PHIL 4110: Problem Set 3

Due on February 26, 2024

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Problem 1

Solution

Part One

$$s(v) = 1, \quad \mathfrak{M}, s \models \exists x(A(f(z), c) \supset \forall y(A(y, x) \vee A(f(y), x)))$$

Let $\varphi_1 \equiv A(f(z), c) \supset \forall y(A(y, x) \vee A(f(y), x))$
$\mathfrak{M}, s \models \exists x \varphi_1$ iff $\mathfrak{M}, s[m_1/x] \models \varphi_1$ for some $m_1 \in |\mathfrak{M}|$
$\mathfrak{M}, s[m_1/x] \models A(f(z), c) \supset \forall y(A(y, x) \vee A(f(y), x))$ for some $m_1 \in |\mathfrak{M}|$
This is equivalent to the expression $\mathfrak{M}, s[m_1/x] \not\models A(f(z), c)$ or $\mathfrak{M}, s[m_1/x] \models \forall y(A(y, x) \vee A(f(y), x))$ for some $m_1 \in |\mathfrak{M}|$.

Evaluating $\mathfrak{M}, s[m_1/x] \not\models A(f(z), c)$ for some $m_1 \in |\mathfrak{M}|$:
$= \langle \mathrm{Val}^{\mathfrak{M}}_{s[m_1/x]}(f(z)), c\rangle \notin A^{\mathfrak{M}}$
$= \langle f^{\mathfrak{M}}(\mathrm{Val}^{\mathfrak{M}}_{s[m_1/x]}(z)), \mathrm{Val}^{\mathfrak{M}}_{s[m_1/x]}(c)\rangle \notin A^{\mathfrak{M}}$
$= \langle f^{\mathfrak{M}}(1), c^{\mathfrak{M}}\rangle \notin A^{\mathfrak{M}}$
$= \langle 2, 3\rangle \notin A^{\mathfrak{M}} =$ False, since $\langle 2, 3\rangle$ is in $A^{\mathfrak{M}}$

Evaluating $\mathfrak{M}, s[m_1/x] \models \forall y(A(y, x) \vee A(f(y), x))$ for some $m_1 \in |\mathfrak{M}|$:
Let $\varphi_2 \equiv A(y, x) \vee A(f(y), x)$
$\mathfrak{M}, s[m_1/x] \models \forall y \varphi_2$ iff $\mathfrak{M}, s[m_1/x][m_2/y] \models \varphi_2$ for some m_1 for any m_2, such that $m_1, m_2 \in |\mathfrak{M}|$
This is just the expression $\mathfrak{M}, s[m_1/x][m_2/y] \models A(y, x)$ or $\mathfrak{M}, s[m_1/x][m_2/y] \models A(f(y), x)$, for some m_1 for any m_2.

Evaluating $\mathfrak{M}, s[m_1/x][m_2/y] \models A(y, x)$, for some m_1 for any m_2:
$= \langle \mathrm{Val}^{\mathfrak{M}}_{s[m_1/x][m_2/y]}(y), \mathrm{Val}^{\mathfrak{M}}_{s[m_1/x][m_2/y]}(x)\rangle \in A^{\mathfrak{M}}$
$= \langle m_2, m_1\rangle \in A^{\mathfrak{M}}$ for some m_1 for any m_2, such that $m_1, m_2 \in |\mathfrak{M}|$
We must check for all possible values of m_2 in the domain:
$= \langle 1, m_1\rangle \in A^{\mathfrak{M}}$ and $\langle 2, m_1\rangle \in A^{\mathfrak{M}}$ and $\langle 3, m_1\rangle \in A^{\mathfrak{M}}$

Evaluating $\mathfrak{M}, s[m_1/x][m_2/y] \models A(f(y), x)$, for some m_1 for any m_2.
$= \langle \mathrm{Val}^{\mathfrak{M}}_{s[m_1/x][m_2/y]}(f(y)), \mathrm{Val}^{\mathfrak{M}}_{s[m_1/x][m_2/y]}(x)\rangle \in A^{\mathfrak{M}}$
$= \langle f^{\mathfrak{M}}(\mathrm{Val}^{\mathfrak{M}}_{s[m_1/x][m_2/y]}(y)), m_1\rangle \in A^{\mathfrak{M}}$
$= \langle f^{\mathfrak{M}}(m_2), m_1\rangle \in A^{\mathfrak{M}}$ for some m_1 for any m_2, such that $m_1, m_2 \in |\mathfrak{M}|$
We must check for all possible values of m_2 in the domain:
$= \langle f^{\mathfrak{M}}(1), m_1\rangle \in A^{\mathfrak{M}}$ and $\langle f^{\mathfrak{M}}(2), m_1\rangle \in A^{\mathfrak{M}}$ and $\langle f^{\mathfrak{M}}(3), m_1\rangle \in A^{\mathfrak{M}}$
$= \langle 2, m_1\rangle \in A^{\mathfrak{M}}$ and $\langle 3, m_1\rangle \in A^{\mathfrak{M}}$ and $\langle 2, m_1\rangle \in A^{\mathfrak{M}}$
$= \langle 2, m_1\rangle \in A^{\mathfrak{M}}$ and $\langle 3, m_1\rangle \in A^{\mathfrak{M}}$

$\mathfrak{M}, s[m_1/x][m_2/y] \models \varphi_2$ becomes the expression:

$$(\langle 1, m_1\rangle \in A^{\mathfrak{M}} \text{ and } \langle 2, m_1\rangle \in A^{\mathfrak{M}} \text{ and } \langle 3, m_1\rangle \in A^{\mathfrak{M}}) \text{ or } (\langle 2, m_1\rangle \in A^{\mathfrak{M}} \text{ and } \langle 3, m_1\rangle \in A^{\mathfrak{M}})$$

for some $m_1 \in |\mathfrak{M}|$
Let $m_1 = 3$, then the second half of the 'or' operation is satisfied, since $\langle 2, 3\rangle, \langle 3, 3\rangle \in A^{\mathfrak{M}}$.
Therefore, $\mathfrak{M}, s[m_1/x][m_2/y] \models \varphi_2$ and $\mathfrak{M}, s[m_1/x] \models \forall y(A(y, x) \vee A(f(y), x))$

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Recall that $\mathfrak{M}, s \models \exists x \varphi_1$ iff $(\mathfrak{M}, s[m_1/x] \not\models A(f(z), c)$ or $\mathfrak{M}, s[m_1/x] \models \forall y(A(y,x) \vee A(f(y),x)))$ for some $m_1 \in |\mathfrak{M}|$.
We have shown the second half of the 'or' operation, therefore $\mathfrak{M}, s \models \exists x \varphi_1$ and $\mathfrak{M}, s$ satisfies the original formula.

Part Two
Use the originial variable assignment and structrue but have $A^{\mathfrak{M}} = \{\langle 1,3\rangle \langle 2,2\rangle \langle 3,3\rangle\}$.

Problem 2

Solution

Part One
$\varphi \equiv \psi \supset \chi$

By definition, $\mathfrak{M}, s_1 \models \varphi$ iff $\mathfrak{M}, s_1 \not\models \psi$ or $\mathfrak{M}, s_1 \models \chi$.
By the IH we have $\mathfrak{M}, s_2 \not\models \psi$ or $\mathfrak{M}, s_2 \models \chi$. Then, $\mathfrak{M}, s_2 \models \varphi$.

Part Two
$\varphi \equiv \forall x \psi$