Seminar 3

Open-Ended Problem: Portfolio Optimization Methods

Mathew Thiel

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1 Introduction

Portfolio Optimization is the process of selecting a portfolio of assets that optimizes a specific return-risk parameter, usually in the form of future returns and variance of returns. Formal Portfolio Optimization methods began with the seminal work *Portfolio Selection*, by Harry Markowitz[1], who paved the way for what is now known as Markowitz Portfolios and Modern Portfolio Theory. The idea behind these methods is to reduce idiosyncratic (non-systematic) risk by creating a diversified portfolio of N assets such that the risk-return ratio of the portfolio is maximized. These seminal works paved the way for other models, such as the Black-Litterman model, which is more robust to various markets, and the Capital Asset Pricing Model, which is a foundational model still used today. However, as will be seen in the seminar, these models rely on various assumptions that may not be practical for use in real markets.

2 Optimization Methods

2.1 Markowitz Portfolios

In 1952, Harry Markowitz published a paper titled "Portfolio Selection", which served as the genesis for modern portfolio optimization methods [1]. Because of this work, Markowitz earned the Nobel Prize with William Sharpe and Merton Miller in 1990 [2]. Markowitz introduced the idea that portfolios should be optimized by future return and variance of the portfolio as a risk factor [1]. This forms an "efficient frontier", or best possible portfolio pertaining to specific risk/return characteristics [2]. There are a few sub-problems that can be attributed to Markowitz Portfolio construction, the first of which is the Risk Minimization problem[2]:

$$\begin{aligned} \min_{w} \quad & w^T \sum w \\ \text{s.t.} \quad & w^T \mu \geq \mu_0 \\ & w^T 1 = 1 \end{aligned} \tag{1}$$

where μ_0 is the target expected return. An inverse problem is the return maximization problem, where [2]:

$$\begin{aligned} \max_{w} \quad & w^T \mu \\ \text{s.t.} \quad & w^T \sum w \leq \sigma_0^2 \\ & w^T 1 = 1 \end{aligned} \tag{2}$$

where σ_0^2 is a variance target. The last optimization problem is the Risk-Adjusted Return Maximation Problem, where [2]:

$$\begin{aligned} \max_{w} \quad & w^T \mu - \lambda w^T \sum w \leq \sigma_0^2 \\ \text{s.t.} \quad & w^T 1 = 1 \end{aligned} \tag{3}$$

where $\lambda >= 0$ is the tradeoff between expected return and variance. The various problem solutions are represented intuitively on the efficient frontier, where the minimum variance and maximum return portfolios will be at their respective axis maxima and minima relative to the problem bounds. Although the formulation is somewhat simple, the Markowitz Portfolio construction method has a few important drawbacks. In practice, variance is not a good risk measurement, and Markowitz Portfolios are too sensitive to parameters, such as μ [2].

2.2 Black-Litterman Model

The Black-Litterman Model is a framework introduced by Fischer Black and Robert Litterman in 1991 [3]. The Black-Litterman model is an alternative that deals with the sensitivity in the Markowitz framework, and it relies on a few assumptions[2]. The first assumption is that a market equilibrium, a balance between supply and demand, can provide an estimate of the expected excess returns μ, which can be expressed as [2]:

$$\begin{aligned} \pi &= \mu + w_\pi \\ w_\pi &\sim \mathcal{N}(0, \tau \sum) \end{aligned} \tag{4}$$

where $\tau > 0$ measures the uncertainty in excess returns π. The second assumption is that there are K investor views, which are implemented in the model as [2]:

$$\begin{aligned} q &= P\mu + w_q \\ w_q &\sim \mathcal{N}(0,\, \Omega) \end{aligned} \tag{5}$$

where $P \in^{K\times N}$ and $q \in^{K}$ describe the views K and $\Omega \in^{K\times N}$ measures the uncertainty in those views[2]. The goal of the Black-Litterman model is to find an optimal market return pertaining to investor views. This model is less sensitive and provides a more precise model for producing excess returns because market equilibrium and investor views are allowed to be formulated in a more general sense than the variance and returns in the Markowitz portfolio[2]. This allows for a more robust model that produces more consistent excess returns.

2.3 Capital Asset Pricing Model (CAPM)

The Capital Asset Pricing Model (CAPM) is an asset pricing model first introduced into literature by the seminal works *Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk* by William Sharpe (1964)[4] and *The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets* by John Lintner (1965)[5]. CAPM is the centerpiece of most MBA courses taught today, as the model offers powerful predictions for not only measuring risk but also the relation between expected return and risk. The CAPM has the following assumptions [6]:

1. Investors make decisions based on expected returns and variance of returns.

2. Investors are rational and risk-averse.

3. Investors use the Markowitz method of portfolio diversification

4. Investors invest for the same amount of time

5. Investors have the same expectations about the expected return and variance of all assets.

6. There is a risk-free asset in which investors can lend or borrow any amount

7. Capital markets are perfectly competitive and friction-less

If constructing a portfolio P of N assets, w_i is the weight of asset i in the portfolio, which must sum to 1 for all assets in the portfolio. The variance is then[6]:

$$\mathrm{var}(R_P) = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \mathrm{cov}(R_i, R_j) \tag{6}$$

We can also use a market portfolio M instead of portfolio P, which results in the equation[6]:

$$\mathrm{var}(R_M) = w_{1M}\mathrm{cov}(R_1, R_M) + ... + w_{NM}\sum_{j=1}^{N} w_{jM}, \mathrm{cov}(R_N, R_j) \tag{7}$$

where the market portfolio is represented as a function of the covariance of each asset in the portfolio. If we assume that the market is in equilibrium, we can then define the expected return $E[R_i]$ as[6]:

$$E[R_i] = R_f + \frac{[E[R_M] - R_f]}{\mathrm{var}(R_M)}\mathrm{cov}(R_i, R_M) \tag{8}$$

with risk-free rate R_f. Equation 8 is called the security market line (SML) and the expected return of each individual security will lie on this line [6]. The final representation of the CAPM is thus[6]:

$$E[R_i] = R_f + \beta_i[E[R_M] - R_f] \tag{9}$$

which states that the expected return of an individual asset is a positive linear function of its systematic risk relative to beta B_i, which is the variance of the asset returns versus the variance of the market's returns [6].

3 Conclusion

The CAPM model, although extremely popular, seems to have a lot of downfalls. Many of its assumptions seem problematic to me. The one that sticks out the most is that the expectations for the expected return and variance of all assets are the same. This is probably a big issue in practice, as not all assets have the same risk-return profile. Plus, there may also be additional needed parameters in the model, such as a re-balancing time for the portfolio. This is because risk factors are likely to change over time, especially for a single asset. If CAPM is so reliant on beta, then there would need to be a period where the portfolio is re-weighted to account for changes in the volatility in not only the individual asset but the market as well. Assume you use the S&P 500 as the market, and an asset like Apple as a portfolio asset, what happens if there is a flash crash or managerial issue within the company? The simple answer is that the portfolio would then need to be adjusted and re-weighted to account for the new market conditions, but this seems sub-optimal for any investor who doesn't want to lose money. Ideally, in my mind, a portfolio optimization method should define or add to an approach for controlling risk and/or execution, so that instead of adjusting the portfolio after events, the portfolio can be optimized for another model or risk control method that adjusts to potential negative events before they happen. This idea is similar to the idea of splitting up side and size when making financial predictions, as it may be beneficial to separate risk estimates and portfolio weighting into two different models. The models introduced in this seminar are some of the most fundamental and widely known in the industry, yet I don't think that they provide enough rigor as-is to be used in my project. If possible, I would like to continue research into this area and attempt to find a better solution that results in a more robust optimization method.

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