6XTVZXww`.