Derivatives of Arctan at 0

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1 Statement of Problem

Define $f(x) := \arctan x$. Find $f^{(n)}(0)$.

2 Background

For this problem we will need two theorems. Firstly, we need Taylor's Theorem.

Theorem 2.1 (Taylor's Theorem on a disk in $\mathbb{C}$)**.** *Let $f : \mathbb{C} \to \mathbb{C}$ be analytic on a disk of radius r about z_0. Then there exists a unique power series such that*

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$$

for all z such that $|z - z_0| < r$.

Furthermore, we need an additional Taylor expansion.

Theorem 2.2. *Define*

$$f(z) := \frac{1}{1-z}.$$

Then,

$$f(z) = \sum_{k=0}^{\infty} z^k$$

for any $z \in \mathbb{C}$ with $|z| < 1$.

3 Solution

Define $g(x) := f'(x)$. We know that

$$g(x) = f'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)}.$$

From here we can invoke Theorem 2.2 to get

$$g(x) = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n (x-0)^{2n}.$$

Furthermore, we know that this power series representation is unique by Theorem 2.1. Thus,

$$g(x) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!}(x-0)^n = \sum_{n=0}^{\infty}(-1)^n(x-0)^{2n}.$$

Using Theorem 2.1, we notice that $g^{(2n+1)}(0) = 0$ since no odd terms appead in the Taylor expansion. Moreover, by Theorem 2.1 we get

$$\frac{g^{(2n)}(0)}{(2n)!} = (-1)^n.$$

Therefore, $g^{(2n)}(0) = (-1)^n(2n)!$. However, by definition of $g = f'$, we have $f^{(2n+2)}(0) = 0$ and $f^{(2n+1)}(0) = (-1)^n(2n)!$.