Generating Monoids from Categories

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Theorem. *Let C be a category and let A be an object in C. Then $C[A, A]$ forms a monoid under arrow composition.*

1 Background

Definition 1 (Category)**.** A *category*, C consists of the following.

1. A class of *objects*, denoted $Obj(C)$.
2. A class of *arrows*, denoted $Arr(C)$. Each arrow $f \in Arr(C)$ has a source object $A \in Obj(C)$, a target object $B \in Obj(C)$, and is denoted $f : A \to B$. We denote the class of all arrows going from $A \in Obj(C)$ to $B \in Obj(C)$ as $C[A, B]$.
3. A partial composition $\circ : Arr(C) \times Arr(C) \to Arr(C)$ such that for any $f : A \to B, g : B \to D$, $gf : A \to D \in Arr(C)$.

Furthermore, the following axioms must hold.

1. For all $f : A \to B, g : B \to D, h : D \to E \in Arr(C)$, $h(gf) = (hg)f$.
2. For all $A \in Obj(C)$, there exists an $id_A \in C[A, A]$ such that for all arrows $f : X \to A$, $g : A \to Y$ $id_A\, f = f$ and $g\, id_A = g$.

Definition 2 (Monoid)**.** Let M be a set, and let $* : M^2 \to M$ be a binary operation. Then $(M, *)$ forms a *monoid* if all of the following are satisfied.

1. For all $a, b, c \in M$, $a * (b * c) = (a * b) * c$.
2. There exists an $e \in M$ such that for all $a \in M$, $e * a = a * e = a$.

2 Solution

Theorem 1. *Let C be a category and let A be an object in C. Then $C[A, A]$ forms a monoid under arrow composition.*

Proof. Let $f, g \in C[A, A]$. Then

$$A \xrightarrow{f} A \xrightarrow{g} A$$

and thus

$$A \xrightarrow{fg} A.$$

Therefore, arrow composition forms a binary operation on $C[A, A]$.

Next, we claim that id_A is the identity for $C[A, A]$ with respect to arrow composition. Let $f \in C[A, A]$. Then, by definition, we know that $f\, id_A = id_A\, f = f$. Thus, id_A is the identity for $C[A, A]$ with respect to arrow composition.

Lastly, we must show that arrow composition is associative for all arrows in $C[A, A]$. Let $f, g, h \in C[A, A]$. Consider $f(gh)$.

$$A \xrightarrow{f(gh)} A = A \xrightarrow{gh} A \xrightarrow{f} = A \xrightarrow{h} A \xrightarrow{g} A \xrightarrow{f} A$$

Now consider $(fg)h$.

$$A \xrightarrow{(fg)h} A = A \xrightarrow{h} A \xrightarrow{fg} A = A \xrightarrow{h} A \xrightarrow{g} A \xrightarrow{f} A$$

Therefore, $(fg)h = f(gh)$ and $C[A, A]$ is associative with respect to arrow composition. Thus, $C[A, A]$ forms a monoid under arrow composition. $\square$