Induction

Part 10

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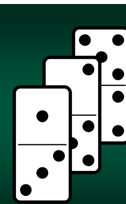

Induction

Proof by Pattern

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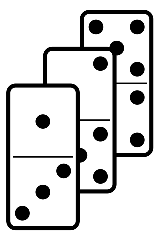
Induction

- Many proofs, in fact a great number of them, are based on "all positive integers"
- *Induction* is a technique of proving a theorem that is based on this criteria



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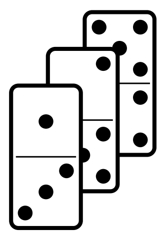
Induction



- The proof by induction is based on the *Well-Ordering Property*
- It states that: given a set of non-negative numbers there is a *least* element

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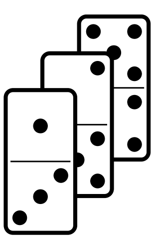
Induction



- Induction basically helps prove $\forall_x P(x)$ where the universe is positive numbers
- *Or any range starting at one point and going off to infinity (positive or negative)*

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How it Works

- It works in 2 steps
  1. proving P(1) and then
  2. proving that $P(n) \rightarrow P(n + 1)$
- As a result...
  - since $P(n) \rightarrow P(n + 1)$
  - then $P(n + 1) \rightarrow P(n + 2)$ and so on…

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Metaphor: Line

- There is a line of people
- First person tells a secret to the second person
- The second person then tells it to the third
- … and so on until everyone knows the secret

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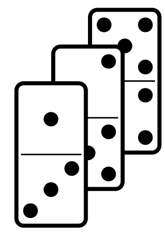
Metaphor: Dominos

- You have a long row of Dominos
- The first domino falls over and hits the second domino
- The second hits the third
- … and so on until they are all knocked over



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Foundation of Induction

- The following summarizes the proof technique
- It is important to understand the approach since it is so commonly used

$$P(1) \land \forall_n \, (P(n) \to P(n+1)) \rightarrow \forall_n \, P(n)$$

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Steps to Proof

- Step 1: *Basis*
  - show the proposition P(1) is true
  - very easy to do – just plug in the values
- Step 2: *Induction*
  - assume P(n) is true (which is your theorem)
  - show that P(n + 1) must be true

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Example: Sum of Odds

Using induction…

Show that the sum of n odd numbers equals n^2

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Example: Sum of Odds

- So, the sum of odd numbers is a square?
- Is that true?
  - 1 + 3 = 4
  - 1 + 3 + 5 = 9
  - 1 + 3 + 5 + 7 = 16
- Okay, that's just odd! *(pun intended)*

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Sum of Odds

```
P(n) is written as:

   1 + 3 + 5 + ... + (2n – 1) = n²

P(n + 1) is written as:

   1 + 3 + … + (2n–1) + (2n+1) = (n+1)²
```


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Basis: Sum of Odds

- The sum of odds, for just 1 number is simply 1
- Of course, this is also 1 squared

$P(1) = 1 = 1^2$

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Induction: Sum of Odds

```
P(n) is written as:

   1 + 3 + 5 + ... + (2n – 1) = n²
```

We assume P(n) is true. So, we are assuming that this equality is valid.

Now we prove $P(n) \rightarrow P(n + 1)$

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Induction: Sum of Odds

```
P(n + 1) is written as:

   1 + 3 + … + (2n–1) + (2n+1) = (n+1)²
```

Can we show this equality is valid?

Let's look at the left side of the equals …

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Show following equals: (n + 1)²

$1 + 3 + \ldots + (2n - 1) + (2n + 1)$

$= 1 + 3 + \ldots + (2n - 1) + (2n + 1)$

$= n^2 + (2n + 1)$

$= (n + 1)^2$

P(n) assumed true, so the equality is true. You can replace!

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Induction: Sum of Odds

So, we have shown that when P(n) is true, then P(n + 1) is true.

$P(n) \rightarrow P(n + 1)$

Since P(1) is true, we have proved $\forall_n P(n)$

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Example: Divisible by 3

Using induction…

Show that $n^3 - n$ is divisible by 3 whenever n is a positive integer

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Basis: Divisible by 3

- For our basis, we plug 1 into our expression and get the result
- The result, 0, is divisible by 3.

$$P(1) = 1^3 - 1 = 1 - 1 = 0$$

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Induction: Divisible by 3

P(n) is written as:

$n^3 - n$

P(n + 1) is written as:

$(n + 1)^3 - (n + 1)$

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Show following is: Divisible by 3

$(n + 1)^3 - (n + 1)$

$= n^3 + 3n^2 + 3n + 1 - (n + 1)$
$= n^3 + 3n^2 + 3n + 1 - n - 1$
$= n^3 + 3n^2 + 3n - n$
$= n^3 - n + 3n^2 + 3n$ ***Rearranged***
$= (n^3 - n) + 3(n^2 + n)$

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Induction: Divisible by 3

So, for $(n^3 - n) + 3(n^2 + n)$

Since we assumed P(n) is true, then $(n^3 - n)$ <u>is</u> divisible by 3.

… and $3(n^2 + n)$ is divisible by 3 since 3 is a factor

Hence, $P(n) \rightarrow P(n + 1)$

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Example: Sum of 2^n

Using induction…

Show that $2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$

whenever n is a positive integer

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Basis: Sum of 2^n

- For our basis, we plug 1 into our expression and get the result
- The result is 1 – which is true

$$P(0) = 2^0 = 1 = 2^1 - 1$$

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Induction: Sum of 2^n

P(n) is written as:

$$2^0 + 2^1 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$$

P(n + 1) is written as:

(n+1) + 1

$$2^0 + 2^1 + \ldots + 2^n + 2^{n+1} = 2^{n+2} - 1$$

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Show following equals: 2^{n+2} - 1

$$\begin{aligned}
&2^0 + 2^1 + \ldots + 2^n + 2^{n+1} \\
&= (2^0 + 2^1 + \ldots + 2^n) + 2^{n+1} \\
&= (2^{n+1} - 1) + 2^{n+1} \\
&= 2^{n+1} + 2^{n+1} - 1 \\
&= 2^n(2^1 + 2^1) - 1 \\
&= 2^n(4) - 1 \\
&= 2^n(2^2) - 1 \\
&= 2^{n+2} - 1
\end{aligned}$$

P(n) assumed true, so the equality is true

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Induction: Sum of 2^n

Since we assumed P(n) is true…

$2^0 + 2^1 + \ldots + 2^n + 2^{n+1}$ is equal to $2^{n+2} - 1$

Hence, $P(n) \rightarrow P(n + 1)$

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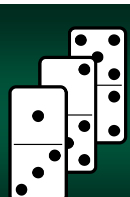

Strong Induction

Another approach

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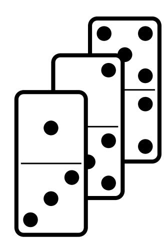
Strong Induction

- *Weak* induction assumes that P(*n*) is true, and then uses that to show P(*n* +1) is true
- *Strong* induction assumes P(1), P(2), …, P(*n*) are all true and then uses that to show that P(*n* +1) is true



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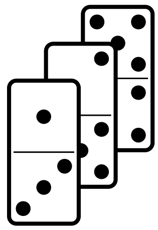
Using the Domino Metaphor…

If all the dominos (1 to n) fell over, will it also have knocked over n+1?



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Strong Induction

- So, strong induction uses more "dominoes" than weak induction – *which just uses one*
- Both proof techniques are equally valid

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( P(1) ∧ P(2) ∧ … ∧ P(k) ) → P(k + 1)
```


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Steps to Proof

- Step 1: *Basis*
  - show the proposition P(1) is true
  - very easy to do – just plug in the values
- Step 2: *Induction*
  - assume that P(1), P(2), …, P(n) are all true
  - show that P(n + 1) is true
  - *or, changing the math slightly: show P(n) is true by assuming P(n – 1), P(n – 2), etc…*

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Example: Product of Primes

Using strong induction…

Show that any number n ≥ 2 can be written as the product of primes

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Basis: Product of Primes

- For 2, we can show that 2 is a product of two primes
- Namely, 1 and itself

```
P(2) = 1 * 2 = 2
```


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Induction: Product of Primes

- There are two cases for n + 1:
- P(n + 1) is prime
- P(n + 1) is composite
  - it can be written as the product of two composites, a and b
  - where 2 ≤ a ≤ b < n + 1

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Induction: Product of Primes

```
n + 1 prime:

   it is a product itself and 1

n + 1 is composite:

   both P(a) and P(b) are assumed to be
   true

   so, there exists primes where
   a * b = n + 1
```


Result

- We showed that any number can be either prime or composite of two numbers
- … and that number holds the same
- As a result, we can keep moving backwards to show that everything must be whittled down to primes