PHIL 4110: Problem Set 3

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Problem 1

Solution

Part One
$$s(v) = 1, \quad \mathfrak{M}, s \models \exists x(A(f(z), c) \supset \forall y(A(y, x) \lor A(f(y), x)))$$

Let $\varphi_1 \equiv A(f(z), c) \supset \forall y(A(y, x) \lor A(f(y), x))$
$\mathfrak{M}, s \models \exists x \varphi_1$ iff $\mathfrak{M}, s[m_1/x] \models \varphi_1$ for some $m_1 \in |\mathfrak{M}|$
$\mathfrak{M}, s[m_1/x] \models A(f(z), c) \supset \forall y(A(y, x) \lor A(f(y), x))$ for some $m_1 \in |\mathfrak{M}|$
This is equivalent to the expression $\mathfrak{M}, s[m_1/x] \not\models A(f(z), c)$ or $\mathfrak{M}, s[m_1/x] \models \forall y(A(y, x) \lor A(f(y), x))$ for some $m_1 \in |\mathfrak{M}|$.

> Evaluating $\mathfrak{M}, s[m_1/x] \not\models A(f(z), c)$ for some $m_1 \in |\mathfrak{M}|$:
> $= \langle \text{Val}^{\mathfrak{M}}_{s[m_1/x]}(f(z)), c \rangle \notin A^{\mathfrak{M}}$
> $= \langle f^{\mathfrak{M}}(\text{Val}^{\mathfrak{M}}_{s[m_1/x]}(z)), \text{Val}^{\mathfrak{M}}_{s[m_1/x]}(c) \rangle \notin A^{\mathfrak{M}}$
> $= \langle f^{\mathfrak{M}}(1), c^{\mathfrak{M}} \rangle \notin A^{\mathfrak{M}}$
> $= \langle 2, 3 \rangle \notin A^{\mathfrak{M}} = $ False, since $\langle 2, 3 \rangle$ is in $A^{\mathfrak{M}}$

> Evaluating $\mathfrak{M}, s[m_1/x] \models \forall y(A(y, x) \lor A(f(y), x))$ for some $m_1 \in |\mathfrak{M}|$:
> Let $\varphi_2 \equiv A(y, x) \lor A(f(y), x)$
> $\mathfrak{M}, s[m_1/x] \models \forall y \varphi_2$ iff $\mathfrak{M}, s[m_1/x][m_2/y] \models \varphi_2$ for some m_1 for all m_2, such that $m_1, m_2 \in |\mathfrak{M}|$
> This is just the expression $\mathfrak{M}, s[m_1/x][m_2/y] \models A(y, x)$ or $\mathfrak{M}, s[m_1/x][m_2/y] \models A(f(y), x)$, for some m_1 for all m_2.

>> Evaluating $\mathfrak{M}, s[m_1/x][m_2/y] \models A(y, x)$, for some m_1 for all m_2:
>> $= \langle \text{Val}^{\mathfrak{M}}_{s[m_1/x][m_2/y]}(y), \text{Val}^{\mathfrak{M}}_{s[m_1/x][m_2/y]}(x) \rangle \in A^{\mathfrak{M}}$
>> $= \langle m_2, m_1 \rangle \in A^{\mathfrak{M}}$ for some m_1 for all m_2, such that $m_1, m_2 \in |\mathfrak{M}|$
>> We must check for all possible values of m_2 in the domain:
>> $= \langle 1, m_1 \rangle \in A^{\mathfrak{M}}$ and $\langle 2, m_1 \rangle \in A^{\mathfrak{M}}$ and $\langle 3, m_1 \rangle \in A^{\mathfrak{M}}$

>> Evaluating $\mathfrak{M}, s[m_1/x][m_2/y] \models A(f(y), x)$, for some m_1 for all m_2.
>> $= \langle \text{Val}^{\mathfrak{M}}_{s[m_1/x][m_2/y]}(f(y)), \text{Val}^{\mathfrak{M}}_{s[m_1/x][m_2/y]}(x) \rangle \in A^{\mathfrak{M}}$
>> $= \langle f^{\mathfrak{M}}(\text{Val}^{\mathfrak{M}}_{s[m_1/x][m_2/y]}(y)), m_1 \rangle \in A^{\mathfrak{M}}$
>> $= \langle f^{\mathfrak{M}}(m_2), m_1 \rangle \in A^{\mathfrak{M}}$ for some m_1 for all m_2, such that $m_1, m_2 \in |\mathfrak{M}|$
>> We must check for all possible values of m_2 in the domain:
>> $= \langle f^{\mathfrak{M}}(1), m_1 \rangle \in A^{\mathfrak{M}}$ and $\langle f^{\mathfrak{M}}(2), m_1 \rangle \in A^{\mathfrak{M}}$ and $\langle f^{\mathfrak{M}}(3), m_1 \rangle \in A^{\mathfrak{M}}$
>> $= \langle 2, m_1 \rangle \in A^{\mathfrak{M}}$ and $\langle 3, m_1 \rangle \in A^{\mathfrak{M}}$ and $\langle 2, m_1 \rangle \in A^{\mathfrak{M}}$
>> $= \langle 2, m_1 \rangle \in A^{\mathfrak{M}}$ and $\langle 3, m_1 \rangle \in A^{\mathfrak{M}}$

> $\mathfrak{M}, s[m_1/x][m_2/y] \models \varphi_2$ becomes the expression:

$$(\langle 1, m_1 \rangle \in A^{\mathfrak{M}} \text{ and } \langle 2, m_1 \rangle \in A^{\mathfrak{M}} \text{ and } \langle 3, m_1 \rangle \in A^{\mathfrak{M}}) \text{ or } (\langle 2, m_1 \rangle \in A^{\mathfrak{M}} \text{ and } \langle 3, m_1 \rangle \in A^{\mathfrak{M}})$$

> for some $m_1 \in |\mathfrak{M}|$
> Let $m_1 = 3$, then the second half of the 'or' operation is satisfied, since $\langle 2, 3 \rangle, \langle 3, 3 \rangle \in A^{\mathfrak{M}}$.
> Therefore, $\mathfrak{M}, s[m_1/x][m_2/y] \models \varphi_2$ and $\mathfrak{M}, s[m_1/x] \models \forall y(A(y, x) \lor A(f(y), x))$

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Recall that $\mathfrak{M}, s \models \exists x \varphi_1$ iff $(\mathfrak{M}, s[m_1/x] \not\models A(f(z), c)$ or $\mathfrak{M}, s[m_1/x] \models \forall y(A(y, x) \vee A(f(y), x)))$ for some $m_1 \in |\mathfrak{M}|$.

We have shown the second half of the 'or' operation, therefore $\mathfrak{M}, s \models \exists x \varphi_1$ and $\mathfrak{M}, s$ satisfies the original formula.

Part Two

Use the originial variable assignment and structrue but have $A^{\mathfrak{M}} = \{\langle 1, 3 \rangle \langle 2, 2 \rangle \langle 3, 3 \rangle\}$.

Problem 2

Proposition 7.14. If the free variables in φ are among $x_1, \ldots, x_n$, and $s_1(x_i) = s_2(x_i)$ for $i = 1, \ldots, n$, then $\mathfrak{M}, s_1 \models \varphi$ iff $\mathfrak{M}, s_2 \models \varphi$.

Induction Hypothesis: Assume $\mathfrak{M}, s_1 \models \varphi$ iff $\mathfrak{M}, s_2 \models \varphi$ for all formulae ψ less complex than φ.

Solution

Part One

$\varphi \equiv \psi \supset \chi$

By definition, $\mathfrak{M}, s_1 \models \varphi$ iff $\mathfrak{M}, s_1 \not\models \psi$ or $\mathfrak{M}, s_1 \models \chi$.

By the IH we have $\mathfrak{M}, s_2 \not\models \psi$ or $\mathfrak{M}, s_2 \models \chi$. Then, $\mathfrak{M}, s_2 \models \varphi$.

The other direction of the proof relies on similar reasoning.

Part Two

$\varphi \equiv \forall x \psi$

If $\mathfrak{M}, s_1 \models \varphi$, then for any variable assignment s_1, $\mathfrak{M}, s_1[m/x] \models \psi$ for all $m \in |\mathfrak{M}|$.

Let $s'_1 = s_1[m/x]$, $s'_2 = s_2[m/x]$, for all $m \in |\mathfrak{M}|$.

By IH we get $s_1(x_i) = s_2(x_i)$ for any x_i in the free variables of φ.

Since s'_1, s'_2 are x-variants of s_1, s_2 respectively, and the free variables of ψ are in those of φ with x, then we know $s'_1(x_j) = s'_2(x_j)$ for any x_j in the free variables of ψ.

Since the x-variants s'_1, s'_2 agree on any variable assignment we can apply the IH, deriving $\mathfrak{M}, s'_2 \models \psi$ from $\mathfrak{M}, s'_1 \models \psi$.

If $\mathfrak{M}, s'_2 \models \psi$ where $s'_2 = s_2[m/x]$, for all $m \in |\mathfrak{M}|$, then $\mathfrak{M}, s_2 \models \varphi$.

The other direction of the proof relies on similar reasoning.

Problem 3

Proposition 7.17. Let $\mathfrak{M}$ be a structure, φ be a sentence, and s a variable assignment.

$\mathfrak{M} \models \varphi$ iff $\mathfrak{M}, s \models \varphi$.

Proof

If $\mathfrak{M} \models \varphi$, then $\mathfrak{M}$ satisfies the sentence φ, and for all variable assignments s, $\mathfrak{M}, s \models \varphi$.

Working in the opposite direction, if $\mathfrak{M}, s \models \varphi$, and φ is a sentence, then corollary 7.15 allows us to

say that $\mathfrak{M}, s' \models \varphi$ for every variable assignment s'.

Since φ is a sentence and is satisfied for every s', then we can write $\mathfrak{M}, s \models \varphi$ for all variable assignments s.

Problem 4

Proposition 7.18. Suppose $\varphi(x)$ only contains x free, and $\mathfrak{M}$ is a structure. Then:

1. $\mathfrak{M} \models \exists x \varphi(x)$ iff $\mathfrak{M}, s \models \varphi(x)$ for at least one variable assignment s.

2. $\mathfrak{M} \models \forall x \varphi(x)$ iff $\mathfrak{M}, s \models \varphi(x)$ for all variable assignments s.

Proof

Part One
If $\mathfrak{M} \models \exists x \varphi(x)$, then by definition $\mathfrak{M}, s[m/x] \models \varphi(x)$ for some variable assignment s from x to some $m \in |\mathfrak{M}|$.
Therefore there must be at least one variable assignment s such that $\mathfrak{M}, s \models \varphi(x)$.

Working in the opposite direction, assume $\mathfrak{M}, s \models \varphi(x)$ for at least one variable assignment s.
Since x is the only free variable and $\mathfrak{M}, s \models \varphi(x)$, we know s must at least assign x to some value $m \in |\mathfrak{M}|$
such that $\varphi(x)$ is satisfied relative to s.
Then we write $\mathfrak{M}, s[m/x] \models \varphi(x)$ for some $m \in |\mathfrak{M}|$, which gives way to $\mathfrak{M} \models \exists x \varphi(x)$.

Part Two
If $\mathfrak{M} \models \forall x \varphi(x)$, then by definition for any variable assignment s, $\mathfrak{M}, s[m/x] \models \varphi(x)$ for all $m \in |\mathfrak{M}|$.
The satisfaction of $\varphi(x)$ relative to any s is not restricted by any particular assignment of x, so we can write
$\mathfrak{M}, s \models \varphi(x)$ for all s.

Working in the opposite direction, assume $\mathfrak{M}, s \models \varphi(x)$ for all variable assignments s.
Then any s could assign x to any member of the domain, and we would still have $\mathfrak{M}, s \models \varphi(x)$.
We can express this property of s as $\mathfrak{M}, s[m/x] \models \varphi(x)$ for all $m \in |\mathfrak{M}|$.
This implies $\mathfrak{M} \models \forall x \varphi(x)$.

Problem 5

Proposition 7.19.
Let φ be a formula, and $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be structures with $|\mathfrak{M}_1| = |\mathfrak{M}_2|$, and s a variable assignment on $|\mathfrak{M}_1| = |\mathfrak{M}_2|$.
If
$$c^{\mathfrak{M}_1} = c^{\mathfrak{M}_2},$$
$$R^{\mathfrak{M}_1} = R^{\mathfrak{M}_2},$$
$$\text{and } f^{\mathfrak{M}_1} = f^{\mathfrak{M}_2}$$
for every constant symbol c, relation symbol R, and function symbol f occurring in φ,
then $\mathfrak{M}_1, s \models \varphi$ iff $\mathfrak{M}_2, s \models \varphi$.

Proof

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Part One: Valuation, Induction on Terms
We must show that for any term t, $\text{Val}_s^{\mathfrak{M}_1}(t) = \text{Val}_s^{\mathfrak{M}_2}(t)$
Base:

1. $t \equiv c$
   $\text{Val}_s^{\mathfrak{M}_1}(c) = c^{\mathfrak{M}_1} = c^{\mathfrak{M}_2} = \text{Val}_s^{\mathfrak{M}_2}(c)$

2. $t \equiv v$, where v is a variable.
   $\text{Val}_s^{\mathfrak{M}_1}(v) = s(v) = \text{Val}_s^{\mathfrak{M}_2}(v)$. s is the same for both domains.

IH: Assume for a term t, for any subterm t_i in $t_1 \ldots t_n$, $\text{Val}_s^{\mathfrak{M}_1}(t_i) = \text{Val}_s^{\mathfrak{M}_2}(t_i)$.
      Let $t \equiv f(t_1 \ldots t_n)$.
      $\text{Val}_s^{\mathfrak{M}_1}(t) = f^{\mathfrak{M}_1}(\text{Val}_s^{\mathfrak{M}_1}(t_1) \ldots \text{Val}_s^{\mathfrak{M}_1}(t_n))$
      Apply the IH, and then
      $\text{Val}_s^{\mathfrak{M}_1}(t) = f^{\mathfrak{M}_1}(\text{Val}_s^{\mathfrak{M}_2}(t_1) \ldots \text{Val}_s^{\mathfrak{M}_2}(t_n))$
      Since $f^{\mathfrak{M}_1} = f^{\mathfrak{M}_2}$ we can write
      $\text{Val}_s^{\mathfrak{M}_1}(t) = f^{\mathfrak{M}_2}(\text{Val}_s^{\mathfrak{M}_2}(t_1) \ldots \text{Val}_s^{\mathfrak{M}_2}(t_n))$
      $\text{Val}_s^{\mathfrak{M}_1}(t) = \text{Val}_s^{\mathfrak{M}_2}(t)$

The reasoning in the other direction is symmetrical.

Part Two: Satisfaction, Induction on Formulae
We must show that for any formula φ, $\mathfrak{M}_1, s \models \varphi$ iff $\mathfrak{M}_2, s \models \varphi$
Base: φ is atomic

1. $\varphi \equiv \bot$
   By definition, any $\mathfrak{M}, s$ satisfies $\bot$, therefore $\mathfrak{M}_1, s \models \varphi$ and $\mathfrak{M}_2, s \models \varphi$

2. $\varphi \equiv R(t_1 \ldots t_n)$
   $\mathfrak{M}_1, s \models \varphi$ iff $\langle \text{Val}_s^{\mathfrak{M}_1}(t_1) \ldots \text{Val}_s^{\mathfrak{M}_1}(t_n) \rangle \in R^{\mathfrak{M}_1}$
   From our valuation proof, we know $\text{Val}_s^{\mathfrak{M}_1}(t_1) \ldots \text{Val}_s^{\mathfrak{M}_1}(t_n) = \text{Val}_s^{\mathfrak{M}_2}(t_1) \ldots \text{Val}_s^{\mathfrak{M}_2}(t_n)$. Rewrite:
   $\langle \text{Val}_s^{\mathfrak{M}_2}(t_1) \ldots \text{Val}_s^{\mathfrak{M}_2}(t_n) \rangle \in R^{\mathfrak{M}_1}$
   Since $R^{\mathfrak{M}_1} = R^{\mathfrak{M}_2}$,
   $\langle \text{Val}_s^{\mathfrak{M}_2}(t_1) \ldots \text{Val}_s^{\mathfrak{M}_2}(t_n) \rangle \in R^{\mathfrak{M}_2}$
   Then $\mathfrak{M}_2, s \models \varphi$

3. $\varphi \equiv\, = (t_1, t_2)$
   $\mathfrak{M}_1, s \models \varphi$ iff $\text{Val}_s^{\mathfrak{M}_1}(t_1) = \text{Val}_s^{\mathfrak{M}_1}(t_2)$
   We know $\text{Val}_s^{\mathfrak{M}_1}(t) = \text{Val}_s^{\mathfrak{M}_2}(t)$, so
   $\text{Val}_s^{\mathfrak{M}_2}(t_1) = \text{Val}_s^{\mathfrak{M}_2}(t_2)$
   Then $\mathfrak{M}_2, s \models \varphi$

IH: Assume the result holds for all formulae less complex than φ.

1. $\varphi \equiv\, \sim \psi$
   $\mathfrak{M}_1, s \models \varphi$ iff $\mathfrak{M}_1, s \not\models \psi$
   If $\mathfrak{M}_1, s \not\models \psi$, then by IH $\mathfrak{M}_2, s \not\models \psi$
   Then $\mathfrak{M}_2, s \models \varphi$

2. $\varphi \equiv \psi \wedge \chi$
   $\mathfrak{M}_1, s \models \varphi$ iff $\mathfrak{M}_1, s \models \psi$ and $\mathfrak{M}_1, s \models \chi$
   Then by IH $\mathfrak{M}_2, s \models \psi$ and $\mathfrak{M}_2, s \models \chi$
   Then $\mathfrak{M}_2, s \models \varphi$

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3. $\varphi \equiv \psi \lor \chi$
   $\mathfrak{M}_1, s \models \varphi$ iff $\mathfrak{M}_1, s \models \psi$ or $\mathfrak{M}_1, s \models \chi$
   Then by IH $\mathfrak{M}_2, s \models \psi$ or $\mathfrak{M}_2, s \models \chi$
   Then $\mathfrak{M}_2, s \models \varphi$

4. $\varphi \equiv \psi \supset \chi$
   $\mathfrak{M}_1, s \models \varphi$ iff $\mathfrak{M}_1, s \not\models \psi$ or $\mathfrak{M}_1, s \models \chi$
   Then by IH $\mathfrak{M}_2, s \not\models \psi$ or $\mathfrak{M}_2, s \models \chi$
   Then $\mathfrak{M}_2, s \models \varphi$

5. $\varphi \equiv \exists x \psi$
   $\mathfrak{M}_1, s \models \varphi$ iff $\mathfrak{M}_1, s[m/x] \models \psi$ for some $m \in |\mathfrak{M}_1|$
   Since $s[m/x](x) = m \in |\mathfrak{M}_1|$, and $|\mathfrak{M}_1| = |\mathfrak{M}_2|$, then with IH we can write:
   $\mathfrak{M}_2, s[m/x] \models \psi$ for some $m \in |\mathfrak{M}_2|$
   Then $\mathfrak{M}_2, s \models \varphi$

6. $\varphi \equiv \forall x \psi$
   $\mathfrak{M}_1, s \models \varphi$ iff $\mathfrak{M}_1, s[m/x] \models \psi$ for all $m \in |\mathfrak{M}_1|$
   Since $s[m/x](x) = m \in |\mathfrak{M}_1|$, and $|\mathfrak{M}_1| = |\mathfrak{M}_2|$, then with IH we can write:
   $\mathfrak{M}_2, s[m/x] \models \psi$ for all $m \in |\mathfrak{M}_2|$
   Then $\mathfrak{M}_2, s \models \varphi$

The reasoning in the other direction is symmetrical.

By induction, we've shown if two structures declare the same domain, constants, functions, relations, and variable assignments, then they must only satisfy the same formulae.