The Mystery of 1^π
Wrestling with Complex Exponentiation

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Problem. *Find what step is wrong in the following statements.*

$$\begin{aligned}
1 &= 1^\pi \\
&= \left(e^{2i\pi}\right)^\pi \\
&= e^{2i\pi^2} \\
&= \cos 2\pi^2 + i \sin 2\pi^2 \\
&\approx 0.6296 + 0.7768i
\end{aligned}$$

1 Background

1.1 Complex Logarithms

In $\mathbb{R}$ we can define logarithms in various ways such as the inverse of the exponential and as

$$\ln x = \int \frac{1}{x} dx.$$

However, in $\mathbb{C}$ the original definition (inverse of the exponential) fails since

$$e^{i\theta} = e^{i\theta + 2ki\pi}$$

where $k \in \mathbb{Z}$. Essentially, the exponential is a bijection between $\mathbb{R}$ and $(0, \infty)$ and is thus invertible. When we move to $\mathbb{C}$, the exponential function loses its injectivity since multiple domain elements get mapped to a single element in the range. However, defining logarithms as the antiderivative of $1/x$ still works.

Definition 1. Let $z \in \mathbb{C}$. Then $\log z$ is defined as

$$\log z := \ln |z| + i \arg z.$$

If $\arg z \in [0, 2\pi)$ we say it is the *principal logarithm* of z.

As we can see by the definition, the complex logarithm is *multivalued*, thus if we want it to be well-defined we need to restrict the argument of the input to an open interval of length 2π.

1.2 Exponentiation with a Complex Base

Because of these problems with the logarithm, exponentiation becomes less straightforward. In the complex world, exponentiation is defined as follows.

Definition 2. Let $z, w \in \mathbb{C}$ with $w \neq 0$. Then

$$w^z = e^{z \log w}.$$

However, this definition is equivalent to

$$w^z = e^{z(\ln |w| + i \arg w)}.$$

Thus, due to the properties of the complex logarithm, exponentiation in the complex plane is also multivalued.

When exponentiating, the following are true.

1. z^w is single-valued iff $w \in \mathbb{Z}$.

2. $z^{m/n}$ has n distinct values for $m, n \in \mathbb{Z}$ with $n > 0$ and $\gcd(m, n) = 1$.

3. z^x has infinitely many values if x is irrational.

2 Solution

Problem. *Find what step is wrong in the following statements.*

$$\begin{aligned} 1 &= 1^\pi \\ &= \left(e^{2i\pi}\right)^\pi \\ &= e^{2i\pi^2} \\ &= \cos 2\pi^2 + i \sin 2\pi^2 \\ &\approx 0.6296 + 0.7768i \end{aligned}$$

The problem is

$$1^\pi = 1.$$

By definition of complex exponentiation we have

$$1^\pi = e^{\pi \log 1}.$$

However

$$\log 1 = \ln |1| + i \arg 1 = 0 + 2ki\pi$$

for $k \in \mathbb{Z}$. Thus,

$$1^\pi = e^{2ki\pi^2}.$$

Since π is irrational, there are infinitely many values that $e^{2ki\pi^2}$ can take only one of which will yield 1 as the answer. If we force $k = 0$, then we get the desired result that $1^\pi = 1$, however this excludes infinitely many other values.

References

[1] John M. Howie. *Complex Analysis*. Springer. ISBN: 978-1-85233-733-9.