

# The Mystery of $1^\pi$

## Wrestling with Complex Exponentiation

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**Problem.** Find what step is wrong in the following statements.

$$\begin{aligned}1 &= 1^\pi \\&= (e^{2i\pi})^\pi \\&= e^{2i\pi^2} \\&= \cos 2\pi^2 + i \sin 2\pi^2 \\&\approx 0.6296 + 0.7768i\end{aligned}$$

## 1 Background

### 1.1 Complex Logarithms

In  $\mathbb{R}$  we can define logarithms in various ways such as the inverse of the exponential and as

$$\ln x = \int \frac{1}{x} dx.$$

However, in  $\mathbb{C}$  the original definition (inverse of the exponential) fails since

$$e^{i\theta} = e^{i\theta + 2ki\pi}$$

where  $k \in \mathbb{Z}$ . Essentially, the exponential is a bijection between  $\mathbb{R}$  and  $(0, \infty)$  and is thus invertible. When we move to  $\mathbb{C}$ , the exponential function loses its injectivity since multiple domain elements get mapped to a single element in the range. However, defining logarithms as the antiderivative of  $1/x$  still works.

**Definition 1.** Let  $z \in \mathbb{C}$ . Then  $\log z$  is defined as

$$\log z := \ln |z| + i \arg z.$$

If  $\arg z \in [0, 2\pi)$  we say it is the *principal logarithm* of  $z$ .

As we can see by the definition, the complex logarithm is *multivalued*, thus if we want it to be well-defined we need to restrict the argument of the input to an open interval of length  $2\pi$ .

### 1.2 Exponentiation with a Complex Base

Because of these problems with the logarithm, exponentiation becomes less straightforward. In the complex world, exponentiation is defined as follows.

**Definition 2.** Let  $z, w \in \mathbb{C}$  with  $w \neq 0$ . Then

$$w^z = e^{z \log w}.$$

However, this definition is equivalent to

$$w^z = e^{z(\ln |w| + i \arg w)}.$$

Thus, due to the properties of the complex logarithm, exponentiation in the complex plane is also multivalued. When exponentiating, the following are true.

1.  $z^w$  is single-valued iff  $w \in \mathbb{Z}$ .
2.  $z^{m/n}$  has  $n$  distinct values for  $m, n \in \mathbb{Z}$  with  $n > 0$  and  $\gcd(m, n) = 1$ .
3.  $z^x$  has infinitely many values if  $x$  is irrational.

## 2 Solution

**Problem.** Find what step is wrong in the following statements.

$$\begin{aligned} 1 &= 1^\pi \\ &= (e^{2i\pi})^\pi \\ &= e^{2i\pi^2} \\ &= \cos 2\pi^2 + i \sin 2\pi^2 \\ &\approx 0.6296 + 0.7768i \end{aligned}$$

The problem is

$$1^\pi = 1.$$

By definition of complex exponentiation we have

$$1^\pi = e^{\pi \log 1}.$$

However

$$\log 1 = \ln |1| + i \arg 1 = 0 + 2ki\pi$$

for  $k \in \mathbb{Z}$ . Thus,

$$1^\pi = e^{2ki\pi^2}.$$

Since  $\pi$  is irrational, there are infinitely many values that  $e^{2ki\pi^2}$  can take only one of which will yield 1 as the answer. If we force  $k = 0$ , then we get the desired result that  $1^\pi = 1$ , however this excludes infinitely many other values.

## References

- [1] John M. Howie. *Complex Analysis*. Springer. ISBN: 978-1-85233-733-9.