The Rocking Block Revisited, Again

Ben Collins
Supervised by Professor John Hogan

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1 Project description & background

The rocking block is a classic problem with roots going back to Housner's [1] 1963 paper. It concerns the dynamics and overturning conditions for a rigid body rocking under an "earthquake-like" forcing. Recently, a number of papers have been published examining the same problem with a pendulum attached to the block. A tuned mass damper such as a pendulum may stabilise the block under the forcing. The skyscraper Taipei 101 employs a similar concept to resist wind loads.

This project revolves around the dynamics of this rocking block with additional pendulum as described in Figure 2. De Leo et al. [2] and Collini et al. [3] derived the equations of motion for the full system as it rocks and undergoes impact. However neither performed much analytical work. This project aims to unify these two papers and to fill the gap in analytical work.

The key goal of the project is to find a set of characteristics describing the ideal tuned mass damper for a specified block and forcing. Addition of this pendulum would decrease the amplitude of oscillations by the maximum possible amount. The system can also be examined for periodic cycles, normal modes, resonance, chaos and other interesting dynamical features drawing parallels with Hogan's [4] analysis of the rocking block.

The rocking block problem has applications to free-standing structures under ground excitation. Examples include nuclear heat-exchange boilers, electric transformers and even ancient columns which all exhibit this rocking behaviour. The inclusion of a stabilising pendulum may be a particularly welcome addition for safety critical nuclear heat-exchange boilers.

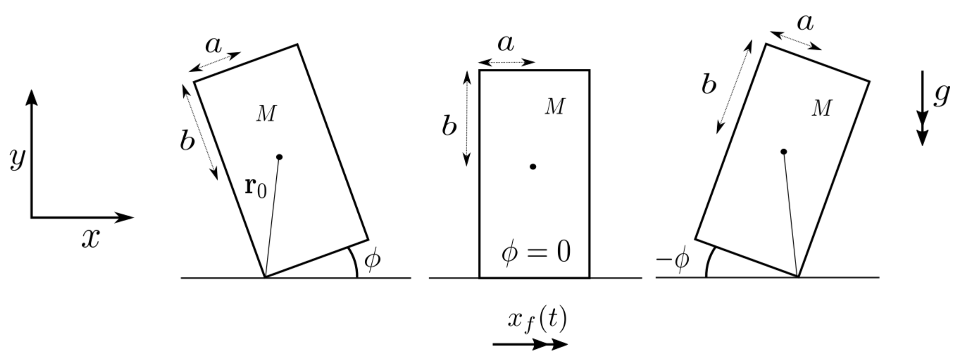

Figure 1: Three cases: Left- left rocking for positive ϕ; Middle- the block impacts the ground ($\phi = 0$); Right- right rocking for negative ϕ. Note the applied forcing $x_f(t)$.

2 Literature review

The rocking block

Many authors have explored the rocking block problem. Figure 1 shows the classic set up. A planar, rectangular, rigid body, the 'block', is subject to a horizontal forcing. The forcing causes the block to rock back and forth about its corners. When the block switches from rotating about one corner to the other, it undergoes an impact with the ground and the governing equations switch. This makes for an interesting and applicable problem which has been studied for over 50 years.

Housner [1], in 1963, examined the rocking block problem as a consequence of strange behaviour observed in the 1960 Chilean Earthquake. With a magnitude of 9.5 this was the strongest earthquake ever recorded. It was observed in the aftermath that certain structures had remained standing whilst other "more stable appearing" structures collapsed. "Box-like" electric transformers rocked and overturned, whereas "golf-ball-on-a-tee" structures, such as water towers, survived. Subsequent analysis of the rocking block by Housner attempted to find reasons why these box-like structures were unstable.

Three different types of forcing and their respective overturning conditions were examined. These were: "constant horizontal acceleration, a single sine pulse and an earthquake type excitation". The conditions obtained for each are particularly illuminating because buildings and structures are typically designed to withstand a certain "percent-g" i.e. a constant acceleration. However, Housner found that blocks can be made to topple under many sequential kicks which are smaller than this constant acceleration limit. The "percent-g" standard therefore might not be so useful.

On examining the latter two forcings Housner discovered a scaling effect that paradoxically means larger blocks are more stable than smaller blocks under certain conditions. It was also shown that these "golf-ball-on-a-tee" structures are surprisingly stable when subjected to an earthquake forcing in comparison to their constant horizontal acceleration response.

Hogan [4] took a different approach. His 1989 paper extended work done by Housner into a more theoretical, dynamical systems setting. A pair of linearised equations describing the block angle were derived and extensively analysed. In contrast to Housner, Hogan stipulated a continuous harmonic forcing. With this, one can identify periodic cycles, chaos and bifurcations transitioning between them. Asymmetric periodic cycles were also observed as an interesting result of the harmonic forcing and non-smoothness of the problem.

The harmonic forcing raises interesting questions about the stability of the blocks with respect to overturning. In more recent work by Zhang et al. [5] three separate outcomes were identified when the block is subject to a single sine pulse. These are 1) no overturning, 2) overturning without impact and 3) overturning with impact. All the cases are analysed separately and 'safe' regions of parameter space can be found where the block does not topple over. When the sine pulse is repeated, as with harmonic forcing, infinite numbers of such outcomes appear. The block is able to overturn after any number of impacts. Hogan introduced the idea of the domain of maximum transients to the rocking block problem in order to examine overturning.

One extremely important feature of the rocking block that has yet to be mentioned in detail is that of impact. This appears to be a rather contentious part of the problem. Many authors such as Housner and Brzeski et al. [6] have looked at impact via conservation of angular momentum before and after impact with the ground. This fixes a value for the coefficient of angular restitution. Although, it is questionable whether or not angular momentum is in fact conserved. Hogan employs a coefficient of restitution, but imposes no constraints on its value. However, simplicity has its limits. Experiments performed by Lipscombe et al. [7] in 1993 show that the block actually bounces and loses contact with the ground. In such a case an angular coefficient of restitution misses vital information about the problem. On the other hand it is noted by Lipscombe et al. that the “simple rocking model” is adequate for sufficiently slender blocks, but one should be aware that small errors accumulate as the number of rocking cycles increases. This does not affect the dynamics of the block whilst in the rocking phase of the motion, only at impact when the rocking angle $\phi = 0$.

More experimental investigation of the rocking block has been performed by Peña et al [8]. A number of blocks of blue granite were subjected to harmonic and random forcing. In addition, their response under free vibration was recorded. Peña et al. determined experimentally estimates for the parameters needed to describe the system analytically. There is some disagreement between the theoretical and experimental results, but this is to be expected with the simplicity of the classical rocking model.

The rocking block with pendulum

Current literature concerning the rocking block has investigated whether the oscillations can be controlled by the addition of a tuned pendulum. The pendulum should oscillate, leaving the block as stationary as possible. A diagram is shown in Figure 2. Collini et al. [3], De Leo et al. [2] and Brzeski et al. [6] have made notable progress in this area.

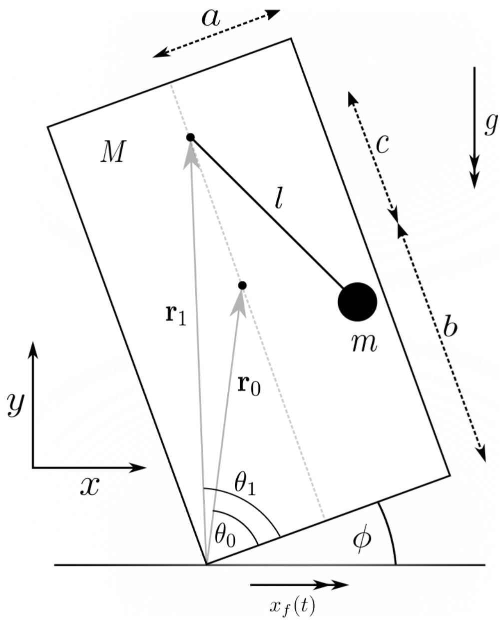

Figure 2: Rocking block with attached pendulum in left rocking motion under forcing $f(t)$. The block angle ϕ is measured from horizontal as in Figure 1. The pendulum angle ψ (not shown) is measured from the downward vertical. Also not shown on the diagram is the polar moment of inertia J about the centre of mass, $\mathbf{r}_0$.

Adding a pendulum to the rocking block significantly increases the complexity of an already subtle problem. Whereas Housner, Hogan and others were able to determine equations of motion through the Newtonian formulation, now it is rather more involved. To this end, Collini et al. and De Leo et al. switched to the Lagrangian formulation with generalised coordinates ϕ and ψ corresponding to the block and the pendulum angle respectively. The block-pendulum system resembles a double pendulum, with an important difference: the block is a rigid body not a point mass. Physically and mathematically, the system described by Figure 2 is closely related to the compound double pendulum. Rafat et al. [9] uncovered two normal modes for the compound double pendulum at lower energies. Correspondingly we should expect to find normal modes in the block-pendulum system in the absence of impact. Notably the compound double pendulum is known to have chaotic dynamics, even in the absence of forcing. The unforced rocking block alone does not exhibit such dynamics. This is evidence for the step up in complexity when transitioning to the block-pendulum system.

A key aim of work performed by Collini et al. was to tune the pendulum to decrease the amplitude of the block oscillations. They first examine the amplitude response of the lone block to a harmonic forcing. It is shown that the block has no resonant frequency. However, when the pendulum is included, resonance becomes an important factor. The question is: can this resonant frequency be shifted away from physically realisable forcings? Collini et al. simulated multiple cases with different pendulum configurations by varying the length and mass. They conclude the pendulum can greatly reduce the oscillations at frequencies away from the resonant frequency. They also concluded that the length of the pendulum makes little difference, whereas its mass makes a large difference. For pendulums with mass greater than 20% of the block mass, oscillations can be removed entirely in special cases.

Whilst Collini et al. examined the block-pendulum system under a harmonic forcing, De Leo et al. performed a similar analysis with a one-sine pulse forcing. They also assumed a more general rigid body shape for the block. A similar conclusion to the one put forward by Collini et al. is reached, namely that the addition of a tuned pendulum improves the blocks response. A pulse that would make the block topple may no longer do so when a pendulum is added. However both Collini et al. and De Leo et al. have eschewed more analytical work in favour of jumping into the numerical simulation. This project aims at filling this gap in analytical work, extending Hogan's work to the new system.

3 Project plan

Action	Time Frame	Relevance to project
Research the problem	Weeks 1-2	Gain a background understanding of the project
Derive equations of motion	Weeks 2-5	These are needed for both analytical and numerical work
Thoroughly check equations for correctness	Weeks 5-6	Very important to begin further work on solid foundations
Solve homogeneous equations	Weeks 6-7	Opens up further analysis of unforced problem.
Find the particular solution	Weeks 7-8	Opens up analysis of the forced problem, i.e. resonant frequencies.
Write interim report	Weeks 8-9	
Look for periodic solutions and normal modes without impact	Weeks 9-10	Important to match up with Hogan's rocking block analysis [4]
Study impact conditions and their effect on solutions	Weeks 10-11	Enables study of the full non-smooth problem
Look for periodic solutions and normal modes with impact.	Weeks 11-12	Again matches up with Hogan's analysis for just the rocking block.
Examine findings for dependencies on pendulum characteristics.	Weeks 12-14	Main aim of the project

Verify this by simulation	Weeks 14-15	Check that analysis is correct and figures
Poster Presentation	Week 15	
Finish technical work, make diagrams and write report	Weeks 15-19	Also provides some leeway in case of setbacks or dead ends
Final Draft Hand-in	Week 19	
Proofreading	Week 19 - End	

4 Progress

Equations of motion

In order to obtain the equations of motion for the block and the pendulum, the Lagrangian formulation was used. Referring to Figure 2, the kinetic and potential energies for the block, of mass M, and pendulum, of mass m, in the positive rocking case ($\phi > 0$) can be calculated:

$$\begin{aligned}
T_M &= \frac{M}{2}\Big\{\big(\dot{x}_f - \dot{\phi}\mathbf{r}_0\sin(\theta_0+\phi)\big)^2 + (\dot{\phi}\mathbf{r}_0\cos(\theta_0+\phi)^2\Big\} + \frac{J\dot{\phi}^2}{2},\\
T_m &= \frac{m}{2}\Big\{\big(\dot{x}_f - \dot{\phi}\mathbf{r}_1\sin(\theta_1+\phi) + l\dot{\psi}\cos(\psi)\big)^2 + \big(\dot{\phi}\mathbf{r}_1\cos(\theta_1+\phi) + l\dot{\phi}\sin(\psi)\big)^2\Big\},\\
V_M &= Mg\,\mathbf{r}_0\sin(\theta_0+\phi),\\
V_m &= mg\big(\mathbf{r}_1\sin(\theta_1+\phi) - l\cos(\psi)\big),
\end{aligned}$$

where x_f is the forcing function. The Lagrangian, $\mathcal{L}$, for the system is the sum $T_M + T_m - V_M - V_m$, the kinetic energy minus the potential energy. Once these quantities have been obtained it's straightforward but laborious to derive the equations of motion for ϕ and ψ

$$\begin{aligned}
\ddot{\phi}(Mr_0^2 + mr_1^2 + J) - \ddot{x}_f(M\mathbf{r}_0\sin(\theta_0+\phi) + m\mathbf{r}_1\sin(\theta_1+\phi)) - ml\mathbf{r}_1\ddot{\psi}\sin(\theta_1+\phi-\psi) \quad (1)\\
+ml\mathbf{r}_1\dot{\psi}^2\cos(\theta_1+\phi-\psi) + Mg\mathbf{r}_0\cos(\theta_0+\phi) + mg\mathbf{r}_1\cos(\theta_1+\phi) = 0,\\
\ddot{\psi}ml^2 + ml\ddot{x}_f\cos(\phi) - ml\mathbf{r}_1\ddot{\phi}\sin(\theta_1+\phi-\psi) - ml\dot{\phi}^2\mathbf{r}_1\cos(\theta_1+\phi-\psi) + mlg\sin(\psi) = 0. \quad (2)
\end{aligned}$$

Simplifying the equations of motion

By restricting our analysis to slender blocks, small angles ϕ, ψ can be assumed. By also assuming a small forcing x_f, Equations (1) and (2) can be linearised as follows

$$\ddot{\phi}(Mr_0^2 + mr_1^2 + J) - ml(b+c)\ddot{\psi} \pm ga(M+m) + g(Mb + m(b+c))\phi = \ddot{x}_f(Mb + m(b+c)), \quad (3)$$

$$ml^2\ddot{\psi} - ml(b+c)\ddot{\phi} + mlg\psi = -ml\ddot{x}_f, \quad (4)$$

where a, b, c are lengths as described in Figure 2. The $\pm$ sign denotes the equations for the rocking cases ($\phi > 0$ and $\phi < 0$) respectively. It is beneficial to stop for a moment and perform some checks on Equations (3) and (4) to give a measure of confidence. Firstly, all terms are dimensionally consistent and have units $\mathrm{kg\,m^2\,s^{-2}}$. When $c = b$ i.e. the pendulum is fixed to the top of the block, these equations match those obtained by Collini et al. [3] and De Leo et al. [2], and when $c = 0$ we obtain very similar equations to those obtained by Scammell [10]. Letting $m = 0$ to remove the pendulum the bottom equation disappears and the top equation becomes that of a forced inverted pendulum as obtained by Hogan. Fixing $\phi = 0$ Equation (3) is undefined and Equation (4) becomes simple harmonic motion as one would expect. It is also possible to linearise the Lagrangian, keeping only the terms that will become linear after differentiating, and then derive the equations of motion. The resulting equations are the same. This is all good evidence that the equations are correct.

To simplify the linearised equations further, let $c = b$ to place the pendulum at the top of the block. Note that the polar moment of inertia for a uniform rectangle with side lengths $2a$ and $2b$ is $J = (M/3)(a^2+b^2)$ and introduce $\mu = m/M$. The co-dependence on the angular accelerations can also be removed. The equations are now only coupled through their angular position terms ϕ and ψ and can be written in matrix form

$$\begin{pmatrix}\ddot{\phi}\\ \ddot{\psi}\end{pmatrix} = \left(\frac{g}{\mu a^2 + 4/3r_0^2}\right)\begin{pmatrix} b(1+2\mu) & -2b\mu \\ \frac{2b^2}{l}(1+2\mu) & -\frac{1}{3l}(3\mu r_0^2 + 4r_0^2)\end{pmatrix}\begin{pmatrix}\phi\\ \psi\end{pmatrix} \tag{5}$$
$$+\left(\frac{g}{\mu a^2 + 4/3r_0^2}\right)\begin{pmatrix} \frac{b}{g} & \mp a(1+\mu) \\ \frac{1}{3gl}(2b^2 - a^2(3\mu+4)) & \mp\frac{2ab}{l}(1+\mu)\end{pmatrix}\begin{pmatrix}\ddot{x}_f\\ 1\end{pmatrix},$$

where $r_0^2 = a^2 + b^2$ and $r_1^2 = a^2 + 4b^2$ and $\mp$ sign corresponds to the positive and negative rocking cases respectively. We could also apply a variable transformation of $\phi' = \phi \pm a(1+\mu)/b(1+2\mu)$ and $\psi' = \psi$ to remove the constant term. This scaling is exactly the steady state solution, $\bar{\phi}$, for the unforced problem and it shifts the dependence on the sign of the rocking angle to the transformation. However this complicates the impact conditions, so the equations are left untransformed.

Equation (5) is a system of second order linear inhomogeneous ODEs, $\ddot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{f}(t)$. In general, the solution to such a problem is the sum of the complementary solution and the particular solution. The complementary solution is the solution of the unforced system, which is useful to uncover periodic solutions and normal modes without forcing. The particular solution will give information about any resonance induced by the forcing.

Solving the homogeneous equation - the unforced system

To solve for the complementary solution, we split the two second order ODEs into 4 first order ODEs and solve the resulting equation, $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. This will have solution

$$\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2 + C_3 e^{\lambda_3 t}\mathbf{v}_3 + C_4 e^{\lambda_4 t}\mathbf{v}_4,$$

where C_i's are determined by initial conditions and λ_i and $\mathbf{v}_i$ are the eigenvalues and eigenvectors of $\mathbf{A}$ respectively. Computing the eigenvalues and eigenvectors of the 4×4 matrix involves solving a quartic equation, which is very difficult to do analytically. However, because there are really two second order ODE's the matrix $\mathbf{A}$ has many zero entries. The characteristic equation of $\mathbf{A}$ is a quadratic in λ^2

$$\lambda^4 + (E-A)\lambda^2 + (BD - AE) = 0$$

with solution

$$\lambda = \pm\sqrt{\frac{1}{2}}\sqrt{-(A-E) \pm \sqrt{(A-E)^2 + 4(AE-BD)}}, \tag{6}$$

where A, B, D, E are positive constants originating from the coefficients in Equation (5)

$$A = \frac{gb(1+2\mu)}{\mu a^2 + 4/3\mathbf{r}_0^2}, \qquad B = \frac{2gb\mu}{\mu a^2 + 4/3\mathbf{r}_0^2}, \tag{7}$$
$$D = \frac{2gb^2(1+2\mu)}{l(\mu a^2 + 4/3\mathbf{r}_0^2)}, \qquad E = \frac{g(3\mu\mathbf{r}_1^2 + 4\mathbf{r}_0^2)}{3l(\mu a^2 + 4/3\mathbf{r}_0^2)},$$

The quantity $AE - BD$ under the square root is positive for all physical parameter values

$$AE - BD = \frac{g^2 b(1+2\mu)}{l(\mu a^2 + 4/3\mathbf{r}_0^2)} > 0,$$

The inner square root in Equation (6) is therefore positive and real. This implies that there are a pair of real eigenvalues, one positive, one negative, λ_1 and $-\lambda_1$, and two purely imaginary conjugate eigenvalues, $i\lambda_3$ and $-i\lambda_3$. Upon further analysis of the homogeneous solution, the real eigenvalues are expected to form non-oscillatory terms through sinh and cosh, while the imaginary eigenvalues should form the basis of an oscillatory solution through sin and cos. This is where periodic solutions will be found.

Solving the inhomogeneous equation - Looking for resonance

Usually the way to solve for the particular solution is with an ansatz. Substituting

$$\begin{pmatrix} \phi(t) \\ \psi(t) \end{pmatrix} = \begin{pmatrix} a_1\cos(\omega t) + b_1\sin(\omega t) \\ a_2\cos(\omega t) + b_2\sin(\omega t) \end{pmatrix} + \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

into Equation (5) with a harmonic forcing $x_f(t) = \beta\cos(\omega t)$ gives the particular solution

$$\begin{pmatrix} \phi(t) \\ \psi(t) \end{pmatrix} = \begin{pmatrix} \frac{\beta\omega^2(FB - C(E-\omega^2)}{BD-(E-\omega^2)(A+\omega^2)} \\ \frac{\beta\omega^2(A+\omega^2)(FB-C(E-\omega^2))}{B(BD-(E-\omega^2)(A+\omega^2))} - \frac{\beta C\omega^2}{B} \end{pmatrix}\cos(\omega t) \pm \begin{pmatrix} \bar{\phi} \\ 0 \end{pmatrix},$$

where A, B, D, E are the same positive constants defined in Equation (7). Similarly, C and F originate from the coefficients of the forcing terms in Equation (5). The additive constant $\bar{\phi}$ is the homogeneous steady state of the block angle. The solutions of the polynomial on the denominator, $BD-(E-\omega^2)(A+\omega^2) = 0$, give the resonant frequencies for the system. Clearly it is exactly the same polynomial as the characteristic equation for the homogeneous solution, meaning that resonance occurs when the forcing frequency equals the real positive eigenvalue of $\mathbf{A}$. Therefore, from Equation (6) there is only one real, positive solution of the polynomial and consequently only one resonant frequency in the system. In addition, if we remove the pendulum the polynomial in ω reduces to $(A+\omega^2)$ which gives no real positive roots. Therefore the block has no resonant frequency and by adding the pendulum to the system we run the risk of encountering one.

Conclusions and future work

This report has extended Scammell's [10] work by deriving the equations of motion for a pendulum attached not at the centre of mass but a distance c above the centre of mass. These equations, (1) and (2), also agree with those of Collini et al. [3] and De Leo et al. [2], a significant step in unifying the three works.

The equations of motion were linearised and the coupling in angular acceleration removed. They were then converted into a system of four coupled linear inhomogeneous ODEs. Looking at the forced system and the unforced system separately, the complementary and particular solutions to the ODEs were found. Upon further analysis the complementary solution will give information about any normal modes present in the system. The particular solution has shown the existence of a single resonant frequency, which will require further examination to see how it can be shifted away from physically realisable frequencies present in earthquake excitations.

With both parts to the general solution now found, it should be possible to determine the trajectories of ϕ and ψ for general initial conditions. Thus far impact has been completely ignored. It would be interesting to examine these solutions with impact to see if any periodic cycles survive the switching of the governing equations. Hogan [4] proved they exist for the rocking block, but will they still exist with the additional pendulum?

Immediate further work will continue to examine the whole analytic solution. Later in the project the analytic findings will be confirmed numerically, and augmented with purely numerical work. The parameters of the tuned pendulum will be determined and their effects on the rocking block quantified for a wide range of scenarios.

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