Finding Trigonometric Identities using DeMoivre's Theorem

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Problem. *For $x, r \in \mathbb{R}$ find*

$$\sum_{k=0}^{n} r^k \cos kx \text{ and } \sum_{k=0}^{n} r^k \sin kx.$$

Then, assuming $|r| < 1$ take their limits as $n \to \infty$.

1 Background

Before beginning, we need to mention DeMoivre's theorem.

Theorem 1.

$$\cos kx + i \sin kx = (\cos x + i \sin x)^k$$

Proof. We know by Euler's formula that

$$(\cos x + i \sin x)^k = e^{ikx} = \cos kx + i \sin kx.$$

□

Another thing to know is that for $z = a + bi \in \mathbb{C}$, the *conjugate* of z, denoted $\overline{z}$, is $\overline{z} = a - bi$. Furthermore, $z \cdot \overline{z} = a^2 + b^2$.

Lastly, we need that

$$\sum_{k=0}^{n} z^k = \frac{1 - z^{k+1}}{1 - z}.$$

2 Solution

Consider the following.

$$\begin{aligned}
\sum_{k=0}^{n} r^k \cos kx + i \sum_{k=0}^{n} r^k \sin kx &= \sum_{k=0}^{n} r^k (\cos kx + i \sin kx) \\
&= \sum_{k=0}^{n} r^k (\cos x + i \sin x)^k \\
&= \frac{1 - r^{n+1} (\cos x + i \sin x)^{n+1}}{1 - r \cos x - ir \sin x} \\
&= \frac{1 - r^{n+1} \cos((n+1)x) - ir^{n+1} \sin((n+1)x)}{(1 - r \cos x) - ir \sin x} \cdot \frac{(1 - r \cos x) + ir \sin x}{(1 - r \cos x) + ir \sin x} \\
&= \frac{1 - r \cos x - r^{n+1} \cos((n+1)x) + r^{n+2} \cos x \cos((n+1)x) + r^{n+2} \sin x \sin((n+1)x)}{1 - 2r \cos x + r^2 \cos^2 x + r^2 \sin^2 x} \\
&\quad + i \frac{r \sin x - r^{n+1} \sin((n+1)x) - r^{n+2} \cos((n+1)x) \sin x + r^{n+2} \cos x \sin((n+1)x)}{1 - 2r \cos x + r^2 \cos^2 x + r^2 \sin^2 x}
\end{aligned}$$

By matching real parts and imaginary parts we get

$$\sum_{k=0}^{n} r^k \cos kx = \frac{1 - r\cos x - r^{n+1}\cos((n+1)x) + r^{n+2}\cos x\cos((n+1)x) + r^{n+2}\sin x\sin((n+1)x)}{1 - 2r\cos x + r^2}$$
$$\sum_{k=0}^{n} r^k \sin kx = \frac{r\sin x - r^{n+1}\sin((n+1)x) - r^{n+2}\cos((n+1)x)\sin x + r^{n+2}\cos x\sin((n+1)x)}{1 - 2r\cos x + r^2}.$$

If we suppose $|r| < 1$ and let $n \to \infty$ we get that

$$\sum_{k=0}^{\infty} r^k \cos kx = \frac{1 - r\cos x}{1 - 2r\cos x + r^2}$$
$$\sum_{k=0}^{\infty} r^k \sin kx = \frac{r\sin x}{1 - 2r\cos x + r^2}.$$