Comparison of Portfolio Optimization Methods

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1 Introduction

Portfolio Optimization is the process of selecting a portfolio of assets that optimizes a specific return-risk parameter, usually in the form of future returns and variance of returns. Formal Portfolio Optimization methods began with the seminal work *Portfolio Selection*, by Harry Markowitz[5], who paved the way for what is now known as Markowitz Portfolios and Modern Portfolio Theory. The idea behind these methods is to reduce idiosyncratic (non-systematic) risk by creating a diversified portfolio of N assets such that the risk-return ratio of the portfolio is maximized. These seminal works paved the way for other models, such as the Black-Litterman model, which is more robust to various markets, and the Capital Asset Pricing Model, which is a foundational model still used today. However, these models rely on various assumptions that are hard to reproduce in practice. Many of the previously mentioned models also rely on the benefits of diversified risk, which picks assets with different characteristics (such as from different sectors) to ”spread out the risk” of sorts. This paradigm has shifted over the years, as recent studies suggest that it may now take over 200 equities in a portfolio to reach the same level of diversification that was historically possible with as few as 20 equities [1]. As recently as 2016, a new method called Heirarchical Risk Parity (HRP) was introduced that combats some of the aforementioned issues and is shown to not only be more robust, but produce better performance than classical Portfolio Optimization methods [4].

2 Optimization Methods

2.1 Markowitz Portfolios and Mean-Variance Optimization

In 1952, Harry Markowitz published a paper titled ”Portfolio Selection”, which served as the genesis for modern portfolio optimization methods [5]. Because of this work, Markowitz earned the Nobel Prize with William Sharpe and Merton Miller in 1990 [2]. Markowitz introduced the idea that portfolios should be optimized by future return and variance of the portfolio as a risk factor [5]. This forms an ”efficient frontier”, or best possible portfolio pertaining to specific risk/return characteristics [2]. There are a few sub-problems that can be attributed to Markowitz Portfolio construction, the first of which is the Risk Minimization problem[2]:

$$\begin{aligned} \min_{w} \quad & w^T \sum w \\ \text{s.t.} \quad & w^T \mu \geq \mu_0 \\ & w^T 1 = 1 \end{aligned} \tag{1}$$

where μ_0 is the target expected return. An inverse problem is the return maximization problem, where [2]:

$$\begin{aligned} \max_{w} \quad & w^T \mu \\ \text{s.t.} \quad & w^T \sum w \leq \sigma_0^2 \\ & w^T 1 = 1 \end{aligned} \tag{2}$$

where σ_0^2 is a variance target. The last optimization problem is the Risk-Adjusted Return Maximation Problem, where [2]:

$$\begin{aligned} \max_{w} \quad & w^T \mu - \lambda w^T \sum w \leq \sigma_0^2 \\ \text{s.t.} \quad & w^T 1 = 1 \end{aligned} \tag{3}$$

where $\lambda >= 0$ is the trade-off between expected return and variance. The various problem solutions are represented intuitively on the efficient frontier, where the minimum variance and maximum return portfolios will be at their respective axis maxima and minima relative to the problem bounds. Although the formulation is somewhat simple, the Markowitz Portfolio construction method has a few important drawbacks. In practice, variance is not a good risk measurement, and Markowitz Portfolios are too sensitive to parameters [2]. Expected return calculations greatly influence performance, and small changes can heavily impact the risk/return profile of the selected portfolio [6].

2.2 Hierarchical Risk Parity (HRP)

In the seminal paper *"Building Diversified Portfolios that Outperform Out-of-Sample"*, Hierarchical Risk Parity is introduced, which is a new portfolio optimization method that solves some of the issues of Markowitz Portfolios and other commonly used portfolio optimization methods, like the Capital Asset Pricing Model (CAPM). HRP uses mathematics and machine learning techniques to build a diversified portfolio based on the covariance matrix of the underlying assets [4]. Although this seems like a similar approach to other methods, it is unique in the fact that the covariance matrix is not required to be invertible, which methods like Markowitz Portfolios rely on [4]. This is an important advantage because many of the instability issues associated with other portfolio optimization methods stem from inversion operations, specifically when the underlying covariance matrix is nearly singular [6]. The approach to diversification used by HRP is also unique in the fact that it instead uses a hierarchy to define which assets are proper substitutes for other assets [6]. In methods like Markowitz Portfolios, it is assumed that each asset is a proper substitution for another, regardless of properties like size. This is the root cause for the previously mentioned instability, and implementing a hierarchical diversification structure that accounts for varying properties, like size, provides more stable results [6]. HRP is a tree-based optimization method that forms clusters from a tree hierarchy, where each cluster is a subset of the covariance matrix. These clusters are combined into a new covariance matrix, which is then sorted to place the largest values on the diagonal [4]. To assign optimal weights to the given assets, the sorted covariance matrix is recursively halved, where an inverse variance approach is used to assign weights to each asset in every bisection. The bisection variance is defined as [4]:

$$V_i^{(j)} = w_i^{(j)'} V_i^{(j)} w_i^{(j)} \tag{4}$$

where $V_i^{(j)}$ is the covariance matrix between the assets, and the weights $w_i^{(j)}$ are defined as:

$$w_i^{(j)} = \frac{\text{diag}[V_i^{(j)}]^{-1}}{\text{trace}(\text{diag}[V_i^{(j)}]^{-1})} \tag{5}$$

Figure 1: Hierarchical Risk Parity Weights found from the testing set

The split factor α is defined as:

$$\alpha = 1 - \frac{V_i^{(1)}}{V_i^{(1)} + V_i^{(2)}} \tag{6}$$

which is used to re-scale the allocation weights after each bisection [4]. The allocation weights can then be used to scale the given baskets of assets into the optimal portfolio.

3 Results: Hierarchical Risk Parity vs. Markowitz Portfolios

In *"Building Diversified Portfolios that Outperform Out-of-Sample"*[4], it is shown that HRP outperforms general portfolio optimization methods like Markowitz Portfolios out of sample. In an attempt to reproduce the findings from this paper, daily data from 21 common country ETFs from August 2000 to January 2008 was used. After splitting the data into training and validation sets, performance is compared between HRP, MP, and Equal Weight portfolios. The HRP weights found from the testing set are shown in Figure 1. The HRP weights are somewhat balanced, aside from large spikes for two assets EWH and EWT.

Weights were then calculated for both the Equal Weights Portfolio and the Markowitz Portfolio. To get the performance of each portfolio out-of-sample, the weights are multiplied by the asset returns of the validation set. The performance of each portfolio is plotted in Figures 2-4.

It is clear that the HRP portfolio produced the best results, with a Sharpe Ratio (SR) of 0.95, which is more than double the SR of MP at 0.46 and about 0.2 greater than the SR of the Equal Weight Portfolio. This improvement out-of-sample is important, because MP wildly overestimates performance in-sample, which can lead to false conclusions about future performance. Figure 5 and 6 show the in-sample performances of MP and HRP Portfolios, with the MP portfolio earning a whopping 300% more return with over double the SR of the HRP Portfolio.

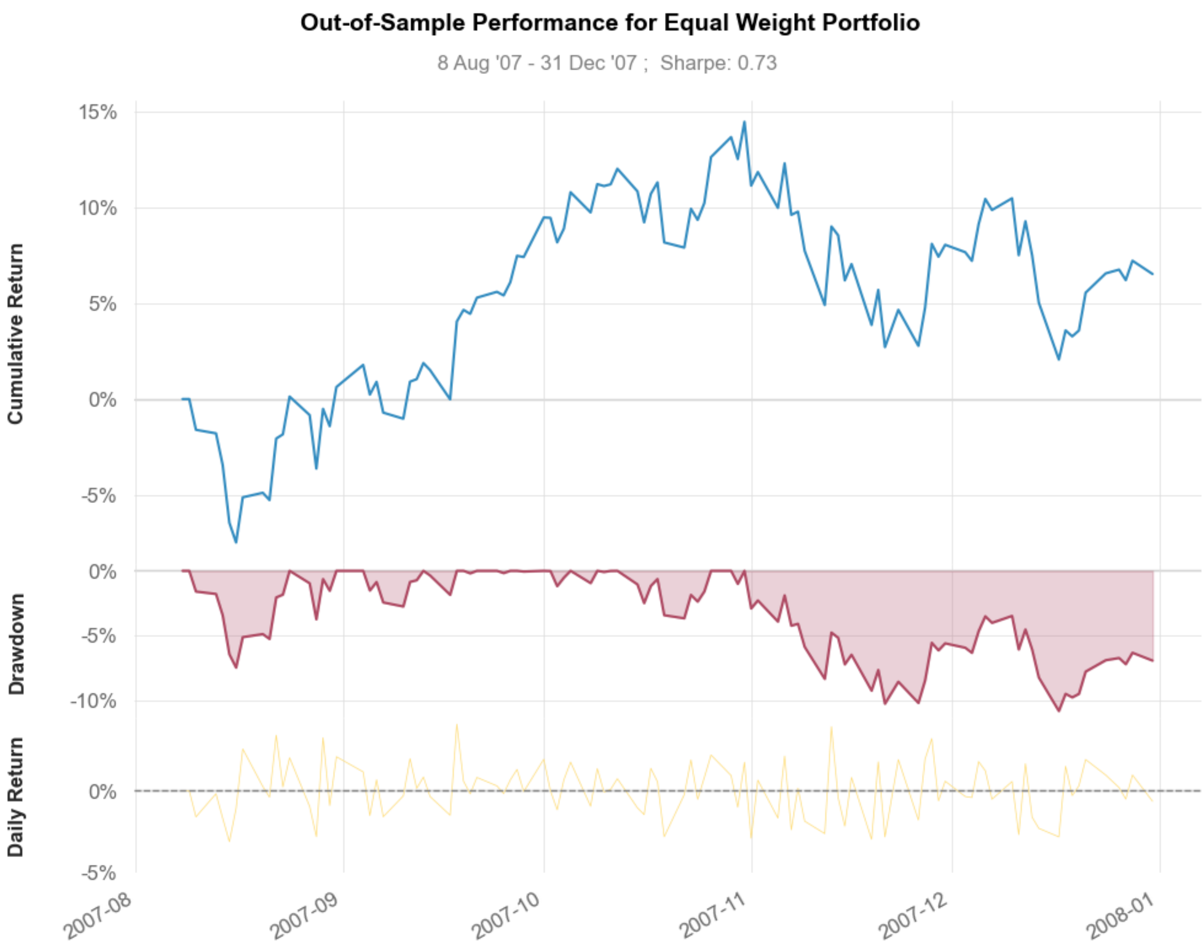

Figure 2: Out-of-sample Equal Weight Portfolio Performance, 2007-08-01 - 2008-01-01

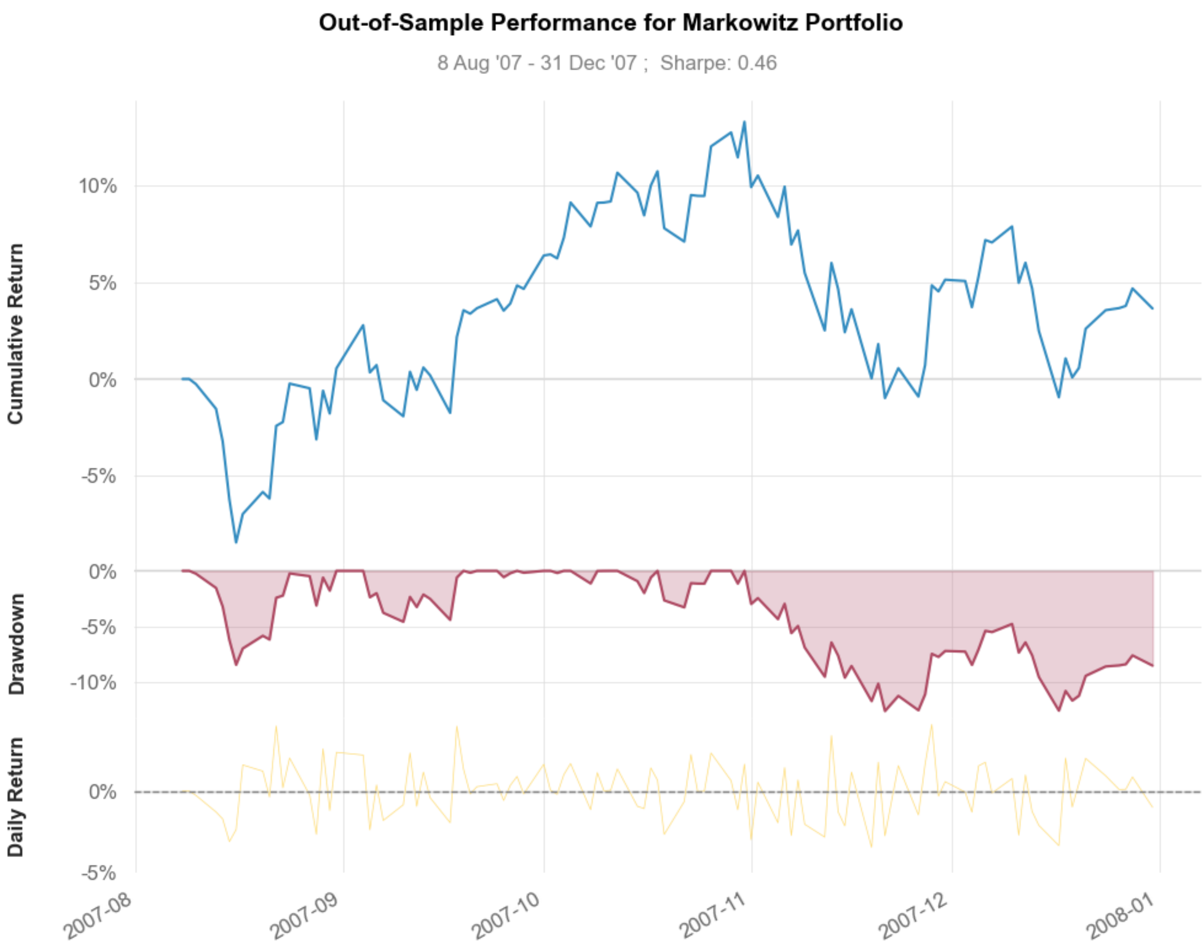

Figure 3: Out-of-sample Markowitz Portfolio Performance, 2007-08-01 - 2008-01-01

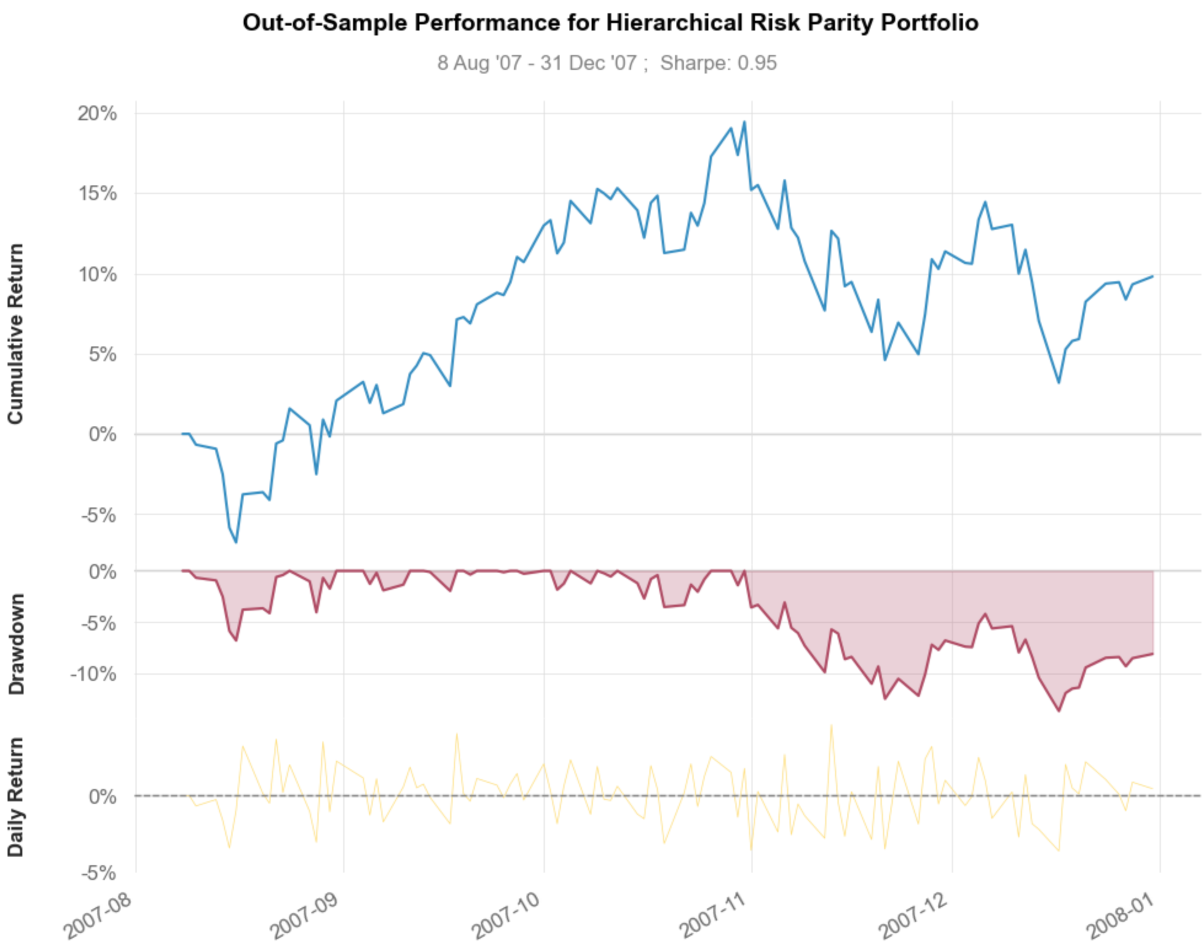

Figure 4: Out-of-sample Hierarchical Risk Parity Portfolio performance, 2007-08-01 - 2008-01-01

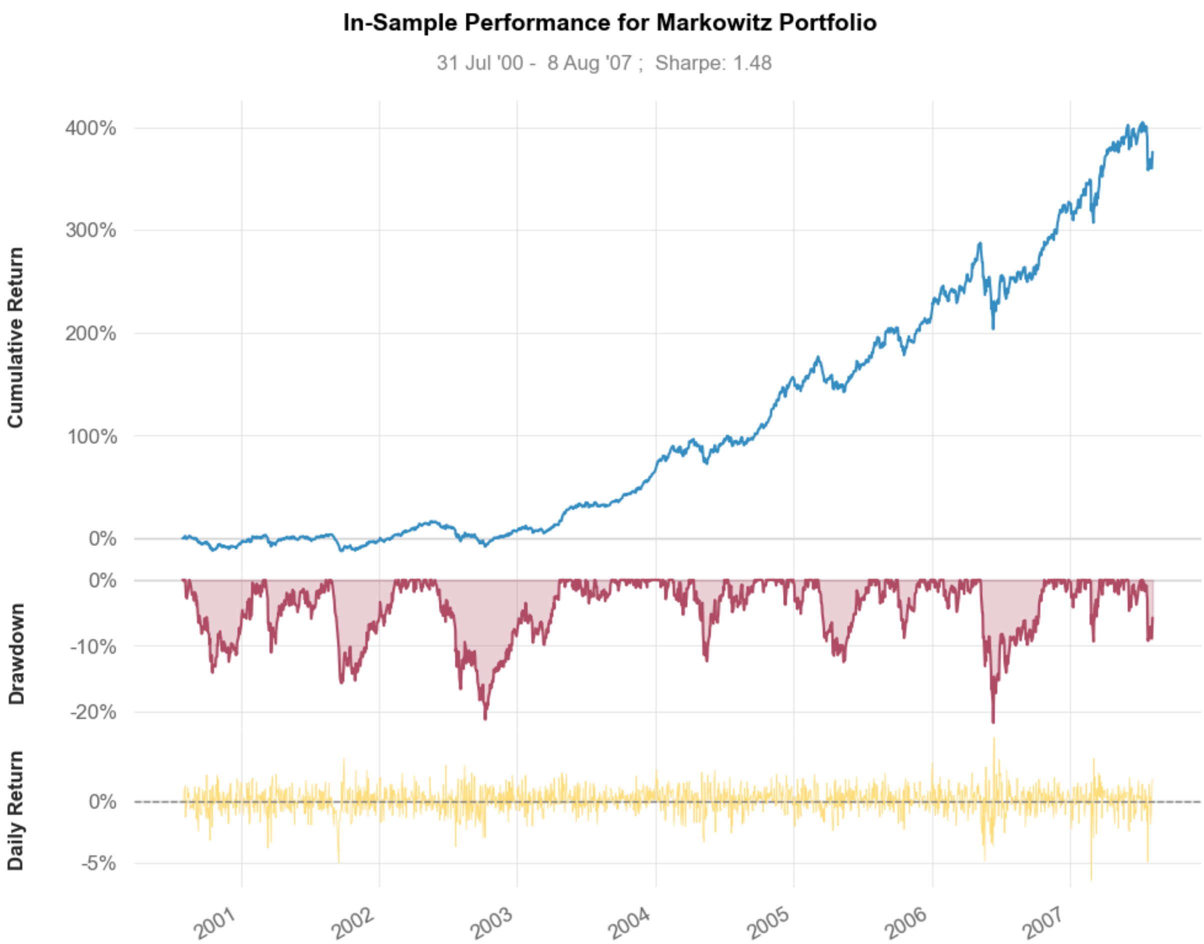

Figure 5: In-sample Markowitz Portfolio Performance, 2007-08-01 - 2008-01-01

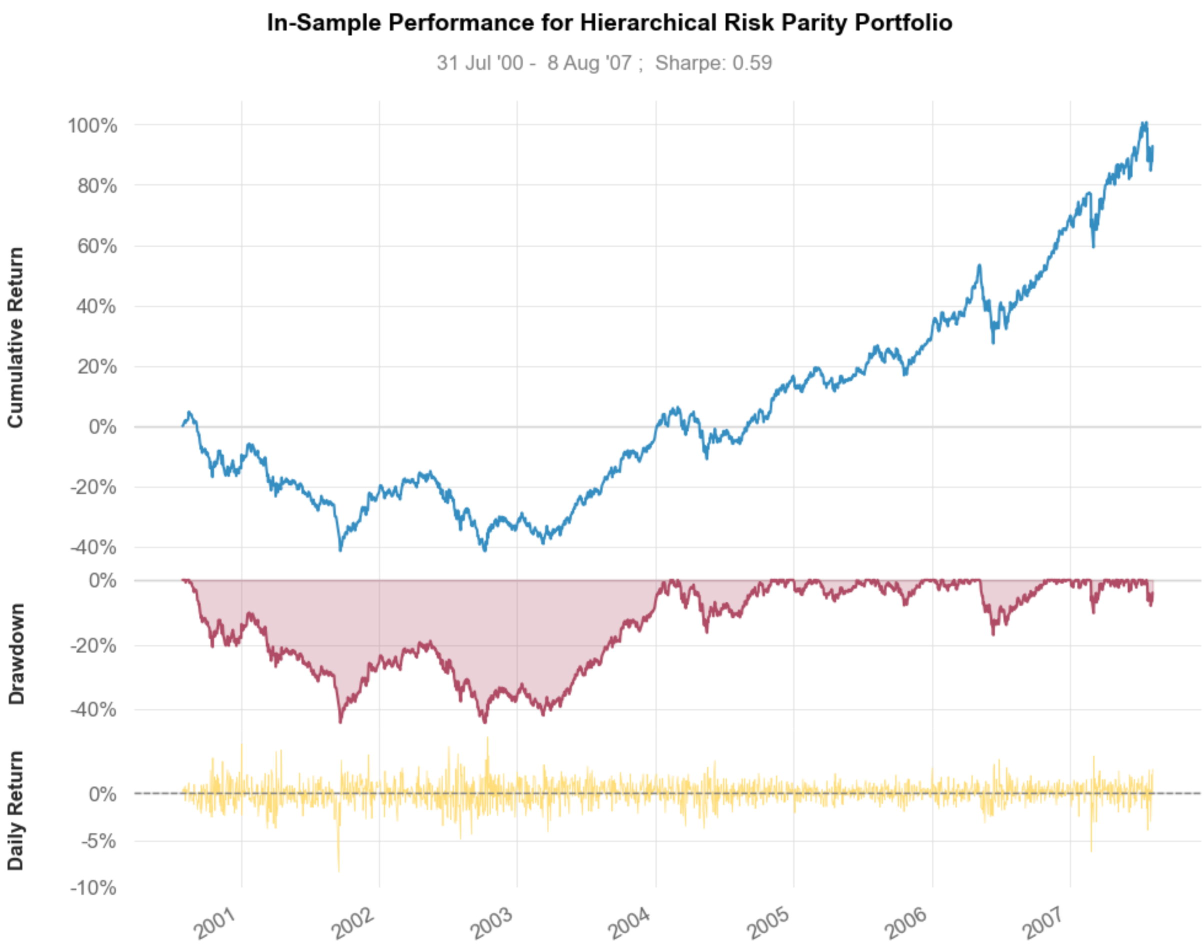

Figure 6: In-sample Hierarchical Risk Parity Portfolio performance, 2007-08-01 - 2008-01-01

4 Analysis

From the results presented in the previous section, it is wildly evident that HRP portfolio optimization is a more robust portfolio optimization method than MP. Of course, there are more complex ways of implementing MP that can improve performance, and this may be another further research direction, but for the scope of this problem and the use case, it does not make sense. For an MP portfolio to perform well out of sample, it would need to be specifically optimized to a basket of assets and their risk characteristics, which would not generalize well between new baskets and require much more user intervention than is desired for this project, and these improvements have no guarantee to produce better results than HRP. From the current literature, HRP seems to be one of the most robust and current portfolio optimization methods available today, and its ease of generalization allows it to be a good fit for this project. The increase in out-of-sample performance over MP is very strong, and the clear improvement over equal weight portfolios justifies HRP as a proper portfolio optimization method. An interesting point to note is that drawdowns did not improve for any of the given methods. This implementation of HRP also only focuses on one form of optimal portfolio calculation, meaning different calculations, such as optimal SR, could be implemented.

5 Conclusion

The findings in *"Building Diversified Portfolios that Outperform Out-of-Sample"* are further confirmed in this work as the HRP Portfolio clearly outperformed both the MP Portfolio and the Equal Weight Portfolio out-of-sample. The HRP Portfolio demonstrated not only higher Sharpe Ratio, but also larger cumulative returns over both methods, and had over double the SR of the MP Portfolio. HRP also showed much more realistic in-sample results than the MP Portfolio, as the MP Portfolio had almost 300% larger cumulative returns and double the SR than HRP in-sample. HRP is robust and generalizes well to new baskets of assets as it requires minimal parameters, which is an important advantage for this project. From the research done in this project, multiple areas of future research have presented themselves. First, implementing different objective portfolios such as maximum SR in HRP calculation could have an interesting result on performance. Drawdown does not improve from any of the optimization methods, so implementing drawdown minimization either inside or outside the HRP calculation could improve Sharpe Ratio. HRP is also able to be parallelized, which could be important for large baskets of assets >1000. The use-case for something like this is universe selection, where a large universe of assets is filtered to produce a smaller basket of assets with stronger characteristics. Another area of portfolio optimization not discussed in this project is Online Portfolio Selection, which attempts to construct optimal portfolios based on decisions such as "Follow-the-Winner" or momentum characteristics [3]. Online Portfolio Selection is a recent and relevant portfolio optimization topic, however research in this area is still developing and there are many open problems [3]. It would be interesting to use a form of regime switching with Online Portfolio Selection, where different portfolios could be specifically selected for various regimes. Another important future research direction is generating strategy portfolios, where a similar to a portfolio of assets, a portfolio of rules/strategies is created and optimized. HRP easily allows for this, since it does not require inverse covariance matrices and is easily generalized to new baskets. The ETF Trick can be used to combine assets into one return path, or "ETF" [6], which would allow strategies to be given different baskets of stocks with minimal changes in code. This has already been implemented outside this project, and this project is an important milestone, as HRP can be used to generate optimal weights for the combination of assets that will be made into an ETF. In summary, the results from this project show that HRP is a successful method for generating optimal portfolio weights, which is very promising for future research.

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