

Fourier Analysis and Musical Signal Processing

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Jan.-Mar. 2022

1 Introduction

2 Musical Preliminaries

2.1 Sound

In physics, **sound** is a series of oscillating low and high air pressures. The oscillations are created when a material vibrates. A **cycle**, or **vibration**, is one high-pressure and one low-pressure area together. The **frequency** or **pitch** is the number of cycles per second, measured in **Hz** [1]. The higher the frequency of the sound, the higher the pitch of the sound heard. The **volume**, “**loudness**”, or **amplitude** of a sound is defined by the difference in air pressure between a low area and high area of a cycle. A quiet sound can be described as a transition between two densities of similar value. Likewise, a loud sound is a transition between a very high density and a very low density [2] [3]. These terms oscillating, cycles, frequencies, etc. are all terms used when dealing with certain functions in mathematics. A diagram of this relation is shown in figure 1. The elementary **sine** function graphs the air density over time. As the density increases, the sine wave increases.

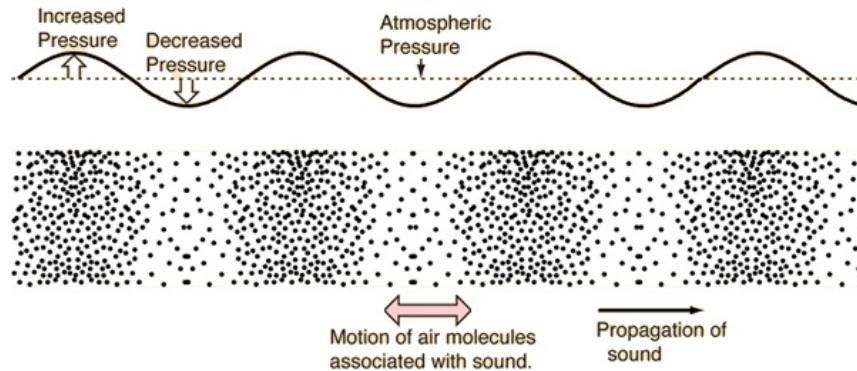


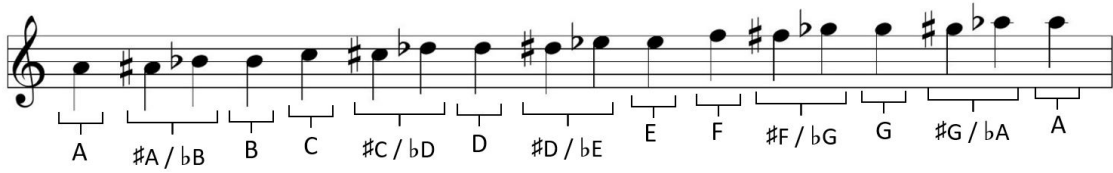
Figure 1: The high and low air pressures of sound can be represented by a wave function [4].

2.2 Music Theory

Different notes are determined by specific frequencies. The lower sounding a note is, the lower the frequency of the sound wave. Similarly with high-pitched sounds, their frequencies are much higher. The most common system in western music for determining the frequency values of musical notes is a 12-tone system of ratios called **equal temperament**. Given a reference note with frequency x , there are 11 notes between x and $2x$. Each of these notes are separated by a ratio equal to the 12th root of 2, ($\sqrt[12]{2} \approx 1.05946$). See figure 2 for a 12-tone scale using equal temperament [5] [2].

Today the note “A”, denoted “Unison” or A4, is commonly set to 440Hz and is used as a reference point for calculating the other frequencies. This standard has changed throughout history and has its origins in medieval Europe. See [8] for further reading.

When a note is played by a computer, the frequency is perfectly calculated and a very artificial sound is heard. With instruments and the human voice however, perfect frequencies are not generated and



12-Tone Equal Temperament			
Note	Interval	Ratio	Frequency(Hz)
A	Unison	1 : 1	440.00
$\sharp A / \flat B$	Minor Second	1 : 1.05946	466.16
B	Major Second	1 : 1.12246	493.88
C	Minor Third	1 : 1.18921	523.25
$\sharp C / \flat D$	Major Third	1 : 1.25992	554.37
D	Fourth	1 : 1.33483	587.33
$\sharp D / \flat E$	Diminished Fifth	1 : 1.41421	622.25
E	Fifth	1 : 1.49831	659.25
F	Minor Sixth	1 : 1.58740	698.46
$\sharp F / \flat G$	Major Sixth	1 : 1.68179	739.99
G	Minor Seventh	1 : 1.78180	783.99
$\sharp G / \flat A$	Major Seventh	1 : 1.88775	830.61
A	Octave	1 : 2	880.00

Figure 2: The following shows the 12 divisions within the equal temperament system. Each ratio is a $\sqrt[12]{2} \approx 1.05946$ apart [2] [6] [7]. Although considered the same in the equal temperament system, the paired accented notes ($\sharp A / \flat B$, etc.) are actually not the same. See [5] for more information on this beyond the scope of this paper.

there are qualities which cause notes to sound unique and beautiful. For example, when a clarinet and a trumpet both play A4, even though they play the same frequency, they sound very different. This is due to differences in air flow and materials, wood vs. metal, lip buzzing vs. reed vibrations, etc. Analyzing the frequencies of computer generated notes and instrument generated notes reveals something amazing behind the physics of musical sound.

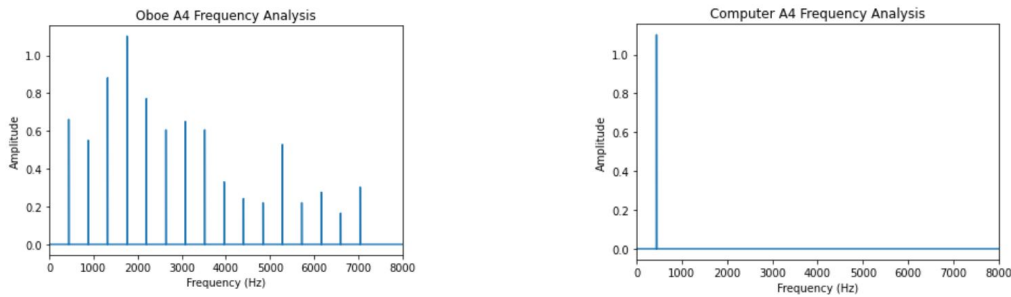


Figure 3: The graph on the left shows the frequency analysis of an oboe playing an A4. The right graph shows a computer playing the same note.

Figure 3 shows the frequency analysis of a computer generated A4 and an A4 played by a oboe. As you can see on the computer graph, the large spike is centered around 440 Hz, which matches the table in figure 2 for the note A4. Notice how the oboe graph contains multiple other spikes decreasing in size from the largest spike at 440 Hz. This phenomena is known as the **overtone** or **harmonic** series and consists of regular intervals from the **fundamental**, or largest, frequency. These successive spikes are created by properties of the instrument used such as material, size, air column, etc., meaning every note played is

slightly unique. Because a computer will always generate the exact same frequency, its notes are never unique. This is one reason for the unique beauty in singing and instrumental music.

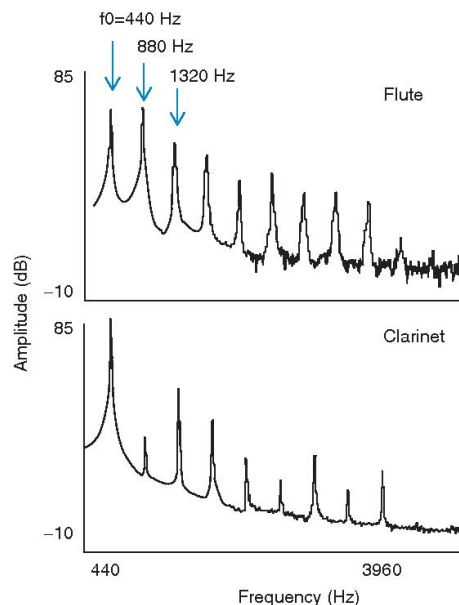


Figure 4: Graphs of a clarinet and a flute both playing the note A4.

Figure 4 shows the frequency analysis of a clarinet and flute both playing the note A4. Notice how although both are playing the same note, the differences in spikes leads to the different sound between the two instruments. The frequencies within the harmonic series are determined by whole number ratios. No matter what the fundamental frequency is, the harmonic series will always follow the ratios; 1:1, 1:2, 1:3, 1:4, ..., 1:n for n harmonics in the series.

2.3 Time and Frequency Domains

There are two common ways of graphing sound. It can be graphed as amplitude over time, which shows the volume of the sound, or as amplitude over frequency, which shows the frequency of the sound. Mathematically, these are referred to as the **time domain representation** and **frequency domain representation**. The time domain shows how the sound physically behaves over time (the x-axis is time and the y-axis is amplitude). The frequency domain shows the frequencies of the sound (the x-axis is frequency and the y-axis is the amplitude, or strength of the frequencies). Time domain data can be generated through a microphone as it is a literal representation of the sound waves. The frequency domain data is much more complicated and requires a bit of math to extract from the time domain data. The rest of this paper will focus on understanding this process of moving from the time domain to the frequency domain.

3 Mathematical Preliminaries

The following sections will serve both as an introduction to the mathematics of music as well as an exploration of various concepts through computer programs. Additionally, the process of moving between the time and frequency domains will be described and explored.

3.1 Periodic Functions

A **periodic function** is any function for which $f(t) = f(t + T)$ for all t . The smallest constant T which satisfies it is called the **period** of the function (it 'repeats' every T) [9]. The sine wave that represented the sound in figure 1 is an example of a periodic function where $f(t) = A\sin(\omega t + \varphi)$. Here A is the amplitude, or the maximum deviation of the function from zero. $\omega = 2\pi f$ where f is the frequency, also denoted $\frac{1}{T}$, multiplied by 2π to convert it to an angular frequency (a measure of radians per second instead of cycles per second). Lastly, φ specifies the phase of the wave (a measure, in radians, of where the oscillation is in within its cycle at $t = 0$) [10] [11].

Showing an example of a sine function that generates a musical note is figure 5

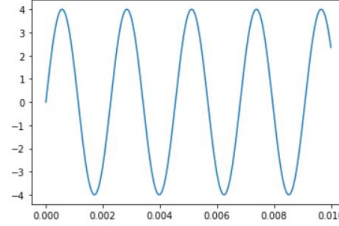


Figure 5: The note A4 created using the sine function: $f(t) = A\sin(\omega t + \varphi)$, where $A = 4$, $\omega = 2\pi 440$, and $\varphi = 0$.

By changing the amplitude $A = 20$ and making the x-axis go from 0 to 10000, the sine function can be calculated and stored into a wav file. This file can then be played on a computer and the note A4 will actually be heard. By adding successive sine functions to the original, a more natural sounding A4 can be generated. These added sine functions represent the additional harmonics.

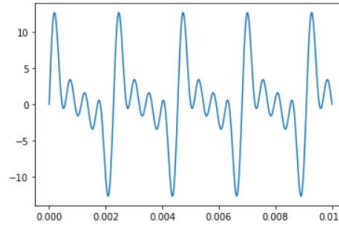


Figure 6: The note A4 created using the sine functions: $f(t) = 4\sin(2\pi 440t + \varphi) + 3.95\sin(2\pi 880t + \varphi) + 3.9\sin(2\pi 1320t + \varphi) + 3.85\sin(2\pi 1760t + \varphi)$.

Notice in figure 6 that the amplitudes (4, 3.95, 3.9, and 3.85) are decreasing and the frequencies (440, 880, 1320, and 1760) are increasing by the before mentioned whole number ratios 1:1, 1:2, 1:3, and 1:4.

Moving now away from sound and music, we will dive into the Fourier analysis, which will allow us to move from the time domain into the frequency domain and vice versa. This exploration begins with Orthogonality.

3.2 Orthogonality of Vectors and Functions

A **vector space** is any set that satisfies the defined axioms of addition and scalar multiplication. Details not listed here. An **inner product** is a function on a vector space which satisfies the following axioms: (u and v and w are vectors in the vector space and c is a scalar) [12].

1. $\langle u, v \rangle = \langle v, u \rangle$
2. $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$
3. $c\langle u, v \rangle = \langle cu, v \rangle$

4. $\langle v, v \rangle \geq 0$, and $\langle v, v \rangle = 0$ if and only if $v = 0$.

An **inner product space** is a vector space with a defined inner product which satisfies the above axioms for every vector within it. Properties & Definitions related to Inner Product Spaces: [12].

1. $\langle 0, v \rangle = \langle v, 0 \rangle = 0$
2. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
3. $c\langle u, v \rangle = \langle u, cv \rangle$
4. The **norm** (or **length**) of u is $\|u\| = \sqrt{\langle u, u \rangle}$
5. The **distance** between u and v is $d(u, v) = \|u - v\|$
6. The **angle** between two nonzero vectors u and v is:

$$\cos(\theta) = \frac{\langle u, v \rangle}{\|u\|\|v\|}, 0 \leq \theta \leq \pi$$
7. Vectors u and v are **orthogonal** if $\langle u, v \rangle = 0$.

When a set of vectors $S = \{u_1, u_2, u_3, \dots\}$ satisfies $\langle u_n, u_m \rangle = 0$ when $n \neq m$, we say that the elements of S are **mutually orthogonal** and that they form an **orthogonal basis** for the space spanned by S . Any vector in that space can be represented as a linear combination of those basis vectors [12].

Sets of functions: The set of functions $\{f_1(t), f_2(t), f_3(t), \dots, f_k(t), \dots\}$ is orthogonal on an interval $a < t < b$ if for any two functions $f_n(t)$ and $f_m(t)$ in the set, [12].

$$\langle f_n, f_m \rangle = \int_a^b f_n(t) f_m(t) dt = \begin{cases} 0 & \text{for } n \neq m \\ r_n & \text{for } n = m \end{cases} \quad (1)$$

This property of sets of functions will provide usefully when working with complex exponentials in series.

3.3 Complex Exponentials

Let z be a complex number where $z = x + jy$ and $j = \sqrt{-1}$. An additional complex number to note is the complex conjugate $x - jy$, denoted z^* .

The **complex exponential** is a formula of sorts which represents the distance traveled around a circle of radius 1. Denoted e^{jx} , this exponential gives us the location on a unit circle after walking x units in a clockwise direction. The exponential e^{-jx} will calculate the same location but after traveling in a counter clockwise direction. By adding $2\pi f$ to the exponential and changing the x for a t we get $e^{2\pi f j t}$. By adding the $2\pi f t$, where f is the frequency and t is time, we now discover that the formula makes f full rotations of the unit circle every second. The counterclockwise rotation of our earlier formula holds here as well. By setting $\omega_0 = 2\pi f$, we have $e^{j\omega_0 t}$.

The complex exponentials $e^{jn\omega_0 t}$ (by adding n the value is magnified) together form the set [14].

$$S = \{\dots, e^{-j3\omega_0 t}, e^{-j2\omega_0 t}, e^{-j\omega_0 t}, 1, e^{j\omega_0 t}, e^{j2\omega_0 t}, \dots, e^{jn\omega_0 t}, \dots\}$$

Theorem 2.1: Orthogonality of the Complex Exponentials

The complex exponentials $e^{jn\omega_0 t}$ of the above set satisfy the orthogonality condition

$$\frac{1}{T_0} \int_{-T_0/2}^{T_0/2} e^{jn\omega_0 t} e^{jm\omega_0 t^*} dt = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

where $T_0 = 2\pi/\omega_0$ and $e^{jm\omega_0 t^*}$ is the complex conjugate $e^{-jm\omega_0 t}$. Here is a proof showing this.

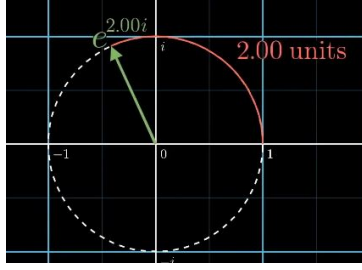


Figure 7: Euler's formula is an incredible connection between complex analysis and trigonometry. For more information, see [13].

Proof:

$$\begin{aligned} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} e^{jn\omega_0 t} e^{jm\omega_0 t^*} dt &= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} e^{jn\omega_0 t} e^{-jm\omega_0 t} dt = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} e^{jn\omega_0 t - jm\omega_0 t} dt \\ &= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} e^{j\omega_0 t(n-m)} dt \end{aligned} \quad (2)$$

When $n \neq m$, then $n - m$ is a nonzero integer which we shall call p , and so (2) continues as

$$\begin{aligned} &= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} e^{jp\omega_0 t} dt = \frac{1}{T_0} \left[\frac{1}{j\omega_0 p} e^{j\omega_0 p t} \right]_{-T_0/2}^{T_0/2} = \frac{1}{T_0} \left(\frac{e^{j\omega_0 p(T_0/2)}}{j\omega_0 p} - \frac{e^{j\omega_0 p(-T_0/2)}}{j\omega_0 p} \right) \\ &= \frac{1}{T_0} \left(\frac{e^{(j\omega_0 p T_0)/2} - e^{(-j\omega_0 p T_0)/2}}{j\omega_0 p} \right) = \frac{1}{T_0} \left(\frac{e^{jp\pi} - e^{-jp\pi}}{j\omega_0 p} \right) = \frac{e^{jp\pi} - e^{-jp\pi}}{T_0 j\omega_0 p} \\ &= \frac{e^{jp\pi} - e^{-jp\pi}}{2\pi jp} = \frac{(\cos p\pi + j \sin p\pi) - (\cos p\pi - j \sin p\pi)}{2\pi jp} = \frac{\cos p\pi + j \sin p\pi - \cos p\pi + j \sin p\pi}{2\pi jp} \\ &= \frac{2j \sin p\pi}{2\pi jp} = \frac{\sin p\pi}{p\pi} = 0 \end{aligned} \quad (3)$$

On the other hand, when $n = m$ then (2) continues as

$$\begin{aligned} &= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} e^{j\omega_0 t(n-m)} dt = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} e^{j\omega_0 t(0)} dt = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} e^0 dt = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} dt \\ &= \frac{1}{T_0} [t]_{-T_0/2}^{T_0/2} = \frac{1}{T_0} \left(\frac{T_0}{2} + \frac{T_0}{2} \right) = \frac{T_0}{T_0} = 1 \end{aligned} \quad (4)$$

And the proof is complete. Q.E.D. [14].

3.4 Fourier Series

Using this property of orthogonality, let $f_p(t)$ be a periodic function with period T_0 , and assume it can be expressed as an infinite sum of complex exponentials, that is, that

$$f_p(t) = \sum_{n=-\infty}^{\infty} F(n) e^{jn\omega_0 t} \quad (5)$$

The complex exponentials on the right-hand side all repeat at least once whenever t is increased by T_0 , so both sides are periodic with period T_0 . To obtain the constants $F(n)$, we first multiply both sides by $e^{-jm\omega_0 t}$ and integrate, obtaining

$$f_p(t) = \sum_{n=-\infty}^{\infty} F(n)e^{jn\omega_0 t}$$

$$\frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f_p(t)e^{-jm\omega_0 t} dt = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \sum_{n=-\infty}^{\infty} F(n)e^{jn\omega_0 t} e^{-jm\omega_0 t} dt \quad (6)$$

Interchanging the order of summation and integration on the right we get

$$\frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f_p(t)e^{-jm\omega_0 t} dt = \sum_{n=-\infty}^{\infty} F(n) \left[\frac{1}{T_0} \int_{-T_0/2}^{T_0/2} e^{jn\omega_0 t} e^{-jm\omega_0 t} dt \right] \quad (7)$$

By theorem 2.1, when $n = m$, the quantity in square brackets becomes 1 and so we are left with

$$\frac{1}{T_0} \int_{-T_0/2}^{T_0/2} e^{jn\omega_0 t} e^{-jm\omega_0 t} dt = F(m) \quad (8)$$

Then replacing m with n we get

$$F(n) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f_p(t)e^{-jn\omega_0 t} dt \quad (9)$$

[14]

This formula essentially describes the inner product of $f_p(t)$ and $e^{-jn\omega_0 t}$. A way to visualize this process is that the formula “projects” the function $f_p(t)$ onto a unit circle in the complex plane. In figure 8 we see this more literally. As mentioned previously, the complex exponential provides rotational movement while the function $g(t)$, substituted for our $f_p(t)$, provides the amplitude of the vector (the green arrow).

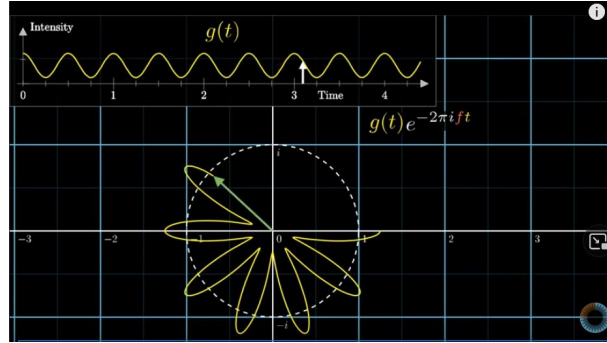


Figure 8: The projection of $g(t)$ into the complex plane. This visualization was done by Grant Sanderson on his YouTube channel 3Blue1Brown. The channel contains tons of incredible videos on mathematics topics, see [15] for more information.

By integrating from $-T_0/2$ to $T_0/2$ and dividing by T_0 , $F(n)$ is essentially the average value along the projection, or a sort of center-of-mass for the unit circle.

By graphing $F(n)$, shown in figure ?? as $\hat{g}(f)$, we have the frequency domain graph. The x axis was the previous

We can summarize all of this as

Theorem 2.2: Complex Fourier Series for Periodic Functions

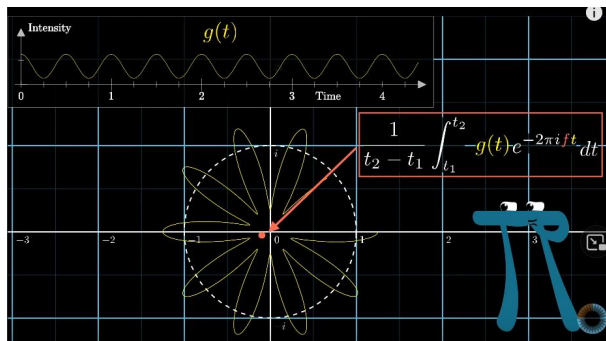


Figure 9: Another visualization showing how the integral calculates a “center-of-mass” for the function $g(t)$ [15].

Let $f_p(t)$ be periodic with period T_0 , defined analytically. Then it can also be represented by the infinite series of complex exponentials

$$f_p(t) = \sum_{n=-\infty}^{\infty} F(n)e^{jn\omega_0 t} \quad (10)$$

where the coefficients $F(n)$ can be found from the analytical definition of $f_p(t)$ as follows:

$$F(n) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f_p(t)e^{-jn\omega_0 t} dt \quad (\forall n) \quad (11)$$

The previous sections will serve as a basis for what the rest of the paper will discuss. To summarize parts of theorem 2.2

- Equation (10) is called the **synthesis equation** because it generates a periodic function from coefficients and complex exponentials.
- Equation (11) is called the **analysis equation** because it extracts frequency coefficients from a periodic function.
- The constants $F(n)$ are called the **complex Fourier coefficients** [14].

4 Extra Information

5 Experiments and Implementations

With an understanding of the Fourier transform, we can use computers to create real-world examples. Starting in python, there are various libraries which facilitate real-time Fourier analysis. The library Scipy contains multiple Fast Fourier Transform functions which allow us to analyze a computer’s microphone input in real-time. Additionally, as has already been mentioned, python has support for creatign and graphing sine functions and then using those functions to create audio files.

5.1 Time Domain of a Real Time Signal

In order to visualize music, the audio signal must be broken down so that various aspects of the sound can be analyzed. The most sensible first steps would be to look at the musical signal in the time and frequency domains. The time domain can be easily obtained without much math with the help of a python library. Using the library PyAudio, the microphone can be accessed and sound input can be read and manipulated. The library takes in the microphone data as samples and graphs the

samples in real time. The x and y axes are time and amplitude. Figure 10 shows work done so far in real-time sound analysis. The code starts off by opening a PyAudio "stream." This object takes in microphone data and organizes it into chunks. The chunks are loaded into a buffer and the graphing library, matplotlib.pyplot, takes the buffer and graphs it. There is also some data manipulation to get the microphone raw data into a graph-able format but details are unimportant for the scope of this paper. The Python, PyAudio, and PyPlot versions used here and throughout the rest of the paper are 3.7.7, 0.2.11, and 3.5.1 respectively [16] [17] [18]. Figure 11 shows another such graph.

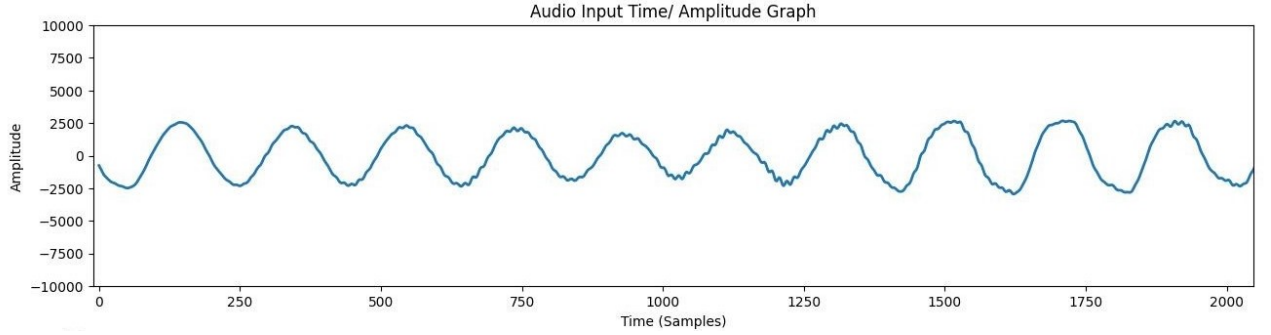


Figure 10: A graph of the amplitude over time of the author singing a note somewhere between G4 and A4 [7]

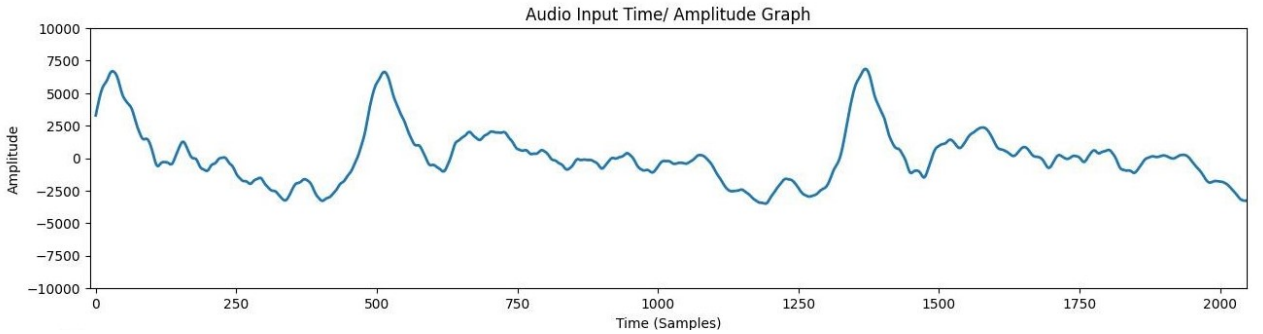


Figure 11: A graph of the amplitude over time of the author and the author's friend trying to harmonize.

5.2 Frequency Domain of a Real Time Signal

The next step in this direction is to look at the frequency domain. Taking the same exact signals as shown in figures 10 and 11 a library called SciPy can be used to go from the time domain to the frequency domain. With the fast Fourier transform (FFT) build into SciPy, the frequencies that make up the signals of figures 10 and 11 can be extracted. This was a bit more complicated than the previous analysis but luckily all of the mathematical intricacies are handled by SciPy. [19].

Figures 12 and 13 show the same note and harmony done by the author but instead of plotting the time on the x-axis, the frequency is plotted instead. This is how the author approximated what note was being sung.

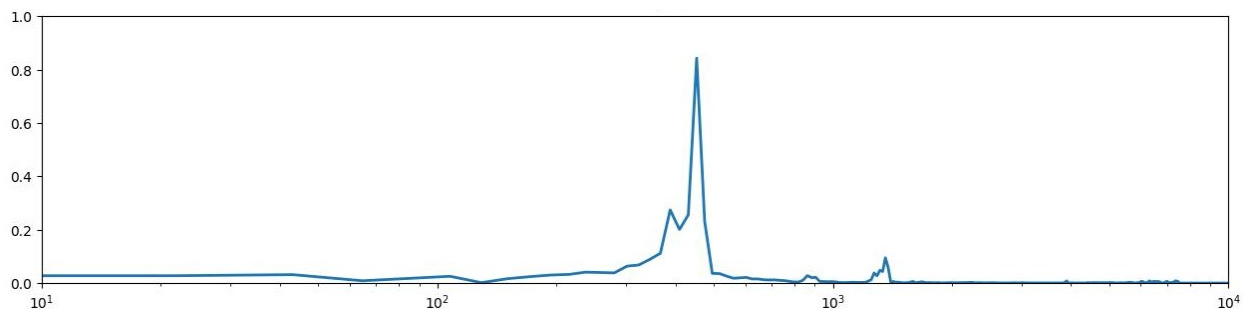


Figure 12: A graph of the frequencies sung by the author in figure 10

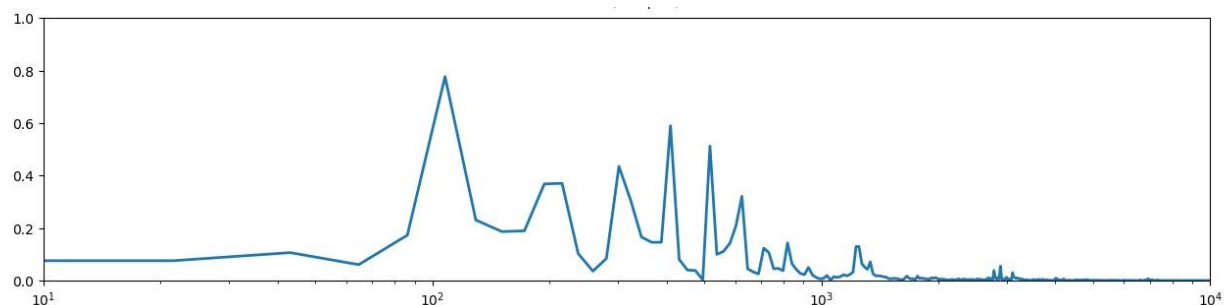


Figure 13: A graph of the frequencies sung by the author and author's friend's attempted harmony in figure 11

6 Expected Outcome

7 Conclusion

8 Acknowledgements

A Special Thank You To:

Dr. Kwadwo Antwi-Fordjour

Dr. Brian Toone

Professor Kevin Kozak

Donny Snyder

Samford University

Mr. and Mrs. Paul and Eileen Lad

Alan Crisologo for being the "author's friend," (See Figure 4)

† JMJ † (Jesus, Mary, Joseph)

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