Predicate Logic

Part 9

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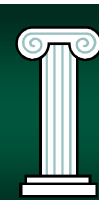

Predicates

Just the facts…

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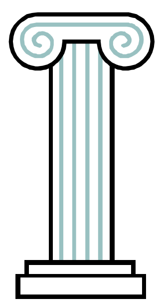
Predicates

- A predicate is a statement about one or more variables
- It is stated as a fact – being true for the data provided



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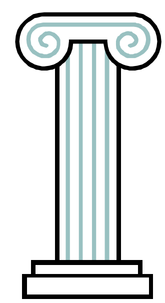
Predicates

- Predicates express *properties*
- These can apply to a single entity or *relations* which may hold on more than one individual



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Predicate Notation

- It follows the same basic syntax as function calls in Java (and most programming languages)
- However, type case is important:
  - constants start with lower case letters
  - predicates start with upper case letters

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Single variable predicate

- Predicates can have one variable (at a minimum)
- The following sentence states one that the cat named Pattycakes has the "sleepy" property

```
"Pattycakes The Cat is sleepy"
```


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Single variable predicate

- Alternatively, we can write that property in predicate form
- "Sleepy" predicate for "Pattycakes" is true
- Note the uppercase and lowercase!

```
Sleepy(pattycakes)
```


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Two Variable Predicate

- Predicates can have multiple variables (unlimited actually… well within reason)
- The following is a classic example of a two-variable relationship

```
x < y
```


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Two Variable Predicate

- The LessThan predicate is true for x, y

```
LessThan(x, y)
```


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Predicates Summary

- 1-place predicates assign properties to individuals:
  - ___ is a cat
  - ___ is sleepy
- 2-place assign relations to a pair
  - ___ is sleeping on ___
  - ___ is the capitol of ___

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Predicates Summary

- 3-place predicates assign relations to triples
  - ___ wants ___ to _____
  - Cat named ____ likes to ___ on ____
- Etc…

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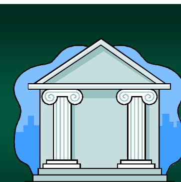

Quantified Statements

More Symbols

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Quantified Statements

- Sometimes we want to say that *every element in the universe* has some property
- Let's say the universe is the people in this Zoom "room" & we want to say "*everyone in the room is awake*"

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Propositional Approach #1

- We can write out a descriptive sentence
- Shortcomings:
  - it is monolithic an inflexible
  - not "mathematical" enough

```
Everyone in this room is awake.
```


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Propositional Approach #2

- We can also write out that sentence using a long list of predicates
- So, we list them all <u>or</u> make a pattern
- Shortcomings: cumbersome & verbose

```
P(moe) and P(larry) and P(curly) and …
```


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Limitations of Propositional Logic

- While propositional logic, which we covered, can express a great deal of complex logical expressions, it ultimately is <u>insufficient</u> for all arguments
- Why? The premises (and conclusions) in propositional logic have <u>no</u> internal structure.

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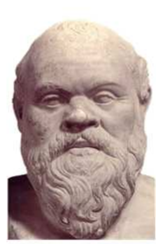
Limitations of Propositional Logic

- For example, it cannot show the validity of Socrates Argument
- This arguments states: *"All humans are mortal. Socrates is a human. Therefore, he is mortal."*



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Socrates Argument

- The following is the propositional logic form of the Socrates Argument
- Can we prove the conclusion?

```
All humans are mortal
Socrates is a human
________________________
Therefore, Socrates is mortal
```


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The Socrates Argument

- The following is the argument in normal form
- A problem arises since the validity of this argument comes from the internal structure which propositional logic cannot "see"

```
H
S
—
M
```


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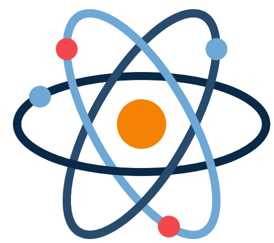
Solution

- It's time to break apart the logic and see the internal structure
- So, ***we are splitting the atom!***



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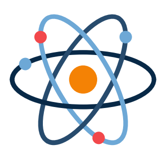
Why Go Nuclear?

1. Expose the internal structure of those "atomic" sentences
2. Create new terminology to describe the semantics
3. Introduce laws to use and manage them



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New Notation: For All

- The "For-All" symbol states every element *x* in the universe makes P(*x*) true
- So, it is true if and only if the every element *x* in the universe has P as true

$\forall_x \; P(x)$

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New Notation: Exists

- The "Exists" States at least one element *x* in the universe makes P(*x*) true
- True if just a single P is true

$\exists_x \; P(x)$

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Example: Pineapple Pizza

- Let's create the quantified statement for *"Someone doesn't like pineapple pizza!"*
- Let's create a predicate P(x) means *"x likes pineapple pizza"*
- What does someone mean? At least one person?

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Pineapple Pizza: Try #1

- How about the following expression?
- It's not true if at least one person likes pineapple
- This means *"nobody likes pineapple pizza"*

$\neg (\exists_x P(x))$

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Pineapple Pizza: Try #2

- We can also negate the predicate
- Means, for at least one person, they dislike pineapple pizza
- *"Someone doesn't like pineapple pizza!"*

$\exists_x \neg P(x)$

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Quantifier Equivalence

Quantifier Conversion

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Equivalence

- Just like propositional logic, quantitative expressions have equivalencies
- They follow the same basic logic we have seen before

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Example: Opposite Expression

- Example: "Everyone in the room is awake"
- Let's create the reverse of this expression *(that still says the same thing)*

"Everyone in the room is awake."

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Example: Opposite Expression

- So, let's just negate the predicate "is awake" into "is asleep"
- Does that work? No.

"Everyone in the room is asleep."

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Example: Opposite Expression

- Now, let's negate the quantifier "everyone'" (for-all) into "someone" (exists)
- The expression below works: almost

"Someone in the room is asleep."

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Example: Opposite Expression

- Well, what if we change the quantifier "everyone'" (for-all) into "someone" (exists)
- The expression below works: yes

"There is no one in the room that is asleep."

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Exists and For-All

- The previous two laws allows us to extrapolate two additional laws
- Note: we simply push the negative and remove the double-negation

$$\forall_x \ P(x) \equiv \neg \exists_x \neg P(x)$$

$$\exists_x \ P(x) \equiv \neg \forall_x \neg P(x)$$

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Equivalence – Moving Negation

- You can push a negation through a quantifier by toggling the quantifier
- Read the expressions below carefully…

$$\neg \exists_x \ P(x) \equiv \forall_x \neg P(x)$$

$$\neg \forall_x \ P(x) \equiv \exists_x \neg P(x)$$

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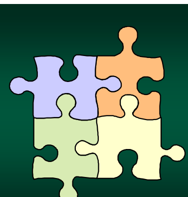

Conjunction & Disjunction

Breaking Apart and Combining

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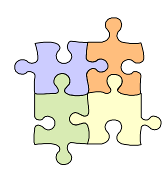
Conjunction & Disjunction

- Both the Exists and For-All quantifiers can be broken apart (and combined)
- This can occur if the expression contains an AND or an OR



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Exists Disjunction

- If the Exists quantifier is used on a disjunction, it can be broken into two Exists
- This only works with $\lor$

$$\exists_x (P(x) \lor Q(x)) \equiv \exists_x P(x) \lor \exists_x Q(x)$$

For-All Conjunction

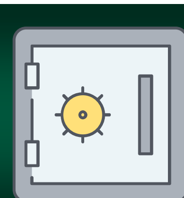
- If the For-All is used on a conjunction (and), it can be broken into two For-All
- This only works with $\land$

$$\forall_x (P(x) \land Q(x)) \equiv \forall_x P(x) \land \forall_x Q(x)$$

Please Wait

CSC 28
will begin shortly

(open the chat window)

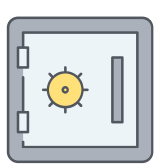


Bound & Free Variables

Some variables are not variable

Bound & Free Variables

- Not all variables used in a quantified expression is treated the same
- Each variable in an expression is either considered "bound" or "free"



Bound & Free Variables

- A variable is *free* if a value must be supplied to it before expression can be evaluated
- A variable is *bound* if it not free (usually a dummy variable) and contains values that are not needed to be evaluated

Example 1

- Which variables need we supply a value before the expression can be evaluated?
- Both x and c
- Without knowing both we cannot evaluate the expression (both are free)

```
(x ^ 2 < 4 * c)
```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Example 2

- Which variables need to be supplied before the expression can be evaluated?
- x: no, it is a dummy variable
- c: yes, once we give a value for c, we can evaluate the expression

$\forall_x$ (x ^ 2 < 4 * c)

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Multiple Quantifiers

Many E's and A's doing headstands

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Multiple Quantifiers

- A quantified statement may have more than one quantifier
- In fact, most of the time, statements will contain several

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Example

- x > y is an expression with two variables
- The expression is true if an x is supplied which is greater than y

$\forall_x \exists_y$ P(x, y)

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Example

- ∃y (x > y) is an expression with one free variable
- Evaluates to true if x is supplied and there is a y greater than the supplied x

$\forall_x \exists_y$ P(x, y)

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Example

- ∀x ∃y (x > y) contains no free variables
- Evaluates to true if ∃y (x > y) is true for every x in the universe.

```
∀x ∃y P(x, y)
```


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Example 2

- The following in an implication with two quantifiers as operands
- It states that whenever "∀x P(x)" is a true statement, then so is "∀x Q(x)".

```
∀x P(x) → ∀x Q(x)
```


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Converting English to Logic

Do it bit by bit

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Difficult Example

- Let's create a quantified statement for the following logical statement
- *We will go slowly, since this is not easy*

Everyone who has a friend who has Covid will have to be quarantined

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Difficult Example

- "Everyone" is a For-All relationship
- What is everyone referring to? People
- So, the abstract object is a person

```
∀x (if x has a friend with Covid,
   then x must be quarantined)
```


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Difficult Example

- So, we can factor it out into the expression below – x is a person
- *Now*, let's look at the sub expression…

```
∀x (if x has a friend with Covid,
   then x must be quarantined)
```


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Difficult Example

- The sentence *" if x has a friend with Covid, x must be quarantined"* is an implication!
- Let's look at the antecedent (hypothesis)

```
if x has a friend with Covid,
   then x must be quarantined
```


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Difficult Example

- How do we write the concept: *"x has **a** friend with Covid"* ?
- They just need a single friend
- So, this is an Exists quantifier

$\exists_y$ (x is friends with y, and y has Covid)

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Difficult Example

- Now that we have a basic form of the final version, let's make some predicates
- We will use single letter names for brevity

```
F(x, y) means "x and y are friends"
C(x)    means "x has Covid"
Q(x)    means "x must be quarantined"
```


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Difficult Example

- This says: "There exists a *person y* where *y* is friends with *person x*, and *y* has Covid"
- Note: *x* is not bound in this expression

$$\exists_y \ (F(x, y) \land C(y))$$

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Difficult Example

- So, what happens if a friend has Covid?
- Then, they must be quarantined
- Note: implication is outside the exists

$$\exists_y (F(x, y) \land C(y)) \rightarrow Q(x)$$

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Difficult Example

- Now we can put it all together…
- The following is the quantified expression for our original statement

$$\forall_x (\exists_y (F(x, y) \land C(y)) \rightarrow Q(x))$$

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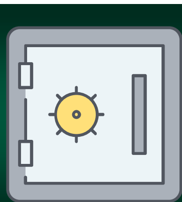

Bounded Quantifiers

Hidden Implication
(for those who hate to type)

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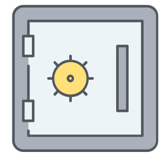
Bound Quantifiers

- Some quantifiers can be more than meets the eye
- For brevity, many predicate and propositional expresses are merged with the ∀ and ∃



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Shorthand Notation

- The following type of expression is quite common
- So much so that a shortcut notation is often employed

$$\forall_{x}\ (R(x) \rightarrow P(x))$$

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Shorthand Notation

- The membership sub-expression is moved to the quantifier's subscript
- This is equivalent to the last

$$\forall_{R(x)} P(x)$$

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Likewise…

- The sub-expression before the implication can be anything
- In this example, x > 5 is moved to the subscript

$$\forall_{x}\ (x > 5 \rightarrow P(x)) \equiv \forall_{x > 5}\ P(x)$$

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