Fibonacci Generating Function

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Problem. *Let $(F_n)_{n\in\mathbb{N}}$ denote the sequence of Fibonacci numbers. Find the closed form of the generating function for F_n and use it to create a closed form for F_n.*

1 Background

Lets begin with the definition of the Fibonacci numbers.

Definition 1. The Fibonacci numbers $(F_n)_{n\in\mathbb{N}}$ are defined by $F_1 = F_2 = 1$, and $F_{n+2} = F_{n+1} + F_n$.

From the Fibonacci numbers we get the golden ratio, φ, and its conjugate, $\overline{\varphi}$.

$$\varphi = \frac{1+\sqrt{5}}{2} \qquad \overline{\varphi} = \frac{1-\sqrt{5}}{2}$$

While the Fibonacci recurrence is simple, its difficult to tell what the nth term is without computing all of the terms before it. Since the Fibonacci's are defined recursively, we can use *generating functions* to extract a closed form for F_n relatively easily.

Definition 2. Let $(a_n)_{n\in\mathbb{N}}$ be a $\mathbb{R}$-valued sequence. Then the generating function for a_n is the power series

$$G(x) = \sum_{n\in\mathbb{N}} a_n x^n.$$

After we create the generating function, we generally want to find its closed form (that is, a representation without using limits). Then, we use that new representation to find a closed form for the terms of the sequence.

2 Solution

Problem. *Let $(F_n)_{n\in\mathbb{N}}$ denote the sequence of Fibonacci numbers. Find the closed form of the generating function for F_n and use it to create a closed form for F_n.*

We begin by setting up a generating function, $F(x)$, for F_n.

$$F(x) = \sum_{n=1}^{\infty} F_n x^n.$$

The next step is to invoke the recurrence, however F_1 and F_2 need to be handled separately since they are not defined recursively. Thus,

$$F(x) = F_1 x + F_2 x^2 + \sum_{n=3}^{\infty} F_n x^n.$$

Now we invoke the definition of the Fibonacci numbers yielding,

$$F(x) = x + x^2 + \sum_{n=3}^{\infty} (F_{n-1} + F_{n-2})x^n.$$

By rearranging we get,

$$F(x) = x + x^2 + \left(\sum_{n=3}^{\infty} F_{n-1}x^n \right) + \left(\sum_{n=3}^{\infty} F_{n-2}x^n \right).$$

We then factor out as many powers of x from the sums and reindex to get

$$F(x) = x + x^2 + x \left(F_1 x - F_1 x + \sum_{n=2}^{\infty} F_n x^n \right) + x^2 \left(\sum_{n=1}^{\infty} F_n x^n \right).$$

A little algebraic manipulation yields,

$$F(x) = x + x^2 - x^2 + x \left(\sum_{n=1}^{\infty} F_n x^n \right) + x^2 \left(\sum_{n=1}^{\infty} F_n x^n \right).$$

Note, that we see the original definition of $F(x)$ in this new form, therefore

$$F(x) = x + xF(x) + x^2 F(x).$$

We now solve for $F(x)$, yielding a closed form of

$$F(x) = \frac{x}{1 - x - x^2}.$$

The next step is to find the closed form of F_n. We know that φ and $\overline{\varphi}$ are roots of the polynomial in the denominator, therefore

$$F(x) = \frac{x}{(1 - \varphi x)(1 - \overline{\varphi} x)}.$$

To do so, we perform partial fraction decomposition of $F(x)$ yielding,

$$F(x) = \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \varphi x} - \frac{1}{1 - \overline{\varphi} x} \right).$$

Using the formula for the power series for $(1 - x)^{-1}$ yields,

$$F(x) = \sum_{k=1}^{\infty} \frac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}} x^n = \sum_{k=1}^{\infty} F_n x^n.$$

By uniqueness of power series, we deduce that

$$F_n = \frac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}}.$$

References

[1] Hongwei Chen. *Excursions in Classical Analysis*. Mathematical Association of America. ISBN: 978-0-88385-768-7.