Goal and idea - Module 14

GOAL:

In the last module, we learned how to solve *nonhomogeneous* linear differential equations with constant coefficients where the nonhomogeneous piece had a certain form. In this module we

- develop another technique for solving nonhomogeneous linear differential equations which solves some cases not covered in the previous module (but also does not cover some of those cases).

One advantage of this method is that it does not necessary require the differential equation to have constant coefficients.

IDEA:

We write the differential equation in standard form, utilize a formula for two new variables, and then integrate these.

Consider: $a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$

In particular before we'd have just constants. Where $a_2(x)y''$, $a_1(x)y'$, $a_0(x)y$, $g(x)$ are functions

The Method:

Step 1 First turn

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

↓ into ↓

$$y'' + P(x)y' + Q(x) = f(x)$$

The idea is to first put it in to this form

$f(x) = \frac{g(x)}{a_2(x)}$, $P(x) = \frac{a_1(x)}{a_2(x)}$ So you divide everything by whatever function is in y''

Step 2 Find our homogeneous peice ($Y_h = c_1y_1 + c_2y_2$), which is the general solution for $y'' + P(x)y' + Q(x)y = 0$

Step 3 Then $Y_p = u_1y_1 + u_2y_2$, depends on the homogeneous peice (Y_1 & Y_2) where u_1 and u_2 are found by integrating

$$u_1' = \frac{-y_2f(x)}{W} \quad \text{and} \quad u_2' = \frac{y_1f(x)}{W},$$

and $W(y_1, y_2)$ is the wronskian of y_1, y_2

$$Y_p = U_1y_1 + U_2y_2$$

Example Solve $y'' + y' = \csc(x)$

Here we have constant coeffcients, so that homogeneous peice we aren't that scared of

Soln

Step 2 Find our homogeneous peice ($Y_h = c_1y_1 + c_2y_2$), which is the general solution for $y'' + P(x)y' + Q(x)y = 0$

Here $y'' + y' = 0$, Plug-in $y = e^{mx}$:

$$m^2 + 1 = 0 \rightarrow m_1 = i,\ m_2 = -i,$$

Thus, $y_h = C_1\underbrace{\cos(x)}_{y_1} + C_2\underbrace{\sin(x)}_{y_2}$

Step 3 Then $Y_p = U_1y_1 + U_2y_2$, depends on the homogeneous peice (Y_1 & Y_2) where u_1 and u_2 are found by integrating

$u_1' = \frac{-y_2f(x)}{W}$ and $u_2' = \frac{y_1f(x)}{W}$, and $W(y_1, y_2)$ is the wronskian of y_1, y_2

Thus,

$$u_1' = \frac{-y_2f(x)}{W} = \frac{-\sin(x)\cdot\csc(x)}{1} = -\frac{\sin(x)}{\sin(x)} = -1$$

Therefore, when we solve we want the constant to be zero

$$u_1 = \int u_1' = -\int 1\,dx = -x \leftarrow \boxed{\text{No Plus C here}}$$

also,

$$u_2' = \frac{y_1f(x)}{W} = \frac{\cos(x)\cdot\csc(x)}{1} = \frac{\cos(x)}{\sin(x)},$$

and

[u-sub: $u = \sin(x)$] No+C

$$u_2 = \int u_2' = \int \frac{\cos(x)}{\sin(x)}dx = \int \frac{1}{u}dx = \ln|\sin(x)|$$

We have our $u_1 = -x$, $u_2 = \ln|\sin(x)|$ and $Y_1 = \cos(x)$, $Y_2 = \sin(x)$.

Thus, we are able to now find the particular solution

$$\begin{aligned} Y_p &= u_1Y_1 + u_2Y_2 \\ &= (-x)(\cos(x)) + (\ln|\sin(x)|)(\sin(x)) \\ &= -x\cos(x) + \sin(x)\ln|\sin(x)| \end{aligned}$$

Thus, our final answer: homogeneous + particular

$$\begin{aligned} y &= C_1\cos(x) + C_2\sin(x) + (-x\cos(x) + \sin(x)\ln|\sin(x)|) \\ &= C_1\cos(x) + C_2\sin(x) - x\cos(x) + \sin(x)\ln|\sin(x)| \end{aligned}$$

To get to our answer we have to NOT include those constant (+C) terms when we take the antiderivatives. Otherwise we'd have additional parameters floating around that we don't want.

This technique doesn't always work nicely because the antiderivative are sometimes difficult.

Solve $y'' + y' = \csc(x)$ The way we approach these in general and ask what is the $f(x)$. If it's a polynomial, if it's sine or cosine, or if it's e^x type stuff GREAT; then, maybe it's undetermined coefficents. If it's anything else then it's probably gonna have to be Variation of Parameters

Expectation checklist - Module 14

At the completion of this module, you should:

- Be able to find the general solution for some additional kinds of 2nd-order nonhomogeneous linear differential equations.

Coming up next, we:

- Solve one additional type of homogeneous linear differential equation, that does not have constant coefficients.