Set Theory in Computer Science

Part 3

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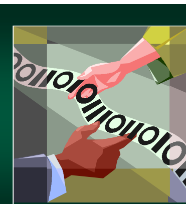

Binary Numbers

Bit of This and a Bit of That

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What is a Number?

- We use the Hindu-Arabic Number System
  - positional grouping system
  - each position is a power of 10
- Binary numbers
  - based on the same system
  - powers of **2** rather than 10
  - each digit is in the set { 0, 1 }

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Base 10 Number

The number 1783 is ...

10^4	10^3	10^2	10^1	10^0
10000	1000	100	10	1
0	**1**	**7**	**8**	**3**

1000 + 700 + 80 + 3 = 1783

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Binary Number Example

The number 0100 1010 is ...

2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
128	64	32	16	8	4	2	1
0	**1**	**0**	**0**	**1**	**0**	**1**	**0**

64 + 8 + 2 = 74

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Binary Number Example

The number 1101 1011 is ...

2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
128	64	32	16	8	4	2	1
1	**1**	**0**	**1**	**1**	**0**	**1**	**1**

128 + 64 + 16 + 8 + 2 + 1 = 219

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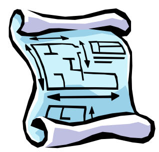
Numbers are Tuples

- In Hindu-Arabic system, the order of the symbols is important – so they are tuples
- e.g. 123 ≠ 321
- Other number styles use sets – i.e. the ancient Egyptian system



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Looking at Numbers

- Numbers are tuples 1947 ≠ 1974
- Members of the decimals number are also members of the set {0, 1, 2, … 9}

```
1947 → (1,9,4,7)
```


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Looking at Binary Numbers

- Binary numbers are tuples 10010100 ≠ 11100000
- Members of the binary number are also members of the set {0, 1}

```
10100111 → (1,0,1,0,0,1,1,1)
```


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Looking at Binary Numbers

- So, for a binary number B, all $x \in B$ holds the following: $x \in \{0, 1\}$

```
10100111 → (1,0,1,0,0,1,1,1)
```


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So….

```
{1776, 1846, 1947} →

{ (1,7,7,6), (1,8,4,6)
  (1,9,4,7) }
```


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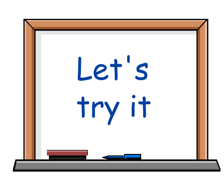
Let's Make a Set-Based System

- We are mostly used to tuple-based number systems
- But, for most of history, people used sets
- Let's create one



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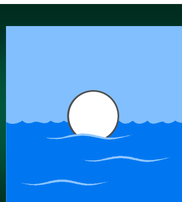

Floating Point Numbers

Real numbers are *real* complex

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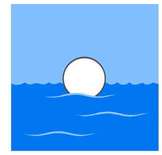
Floating Point Numbers

- Often, programs need to perform mathematics on *real* numbers
- *Floating point numbers* are used to represent quantities that cannot be represented by integers



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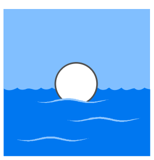
Why use them?



- Regular binary numbers can only store whole positive and negative values
- Many numbers outside the range representable within the system's bit width (too large/small)

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IEEE 754

- Practically modern computers use the *IEEE 754 Standard* to store floating-point numbers
- Represent by a mantissa and an exponent
  - similar to scientific notation
  - the value of a number is: *mantissa* $\times$ $2^{exponent}$
  - uses signed magnitude

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IEEE 754

- Comes in three forms:
  - single-precision: 32-bit
  - double-precision: 64-bit
  - quad-precision: 128-bit
- Also supports special values:
  - negative and positive *infinity*
  - and "not a number" for errors (e.g. 1/0)

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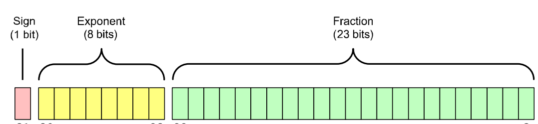
IEEE 754 Single Precision (32 bit)



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Fractional Field

- The fraction field number that represents part of the mantissa
- If a number is in proper scientific notation…
  - it always has a single digit before the decimal place
  - for decimal numbers, this is 1..9 (never zero)
  - for base-2 numbers, it is always 1

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Fractional Field

- So, do we need to store the leading 1? It will always be a 1
- The faction field, therefore…
  - only represents the fractional portion of a binary number
  - the integer portion is assumed to be 1
  - this increases the number of significant digits that can be represented (by not wasting a bit)

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Exponent Field

- The exponent field supports negative and positive values but does not use sign-magnitude or 2's complement
- Uses a "biased" integer representation
  - fixed value is added to the exponent before storing it
  - when interpreting the stored data, this fixed value is then subtracted

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Exponent Field

- Bias is different depending on precision
  - single precision: 127
  - double precision: 1023
  - quad precision: 16383
- For example, for single precision…
  - exponent of 12 stored as (+12 + 127) → 139
  - exponent of -56 stored as (-56 + 127) → 71

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Interpretation: Normal Case

- Exponent Field: not all 0's or all 1's
- Fraction Field: Any

$$\pm\ (1.\mathit{fraction}) \times 2^{(\mathrm{exponent}\ -\ \mathrm{bias})}$$

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Interpretation: Zero

- Exponent Field: all 0's
- Fraction Field: all 0's

0

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Interpretation: Tiny Numbers

- Exponent Field: all 0's
- Fraction Field: Any

$\pm\ (0.\textit{fraction}) \times 2^{(1 - bias)}$

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Interpretation: Infinity

- Exponent Field: All 1's
- Fraction Field: 0

± ***infinity***

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Interpretation: Invalid Numbers

- Exponent Field: All 1's
- Fraction Field: Not 0

Not a number (NaN)

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Interpretation: Invalid Numbers

NaN → **1 / 0**

Naan → 

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More Single-Precision Examples

```
Zero:
0 00000000 00000000000000000000000

Positive Infinity:
0 11111111 00000000000000000000000

Negative Infinity:
1 11111111 00000000000000000000000
```


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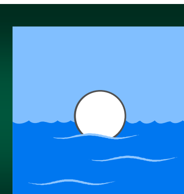

Let's Encode Some Numbers!

This is actually fun!

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Something Else About Numbers…

The number 36.74 is ...

10^2	10^1	10^0	10^{-1}	10^{-2}
100	10	1	1/10	1/100
0	3	6	7	4

= (3×10) + (6×1) + 7/10 + 4/100

Binary Fractions!

The number **101.011** is ...

2^4	2^3	2^2	2^1	2^0	2^{-1}	2^{-2}	2^{-3}
16	8	4	2	1	1/2	1/4	1/8
0	0	1	0	1	0	1	1

= 4 + 1 + 1/4 + 1/8 = 5.375

Let's Encode 14.25 (32 bit)

- First, we need to convert 14.25 to its binary equivalent
- So, we need to calculate the integer part and fraction part of the number
- Everything after the decimal is a fraction
  - 0.25 is actually 25 / 100
  - we need to find the base 2 equivalent (1/4)

Step 1: Convert to binary

```
14 → 1110

0.25 → 1/4 → 0.01

Hence:
14.25 → 1110.01
```

binary 01 / 100

Step 2: Scientific Notation

- IEEE stores the data in scientific notation
- So we move the "binary point" over

```
1110.01 → 1.11001 × 2³
```

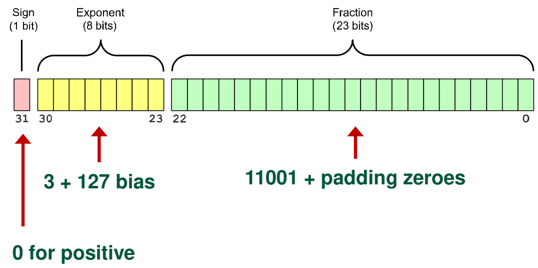

Step 2: Scientific Notation

- In binary scientific notation, the leading digit is always going to be 1
- Why store it? IEEE doesn't.
- Only data after the point is encoded

Fraction

```
1.11001 × 2³ → (1 + .11001) × 2³
```

Step 3: Encode



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Result: 14.25 (32 bit)

- The following is the encoded version of 14.25
- The rules are similar for double-precision

```
0 10000010 11001000000000000000000
```


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Result: 14.25 (32 bit)

- We can also convert this into bytes
- Note: the floating point fields don't really "fit" cleanly into actual bytes

```
01000001 01100100 00000000 00000000
   41       64       00       00
```


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Example 2: Encode 13.75 (32 bit)

- First, we need to convert 13.75 to its binary equivalent
- So, we need to calculate the integer part and fraction part of the number
- Everything after the decimal is a fraction
  - 0.75 is actually 75 / 100
  - we need to find the base 2 equivalent (3/4)

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Step 1: Convert to binary

```
13 → 1101

0.75 → 3/4 → 0.11

Hence:
13.75 → 1101.11
```

binary 11 / 100

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Step 2: Scientific Notation

- IEEE stores the data in scientific notation
- So we move the "binary point" over

$1101.11 \rightarrow 1.10111 \times 2^3$

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Step 2: Scientific Notation

- In binary scientific notation, the leading digit is always going to be 1
- Why store it? IEEE doesn't.
- Only data after the point is encoded

Fraction

```
1.10111 × 2³ → (1 + .10111) × 2³
```


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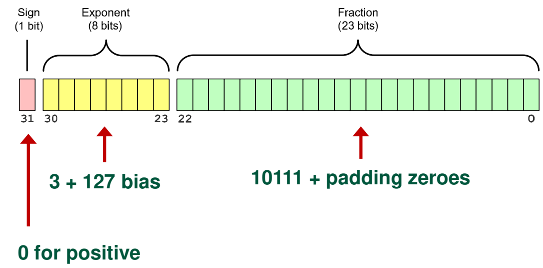
Step 3: Encode



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Result: 13.75 (32 bit)

- The following is the encoded version of 13.75
- The rules are similar for double-precision

```
0 10000010 10111000000000000000000
```


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Result: 13.75 (32 bit)

- We can also convert this into bytes
- Note: the floating point fields don't really "fit" cleanly into actual bytes

```
01000001 01011100 00000000 00000000
   41       5C       00       00
```


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Tuples and Floats

Its all set theory, folks!

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Floats Are Tuples

- Just like regular binary numbers, floating-point numbers of tuples
- They consist of three fields making them 3-tuples

```
(sign, exponent, fraction)
```


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Encoding of 14.25

- Sign is a 1-tuple
- Exponent is a 8-tuple
- Fraction is a 23-tuple

```
0 10000010 11001000000000000000000
```


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Set Notation of 14.25

```
(  (0),
   (1,0,0,0,0,0,1,0),
   (1,1,0,0,1,0,0,0,
    0,0,0,0,0,0,0,0,
    0,0,0,0,0,0,0)    )
```


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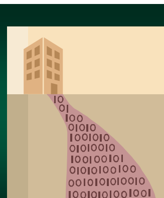

Bit Vectors

Sets and Bits

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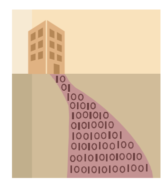
Bit Vectors

- A *bit vector* is a way to store countable sets using bits
- Also known as a *bit array*, *bit set*, and *bit map*
- Compact format that can perform a set operations with a single operation *(fast!)*



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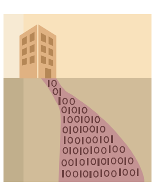
Bit Vectors



- Each object in the universe is given a single bit in the bit array
- If the x ∈ A, then the bit is 1, otherwise 0
- Order is important, so this is a tuple approach

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Example 1

- U = { fry, bender, farnsworth, leela, zoidberg}
- A = { fry, bender, leela }

```
U = 11111
A = 11010
```


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Example 2

- U = { 2, 3, 5, 7, 11, 13, 17, 19}
- A = { 3, 5, 11, 19 }

```
U = 11111111
A = 01101001
```


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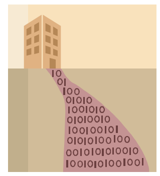
Why this is useful



- Computers can easily perform and & or operations on bytes (or multiple bytes)
- This means set operations can be performed amazingly fast

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Let's look at the definitions again…

- The definitions of union and intersection are nearly identical
- The relationship between the elements is defined using an *and* or *or*

A ∪ B = { x | x ∈ A or x ∈ B }
A ∩ B = { x | x ∈ A and x ∈ B }

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Let's look at the definitions again…

- We can apply a *bit-wise-and* & a *bit-wise-or* to our bit array
- It will apply the operation to each of the bits in matching columns

A ∪ B = { x | x ∈ A or x ∈ B }
A ∩ B = { x | x ∈ A and x ∈ B }

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Let's look at the definitions again…

- So, each bit in A will be compared to its matching bit in B
- Bit match can do sets!

A ∪ B = { x | x ∈ A or x ∈ B }
A ∩ B = { x | x ∈ A and x ∈ B }

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Example: Union (using or)

```
U = {a,b,c,d,e,f,g}
A = {b,c,d} = 0111000
B = {d,e,f} = 0001110

    0111000
 or 0001110
    0111110 = {b,c,d,e,f}
```


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Example: Intersection (using and)

```
U = {a,b,c,d,e,f,g}
A = {b,c,d} = 0111000
B = {d,e,f} = 0001110

    0111000
and 0001110
    0001000 = {d}
```


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Complement

- How do we do a complement of a set A?
- We must flip all the bits from 1 to 0, and 0 to 1
- We can use a *binary-not* or the *XOR* operation

$$A' = \{ x \mid x \notin A \}$$

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Java/C Code

&& and || are Boolean.
These are bit-wise.

```
Intersection :  a & b
Union        :  a | b
Complement   :  ~a
```

The tilde ~ is a bitwise not.

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Exclusion

- Finally, how do we do set difference?
- The "subtract" operator will not work
- Let's look at the definition a bit more closely

$$A \setminus B = \{ x \mid x \in A \text{ and } x \notin B \}$$

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Exclusion

- It's essentially the definition of *intersection*
- Except, the second operand is the definition of *complement*.

$$A \setminus B = \{ x \mid x \in A \text{ and } x \notin B \}$$

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Java/C Code

Just complement the second operand

```
Exclusion :  a & ~b
```


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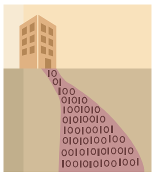
Limits of Bit Vectors



- Bit vectors, while useful, do have some notable limitations
- They only work on finite, countable sets
- For all other cases, you will have to work use a more advanced ADT

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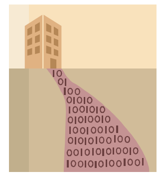
CSC 130 is waiting for you!



- For most cases, a very sophisticated list or tree can be used
- You will need to know:
  - lists / trees
  - sorting
  - binary-searches
  - Big-O

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