Vector Space Isomorphisms

Matt McCarthy

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Theorem 1. *Let V, W be vector spaces over a field $\mathbb{F}$ and $\tau : V \to W$ be a homomorphism. Then τ is an isomorphism if and only if τ takes a basis of V to a basis of W.*

Proof. Let B be a basis of V.

$\Rightarrow$ Assume τ is an isomorphism. We want to show $\tau(B)$ is a basis of W. Therefore we need to show that $\tau(B)$ is linearly independent and spans W. Let $\{\tau(v_1), \tau(v_2), \ldots, \tau(v_n)\} \subseteq \tau(B)$ and $\{a_1, a_2, \ldots, a_n\} \subseteq \mathbb{F}$ such that

$$a_1\tau(v_1) + a_2\tau(v_2) + \ldots + a_n\tau(v_n) = 0.$$

By properties of vector space homomorphisms, we know that

$$\tau(a_1v_1 + \ldots + a_nv_n) = 0$$

and thus,

$$a_1v_1 + \ldots + a_nv_n \in \ker \tau.$$

However, since τ is an isomorphism, τ is injective which directly implies that $\ker \tau = \{0\}$. Therefore,

$$a_1v_1 + \ldots + a_nv_n = 0$$

and since B is linearly independent, $\{a_1, \ldots, a_n\} = \{0\}$. Therefore, $\tau(B)$ is linearly independent. Let $w \in W$. Since τ is an isomorphism, τ is surjective. Therefore, there exists a $v \in V$ such that $w = \tau(v)$. However B spans V and thus, there exist $\{v_1, \ldots, v_n\} \subseteq B$ and $\{a_1, \ldots, a_n\} \subseteq \mathbb{F}$ such that

$$v = a_1v_1 + \ldots + a_nv_n.$$

Therefore,

$$\begin{aligned} w &= \tau(v) \\ &= \tau(a_1v_1 + \ldots + a_nv_n) \\ &= a_1\tau(v_1) + \ldots + a_n\tau(v_n). \end{aligned}$$

Ergo, $\tau(B)$ spans W.

$\Leftarrow$ Assume $\tau(B)$ is a basis. We need to show that τ is injective and surjective. Let $v \in \ker \tau$. Since $v \in V$,

$$v = a_1v_1 + \ldots + a_nv_n$$

for some $\{a_1, \ldots, a_n\} \subseteq \mathbb{F}$ and $\{v_1, \ldots, v_n\} \subseteq V$. Therefore,

$$0 = \tau(v) = \tau(a_1v_1 + \ldots + a_nv_n) = a_1\tau(v_1) + \ldots + a_n\tau(v_n).$$

Since $\tau(B)$ is a basis, we know that $\{a_1, \ldots, a_n\} = \{0\}$. Thus, $\ker \tau = \{0\}$ and τ is injective. Let $w \in W$. Since $\tau(B)$ is a basis, there exist $\{a_1, \ldots, a_n\} \subseteq \mathbb{F}$ and $\{\tau(v_1), \ldots, \tau(v_n)\} \subseteq \tau(B)$ such that

$$w = a_1\tau(v_1) + \ldots + a_n\tau(v_n) = \tau(a_1v_1 + \ldots + a_nv_n)$$

which is an element of V. Ergo, τ is surjective and thus τ is a vector space isomorphism. $\square$