Roots of Unity

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Problem. *Find the roots of the equation*

$$z^5 - 1 = 0.$$

1 Background

The complex numbers, denoted as $\mathbb{C}$ are defined as follows.

Definition 1. The set of *complex numbers* is the following two-dimensional vector space over the real numbers.

$$\mathbb{C} := \{a + bi \,:\, a, b \in \mathbb{R}\}$$

Furthermore, all $z \in \mathbb{C}$ have the form

$$r(\cos\theta + i\sin\theta)$$

where $r \in \mathbb{R}$ with $r \geq 0$ and $\theta \in \mathbb{R}$. Additionally, we have Euler's formula which will allow us to write the polar form of a complex number more concisely.

Theorem 1. *For all* $\theta \in \mathbb{R}$,

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Thus, any complex number, z, can be written as

$$z = re^{i\theta}.$$

Moreover, for any $z = a + bi = re^{i\theta} \in \mathbb{C}$, the *conjugate* of z, denoted $\bar{z}$, is defined as

$$\bar{z} := a - bi = re^{-i\theta}$$

and the *modulus* of a z is defined as

$$|z| := \sqrt{a^2 + b^2} = r.$$

Thus,

$$|e^{i\theta}| = 1.$$

Lastly, an *nth root of unity* is a solution to the equation

$$z^n = 1.$$

2 Solution

Problem. *Find the roots of the equation*

$$z^5 - 1 = 0.$$

We want to find the roots of the equation,

$$z^5 - 1 = 0.$$

Equivalently, we will find the 5th roots of unity, or the solutions to

$$z^5 = 1.$$

To begin, we know that $z = re^{i\theta}$ for some $r \geq 0$ and $\theta \in \mathbb{R}$. Furthermore, we know that $1 = e^{2ki\pi}$ for all $k \in \mathbb{Z}$. Thus,

$$r^5 e^{5i\theta} = e^{2ki\pi}$$

for all $k \in \mathbb{Z}$. Since $|e^{i\phi}| = 1$ for all $\phi \in \mathbb{R}$, we know that $r^5 = 1$. Thus, $r = 1$ because r is a positive real. Thus, we are left with

$$e^{5i\theta} = e^{2ki\pi}.$$

Ergo,

$$\theta = 2ki\pi/5$$

for all $k \in \mathbb{Z}$. However, this is an infinite solution set and there are only 5 *distinct* solutions. To find these distinct solutions, we use the fact that

$$e^{i\theta} = e^{i(\theta + 2k\pi)}$$

for any $k \in \mathbb{Z}$. Thus, our distinct values for θ are as follows.

$$\theta \in \{0, 2\pi/5, 4\pi/5, 6\pi/5, 8\pi/5\}$$

When $\theta = 10\pi/5 = 2\pi$, we get the same result as when $\theta = 0$ since $2\pi \equiv 0 \mod 2\pi$. Therefore,

$$z \in \{1, e^{2\pi/5}, e^{4\pi/5}, e^{6\pi/5}, e^{8\pi/5}\}.$$