Please Wait

CSC 28
will begin shortly

(open the chat window)

1

Basic Proof

Part 8

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Theorems

The Big Bang Theory

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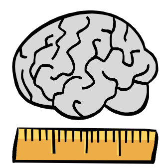
Theorems

- A *theorem* is a statement we intend to prove using existing known facts (called *axioms* or *lemmas*)
- Used extensively in all mathematical proofs – which should be obvious



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Example

- Most theorems are of the form: If A, then B
- The theorem below is very easy to interpret

If a and b are even integers
then a × b is an even integer

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Example

- Theorems are arguments
- They can be structured as such

```
a is even
b is even
_____________
a × b is even
```


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Example

- Sometimes it is hard to see
- Below, the <u>same</u> theorem is written using different language

x and y are even integers and the product is even.

The product xy is even when x and y are both even.

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Some Basic Definitions

Abstract? Not really.

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Definition: x is even

a is even if and only if…

a = 2n for some integer n

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Definition: x is odd

a is odd if and only if…

a = 2n + 1 for some integer n

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Definition: x | y (x divides y)

a | b (a divides b) iff…

b = k × a for some integer k

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Definition: x ∈ Q (x is a rational)

a is a rational number iff…

a = b / c for some integers b and c, and c ≠ 0

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Proving $A \rightarrow B$

Modus Ponens

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Proving $A \rightarrow B$

- The Boolean operator $A \rightarrow B$ is true except when A is true and B is false
- So, we can logically prove that $A \rightarrow B$ by showing that *whenever A is true, B **must** also be true*
- *… so $T \rightarrow F$ isn't possible*

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Proving $A \rightarrow B$

- This is essentially a *Modus Ponens* proof
- You are showing that if A is true, and $A \rightarrow B$ is true, then B **must** be true
- Also note that "A" and "B" can be compound statements

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Modus Ponens

$A \rightarrow B$ — Prove this is always true

A — If this is true

B

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The Steps

- There are basically just two steps to follow:
  1. Assume A is true
  2. Show that B must be true
- This shows that B is true whenever A is true

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Example 1

- Let's prove the following theorem from before
- This is actually quite easy

If a and b are even integers
then $a \times b$ is an even integer

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Example 1 – the argument

```
a is even
b is even
____________
a × b is even
```


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Example 1 – internal structure

- Remember: all proofs are implications
- So, we will assume the both premises are true and show the conclusion must be true

```
a is even ∧ b is even → a × b is even
```


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Example 1

```
Assume that x and y are even integers.

So, by the definition…

a = 2i and b = 2j (for some i, j)

Note: use different arbitrary variables
or you are assuming they are equal!
```


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Example 1 - Proof

```
So, the product is:

a × b = 2i × 2j
      = 4 × i × j
      = 2 × (2 × i × j)

So, by definition, a × b is even
```


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Example 2

- The following is a theorem about the product of an odd and even number
- The proof is straight-forward using the definitions

If a is even and b is odd, then a × b is even

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Example 2

```
Assume:
a is an even integer and
b is an odd integer.

Then a = 2i and b = 2j+1 for some
integers i and j
```


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Example 2

```
Multiplying, we get:

a × b = (2i) × (2j + 1)
      = 4ij + 2i
      = 2 × (2ij + i)

...which is even
```


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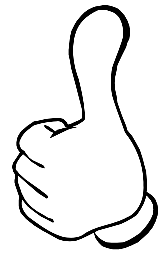
Proof Tips

- Begin your proof with what you assume to be true (the hypothesis)
- Don't argue the truth of a theorem *by example*
  - stay abstract
  - e.g. you know x and y are even integers – that's all you know



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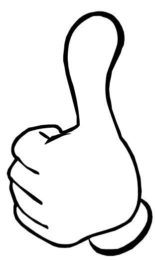
Proof Tips

- Your proof must work forward from your assumptions to your goal
- You may work backward on scratch paper to help you figure out how to work forward



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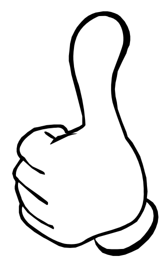
Proof Tips

- Write your proof in prose
- Then read out loud, it should sound like well-written paragraph
- Quite often you'll use definitions to make progress



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Proof by Contrapositive

Proof with inverse logic

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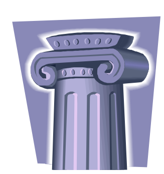
Proof by Contrapositive

- There are several techniques that can be employed to prove an theorem
- The direct approach, like before, is quite common, but its not the only path you can take



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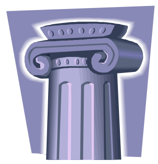
Proof by Contrapositive

- *Proof by Contrapositive* has you prove the opposite of the original theorem
- Quite impressively, this will also prove the original theorem



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Getting the Contrapositive

- First…
  - negate both the assertion and conclusion of the implication
  - so, basically, put "not" in front of both operands
- Second…
  - reverse the implication
  - you basically swap the left-hand and right-hand operand of the implication

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Getting the Contrapositive

- So, both operands swap positions and are negated
- Are they equal? Let's confirm in a Truth Table

```
for p → q
contrapositive is ¬q → ¬p
```


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Contrapositive Truth Table

p	q	¬ q	¬ p	p → q	¬ q → ¬ p
T	T	F	F	T	T
T	F	T	F	F	F
F	T	F	T	T	T
F	F	T	T	T	T

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Modus Ponens Contrapositive

Prove always true

```
¬ B → ¬ A
¬ B
_________
¬ A
```

If this is true (i.e. B is false)

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How it Works

- So, if we prove the contrapositive, we also prove the original theorem
- For the original A → B
  - suppose that if B is false
  - show that A must be false
- It does make sense, if you think about it

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Example

- The following theorem should look familiar
- This theorem states that the square of a odd number is also odd
- *Direct proof is near impossible!*

If x^2 is odd then x is odd

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Example Contrapositive

- The contrapositive negates each operand in the implication A → B
- The following shows the reverse of each

A = x^2 is odd

¬A = x^2 is not odd = x^2 is even

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Example Contrapositive

- The contrapositive negates each operand in the implication A → B
- The following shows the reverse of each

B = x is odd

¬B = x is not odd = x is even

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Example Contrapositive

- Finally, we reconstruct our theory with B → A rather than A → B
- This expression is equivalent to the original

if x is not odd then x^2 is not odd

or rewritten as…

if x is even then x^2 is even

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Example Contrapositive

We assume x is not odd

x is not odd means x is even

x = 2k for some integer k

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Example Contrapositive

We assume x is not odd (even)

$$x^2 = (2k)^2 = 4k^2 = 2(2k^2)$$

So, x^2 is even which is *not odd*

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Example Result

We proved:

if x is even then x^2 is even

By contrapositive, we proved:

if x^2 is odd then x is odd

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Proof by Contradiction

Welcome down the rabbit hole

Proof by Contradiction

- *Proof by Contradiction* takes a novel approach
- It uses the approach of *reductio ad absurdum*
- So what is it? Well, it proves the theorem by showing it can't be false

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But, How?

- Assume it is false
- Show that (if it is false) something impossible results
- Therefore, it can't be false and, thus, true!

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Proving $A \rightarrow B$

- Argue: $A \land \neg B$
  …which is $\neg(A \rightarrow B)$
- Show that something *impossible* results!
- Since $A \rightarrow B$ cannot be false, it **must** be true

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Contradiction

A	B	¬ B	A ∧ ¬B	¬ (A → B)
T	T	F	F	F
T	F	T	T	T
F	T	F	F	F
F	F	T	F	F

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Example

- The following is a classic Proof By Contradiction
- The theorem covers if the square-root of 2, is an irrational number

The square root of 2 is irrational

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Example

- To prove by contradiction, we need to show that the opposite cannot be true (i.e. false)
- The sentence bellow is the theorem negated
- So, how do we go about proving this?

The square root of 2 *is rational*

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Example

- Well, what is a rational number?
- A rational number can be expressed as "a / b" where a and b are *integers with no common factors* (aka "lowest terms")

The square root of 2 *is rational*

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Examples of Rational Numbers

- **1 / 3**
- **7 / 1**
- **22 / 7**
- **7734 / 10001**
- **1 / 123456789**
- **5 / 3**

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The Proof: Prove Opposite

$\sqrt{2}$ is rational

$\sqrt{2} = a / b$ where $a, b \in \mathbb{Z}$
and $b \neq 0$
and a, b have no common factors

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Analyzing a

$\sqrt{2} = a / b$

Let's look at the properties of a

$\sqrt{2} \times b = a$

$2 \times b^2 = a^2$

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Analyzing a – It is even

$2 \times b^2 = a^2$

So… a^2 is an even number

therefore, we know a is also even (previous proof – even × even)

Analyzing b

Since a is even and a / b is in lowest terms, then b must be odd

Why? If b is even, then a / b would have common factors – namely 2.

Example: Oh ohhhhh

However… look again at $2 \times b^2 = a^2$

Since a is even, we can use the definition. So…

$2 \times b^2 = (2k)^2$
$= 4k^2$

Example: Oh ohhhhh

Solving for b^2…

$2 \times b^2 = 4k^2$
$b^2 = 2k^2$

Since b^2 is even, b is even

Result

Since b must be both odd and even, we have a contradiction

The theorem "square root of 2 is rational" cannot be true

Therefore, "square root of 2 is irrational" is true