Set Theory

Part 1

1

What is a Set?

Organizing Information

2

What is a Set?

- A *set* is an unordered collection of “objects”
- The collection objects are also called “members” or "elements"
- One of the most fundamental structures in mathematics

3

Set Notation

- We typically denote a set name using capital letter
- Members are separated with commas and encapsulated within curly brackets

4

Standard Sets

Letter	Name	Members
Z	Integers	…, -2, -1, 0, 1, 2, 3, …
N	Natural Numbers	1, 2, 3, 4, …
Q	Rational Numbers	a / b where both a and b are integers and b is not 0

5

Standard Sets

Letter	Name	Members
R	Real Numbers	All non-imaginary numbers. e.g. 1, 2.5, 3.1415….
U	Universal Set	All values of potential interest (U depends on context)

6

Set Notation: Membership

- Set notation uses a special symbol to denote if an object is a member of a set
- Below, the set V contains vegetables

```
potato ∈ V
```


7

Set Notation: Membership

- This is read as "potato is an element of V"
- …or "potato is a member of V"

```
potato ∈ V
```


8

Set Notation: Not a Member

- There is another special symbol that denotes an object is not a member of a set
- In the example below, the set F contains fluffy animals

```
lizard ∉ F
```


9

Multiple members can be listed

- Multiple elements can be listed
- The expression below states that both *potato* and *carrot* are in the set V.

```
potato, carrot ∈ V
```


10

{ }

Defining Sets

How to specify items

11

Defining Sets

- Sets can be defined a number of different ways
- Each competing notation has advantages & disadvantages – depending on what you are defining

{ }

12

Set Notation: Explicit

- We can *explicitly* define this by listing each element
- For example, we can define a set *S* for members of the Three Stooges

```
S = {moe, larry, curly, shemp}
```


13

Set Notation: Pattern

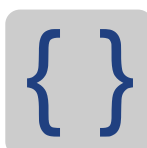
- We can also specify a set by using a *pattern*.
- In the example below we are define a set of integers between 0 and 9.

```
A = {0, 1, 2, ... 9}
```


14

Set Builder Notation

- A set can also be defined using *set builder notation*
- Consists of a variable name, a pipe symbol, and an true/false expression



15

By Characteristic

- The most basic form consists of a variable and an true/false statement
- In this example, everything that satisfies "x is an even integer" will be the set

```
{x | x is a even integer}
```


16

By Characteristic Examples

Expression	Result
{ x | x is an integer }	{ …, -1, 0, 1, 2, 3, … }
{ x | x is an even integer }	{ …, -2, 0, 2, 4, 6, … }
{ x | x is odd natural number}	{ 1, 3, 5, 7, 9, … }

17

Shorthand Notation

- Definitions can also be restricted by another set
- There are two different notations that *mean the same thing*

```
{x ∈ S | true/false expression on x}

{x | x ∈ S and true/false expression on x}
```


18

Characteristic Example

- Remember, Z is the set of all integers
- It reads: *"All x where x is in Z and x is even"*

```
A = {x | x ∈ Z and x is even}
```


19

By Characteristic Examples

Expression	Result
$\{ x \in Z \mid 0 < x < 5 \}$	{1, 2, 3, 4}
$\{ x \mid x \in N$ and $x < 7 \}$	{1, 2, 3, 4, 5, 6}

20

Characteristic with Structure

- The left-hand-side (before the pipe) doesn't have to be a simple variable name
- It can also be any mathematical expression

```
{f(x) | true/false expression using x}

{y | y = f(x) and true/false using x}
```


21

Let's Try One…

- The second part of the notation must always be a true/false expression
- So, how do we create a set that contains:

{2, 4, 6, 8, 10, …}

22

Let's Try One…

```
First approach:
A = {x | x ∈ N and x is even}

Second approach:
A = {2x | x ∈ N}
```


23

How Does It Evaluate?

- Basically, when you look at something like: $\{ 2x \mid x \in N \}$, you should do the following
- Steps:
  1. Identify which variables make the right-hand-side true
  2. Plug them into the left-hand-side. These are the values in the set.

24

More Examples

Expression	Result
{ 2x + 1 | x ∈ Z }	{…, -3, -1, 1, 3, 5, …}
{ x ∈ Z | sqrt(x) ∈ Z }	{0, 1, 4, 9, 16, 25, …}

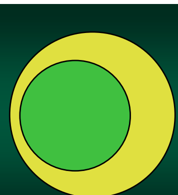
25

Empty Set

- An *empty set* contains no elements
- Can be represented with two curly-brackets (nothing in between)
- There is also a special symbol for empty sets

```
A = { }
A = ∅
```


26



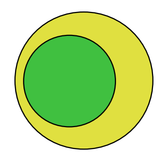
Subsets

Set Your Sets to Fun

27

Subsets

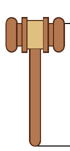
- Commonly, sets are compared to one another using set relationship operators
- Basically, set are defined on elements which they may have in common



28

Subsets

- Set A is considered a subset of set B if <u>all</u> the members of A are also members of B
- The subset operator is similar looking to the member operator



A ⊆ B if and only if:
for all x ∈ A there is x ∈ B

29

Subsets

- In the example below, the set {1, 4} is a subset of the set {1, 3, 4, 5}
- Note that the reverse is not true.

```
{ 1, 4 } ⊆ { 1, 3, 4, 5 }
```


30

Subsets

- To denote a set is not a subset, we use the subset operator and add a slash
- Below, the set {3, 5} is not a subset of {3, 7} because {3, 7} does not contain 5.

```
{ 3, 5 } ⊄ { 3, 7 }
```


31

Null is Always a Subset

- A null set contains no elements
- Hence, the null set is always a subset

```
∅  ⊆ { 2, 3 }
{} ⊆ { 2, 3 }
```


32

Proper Subsets

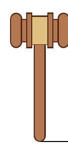
- Set A is a *proper subset* of B if A is a subset of B, but **not** equal to B
- Note: the notation lacks the underline – it is consistent with other operators like < and ≤

```
{ 3, 5 } ⊂ { 3, 5, 7 }
{ 1, 2 } ⊄ { 1, 2 }
```


33

Equality

- Sets A and B are considered equal if-and-only-if… each contain the same elements
- … remember, duplicates don't count



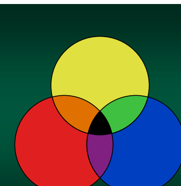
A = B if and only if:
all $x \in A$ there is $x \in B$ and
all $y \in B$ there is $y \in A$

34

Equality

- So, are { 1, 2, 3 } and { 2, 1, 3 } equal?
- How about { 1, 1, 2, 3, 3 } and { 3, 2, 1 }
- Answer is **yes!**
  - order does not matter in a set
  - multiple occurrences does not change if an element is a member

35



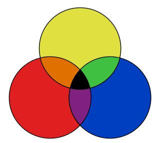
Venn Diagrams

Graphically Representing Sets

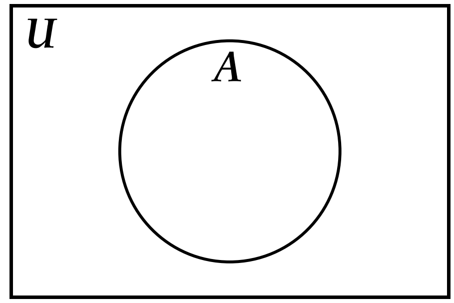
36

Venn Diagrams

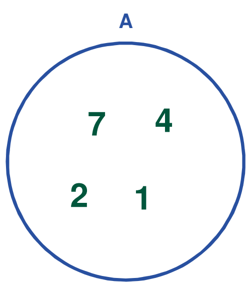
- Sets can also be abstractly representing graphically using Venn Diagrams
- Each set is represented by circle
- Overlaps between each set can show logical relations with set members



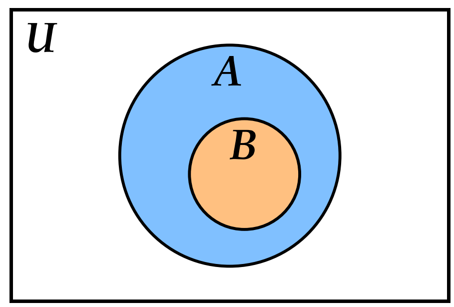
Basic Venn Diagram



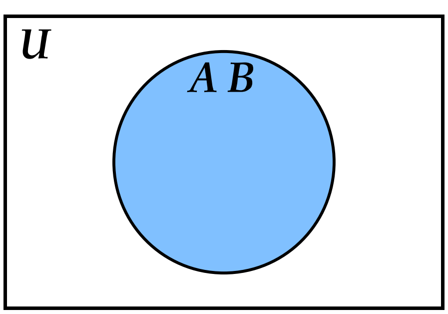
Example: A = {2, 7, 1, 4}



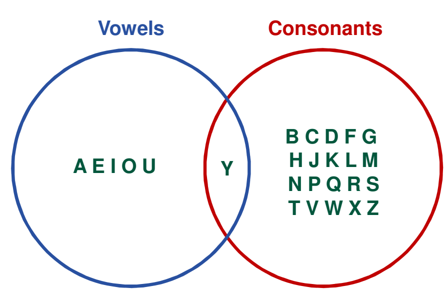
Subset Venn Diagram

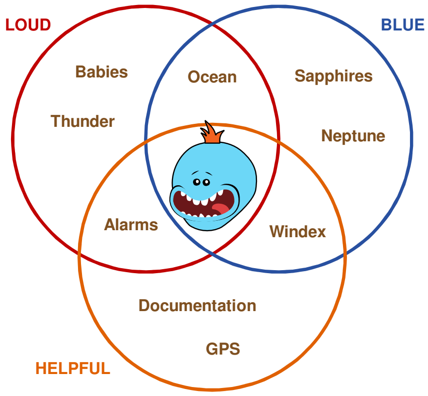


Equality



Example: Letters in English





43

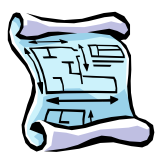
Tuples

Order is Important

44

Tuples & Sets

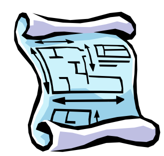
- To denote sets, we delimit the list of members with curly brackets
- For example, the prime numbers between 1 and 10 is {2, 3, 5, 7}
- Order does not matter, so {2, 3, 5, 7} = {7, 5, 3, 2}



45

Tuples

- However, in many cases the *order is important*
- These are called *n-tuples* where "n" is the number of elements
- 2-tuples are also called *ordered pairs*



46

Tuple Notation

- To denote a tuple – we delimit the elements by either parenthesis, angle brackets or square brackets
- Curly-brackets are never used to avoid obvious confusion

```
( 1, 2, 3 )
< 1, 2, 3 >
[ 1, 2, 3 ]
```


47

Tuple Examples

- Order is important, so any element out of position will cause inequality

```
( 1, 2, 3 ) ≠ ( 3, 2, 1 )
```

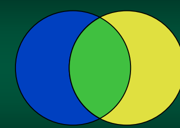

48

Tuple Examples

- Logic generally applies to algorithms since, in procedural programming, order is important
- The following is a tuple of events in California History

```
( Sutter's Fort Built, Bear Flag Revolt,
  Gold Rush, California Joins Union )
```


49



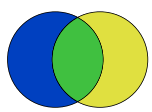
Set Operators

Defining Sets Using Sets

50

Operations on Sets

- New sets can be made from old sets using set operators.
- Just like new numbers can be created from old numbers: 1 + 2 = 3
- So, for the rest of this section, let **U** be the universe, and let A and B be sets



51

Union

- A union of two sets combines all members of each set into a new one
- So, the result is two merged sets

$$A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$$

52

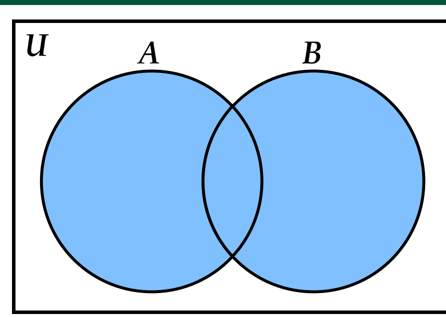
Union

- The symbol ∪ looks like **U**
  - which is also used for the "universe set"
  - be careful not the confuse the two

$$A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$$

53

A ∪ B



54

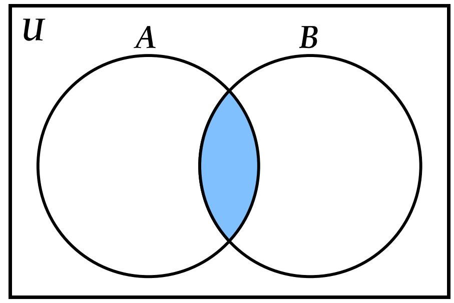
Intersection

- The intersection of two sets contains only those elements that are found in both sets
- So, the result is where the two sets overlap

$A \cap B = \{ x \mid x \in A \text{ and } x \in B \}$

55

A ∩ B



56

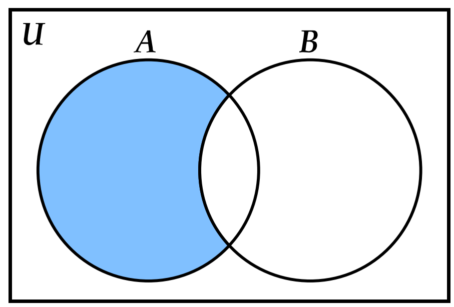
Difference

- *Difference* excludes all items found in one set from another
- Also known as the *relative complement*

$A \setminus B = \{ x \mid x \in A \text{ and } x \notin B \}$

57

A \ B



58

Difference – So Many Notations

- Difference can be written A \ B or A – B (*even though it is not the same as subtraction)*
- Both notations are valid, but some mathematians prefer one over another

```
A - B
A \ B
```


59

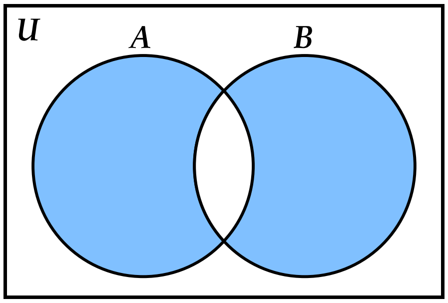
Symmetric Difference

- The *Symmetric Difference* is all the items that are in either of two sets, but not both
- It can be defined two different ways

$$A \oplus B = (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$$

60

A ⊕ B



61

Let's Draw Some…

Let's try it

- So, A ⊕ B has two definitions
- Let's test if both definitions create the same result

$(A \cup B) \setminus (A \cap B)$
$(A \setminus B) \cup (B \setminus A)$

62

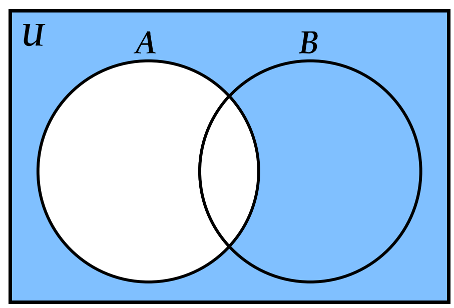
Complement

- The *complement* of a set A, is all elements in the Universe, not in A
- Remember: what elements are in the the Universe depends on the sets

$A' = \{ x \mid x \notin A \}$

63

A'



64

Different Notations Used

- Single postfix apostrophe
- An "over bar" (which is underlining on top)
- Superscript "c" for complement

$A' \equiv \overline{A} \equiv A^c$

65

Complement Example

- If set A is a subset of a set B, then the complement of A is all elements not in A but still in B
- Look at the following:

```
A ⊂ B
A = {1, 2, 3}
B = {1, 2, … 10}
```


66

Complement Example

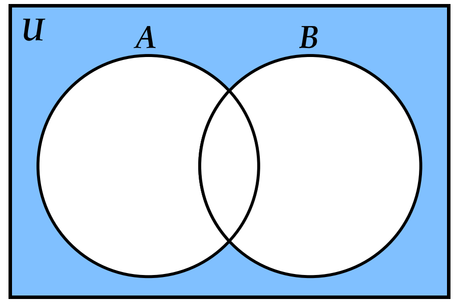
- Set A is a subset of a set B
- Therefore its "universe" is defined as the set of B

```
Therefore…

A' = {4, 5, 6, 7, 8, 9, 10}
```

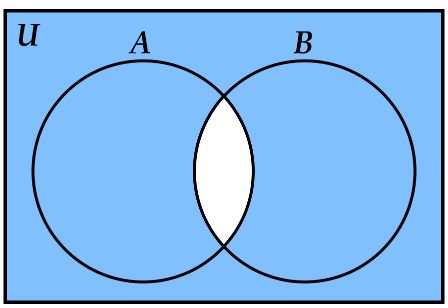

67

(A ∪ B)'



68

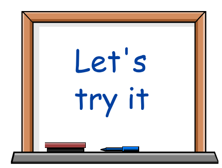
(A ∩ B)'



69

Let's Draw Some…

- Let's draw some Venn Diagrams using a several sets
- Using set operators we can highlight any area we want



70

Let's Draw Some


```
U = {a, b, c, d, e, f}

A = {a, b, c}
B = {b, c, d, e}
```

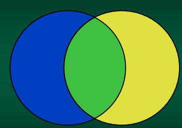

71

Let's Draw Some

```
Two sets:

In A but not in B: {a}
In B but not in A: {d, e}
In both A and B: {b, c}
In neither A nor B: {f}
```


72



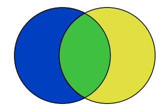
Set Algebra

Just a preview…

73

Set Algebra

- Sets share the same principles as basic math
- You can visually treat the union as an * and the intersection as a +
- You can then factor out sets



74

Communitive Law

- Both ∩ and ∪ are communitive
- This means the left-hand and right-hand operands can be switched

$$A \cap B \equiv B \cap A$$
$$A \cup B \equiv B \cup A$$

75

Idempotent Law

- When a set is combined with itself, it is equivalent to just the statement (no duplicate)
- This applies to both ∩ and ∪

$$A \cap A \equiv A$$
$$A \cup A \equiv A$$

76

Involution Law

- One of the most basic equivalences in logic is the *double negation*
- It is fairly obvious, so not more needs to be said

$$(A')' \equiv A$$

77

Complement Law

- When a set is used with its complement it will result in either the universe or the empty set

$$A \cap A' \equiv \varnothing$$
$$A \cup A' \equiv U$$

78

Complement Law

- Complement Law also can be applied to the Universal Set and Empty Set
- The results should be fairly obvious

$\varnothing' \equiv U$

$U' \equiv \varnothing$

79

Identity Law

- Some operands will have no effect on the truth table of a statement
- In this case, the statement can be simplified

$A \cap U \equiv A$

$A \cup \varnothing \equiv A$

80

Domination Law

- This might look similar to the identity law, but look very careful at which operator is being used
- These will result in either the universe or the empty set

$A \cup U \equiv U$

$A \cap \varnothing \equiv \varnothing$

81

Associative Law

- Some operators in math are *associative*
- For example: (a + b) + c = a + (b + c)
- Same applies to ∩ and ∪

$A \cap (B \cap C) \equiv (A \cap B) \cap C$

$A \cup (B \cup C) \equiv (A \cup B) \cup C$

82

Distributive Law

- Math has operators that are *distributive*
- For example: a * (b + c) = (a * b) + (a * c)
- Works for both ∩ and ∪

$A \cap (B \cup C) \equiv (A \cap B) \cup (A \cap C)$

$A \cup (B \cap C) \equiv (A \cup B) \cap (A \cup C)$

83

Another Look

(A ∪ B) ∩ (A ∪ C)

→ (A * B) + (A * C) =
A * (B + C)

A ∪ (B ∩ C)

84

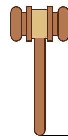
DeMorgan's Law

- *DeMorgan's Law* states important rule for logical equivalency
- These are used to convert $\cap$ to $\cup$ and vice-versa

85

DeMorgan's Law

- So, it states you can change the operator from $\cap$ to $\cup$ or vice-versa
- If you negate both operands

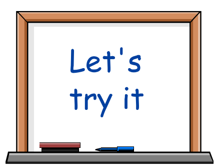


$$(A \cap B)' \equiv A' \cup B'$$
$$(A \cup B)' \equiv A' \cap B'$$

86

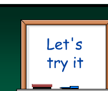
Analyzing DeMorgan's Law

- Let's draw some Venn Diagrams and analyze if DeMorgan's Law works
- First, let's look the logic of what an expression says



87

Let's Draw These…



Let's see if the following are the same:

$A \cup (B \cap C)$

$(A \cup B) \cap (A \cup C)$

88

Algebra: Example 1

$(A \cup B)' \cap B$

$A' \cap B' \cap B$ — After using DeMorgan's Law

$A' \cap \varnothing =$ — After using Complement Law

$\varnothing$ — After using Domination Law

89

Algebra: Example 2

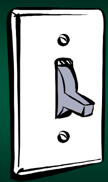
$(A \cap B) \cup (A' \cap B)$

$(A \cup A') \cap B$ — After using Distributive Law

$U \cap B$ — After using Complement Law

B — After using Identify Law

90



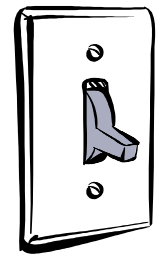
Duals

Don't "Flip" Out

91

Duels

- The dual of a theorem is created by:
  - flip all **U** and ∅
  - flip all ∪ and ∩
- Important property:
  if an expression is an identity then its duel is an identity



92

Example Duels

- The two expressions (identities) below are duels of each other
- Both expressions are true, but let's test

```
1. A = (U ∩ A) ∪ (A ∩ B)
2. A = (∅ ∪ A) ∩ (A ∪ B)
```


93

Example: Expression 1

```
A = (U ∩ A) ∪ (A ∩ B)
  = A ∩ (U ∪ B)       Distributive
  = A ∩ U             Domination
  = A                 Identity
```

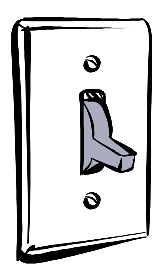

94

Example: Expression 2

```
A = (∅ ∪ A) ∩ (A ∪ B)
  = A ∪ (∅ ∩ B)       Distributive
  = A ∪ ∅             Domination
  = A                 Identity
```


95

Duels



- Also note that the proof (i.e. the reduction) of the expressions was identical
- The number of steps (as well as each law) was the same

96