Set Attributes

Part 2

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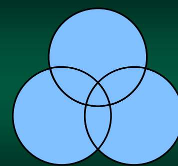

Fundamental Products

How Many Subsets Are There?

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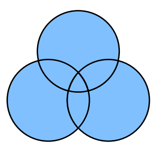
Fundamental Products

- *Fundamental Product* is an intersection of each set *(or it's complement)*
- They reveal <u>all</u> the base subsets of interest
- …since, each fundamental product is unique



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Fundamental Products

For each set $S_{1..n}$ in the universe, each product, P, is defined:

$$P = A_1 \cap A_2 \cap A_3 \cap A_3 \cap \ldots \cap A_n$$

where A_i is the set S_i <u>or</u> S'_i

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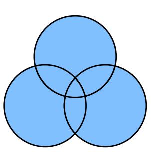
Some Attributes



- There are few properties that can be observed from fundamental products
- These will be important in other areas of discrete mathematics

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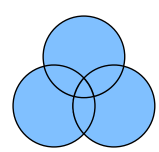
Three Major Attributes

1. There are $m = 2^n$ such fundamental products
2. Any two such fundamental products are disjoint
3. The universal set U is the union of all fundamental products



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#1. Number of Products

- Number of fundamental products *m* grows exponentially in relation to the number of sets *n*
- Observe: this is beginning to look "binary"

$$m = 2^n$$

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With 1 Set

- For a Universe with a single set, A, it results in 2^1 products
- Namely A and A'

$P_1 = A$
$P_2 = A'$

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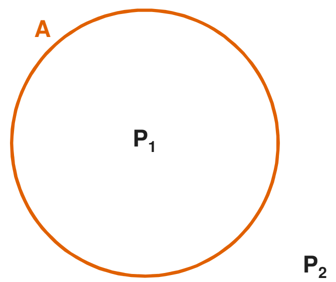

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With 2 Sets

- With two sets, A and B, there are a total of $2^2 = 4$ products

$P_1 = A \cap B$
$P_2 = A \cap B'$
$P_3 = A' \cap B$
$P_4 = A' \cap B'$

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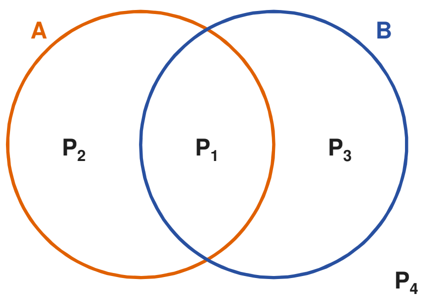

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With 3 Sets: $2^3 = 8$

$P_1 = A \cap B \cap C$
$P_2 = A \cap B \cap C'$
$P_3 = A \cap B' \cap C$
$P_4 = A \cap B' \cap C'$
$P_5 = A' \cap B \cap C$
$P_6 = A' \cap B \cap C'$
$P_7 = A' \cap B' \cap C$
$P_8 = A' \cap B' \cap C'$

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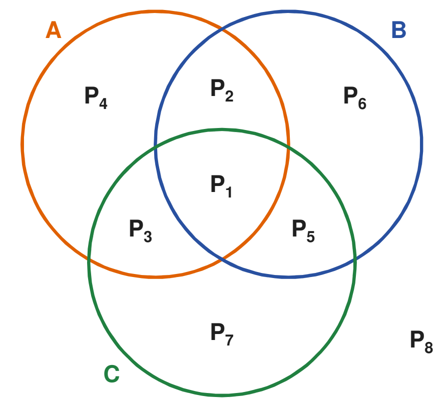

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#2. All Products are Disjoint

- Any two different fundamental products are disjoint
- Which means, they have no elements in common

$$P_i \cap P_j = \varnothing \quad \text{when } i \neq j$$

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#2. All Products are Disjoint

Let's try it

- We can use set algebra to show *fundamental products* can be unioned into the original sets

$$P_i \cap P_j = \varnothing \quad \text{when } i \neq j$$

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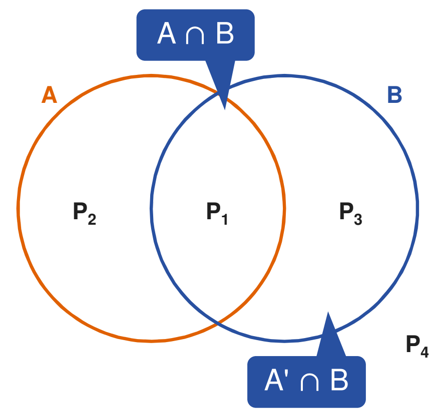

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#3. Union of all products is U

- The union of all fundamental products is the universe set U
- This should be fairly obvious from what we observed

$$U = P_1 \cup P_2 \cup P_3 \cup P_3 \cup \ldots P_n$$

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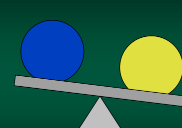

Cardinality

Counting Sets

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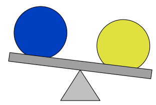
Cardinality of a Set

- The *cardinality* of a set is the number of *distinct* elements
- This information is used in counting – the classification of the set's contents



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Different Notations Used

- There are two different notations used
- The most common is the | pipe delimiters
- Alternatively, the "n" function is used

$$|A| \equiv n(A)$$

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Examples

```
A = {1, 3, 5, 7}

|A| = 4
```


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Examples

Duplicates don't count

```
B = {1, 2, 3, 3, 3, 4}

|B| = 4
```


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Counting

- If the set contains a finite number of elements, it is said to be *countable* – i.e. the cardinality is knowable
- If the set is infinitely large, but the elements can be uniquely identified, then it is *countably infinite*
- Otherwise it is said to be *uncountable*

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Countable Examples

Set	Result
$\{ x \mid x \in N \text{ and } x \le 100\}$	Countable
$\{ 2x \mid x \in N \}$	Countably Infinite
$\{ x \mid x \in R \text{ and } 0 < x < 1\}$	Uncountable

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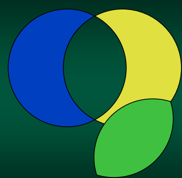

Inclusion-Exclusion

Counting by Subtracting!

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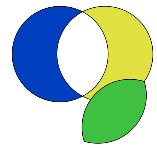
Inclusion-Exclusion

- Sets can overlap – and can contain the same elements
- So, when counting items in sets, you must be careful not to count an item twice
- *Inclusion-exclusion* principle, can get the correct count



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Reasoning with Venn Diagrams

Say 100 people receive a questionnaire with two questions:

1. Do you watch The Orville
2. Do you watch Rick & Morty?

60 said 'yes' to The Orville.
35 said 'yes' to **both**.
How many people watch Rick & Morty?

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Disjoint Set Cardinality

- If sets A and B are disjoint then they have no elements in common
- Cardinality of the union is the sum of the cardinality of both A and B

$$|A \cup B| = |A| + |B|$$

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Set Exclusion

- If sets A and B overlap they have elements in common
- We cannot simply add $|A| + |B|$
- Why? $|A| + |B|$ counts the intersection twice!

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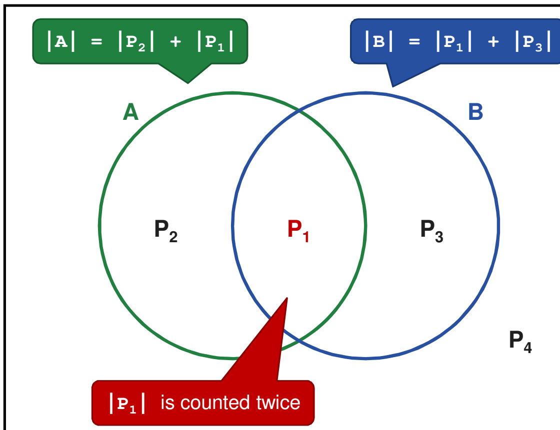

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Set Exclusion

- So, we need to remove the duplicate count
- The cardinality of the union is the sum of A and B excluding the intersection

```
|A ∪ B| = |A| + |B| – |A ∩ B|
```


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Set Exclusion

- Note: this is the same equation for disjoint sets
- If disjoint, the intersection is ∅
- So, this formula works in all cases

```
|A ∪ B| = |A| + |B| – |A ∩ B|
```


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Revisit that Question

Say 100 people receive a questionnaire with two questions:

1. Do you watch The Orville
2. Do you watch Rick & Morty?

60 said 'yes' to The Orville.
35 said 'yes' to both.
How many people watch Rick & Morty?

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Using the Formula…

- Union of The Orville (T) and Rick & Morty (R) contains 100
- The Orville set contains 60
- The intersection contains 35

```
|T ∪ R| = |T| + |R| – |T ∩ R|
```


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Using the Formula…

- Union of The Orville (T) and Rick & Morty (R) contains 100
- The Orville set contains 60
- The intersection contains 35

```
100 = 60 + |R| – 35
```


35

Using the Formula…

- Union of The Orville (T) and Rick & Morty (R) contains 100
- The Orville set contains 60
- The intersection contains 35

```
|R| = 100 – 60 + 35
    = 75
```


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Power Series

All the combinations

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Power Series

- A *power set* of a set S is a set of all the subsets of S
- This also, obviously, contains the null set
- The notation for the power set S is *P*(S)

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Power Set Example

```
G = {a, b}

P(G) = { ø, {a}, {b}, {a,b} }
```


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Power Set Example 2

```
H = {a, b, c}

P(H) = { ø, {a}, {b}, {c},
         {a,b}, {a,c}, {b,c}
         {a,b,c} }
```


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Power Set Example 3

```
I = {a, b, c, d}

P(I) = {  ø,
          {a}, {b}, {c}, {d}
          {a,b}, {a,c}, {a,d},
          {b,c}, {b,d}, {c,d},
          {a,b,c}, {a,b,d}, {b,c,d}, {a,c,d}
          {a,b,c,d} }
```


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Cardinality of Power Sets

- What is the cardinality of power sets?
- Is it possible to determine | P(S) | if we know | S |
- This will be important later…

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Let's Look at the Examples

```
G = {a, b}

P(G) = { ø, {a}, {b}, {a,b} }

|G|    = 2
|P(G)| = 4
```


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Power Set Example 2

```
H = {a, b, c}

P(H) = {  ø, {a}, {b}, {c},
          {a,b}, {a,c}, {b,c}
          {a,b,c} }

|H|    = 3
|P(H)| = 8
```


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Power Set Example 3

```
I = {a, b, c, d}

P(I) = {  ø, {a}, {b}, {c}, {d}
          {a,b}, {a,c}, {a,d},
          {b,c}, {b,d}, {c,d},
          {a,b,c}, {a,b,d}, {b,c,d}, {a,c,d}
          {a,b,c,d} }

|I|    = 4
|P(I)| = 16
```


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Cardinality of Power Set

- The cardinality of a power set is 2^n where n is the cardinality of the original set
- This is used in statistics… covered later

$$|P(S)| = 2^{|S|}$$

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Partitions

Cutting a Set Into Pieces

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Partitions

- A *partition* of a set A is a collection of non-empty disjoint sets whose union is A
- So, it is like the set A was "chopped", cleanly, into subsets

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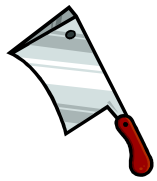
Requirements

- Each subset must be mutually exclusive
- … unless they are identical *(because duplicates don't count in sets)*



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Partition Example

- The following is a valid partition of the set {1, 2, 3, … 9}

```
{ {1}, {2,3,5,7}, {4,6}, {8,9} }
```


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Partition Example

- The following is a partition of N.

```
N = { {1}, {2,3}, {4,5,6}, ... }
```


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Partition Examples

For the set {1, 2, 3, 4}…

Set	Partition?
`{ {1}, {2}, {3}, {4} }`	**Yes**
`{ {1,2}, {1,2}, {3,4} }`	**Yes.** {1,2} is duplicate
`{ {1,2,3}, {2,4} }`	**No.**

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