Set Theory

Part 1

1

What is a Set?

Organizing Information

2

What is a Set?

- A *set* is an unordered collection of “objects”
- The collection objects are also called “members” or "elements"
- One of the most fundamental structures in mathematics

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Set Notation

- We typically denote a set name using capital letter
- Members are separated with commas and encapsulated within curly brackets

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Standard Sets

Letter	Name	Members
Z	Integers	…, -2, -1, 0, 1, 2, 3, …
N	Natural Numbers	1, 2, 3, 4, …
Q	Rational Numbers	a / b where both a and b are integers and b is not 0

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Standard Sets

Letter	Name	Members
R	Real Numbers	All non-imaginary numbers. e.g. 1, 2.5, 3.1415….
U	Universal Set	All values of potential interest (U depends on context)

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Set Notation: Membership

- Set notation uses a special symbol to denote if an object is a member of a set
- Below, the set V contains vegetables

```
potato ∈ V
```


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Set Notation: Membership

- This is read as "potato is an element of V"
- …or "potato is a member of V"

```
potato ∈ V
```


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Set Notation: Not a Member

- There is another special symbol that denotes an object is not a member of a set
- In the example below, the set F contains fluffy animals

```
lizard ∉ F
```


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Multiple members can be listed

- Multiple elements can be listed
- The expression below states that both *potato* and *carrot* are in the set V.

```
potato, carrot ∈ V
```


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{ }

Defining Sets

How to specify items

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Defining Sets

- Sets can be defined a number of different ways
- Each competing notation has advantages & disadvantages – depending on what you are defining

{ }

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Set Notation: Explicit

- We can *explicitly* define this by listing each element
- For example, we can define a set *S* for members of the Three Stooges

```
S = {moe, larry, curly, shemp}
```


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Set Notation: Pattern

- We can also specify a set by using a *pattern*.
- In the example below we are define a set of integers between 0 and 9.

```
A = {0, 1, 2, ... 9}
```


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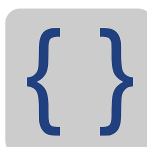
Set Builder Notation

- A set can also be defined using *set builder notation*
- Consists of a variable name, a pipe symbol, and an true/false expression



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By Characteristic

- The most basic form consists of a variable and an true/false statement
- In this example, everything that satisfies "x is an even integer" will be the set

```
{x | x is a even integer}
```


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By Characteristic Examples

Expression	Result
{ x | x is an integer }	{ …, -1, 0, 1, 2, 3, … }
{ x | x is an even integer }	{ …, -2, 0, 2, 4, 6, … }
{ x | x is odd natural number}	{ 1, 3, 5, 7, 9, … }

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Shorthand Notation

- Definitions can also be restricted by another set
- There are two different notations that *mean the same thing*

```
{x ∈ S | true/false expression on x}

{x | x ∈ S and true/false expression on x}
```


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Characteristic Example

- Remember, Z is the set of all integers
- It reads: *"All x where x is in Z and x is even"*

```
A = {x | x ∈ Z and x is even}
```


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By Characteristic Examples

Expression	Result
{ x ∈ Z | 0 < x < 5 }	{1, 2, 3, 4}
{ x | x ∈ N and x < 7 }	{1, 2, 3, 4, 5, 6}

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Characteristic with Structure

- The left-hand-side (before the pipe) doesn't have to be a simple variable name
- It can also be any mathematical expression

```
{f(x) | true/false expression using x}

{y | y = f(x) and true/false using x}
```


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Let's Try One…

- The second part of the notation must always be a true/false expression
- So, how do we create a set that contains:

{2, 4, 6, 8, 10, …}

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Let's Try One…

```
First approach:
A = {x | x ∈ N and x is even}

Second approach:
A = {2x | x ∈ N}
```


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How Does It Evaluate?

- Basically, when you look at something like: { 2x | x ∈ N }, you should do the following
- Steps:
  1. Identify which variables make the right-hand-side true
  2. Plug them into the left-hand-side. These are the values in the set.

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More Examples

Expression	Result
$\{ 2x + 1 \mid x \in Z \}$	$\{\ldots, -3, -1, 1, 3, 5, \ldots\}$
$\{ x \in Z \mid \text{sqrt}(x) \in Z \}$	$\{0, 1, 4, 9, 16, 25, \ldots\}$

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Empty Set

- An *empty set* contains no elements
- Can be represented with two curly-brackets (nothing in between)
- There is also a special symbol for empty sets

```
A = { }
A = ∅
```


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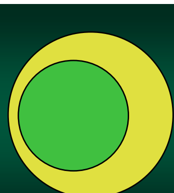

Subsets

Set Your Sets to Fun

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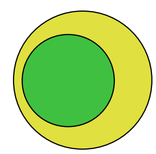
Subsets

- Commonly, sets are compared to one another using set relationship operators
- Basically, set are defined on elements which they may have in common



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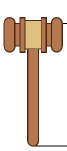
Subsets

- Set A is considered a subset of set B if all the members of A are also members of B
- The subset operator is similar looking to the member operator



$A \subseteq B$ if and only if:
for all $x \in A$ there is $x \in B$

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Subsets

- In the example below, the set {1, 4} is a subset of the set {1, 3, 4, 5}
- Note that the reverse is not true.

```
{ 1, 4 } ⊆ { 1, 3, 4, 5 }
```


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Subsets

- To denote a set is not a subset, we use the subset operator and add a slash
- Below, the set {3, 5} is not a subset of {3, 7} because {3, 7} does not contain 5.

$$\{ 3, 5 \} \not\subseteq \{ 3, 7 \}$$

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Null is Always a Subset

- A null set contains no elements
- Hence, the null set is always a subset

$$\emptyset \subseteq \{ 2, 3 \}$$
$$\{\} \subseteq \{ 2, 3 \}$$

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Proper Subsets

- Set A is a *proper subset* of B if A is a subset of B, but **not** equal to B
- Note: the notation lacks the underline – it is consistent with other operators like < and ≤

$$\{ 3, 5 \} \subset \{ 3, 5, 7 \}$$
$$\{ 1, 2 \} \not\subset \{ 1, 2 \}$$

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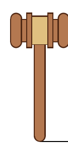
Equality

- Sets A and B are considered equal if-and-only-if… each contain the same elements
- … remember, duplicates don't count



A = B if and only if:
all $x \in A$ there is $x \in B$ and
all $y \in B$ there is $y \in A$

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Equality

- So, are { 1, 2, 3 } and { 2, 1, 3 } equal?
- How about { 1, 1, 2, 3, 3 } and { 3, 2, 1 }
- Answer is **yes!**
  - order does not matter in a set
  - multiple occurrences does not change if an element is a member

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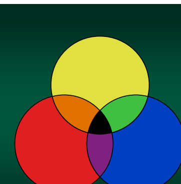

Venn Diagrams

Graphically Representing Sets

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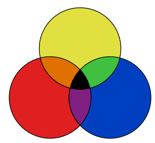
Venn Diagrams

- Sets can also be abstractly representing graphically using Venn Diagrams
- Each set is represented by circle
- Overlaps between each set can show logical relations with set members



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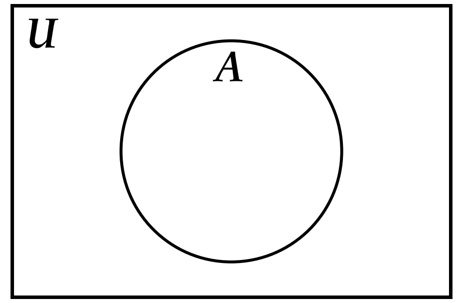
Basic Venn Diagram



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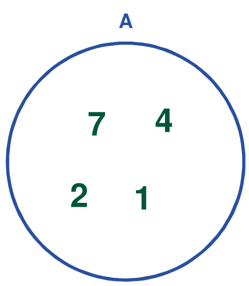
Example: A = {2, 7, 1, 4}



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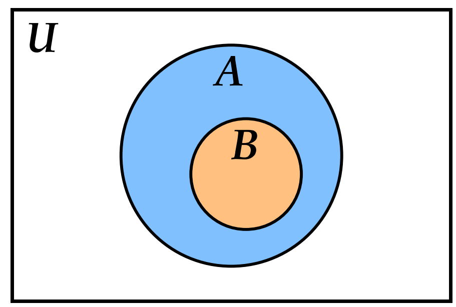
Subset Venn Diagram



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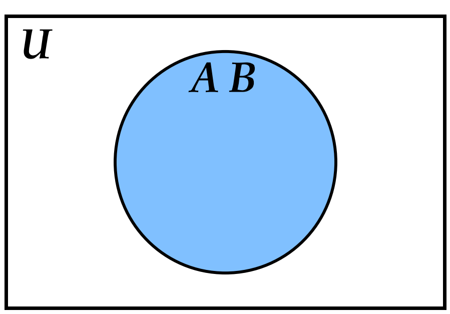
Equality



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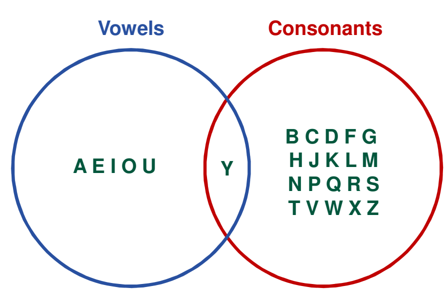
Example: Letters in English



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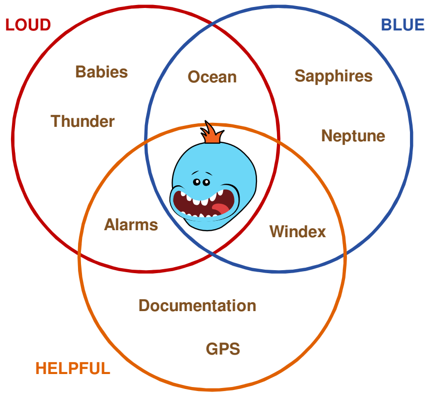

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Tuples

Order is Important

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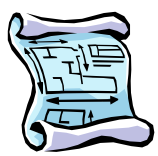
Tuples & Sets

- To denote sets, we delimit the list of members with curly brackets
- For example, the prime numbers between 1 and 10 is {2, 3, 5, 7}
- Order does not matter, so {2, 3, 5, 7} = {7, 5, 3, 2}



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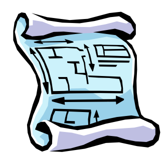
Tuples

- However, in many cases the *order is important*
- These are called *n-tuples* where "n" is the number of elements
- 2-tuples are also called *ordered pairs*



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Tuple Notation

- To denote a tuple – we delimit the elements by either parenthesis, angle brackets or square brackets
- Curly-brackets are never used to avoid obvious confusion

```
( 1, 2, 3 )
< 1, 2, 3 >
[ 1, 2, 3 ]
```


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Tuple Examples

- Order is important, so any element out of position will cause inequality

```
( 1, 2, 3 ) ≠ ( 3, 2, 1 )
```


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Tuple Examples

- Logic generally applies to algorithms since, in procedural programming, order is important
- The following is a tuple of events in California History

```
( Sutter's Fort Built, Bear Flag Revolt,
  Gold Rush, California Joins Union )
```


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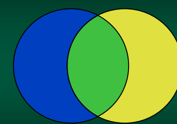

Set Operators

Defining Sets Using Sets

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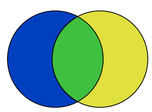
Operations on Sets

- New sets can be made from old sets using set operators.
- Just like new numbers can be created from old numbers: 1 + 2 = 3
- So, for the rest of this section, let **U** be the universe, and let A and B be sets



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Union

- A union of two sets combines all members of each set into a new one
- So, the result is two merged sets

$$A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$$

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Union

- The symbol ∪ looks like **U**
  - which is also used for the "universe set"
  - be careful not the confuse the two

$$A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$$

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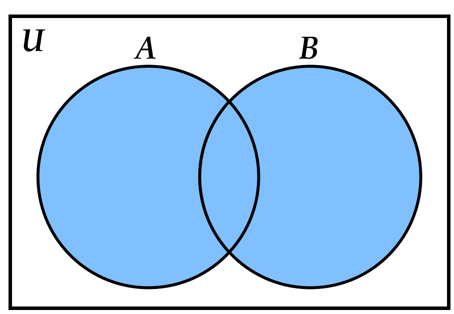
A ∪ B



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Intersection

- The intersection of two sets contains only those elements that are found in both sets
- So, the result is where the two sets overlap

$A \cap B = \{ x \mid x \in A \text{ and } x \in B \}$

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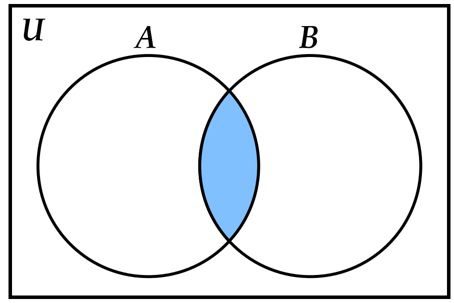
A ∩ B



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Difference

- *Difference* excludes all items found in one set from another
- Also known as the *relative complement*

$A \setminus B = \{ x \mid x \in A \text{ and } x \notin B \}$

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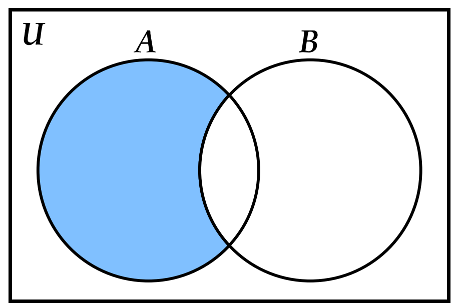
A \ B



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Difference – So Many Notations

- Difference can be written A \ B or A – B (*even though it is not the same as subtraction)*
- Both notations are valid, but some mathematians prefer one over another

```
A - B
A \ B
```


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Symmetric Difference

- The *Symmetric Difference* is all the items that are in either of two sets, but not both
- It can be defined two different ways

$$A \oplus B = (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$$

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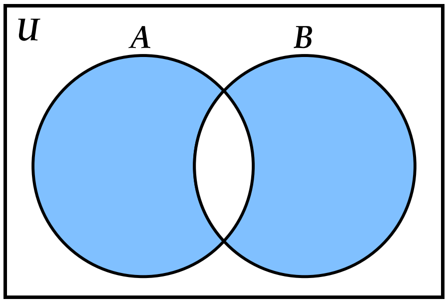
A ⊕ B



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Let's Draw Some…

Let's try it

- So, A ⊕ B has two definitions
- Let's test if both definitions create the same result

$(A \cup B) \setminus (A \cap B)$

$(A \setminus B) \cup (B \setminus A)$

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Complement

- The *complement* of a set A, is all elements in the Universe, not in A
- Remember: what elements are in the Universe depends on the sets

$A' = \{ x \mid x \notin A \}$

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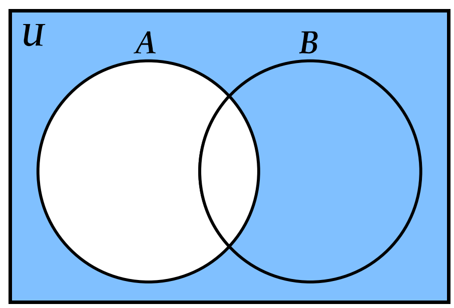
A'



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Different Notations Used

- Single postfix apostrophe
- An "over bar" (which is underlining on top)
- Superscript "c" for complement

$A' \equiv \overline{A} \equiv A^c$

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Complement Example

- If set A is a subset of a set B, then the complement of A is all elements not in A but still in B
- Look at the following:

```
A ⊂ B
A = {1, 2, 3}
B = {1, 2, … 10}
```


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Complement Example

- Set A is a subset of a set B
- Therefore its “universe” is defined as the set of B

```
Therefore…

A' = {4, 5, 6, 7, 8, 9, 10}
```


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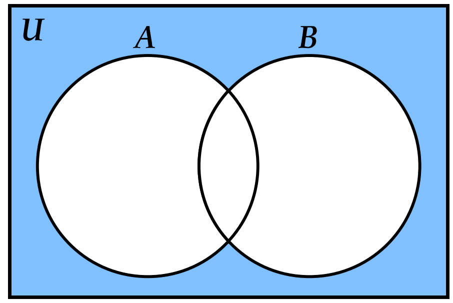
(A ∪ B)'



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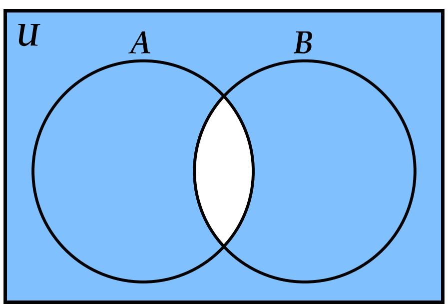
(A ∩ B)'



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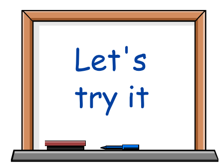
Let’s Draw Some…

- Let’s draw some Venn Diagrams using a several sets
- Using set operators we can highlight any area we want



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Let’s Draw Some


```
U = {a, b, c, d, e, f}

A = {a, b, c}
B = {b, c, d, e}
```


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Let’s Draw Some

```
Two sets:

In A but not in B: {a}
In B but not in A: {d, e}
In both A and B: {b, c}
In neither A nor B: {f}
```


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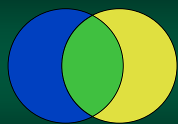

Set Algebra

Just a preview…

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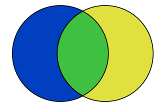
Set Algebra

- Sets share the same principles as basic math
- You can visually treat the union as an * and the intersection as a +
- You can then factor out sets



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Communitive Law

- Both ∩ and ∪ are communitive
- This means the left-hand and right-hand operands can be switched

$$A \cap B \equiv B \cap A$$
$$A \cup B \equiv B \cup A$$

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Idempotent Law

- When a set is combined with itself, it is equivalent to just the statement (no duplicate)
- This applies to both ∩ and ∪

$$A \cap A \equiv A$$
$$A \cup A \equiv A$$

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Involution Law

- One of the most basic equivalences in logic is the *double negation*
- It is fairly obvious, so not more needs to be said

$$(A')' \equiv A$$

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Complement Law

- When a set is used with its complement it will result in either the universe or the empty set

$$A \cap A' \equiv \varnothing$$
$$A \cup A' \equiv U$$

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Complement Law

- Complement Law also can be applied to the Universal Set and Empty Set
- The results should be fairly obvious

$\varnothing' \equiv U$
$U' \equiv \varnothing$

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Identity Law

- Some operands will have no effect on the truth table of a statement
- In this case, the statement can be simplified

$A \cap U \equiv A$
$A \cup \varnothing \equiv A$

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Domination Law

- This might look similar to the identity law, but look very careful at which operator is being used
- These will result in either the universe or the empty set

$A \cup U \equiv U$
$A \cap \varnothing \equiv \varnothing$

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Associative Law

- Some operators in math are *associative*
- For example: (a + b) + c = a + (b + c)
- Same applies to ∩ and ∪

$A \cap (B \cap C) \equiv (A \cap B) \cap C$
$A \cup (B \cup C) \equiv (A \cup B) \cup C$

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Distributive Law

- Math has operators that are *distributive*
- For example: a * (b + c) = (a * b) + (a * c)
- Works for both ∩ and ∪

$A \cap (B \cup C) \equiv (A \cap B) \cup (A \cap C)$
$A \cup (B \cap C) \equiv (A \cup B) \cap (A \cup C)$

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Another Look

(A ∪ B) ∩ (A ∪ C)

→ *(A * B) + (A * C) =*
*A * (B + C)*

A ∪ (B ∩ C)

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DeMorgan's Law

- *DeMorgan's Law* states important rule for logical equivalency
- These are used to convert ∩ to ∪ and vice-versa

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DeMorgan's Law

- So, it states you can change the operator from ∩ to ∪ or vice-versa
- If you negate both operands

$(A \cap B)' \equiv A' \cup B'$

$(A \cup B)' \equiv A' \cap B'$

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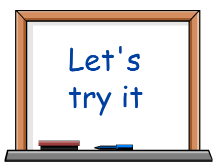
Analyzing DeMorgan's Law

- Let's draw some Venn Diagrams and analyze if DeMorgan's Law works
- First, let's look the logic of what an expression says



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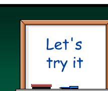
Let's Draw These…



Let's see if the following are the same:

$A \cup (B \cap C)$

$(A \cup B) \cap (A \cup C)$

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Algebra: Example 1

(A ∪ B)' ∩ B

A' ∩ B' ∩ B — After using DeMorgan's Law

A' ∩ ∅ = — After using Complement Law

∅ — After using Domination Law

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Algebra: Example 2

(A ∩ B) ∪ (A' ∩ B)

(A ∪ A') ∩ B — After using Distributive Law

U ∩ B — After using Complement Law

B — After using Identify Law

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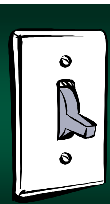
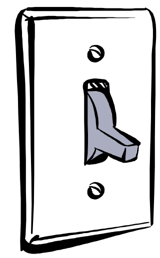


Duals

Don't "Flip" Out

Duels

- The dual of a theorem is created by:
  - flip all **U** and ∅
  - flip all ∪ and ∩
- Important property: *if an expression is an identity then its duel is an identity*



Example Duels

- The two expressions (identities) below are duels of each other
- Both expressions are true, but let's test

```
1. A = (U ∩ A) ∪ (A ∩ B)
2. A = (∅ ∪ A) ∩ (A ∪ B)
```

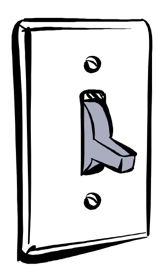

Example: Expression 1

```
A = (U ∩ A) ∪ (A ∩ B)
  = A ∩ (U ∪ B)       Distributive
  = A ∩ U             Domination
  = A                 Identity
```


Example: Expression 2

```
A = (∅ ∪ A) ∩ (A ∪ B)
  = A ∪ (∅ ∩ B)       Distributive
  = A ∪ ∅             Domination
  = A                 Identity
```


Duels



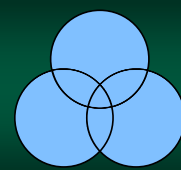
- Also note that the proof (i.e. the reduction) of the expressions was identical
- The number of steps (as well as each law) was the same

Set Attributes

Part 2

1

Fundamental Products

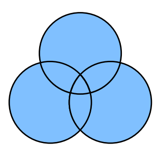


How Many Subsets Are There?

2

Fundamental Products

- *Fundamental Product* is an intersection of each set *(or it's complement)*
- They reveal all the base subsets of interest
- …since, each fundamental product is unique



3

Fundamental Products

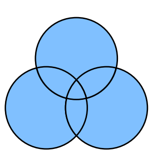
For each set $S_{1..n}$ in the universe, each product, P, is defined:

$$P = A_1 \cap A_2 \cap A_3 \cap A_3 \cap \ldots \cap A_n$$

where A_i is the set S_i or S'_i

4

Some Attributes

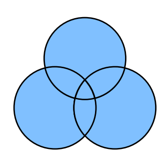


- There are few properties that can be observed from fundamental products
- These will be important in other areas of discrete mathematics

5

Three Major Attributes

1. There are $m = 2^n$ such fundamental products
2. Any two such fundamental products are disjoint
3. The universal set U is the union of all fundamental products



6

#1. Number of Products

- Number of fundamental products *m* grows exponentially in relation to the number of sets *n*
- Observe: this is beginning to look "binary"

$$m = 2^n$$

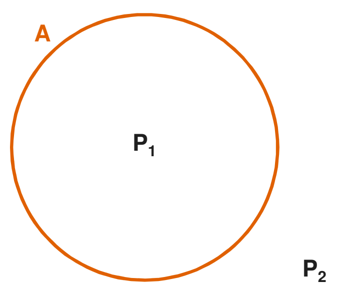
7

With 1 Set

- For a Universe with a single set, A, it results in 2^1 products
- Namely A and A'

P_1 = A
P_2 = A'

8



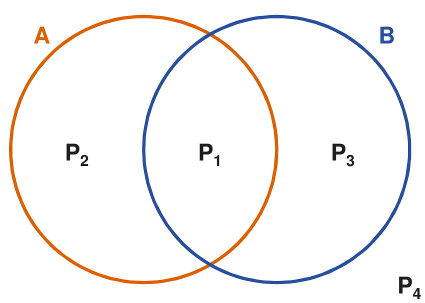
9

With 2 Sets

- With two sets, A and B, there are a total of $2^2 = 4$ products

P_1 = A ∩ B
P_2 = A ∩ B'
P_3 = A' ∩ B
P_4 = A' ∩ B'

10

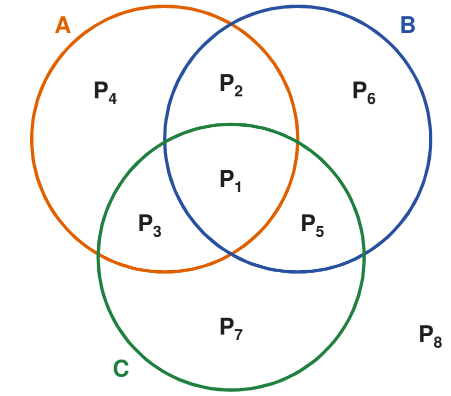


11

With 3 Sets: $2^3 = 8$

P_1 = A ∩ B ∩ C
P_2 = A ∩ B ∩ C'
P_3 = A ∩ B' ∩ C
P_4 = A ∩ B' ∩ C'
P_5 = A' ∩ B ∩ C
P_6 = A' ∩ B ∩ C'
P_7 = A' ∩ B' ∩ C
P_8 = A' ∩ B' ∩ C'

12



13

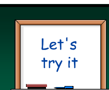
#2. All Products are Disjoint

- Any two different fundamental products are disjoint
- Which means, they have no elements in common

$$P_i \cap P_j = \varnothing \quad \textbf{when } i \neq j$$

14

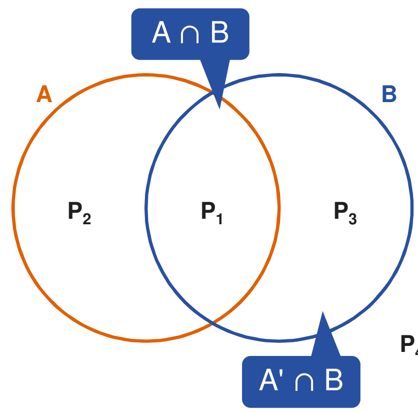
#2. All Products are Disjoint



- We can use set algebra to show *fundamental products* can be unioned into the original sets

$$P_i \cap P_j = \varnothing \quad \textbf{when } i \neq j$$

15



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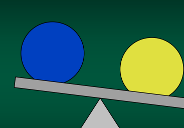
#3. Union of all products is U

- The union of all fundamental products is the universe set U
- This should be fairly obvious from what we observed

$$U = P_1 \cup P_2 \cup P_3 \cup P_3 \cup \ldots P_n$$

17



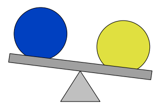
Cardinality

Counting Sets

18

Cardinality of a Set

- The *cardinality* of a set is the number of *distinct* elements
- This information is used in counting – the classification of the set's contents



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Different Notations Used

- There are two different notations used
- The most common is the | pipe delimiters
- Alternatively, the "n" function is used

$$|A| \equiv n(A)$$

20

Examples

```
A = {1, 3, 5, 7}

|A| = 4
```


21

Examples

Duplicates don't count

```
B = {1, 2, 3, 3, 3, 4}

|B| = 4
```


22

Counting

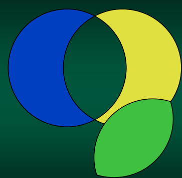
- If the set contains a finite number of elements, it is said to be *countable* – i.e. the cardinality is knowable
- If the set is infinitely large, but the elements can be uniquely identified, then it is *countably infinite*
- Otherwise it is said to be *uncountable*

23

Countable Examples

Set	Result
$\{ x \mid x \in N \text{ and } x \leq 100\}$	Countable
$\{ 2x \mid x \in N \}$	Countably Infinite
$\{ x \mid x \in R \text{ and } 0 < x < 1\}$	Uncountable

24



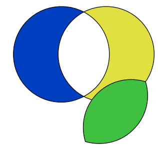
Inclusion-Exclusion

Counting by Subtracting!

25

Inclusion-Exclusion

- Sets can overlap – and can contain the same elements
- So, when counting items in sets, you must be careful not to count an item twice
- *Inclusion-exclusion* principle, can get the correct count



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Reasoning with Venn Diagrams

Say 100 people receive a questionnaire with two questions:

1. Do you watch The Orville
2. Do you watch Rick & Morty?

60 said 'yes' to The Orville.
35 said 'yes' to **both**.
How many people watch Rick & Morty?

27

Disjoint Set Cardinality

- If sets A and B are disjoint then they have no elements in common
- Cardinality of the union is the sum of the cardinality of both A and B

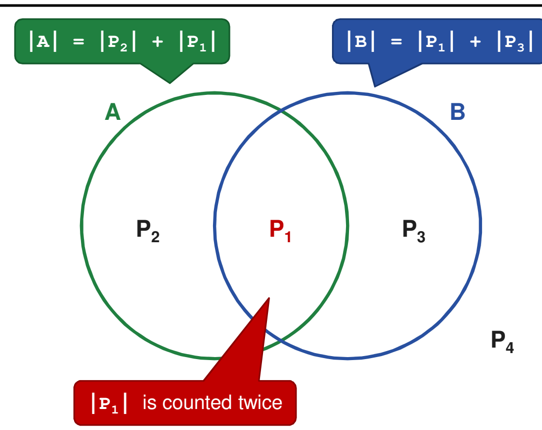
$$|A \cup B| = |A| + |B|$$

28

Set Exclusion

- If sets A and B overlap they have elements in common
- We cannot simply add $|A| + |B|$
- Why? $|A| + |B|$ counts the intersection twice!

29



30

Set Exclusion

- So, we need to remove the duplicate count
- The cardinality of the union is the sum of A and B excluding the intersection

$$|A \cup B| = |A| + |B| - |A \cap B|$$

31

Set Exclusion

- Note: this is the same equation for disjoint sets
- If disjoint, the intersection is ∅
- So, this formula works in all cases

$$|A \cup B| = |A| + |B| - |A \cap B|$$

32

Revisit that Question

Say 100 people receive a questionnaire with two questions:

1. Do you watch The Orville
2. Do you watch Rick & Morty?

60 said 'yes' to The Orville.
35 said 'yes' to both.
How many people watch Rick & Morty?

33

Using the Formula…

- Union of The Orville (T) and Rick & Morty (R) contains 100
- The Orville set contains 60
- The intersection contains 35

$$|T \cup R| = |T| + |R| - |T \cap R|$$

34

Using the Formula…

- Union of The Orville (T) and Rick & Morty (R) contains 100
- The Orville set contains 60
- The intersection contains 35

$$100 = 60 + |R| - 35$$

35

Using the Formula…

- Union of The Orville (T) and Rick & Morty (R) contains 100
- The Orville set contains 60
- The intersection contains 35

$$\begin{aligned} |R| &= 100 - 60 + 35 \\ &= 75 \end{aligned}$$

36

Power Series

All the combinations

37

Power Series

- A *power set* of a set S is a set of all the subsets of S
- This also, obviously, contains the null set
- The notation for the power set S is *P*(S)

38

Power Set Example

```
G = {a, b}

P(G) = { ø, {a}, {b}, {a,b} }
```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Power Set Example 2

```
H = {a, b, c}

P(H) = { ø, {a}, {b}, {c},
         {a,b}, {a,c}, {b,c}
         {a,b,c} }
```


40

Power Set Example 3

```
I = {a, b, c, d}

P(I) = {  ø,
         {a}, {b}, {c}, {d}
         {a,b}, {a,c}, {a,d},
         {b,c}, {b,d}, {c,d},
         {a,b,c}, {a,b,d}, {b,c,d}, {a,c,d}
         {a,b,c,d} }
```


41

Cardinality of Power Sets

- What is the cardinality of power sets?
- Is it possible to determine | P(S) | if we know | S |
- This will be important later…

42

Let's Look at the Examples

```
G = {a, b}

P(G) = { ø, {a}, {b}, {a,b} }

|G|    = 2
|P(G)| = 4
```


43

Power Set Example 2

```
H = {a, b, c}

P(H) = {  ø, {a}, {b}, {c},
          {a,b}, {a,c}, {b,c}
          {a,b,c} }

|H|    = 3
|P(H)| = 8
```


44

Power Set Example 3

```
I = {a, b, c, d}

P(I) = {  ø, {a}, {b}, {c}, {d}
          {a,b}, {a,c}, {a,d},
          {b,c}, {b,d}, {c,d},
          {a,b,c}, {a,b,d}, {b,c,d}, {a,c,d}
          {a,b,c,d} }

|I|    = 4
|P(I)| = 16
```


45

Cardinality of Power Set

- The cardinality of a power set is 2^n where n is the cardinality of the original set
- This is used in statistics… covered later

$$|P(S)| = 2^{|S|}$$

46

Partitions

Cutting a Set Into Pieces

47

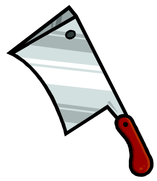
Partitions

- A *partition* of a set A is a collection of non-empty disjoint sets whose union is A
- So, it is like the set A was "chopped", cleanly, into subsets

48

Requirements

- Each subset must be mutually exclusive
- … unless they are identical *(because duplicates don't count in sets)*



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Partition Example

- The following is a valid partition of the set {1, 2, 3, … 9}

```
{ {1}, {2,3,5,7}, {4,6}, {8,9} }
```


50

Partition Example

- The following is a partition of N.

```
N = { {1}, {2,3}, {4,5,6}, ... }
```


51

Partition Examples

For the set {1, 2, 3, 4}…

Set	Partition?
`{ {1}, {2}, {3}, {4} }`	**Yes**
`{ {1,2}, {1,2}, {3,4} }`	**Yes.** {1,2} is duplicate
`{ {1,2,3}, {2,4} }`	**No.**

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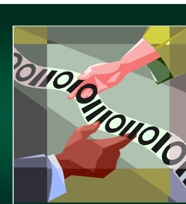
Set Theory in Computer Science

Part 3

1

Binary Numbers



Bit of This and a Bit of That

2

What is a Number?

- We use the Hindu-Arabic Number System
  - positional grouping system
  - each position is a power of 10
- Binary numbers
  - based on the same system
  - powers of **2** rather than 10
  - each digit is in the set { 0, 1 }

3

Base 10 Number

The number 1783 is ...

10^4	10^3	10^2	10^1	10^0
10000	1000	100	10	1
0	**1**	**7**	**8**	**3**

1000 + 700 + 80 + 3 = 1783

4

Binary Number Example

The number 0100 1010 is ...

2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
128	64	32	16	8	4	2	1
0	**1**	**0**	**0**	**1**	**0**	**1**	**0**

64 + 8 + 2 = 74

5

Binary Number Example

The number 1101 1011 is ...

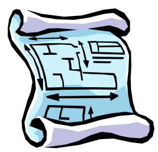
2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
128	64	32	16	8	4	2	1
1	**1**	**0**	**1**	**1**	**0**	**1**	**1**

128 + 64 + 16 + 8 + 2 + 1 = 219

6

Numbers are Tuples

- In Hindu-Arabic system, the order of the symbols is important – so they are tuples
- e.g. 123 ≠ 321
- Other number styles use sets – i.e. the ancient Egyptian system



7

Looking at Numbers

- Numbers are tuples 1947 ≠ 1974
- Members of the decimals number are also members of the set {0, 1, 2, … 9}

```
1947 → (1,9,4,7)
```


8

Looking at Binary Numbers

- Binary numbers are tuples 10010100 ≠ 11100000
- Members of the binary number are also members of the set {0, 1}

```
10100111 → (1,0,1,0,0,1,1,1)
```


9

Looking at Binary Numbers

- So, for a binary number B, all $x \in B$ holds the following: $x \in \{0, 1\}$

```
10100111 → (1,0,1,0,0,1,1,1)
```


10

So….

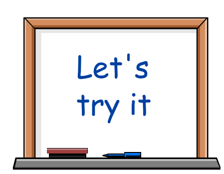
```
{1776, 1846, 1947} →

{ (1,7,7,6), (1,8,4,6)
  (1,9,4,7) }
```


11

Let's Make a Set-Based System

- We are mostly used to tuple-based number systems
- But, for most of history, people used sets
- Let's create one



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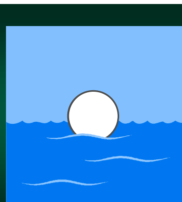


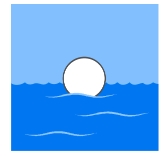
Floating Point Numbers

Real numbers are *real* complex

13

Floating Point Numbers

- Often, programs need to perform mathematics on *real* numbers
- *Floating point numbers* are used to represent quantities that cannot be represented by integers



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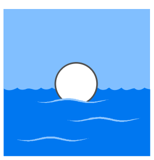
Why use them?



- Regular binary numbers can only store whole positive and negative values
- Many numbers outside the range representable within the system's bit width (too large/small)

15

IEEE 754

- Practically modern computers use the *IEEE 754 Standard* to store floating-point numbers
- Represent by a mantissa and an exponent
  - similar to scientific notation
  - the value of a number is: $mantissa \times 2^{exponent}$
  - uses signed magnitude

16

IEEE 754

- Comes in three forms:
  - single-precision: 32-bit
  - double-precision: 64-bit
  - quad-precision: 128-bit
- Also supports special values:
  - negative and positive *infinity*
  - and "not a number" for errors (e.g. 1/0)

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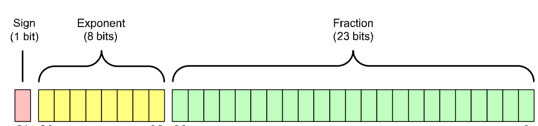
IEEE 754 Single Precision (32 bit)



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Fractional Field

- The fraction field number that represents part of the mantissa
- If a number is in proper scientific notation…
  - it always has a single digit before the decimal place
  - for decimal numbers, this is 1..9 (never zero)
  - for base-2 numbers, it is always 1

19

Fractional Field

- So, do we need to store the leading 1? It will always be a 1
- The faction field, therefore…
  - only represents the fractional portion of a binary number
  - the integer portion is assumed to be 1
  - this increases the number of significant digits that can be represented (by not wasting a bit)

20

Exponent Field

- The exponent field supports negative and positive values but does not use sign-magnitude or 2's complement
- Uses a "biased" integer representation
  - fixed value is added to the exponent before storing it
  - when interpreting the stored data, this fixed value is then subtracted

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Exponent Field

- Bias is different depending on precision
  - single precision: 127
  - double precision: 1023
  - quad precision: 16383
- For example, for single precision…
  - exponent of 12 stored as (+12 + 127) → 139
  - exponent of -56 stored as (-56 + 127) → 71

22

Interpretation: Normal Case

- Exponent Field: not all 0's or all 1's
- Fraction Field: Any

$$\pm\ (1.\mathit{fraction}) \times 2^{(\text{exponent} - \text{bias})}$$

23

Interpretation: Zero

- Exponent Field: all 0's
- Fraction Field: all 0's

0

24

Interpretation: Tiny Numbers

- Exponent Field: all 0's
- Fraction Field: Any

$$\pm\ (0.\mathit{fraction}) \times 2^{(1 - \text{bias})}$$

25

Interpretation: Infinity

- Exponent Field: All 1's
- Fraction Field: 0

± ***infinity***

26

Interpretation: Invalid Numbers

- Exponent Field: All 1's
- Fraction Field: Not 0

Not a number (NaN)

27

Interpretation: Invalid Numbers

NaN → **1 / 0**

Naan →

28

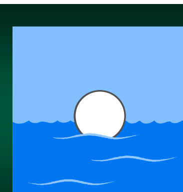
More Single-Precision Examples

```
Zero:
0 00000000 00000000000000000000000

Positive Infinity:
0 11111111 00000000000000000000000

Negative Infinity:
1 11111111 00000000000000000000000
```


29



Let's Encode Some Numbers!

This is actually fun!

30

Something Else About Numbers…

The number 36.74 is ...

10^2	10^1	10^0	10^{-1}	10^{-2}
100	10	1	1/10	1/100
0	3	6	7	4

= (3×10) + (6×1) + 7/10 + 4/100

31

Binary Fractions!

The number **101.011** is ...

2^4	2^3	2^2	2^1	2^0	2^{-1}	2^{-2}	2^{-3}
16	8	4	2	1	1/2	1/4	1/8
0	0	1	0	1	0	1	1

= 4 + 1 + 1/4 + 1/8 = 5.375

32

Let's Encode 14.25 (32 bit)

- First, we need to convert 14.25 to its binary equivalent
- So, we need to calculate the integer part and fraction part of the number
- Everything after the decimal is a fraction
  - 0.25 is actually 25 / 100
  - we need to find the base 2 equivalent (1/4)

33

Step 1: Convert to binary

```
14 → 1110

0.25 → 1/4 → 0.01

Hence:
14.25 → 1110.01
```

binary 01 / 100

34

Step 2: Scientific Notation

- IEEE stores the data in scientific notation
- So we move the "binary point" over

1110.01 → 1.11001 × 2^3

35

Step 2: Scientific Notation

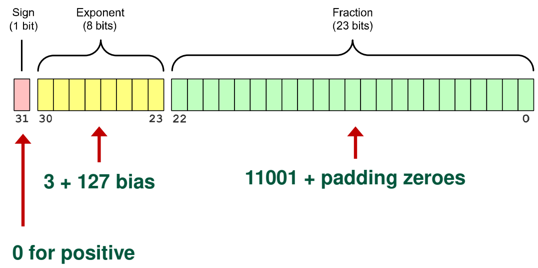
- In binary scientific notation, the leading digit is always going to be 1
- Why store it? IEEE doesn't.
- Only data after the point is encoded

Fraction

1.11001 × 2^3 → (1 + .11001) × 2^3

36

Step 3: Encode



Result: 14.25 (32 bit)

- The following is the encoded version of 14.25
- The rules are similar for double-precision

```
0 10000010 11001000000000000000000
```


Result: 14.25 (32 bit)

- We can also convert this into bytes
- Note: the floating point fields don't really "fit" cleanly into actual bytes

```
01000001 01100100 00000000 00000000
   41       64       00       00
```


Example 2: Encode 13.75 (32 bit)

- First, we need to convert 13.75 to its binary equivalent
- So, we need to calculate the integer part and fraction part of the number
- Everything after the decimal is a fraction
  - 0.75 is actually 75 / 100
  - we need to find the base 2 equivalent (3/4)

Step 1: Convert to binary

```
13 → 1101

0.75 → 3/4 → 0.11

Hence:
13.75 → 1101.11
```

binary 11 / 100

Step 2: Scientific Notation

- IEEE stores the data in scientific notation
- So we move the "binary point" over

$$1101.11 \rightarrow 1.10111 \times 2^3$$

Step 2: Scientific Notation

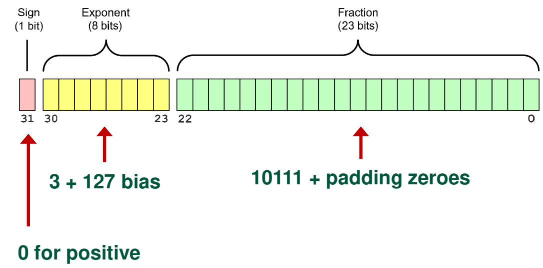
- In binary scientific notation, the leading digit is always going to be 1
- Why store it? IEEE doesn't.
- Only data after the point is encoded

Fraction

```
1.10111 × 2³ → (1 + .10111) × 2³
```


43

Step 3: Encode



44

Result: 13.75 (32 bit)

- The following is the encoded version of 13.75
- The rules are similar for double-precision

```
0 10000010 10111000000000000000000
```


45

Result: 13.75 (32 bit)

- We can also convert this into bytes
- Note: the floating point fields don't really "fit" cleanly into actual bytes

```
01000001 01011100 00000000 00000000
   41       5C       00       00
```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Tuples and Floats

Its all set theory, folks!

47

Floats Are Tuples

- Just like regular binary numbers, floating-point numbers of tuples
- They consist of three fields making them 3-tuples

```
(sign, exponent, fraction)
```


48

Encoding of 14.25

- Sign is a 1-tuple
- Exponent is a 8-tuple
- Fraction is a 23-tuple

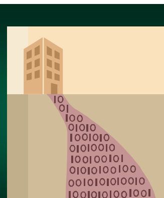
```
0 10000010 11001000000000000000000
```


49

Set Notation of 14.25

```
(  (0),
   (1,0,0,0,0,0,1,0),
   (1,1,0,0,1,0,0,0,
    0,0,0,0,0,0,0,0,
    0,0,0,0,0,0,0)   )
```


50



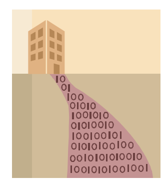
Bit Vectors

Sets and Bits

51

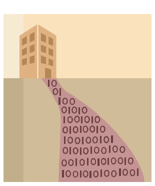
Bit Vectors

- A *bit vector* is a way to store countable sets using bits
- Also known as a *bit array*, *bit set*, and *bit map*
- Compact format that can perform a set operations with a single operation *(fast!)*



52

Bit Vectors



- Each object in the universe is given a single bit in the bit array
- If the $x \in A$, then the bit is 1, otherwise 0
- Order is important, so this is a tuple approach

53

Example 1

- U = { fry, bender, farnsworth, leela, zoidberg}
- A = { fry, bender, leela }

```
U = 11111
A = 11010
```

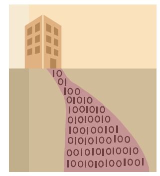

54

Example 2

- U = { 2, 3, 5, 7, 11, 13, 17, 19}
- A = { 3, 5, 11, 19 }

```
U = 11111111
A = 01101001
```


Why this is useful



- Computers can easily perform and & or operations on bytes (or multiple bytes)
- This means set operations can be performed amazingly fast

Let's look at the definitions again…

- The definitions of union and intersection are nearly identical
- The relationship between the elements is defined using an *and* or *or*

$A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$
$A \cap B = \{ x \mid x \in A \text{ and } x \in B \}$

Let's look at the definitions again…

- We can apply a *bit-wise-and* & a *bit-wise-or* to our bit array
- It will apply the operation to each of the bits in matching columns

$A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$
$A \cap B = \{ x \mid x \in A \text{ and } x \in B \}$

Let's look at the definitions again…

- So, each bit in A will be compared to its matching bit in B
- Bit match can do sets!

$A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$
$A \cap B = \{ x \mid x \in A \text{ and } x \in B \}$

Example: Union (using or)

```
U = {a,b,c,d,e,f,g}
A = {b,c,d} = 0111000
B = {d,e,f} = 0001110

    0111000
 or 0001110
    0111110 = {b,c,d,e,f}
```

Example: Intersection (using and)

```
U = {a,b,c,d,e,f,g}
A = {b,c,d} = 0111000
B = {d,e,f} = 0001110

    0111000
and 0001110
    0001000 = {d}
```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Complement

- How do we do a complement of a set A?
- We must flip all the bits from 1 to 0, and 0 to 1
- We can use a *binary-not* or the *XOR* operation

$$A' = \{ x \mid x \notin A \}$$

62

Java/C Code

&& and || are Boolean.
These are bit-wise.

```
Intersection :  a & b
Union        :  a | b
Complement   :  ~a
```

The tilde ~ is a bitwise not.

63

Exclusion

- Finally, how do we do set difference?
- The "subtract" operator will not work
- Let's look at the definition a bit more closely

$$A \setminus B = \{ x \mid x \in A \text{ and } x \notin B \}$$

64

Exclusion

- It's essentially the definition of *intersection*
- Except, the second operand is the definition of *complement*.

$$A \setminus B = \{ x \mid x \in A \text{ and } x \notin B \}$$

65

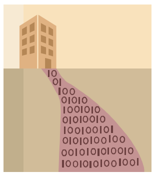
Java/C Code

Just complement the second operand

```
Exclusion :  a & ~b
```


66

Limits of Bit Vectors



- Bit vectors, while useful, do have some notable limitations
- They only work on finite, countable sets
- For all other cases, you will have to work use a more advanced ADT

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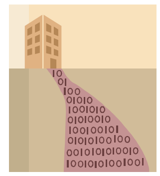
CSC 130 is waiting for you!



- For most cases, a very sophisticated list or tree can be used
- You will need to know:
  - lists / trees
  - sorting
  - binary-searches
  - Big-O

68

Relations

Part 4

1

Cross Products

Databases use this…

2

Cross Products

- Sets can be multiplied, which will result in a set of tuples
- Well, a set of ordered pairs, to be more specific
- Cross products are important in databases and counting *(to name a few)*

3

Products

- A *cross product* is a set of ordered pairs
- Note: Unlike multiplication, the order of the operands is important

$$A \times B = \{ (x, y) \mid x \in A \text{ and } y \in B \}$$

4

Example Product

```
Given:
  A = {small, big}
  B = {cat, dog}

A × B = { (small, cat), (small, dog),
          (big, cat), (big, dog) }
```


5

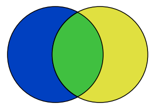
Binary Relations

How Stuff Compares to Stuff

6

Relations

- A *binary relation* is a stated fact between on two objects
- A "fact" is called a *predicate*
- Evaluates to true or false
- These are the foundation of most programming tasks



7

Example Relations

- "x is bigger than y"
- "x lives less than 50 miles from y"
- "x ≤ y"
- "x and y are siblings"
- "x has a y"

8

Relations

- A *binary relation* from A to B is a subset of the cross-product $A \times B$
- A relation from A to B is a set of ordered pairs (a, b) where $a \in A$ and $b \in B$
- We can use the shorthand notation of: a R b to denote that $(a, b) \in R$

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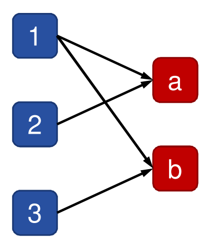
Relationship Chart



	a	b
1	✓	✓
2	✓	
3		✓

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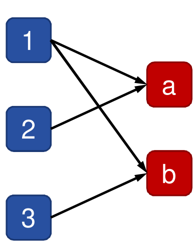
Relationship Chart




```
{ (1, a), (1, b),
  (2, a), (3, b) }
```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Example: Capitols

- A is a set of all cities in the World
- B is a set of all states in the World
- The relation *a* R *b* specifies that *a* is the capitol of *b*

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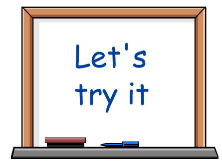
Example: Capitol Members

- (London, Britain)
- (Sacramento, California)
- (Madrid, Spain)
- (Tokyo, Japan)
- (New Delhi, India)
- (Albany, New York)

13

Let's Draw One!

- Let's draw a relationship graph
- It will be between students and shows/movies they enjoy



14

Relation Domain

- The *domain* of a relation is a set of all the first elements of each tuple
- So, it is the elements that the make up the left-hand side of *a* R *b*

$\{ a \mid (a, b) \in R \text{ for some } b \}$

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Relation Range

- The *range* of a relation is a set of all second elements from each tuple
- So, it is the elements that the make up the right-hand side of *a* R *b*

$\{ b \mid (a, b) \in R \text{ for some } a \}$

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Example

```
R = { (1,1), (1,2), (1,3),
      (1,4), (2,4), (4,4) }

Domain of R = { 1, 2, 4 }

Range of R = { 1, 2, 3, 4 }
```


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Inverse Relation

- The *inverse* of a relation swaps first and last element for each tuple
- The number of elements are the same, but the range and domain are reversed

$R^{-1} = \{ (b, a) \mid (a, b) \in R \}$

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Inverse Example

```
R = { (1,1), (1,2), (1,3),
      (1,4), (2,4), (4,4) }

R-1 = { (1,1), (2,1), (3,1),
        (4,1), (4,2), (4,4) }
```


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Relations can be infinitely large

- On a finite set, relations are quite simple…
- For a set with n elements, the maximum number of relations is simply $n \times n = n^2$
- However, many relations are defined over an infinite set – e.g. all integers, reals, etc…

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Representing Relations

- We can represent a relation using set notation
- However, some are required to be denoted using set builder notation

```
R1 = { (a, b) | a is bigger than b }
R2 = { (a, b) | a ≤ b }
```


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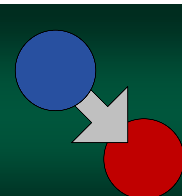

Types of Relations

How a Set Sees Itself

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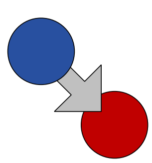
"Relation On"

- Some relations of a set A are upon itself
- In other words, each object in the related to the same "type" of object
- This is called a *relation on* A
- …and it is a important to examine its properties



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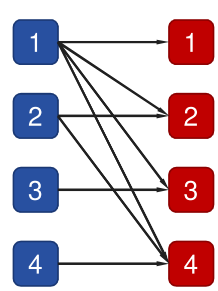
Example Relationship Chart



	1	2	3	4
1	✓	✓	✓	✓
2		✓		✓
3			✓	
4				✓

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Example Relation Chart

- The previous chart represents when *a* divides *b*
- In other words, *a* times some integer equals the value *b*
- So, R = { (1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4) }

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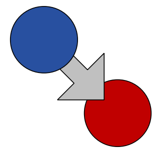
Reflexive Relations

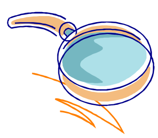


- A *reflexive* relationship means that there is *a*R*a* for every *a*
- Basically, everything has to be related to itself
- *Every element, in the domain, must be related to itself*

26

To Determine Reflexive…

- Look for some $a \in A$ where there isn't a *a*R*a*
- If found, not reflexive
- Otherwise reflexive



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Reflexive Example

```
Relation on set {1, 2, 3, 4}

R = { (1,1), (1,4), (2,2),
      (2,3), (3,3), (4,4) }
```


28

Reflexive Example

```
Relation on set {1, 2, 3, 4}

R = { (1,1), (1,4), (2,2),
      (2,3), (3,3), (4,4) }
```


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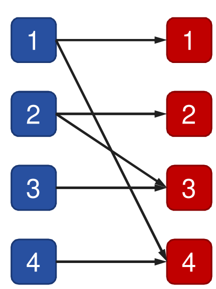
Reflexive Example



	1	2	3	4
1	✓			✓
2		✓	✓	
3			✓	
4				✓

30

Nonreflexive Example

```
Relation on set {1, 2, 3, 4}

R = { (1,1), (1,4), (2,2),
      (2,3), (3,1), (4,4) }
```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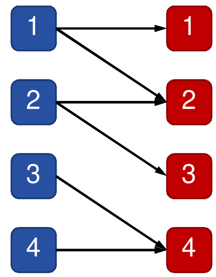
Nonreflexive Example



	1	2	3	4
1	✓	✓		
2		✓	✓	
3				✓
4				✓

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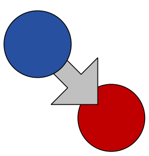
Symmetric Relations



- A *symmetric* relationship means that for every *a*R*b* there is a *b*R*a*
- So, if (a, b) exists in the relation, so must (b, a)

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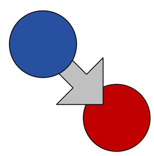
Symmetric Relations

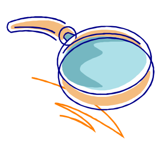


- Note: Unlike the definition for reflexive, not every element in the domain needs to exist
- If a relation is not symmetric, it is *antisymmetric*

34

To Determine Symmetric…

- Look for an *a* and *b* where there is a *a*R*b* but no *b*R*a*
- If found, nonsymmetric
- Otherwise symmetric



35

Symmetric Example

```
Relation on set {1, 2, 3, 4}

R = { (1,1), (1,2), (2,1)
      (2,4), (3,3), (4,2) }
```


36

Symmetric Example

```
Relation on set {1, 2, 3, 4}

R = { (1,1), (1,2), (2,1)
      (2,4), (3,3), (4,2) }
```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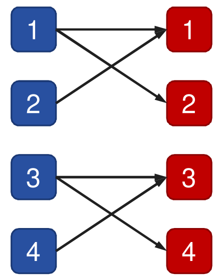
Symmetric Example



	1	2	3	4
1	✓	✓		
2	✓			✓
3			✓	
4		✓		

38

Nonsymmetric Example

```
Relation on set {1, 2, 3, 4}

R = { (1,1), (1,4), (2,2),
      (2,3), (3,3), (4,4) }
```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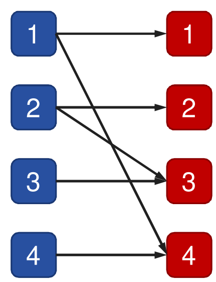
Nonsymmetric Example



	1	2	3	4
1	✓			✓
2		✓	✓	
3			✓	
4				✓

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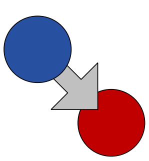
Transitive Relations

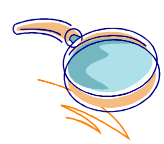


- A *transitive* relationship means that for every *a*R*b* and *b*R*c* that also *a*R*c*
- So, if (a, b) and (b, c) exists in the relation, so must (a, c)

41

To Determine Transitive…

- Look for an *a*, *b*, *c* where there is a *a*R*b* and *b*R*c* but no *a*R*c*
- If found, non transitive
- Otherwise transitive



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Transitive Example

```
Relation on set {1, 2, 3, 4}

R = { (2,1), (3,1), (3,2),
      (4,1), (4,2), (4,3) }
```


43

Transitive Example

```
Relation on set {1, 2, 3, 4}

R = { (2,1), (3,1), (3,2),
      (4,1), (4,2), (4,3) }
```

Starting with (4,3)

44

Transitive Example

```
Relation on set {1, 2, 3, 4}

R = { (2,1), (3,1), (3,2),
      (4,1), (4,2), (4,3) }
```

Starting with (3,2)

45

Transitive Example

```
Relation on set {1, 2, 3, 4}

R = { (2,1), (3,1), (3,2),
      (4,1), (4,2), (4,3) }
```

Starting (again) with (4,3)

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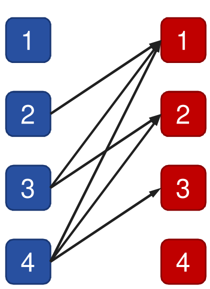
Transitive Example



	1	2	3	4
1				
2	✓			
3	✓	✓		
4	✓	✓	✓	

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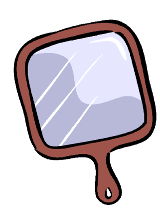
Equivalence Relations

- If a relation has all three properties:
  - reflexive
  - symmetric
  - transitive
- Then, and only then, it is an *equivalence*



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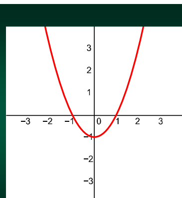
Example Table

Relation	Reflexive	Symmetric	Transitive	Equivalent
≤				
⊂				
Perpendicular lines				
Parallel lines				

Example Table

Relation	Reflexive	Symmetric	Transitive	Equivalent
≤	✓	✗	✓	✗
⊂	✗	✗	✓	✗
Perpendicular lines	✗	✓	✗	✗
Parallel lines	✓	✓	✓	✓

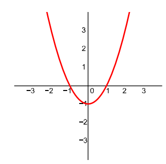


Functions

Math Friendly Relations

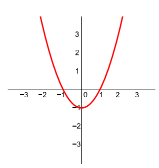
Functions

- We have all seen functions – which take inputs and produce output
- Example: $f(x) = x^2$
  - $f(1) = 1$
  - $f(2) = 4$
  - $f(3) = 9$ ...

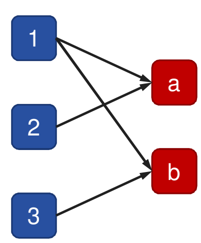


Functions

- Sets give a way to document "types" in mathematical functions
- A *function* from set X to set Y is a mapping from <u>each</u> element in set X to elements in set Y



Relationship Review




```
{ (1, a), (1, b),
  (2, a), (3, b) }
```

Function Attributes

- Function Rules:
  - must be defined for *every element in domain*
  - each value in domain *maps to one element*
- Notice that a function defines a set of ordered pairs: e.g. (1,1) (2,4) (3,9) ...
- We can therefore think of a function as a special kind of relation.

Relations vs. Functions

- Each domain element, in a relation, can specify *many* relationships
- While, each element in a function domain only specifies *one* relationship
- So….
  - every function is a relation
  - but not every relation is a function

Definition of a Function

Let f be a relation from A → B

f is a function if and only if:

each a ∈ A appears exactly once in an ordered pair (a, b) ∈ f for some b

Function Signature

- We will restrict a functions inputs and outputs by giving a "signature" for it
- f is the function name

```
f: N → N
```


Function Signature

- The first N is the function domain
- The second N is the function range (codomain)

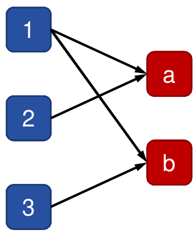
```
f: N → N
```


Domain and Range Definitions

- The domain and range of f is defined exactly as we saw for relations
- Which is not surprising given what a function really is

domain(f) = { x | (x, y) ∈ f for some y }
range(f) = { y | (x, y) ∈ f for some x }

Let's Look At This Again….




```
{ (1, a), (1, b),
  (2, a), (3, b) }
```


61

Relations vs. Functions

- Not that in the example (with 1,2,3 and a, b) that some elements in A had multiple values in B
- In a function, each member in A maps to exactly one value in B
- So, that relation was not a function!

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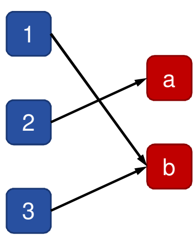
Function Example




```
{ (1, b), (2, a),
  (3, b) }
```


63

Examples

- For the following examples, let each example be defined as a relation from A to B
- Domain and range (codomain) are defined as:

```
A = {1, 2, 3}
B = {x, y, z}
```


64

Is This a Function?

```
Let f = { (1,x), (2,y) }

No, the domain value 3 is missing
as a first ordered-pair element
```


65

Is This a Function?

```
Let g = { (1,x), (2,y),
          (3,z), (1,y) }

No, the domain element 1 is listed
twice.
```


66

Is This a Function?

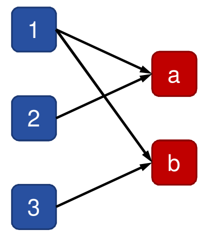
```
Let h = { (1,x), (2,y), (3,x) }

Yes, each domain element of A is
the first element once.

(There is no restriction on
 the second element)
```


67

Is This a Function?

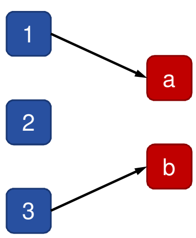


No!

(1,a) and (1,b) both exist

68

Is This a Function?

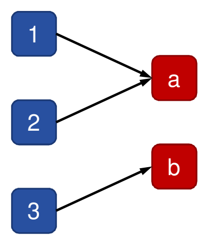


No!

The domain element 2 is not mapped to a or b

69

Is This a Function?



Yes!

70

Function Definitions

The Mapping of Sets

71

Function Definitions

- Functions are usually defined using a formula
- You should be able to tell that these match a Java method definition – header and body

```
f: Z → Z
f(x) = x * x
```


72

Function Definitions

- First part tells us that f maps every integer to an integer
- Second part tells us f(x) and x^2 are the same thing

```
f: Z → Z
f(x) = x * x
```


73

Example

- In the following, is *g* a valid function?
- R is a set of reals
- √ is the square root function

```
g: R → R
g(x) = √x
```


74

Example

- **No.**
- Not every element of R maps to something in R
- For example, g(-1) $\notin$ R

```
g: R → R
g(x) = √x
```


75

Manipulating Relations

How a Set Sees Itself

76

Manipulating Relations

- Because relations are representable as sets, we can use set notation to define them
- We can also use set notation to manipulate them

77

Example

```
A = { (1,1), (2,2), (3,3) }
B = { (1,1), (2,4), (3,9) }
```


78

Example

```
A ∪ B =   { (1,1), (2,2), (2,4),
            (3,3), (3,9) }

A ∩ B =   { (1,1) }

A \ B =   { (2,2), (3,3) }

B \ A =   { (2,4), (3,9) }
```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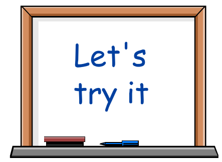
Let's Examine Some…

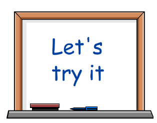
- Let's use students to create two relations
- Classes you plan to take
- Classes you might/did enjoy



80

Let's Examine Some…

- Let's examine two relations over a simple set {1, 2, 3} using set operators
- We can check if it is:
  - reflexive
  - symmetric
  - transitive



81

Let's Examine Some…

```
A = {1,2,3}: R, S relations.

R = { (1,1), (1,2), (2,2),
      (2,3), (3,1), (3,3)}

S = { (1,1), (1,2), (1,3),
      (2,1), (2,3), (3,2) }
```


82

Closures

Making a relation "complete"

83

Closure

- *Closure* of relation R is the smallest set (when unioned) gives R the desired property
- So, the closure of R is R ∪ C, where C is the smallest set giving R ∪ C the desired property

84

Some Examples

- For the following examples, the relation is over the set {1, 2, 3, 4}
- The slides will show how to make the closures for reflexive, symmetric, and (the hard one) transitive

85

Example Reflexive Closure

```
R = { (1,2), (2,3), (3,4) }

C = { (1,1), (2,2), (3,3), (4,4) }
```

Missing (1,1) (2,2), (3,3) and (4,4)

86

Example Reflexive Closure

```
R ∪ C = { (1,2), (2,3), (3,4),
          (1,1), (2,2), (3,3),
          (4,4) }
```


87

Example Symmetric Closure

```
R = { (1,2), (2,3), (3,4) }

C = { (2,1), (3,2), (4,3) }
```


88

Example Symmetric Closure

```
R ∪ C = { (1,2), (2,3), (3,4),
          (2,1), (3,2), (4,3) }
```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Example Transitive Closure (1 of 3)

```
R = { (1,2), (2,3), (3,4) }

C = { (1,3)                    }
```

Added due to (1,2) and (2,3)

90

Example Transitive Closure (2 of 3)

```
R = { (1,2), (2,3), (3,4) }

C = { (1,3), (2,4)        }
```

Added due to (2,3) and (3,4)

Example Transitive Closure (3 of 3)

```
R = { (1,2), (2,3), (3,4) }

C = { (1,3), (2,4), (1,4) }
```

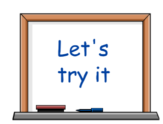
Had to add after we added (2,4) since R contains (1,2)

Example Transitive Closure

```
R ∪ C = { (1,2), (2,3), (3,4),
          (1,3), (2,4), (1,4) }
```


Let's Try Some

- Set is over {1, 2, 3, 4, 5}
- R = { (1,2), (2,3), (3,5), (4,1) }
- What are the closures for this relation?



Composition

The Inception of Functions

Composition

- *Composition* of two functions means the output of one function is used as the input as another
- This is very common in programming – you use the result of one expression as input to another

Notation

- Notation for composition is straight forward – it simply consists of a empty circle operator
- Sometimes the (x) is put in front of the first function, but this is not always the case

```
f ∘ g(x)   ≡   f(g(x))
```


97

Composition Example

```
f(x)  = x + 4
g(x)  = x²

f ∘ g(z)  = f(g(z))
          = f(z²)
          = z² + 4
```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Composition Example 2

```
f(x)  = x + 4
g(x)  = x²

g ∘ f(z)  = g(f(z))
          = g(z + 4)
          = z² + 8z + 16
```


99

Composite Example

```
R = { (1,2), (3,1), (5,3) }
S = { (2,3), (2,6), (3,9) }

R ∘ S = { (1,3), (1,6), (5,9) }
```


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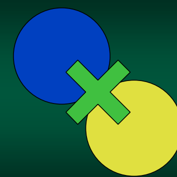
Relations in Computer Science

Part 5

1



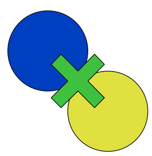
Cross Products & Databases

SQL <u>is</u> set notation

2

Cross Products & Databases

- We are in the "Information Age" where knowledge is now computerized
- Information is stored in databases
- These systems are based on tuples and sets



3

Fields

- *Fields* contain the smallest unit of data
  - e.g. Number, Text
  - So, each can be seen as a tuple (it can be a set, but rarely so)
- Each field has a unique field name
  - Name
  - Age
  - etc....

4

Records

- A *record* is a set of data fields
  - represents a logical group of data
  - these include related numbers, text, images, etc...
- Examples
  - Course: department, number, section
  - Student: name, age, class
  - Computer: brand, speed, cost, etc...

5

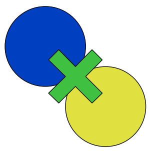
Database Example

First	Last	Major	Greek
Peter	Griffen	Und	Tappa Kegga Bru
Joe	Gunchy	CSc	Cuppa Kappa Chino
Rick	Sanchez	Sci	elta Phart
Eric	Cartman	Bus	Eta Lotta Pi

Record

6

Relationships & Cardinality



- *Relational Databases* allow the user to query multiple related tables
- Related tables are *joined* which performs a cross product on two tables
- Restrictions are used to eliminate unneeded records

7

Locating Specific Data

- A *query language* is used to:
  - locate information
  - sort records
  - change data in records
- Examples:
  - SQL (Structured Query Language)
  - Natural language queries – not used often

8

SQL Inner Join

```
SELECT Student.name,
       Course.grade
FROM Student
INNER JOIN Course
ON Student.sid = Course.sid
WHERE Course.department = "csc"
```


9

SQL Inner Join - Sets (simplified)

$$\{\ (s_{name},\ c_{grade})\ |\ s \in \text{Student and } c \in \text{Course and } s_{sid} = c_{sid} \text{ and } c_{department} = \text{"csc"}\ \}$$

10

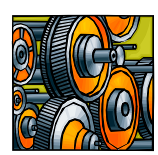
Abstract Data Types

What *int* really means

11

Application of Sets

- An *abstract data type (ADT)* defines:
  - a set of possible values and
  - operations (functions) that an be performed on those values
- These are the basis for all classes and data structures in programming languages



12

Integer Example

- In the code below, *int* is an ADT (found in most programming languages)
- It declares a variable *n* of type *int*
- n represents a value from int's set of values

```
int n;
```

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Domain of 'int'

- int is defined (normally) as 32-bit
- So, the set is { -2^{31}, …, (2^{31} – 1) }
- Also note: int $\subset$ Z
  - this means that it cannot store any integer
  - … just within a small range subset of Z

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What Java's notation means

- When you declare a variable, you are stating that it is a member of a set
- In the example below, the two statements mean same thing: set notation vs. Java notation

```
n ∈ int
int n;
```

Set notation

Java notation

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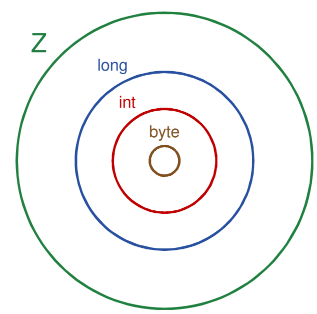
Java Integer Sets



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Possible Set Violation

```
void test(int x)
{
    ...
}

int main()
{
    long n;
    test(n);
}
```

long is a superset of int

int may not able be able to store n

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No Set Violation

```
void test(long x)
{
    ...
}

int main()
{
    int n;
    test(n);
}
```

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Operations

- ADT also defines that n can be manipulated by via functions +, -, ×, ÷
- Sometimes, programming languages are different (division for example)

```
÷ : Z, Z → Z  in Java, C++, C#
÷ : Z, Z → R  in Visual Basic
```


19

Functions

Notation varies, but logic the same

20

Functions

- Many programming languages support customized functions
- The format often mirrors the notation used in discrete mathematics

21

C Family

- C programming language family includes: C, C++, Java, and C#
- The notation is very terse, but includes all the same information

```
name : int → int
int name(int n)
```

Discrete math

22

Visual Basic

- Visual Basic evolved from the original BASIC programming language
- The notation is far more verbose

```
name : Integer → Integer
Function name(n As Integer) As Integer
```


23

Pascal

- Pascal was very popular in the 1980's and 90's
- Created many concepts that were integrated into other languages

```
name : integer → integer
function name(n : integer) : integer
```


24

Fortran

- Fortran was the first third-generation programming language
- It has evolved over 50 years

```
name : integer → integer

function name(n) result(x)
    integer, intent(in) :: n
    integer :: x
```


25

Swift

- Swift was created by Apple to replace older Objective-C
- Influenced by multiple languages

```
name : Int → Int

func name(n : Int) -> Int
```


26

Swift

- Swift was created by Apple to replace older Objective-C
- Influenced by multiple languages

```
name : Int → Int

func name(n : Int) -> Int
```

An arrow!

OMG! An arrow!

Wow! Correct notation!

27

Boolean Logic

Part 6

1

Logic Statements

Make Mr. Spock Proud

2

Logic Statements

- Logic is used to construct all proofs and computer systems
- A *statement* is any declarative sentence that results in either true or false

3

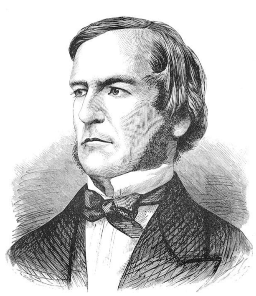
Examples of Statements

- *There are exactly 35 people in this room*
- *Sacramento State is located next to a river*
- *10 + 2 = 11*
- *We have great choices for the 2020 Election*

4

Boolean Logic

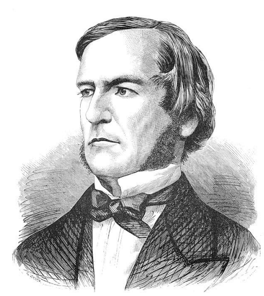
- Discovered by George Boole
- First published in *The Mathematical Analysis of Logic (1847)*



5

Boolean Logic

- Revolutionized logic & proofs and is part of framework of modern of computer science



6

Boolean Operators

- Statements can be combined in *compound statements* using Boolean operators
- After statements are combined, they are still statements
- For example: "p and q"
  - given that p and q are both statements
  - then "p and q" is also a statement

7

Let's Review Boolean Operators

Operator	Name
`p and q`	True only if both p and q are true
`p or q`	True if either p or q true
`not p`	True if p false
`p xor q`	True if p and q are different
`p implies q`	True unless p is true and q is false

8

Logic Notation of Operators

Logic	C Family	Visual Basic
$p \wedge q$	`p && q`	`p and q`
$p \vee q$	`p || q`	`p or q`
$\neg p$	`!p`	`not p`
$p \oplus q$	*none*	`p xor q`

9

Truth Tables

- Truth tables are useful tools for analyzing a large Boolean expression
- The table includes all the possible combinations of True and False for each input into the equation
- This results in 2^n rows where n is the number of inputs

10

Truth Table

p	q	$\neg p$	$p \vee q$	$p \wedge q$	$p \rightarrow q$
T	T	F	T	T	T
T	F	F	T	F	F
F	T	T	T	F	T
F	F	T	F	F	T

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AND Example

True True

$1 < 2 \wedge 10 < 40$

Result: True

12

AND Example 2

True — False

$1 < 2 \land 12 < 10$

Result: False

13

OR Example

True — True

$1 < 2 \lor 10 < 40$

Result: True

14

OR Example 2

True — False

$5 > 3 \lor 44 < 8$

Result: True

15

OR Example 3

False — False

$49 < 47 \lor 99 < 10$

Result: False

16

NOT Example

True

$\neg(1 < 2)$

Result: False

17

NOT Example

False

$\neg(1 = 2)$

Result: True

18

Examples

$1 < 3 \land 10 < 40$	True
$1 = 3 \land 10 < 40$	False
$\neg (12 \neq 12)$	True
$1 > 3 \lor 30 < 20$	False

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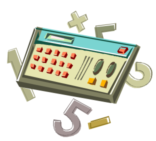
Logical Operator Precedence

Yup, we have that here too!

20

Logical Operator Precedence

- In *propositional logic*, statements can be combined with other statements using logical operators
- So, they can be chained together to form complex logic



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Algebra: Order of Operations

- Some mathematical operators have a high "precedence" than others
- They are computed first

```
3 + 6 / 3 * 2
```


22

Algebra: Order of Operations

- Knowing the correct order is vital
- For example, what is the result of the expression below?

```
3 + 6 / 3 * 2
```


23

Algebra: Order of Operations

- It is 7
- Divide and multiply are equal (and then done left to right), addition is done last

```
3 + 6 / 3 * 2 = 7
```


24

Many try, many fail

- Many students, using "PEMDAS" , think multiply is done before divide (M is before D)
- … or they just go left to right with no regard to precedence

```
3 + 6 / 3 * 2  =  4    WRONG! F-!
```


25

Standard Precedence Levels

1	¬
2	∧
3	∨ ⊕

Highest Level

Lowest Level

26

Defining Boolean Logic

"Want to define it?"
"True."

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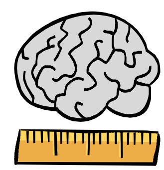
Defining Boolean Logic

- Let's look at what exactly Boolean logic is in context of data types and functions
- Once we define the Boolean Data type, we can apply it to other systems



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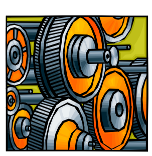
Functions

- Also recall functions from earlier
- An *abstract data type* is a set of values and functions on those values
- So, we can define the data type for Boolean values



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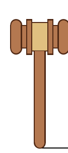
Extending Boolean to Other Types

- The abstract data type for a Boolean Data Type can be written as a 6-tuple



$$B = (S, \vee, \wedge, \neg, \bot, \top)$$

30

Extending Boolean to Other Types

- The first property is a set S contains two elements
- These are: $\bot$ (smaller) and $\top$ (bigger)

$$B = (S, \vee, \wedge, \neg, \bot, \top)$$

31

Extending Boolean to Other Types

- 3 operations on the set S: $\vee$, $\wedge$, $\neg$
- Must follow the 4 primary axioms: Identity, Communitive, Distributive, Complement

$$B = (S, \vee, \wedge, \neg, \bot, \top)$$

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Defining Boolean Logic

- Boolean Logic is closely related to Set Theory
- So much, in fact, that Boolean Logic can be considered as a special case of sets

{ }

33

Defining Boolean Logic

- We can show that the behavior of Boolean Logic can be created in sets
- It's not surprised that many of the laws for sets work for Boolean Logic

{ }

34

Set Theory & Boolean Logic

- First, we can define True as the Universe
- Remember that in binary logic, it simply *is* or it *isn't*
- So, the Universe means 1 or true

$$T = \mathbf{U}$$
$$F = \varnothing$$

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Set Theory & Boolean Logic

- The complement of $\mathbf{U}$ is $\varnothing$
- So, naturally, False is represented with $\varnothing$

$$T = \mathbf{U}$$
$$F = \varnothing$$

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Boolean Operators with Sets

- The Union operator is analogous to the And operator
- Likewise, Intersection is analogous to Or.

$$a \wedge b = A \cap B$$
$$a \vee b = A \cup B$$

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And-Intersection Comparison

Boolean Logic	Set Theory
T ∧ T = T	U ∩ U = U
T ∧ F = F	U ∩ ∅ = ∅
F ∧ T = F	∅ ∩ U = ∅
F ∧ F = F	∅ ∩ ∅ = ∅

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Or-Intersection Comparison

Boolean Logic	Set Theory
T ∨ T = T	U ∪ U = U
T ∨ F = T	U ∪ ∅ = U
F ∨ T = T	∅ ∪ U = U
F ∨ F = F	∅ ∪ ∅ = ∅

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Boolean Not with Complement

- The Boolean Not operator can also be implemented using set theory
- In this case, complement

$$\neg a = A'$$

40

Or-Intersection Comparison

Boolean Logic	Set Theory
¬T = F	U' = ∅
¬F = T	∅' = U

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The Axioms

- The Axioms are:
  - Identity
  - Communitive
  - Distributive
  - Complement
- The axioms from Set Theory apply to Boolean Logic

42

Tautology & Contradiction

When the logic is quite elementary

43

Tautology & Contradictions

- Some statements are always true or false regardless of the variables used
- If the statement is always true, it is called *tautology*
- If the statement is always false, it is called a *contradiction*

44

Example Tautologies

- The following are examples of tautologies
- The result will always be **true**

$p \lor \neg p$
$p \rightarrow p$
$p = p$

45

Example Contradictions

- The following are examples of contradictions
- The result will always be **false**

$p \land \neg p$
$p \oplus p$

46

Example

- So, what is the truth table for the example below?
- Let's create a truth table

$(\neg p \land q) \land (p \lor \neg q)$

47

Logic Equivalence

Same meaning, different form

48

Logical Equivalence

- *Logical equivalence* is when two different statements are the same
- The truth tables for both statements are identical

49

First Four Axioms

- The first four fundamental axioms were developed by *Edward Huntington* in 1904
- Other rules are derived from these

50

Communitive Law

- Both $\wedge$ and $\vee$ are communitive
- This means the left-hand and right-hand operands can be switched (symmetric relation)

$$a \wedge b \equiv b \wedge a$$
$$a \vee b \equiv b \vee a$$

51

Identity Law

- Some operands will have no effect on the truth table of a statement
- In this case, the statement can be simplified

$$a \wedge \text{true} \equiv a$$
$$a \vee \text{false} \equiv a$$

52

Complement Law

- When a statement is used with its complement (itself negated), it will result in either true or false
- So, it is always as tautology or contradiction

$$a \wedge \neg a \equiv \text{false}$$
$$a \vee \neg a \equiv \text{true}$$

53

Distributive Law

- Math has operators that are *distributive*
- For example: a * (b + c) = (a * b) + (a * c)
- Works for both $\wedge$ and $\vee$

$$a \wedge (b \vee c) \equiv (a \wedge b) \vee (a \wedge c)$$
$$a \vee (b \wedge c) \equiv (a \vee b) \wedge (a \vee c)$$

54

Logical Equivalence

- A number of useful laws can be derived from the first four
- It is vital to remember all of these when solving complex Boolean equations

55

Associative Law

- Some operators in math are *associative*
- For example: (a + b) + c = a + (b + c)
- Same applies to ∧ and ∨

$$a \wedge (b \wedge c) \equiv (a \wedge b) \wedge c$$
$$a \vee (b \vee c) \equiv (a \vee b) \vee c$$

56

Absorption Law

- There is a special case of the Distributive Law where one variable is *absorbed* (i.e. eliminated)
- Applies to both ∧ and ∨

$$a \wedge (a \vee b) \equiv a$$
$$a \vee (a \wedge b) \equiv a$$

57

Idempotent Law

- When a statement is combined with itself, it is equivalent to just the statement (no duplicate)
- This applies to both ∧ and ∨

$$a \wedge a \equiv a$$
$$a \vee a \equiv a$$

58

Involution Law

- One of the most basic equivalences in logic is the *double negation*
- It is fairly obvious, so not more needs to be said

$$\neg\neg a \equiv a$$

59

Domination Law

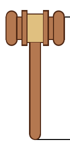
- This might look similar to the identity law, but look very careful at which operator is being used
- These will result in either true or false.

$$a \vee \text{true} \equiv \text{true}$$
$$a \wedge \text{false} \equiv \text{false}$$

60

DeMorgan's Law

- So, it states you can change the operator from $\wedge$ to $\vee$ or vice-versa
- If you negate both operands



$\neg (a \wedge b) \equiv \neg a \vee \neg b$

$\neg (a \vee b) \equiv \neg a \wedge \neg b$

61

Truth Table – Testing Not-Or

a	b	¬ a	¬ b	¬ a ∧ ¬ b	¬ (a ∨ b)
T	T	F	F	F	F
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

62

Truth Table – Testing Not-And

a	b	¬ a	¬ b	¬ a ∨ ¬ b	¬ (a ∧ b)
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

63

Example

- So, can we simplify the expression below?
- Let's create a truth table

$(a \wedge b) \vee (\neg a \wedge b)$

64

Boolean Algebra

- Truth tables become unwieldy as the number of variables increase
- Logical algebra is another way to evaluate equivalence
- Equivalences can be used to generate one expression from another

65

Example Simplification

$(a \wedge b) \vee (\neg a \wedge b) =$

$(a \vee \neg a) \wedge b =$ After using Distributive Law

$\text{true} \wedge b =$ After using Complement Law

b After using Identify Law

66

Implication

The operator of science

67

Implication

- The only Boolean operator that causes confusion is implication
- However, its usage is ***vital*** to understand – since it is used in your programs (even if you might not see it)

68

Implication

- For "p implies q"…
- p is called the *antecedent* (or hypothesis or assumption)
- q is called the *consequent* (or conclusion)

69

Implication

- "p implies q" is contradicted (false) **only** when…

  p is true and **q is false**

- In all other cases, it is true

70

Standard Precedence Levels

1	¬
2	∧
3	∨ ⊕
4	→

Highest Level

Lowest Level

71

Implies Example

True True

1 < 2 → 10 < 40

Result: True

72

Implies Example 2

True

False

5 > 3 → 44 < 8

Result: False

73

Implies Example 3

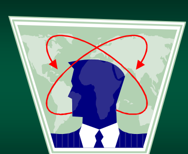
False

False

10 < 2 → 12 < 10

Result: True

74



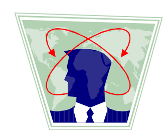
Analyzing Implication

Understanding is Power

75

Analyzing Implication

- Implication is both simple and complex
- It is used in all aspects of logical proof and the basis of all programs
- Understanding is complexity is essential to understanding logic (and discrete math)



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Implication Examples

- *If the Moon is made of cheese then the Moon is a tasty snack.*
- *If the flag has a bear then it's the Flag of California.*
- *If it is a fish then it is lives in water.*
- *If the university is Sacramento State then the mascot is a hornet.*

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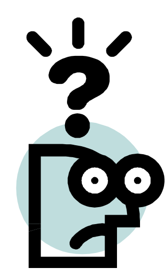
Many Ways to Say "Implies"

- *A implies B*
- *A → B*
- *B if A*
- *If A then B*
- *B given A*



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Example From History

- Let's look at a implication from California history
- During the Gold Rush, people were inspired by a simple idea…
- *"If I pan for gold then I'll get rich!"*

79

"If I pan for gold then I'll get rich"

- Let's look at this statement closer
- It can be rewritten: *"Pan of Gold→ Get Rich"* or, very tersely, *"P → R"*

P → R

80

"If I pan for gold then I'll get rich"

- There four combinations of the truth table
- Which of these combinations would invalidate the statement

P → R

81

True → True

- If P is true, and R is true…
- *"We panned for gold and got rich"*
- Statement is true
  - we asserted that if P is true then R is true
  - since both are true, the statement is affirmed
  - true → true = true

82

False → True

- What if P is false and R is true
- *"We didn't pan for gold and got rich"*
- Statement is true
  - the fact we got rich (without panning for gold), doesn't mean that the statement is false
  - it has not contradicted the statement
  - false → true = true

83

False → False

- What if P is false and R is false?
- *"We didn't pan for gold and didn't get rich"*
- Statement is true
  - the fact that both are false, still does not contraction our original statement
  - it stated "IF we pan for gold **then** we get rich"
  - false → false = true

84

True → False

- Finally what if P is true and R is false?
- *We panned for gold, but didn't get rich*
- Statement is false
  - we asserted if P is true then R must be true
  - however, since this contradicts the assertion, the result of the implication is false
  - true → false = false

85

Implication Hiding in Plain Sight

- Consider the expression below
- The word "then" is alternative way of saying "implies"
- So, Is it True? False?

if x > 2 then x^2 > 4

86

Implication Hiding in Plain Sight

- There are different values of x that will make the antecedent and consequent both true and false
- If both are true, then the statement is correct

if x > 2 then x^2 > 4

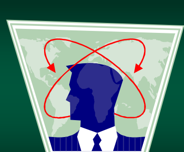
87

Implication Hiding in Plain Sight

- If **x > 2** is false, then we don't care about the consequent

if x > 2 then x^2 > 4

88



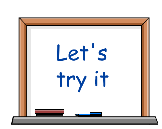
Deconstructing Implication

Other operators; same logic

89

Deconstructing Implication

- The implication logic can be broken down into the forms that are easier to remember
- This is actually quite important when we cover a few logical tricks later one
- So, let's look at the truth table for other operators



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Analyzing Implication

- So, can implication be written using just logical "and", "or", or "not"?
- Yes, we can!

$p \rightarrow q$

91

Truth Table – One way to do it

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

True whenever q is true

True whenever p is false

92

Analyzing Implication

- So, we can definite $p \rightarrow q$ as "not p or q"
- Like before, let's prove in our truth table

$p \rightarrow q \equiv \neg p \lor q$

93

Truth Table – Or Logic

p	q	$\neg p$	$\neg p \lor q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

94

Truth Table – One way to look at it

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Only false for:

$p \land \neg q$

We can negate this.

95

Analyzing Implication

- So, we can definite $p \rightarrow q$ as "not (p and not q)"
- It doesn't look quite right, let's test it out

$p \rightarrow q \equiv \neg (p \land \neg q)$

96

Truth Table – And Logic

p	q	¬ q	p ∧ ¬ q	¬ (p ∧ ¬ q)	p → q
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	F	T	T
F	F	T	F	T	T

Analyzing Implication

$$p \rightarrow q \equiv \neg(p \land \neg q)$$
$$\equiv \neg p \lor q$$

Arguments

Part 7

1

Arguments

Proving a Point

2

Arguments

- A combination of true statements can be used to claim another as true
- An *argument* is a collection of statements (called *premises*), which, when all are true, imply a consequence

3

Example Argument

```
raining → wet outside
not wet outside
_____________________
?
```

Our conclusion goes here

4

Example Argument 2

```
raining → wet outside
not raining
_____________________
?
```

What conclusion goes here?
Does it work?

5

Example Argument 3

```
x is duck or x is swan
x isn't a swan
_____________________
?
```

Obvious! But why?

6

When an Argument is Valid

- When all the premises are true then the consequence must be true
- If all the premises are true, but the conclusion can be false, the argument is disproven

7

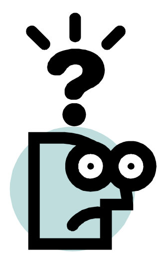
When an Argument is Valid

- However, if any premise is false, then the argument is not disproven – *it is still valid*
- We can often prove arguments by building truth tables

8

Argument Notation

- Arguments can be written out several ways
- The most common approach is to write each premise on a different line
- The consequence is written below the premises separated with horizontal line



9

Common Notation

$p \rightarrow q$

$q \rightarrow r$ — Premises

$p \rightarrow r$ — Consequence

10

Another Argument Notation

- Arguments can be written on a single line
- Premises are separated with commas
- Consequence can use the symbol ⊦ or ∴

"Therefore"

$p \rightarrow q,\ q \rightarrow r\ \vdash\ p \rightarrow r$

11

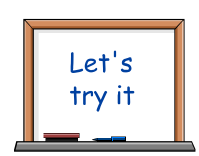
Let's Try This

$p \rightarrow q$

p

---

q — Valid!



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Arguments and Implication

- Arguments are actually implications with each premise connected with $\wedge$
- So, if you have premises A, premise B, and conclusion C, then it has the following form

$$A \wedge B \rightarrow C$$

13

Valid Arguments

Proving a Point

14

Valid Arguments

- *Rules of Inference* are valid arguments that are commonly used in proofs
- Most of these are obvious to you… it is natural logical thought

15

Rules of Inference

- *Modus Pones*
  (aka Law of Detachment)
- *Modus Tollens*
- *Disjunctive Syllogism*
- *Hypothetical Syllogism*

16

Modus Ponens

- *Modus Ponens* is the most basic Rule of Inference
- Based on the logic that if:
  - an implication is true
  - implication's hypothesis is true
  - then the implication's conclusion must be true

17

Modus Ponens Example

If it is a fish, then it lives in water.

It is a fish.

Therefore, it lives in water!

18

Modus Ponens

Rules of Inference

$p \rightarrow q$
p

---

q

19

Modus Ponens

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

20

Modus Tollens

- *Modus Tollens* is closely relate to modus ponens
- Based on the logic that if:
  - an implication is true
  - implication's conclusion is false
  - then the implication's hypothesis must be false

21

Modus Tollens Example

If it is a fish, then it lives in water.

It doesn't live in water.

Therefore, it is <u>not</u> a fish!

22

Modus Tollens

Rules of Inference

$p \rightarrow q$
$\neg q$

$\neg p$

23

Modus Tollens

p	q	p → q	¬ p	¬ q
T	T	T	F	F
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

24

Disjunctive Syllogism

- *Disjunctive Syllogism* is based on the ∨ operator that
- Based on the logic that if:
  - an or-statement is true
  - one of the operands is false
  - then, the other operand must be true

Disjunctive Syllogism Example

It breathes water or air.

It doesn't breath water.

Therefore, it breathes air.

Disjunctive Syllogism

$p \lor q$
$\neg p$

---

q

Rules of Inference

Hypothetical Syllogism

- *Hypothetical Syllogism* is based on an implication chain
- Gives a logical "chain" of events
- So, if a → b and b → c then a → c

Hypothetical Syllogism Example

If is a trout, then it is a fish

If it is a fish, then it lives in water.

Therefore, a trout lives in water!

Hypothetical Syllogism

$p \rightarrow q$
$q \rightarrow r$

$p \rightarrow r$

Rules of Inference

Let's Apply The Logic

- Let's these rules on the argument below
- We can use either a truth table or logical deduction

```
If I study then I will an A
If I don't watch Netflix then I will study
I didn't get an A
________________________
I watched Netflix
```


31

Simply Logic: letters

- First, let's simply the structure of the argument so we can see the logic
- We will assign each part a letter

```
s = studied
a = got an A
n = watched Netflix
```


32

Simply Logic: New Form

```
1.  s → a
2.  ¬ n → s
3.  ¬ a
    ___________
    n
```


33

Modus Tollens – study, no A

```
1.  s → a
2.  ¬ n → s
3.  ¬ a
    ___________
    n
```

1 and 3:
Modus Tollens

34

Modus Tollens – study, no A

```
1.  ¬ s
2.  ¬ n → s
3.  ¬ a
    ___________
    n
```

1 and 3:
Modus Tollens

"Did not study"

35

Modus Tollens – study, Netflix

```
1.  ¬ s
2.  ¬ n → s
3.  ¬ a
    ___________
    n
```

1 and 2:
Modus Tollens

36

Modus Tollens – study, Netflix

1. ¬ s
2. ¬ ¬ n
3. ¬ a

---

n

1 and 2:
Modus Tollens

"Did not *not* watch Netflix"

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Double Negation

1. ¬ s
2. ¬ ¬ n
3. ¬ a

---

n

Double negation

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Double Negation

1. ¬ s
2. n
3. ¬ a

---

n

2:
Double Negation

"Watched Netflix"

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Logical Fallacies

"This is most illogical" – Mr. Spock

40

Logical Fallacies

- There are a number of fallacious arguments that, while they might look logical, are wrong
- The following slides contain some of them
- For fun, apply them to current political discourse or *History Channel 2*

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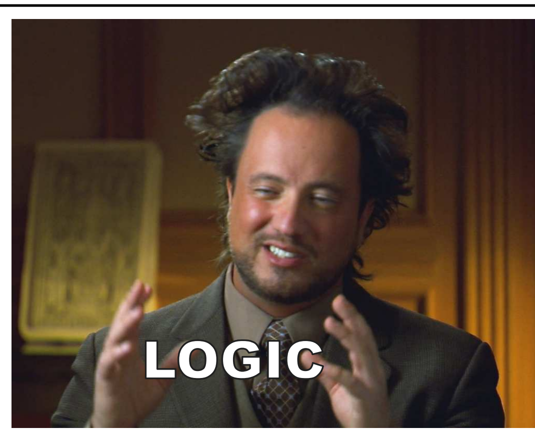

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Fallacy of the Converse

- *Fallacy of the Converse* is based on assumption that if the conclusion is true then the hypothesis is true
- Also called:
  - *affirming the consequent*
  - *converse error*

43

Fallacy of the Converse Example

If it is a fish, then it lives in water.

It lives in water.

Therefore, it is a fish!

44

Fallacy of the Converse

$p \rightarrow q$
q

---

p

p can still be false

45

Fallacy of the Converse

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

46

Fallacy of the Inverse

- *Fallacy of the Inverse* is based on assumption that if the hypothesis is false, then the conclusion is also false
- Also called:
  - denying the antecedent
  - inverse error

47

Fallacy of the Inverse Example

If it is a cat, then it is furry.

It is not a cat.

Therefore, it is not furry!

48

Fallacy of the Inverse

$p \rightarrow q$

$\neg p$

$\neg q$

q can be either true or false

Fallacy of the Converse

p	q	p → q	¬ p	¬ q
T	T	T	F	F
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

Fallacy of Affirming a Disjunct

- *Fallacy of Affirming a Disjunct* is based on assumption that if there are two attributes and one is true, the other must be false
- Other names:
  - *fallacy of the alternative*
  - *false exclusionary disjunct*

Affirming a Disjunct Example

Suspect is either a politician or a lawyer.

Suspect is a politician.

Therefore, the suspect isn't a lawyer.

Fallacy of Affirming a Disjunct

$p \vee q$

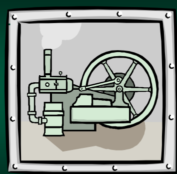
p

$\neg q$

Just because p is true, doesn't mean q has to be false.

Fallacy of the Converse

p	q	p ∨ q	¬ q
T	T	T	F
T	F	T	T
F	T	T	F
F	F	F	T



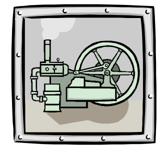
Boolean Algebra & Proofs

Proofs and Logic are one

55

Boolean Algebra

- Remember Boolean algebra laws: Associative, Commutative, etc…
- These can be used to expand an expression… and then simplify it in a different form



56

Example

if "a or b" is true and

"a and b" is false

then

a = not b

57

Example (with a rewrite)

We can rewrite it as an argument:

```
a ∨ b = true
a ∧ b = false
_____________
a = ¬b
```


58

The Strategy

- We could use a Truth Table to prove this
- Let's use Boolean Algebra to prove if this is correct

```
a ∨ b = true
a ∧ b = false
_____________
a = ¬b
```


59

The Approach

- Start with a
- Try to change it into ¬ b
- Use the premises to replace values

```
a ∨ b = true
a ∧ b = false
_____________
a = ¬b
```


60

$a \lor b = \text{true}$
$a \land b = \text{false}$

$a =$	a	
$=$	$a \land \text{true}$	Identity
$=$	$a \land (b \lor \neg b)$	Complement
$=$	$a \land b \lor a \land \neg b$	Distributive
$=$	$\text{false} \lor a \land \neg b$	Premise
$=$	$b \land \neg b \lor a \land \neg b$	Complement
$=$	$\neg b \land (b \lor a)$	Distributive
$=$	$\neg b \land \text{true}$	Premise
$=$	$\neg b$	Identity

61

Please Wait

CSC 28
will begin shortly

(open the chat window)

1

Basic Proof

Part 8

2

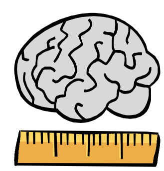
Theorems

The Big Bang Theory

3

Theorems

- A *theorem* is a statement we intend to prove using existing known facts (called *axioms* or *lemmas*)
- Used extensively in all mathematical proofs – which should be obvious



4

Example

- Most theorems are of the form: If A, then B
- The theorem below is very easy to interpret

If a and b are even integers
then a × b is an even integer

5

Example

- Theorems are arguments
- They can be structured as such

```
a is even
b is even
_____________
a × b is even
```


6

Example

- Sometimes it is hard to see
- Below, the same theorem is written using different language

x and y are even integers and the product is even.

The product xy is even when x and y are both even.

7

Some Basic Definitions

Abstract? Not really.

8

Definition: x is even

a is even if and only if…

a = 2n for some integer n

9

Definition: x is odd

a is odd if and only if…

a = 2n + 1 for some integer n

10

Definition: x | y (x divides y)

a | b (a divides b) iff…

b = k × a for some integer k

11

Definition: x ∈ Q (x is a rational)

a is a rational number iff…

a = b / c for some integers b and c, and c ≠ 0

12

Proving A → B

Modus Ponens

13

Proving A → B

- The Boolean operator A → B is true except when A is true and B is false
- So, we can logically prove that A → B by showing that *whenever A is true, B **must** also be true*
- *… so T → F isn't possible*

14

Proving A → B

- This is essentially a *Modus Ponens* proof
- You are showing that if A is true, and A → B is true, then B **must** be true
- Also note that "A" and "B" can be compound statements

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Modus Ponens

A → B — Prove this is always true

A — If this is true

B

16

The Steps

- There are basically just two steps to follow:
  1. Assume A is true
  2. Show that B must be true
- This shows that B is true whenever A is true

17

Example 1

- Let's prove the following theorem from before
- This is actually quite easy

If a and b are even integers
then a × b is an even integer

18

Example 1 – the argument

```
a is even
b is even
──────────
a × b is even
```


19

Example 1 – internal structure

- Remember: all proofs are implications
- So, we will assume the both premises are true and show the conclusion must be true

```
a is even ∧ b is even → a × b is even
```


20

Example 1

```
Assume that x and y are even integers.

So, by the definition…

a = 2i and b = 2j (for some i, j)

Note: use different arbitrary variables
or you are assuming they are equal!
```


21

Example 1 - Proof

```
So, the product is:

a × b = 2i × 2j
      = 4 × i × j
      = 2 × (2 × i × j)

So, by definition, a × b is even
```


22

Example 2

- The following is a theorem about the product of an odd and even number
- The proof is straight-forward using the definitions

If a is even and b is odd, then a × b is even

23

Example 2

```
Assume:
a is an even integer and
b is an odd integer.

Then a = 2i and b = 2j+1 for some
integers i and j
```


24

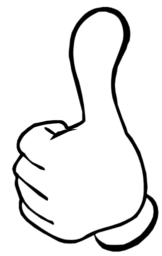
Example 2

```
Multiplying, we get:

a × b = (2i) × (2j + 1)
      = 4ij + 2i
      = 2 × (2ij + i)

...which is even
```


25

Proof Tips

- Begin your proof with what you assume to be true (the hypothesis)
- Don't argue the truth of a theorem *by example*
  - stay abstract
  - e.g. you know x and y are even integers – that's all you know



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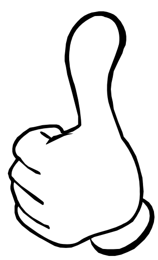
Proof Tips

- Your proof must work forward from your assumptions to your goal
- You may work backward on scratch paper to help you figure out how to work forward



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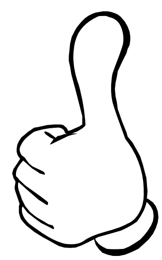
Proof Tips

- Write your proof in prose
- Then read out loud, it should sound like well-written paragraph
- Quite often you'll use definitions to make progress



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Proof by Contrapositive

Proof with inverse logic

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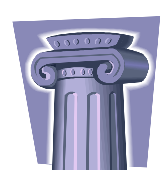
Proof by Contrapositive

- There are several techniques that can be employed to prove an theorem
- The direct approach, like before, is quite common, but its not the only path you can take



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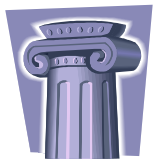
Proof by Contrapositive

- *Proof by Contrapositive* has you prove the opposite of the original theorem
- Quite impressively, this will also prove the original theorem



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Getting the Contrapositive

- First…
  - negate both the assertion and conclusion of the implication
  - so, basically, put "not" in front of both operands
- Second…
  - reverse the implication
  - you basically swap the left-hand and right-hand operand of the implication

32

Getting the Contrapositive

- So, both operands swap positions and are negated
- Are they equal? Let's confirm in a Truth Table

```
for p → q
contrapositive is ¬q → ¬p
```


33

Contrapositive Truth Table

p	q	¬ q	¬ p	p → q	¬ q → ¬ p
T	T	F	F	T	T
T	F	T	F	F	F
F	T	F	T	T	T
F	F	T	T	T	T

34

Modus Ponens Contrapositive

Prove always true

$\neg$ B → $\neg$ A

$\neg$ B

If this is true (i.e. B is false)

$\neg$ A

35

How it Works

- So, if we prove the contrapositive, we also prove the original theorem
- For the original A → B
  - suppose that if B is false
  - show that A must be false
- It does make sense, if you think about it

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Example

- The following theorem should look familiar
- This theorem states that the square of a odd number is also odd
- *Direct proof is near impossible!*

If x^2 is odd then x is odd

37

Example Contrapositive

- The contrapositive negates each operand in the implication A → B
- The following shows the reverse of each

A = x^2 is odd

¬A = x^2 is not odd = x^2 is even

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Example Contrapositive

- The contrapositive negates each operand in the implication A → B
- The following shows the reverse of each

B = x is odd

¬B = x is not odd = x is even

39

Example Contrapositive

- Finally, we reconstruct our theory with B → A rather than A → B
- This expression is equivalent to the original

if x is not odd then x^2 is not odd

or rewritten as…

if x is even then x^2 is even

40

Example Contrapositive

We assume x is not odd

x is not odd means x is even

x = 2k for some integer k

41

Example Contrapositive

We assume x is not odd (even)

$$x^2 = (2k)^2 = 4k^2 = 2(2k^2)$$

So, x^2 is even which is *not odd*

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Example Result

We proved:

if x is even then x^2 is even

By contrapositive, we proved:

if x^2 is odd then x is odd

Proof by Contradiction

Welcome down the rabbit hole

Proof by Contradiction

- *Proof by Contradiction* takes a novel approach
- It uses the approach of *reductio ad absurdum*
- So what is it? Well, it proves the theorem by showing it can't be false

But, How?

- Assume it is false
- Show that (if it is false) something impossible results
- Therefore, it can't be false and, thus, true!

Proving $A \rightarrow B$

- Argue: $A \land \neg B$ …which is $\neg(A \rightarrow B)$
- Show that something *impossible* results!
- Since $A \rightarrow B$ cannot be false, it **must** be true

Contradiction

A	B	¬ B	A ∧ ¬B	¬ (A → B)
T	T	F	F	F
T	F	T	T	T
F	T	F	F	F
F	F	T	F	F

Example

- The following is a classic Proof By Contradiction
- The theorem covers if the square-root of 2, is an irrational number

The square root of 2 is irrational

49

Example

- To prove by contradiction, we need to show that the opposite cannot be true (i.e. false)
- The sentence bellow is the theorem negated
- So, how do we go about proving this?

The square root of 2 *is rational*

50

Example

- Well, what is a rational number?
- A rational number can be expressed as "a / b" where a and b are *integers with no common factors* (aka "lowest terms")

The square root of 2 *is rational*

51

Examples of Rational Numbers

- **1 / 3**
- **7 / 1**
- **22 / 7**
- **7734 / 10001**
- **1 / 123456789**
- **5 / 3**

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The Proof: Prove Opposite

$\sqrt{2}$ is rational

$\sqrt{2}$ = a / b where a, b $\in$ Z
and b $\neq$ 0
and a, b have no common factors

53

Analyzing a

$\sqrt{2} = a / b$

Let's look at the properties of a

$\sqrt{2} \times b = a$

$2 \times b^2 = a^2$

54

Analyzing a – It is even

$2 \times b^2 = a^2$

So… a^2 is an even number

therefore, we know a is also even (previous proof – even × even)

Analyzing b

Since a is even and a / b is in lowest terms, then b must be odd

Why? If b is even, then a / b would have common factors – namely 2.

Example: Oh ohhhhh

However… look again at $2 \times b^2 = a^2$

Since a is even, we can use the definition. So…

$2 \times b^2 = (2k)^2$
$= 4k^2$

Example: Oh ohhhhh

Solving for b^2…

$2 \times b^2 = 4k^2$
$b^2 = 2k^2$

Since b^2 is even, b is even

Result

Since b must be both odd and even, we have a contradiction

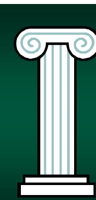
The theorem "square root of 2 is rational" cannot be true

Therefore, "square root of 2 is irrational" is true

Predicate Logic

Part 9

1



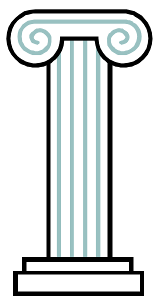
Predicates

Just the facts…

2

Predicates

- A predicate is a statement about one or more variables
- It is stated as a fact – being true for the data provided



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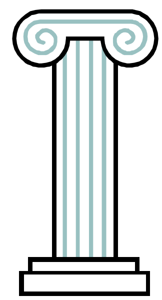
Predicates

- Predicates express *properties*
- These can apply to a single entity or *relations* which may hold on more than one individual



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Predicate Notation

- It follows the same basic syntax as function calls in Java (and most programming languages)
- However, type case is important:
  - constants start with lower case letters
  - predicates start with upper case letters

5

Single variable predicate

- Predicates can have one variable (at a minimum)
- The following sentence states one that the cat named Pattycakes has the "sleepy" property

```
"Pattycakes The Cat is sleepy"
```


6

Single variable predicate

- Alternatively, we can write that property in predicate form
- "Sleepy" predicate for "Pattycakes" is true
- Note the uppercase and lowercase!

```
Sleepy(pattycakes)
```


7

Two Variable Predicate

- Predicates can have multiple variables (unlimited actually… well within reason)
- The following is a classic example of a two-variable relationship

```
x < y
```


8

Two Variable Predicate

- The LessThan predicate is true for x, y

```
LessThan(x, y)
```


9

Predicates Summary

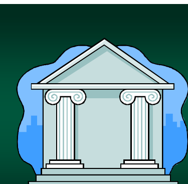
- 1-place predicates assign properties to individuals:
  - ___ is a cat
  - ___ is sleepy
- 2-place assign relations to a pair
  - ___ is sleeping on ___
  - ___ is the capitol of ___

10

Predicates Summary

- 3-place predicates assign relations to triples
  - ___ wants ___ to _____
  - Cat named _____ likes to ___ on _____
- Etc…

11



Quantified Statements

More Symbols

12

Quantified Statements

- Sometimes we want to say that *every element in the universe* has some property
- Let's say the universe is the people in this Zoom "room" & we want to say "*everyone in the room is awake*"

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Propositional Approach #1

- We can write out a descriptive sentence
- Shortcomings:
  - it is monolithic an inflexible
  - not "mathematical" enough

```
Everyone in this room is awake.
```

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Propositional Approach #2

- We can also write out that sentence using a long list of predicates
- So, we list them all or make a pattern
- Shortcomings: cumbersome & verbose

```
P(moe) and P(larry) and P(curly) and …
```

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Limitations of Propositional Logic

- While propositional logic, which we covered, can express a great deal of complex logical expressions, it ultimately is insufficient for all arguments
- Why? The premises (and conclusions) in propositional logic have no internal structure.

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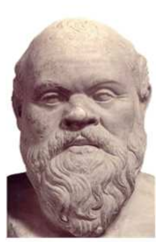
Limitations of Propositional Logic

- For example, it cannot show the validity of Socrates Argument
- This arguments states: *"All humans are mortal. Socrates is a human. Therefore, he is mortal."*



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Socrates Argument

- The following is the propositional logic form of the Socrates Argument
- Can we prove the conclusion?

```
All humans are mortal
Socrates is a human
______________________________
Therefore, Socrates is mortal
```

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The Socrates Argument

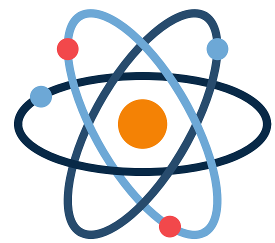
- The following is the argument in normal form
- A problem arises since the validity of this argument comes from the internal structure which propositional logic cannot "see"

```
H
S
───
M
```


19

Solution

- It's time to break apart the logic and see the internal structure
- So, ***we are splitting the atom!***



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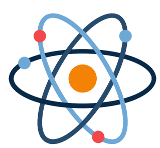
Why Go Nuclear?

1. Expose the internal structure of those "atomic" sentences
2. Create new terminology to describe the semantics
3. Introduce laws to use and manage them



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New Notation: For All

- The "For-All" symbol states every element *x* in the universe makes P(*x*) true
- So, it is true if and only if the every element *x* in the universe has P as true

$\forall_x$ **P(x)**

22

New Notation: Exists

- The "Exists" States at least one element *x* in the universe makes P(*x*) true
- True if just a single P is true

$\exists_x$ **P(x)**

23

Example: Pineapple Pizza

- Let's create the quantified statement for *"Someone doesn't like pineapple pizza!"*
- Let's create a predicate P(x) means *"x likes pineapple pizza"*
- What does someone mean? At least one person?

24

Pineapple Pizza: Try #1

- How about the following expression?
- It's not true if at least one person likes pineapple
- This means *"nobody likes pineapple pizza"*

$\neg (\exists_x P(x))$

25

Pineapple Pizza: Try #2

- We can also negate the predicate
- Means, for at least one person, they dislike pineapple pizza
- *"Someone doesn't like pineapple pizza!"*

$\exists_x \neg P(x)$

26

Quantifier Equivalence

Quantifier Conversion

27

Equivalence

- Just like propositional logic, quantitative expressions have equivalencies
- They follow the same basic logic we have seen before

28

Example: Opposite Expression

- Example: "Everyone in the room is awake"
- Let's create the reverse of this expression *(that still says the same thing)*

"Everyone in the room is awake."

29

Example: Opposite Expression

- So, let's just negate the predicate "is awake" into "is asleep"
- Does that work? No.

"Everyone in the room is asleep."

30

Example: Opposite Expression

- Now, let's negate the quantifier "everyone'" (for-all) into "someone" (exists)
- The expression below works: almost

"Someone in the room is asleep."

31

Example: Opposite Expression

- Well, what if we change the quantifier "everyone'" (for-all) into "someone" (exists)
- The expression below works: yes

"There is no one in the room that is asleep."

32

Exists and For-All

- The previous two laws allows us to extrapolate two additional laws
- Note: we simply push the negative and remove the double-negation

$\forall_x\ P(x) \equiv \neg \exists_x \neg P(x)$

$\exists_x\ P(x) \equiv \neg \forall_x \neg P(x)$

33

Equivalence – Moving Negation

- You can push a negation through a quantifier by toggling the quantifier
- Read the expressions below carefully…

$\neg \exists_x\ P(x) \equiv \forall_x \neg P(x)$

$\neg \forall_x\ P(x) \equiv \exists_x \neg P(x)$

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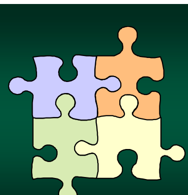

Conjunction & Disjunction

Breaking Apart and Combining

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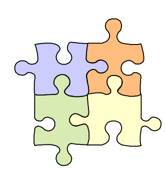
Conjunction & Disjunction

- Both the Exists and For-All quantifiers can be broken apart (and combined)
- This can occur if the expression contains an AND or an OR



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Exists Disjunction

- If the Exists quantifier is used on a disjunction, it can be broken into two Exists
- This only works with $\lor$

$$\exists_x (P(x) \lor Q(x)) \equiv \exists_x P(x) \lor \exists_x Q(x)$$

37

For-All Conjunction

- If the For-All is used on a conjunction (and), it can be broken into two For-All
- This only works with $\land$

$$\forall_x (P(x) \land Q(x)) \equiv \forall_x P(x) \land \forall_x Q(x)$$

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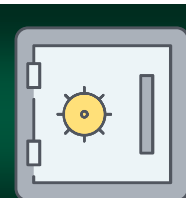
Please Wait

CSC 28
will begin shortly

(open the chat window)

39



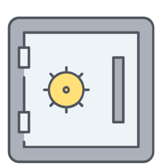
Bound & Free Variables

Some variables are not variable

40

Bound & Free Variables

- Not all variables used in a quantified expression is treated the same
- Each variable in an expression is either considered "bound" or "free"



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Bound & Free Variables

- A variable is *free* if a value must be supplied to it before expression can be evaluated
- A variable is *bound* if it not free (usually a dummy variable) and contains values that are not needed to be evaluated

42

Example 1

- Which variables need we supply a value before the expression can be evaluated?
- Both x and c
- Without knowing both we cannot evaluate the expression (both are free)

```
(x ^ 2 < 4 * c)
```


43

Example 2

- Which variables need to be supplied before the expression can be evaluated?
- x: no, it is a dummy variable
- c: yes, once we give a value for c, we can evaluate the expression

$\forall_x$ `(x ^ 2 < 4 * c)`

44

Multiple Quantifiers

Many E's and A's doing headstands

45

Multiple Quantifiers

- A quantified statement may have more than one quantifier
- In fact, most of the time, statements will contain several

46

Example

- x > y is an expression with two variables
- The expression is true if an x is supplied which is greater than y

$\forall_x \exists_y$ `P(x, y)`

47

Example

- ∃y (x > y) is an expression with <u>one</u> free variable
- Evaluates to true if <u>x is supplied</u> and there is a y greater than the supplied x

$\forall_x \exists_y$ `P(x, y)`

48

Example

- $\forall x\ \exists y\ (x > y)$ contains no free variables
- Evaluates to true if $\exists y\ (x > y)$ is true for every x in the universe.

$\forall_x\ \exists_y$ `P(x, y)`

49

Example 2

- The following in an implication with two quantifiers as operands
- It states that whenever "$\forall x$ P(x)" is a true statement, then so is "$\forall x$ Q(x)".

$\forall_x$ `P(x)` $\rightarrow$ $\forall_x$ `Q(x)`

50

Converting English to Logic

Do it bit by bit

51

Difficult Example

- Let's create a quantified statement for the following logical statement
- *We will go slowly, since this is not easy*

Everyone who has a friend who has Covid will have to be quarantined

52

Difficult Example

- "Everyone" is a For-All relationship
- What is everyone referring to? People
- So, the abstract object is a person

$\forall_x$ `(if x has a friend with Covid,`
`then x must be quarantined)`

53

Difficult Example

- So, we can factor it out into the expression below – x is a person
- *Now*, let's look at the sub expression…

$\forall_x$ `(if x has a friend with Covid,`
`then x must be quarantined)`

54

Difficult Example

- The sentence *" if x has a friend with Covid, x must be quarantined"* is an implication!
- Let's look at the antecedent (hypothesis)

```
if x has a friend with Covid,
   then x must be quarantined
```


Difficult Example

- How do we write the concept: *"x has **a** friend with Covid"* ?
- They just need a single friend
- So, this is an Exists quantifier

```
∃y (x is friends with y, and y has Covid)
```


Difficult Example

- Now that we have a basic form of the final version, let's make some predicates
- We will use single letter names for brevity

```
F(x, y) means "x and y are friends"
C(x)    means "x has Covid"
Q(x)    means "x must be quarantined"
```


Difficult Example

- This says: "There exists a *person y* where *y* is friends with *person x*, and *y* has Covid"
- Note: *x* is not bound in this expression

$$\exists_y \ (F(x, y) \wedge C(y))$$

Difficult Example

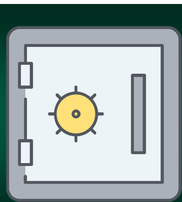
- So, what happens if a friend has Covid?
- Then, they must be quarantined
- Note: implication is outside the exists

$$\exists_y \ (F(x, y) \wedge C(y)) \rightarrow Q(x)$$

Difficult Example

- Now we can put it all together…
- The following is the quantified expression for our original statement

$$\forall_x \ (\exists_y \ (F(x, y) \wedge C(y)) \rightarrow Q(x))$$



Bounded Quantifiers

Hidden Implication
(for those who hate to type)

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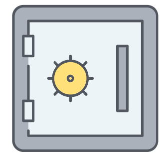
Bound Quantifiers

- Some quantifiers can be more than meets the eye
- For brevity, many predicate and propositional expresses are merged with the $\forall$ and $\exists$



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Shorthand Notation

- The following type of expression is quite common
- So much so that a shortcut notation is often employed

$$\forall_x \ (\mathbf{R(x)} \rightarrow \mathbf{P(x)})$$

63

Shorthand Notation

- The membership sub-expression is moved to the quantifier's subscript
- This is equivalent to the last

$$\forall_{\mathbf{R(x)}} \mathbf{P(x)}$$

64

Likewise…

- The sub-expression before the implication can be anything
- In this example, x > 5 is moved to the subscript

$$\forall_x \ (\mathbf{x} > 5 \rightarrow \mathbf{P(x)}) \equiv \forall_{x > 5} \ \mathbf{P(x)}$$

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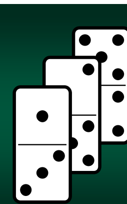
Induction

Part 10

1



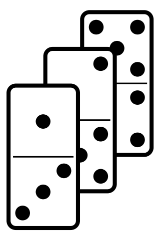
Induction

Proof by Pattern

2

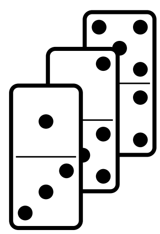
Induction

- Many proofs, in fact a great number of them, are based on "all positive integers"
- *Induction* is a technique of proving a theorem that is based on this criteria



3

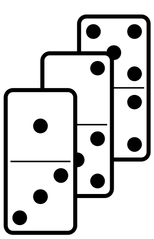
Induction



- The proof by induction is based on the *Well-Ordering Property*
- It states that: given a set of non-negative numbers there is a *least* element

4

Induction



- Induction basically helps prove $\forall_x P(x)$ where the universe is positive numbers
- *Or any range starting at one point and going off to infinity (positive or negative)*

5

How it Works

- It works in 2 steps
  1. proving P(1) and then
  2. proving that $P(n) \rightarrow P(n + 1)$
- As a result...
  - since $P(n) \rightarrow P(n + 1)$
  - then $P(n + 1) \rightarrow P(n + 2)$ and so on…

6

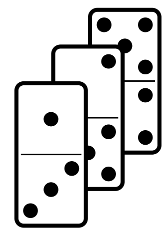
Metaphor: Line

- There is a line of people
- First person tells a secret to the second person
- The second person then tells it to the third
- … and so on until everyone knows the secret

7

Metaphor: Dominos

- You have a long row of Dominos
- The first domino falls over and hits the second domino
- The second hits the third
- … and so on until they are all knocked over



8

Foundation of Induction

- The following summarizes the proof technique
- It is important to understand the approach since it is so commonly used

$$P(1) \land \forall_n \, (P(n) \rightarrow P(n+1)) \rightarrow \forall_n \, P(n)$$

9

Steps to Proof

- Step 1: *Basis*
  - show the proposition P(1) is true
  - very easy to do – just plug in the values
- Step 2: *Induction*
  - assume P(n) is true (which is your theorem)
  - show that P(n + 1) must be true

10

Example: Sum of Odds

Using induction…

Show that the sum of n odd numbers equals n^2

11

Example: Sum of Odds

- So, the sum of odd numbers is a square?
- Is that true?
  - 1 + 3 = 4
  - 1 + 3 + 5 = 9
  - 1 + 3 + 5 + 7 = 16
- Okay, that's just odd! *(pun intended)*

12

Sum of Odds

```
P(n) is written as:

   1 + 3 + 5 + ... + (2n – 1) = n²

P(n + 1) is written as:

   1 + 3 + … + (2n–1) + (2n+1) = (n+1)²
```


13

Basis: Sum of Odds

- The sum of odds, for just 1 number is simply 1
- Of course, this is also 1 squared

```
P(1) = 1 = 1²
```


14

Induction: Sum of Odds

```
P(n) is written as:

   1 + 3 + 5 + ... + (2n – 1) = n²
```

We assume P(n) is true. So, we are assuming that this equality is valid.

```
Now we prove P(n) → P(n + 1)
```


15

Induction: Sum of Odds

```
P(n + 1) is written as:

   1 + 3 + … + (2n–1) + (2n+1) = (n+1)²

Can we show this equality is valid?

Let's look at the left side of the equals …
```


16

Show following equals: **$(n + 1)^2$**

```
1 + 3 + … + (2n – 1) + (2n + 1)

  = 1 + 3 + … + (2n – 1)  + (2n + 1)

  = n² + (2n + 1)

  = (n + 1)²
```

P(n) assumed true, so the equality is true. You can replace!

17

Induction: Sum of Odds

So, we have shown that when P(n) is true, then P(n + 1) is true.

P(n) → P(n + 1)

Since P(1) is true, we have proved $\forall_n$ P(n)

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Example: Divisible by 3

Using induction…

Show that $n^3 - n$ is divisible by 3 whenever n is a positive integer

Basis: Divisible by 3

- For our basis, we plug 1 into our expression and get the result
- The result, 0, is divisible by 3.

$$P(1) = 1^3 - 1 = 1 - 1 = 0$$

Induction: Divisible by 3

P(n) is written as:

$$n^3 - n$$

P(n + 1) is written as:

$$(n + 1)^3 - (n + 1)$$

Show following is: Divisible by 3

$(n + 1)^3 - (n + 1)$

$= n^3 + 3n^2 + 3n + 1 - (n + 1)$

$= n^3 + 3n^2 + 3n + 1 - n - 1$

$= n^3 + 3n^2 + 3n - n$

$= n^3 - n + 3n^2 + 3n$ *Rearranged*

$= (n^3 - n) + 3(n^2 + n)$

Induction: Divisible by 3

So, for $(n^3 - n) + 3(n^2 + n)$

Since we assumed P(n) is true, then $(n^3 - n)$ is divisible by 3.

… and $3(n^2 + n)$ is divisible by 3 since 3 is a factor

Hence, $P(n) \rightarrow P(n + 1)$

Example: Sum of 2^n

Using induction…

Show that $2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$ whenever n is a positive integer

Basis: Sum of 2^n

- For our basis, we plug 1 into our expression and get the result
- The result is 1 – which is true

$$P(0) = 2^0 = 1 = 2^1 - 1$$

25

Induction: Sum of 2^n

P(n) is written as:

$$2^0 + 2^1 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$$

P(n + 1) is written as:

(n+1) + 1

$$2^0 + 2^1 + \ldots + 2^n + 2^{n+1} = 2^{n+2} - 1$$

26

Show following equals: $2^{n+2} - 1$

$$\begin{aligned}
&2^0 + 2^1 + \ldots + 2^n + 2^{n+1} \\
&= (2^0 + 2^1 + \ldots + 2^n) + 2^{n+1} \\
&= (2^{n+1} - 1) + 2^{n+1} \\
&= 2^{n+1} + 2^{n+1} - 1 \\
&= 2^n(2^1 + 2^1) - 1 \\
&= 2^n(4) - 1 \\
&= 2^n(2^2) - 1 \\
&= 2^{n+2} - 1
\end{aligned}$$

P(n) assumed true, so the equality is true

27

Induction: Sum of 2^n

Since we assumed P(n) is true…

$2^0 + 2^1 + \ldots + 2^n + 2^{n+1}$ **is equal to** $2^{n+2} - 1$

Hence, $P(n) \rightarrow P(n + 1)$

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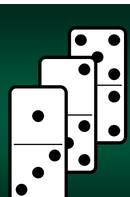

Strong Induction

Another approach

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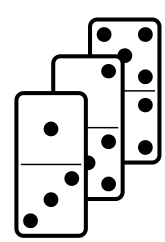
Strong Induction

- *Weak* induction assumes that P(*n*) is true, and then uses that to show P(*n* +1) is true
- *Strong* induction assumes P(1), P(2), …, P(*n*) are all true and then uses that to show that P(*n* +1) is true



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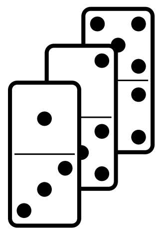
Using the Domino Metaphor…

If all the dominos (1 to n) fell over, will it also have knocked over n+1?



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Strong Induction

- So, strong induction uses more "dominoes" than weak induction – *which just uses one*
- Both proof techniques are equally valid

```
( P(1) ∧ P(2) ∧ … ∧ P(k) ) → P(k + 1)
```


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Steps to Proof

- Step 1: *Basis*
  - show the proposition P(1) is true
  - very easy to do – just plug in the values
- Step 2: *Induction*
  - assume that P(1), P(2), …, P(n) are all true
  - show that P(n + 1) is true
  - *or, changing the math slightly: show P(n) is true by assuming P(n – 1), P(n – 2), etc…*

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Example: Product of Primes

Using strong induction…

Show that any number $n \geq 2$ can be written as the product of primes

34

Basis: Product of Primes

- For 2, we can show that 2 is a product of two primes
- Namely, 1 and itself

```
P(2) = 1 * 2 = 2
```


35

Induction: Product of Primes

- There are two cases for n + 1:
- P(n + 1) is prime
- P(n + 1) is composite
  - it can be written as the product of two composites, a and b
  - where $2 \leq a \leq b < n + 1$

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Induction: Product of Primes

```
n + 1 prime:

   it is a product itself and 1

n + 1 is composite:

   both P(a) and P(b) are assumed to be
   true

   so, there exists primes where
   a * b = n + 1
```


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Result

- We showed that any number can be either prime or composite of two numbers
- … and that number holds the same
- As a result, we can keep moving backwards to show that everything must be whittled down to primes

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Circuits

Part 11

1

Circuits

Boolean Hardware

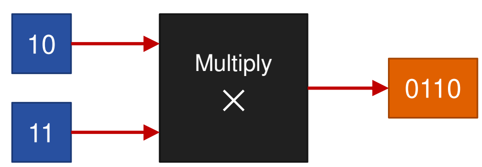
2

Circuits

- Boolean algebra gives designers tools to design & analyze complex solutions
- Can we create hardware that performs solutions?
- Can electronic circuits allow Boolean logic to exist in the physical world?

3

Two Bit Multiplier? Can we make it?



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Designing It

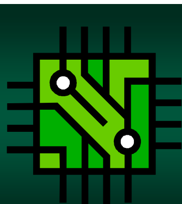
- To design a circuit that multiplies two 2-bit numbers, we can use Boolean algebra
- We need to figure the logic – given that bits of 1 and 0 will map directly to truth values
- The result of the algebra will be the desired output

5

It Takes the Following Skills

1. Design a truth-table to represent the different inputs and the desired output
2. Convert the truth-table into a Boolean function
3. Simplify the Boolean function
4. Finally, convert it into a circuit

6



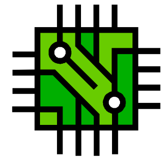
Gates

Boolean Hardware

7

Gates

- Electronic devices are made up of *gates*
- Gates take in two inputs and produce a single output
- This is how hardware is used to implement Boolean logic (or any logic)



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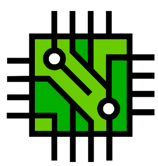
Gates

- Gates can be combined into circuits with *any number of input wires* and a single output wire
- We will chain these together much like subexpressions can be combined into larger expressions



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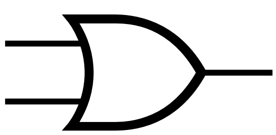
Graphical Representation

- Gates are typically represented using graphical shapes – much like flowcharts
- There are two different competing symbol standards
- We will use the standard, distinct, symbols rather than the IEC (European) ones

10

Or Gate



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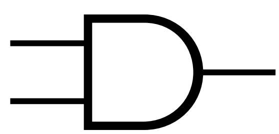
And Gate



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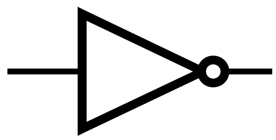
Not Gate



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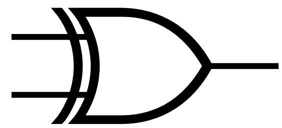
Exclusive Or Gate (aka XOR)



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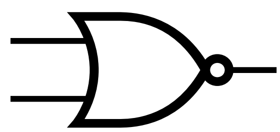
Some Other Gate Symbols

- There are also gate symbols for negated operators
- I won't use these much in class, but it's good to be aware of them (since they are quite common in computer engineering
- For each, note the circle on the output line – it means "not"

15

Not Or Gate (aka NOR)



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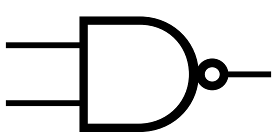
Not And Gate (aka NAND)



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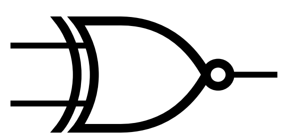
Not Exclusive Or Gate (aka XNOR)



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Computer Engineering Notation

Same thing, different paint job

19

Computer Engineering Notation

- Gates, used to computer engineering, form a Boolean Algebra
- i.e. they can define the three operations (and, or, not) and share the same axioms

20

Computer Engineering Notation

- The notation used in computer engineering is a *tad different* that what we have covered
- … and certainly different than any programming language you have used

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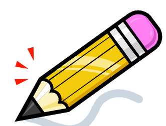
Computer Engineering Notation

- But is serves the same purpose
- And, not surprisingly, it works better for writing expressions in this discipline



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Computer Engineering Notation

1	≡	T
0	≡	F
*	≡	∧
+	≡	∨
'	≡	¬

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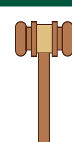
Commutative Law



$$x + y \equiv y + x$$

$$x * y \equiv y * x$$

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Identity Law

$$x * 1 \equiv x$$

$$x + 0 \equiv x$$

25

Complement Law

$$x * x' \equiv 0$$

$$x + x' \equiv 1$$

26

Distributive Law

$$x * (y + z) \equiv (x * y) + (x * z)$$

$$x + (y * z) \equiv (x + y) * (x + z)$$

27

Associative Law

$$(x + y) + z \equiv x + (y + z)$$

$$(x * y) * z \equiv x * (y * z)$$

28

Absorption Law

$$x * (x + y) \equiv x$$

$$x + (x * y) \equiv x$$

29

Idempotent Law

$$x * x \equiv x$$

$$x + x \equiv x$$

30

Involution Law

$x'' \equiv x$

31

Domination Law

$x + 1 \equiv 1$

$x * 0 \equiv 0$

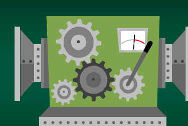
32

DeMorgan's Law

$(x * y)' \equiv x' + y'$

$(x + y)' \equiv x' * y'$

33



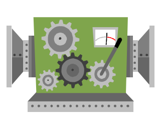
Converting Boolean to Circuits

From Logic to Wires

34

Converting Boolean to Circuits

- Converting from Boolean to circuits maintains a *one-to-one* correspondence between gates and operators in the equation
- But, given an *arbitrary* Boolean expression, how do we realize a circuit for it?



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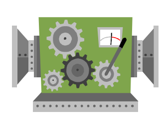
Steps

1. Choose the last operation evaluated
2. Draw a gate and hook up its output
3. Goto 1 until all operations have associated gates
4. Attach the expression inputs



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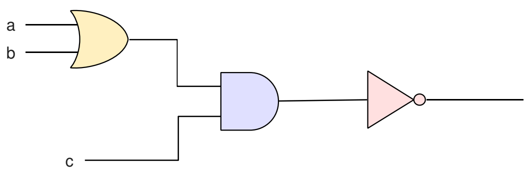
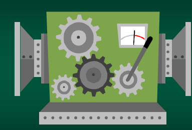
((a + b) * c)'



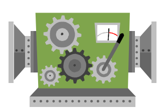


Converting Circuits to Boolean

From Logic to Wires

Converting Circuits to Boolean

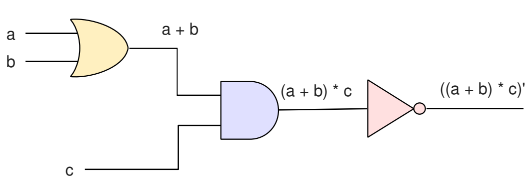
- The other direction is easy too
- Any circuit can be realized as a Boolean expression using the same basic algorithm



Converting Circuits to Boolean

1. Pick a wire that has known Boolean values
2. Write *on the wire* a Boolean expression for its value
3. Goto 1 until all wires are complete
4. Circuit's expression written on the circuit's output wire

Example Circuit

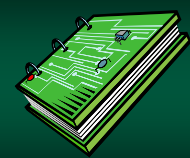


Arbitrary Circuits

Part 12

1

Creating an Arbitrary Circuit

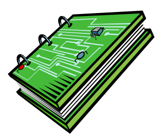


From Truth Table to Wires

2

Creating an Arbitrary Circuit

- We converted between Boolean expressions and circuits
- It maintained a one-to-one correspondence between gates in the circuit and operators in the equation



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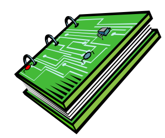
Creating an Arbitrary Circuit

- Given an arbitrary logic table, how do we realize a circuit for it?
- Simple, we look at the inputs that make it true, and write them out in an expression using or's.



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Example: 1 Bit Add Mod 2

5

Example: 1 Bit Add Mod 2

a	b	out
0	0	0
0	1	1
1	0	1
1	1	0

6

Example: 1 Bit Add Mod 2

```
We want a circuit that is true
when:

(a = F and b = T) or
(a = T and b = F)

out = a' * b + a * b'
```


7

Example 2: One Bit Adder

8

Example 2: One Bit Adder

a	b	Out $_1$	Out $_0$
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

9

Example: One Bit Adder (Logic)

```
out1 = (a = T and b = T)

out0 = (a = F and b = T) or
       (a = T and b = F)
```

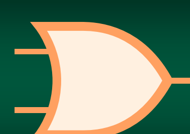

10

Example: One Bit Adder (algebra)

```
out1 = a * b

out0 = a' * b + a * b'
```


11



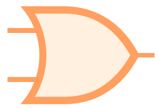
Disjunctive Normal Form

Express Logic With Ease

12

Disjunctive Normal Form

- Best approach to converting tables into circuits is use *Disjunctive Normal Form*
- In this form, the expressions consists of OR's (disjuncts) connecting AND sub-expressions



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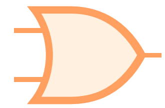
Definitions

- A *literal* is a Boolean variable *v* or its complement (e.g. v *or* v')
- A *minterm* of Boolean product $v_1 * v_2 * \ldots v_n$



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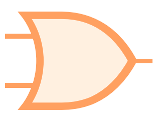
Definitions



- Hence, a minterm is a “product” of *n* literals, with one literal for each variable
- An equation written only as the "OR" of minterms is in *disjunctive normal form* (also called *sum-of-products* form)

15

Algorithm

1. Find the rows that indicates a 1 for output
   - ignore the ones with 0 as output
   - we are making an equation based on true
2. Write a minterm for each of them
3. "OR" all the minterms

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Example

a	b	y (out)
0	0	1
0	1	1
1	0	0
1	1	0

17

Example

```
DNF of the table is:

y = (a' * b') + (a' * b)

For brevity, for this point on, let's
write as:

y = a'b' + a'b
```


18

Example

```
We can simply using Boolean algebra:

y = a'b' + a'b
  = a' (b' + b)         Distributive
  = a' (1)              Complement
  = a'                  Identity
```


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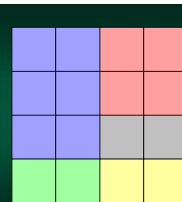

Karnaugh Maps

The Right-Brain Gets to Help

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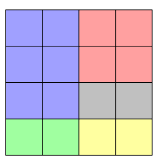
Karnaugh Maps

- A *Karnaugh Map* (pronounced "car-no") is a visual tool to help see relations between minterms.
- A K-Map for *n* variables is a grid of 2^n squares



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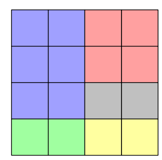
Karnaugh Maps

- Every possible minterm of *n* variables is represented
- *Every square is a minterm*
- It is arranged is such a way that we can simplify our table



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Gray Code

- Literals are ordered using *gray code*
  - values in the table are not ordered in normal ascending order
  - each square differs in exactly one literal
  - why? we will cover this later
- NOTE: squares wrap-around to the sides

23

Two Variable Example

a	b	out
0	0	1
0	1	1
1	0	1
1	1	0

24

Two Variable K-Map

		A: 0	A: 1
B	0	1	1
	1	1	0

Each combination of A

Squares have the output of the circuit

25

Three Variable Example

a	b	c	out
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

26

Three Variable K-Map

		AB: 00	01	11	10
C	0	1	0	1	1
	1	0	0	1	1

27

Four Variable K-Map

		AB: 00	01	11	10
CD	00	0	0	1	1
	01	1	1	0	0
	11	1	1	1	0
	10	0	1	1	0

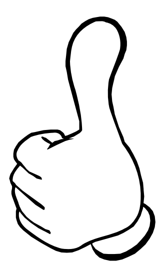
28

How to Use a K-Map

1. Mark the squares of a K-map corresponding to the function
2. Select a minimal set of rectangles where
   - each rectangle has a power-of-two area and is as large as possible
   - cover every marked square
3. Translate each rectangle into a single midterm and sum (or) all these

29

Converting a Rectangle to Minterm



- If any literal contains both 1 and 0, in the rectangle, it is eliminated
- The goal is to draw the **biggest** rectangles possible

30

Example Square: 1×1

CD \ AB	00	01	11	10
00	1	0	0	1
01	1	1	1	1
11	1	1	1	1
10	1	0	0	1

A' B C' D

31

Example Square: 2×1

CD \ AB	00	01	11	10
00	1	0	0	1
01	1	1	1	1
11	1	1	1	1
10	1	0	0	1

B C' D

32

Example Square: 1×4

CD \ AB	00	01	11	10
00	1	0	0	1
01	1	1	1	1
11	1	1	1	1
10	1	0	0	1

C' D

33

Example Square: 2×2

CD \ AB	00	01	11	10
00	1	0	0	1
01	1	1	1	1
11	1	1	1	1
10	1	0	0	1

A' D

34

Example Square: 2×2: Wrapped

CD \ AB	00	01	11	10
00	1	0	0	1
01	1	1	1	1
11	1	1	1	1
10	1	0	0	1

B' D

35

Example Square: 2×2: Wrapped

CD \ AB	00	01	11	10
00	1	0	0	1
01	1	1	1	1
11	1	1	0	1
10	1	0	0	1

B' D'

36

Example Square: 4×2

CD \ AB	00	01	11	10
00	1	0	0	1
01	1	1	1	1
11	1	1	1	1
10	1	0	0	1

D

37

Tips

- There is no magic way to do Step 2. Look and play around until you find the answer
- You can overlap squares – just as long as you "cover" all the 1's

38

Four-Variable K-Map

CD \ AB	00	01	11	10
00	0	0	1	1
01	1	1	0	0
11	1	1	1	0
10	0	1	1	0

A'D + BC + AC'D'

39

Efficiency of K-Maps

- A K-Map does not necessarily make the *best* expression/circuit
- All expressions made this way are sums-of-products and some can be made simpler
- For example: a(b+c) is the same as ab+ac, but uses fewer gate inputs

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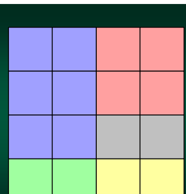


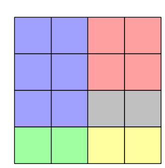
How K-Maps Work

You are doing more than you think

41

How K-Maps Work

- The order of gray code, and the 2^n squares allow us to factor out literals
- Every time you eliminate a literal, you are performing **three** Boolean algebra laws
- This is done visually, so it is invisible!



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How K-Maps Work

1. First you use the *Distribution Law* on the minterms leaving **(v + v')** - which is the terminal that *changed*
2. You then use the *Complement Law* on **(v + v')** leaving 1
3. Finally, you remove the 1 using the *Identity Law*

43

Let's Look at This Again...

CD \ AB	00	01	11	10
00	0	0	1	1
01	1	1	0	0
11	1	1	1	0
10	0	1	1	0

ABC'D' + AB'C'D'

B changes

44

Let's Look at This Again...

```
ABC'D' + AB'C'D'

AC'D'(B + B')      Distributive

AC'D'(1)           Complement

AC'D               Identity
```


45

How About Another Rectangle?

CD \ AB	00	01	11	10
00	0	0	1	1
01	1	1	0	0
11	1	1	1	0
10	0	1	1	0

Both A and D change

A'BCD + ABCD +
A'BCD' + ABCD'

46

How About Another Rectangle?

```
A'BCD  + ABCD + A'BCD' + ABCD'

BCD(A' + A) + BCD'(A' + A)

BCD(1) + BCD'(1)

BCD + BCD'
```

A eliminated

47

… and it keeps going…

```
BCD + BCD'

BC(D + D')

BC(1)

BC
```

D eliminated

48

Please Wait

CSC 28
will begin shortly

Please open the chat window.

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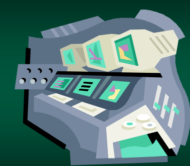

K-Maps and Programming

Using it to simplify code

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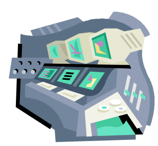
K-Maps and Programming

- The Boolean expressions, that you use in your Java programs, are the same as the expressions we cover
- So, you can apply K-Maps to your Java code to simplify expressions



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K-Maps Can Simplify Expressions

- The following is a complex expression that, on the surface, looks difficult to simplify
- K-Maps can help

```
if (a && !b && c  || a && b && !c ||
    a && !b && !c || a && b && c)
```


52

K-Maps Can Simplify Expressions

- First, let's put the expression in the Computer Engineer notation
- Ah, we can see the structure now!

```
ab'c' + abc' + ab'c + abc
```


53

K-Maps Can Simplify Expressions

ab

c	00	01	11	10
0				
1				

ab'c' + abc' + ab'c + abc

54

K-Maps Can Simplify Expressions

ab

c \ ab	00	01	11	10
0				1
1				

ab'c' + abc' + ab'c + abc

55

K-Maps Can Simplify Expressions

ab

c \ ab	00	01	11	10
0			1	1
1				

ab'c' + abc' + ab'c + abc

56

K-Maps Can Simplify Expressions

ab

c \ ab	00	01	11	10
0			1	1
1				1

ab'c' + abc' + ab'c + abc

57

K-Maps Can Simplify Expressions

ab

c \ ab	00	01	11	10
0			1	1
1			1	1

ab'c' + abc' + ab'c + abc

58

K-Maps Can Simplify Expressions

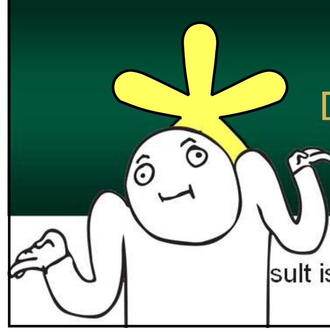
ab

c \ ab	00	01	11	10
0			1	1
1			1	1

ab'c' + abc' + ab'c + abc = a

59

Don't Care

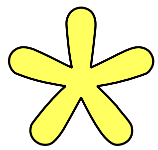


sult is Meaningless

60

Don't Care

- Sometimes *we don't really care* what output the circuit generates for some combinations of inputs
- So, for those inputs, the results are simply not significant



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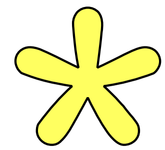
Don't Care

- In truth tables, the value "Don't Care" is represented with an asterisk
- It can be considered True or False – whichever is more *convenient* for the circuit



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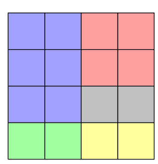
Karnaugh Maps and Don't Care

- We can construct a Karnaugh Map like before
- Except the squares corresponding to don't care outputs are marked (with an asterisk)



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Karnaugh Maps and Don't Care

- Then, when outlining blocks, we can (at our convenience) consider the "don't care" squares as either 0 or 1
- Since we want to make the largest outlines possible, we will sometimes consider a don't care to be true, and sometimes false

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Example

- We want to guarantee that the output of a circuit is 1 if both inputs are 1
- And 0 when both inputs are 0
- But otherwise we do not care

65

Example

a	b	out
0	0	0
0	1	*
1	0	*
1	1	1

66

K-Map For The Example

	A = 0	A = 1
B = 0	0	*
B = 1	*	1

We don't need B!

out = A

67

… or we can do this

	A = 0	A = 1
B = 0	0	*
B = 1	*	1

Just B!

out = B

68

Four-Variable (with Don't Care)

CD \ AB	00	01	11	10
00	0	1	1	*
01	1	1	*	0
11	1	1	1	0
10	*	1	1	0

out = B + A'D

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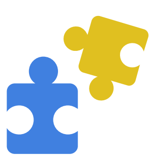
Functional Completeness

Just How Much Do We Need?

70

Functional Completeness

- We can construct a circuit for any Boolean expression using and / or / not
- This means the set of gates {and, or, not} is *functionally complete*



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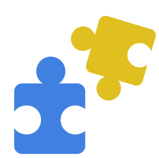
Function Completeness

- However, we don't need all three gates
- DeMorgan's laws shows us that we can construct:
  - an OR using an AND
  - and AND using an OR



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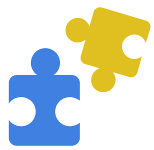
We Don't Need Or!

- So {and, not} are also complete because by DeMorgan's Law: $x + y = (x'y')'$
- So, any expression that can be written using {and, or, not} can be written using just {and, not}



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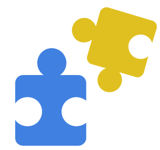
or… We Don't Need And!

- Also {or, not} is functionally complete since $xy = (x'+y')'$
- So, any expression that can be written using {and, or, not} can be written using just {or, not}



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Functional Completeness

- So, are any of the singular sets {and}, {or}, {not} functionally complete?
- In other words, can and/or/not all be converted into a single type of gate?
- **No.** Neither {and} or {or} can be converted to a {not}

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NAND

- So, is there a gate that can, alone, be functional complete?
- What about NAND (negated And)?
  - x nand y = $(xy)'$
  - Note: the NAND gate is not implemented with an AND gate and a NOT gate. It just has the same truth table as $(xy)'$

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NAND

- To show that {nand} is functionally complete, we need to show that we can implement {and, or, not} using it
- The result would be greatly beneficial!
  - we would have to just construct 1 gate to create any circuit
  - this would greatly aid construction

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Not → Nand

```
Converting not to nand:

x'  =  x'
    =  (xx)'          Idempotent
    =  x nand x       nand format
```

We can implement NOT by using a NAND. Both input will be x

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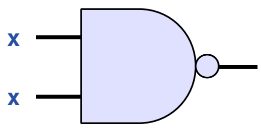
Not → Nand



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Or → Nand

```
Note: x' = x nand x

x + y =  x + y
      =  (x'y')'                DeMorgan
      =  x' nand y'             nand format
      =  (x nand x) nand (y nand y)
```

Last proof let us convert NOT into NAND

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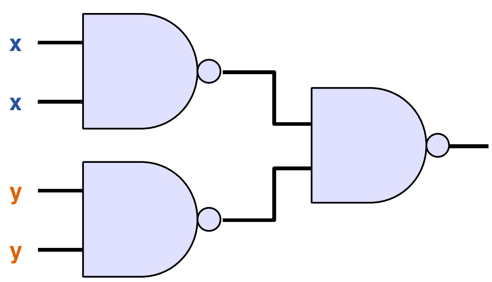
Or → Nand



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And → Nand

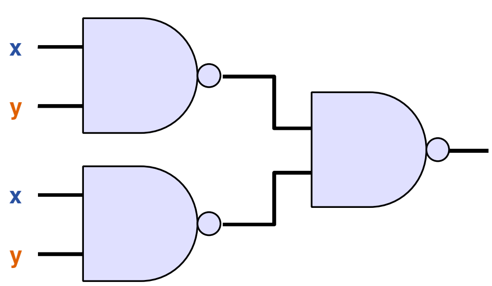
```
Note: x' = x nand x

xy    =  xy
      =  ( (xy)')'              Involution
      =  (x nand y)'            Negate nand
      =  (x nand y) nand (x nand y)
```

Last proof let us convert NOT into NAND

82

And → Nand



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Summary

- The expressions below show that nand can be used to implement NOT, OR, AND
- So, we can just use NAND since it is *functionally complete*

```
x'    =  x nand x
xy    =  (x nand y) nand (x nand y)
x + y =  (x nand x) nand (y nand y)
```


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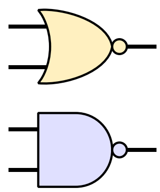
How Hardware Works

- Also NOR is functionally complete
- P NOR Q = (P + Q)'
- Hardware can alternatively use this gate rather than NAND



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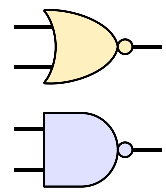
How Hardware Works

- If our hardware can just implement NAND or NOR, then we can create a circuit with just <u>one gate</u>
- In fact, many fabrication processes use only NAND or NOR gates



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