Fourier Analysis and Musical Signal Processing

Matthew Lad

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1 Introduction

The goal of this paper is to explore the application of Fourier analysis in signal processing and specifically the analysis of music. The mathematical background of music and Fourier analysis will be covered as well as the Fast Fourier Transform (FFT) algorithm and applications of the FFT on musical signals. Along with the mathematics, explorations will be done through computer algorithms.

2 Musical and Mathematical Preliminaries

2.1 Musical Preliminaries

2.1.1 Sound

Although **sound** is a aural concept, for this paper we will define it as a series of oscillating low and high air pressures. A **cycle**, or **vibration**, is one high-pressure and one low-pressure area together. The **frequency** or **pitch** is the number of cycles per second, measured in **Hz** [1]. The higher the frequency of the sound, the higher the pitch of the sound heard. The **volume**, "**loudness**", or **amplitude** of a sound is defined by the difference in air pressure between a low area and high area of a cycle. A quiet sound can be described as a transition between two densities of similar value. Likewise, a loud sound is a transition between a very high density and a very low density [2] [3]. These terms oscillating, cycles, frequencies, etc. are all terms used when dealing with certain functions in mathematics. A diagram of this relation is shown in figure 1. The elementary **sine** function graphs the air density over time. As the density grows, the sine wave likewise increases.

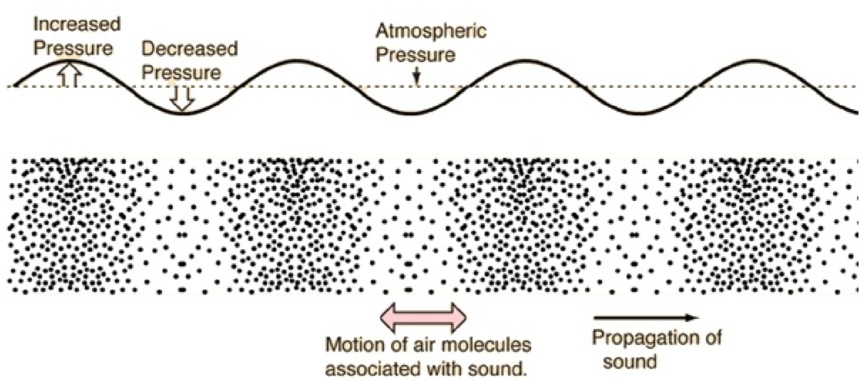
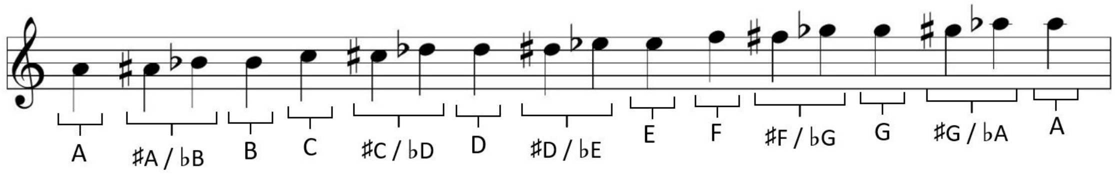


Figure 1: The high and low air pressures of sound can be represented by a wave function [4].

Different notes are determined by specific frequencies. The lower sounding a note is, the lower the frequency the sound wave. Similarly with high-pitched sounds, their frequencies are much higher. The most common system in western music for determining the frequency values of musical notes is a 12-tone system of ratios called **equal temperament**. Given a reference note with frequency x, there are 11 notes between x and $2x$. Each of these notes are separated by a ratio equal to the 12th root of 2, ($\sqrt[12]{2} \approx 1.05946$). See figure ?? for a 12-tone scale using equal temperament.



12-Tone Equal Temperament			
Note	ISO ALPHA 2 Code	ISO ALPHA 3 Code	ISO numeric Code
A	AF	AFG	004
	AX	ALA	248
B	AL	ALB	008
C	DZ	DZA	012
	AS	ASM	016
Andorra	AD	AND	020
Angola	AO	AGO	024

Figure 2: The high and low air pressures of sound can be represented by a wave function [4].

The process of determining the frequency value of each note is known as **tuning**. Western music commonly takes the note $A4 = 440$Hz as a reference value and then takes ratios from A4 to determine the other frequencies. For example, the note $A4$ has a ratio of $1:1$ with itself, which is 440Hz. Going to the next whole note, $B5$, the ratio becomes $2:1$ and so $A5 = 880$Hz. The notes $B4, C5, D5, E5, F5, andG5$ lie between $A4$ and $A5$ and are determined by similar ratios. See [**?**], for more information on tuning.

2.2 Mathematical Preliminaries

2.2.1 Periodic Functions

A **periodic function** is any function for which $f(t) = f(t+T)$ for all t. The smallest constant T which satisfies it is called the **period** of the function (it 'repeats' every T) [5].

2.2.2 Time and Frequency Domains

Remembering the sine wave function from figure 1, we will refer to it with $f(t)$. We see that t is time and the value $f(t)$ represents the motion of the low and high-pressure areas. This function $f(t)$ is called the **time domain representation** because it shows how the sound behaves over time (the x-axis is time and the y-axis is amplitude). Now looking at the graphs in figure **??**, we see that the y-axis is also amplitude but the x-axis here is frequency. The functions that create these graphs (which we can arbitrarily call $f(x)$ and $g(x)$) are called **frequency domain representations** [6]. As we will see later in this paper, the two domains are closely connected and important through the paper.

2.2.3 Orthogonality of Vectors and Functions

A **vector space** is any set that satisfies the defined axioms of addition and scalar multiplication. Details not listed here. An **inner product** is a function on a vector space which satisfies the following axioms: (u and v and w are vectors in the vector space and c is a scalar) [7].

1. $\langle u, v\rangle = \langle v, u\rangle$
2. $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$
3. $c\langle u, v\rangle = \langle cu, v\rangle$
4. $\langle v, v\rangle \geq 0$, and $\langle v, v\rangle = 0$ if and only if $v = 0$.

An **inner product space** is a vector space with a defined inner product which satisfies the above axioms for every vector within it. Properties & Definitions related to Inner Product Spaces: [7].

1. $\langle 0, v\rangle = \langle v, 0\rangle = 0$
2. $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$
3. $c\langle u, v\rangle = \langle u, cv\rangle$
4. The **norm** (or **length**) of u is $||u|| = \sqrt{\langle u, u\rangle}$
5. The **distance** between u and v is $d(u, v) = ||u - v||$
6. The **angle** between two nonzero vectors u and v is:
   $\cos(\theta) = \frac{\langle u,v\rangle}{\|u\|\|v\|}$, $0 \leq \theta \leq \pi$
7. Vectors u and v are **orthogonal** if $\langle u, v\rangle = 0$.

When a set of vectors $S = \{u_1, u_2, u_3, ...\}$ satisfies $(u_n, u_m) = 0$ when $n \neq m$, we say that the elements of S are **mutually orthogonal** and that they form an **orthogonal basis** for the space spanned by S. Any vector in that space can be represented as a linear combination of those basis vectors [7].

Sets of functions: The set of functions $\{f_1(t),\, f_2(t),\, f_3(t),\, ...,\, f_k(t),\, ...\}$ is orthogonal on an interval $a < t < b$ if for any two functions $f_n(t)$ and $f_m(t)$ in the set, [7].

$$\langle f_n, f_m\rangle = \int_a^b f_n(t) f_m(t)\, dt = \begin{cases} 0 & \text{for } n \neq m \\ r_n & \text{for } n = m \end{cases} \tag{1}$$

2.2.4 Complex Exponentials

Let $z = x + jy$ where $j = \sqrt{-1}$. Some facts from complex algebra used in this paper: [6].

1. $z^* = x - jy$ (complex conjugate)

The complex exponentials $e^{jn\omega_0 t}$ together form the set [6].

$$S = \{..., e^{-j3\omega_0 t}, e^{-j2\omega_0 t}, e^{-j\omega_0 t}, 1, e^{j\omega_0 t}, e^{j2\omega_0 t}, ..., e^{jn\omega_0 t}, ...\}$$

<u>Theorem 2.1:</u> **Orthogonality of the Complex Exponentials**

The complex exponentials $e^{jn\omega_0 t}$ of the above set satisfy the orthogonality condition

$$\frac{1}{T_0} \int_{-T_0/2}^{T_0/2} e^{jn\omega_0 t} e^{jm\omega_0 t^*}\, dt = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

where $T_0 = 2\pi/\omega_0$ and $e^{jm\omega_0 t^*}$ is the complex conjugate $e^{-jm\omega_0 t}$.

Proof:

$$\begin{aligned}\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} e^{jn\omega_0 t}e^{jm\omega_0 t^*}\,dt \;=\; \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} e^{jn\omega_0 t}e^{-jm\omega_0 t}\,dt \;&=\; \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} e^{jn\omega_0 t - jm\omega_0 t}\,dt \\ &=\; \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} e^{jw_0 t(n-m)}\,dt\end{aligned} \tag{2}$$

When $n \neq m$, then $n-m$ is a nonzero integer which we shall call p, and so (2) continues as

$$\begin{aligned} &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} e^{jp\omega_0 t}\,dt \;=\; \frac{1}{T_0}\Big[\frac{1}{j\omega_0 p}e^{j\omega_0 pt}\Big]_{-T_0/2}^{T_0/2} \;=\; \frac{1}{T_0}\Big(\frac{e^{j\omega_0 p(T_0/2)}}{j\omega_0 p} - \frac{e^{j\omega_0 p(-T_0/2)}}{j\omega_0 p}\Big) \\ &= \frac{1}{T_0}\Big(\frac{e^{(j\omega_0 pT_0)/2} - e^{(-j\omega_0 pT_0)/2}}{j\omega_0 p}\Big) \;=\; \frac{1}{T_0}\Big(\frac{e^{jp\pi} - e^{-jp\pi}}{j\omega_0 p}\Big) \;=\; \frac{e^{jp\pi} - e^{-jp\pi}}{T_0 j\omega_0 p} \\ &= \frac{e^{jp\pi} - e^{-jp\pi}}{2\pi jp} \;=\; \frac{(\cos p\pi + j\sin p\pi) - (\cos p\pi - j\sin p\pi)}{2\pi jp} \;=\; \frac{\cos p\pi + j\sin p\pi - \cos p\pi + j\sin p\pi}{2\pi jp} \\ &= \frac{2j\sin p\pi}{2\pi jp} \;=\; \frac{\sin p\pi}{p\pi} \;=\; 0\end{aligned} \tag{3}$$

On the other hand, when $n = m$ then (2) continues as

$$\begin{aligned} &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} e^{jw_0 t(n-m)}\,dt \;=\; \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} e^{j\omega_0 t(0)}\,dt \;=\; \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} e^{0}\,dt \;=\; \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} dt \\ &= \frac{1}{T_0}[t]\Big|_{-T_0/2}^{T_0/2} \;=\; \frac{1}{T_0}\Big(\frac{T_0}{2} + \frac{T_0}{2}\Big) \;=\; \frac{T_0}{T_0} \;=\; 1\end{aligned} \tag{4}$$

And the proof is complete. Q.E.D. [6].

2.2.5 Fourier Series

Using this property of orthogonality, let $f_p(t)$ be a periodic function with period T_0, and assume it can be expressed as an infinite sum of complex exponentials, that is, that

$$f_p(t) \;=\; \sum_{n=-\infty}^{\infty} F(n)e^{jn\omega_0 t} \tag{5}$$

The complex exponentials on the right-hand side all repeat at least once whenever t is increased by T_0, so both sides are periodic with period T_0. To obtain the constants $F(n)$, we first multiply both sides by $e^{-jm\omega_0 t}$ and integrate

$$\begin{aligned} f_p(t) \;&=\; \sum_{n=-\infty}^{\infty} F(n)e^{jn\omega_0 t} \\ \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f_p(t)e^{-jm\omega_0 t}\,dt \;&=\; \frac{1}{T_0}\int_{-T_0/2}^{T_0/2}\sum_{n=-\infty}^{\infty} F(n)e^{jn\omega_0 t}e^{-jm\omega_0 t}\,dt \\ \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f_p(t)e^{-jm\omega_0 t}\,dt \;&=\; \sum_{n=-\infty}^{\infty} F(n)\Big[\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} e^{jn\omega_0 t}e^{-jm\omega_0 t}\,dt\Big]\end{aligned} \tag{6}$$

By Theorem 2.1;

when $n \neq m$

$$\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f_p(t)e^{-jm\omega_0 t}\,dt \;=\; \sum_{n=-\infty}^{\infty} F(n)\,[0] \;=\; 0 \tag{7}$$

when $n = m$

$$\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f_p(t)e^{-jm\omega_0 t}\,dt \;=\; \sum_{n=-\infty}^{\infty} F(n)\,[1] \;=\; F(m) \tag{8}$$

Then replacing m with n we get

$$F(n) \;=\; \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f_p(t)e^{-jn\omega_0 t}\,dt \tag{9}$$

[6]

We can summarize all this into

<u>Theorem 2.2:</u> Complex Fourier Series for Periodic Functions

Let $f_p(t)$ be periodic with period T_0, defined analytically. Then it can also be represented by the infinite series of complex exponentials

$$f_p(t) \;=\; \sum_{n=-\infty}^{\infty} F(n)e^{jn\omega_0 t} \tag{10}$$

where the coefficients $F(n)$ can be found from the analytical definition of $f_p(t)$ as follows:

$$F(n) \;=\; \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f_p(t)e^{-jn\omega_0 t}\,dt \qquad (\forall n) \tag{11}$$

The beginning of section 2.2 to now constitutes the basis for what will be covered in the rest of the paper. To summarize parts of theorem 2.2

- Equation (10) is called the **synthesis equation**.
- Equation (11) is called the **analysis equation**.
- The constants $F(n)$ are called the **complex Fourier coefficients** [6].

2.2.6 Fourier Transform

Until now, we have looked exclusively at periodic functions of which we can create a Fourier series representation. The question arises if single pulse functions which do not repeat over a period can be represented by Fourier series as well. The side of Fourier analysis dealing with single pulses primarily deals with the **Fourier transform**.

Given a single pulse, we can think of it as being embedded in a single period of a periodic function. We can then have the period T_0 tend towards infinity. Even though we have a periodic function, because the period tends toward infinity we essentially eliminate all other pulses but our original single pulse [6]. There is a great deal of complexity beyond this, but for the purposes of this paper, we will move straight into theorem 3.1.

<u>Theorem 3.1:</u> Fourier Transform for a Single Pulse

(provided that the integrals exist:)

Synthesis:

$$f(t) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega \tag{12}$$

Analysis:

$$F(\omega) \;=\; \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\, d\omega \tag{13}$$

Equation (13) shows that a time domain function $f(t)$ can be **analyzed** to give an associated frequency domain function $F(\omega)$ that contains all the information in $f(t)$. $F(\omega)$ is known as the **Fourier transform** of $f(t)$. Likewise, in equation (12) we can see that we can **synthesize** a pulse given it's Fourier transform function. $f(t)$ then is called the **inverse Fourier transform** of $F(\omega)$ [6].

An amazing feature of the Fourier transform/ inverse Fourier transform is they can be used to operate on all pulse cases. One-time pulses of finite duration, one-time pulses of infinite duration, and periodic functions that inherently have infinite duration can all be analyzed/ synthesized.

2.3 Sampling

In order to store music and sound on computers, the entire signal must be broken down into a digital format. This consists of sampling the signal at a regular interval, essentially converting the continuous signal into a discrete signal. Saying $s(t)$ is a signal input function, the sampled version of it is a list of $s(t)$ values taken every T seconds. $s(nT)$ is the sampled function, where$T$ is the **sampling period** and n is an integer. The average number of samples taken in one second is the **sampling frequency** $f_s = 1/T$.

2.4 Discrete Fourier Transform

Combining the concepts of the continuous Fourier Transform in section 4.1 with sampling, there is a Fourier Transform that works on sampled functions. Before jumping in, there is some background math that needs to be covered.

Theorem?:

Analysis:

$$F_k \;=\; \sum_{n=-\frac{N}{2}+1}^{N/2} f_n e^{-j2\pi nk/N} \tag{14}$$

Synthesis:

$$f_k \;=\; \sum_{n=-\frac{N}{2}+1}^{N/2} F_n e^{j2\pi nk/N} \tag{15}$$

2.5 Dirac Delta Function

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

3 Methodology, Algorithm, Etc.

3.1 Time and Frequency Domain Analysis

3.1.1 Time Domain of a Periodic Signal

(more to be added)

3.1.2 Time Domain of a Periodic Signal

(more to be added)

3.1.3 Time Domain of a Real Time Signal

In order to visualize music, the audio signal must be broken down so that various aspects of the sound can be analyzed. The most sensible first steps would be to look at the musical signal in the time and frequency domains. The time domain can be easily obtained without much math with the help of a python library. Using the library PyAudio, the microphone can be accessed and sound input can be read and manipulated. The library takes in the microphone data as samples and graphs the samples in real time. The x and y axes are time and amplitude. Figure 3 shows work done so far in real-time sound analysis. The code starts off by opening a PyAudio "stream." This object takes in microphone data and organizes it into chunks. The chunks are loaded into a buffer and the graphing library, matplotlib.pyplot, then takes the buffer and graphs it. There is also some data manipulation to get the microphone raw data into a graph-able format but details are unimportant for the scope of this paper. The Python, PyAudio, and PyPlot versions used here and throughout the rest of the paper are 3.7.7, 0.2.11, and 3.5.1 respectively [8] [9] [10]. Figure 4 shows another such graph.

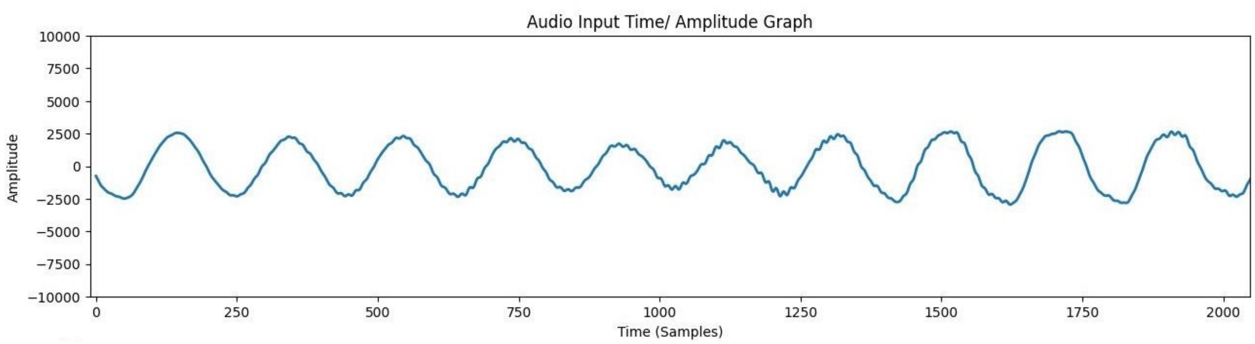


Figure 3: A graph of the amplitude over time of the author singing a note somewhere between G4 and A4 [11]

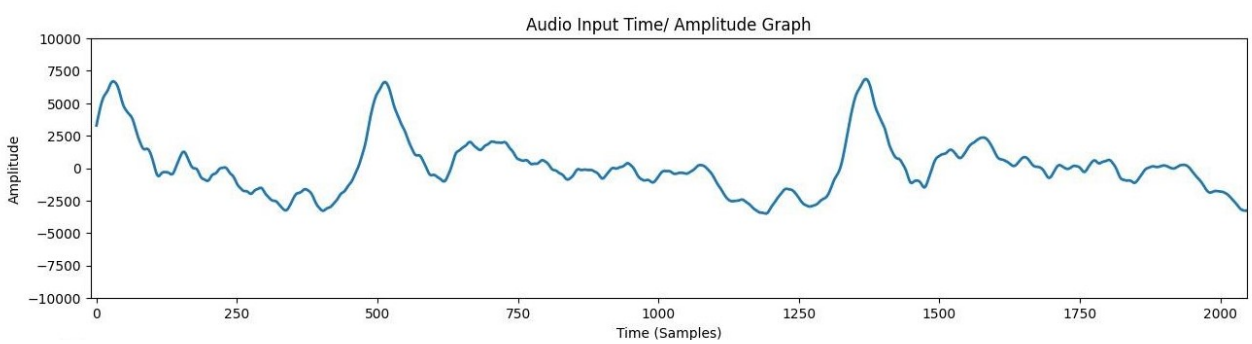


Figure 4: A graph of the amplitude over time of the author and the author's friend trying to harmonize.

3.1.4 Frequency Domain of a Real Time Signal

The next step in this direction is to look at the frequency domain. Taking the same exact signals as shown in figures 3 and 4 a library called SciPy can be used to go from the time domain to the frequency domain. With the fast Fourier trasform (FFT) build into SciPy, the frequencies that make up the signals of figures 3 and 4 can be extracted. This was a bit more complicated than the previous analysis but luckily all of the mathematical intricacies are handled by SciPy. [12]. Figures 5 and 6 show the same note and harmony done by the author but instead of plotting the time on the x-axis, the frequency is plotted instead. This is how the author approximated what note was being sung.

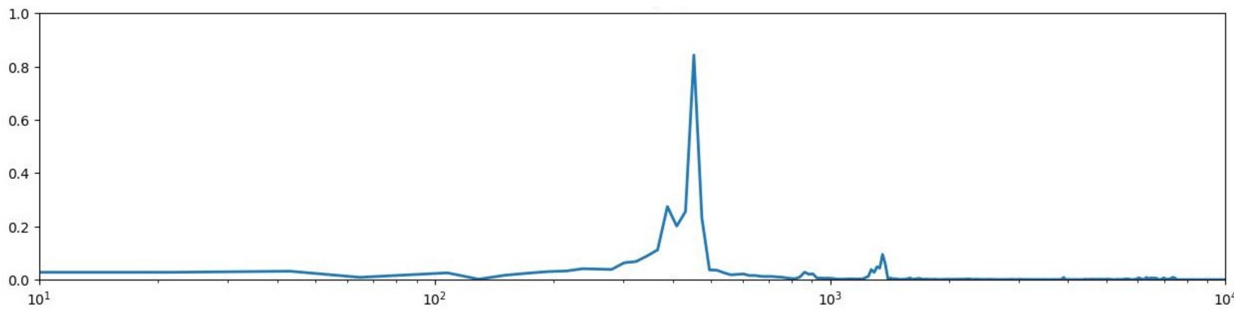


Figure 5: A graph of the frequencies sung by the author in figure 3

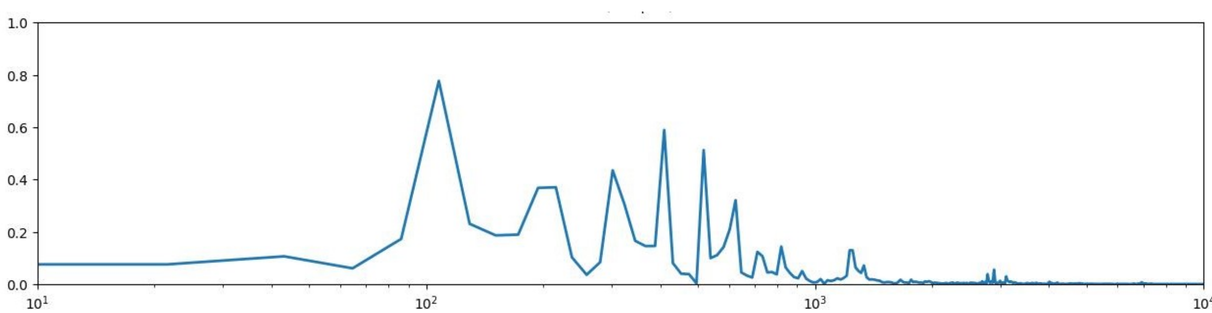


Figure 6: A graph of the frequencies sung by the author and author's friend's attempted harmony in figure 4

(more to be added)

4 Expected Outcome

The expected outcome of this project is to develop a computer program which can take in microphone input, analyze it using Fourier analysis, and then display some form of colorful shapes which represent the musical input. I do not expect the program to be complete by the end of the semester and my main focus of developing this project is in understanding the mathematics behind Fourier analysis and how that math is used in programming, specifically python.

5 Conclusion

(more to be added)

6 Acknowledgements

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✝ JMJ ✝ (Jesus, Mary, Joseph)

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