Goal and idea - Module 6

GOAL:

Yet another type of 1st-order DE that we would like to understand and solve are those known as exact equations.

To do so, we:

- Define what exact equations are, and determine whether a DE is an exact equation.
- Discuss how to solve such differential equations.
- Introduce differentials.

IDEA:

We will find that exact equations are differential equations which have a very distinct form and satisfy a certain partial derivative identity. Checking this identity is our route to knowing whether a DE is exact. From there, we must take antiderivatives--with respect to various variables--as well as partial derivatives, to aid us in solving the DEs.

Exact Equations

Definition

(a) A differental equation $M(x,y)dx + N(x,y)dy = 0$ is an **exact equation** if $\frac{\partial}{\partial y}M(x,y) = \frac{\partial}{\partial x}N(x,y)$

Ex $\underbrace{y}_{M}dx + \underbrace{x}_{N}dy = 0$

$\frac{\partial}{\partial y}y = 1$, $\frac{\partial}{\partial x}x = 1$

$\rightarrow \frac{\partial}{\partial y}y = 1 = \frac{\partial}{\partial x}x = 1$ ✓

Since they are equal to eachother this IS an exact equation.

* Note: The differential of $y=f(x)$ is
$dy = f'(x)dx = \frac{df}{dx}dx$
$\rightarrow dy = f'(x)dx = \frac{df}{\cancel{dx}}\cancel{dx}$
$\rightarrow dy = f'(x)dx = df$

(b) For a function $z=f(x,y)$ the differential of z is $dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$

① $M(x,y)dx + N(x,y)dy = 0$ ② $dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$

when is this partial derivative of "f" with respects to "x" the "M", and when derivative of "f" with respects to "y" the "N", and when is the "dz" equal to zero

And which case the differential of a function ② $dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ is percisely this exact form ① $M(x,y)dx + N(x,y)dy = 0$ with this additional condition $\frac{\partial}{\partial y}M(x,y) = \frac{\partial}{\partial x}N(x,y)$

Ex Consider an expression $c=f(x,y)$, say
$e^x + x^2y^2 - \ln|y| = C$. * "C" is a constant *

(Our function "f" is this whole term "$e^x + x^2y^2 - \ln|y| = C$" $f(x,y) = e^x + x^2y^2 - \ln|y| = C$)

What we want to do is we compute the differential
Then $dz = dc = 0$.
And $0 = dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$

Because of that when we take the derivative of this function with respects to "x" we get and with respects to "y" we get

$= (e^x + 2xy^2)dx + (2x^2y - \frac{1}{y})dy$

In other words we have zero equal to this big expression
That is $e^x + x^2y^2 - \ln|y| = C$ satisfies the DE $(e^x + 2xy^2)dx + (2x^2y - \frac{1}{y})dy = 0$

Ex Find the exact equation solved by $\frac{1}{3}x^3 + \frac{1}{2}x^2y - xy^2 + C = 0$

Solution Two ways:

ONE Using the differential: The expression gives
$\frac{1}{3}x^3 + \frac{1}{2}x^2y - xy^2 = C$
Taking the differential:

$0 = dz = \frac{df}{dx}(\frac{1}{3}x^3 + \frac{1}{2}x^2y - xy^2)dx + \frac{df}{dy}(\frac{1}{3}x^3 + \frac{1}{2}x^2y - xy^2)dy$

$\rightarrow (\frac{1}{3}(3)x^{3-1} + \frac{1}{2}(2)x^{2-1}y - x^0y^2)dx + \frac{df}{dy}(\cancel{\frac{1}{3}x^3}^{0} + \frac{1}{2}x^2y^0 - x(2)y^{2-1})dy$

$\rightarrow (\cancel{\frac{1}{3}}\cancel{(3)}x^2 + \cancel{\frac{1}{2}}\cancel{(2)}xy - (1)y^2)dx + \frac{df}{dy}(\frac{1}{2}x^2(1) - x(2)y^1)dy$

$\rightarrow 0 = dz = (x^2 + xy - y^2)dx + (\frac{1}{2}x^2 - 2xy)dy$

Thus, $(x^2 + xy - y^2)dx + (\frac{1}{2}x^2 - 2xy)dy = 0$

Is this an exact equation? If we the derivative
$(x^2 + xy - y^2)dx + (\frac{1}{2}x^2 - 2xy)dy = 0$
With respects to "y" With respects to "x"

Gives us $(x-2y) + (x-2y)$.
Because it gives us the exact same expression it is an exact equation

Two We need $[???]dx + [???]dy = 0$

$\int dx$ $[???]dx + [???]dy = 0$ $\int dy$

Where, the partial derivative of "x" $(\frac{d}{dx})$ of and "y" $(\frac{d}{dy})$ of

$\frac{d}{dx}(\frac{1}{3}x^3 + \frac{1}{2}x^2y - xy^2)$

$\frac{d}{dy}(\frac{1}{3}x^3 + \frac{1}{2}x^2y - xy^2)$

This gives
$(x^2 + xy - y^2)dx + (\frac{1}{2}x^2 - 2xy)dy = 0$

Exact since:
$\frac{d}{dy}(x^2 + xy - y^2) = x - 2y = \frac{d}{dx}(\frac{1}{2}x^2 - 2xy)$

Solving Exact Equations

Ex | Solve $(e^x+2xy^2)dx+(2x^2y-\frac{1}{y})dy=0$

We do know it is in an exact form, but is it an exact differential equation?

Step 0 | Check if it's exact:

$(e^x+2xy^2)dx+(2x^2y-\frac{1}{y})dy=0$

$$\left.\begin{aligned}\frac{\partial}{\partial y}(e^x+2xy^2)&=4xy\\ \frac{\partial}{\partial x}(2x^2y-\frac{1}{y})&=4xy\end{aligned}\right\}\text{ Yes, exact ✓}$$

Using respective partial derivatives

$\boxed{(e^x+2xy^2)}dx+\boxed{(2x^2y-\frac{1}{y})}dy=0$

We need this term to be the partial derivative of whatever we are looking for with respect to "x" and the same for with respect to "y". This is done by taking the antiderivatives of the two terms (e^x+2xy^2) and $(2x^2y-\frac{1}{y})$

Step 1 | Find (antiderivative with respect to "x" or "y")

Finding an antiderivative with respects to a variable anything that isn't that variable we treat it like a constant

With respect to "x"

$F(x,y)=\int(e^x+2xy^2)dx$

$\rightarrow e^x+x^2y^2$

$\rightarrow e^x+x^2y^2+g(y)$

You may feel compelled to add "+C" when finding the antiderivative "$e^x+x^2y^2$", but you can't. It's different because it's a constant with respect to "x", so what is really being added is a function that could include y-terms "$e^x+x^2y^2+g(y)$" the constant could include pieces of "y"

Step 2 | Take the partial derivative of F (with respect to "y")

$\frac{\partial}{\partial y}(e^x+x^2y^2+g(y))$

$\rightarrow 2x^2y+g'(y)$

Idea: We want to figure-out what "g(y)" actully was.

Step 3 | Set the partial derivative of F (with respect to "y") equal $N(x,y)=2x^2y-1/y$ and solve for $g'(y)$:

$2x^2y+g'(y)=2x^2y-\frac{1}{y}$

$\rightarrow \frac{2x^2y}{2x^2y}+g'(y)=\frac{2x^2y}{2x^2y}-\frac{1}{y}$

$\rightarrow g'(y)=-\frac{1}{y}$

Step 4 | Integrate $g'(y)$:

$g'(y)=-\frac{1}{y}$

$\rightarrow g(y)=-\ln|y|+C$

Step 5 | Plug Step 4 into Step 1's antiderivative with respect to "x"

$g(y)=-\ln|y|+C$

$F(x,y)=e^x+x^2y^2+g(y)$

$\rightarrow F(x,y)=e^x+x^2y^2+(-\ln|y|+C)$

$\rightarrow F(x,y)=e^x+x^2y^2-\ln|y|+C$

Step 6 | Thus we have:

$F(x,y)=e^x+x^2y^2-\ln|y|+C$

In form $\boxed{F(x,y)=C}$

$\rightarrow e^x+x^2y^2-\ln|y|=C$

"$e^x+x^2y^2-\ln|y|=C$" is a solution

Note: Can't solve for "y", thus this is an implicit solution

Ex | Check if exact, and solve $(y^2+1)dx+(2xy)dy=0$

This time step 1 respect to "y"

[0] $\frac{\partial}{\partial y}(y^2+1)=2y$ yes ✓
$\frac{\partial}{\partial x}(2xy)=2y$

[1] $F(x,y)=\int 2xy\,dy=xy^2+g(x)$

[2] $\frac{d}{dx}(xy^2+g(x))=y^2+g'(x)$

[3] $y^2+g'(x)=M(x,y)=(y^2+1)$
$\rightarrow g'(x)=1$

[4] $g(x)=x+C$

[5] Plug Step 4 into Step 1's anti w/r to "y"

$F(x,y)=xy^2+g(x)$

$g(x)=x+C$

$\rightarrow xy^2+x+C$

[6] Thus, $xy^2+x=C$

Expectation checklist - Module 6

At the completion of this module, you should:

- know the definitions introduced, and in particular those of
  - en exact equation, and
  - the differential of a 2-variable function;
- verify a given expression is a solution to an exact equation;
- compute the differential of a 2-variable expression; and
- if given an exact equation, solve it.

You will be assessed on your understanding of these concepts:

- within the homework,
- on the quizzes, and
- later, on the exam.

Coming up next, we:

- introduce first order homogeneous differential equations.