

# Vector Space Isomorphisms

Matt McCarthy

February 2016

**Theorem.** Let  $V, W$  be vector spaces over a field  $\mathbb{F}$  and  $\tau : V \rightarrow W$  be a homomorphism. Then  $\tau$  is an isomorphism if and only if  $\tau$  takes a basis of  $V$  to a basis of  $W$ .

Before we can talk about vector spaces, let alone vector space isomorphisms and bases, we need to define the terms field and vector space.

**Definition 1** (Field). Let  $\mathbb{F}$  be a set and let  $+$  and  $\cdot$  be binary operations on  $\mathbb{F}$ . Then we say  $\mathbb{F}$  is a *field* if and only if all of the following hold.

1.  $(\mathbb{F}, +)$  forms an abelian group (we call its identity 0).
2.  $(\mathbb{F} \setminus \{0\}, \cdot)$  forms an abelian group (we call its identity 1).
3. For any  $x \in \mathbb{F}$ ,  $0 \cdot x = 0$ .
4. For any  $x, y, z \in \mathbb{F}$ ,  $(x + y) \cdot z = (x \cdot z) + (y \cdot z)$ .

In short, a field is a set with two operations that behaves like the rational numbers. That is, we can add, subtract, multiply, and divide as we see fit. A vector space “extends” the properties of a field further.

**Definition 2** (Vector Space). A *vector space*  $V$  over a field  $\mathbb{F}$  is a set with two binary operations,  $+$  :  $V \times V \rightarrow V$  and  $\cdot$  :  $V \times \mathbb{F} \rightarrow V$  such that all of the following hold.

1.  $(V, +)$  forms an abelian group with identity 0.
2. For all  $x \in V$ ,  $1x = x$ .
3. For all  $a, b \in \mathbb{F}$  and  $x \in V$ ,  $a(bx) = (ab)x$ .
4. For all  $a \in \mathbb{F}$  and  $x, y \in V$ ,  $a(x + y) = ax + ay$ .
5. For all  $a, b \in \mathbb{F}$  and  $x \in V$ ,  $(a + b)x = ax + bx$ .

Furthermore,  $x + y$  is called the *sum of  $x$  and  $y$*  while  $ax$  is called the *product of  $x$  and  $a$* . Moreover, each  $x \in V$  is called a *vector* and each  $a \in \mathbb{F}$  is called a *scalar*.

Now that we have a definition for vector space, we can build the definition of a basis. To get there, we must first talk about sets that span a vector space and linearly independent sets.

**Definition 3** (Spanning Set). Let  $V$  be a vector space over a field  $\mathbb{F}$  and let  $S$  be a nonempty subset of  $V$ . Then, the *span of  $S$* , denoted  $\text{span } S$ , is the set

$$\text{span } S = \left\{ \sum_{j=1}^n a_j s_j \mid \{a_j\}_{j=1}^n \subseteq \mathbb{F}, \{s_j\}_{j=1}^n \subseteq S, n < \infty \right\}$$

or the set of linear combinations of elements of  $S$ . We define  $\text{span } \emptyset = \{0\}$ .

**Definition 4** (Span). A subset  $S$  of a vector space  $V$  *spans*  $V$  if and only if  $\text{span } S = V$ .

Essentially, a set spans the space if every element in the space can be written as a linear combination of elements in the set. Now that we can span a space, we need the spanning set to be what we call linearly independent.

**Definition 5** (Linear Independence). A subset  $S$  of a vector space  $V$  over a field  $\mathbb{F}$  is *linearly independent* if and only if for any  $\{x_j\}_{j=1}^n \subseteq S$  where  $n < \infty$  the statement

$$\sum_{j=1}^n a_j x_j = 0$$

implies that  $\{a_j\} = \{0\}$ , where  $\{a_j\} \subseteq \mathbb{F}$ . Furthermore, if  $S$  is not linearly independent, we say that  $S$  is *linearly dependent*.

**Definition 6** (Basis of a Vector Space). A *basis*  $B$  for a vector space  $V$  is a linearly independent subset of  $V$  that spans  $V$ .

Bases are important because they allow us to represent the entire space (which is potentially uncountably infinite) using a countable number of vectors. Moreover, every vector space has a basis which means we can do this for any space.

We now switch gears and talk about homomorphisms and isomorphisms of vector spaces.

**Definition 7** (Homomorphism of Vector Spaces). Let  $V, W$  be vector spaces over a field  $\mathbb{F}$  and let  $\tau : V \rightarrow W$  be a map. We say  $\tau$  is a *homomorphism* if and only if for all  $v, w \in V$  and  $a, b \in \mathbb{F}$ ,

$$\tau(av + bw) = a\tau(v) + b\tau(w).$$

**Definition 8** (Kernel of a Homomorphism). Let  $\tau : V \rightarrow W$  be a vector space homomorphism. Then the *kernel* of  $\tau$  is defined as,

$$\ker \tau = \{x \in V \text{ s.t. } \tau(x) = 0\}.$$

More generally, a morphism in algebra is a map that “preserves operation” in some fashion. Furthermore, the kernel of a homomorphism is simply the set of everything in the domain that gets mapped to zero. However, homomorphisms are not quite as strong as what we want. Therefore, we define the following properties.

**Definition 9** (Surjective and Injective Maps). Let  $S, R$  be nonempty sets and let  $f : S \rightarrow R$  be a map. We say  $f$  is *injective* if and only if the equation

$$f(x) = f(y)$$

implies that  $x = y$ . We say  $f$  is *surjective* if and only if for each  $r \in R$ , there exists an  $s \in S$  such that  $r = f(s)$ . If  $f$  is both injective and surjective, we say that  $f$  is *bijective*.

Essentially, an injective map takes each input to a different output while a surjective map has an input for each output. If we let our homomorphism be bijective, then it is an isomorphism.

**Definition 10** (Isomorphism of Vector Spaces). Let  $V, W$  be vector spaces over a field  $\mathbb{F}$  and let  $\tau : V \rightarrow W$  be a homomorphism. We say that  $\tau$  is an *isomorphism* if and only if  $\tau$  is bijective.

If two vector spaces are isomorphic, they are essentially a relabeling of each other. That is the elements of each vector space are the same object in different clothes.

To get to the proof of the original theorem, we need to show a couple of propositions. The first of which is that any homomorphism maps 0 to 0.

**Proposition 1.** *Let  $\tau : V \rightarrow W$  be a vector space homomorphism. Then  $\tau(0) = 0$ .*

*Proof.*

$$\tau(0) = \tau(0 + 0) = \tau(0) + \tau(0).$$

If we subtract  $\tau(0)$  from both sides we get,

$$\tau(0) = 0.$$

□

Additionally, for homomorphisms we have an equivalent condition for injectivity.

**Proposition 2.** *Let  $\tau : V \rightarrow W$  be a vector space homomorphism. Then  $\tau$  is injective if and only if  $\ker \tau = \{0\}$ .*

*Proof.*  $\Rightarrow$  Assume  $\tau$  is injective. Let  $x \in \ker \tau$ . Then  $\tau(x) = 0$ . Furthermore, we know  $\tau(0) = 0$ . Therefore,  $\tau(x) = \tau(0)$  and since  $\tau$  is injective,  $x = 0$ . Ergo,  $\ker \tau = \{0\}$ .

$\Leftarrow$  Assume  $\ker \tau = \{0\}$ . Let  $x, y \in V$  such that  $\tau(x) = \tau(y)$ . Then

$$\tau(x) - \tau(y) = 0$$

and by properties of homomorphisms,

$$\tau(x - y) = 0.$$

Therefore  $x - y \in \ker \tau$  and  $x - y = 0$ . Thus,  $x = y$  and  $\tau$  is injective. □

With these two propositions, we can begin the proof of the main theorem.

**Theorem 3.** *Let  $V, W$  be vector spaces over a field  $\mathbb{F}$  and  $\tau : V \rightarrow W$  be a homomorphism. Then  $\tau$  is an isomorphism if and only if  $\tau$  takes a basis of  $V$  to a basis of  $W$ .*

*Proof.* Let  $B$  be a basis of  $V$ .

$\Rightarrow$  Assume  $\tau$  is an isomorphism. We want to show  $\tau(B)$  is a basis of  $W$ . Therefore we need to show that  $\tau(B)$  is linearly independent and spans  $W$ . Let  $\{\tau(v_1), \tau(v_2), \dots, \tau(v_n)\} \subseteq \tau(B)$  and  $\{a_1, a_2, \dots, a_n\} \subseteq \mathbb{F}$  such that

$$a_1\tau(v_1) + a_2\tau(v_2) + \dots + a_n\tau(v_n) = 0.$$

By properties of vector space homomorphisms, we know that

$$\tau(a_1v_1 + \dots + a_nv_n) = 0$$

and thus,

$$a_1v_1 + \dots + a_nv_n \in \ker \tau.$$

However, since  $\tau$  is an isomorphism,  $\tau$  is injective which directly implies that  $\ker \tau = \{0\}$ . Therefore,

$$a_1v_1 + \dots + a_nv_n = 0$$

and since  $B$  is linearly independent,  $\{a_1, \dots, a_n\} = \{0\}$ . Therefore,  $\tau(B)$  is linearly independent. Let  $w \in W$ . Since  $\tau$  is an isomorphism,  $\tau$  is surjective. Therefore, there exists a  $v \in V$  such that  $w = \tau(v)$ . However  $B$  spans  $V$  and thus, there exist  $\{v_1, \dots, v_n\} \subseteq B$  and  $\{a_1, \dots, a_n\} \subseteq \mathbb{F}$  such that

$$v = a_1v_1 + \dots + a_nv_n.$$

Therefore,

$$\begin{aligned} w &= \tau(v) \\ &= \tau(a_1v_1 + \dots + a_nv_n) \\ &= a_1\tau(v_1) + \dots + a_n\tau(v_n). \end{aligned}$$

Ergo,  $\tau(B)$  spans  $W$ .

$\Leftarrow$  Assume  $\tau(B)$  is a basis. We need to show that  $\tau$  is injective and surjective. Let  $v \in \ker \tau$ . Since  $v \in V$ ,

$$v = a_1v_1 + \dots + a_nv_n$$

for some  $\{a_1, \dots, a_n\} \subseteq \mathbb{F}$  and  $\{v_1, \dots, v_n\} \subseteq V$ . Therefore,

$$0 = \tau(v) = \tau(a_1v_1 + \dots + a_nv_n) = a_1\tau(v_1) + \dots + a_n\tau(v_n).$$

Since  $\tau(B)$  is a basis, we know that  $\{a_1, \dots, a_n\} = \{0\}$ . Thus,  $\ker \tau = \{0\}$  and  $\tau$  is injective. Let  $w \in W$ . Since  $\tau(B)$  is a basis, there exist  $\{a_1, \dots, a_n\} \subseteq \mathbb{F}$  and  $\{\tau(v_1), \dots, \tau(v_n)\} \subseteq \tau(B)$  such that

$$w = a_1\tau(v_1) + \dots + a_n\tau(v_n) = \tau(a_1v_1 + \dots + a_nv_n)$$

which is an element of  $V$ . Ergo,  $\tau$  is surjective and thus  $\tau$  is a vector space isomorphism.  $\square$