PHIL 4110: Problem Set 3

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Problem 1

Solution

Part One

$$s(v) = 1, \quad \mathfrak{M}, s \models \exists x(A(f(z), c) \supset \forall y(A(y, x) \vee A(f(y), x)))$$

Let $\varphi_1 \equiv A(f(z), c) \supset \forall y(A(y, x) \vee A(f(y), x))$
$\mathfrak{M}, s \models \exists x \varphi_1$ iff $\mathfrak{M}, s[m_1/x] \models \varphi_1$ for some $m_1 \in |\mathfrak{M}|$
$\mathfrak{M}, s[m_1/x] \models A(f(z), c) \supset \forall y(A(y, x) \vee A(f(y), x))$ for some $m_1 \in |\mathfrak{M}|$
This is equivalent to the expression $\mathfrak{M}, s[m_1/x] \not\models A(f(z), c)$ or $\mathfrak{M}, s[m_1/x] \models \forall y(A(y, x) \vee A(f(y), x))$ for some $m_1 \in |\mathfrak{M}|$.

Evaluating $\mathfrak{M}, s[m_1/x] \not\models A(f(z), c)$ for some $m_1 \in |\mathfrak{M}|$:
$= \langle \mathrm{Val}^{\mathfrak{M}}_{s[m_1/x]}(f(z)), c\rangle \notin A^{\mathfrak{M}}$
$= \langle f^{\mathfrak{M}}(\mathrm{Val}^{\mathfrak{M}}_{s[m_1/x]}(z)), \mathrm{Val}^{\mathfrak{M}}_{s[m_1/x]}(c)\rangle \notin A^{\mathfrak{M}}$
$= \langle f^{\mathfrak{M}}(1), c^{\mathfrak{M}}\rangle \notin A^{\mathfrak{M}}$
$= \langle 2, 3\rangle \notin A^{\mathfrak{M}} =$ False, since $\langle 2, 3\rangle$ is in $A^{\mathfrak{M}}$

Evaluating $\mathfrak{M}, s[m_1/x] \models \forall y(A(y, x) \vee A(f(y), x))$ for some $m_1 \in |\mathfrak{M}|$:
Let $\varphi_2 \equiv A(y, x) \vee A(f(y), x)$
$\mathfrak{M}, s[m_1/x] \models \forall y \varphi_2$ iff $\mathfrak{M}, s[m_1/x][m_2/y] \models \varphi_2$ for some m_1 for all m_2, such that $m_1, m_2 \in |\mathfrak{M}|$
This is just the expression $\mathfrak{M}, s[m_1/x][m_2/y] \models A(y, x)$ or $\mathfrak{M}, s[m_1/x][m_2/y] \models A(f(y), x)$, for some m_1 for all m_2.

Evaluating $\mathfrak{M}, s[m_1/x][m_2/y] \models A(y, x)$, for some m_1 for all m_2:
$= \langle \mathrm{Val}^{\mathfrak{M}}_{s[m_1/x][m_2/y]}(y), \mathrm{Val}^{\mathfrak{M}}_{s[m_1/x][m_2/y]}(x)\rangle \in A^{\mathfrak{M}}$
$= \langle m_2, m_1\rangle \in A^{\mathfrak{M}}$ for some m_1 for all m_2, such that $m_1, m_2 \in |\mathfrak{M}|$
We must check for all possible values of m_2 in the domain:
$= \langle 1, m_1\rangle \in A^{\mathfrak{M}}$ and $\langle 2, m_1\rangle \in A^{\mathfrak{M}}$ and $\langle 3, m_1\rangle \in A^{\mathfrak{M}}$

Evaluating $\mathfrak{M}, s[m_1/x][m_2/y] \models A(f(y), x)$, for some m_1 for all m_2.
$= \langle \mathrm{Val}^{\mathfrak{M}}_{s[m_1/x][m_2/y]}(f(y)), \mathrm{Val}^{\mathfrak{M}}_{s[m_1/x][m_2/y]}(x)\rangle \in A^{\mathfrak{M}}$
$= \langle f^{\mathfrak{M}}(\mathrm{Val}^{\mathfrak{M}}_{s[m_1/x][m_2/y]}(y)), m_1\rangle \in A^{\mathfrak{M}}$
$= \langle f^{\mathfrak{M}}(m_2), m_1\rangle \in A^{\mathfrak{M}}$ for some m_1 for all m_2, such that $m_1, m_2 \in |\mathfrak{M}|$
We must check for all possible values of m_2 in the domain:
$= \langle f^{\mathfrak{M}}(1), m_1\rangle \in A^{\mathfrak{M}}$ and $\langle f^{\mathfrak{M}}(2), m_1\rangle \in A^{\mathfrak{M}}$ and $\langle f^{\mathfrak{M}}(3), m_1\rangle \in A^{\mathfrak{M}}$
$= \langle 2, m_1\rangle \in A^{\mathfrak{M}}$ and $\langle 3, m_1\rangle \in A^{\mathfrak{M}}$ and $\langle 2, m_1\rangle \in A^{\mathfrak{M}}$
$= \langle 2, m_1\rangle \in A^{\mathfrak{M}}$ and $\langle 3, m_1\rangle \in A^{\mathfrak{M}}$

$\mathfrak{M}, s[m_1/x][m_2/y] \models \varphi_2$ becomes the expression:

$$(\langle 1, m_1\rangle \in A^{\mathfrak{M}} \text{ and } \langle 2, m_1\rangle \in A^{\mathfrak{M}} \text{ and } \langle 3, m_1\rangle \in A^{\mathfrak{M}}) \text{ or } (\langle 2, m_1\rangle \in A^{\mathfrak{M}} \text{ and } \langle 3, m_1\rangle \in A^{\mathfrak{M}})$$

for some $m_1 \in |\mathfrak{M}|$
Let $m_1 = 3$, then the second half of the 'or' operation is satisfied, since $\langle 2, 3\rangle, \langle 3, 3\rangle \in A^{\mathfrak{M}}$.
Therefore, $\mathfrak{M}, s[m_1/x][m_2/y] \models \varphi_2$ and $\mathfrak{M}, s[m_1/x] \models \forall y(A(y, x) \vee A(f(y), x))$

Recall that $\mathfrak{M}, s \models \exists x \varphi_1$ iff $(\mathfrak{M}, s[m_1/x] \not\models A(f(z), c)$ or $\mathfrak{M}, s[m_1/x] \models \forall y(A(y, x) \vee A(f(y), x)))$ for some $m_1 \in |\mathfrak{M}|$.
We have shown the second half of the 'or' operation, therefore $\mathfrak{M}, s \models \exists x \varphi_1$ and $\mathfrak{M}, s$ satisfies the original formula.

Part Two
Use the originial variable assignment and structrue but have $A^{\mathfrak{M}} = \{\langle 1, 3\rangle \langle 2, 2\rangle \langle 3, 3\rangle\}$.

Problem 2

Proposition 7.14. If the free variables in φ are among $x_1, \ldots, x_n$, and $s_1(x_i) = s_2(x_i)$ for $i = 1, \ldots, n$, then $\mathfrak{M}, s_1 \models \varphi$ iff $\mathfrak{M}, s_2 \models \varphi$.
Induction Hypothesis: Assume $\mathfrak{M}, s_1 \models \varphi$ iff $\mathfrak{M}, s_2 \models \varphi$ for all formulae ψ less complex than φ.

Solution

Part One
$\varphi \equiv \psi \supset \chi$

By definition, $\mathfrak{M}, s_1 \models \varphi$ iff $\mathfrak{M}, s_1 \not\models \psi$ or $\mathfrak{M}, s_1 \models \chi$.
By the IH we have $\mathfrak{M}, s_2 \not\models \psi$ or $\mathfrak{M}, s_2 \models \chi$. Then, $\mathfrak{M}, s_2 \models \varphi$.

The other direction of the proof relies on similar reasoning.

Part Two
$\varphi \equiv \forall x \psi$

If $\mathfrak{M}, s \models \varphi$, then for any variable assignment s, $\mathfrak{M}, s[m/x] \models \psi$ for all $m \in |\mathfrak{M}|$.

Problem 3

Proposition 7.17. Let $\mathfrak{M}$ be a structure, φ be a sentence, and s a variable assignment.
$\mathfrak{M} \models \varphi$ iff $\mathfrak{M}, s \models \varphi$.

Proof
If $\mathfrak{M} \models \varphi$, then $\mathfrak{M}$ satisfies the sentence φ, and for all variable assignments s, $\mathfrak{M}, s \models \varphi$.

Working in the opposite direction, if $\mathfrak{M}, s \models \varphi$, and φ is a sentence, then corollary 7.15 allows us to say that $\mathfrak{M}, s' \models \varphi$ for every variable assignment s'.
Since φ is a sentence and is satisfied for every s', then we can write $\mathfrak{M}, s \models \varphi$ for all variable assignments s.

Problem 4

Proposition 7.18. Suppose $\varphi(x)$ only contains x free, and $\mathfrak{M}$ is a structure. Then:

1. $\mathfrak{M} \models \exists x \varphi(x)$ iff $\mathfrak{M}, s \models \varphi(x)$ for at least one variable assignment s.
2. $\mathfrak{M} \models \forall x \varphi(x)$ iff $\mathfrak{M}, s \models \varphi(x)$ for all variable assignments s.

Proof

Part One
If $\mathfrak{M} \models \exists x \varphi(x)$, then by definition $\mathfrak{M}, s[m/x] \models \varphi(x)$ for some variable assignment s from x to some $m \in |\mathfrak{M}|$. Therefore there must be at least one variable assignment s such that $\mathfrak{M}, s \models \varphi(x)$.

Working in the opposite direction, assume $\mathfrak{M}, s \models \varphi(x)$ for at least one variable assignment s.
Since x is the only free variable and $\mathfrak{M}, s \models \varphi(x)$, we know s must at least assign x to some value $m \in |\mathfrak{M}|$ such that $\varphi(x)$ is satisfied relative to s.
Then we write $\mathfrak{M}, s[m/x] \models \varphi(x)$ for some $m \in |\mathfrak{M}|$, which gives way to $\mathfrak{M} \models \exists x \varphi(x)$.

Part Two
If $\mathfrak{M} \models \forall x \varphi(x)$, then by definition for any variable assignment s, $\mathfrak{M}, s[m/x] \models \varphi(x)$ for all $m \in |\mathfrak{M}|$.
The satisfaction of $\varphi(x)$ relative to any s is not restricted by any particular assignment of x, so we can write $\mathfrak{M}, s \models \varphi(x)$ for all s.

Working in the opposite direction, assume $\mathfrak{M}, s \models \varphi(x)$ for all variable assignments s.
Then any s could assign x to any member of the domain, and we would still have $\mathfrak{M}, s \models \varphi(x)$.
We can express this property of s as $\mathfrak{M}, s[m/x] \models \varphi(x)$ for all $m \in |\mathfrak{M}|$.
This implies $\mathfrak{M} \models \forall x \varphi(x)$.

Problem 5

Proposition 7.19.

Proof