Goal and idea - Module 9

GOAL:

Our goal is to understand the nature of solutions. And in particular, to find some "basic" or "core" solutions which generate *all* other solutions.To do so we:

- Discuss the 'linear' properties of solutions;
- Define the "fundamental set of solutions" as well as what it means for solutions to be linearly independent;
- Define the Wronskian of functions;
- Develop way to use the Wronskian to determine if solutions for a differential equation are linearly independent; and
- Learn how to determine if a given set of functions is a fundamental set of solutions for a differential equation.

IDEA:

It ends up we can add any solutions together and again get a solution. We can also multiply a solution by any constant and get another solution. These are the critical properties of what is known as a vector space (if you've taken linear algebra!). Thus, we want to find a "basis" of this space. Finding such a "fundamental" set of solutions requires discussing the linear independence of functions, and for this we introduce and employ the Wronskian.

Warm-up We have $y_1(x)=e^x$ and $y_2(x)=e^{-x}$ are both solutions for $y''-y=0$. Can you show $2y_1+3y_2$ is a solution without taking derivatives?

soln Plug $2y_1+3y_2$ into the equation $y''-y=0$. If it equals zero it is a solution.

$$(2y_1+3y_2)''-(2y_1+3y_2)\stackrel{?}{=}0$$

Free clues

$$\underbrace{(2y_1+3y_2)''}-(2y_1+3y_2)$$

$$\Rightarrow (2y_1)''+(3y_2)''+(-2y_1)+(-3y_2)$$

$$\Rightarrow (2y_1)''+(3y_2)''+(-2y_1)+(-3y_2)$$

$$\Rightarrow 2(y_1)''+3(y_2)''+(-2y_1)+(-3y_2)$$

$$\Rightarrow 2(y_1)''+(-2y_1)+3(y_2)''+(-3y_2)$$

$$\Rightarrow [2(y_1)''-(2y_1)]+[3(y_2)''-(3y_2)]$$

$$\Rightarrow [2(y_1)''-(2y_1)]+[3(y_2)''-(3y_2)]=0+0=0$$

Derivatives, including the second derivative, using the "Linear Operator" which allows us to break-out sums and scalers

Practically differential equation $\boxed{y''-y}=0$ with y_1 plugged in, $\boxed{y''-y}=0$ with y_2 plugged in

Because $y_1(x)=e^x$ & $y_2(x)=e^{-x}$ are solutions, so when we plug-in y_1 and y_2.

Theorem Suppose $y_1,\dots,y_n$ are solutions for a homogeneous linear differential equation (on I), then $y=c_1y_1+\dots+c_ny_n$ is a solution for the differential equation (on I) for any constants $c_1,c_2,\dots,c_n$.

Note:

(a) If "y" is a solution, so $3y, 4y, 5y$, ect. In fact "cy" is a solution for any constant "c".

Good to Know

(b)* A homogeneous linear differential equation always has $y=0$ as a solution.

Ex

(i) Consider $y''+y=0$. Recall $y_1=\cos(t)$, $y_2=\sin(t)$ are solutions. Thus, so is $y=c_1\cos(t)+c_2\sin(t)$

(ii) We could have taken $y_1=\cos(t)$, $y_2=3\cos(t)$ as solutions.

Then $y=c_1\cos(t)+c_2\cdot 3\cos(t)=(c_1+3c_2)\cos(t)$

$=C\cos(t)$

→Less interesting...

Definition

Functions $f_1(x), \dots, f_n(x)$ are:

(a) Linearly Independent if the only constants $c_1, \dots, c_n$ such that $c_1 f_1(x) + \dots + c_n f_n(x) = 0$ are $c_1 = c_2 = \dots = c_n = 0$

(b) Linearly Dependent if they are not linearly independent.

* Note: (a) equivalent to notion that no "f_j" can be expressed as a linear combination of the other functions.

Ex

(a) $\sin(x), \cos(x)$

Linearly Independent

- No constants
  - $c_1 \cos(x) + c_2 \sin(x) = 0$

    $c_1 = 0 = c_2$

    - Only way to get zero for all "x" is both "c_1" and "c_2" have to be zero

(b) e^x, e^{-x}

Linearly Independent

(c) $x, \ln(x)$

Linearly Independent

(d) $x^2+2x+1, \ x+1, \ x$

Can you think of non-zero number that gives you zero.

$$\underline{1}(x^2+2x+1) + \underline{-1}(x+1) + \underline{-1}(x) = 0$$

You start to see you can't get rid of the x^2

$$\underline{1}(x^2+2x+1) + \underline{-1}(x+1) + \underline{-1}(x) = 0; \ x^2 \neq 0$$

Attemp 2

$$\underline{0}(x^2+2x+1) + \underline{1}(x+1) + \underline{-1}(x) = 0$$

But your still left with the one

$$\underline{0}(x^2+2x+1) + \underline{1}(x+1) + \underline{-1}(x) = 0; \ 1 \neq 0$$

Realization: No matter how you play around with the numbers the left side won't equal zero on the right.

Therefore (d) is Linearly Independent

(e) $x^2+2x+1, \ x^2+x+1, \ x$

Can you think of non-zero number that gives you zero.

$$\underline{1}(x^2+2x+1) + \underline{-1}(x^2+x+1) + \underline{-1}(x) = 0$$

Linearly Dependent because we found some numbers that we can plug-in not all of them being zero.

Summary

Introduce the idea of Linearly Independent & Dependent functions, but moving forward we care about solutions. It turns out that there is more information if you have solutions for differental equations and you want to figure out if they are Linearly Independent or Dependent you can skip definition (a).

Q: When are functions linearly independent?

$y = 1+x \qquad y = x \qquad y = 1+2x$

Linearly independent it MUST be $C_1 = C_2 = C_3 = 0$

Solution | Find C_1, C_2, C_3 such that

$$C_1(1+x) + C_2(x) + C_3(1+2) = 0$$

$\mapsto C_1 + C_1(x) + C_2(x) + C_3 + 2C_3(x) = 0$

$\mapsto x\underbrace{(C_1 + C_2 + C_3)}_{=0} + \underbrace{(C_1 + C_3)}_{=0} = 0$

$C_3 = 5$
$C_1 = -5$
$C_2 = -5$

System of equations:

$C_1 + C_2 + C_3 = 0$ and $C_1 + C_3 = 0$

$\mapsto -C_3 + C_2 + 2C_3 = 0$ $\qquad \mapsto C_1 = -C_3$

$\mapsto C_2 + C_3 = 0$

$\mapsto C_2 = -C_3$

Therefore

$C_1 = -C_3$ and $C_2 = -C_3$

Definition

Any set $y_1, \ldots, y_n$ of n-many linearly independent solutions for a homogeneous n^{th}-order linear DE $a_n(x)y^n + \ldots + a_1(x)y' + a_0(x)y = 0$ on an interval "I" is called a Fundamental Set of Solutions.

Note: Two Things

- →Number of solutions (equal to the order) of the differential equation.
- →The solutions need to be linearly independent.

Key Components to the definition *

(a) How many do you have?

(b) Are they linearly independent?

So if you have n-many, they are linearly independent, and the n-many matches the order then you have what is called a Fundamental Set of Solutions.

Ex $y_1 = \cos(t)$, $y_2 = \sin(t)$ is a fundamental set of solutions for $y'' + y = 0$. **Why?**

Because

→Seen/discussed they are linearly independent, and

→We have two solutions and 2^{nd}-order linear homogeneous differential equation

- Two solutions: $y_1 = \cos(t)$, $y_2 = \sin(t)$
- Number of solution matches the order
    - Second order linear homogeneous DE
        - $y'' + y = 0 \rightarrow \underbrace{y_1 = \cos(t)}_{\text{one}}, \underbrace{y_2 = \sin(t)}_{\text{two}}$

Question: Does there always exist a fundamental set of solutions?

Answer: Theorem – There exist a fundamental set of solutions for $a_n(x)y^{(n)} + \ldots + a_1(x)y' + a_0(x)y = 0$ on some interval "I"

- If it's homogeneous and it's linear; then, there exist a fundamental set of solutions on some interval "I"

Question: How do we know if solutions are linear independent?

Answer: Wronskian

Module 09 SUBJECT: # The Wronskian

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Definition

Suppose we can take the first n-1 many derivatives of functions $f_1(x), \ldots, f_n(x)$. The <u>Wronskian</u> of $f_1(x), \ldots, f_n(x)$ is the determinant

$$\underbrace{w(f_1,\ldots,f_n)}_{\text{To denote Wronskian}} = \det \underbrace{\begin{pmatrix} f_1 & f_2 & f_n \\ f_1' & f_2' & f_n' \\ f_1'' & f_2'' & f_n'' \\ \vdots & \vdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & f_n^{(n-1)} \end{pmatrix}}_{n\times n \text{ matrix}}$$

*Finds out if things are linearly independent.

Question: How does the Wronskian help?

Answer:

Theorem

Suppose $y_1, \ldots, y_n$ are solutions to a homogenous linear differential equation of n^{th}-order. Then $y_1, \ldots, y_n$ are linearly independent on an interval "I" if and only if: the Wronskian is not zero $[W(y_1, y_2, \ldots, y_n) \neq 0]$ for every "x" in "I"

- $W = 0$ anywhere on the interval it's linearly dependent
- $W \neq 0$ on the interval it's linearly independent

Case n=2:

$$w(f_1,f_2) = \det\begin{pmatrix} f_1 & f_2 \\ f_1' & f_2' \end{pmatrix}$$

$$\rightarrow \det\begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{pmatrix} f_1 & f_2 \\ f_1' & f_2' \end{pmatrix} \\ \begin{array}{cc} c & d \end{array} \end{array} = \det\begin{pmatrix} f_1 & f_2 \\ f_1' & f_2' \end{pmatrix}$$

$[a \cdot d] - [c \cdot b]$

$$\rightarrow \det\begin{pmatrix} f_1 & f_2 \\ f_1' & f_2' \end{pmatrix} = (f_1)(f_2') - (f_1')(f_2)$$

Case n=3:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Determinates of n=3+ take the first piece(a) and multiply by the determinate of what remains while alternating between addition and subtraction starting first with subtraction

$$w(f_1,f_2,f_3) = \det\begin{pmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{pmatrix}$$

$$\begin{pmatrix} f_1 & f_2 & f_3 \\ f_1' & \boxed{f_2' \ f_3'} \\ f_1'' & \boxed{f_2'' \ f_3''} \end{pmatrix} \quad \begin{pmatrix} f_1 & f_2 & f_3 \\ \boxed{f_1'} & f_2' & \boxed{f_3'} \\ \boxed{f_1''} & f_2'' & \boxed{f_3''} \end{pmatrix} \quad \begin{pmatrix} f_1 & f_2 & f_3 \\ \boxed{f_1' \ f_2'} & f_3' \\ \boxed{f_1'' \ f_2''} & f_3'' \end{pmatrix}$$

$$\rightarrow f_1 \det\begin{pmatrix} f_2' & f_3' \\ f_2'' & f_3'' \end{pmatrix} - f_2 \det\begin{pmatrix} f_1' & f_3' \\ f_1'' & f_3'' \end{pmatrix} + f_3 \det\begin{pmatrix} f_1' & f_2' \\ f_1'' & f_2'' \end{pmatrix}$$

$$\rightarrow f_1(f_2' \cdot f_3'' - f_2'' f_3') - f_2(f_1' \cdot f_3'' - f_1'' f_3') + f_3(f_1' \cdot f_2'' - f_1'' f_2')$$

$$f_1(f_2' \cdot f_3'' - f_2'' f_3') - f_2(f_1' \cdot f_3'' - f_1'' f_3') + f_3(f_1' \cdot f_2'' - f_1'' f_2')$$

Example

(a) $y_1 = e^x, \ y_2 = e^{-x}$

$$w(e^x, e^{-x}) = \det\begin{pmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{pmatrix}$$

$$= -e^x e^{-x} - e^x e^{-x}$$

$$= -1 - 1 = -2$$

$\neq 0$ on $(-\infty, \infty)$

$\therefore$ Linearly Independent on $(-\infty, \infty)$

(b) $\det\begin{pmatrix} x & 3x \\ 1 & 3 \end{pmatrix}$

$$\det\begin{pmatrix} x & 3x \\ 1 & 3 \end{pmatrix} = (x)(3) - (1)(3x)$$

$$= 3x - 3x$$

$$= 0$$

zero everywhere

$\therefore$ Linearly dependent

Notes: In terms of calculating Wronskain
→ I need to know the n=2 case
→ Prof would give n=3 case on exams

Definition Fundamental Set of Solutions

Any set $y_1, \dots, y_n$ of n-many linearly independent solutions for a homogeneous n^{th}-order linear DE $a_n(x)y^n + \dots + a_1(x)y' + a_0(x)y = 0$ on an interval "I" is called a Fundamental Set of Solutions.

Note: Two Things

→ Number of solutions (equal to the order) of the differential equation.
→ The solutions need to be linearly independent.

Question: How do we know if solutions are linear independent?
Answer: Wronskian

Recall

Warning this theorem is FALSE for regular functions

Question: How does the Wronskian help?
Answer:

Theorem

Suppose $y_1, \dots, y_n$ are solutions to a homogenous linear differential equation of n^{th}-order. Then $y_1, \dots, y_n$ are linearly independent on an interval "I" if and only if: the Wronskian is not zero $[W(y_1, y_2, \dots, y_n) \neq 0]$ for every "x" in "I"

- $W = 0$ anywhere on the interval it's
- $W \neq 0$ on the interval it's linearly independent

What is TRUE is that if the Wronskian is NOT ZERO for some "x" then they are linearly dependent, but if the Wronskian is ZERO it's INCONCLUSIVE.

Example

Recall $y'' + y = 0$. Note $y_1 = \cos(t), y_2 = \sin(t)$ are solutions.

How many solutions?

We need two because of the second order, and we have two $y_1 = \cos(t), y_2 = \sin(t)$

Are the solutions linearly independent?

We need the wronskian

$$w(\cos(t), \sin(t)) = \det\left(\begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}\right)$$

$\to \cos(t)\cos(t) - (-\sin(t))\sin(t)$

$\to \cos^2(t) + \sin^2(t) = 1$

$\cos^2(t) + \sin^2(t) = 1$ for any "t" the equation equals one and inparticular it's NEVER zero on a huge interval no matter what you plug-in it's never zero

$\neq 0$ on $(-\infty, \infty)$

∴ The solutions are linearly independent

So y_1, y_2 fundamental set of solutions

Thus!!! $\boxed{y = c_1 \cos(t) + c_2 \sin(t)}$ is a general solution.

Q: How does the theom of the Wronskian help us moving forward?

A: THEOREM

If $y_1, \dots, y_n$ form a fundamental set of solutions for an n^{th}-order homogenous linear differential equation on "I", then $y = c_1y_1 + c_2y_2 + \dots + c_ny_n$ for arbitrary constants $c_1, c_2, \dots, c_n$.

Summary

If we have n-many solutions $(y_1, \dots, y_n)$ and they are linearly independent (fundamental set of solutions) if we have that for an n^{th}-order homogenous linear differential equation (the wronskian, linear independence, and eet.).

All this builds up to so can find the functions

Once we have them we need to ask

- Are they linearly indendent
- How many do we have
- Do we have a fundamental set of solutions

If we do have a fundamental set of solutions for a homogenous linear differential equation on "I"

- Then the general solution $(y = c_1y_1 + c_2y_2 + \dots + c_ny_n)$ is given as just taking these arbitrary linear combination

General Solutions for Homogeneous Linear Differential Equations

Boils down to is finding a fundamental set of solutions

Expectation checklist - Module 9

At the completion of this module, you should:

- Understand the definitions, and namely
  - linear combination,
  - linear independence,
  - fundamental set of solutions,
  - determinant of 2×2 and 3×3 matrices, and
  - Wronskian;
- if given a set of solutions to a homogeneouns linear DE be able to determine if they
  - are solutions to the DE,
  - are linearly independent,
  - and form a fundamental set of solutions for the DE; and
- if given a fundamental set of solutions for a homogeneous linear DE, be able to provide the general solution.

You will be assessed on your understanding of these concepts:

- within the homework,
- on the quizzes, and
- later, on the exam.

Coming up next, we:

- Fundamental sets of solutions for *nonhomogeneous* linear DEs!