Completeness of $\mathbb{R}$

Matt McCarthy

Februrary 2016

Theorem. *The real numbers form a complete metric space.*

Notation: We denote the set of non-negative real numbers as $\mathbb{R}^+$.
We begin by providing a way to measure distances in a space.

Definition 1 (Metric)**.** Let X be a set and $d : X \times X \to \mathbb{R}^+$ be a map. We say d is a *metric* if and only if all of the following hold.

1. For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$.
2. For all $x, y \in X$, $d(x, y) = d(y, x)$ (symmetric property).
3. For all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, x)$ (triangle inequality).

If d is a metric on X, we say (X, d) forms a *metric space.*

For example, $(\mathbb{Q}, d)$ and $(\mathbb{R}, d)$ where $d(x, y) = |x - y|$ is a metric space, and more generally, $(\mathbb{R}^n, d_n)$ where d_n is the Euclidean distance is also a metric space. Now that we have a way to talk about distances in spaces we can talk about convergence of sequences in those spaces.

Definition 2 (Convergent Sequence)**.** Let (X, d) be a metric space and let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence in X. We say $\{a_n\}_{n\in\mathbb{N}}$ *converges* to some $a \in X$, denoted $a_n \to a$ if and only if for all $\varepsilon > 0$ there exists a $N_\varepsilon \in \mathbb{N}$ such that $d(a_n, a) < \varepsilon$ for all $n \geq N_\varepsilon$.

However, convergence is often too strong a condition to prove, and moreover convergence also requires a proposed limit. Therefore, we introduce the notion of a *Cauchy Sequence* where the terms get arbitrarily close to each other.

Definition 3 (Cauchy Sequence)**.** Let (X, d) be a metric space and let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence in X. We say $\{a_n\}_{n\in\mathbb{N}}$ is a *Cauchy sequence* if and only if for any $\varepsilon > 0$, there exists a $N_\varepsilon \in \mathbb{N}$ such that $d(a_n, a_m) < \varepsilon$ for all $n, m \geq N_\varepsilon$.

Note, however that Cauchy sequences are not always convergent in their space. For example consider the sequence given by

$$a_n = \left(1 + \frac{1}{n}\right)^n.$$

We know these terms are rational and thus a_n is a $\mathbb{Q}$-valued sequence. Furthermore, the sequence is in fact Cauchy. However, the limit of this sequence as n approaches infinity is e and thus, this Cauchy sequence does not converge in $\mathbb{Q}$, but it does in $\mathbb{R}$.
This leads us to a notion of *completeness* in a metric space.

Definition 4 (Complete Metric Space)**.** Let (X, d) be a metric space. Then we say (X, d) is *complete* if and only if every Cauchy sequence in X converges in X.

By this definition of completeness, $\mathbb{Q}$ is not complete as shown by our prior example. However, $\mathbb{R}$ is complete.

Before we get into any theorems here, we need a notion of *boundedness* for subsets of the real numbers.

Definition 5 (Bounded)**.** Let S be a subset of the real numbers. Then S is *bounded* if and only if there exists an $M > 0$ such that $S \subseteq [-M, M]$.

We also need to invoke a famous theorem due to Bolzano and Weierstrass, which states that any bounded sequence in $\mathbb{R}$ has a subsequence that is convergent in $\mathbb{R}$.

Theorem 1 (Bolzano-Weierstrass)**.** *Let $\{a_n\}_{n\in\mathbb{N}}$ be a bounded $\mathbb{R}$-valued sequence. Then there exists a $\{a_{n_k}\} \subseteq \{a_n\}$ such that a_{n_k} converges to some $a \in \mathbb{R}$.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be bounded. Then there exists an $M \in \mathbb{N}$ such that $\{x_n\}_{n\in\mathbb{N}} \subset [-M, M] \subset \mathbb{R}$. Bisect $[-M, M]$ into $[-M, 0], [0, M]$. At least one half has infinitely many sequence points, call this one I_1. Pick a sequence point $x_{n_1} \in I_1$. Bisect I_1 and call the half with infinitely many sequence points I_2. Pick a sequence point $x_{n_2} \in I_2$ such that the index $n_2 > n_1$. Iterate this process, that is bisect I_k and call the half with infinitely many points I_{k+1} Then choose a sequence point $x_{n_{k+1}} \in I_{k+1}$ with the property that the index $n_{k+1} > n_k > \ldots > n_1$. Moreover, by nested interval property, $\bigcap_{k\in\mathbb{N}} I_k \neq \emptyset$. Thus there exists at least one $x \in \bigcap_{k\in\mathbb{N}} I_k$. We want to show that $x_{n_k} \to x$. Let $\varepsilon > 0$ be given. Choose a $K_\varepsilon \in \mathbb{N}$ such that $2^{-K_\varepsilon} M < \varepsilon$. Note that for all $k \geq K_\varepsilon$, $2^{-k}M < 2^{-K_\varepsilon}M < \varepsilon$. Since $x \in \bigcap_{k\in\mathbb{N}} I_k$, $x \in I_k$ for all $k \in \mathbb{N}$. Thus for all $k \in \mathbb{N}$, $|x_{n_k} - x| \leq 2^{-k}M$. Therefore, when we force $k \geq K_\varepsilon$ we have $I_k \subset I_{K_\varepsilon}$ and

$$|x_{n_k} - x| \leq 2^{-k}M \leq 2^{-K_\varepsilon}M < \varepsilon.$$

Thus, $\{x_{n_k}\}_{k\in\mathbb{N}}$ is a subsequence of $\{x_n\}_{n\in\mathbb{N}}$ that is convergent in $\mathbb{R}$. ☐

Furthermore, we need the following lemma.

Lemma 2. *Any $\mathbb{R}$-valued Cauchy sequence is bounded.*

Proof. Let $\varepsilon > 0$ be given and let $\{a_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{R}$. Therefore, there exists an N_ε such that $|a_n - a_m| < \varepsilon$ for all $n, m \geq N_\varepsilon$. Therefore, for all $n \geq N_\varepsilon$, $|a_{N_\varepsilon} - a_n| < \varepsilon$. Thus for each $n \geq N_\varepsilon$, $a_n \in [a_{N_\varepsilon} - \varepsilon, a_{N_\varepsilon} + \varepsilon]$. Take $M_1 = \max\{a_{N_\varepsilon} - \varepsilon, a_{N_\varepsilon} + \varepsilon\}$, which yields $\{a_n\}_{n=N_\varepsilon}^{\infty} \subset [-M_1, M_1]$.

Furthermore, since $\{a_n\}_{n=1}^{N_\varepsilon - 1}$ is finite, take $M_2 = \max\{|a_n|\}_{n=1}^{N_\varepsilon - 1}$. Thus, $\{a_n\}_{n=1}^{N_\varepsilon - 1} \subset [-M_2, M_2]$. If we take $M = \max\{M_1, M_2\}$, then we have $\{a_n\}_{n\in\mathbb{N}} \subset [-M, M]$. Therefore, the sequence is bounded. ☐

And now we provide the proof.

Theorem 3. *The real numbers form a complete metric space.*

Proof. Let $\{a_n\}_{n\in\mathbb{N}}$ be Cauchy in $\mathbb{R}$. Let $\varepsilon > 0$ be given. Since $\{a_n\}_{n\in\mathbb{N}}$ is Cauchy there exists an $N_{\varepsilon/2} \in \mathbb{N}$ such that $|a_n - a_m| < \varepsilon/2$ for all $n, m \geq N_{\varepsilon/2}$. Furthermore, $\{a_n\}_{n\in\mathbb{N}}$ is bounded and thus by Bolzano-Weierstrass, there exists a $\{a_{n_k}\}_{k\in\mathbb{N}} \subset \{a_n\}_{n\in\mathbb{N}}$ such that a_{n_k} converges to some $a \in \mathbb{R}$.

We claim that a_n converges to a. Since a_{n_k} converges to a, there exists a $K_{\varepsilon/2} \in \mathbb{N}$ such that $|a_{n_k} - a| < \varepsilon/2$ for all $k \geq K_{\varepsilon/2}$. Take $N = \max\{N_{\varepsilon/2}, n_{K_{\varepsilon/2}}\}$ and force n, k large enough such that $n, n_k \geq N$. Consider $|a_n - a|$.

$$|a_n - a| = |a_n - a_{n_k} + a_{n_k} - a| \leq |a_n - a_{n_k}| + |a_{n_k} - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ for all $n \geq N$. Therefore a_n converges to a and $\mathbb{R}$ is a complete metric space. ☐