

# Completeness of $\mathbb{R}$

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**Theorem.** *The real numbers form a complete metric space.*

We denote the set of non-negative real numbers as  $\mathbb{R}^+$ .

**Definition 1** (Metric). Let  $X$  be a set and  $d : X \times X \rightarrow \mathbb{R}^+$  be a map. We say  $d$  is a *metric* if and only if all of the following hold.

1. For all  $x, y \in X$ ,  $d(x, y) = 0$  if and only if  $x = y$ .
2. For all  $x, y \in X$ ,  $d(x, y) = d(y, x)$  (symmetric property).
3. For all  $x, y, z \in X$ ,  $d(x, z) \leq d(x, y) + d(y, z)$  (triangle inequality).

If  $d$  is a metric on  $X$ , we say  $(X, d)$  forms a *metric space*.

For example,  $(\mathbb{Q}, d)$  and  $(\mathbb{R}, d)$  where  $d(x, y) = |x - y|$  is a metric space, and more generally,  $(\mathbb{R}^n, d_n)$  where  $d_n$  is the Euclidean distance is also a metric space.

**Definition 2** (Convergent Sequence). Let  $(X, d)$  be a metric space and let  $\{a_n\}_{n \in \mathbb{N}}$  be a sequence in  $X$ . We say  $\{a_n\}_{n \in \mathbb{N}}$  *converges* to some  $a \in X$ , denoted  $a_n \rightarrow a$  if and only if for all  $\varepsilon > 0$  there exists a  $N_\varepsilon \in \mathbb{N}$  such that  $d(a_n, a) < \varepsilon$  for all  $n \geq N_\varepsilon$ .

**Definition 3** (Cauchy Sequence). Let  $(X, d)$  be a metric space and let  $\{a_n\}_{n \in \mathbb{N}}$  be a sequence in  $X$ . We say  $\{a_n\}_{n \in \mathbb{N}}$  is a *Cauchy sequence* if and only if for any  $\varepsilon > 0$ , there exists a  $N_\varepsilon \in \mathbb{N}$  such that  $d(a_n, a_m) < \varepsilon$  for all  $n, m \geq N_\varepsilon$ .

Note, however that Cauchy sequences are not always convergent in their space. For example consider the sequence given by

$$a_n = \left(1 + \frac{1}{n}\right)^n.$$

We know these terms are rational and thus  $a_n$  is a  $\mathbb{Q}$ -valued sequence. Furthermore, the sequence is in fact Cauchy. However, the limit of this sequence as  $n$  approaches infinity is  $e$  and thus, this Cauchy sequence does not converge in  $\mathbb{Q}$ , but it does in  $\mathbb{R}$ .

This leads us to a notion of *completeness* in a metric space.

**Definition 4** (Complete Metric Space). Let  $(X, d)$  be a metric space. Then we say  $(X, d)$  is *complete* if and only if every Cauchy sequence in  $X$  converges in  $X$ .

By this definition of completeness,  $\mathbb{Q}$  is not complete as shown by our prior example. However,  $\mathbb{R}$  is complete.

Before we get into any theorems here, we need a notion of *boundedness* for subsets of the real numbers.

**Definition 5** (Bounded). Let  $S$  be a subset of the real numbers. Then  $S$  is *bounded* if and only if there exists an  $M > 0$  such that  $S \subseteq [-M, M]$ .

**Theorem 1** (Bolzano-Weierstrass). *Let  $\{a_n\}_{n \in \mathbb{N}}$  be a bounded  $\mathbb{R}$ -valued sequence. Then there exists a  $\{a_{n_k}\} \subseteq \{a_n\}$  such that  $a_{n_k}$  converges to some  $a \in \mathbb{R}$ .*

*Proof.* □

**Lemma 2.** *Any  $\mathbb{R}$ -valued Cauchy sequence is bounded.*

*Proof.* Let  $\varepsilon > 0$  be given and let  $\{a_n\}_{n \in \mathbb{N}}$  be a Cauchy sequence in  $\mathbb{R}$ . Therefore, there exists an  $N_\varepsilon$  such that  $|a_n - a_m| < \varepsilon$  for all  $n, m \geq N_\varepsilon$ . Therefore, for all  $n \geq N_\varepsilon$ ,  $|a_{N_\varepsilon} - a_n| < \varepsilon$ . Thus for each  $n \geq N_\varepsilon$ ,  $a_n \in [a_{N_\varepsilon} - \varepsilon, a_{N_\varepsilon} + \varepsilon]$ . Take  $M_1 = \max\{a_{N_\varepsilon} - \varepsilon, a_{N_\varepsilon} + \varepsilon\}$ , which yields  $\{a_n\}_{n=N_\varepsilon}^\infty \subset [-M_1, M_1]$ .

Furthermore, since  $\{a_n\}_{n=1}^{N_\varepsilon-1}$  is finite, take  $M_2 = \max\{|a_n|\}_{n=1}^{N_\varepsilon-1}$ . Thus,  $\{a_n\}_{n=1}^{N_\varepsilon-1} \subset [-M_2, M_2]$ . If we take  $M = \max\{M_1, M_2\}$ , then we have  $\{a_n\}_{n \in \mathbb{N}} \subset [-M, M]$ . Therefore, the sequence is bounded. □

**Theorem 3.** *The real numbers form a complete metric space.*

*Proof.* Let  $\{a_n\}_{n \in \mathbb{N}}$  be Cauchy in  $\mathbb{R}$ . Let  $\varepsilon > 0$  be given. Since  $\{a_n\}_{n \in \mathbb{N}}$  is Cauchy there exists an  $N_{\varepsilon/2} \in \mathbb{N}$  such that  $|a_n - a_m| < \varepsilon/2$  for all  $n, m \geq N_{\varepsilon/2}$ . Furthermore,  $\{a_n\}_{n \in \mathbb{N}}$  is bounded and thus by Bolzano-Weierstrass, there exists a  $\{a_{n_k}\}_{k \in \mathbb{N}} \subset \{a_n\}_{n \in \mathbb{N}}$  such that  $a_{n_k}$  converges to some  $a \in \mathbb{R}$ .

We claim that  $a_n$  converges to  $a$ . Since  $a_{n_k}$  converges to  $a$ , there exists a  $K_{\varepsilon/2} \in \mathbb{N}$  such that  $|a_{n_k} - a| < \varepsilon/2$  for all  $k \geq K_{\varepsilon/2}$ . Take  $N = \max\{N_{\varepsilon/2}, n_{K_{\varepsilon/2}}\}$  and force  $n, k$  large enough such that  $n, n_k \geq N$ . Consider  $|a_n - a|$ .

$$|a_n - a| = |a_n - a_{n_k} + a_{n_k} - a| \leq |a_n - a_{n_k}| + |a_{n_k} - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus for any  $\varepsilon > 0$ , there exists an  $N \in \mathbb{N}$  such that  $|a_n - a| < \varepsilon$  for all  $n \geq N$ . Therefore  $a_n$  converges to  $a$  and  $\mathbb{R}$  is a complete metric space. □