CHAPTER 1

Definitions and classifications

A **differential equation** is an equation which relates a function with its derivatives. This is a broad definition, and already raises many questions. For example, does this mean higher derivatives, partial derivatives, or both? The answer is both. In fact, the broad definition means that differential equations occur in many different forms. Therefore, learning the terminology and mathematical definitions surrounding common different types of differential equations, as well as their solutions, is imperative, and is thus the main goal of this first chapter. Later, we focus on a few important classes of differential equations to study in more depth.

1.1. Classifications of Differential Equations

Mathematical equations often have variables. These can be denoted as x, y, or many other symbols. Often these variables occur at the same time. For example, $y = x^2$. A symbol which is a function of other variables is called a **dependent variable**. Any variable that is not a dependent variable is called an **independent variable**. These definitions can loosely be thought of as 'inputs' and 'outputs.' The output (which is dependent on the input) is the dependent variable, while the input is the independent variable. Returning to $y = x^2$, we find that x is the independent variable, while y is the dependent variable. Sometime, to highlight when a variable is dependent we use clear notation. For example, in the equation $f(x) = x^2$, we have $f(x)$ represents the dependent variable, which is dependent on the independent variable x. Finally, we note that a **multivariable function** is a function which consists of more than one independent variables.

Example 1.1.

(a) The function $f(x, y, z) = xy\sin(z)$ is a multivariable function as x, y, and z are all independent variables since they do not themselves change when another variable is changed.
(b) In the equation $w = xy\sin(z)$ we could say that x, y, and z are again independent variables, while w is a dependent variable. This is because w is dependent on the x, y, and z plugged in. In assigning these definitions we have assumed w is a multivariable function of x, y, and z.
(c) If we consider y as a function of x, then the equation $y' = 7xy$ is a differential equation. Moreover, x is an independent variable, while y is a dependent variable.

We have introduce dependent and independent variables so that we may classify differential equations more readily. We introduce our first form of 'classifying' differential equations to be based on how many variables are in the equation.

1.1.1. Classification based on variables. An **ordinary differential equation (ODE)** is a differential equation consisting of at least one function of one independent variable and its derivatives.

EXAMPLE 1.2.

(a) The equation $y'' + y' + y = 0$ is an ordinary differential equation.
(b) The equation $y = x^2$ is not an ODE since it is not a differential equation.
(c) The equation $f^{(7)}(x) + f'(x) - g'(x) = 0$ is an ODE. Note that there are two functions, but both have only one independent variable.
(d) Recall that a partial derivative of a multivariable function is a derivative with respect to only one variable (treating the other variables as constants). The equation $\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y} = 0$ is a differential equation, but not an ODE. This is because f is a function of at least two independent variables (since their are partial derivatives with respect to x and y).

A **partial differential equation (PDE)** is a differential equation consisting of at least one multivariable function and its partial derivatives. To examine this farther, we must understand what a partial derivative is. A partial derivative of a multivariable function is a derivative with respect to only one variable (treating the other variables as constants). For example, the derivative of $f(x, y) = x^2y^3$ with respect to x is denoted by $\frac{\partial f}{\partial x}$ and would be $2xy^3$. On the other hand, we have $\frac{\partial}{\partial y}(x^2y^3) = 3x^2y^2$. We will discuss this more later, but for now it is sufficient to understand that it simply means we can take derivative with respect to different variables.

EXAMPLE 1.3.

(a) The equation $\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y} = 0$ is a differential equation, but not an ODE. This is because f is a function of at least two independent variables (since their are partial derivatives with respect to x and y). The equation is, however, a PDE.
(b) The equation $\frac{\partial}{\partial x} g(x, y, z) = 7$ is a PDE.
(c) The equation $\frac{dy}{dx} + \sin(xy) = 0$ is more difficult to determine. We see that the derivative of y is with respect to x, and so we are viewing y as a function of x. Therefore, there is one function (y) with one dependent variable (x), and we conclude the equation is an ODE. It is not a PDE.
(d) For another hard example, consider $\frac{dx}{dt} + \frac{dy}{dt} + xy = 0$. Notice that there is only one dependent variable (in this case, t). Both x and y are functions dependent on t. Therefore, this equation is an ODE. It is not a PDE.

The difference between ODEs and PDEs centers around how many dependent variables there are. In this text, we will focus more on ODEs. There are other ways to classify differential equations, however.

1.1.2. Classification based on linearity. We recall that the degree of a polynomial is the largest product of its determinants. For example, $3+x-5x^2-2x^5$ has degree 5, while $3 + xy - 2x^2y^5$ has degree y.

A **linear differential equation** is a differential equation whose dependent functions and derivatives have degree one. That is, no products of the function or its derivatives are allowed, even with themselves. On the other hand, we say a differential equation is a **nonlinear differential equation** if it contains products of functions and its derivatives that have degree greater than one.

EXAMPLE 1.4.

(a) The equation $(y')y = ye^x$ is a differential equation. It is not a linear differential equation, however, due to the product of y' and y.
(b) On the other hand, $y' + y = ye^x$ is a linear differential equation. In fact, it is a linear ODE.
(c) Is $y^3 - y^2 + y' = 0$ a linear differential equation? It depends on if we mean y^n to be the product of y with itself n-many times of if it denotes the nth derivative. For this text, we will always mean y^n to be the product of y with itself n many times. That is, $y^n = y \cdot y \cdots y$, n-many times. To denote the nth derivative of y, we will write $y^{(n)}$. Thus, $y^3 - y^2 + y' = 0$ is a nonlinear (ordinary) differential equation, while $y^{(3)} - y^{(2)} + y' = 0$ is a linear (ordinary) differential equation. It is important to understand difference between these two ODEs.
(d) The PDE $\frac{\partial^2 f}{\partial x} = f$ is linear while $\left(\frac{\partial f}{\partial x}\right)^2 = f$ is nonlinear.

This text will largely focus on linear differential equations. Combined with the previous subsections, we find that we will spend most of our energies studying linear ODEs. However, there is an additional way we would like to classify differential equations.

1.1.3. Classification based on order. In the previous subsection we utilized the notion of degree. Here we recall that the order of a derivative is the number of compositions of the derivative. That is, $f^{(n)}(x) = \frac{d^n f}{dx^n} = \left(\frac{d}{dx}\right)^n f$ is the derivative of order n. Note the strong difference between $\left(\frac{d}{dx}\right)^n f$ and $\left(\frac{df}{dx}\right)^n$. Indeed, the former is linear and the latter is nonlinear.

An **nth order differential equation** is a differential equation whose highest derivative has order n.

EXAMPLE 1.5.

(a) The differential equation $(y')y = ye^x$ has order 1. Note that order does not require the differential equation to be linear.
(b) The linear ODE $y^{(3)} - y^{(2)} + y' = 0$ has order 3.
(c) The PDE $\frac{\partial^2 f}{\partial x} - \frac{\partial^5 f}{\partial x} = f$ has order 5.

This text will focus on smaller order differential equations. However, larger order differential equations will be considered at times as well.

At this point, we conclude our prelude to different types of differential equations. However, we will add more terms and distinctions to our vocabulary later.

1.2. Expressions of ODEs

Here we want to introduce some terminology regarding how we see, notationally, the differential equations. For example, consider the ODE $5x\frac{dy}{dx} - e^x + y = 0$. Besides writing it this way, we could express this ODE as $\frac{dy}{dx} = \frac{e^x - y}{5x}$. Additionally, if we view this all as formal notation, we could multiply the original expression by dx and write the equation as $5xdy + (y - e^x)dy = 0$. It ends up that there are many different ways to express a differential equation, and often times we want to write them in a certain way. To differentiate between the most useful ways to express a differential equation, we provide them with names.

If a differential equation is written in a form such as

$$F(x, y)\,dx + G(x, y)\,dy = 0, \tag{1.1}$$

where F and G are functions of x and y, we say it is in **differential form**. For example, $5xdy + (y - e^x)dy = 0$ above is the differential form. In the differential form of a differential equation, we are utilizing the notion of the 'differentials' dx and dy. Much could be said about the true mathematical definition of such things, but for this class it suffices to view them as terms which arise so that the expressions $\frac{dy}{dx} = f'(x)$ and $dy = f'(x)dx$ are compatible.

EXAMPLE 1.6. Consider the differential equation $\sin(\theta)\frac{dq}{d\theta} = \ln(\theta)$. To place this into differential form we can first 'multiply' the expression by $d\theta$ so that we get $\sin(\theta)\,dq = \ln(\theta)\,d\theta$. Then we can rewrite this as $\sin(\theta)\,dq - \ln(\theta)\,d\theta = 0$ to obtain the differential form.

Meanwhile if we gather a differential equation so that the highest derivative is on one side of the equation and all other terms are on the other we say the differential equation is in **normal form**. It is useful to introduce notation for this. Suppose the differential equation consists of a function y which is in terms of the independent variable x, along with derivatives of y, the highest order being n. Then $y^{(n)}$ is the highest order derivative. If we 'solve' for $y^{(n)}$ and let $F(x, y, y', y'', \ldots, y^{(n-1)})$ denote the expression which $y^{(n)}$ then equals we can write this differential equation as

$$y^{(n)} = F(x, y, y', y'', \ldots, y^{(n-1)}). \tag{1.2}$$

This is the normal form. For example, for $5x\frac{dy}{dx} - e^x + y = 0$ above, we have its normal form is $\frac{dy}{dx} = \frac{e^x - y}{5x}$. In the notation of (1.2) we have $y' = F(x, y)$, where $F(x, y) = \frac{e^x - y}{5x}$.

EXAMPLE 1.7. Consider the differential equation $y''' - t^3y'' + \sin(t)y' - y = 0$. If we want to place this differential equation into normal form, we would 'solve' for y''' so that $y''' = t^3y'' - \sin(t)y' + y$. This is the normal form, and in the notation of of (1.2) we have $y''' = F(t, y, y', y'')$, where $F(t, y, y', y'') = t^3y'' - \sin(t)y' + y$. We could then go about trying to place this differential equation into differential form. However, we find that when we rewrite y''' into $\frac{d^3y}{dx^3}$ that we would get expressions like dx^3 and this was not accounted for in our discussion above. Indeed, the differential form of a differential equation only makes sense when the differential equation is of order 1!

We note that we often want to consider some arbitrary nth order differential equation. It is painstaking to write such a thing as, for example (if it is linear),

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0, \tag{1.3}$$

where the $a_j(x)$ are coefficient functions of in terms of x. Instead, we sometimes will simply refer to such an arbitrary nth order differential equation by

$$F\left(x, y, y', \ldots, y^{(n)}\right) = 0,$$

where it is understood that the left side represents something like the left side of (1.3).

1.3. Solutions of differential equations

A function ϕ that satisfies a differential equation for all input of ϕ is called a solution to the differential equation. Notice that we have not yet made bold the definition of a 'solution.' This is because the definition is chocked full of subtlety which we would like to discuss first. The core subtlety to notice is that the definition of a solution for a differential equation is dependent on the inputs of the function. We keep this in mind as we begin to explore solutions.

EXAMPLE 1.8. Consider the differential equation $y' = \cos(x)y$. We claim that the function $\phi(x) = e^{\sin(x)}$ is a solution *for all real numbers* x. First, we note that $\phi'(x) = \cos(x)e^{\sin(x)}$. This is the left side of the differential equation. Meanwhile, the right side is $\cos(x)y = \cos(x)e^{\sin(x)}$, where we have plugged in ϕ for y. Thus, $\phi(x)$ satisfies the differential equation. Moreover, the function $\phi(x)$ is defined for all x, and so is a solution for any real number x.

Since there is a subtlety of a function being a solution for certain values, let us look at another example, where a function is only a solution for certain values.

EXAMPLE 1.9. Consider the differential equation $y' = -y^2$. Taking $y = \frac{1}{x}$, we find that $\phi(x) = \frac{1}{x}$ satisfies the equation $y' = -y^2$. However, $\frac{1}{x}$ is not defined for $x = 0$. In other words, $\frac{1}{x}$ is not a solution for $x = 0$. Thus we need to be careful when discussing whether $\frac{1}{x}$ is a solution of $y' = -y^2$ or not.

We have found that when stating a function is a solution, it is important to state for what input it is a solution. This leads to the notion of an interval of existence. However, first, we provide a more accurate definition of a solution for a differential equation. Indeed, a function ϕ defined on an interval I that satisfies a differential equation on I is called a **solution** of the differential equation on the interval I. We emphasis here that the phrase 'on the interval I' is very important when discussion a solution of a differential equation.

EXAMPLE 1.10. Returning to our previous example, we find that $y = \frac{1}{x}$ is a solution to the differential equation $y' = -y^2$ on $(-\infty, 0)$. Also, $y = \frac{1}{x}$ is a solution to the differential equation $y' = -y^2$ on $(0, \infty)$.

The interval I for which a function is a solution for a differential equation is called the **interval of existence** (or the interval of **definition**, or **validity**). It is also called the **domain of the solution**.

Let us return to the example of $y' = -y^2$. Does our discussion above mean that this differential equation does not have a solution at $x = 0$? It cannot be the function $\frac{1}{x}$. Upon further inspection, however, we find that $y = 0$ satisfies the differential equation $y' = -y^2$. Moreover, as a function, $y = 0$ is defined for all real x. That is, $y = 0$ is a solution for the differential equation $y' = -y^2$ on the interval $I = (-\infty, \infty)$.

If the function $y = 0$ is a solution on an interval I, we call this the **trivial solution**. Therefore, in our example above we have that $y = 0$ is the trivial solution for $y' = -y^2$. We have found two solutions for the differential equation $y' = -y^2$. Are there others? We could note that the solution $y = \frac{1}{x}$ works when multiplied by any constant c. That is, $y = \frac{c}{x}$ is a solution of $y' = -y^2$ for all constants c.

There is many solutions for a differential equation $F(x, y, y', \dots, y^{(n)}) = 0$. If a set of solutions for a differential equation differs by a constant c, we call the constant c the **parameter** and the set of solutions a **one-parameter family of solutions**.

EXAMPLE 1.11. The solutions $y = \frac{c}{x}$ is a one-parameter family of solutions of the differential equation $f' = -y^2$. Note that this family includes the trivial solution (taking $c = 0$).

In the abstract notation[1], we could have said for a first-order differential equation $F(x, y, y') = 0$ that a set of solutions $G(x, y, c) = 0$ is a one-parameter family of solutions. In our example above, we would have $F(x, y, y') = y' + y^2$ and $G(x, y, c) = y - \frac{c}{x}$. This abstract notation allows us to further discuss families of solutions. Indeed, for a second-order differential equation $F(x, y, y', y'') = 0$ we call a set of solutions $G(x, y, c_1, c_2)0$ dependent on two constants a **two-parameter family of solutions** to $F(x, y, y', y'') = 0$. More generally, we would call a set dependent on n-many constants $G(x, y, c_1, \dots, c_n)0$ an n-parameter family of solutions for a differential equation.

EXAMPLE 1.12. Consider the second order differential equation $\frac{d^2y}{dx^2} = y$. In future chapters we will examine how to go about solving such differential equations. For now, however, we restrict our attention to verifying that a given function is a solution, describing it as a solution, and discussing its interval of definition. Consider there the two-parameter family of solutions $y = c_1e^x + c_2e^{-x}$. To verify this is in fact a family of solutions we see that $\frac{dy}{dx} = c_1e^x - c_2e^{-x}$ and $\frac{d^2y}{dx^2} = c_1e^x + c_2e^{-x}$. Thus, plugging all of this into $\frac{d^2y}{dx^2} = y$ we find the two sides do equal one another. Since $y = c_1e^x + c_2e^{-x}$ is a solution for any constants c_1 and c_2 it is a two-parameter family of solutions. Finally, we note that since e^x and e^{-x} are defined and continuous for all real numbers we have that the largest interval of definition is $I = (-\infty, \infty)$.

We add two more definitions surrounding family of solutions. For starters, we say a solution for a differential equation that has no parameters a **particular solutions**. Meanwhile, an n-parameter family of solutions which gives rise to all solutions of a differential equation (by ranging the parameters) is called a **general solution**. A core goal for us throughout these notes is to learn how to first find a family of solutions to a differential equation, and then to determine whether this family is a general solution.

EXAMPLE 1.13. Consider the first-order differential equation $y' = e^xy^2$. Plugging in $y = -\frac{1}{5+e^x}$ and $y' = \frac{e^x}{(5+e^x)^2}$ shows that y is a particular solution. Meanwhile, plugging in $y = -\frac{1}{c+e^x}$ and $y' = \frac{e^x}{(c+e^x)^2}$ shows that y is a one-parameter family of solutions. Is this a general solutions? Though it may not be obvious, no, it is not. To see this, note that $y = 0$ is the trivial solution, however, $y = 0$ does not come from the one-parameter family of solutions $y = -\frac{1}{c+e^x}$.

EXAMPLE 1.14. Later in these notes we will find that $y = c_1e^x + c_2e^{-x}$ is a general solution to the differential equation $\frac{d^2y}{dx^2} = y$.

[1] When we don't have a specific example in mind.

We conclude this section discussing one more subtle issue pertaining to solutions. That is the notion of implicit and explicit solutions. These concepts lie deeply in the concept of implicit differentiation covered in Calculus I. Let us first recall an example of implicit differentiation.

EXAMPLE 1.15. Consider the equation $y^2 = ye^x$ and let us find $\frac{dy}{dx}$. Due to the y^2 term we cannot *explicitly* solve for y, i.e., isolate y uniquely. On the other hand, we can apply $\frac{d}{dx}$ to both sides of the equation

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dy}\left(ye^x\right).$$

Then treating y as a function of x and recalling that the chain rule would read $\frac{d}{dx}g(y) = g'(y)\frac{dy}{dx}$ for a function which contains y, we find

$$2y\frac{dy}{dx} = \frac{dy}{dx}e^x + ye^x,$$

where we also used the product rule on the right side. Grouping the terms for $\frac{dy}{dx}$ yields

$$\frac{dy}{dx}\left(2y - e^x\right) = ye^x,$$

and thus we find

$$\frac{dy}{dx} = \frac{ye^x}{2y - e^x}.$$

We note that the end result is dependent on y. This derivative, while not explicitly solved for y, is an implicit derivative.

In the example above, we see that we may have an expression y' even if we don't have an explicit expression for y. Therefore, it is possible that we have a solution for a differential equation which is not explicitly expressed in terms of y. Instead, it may be implicitly expressed in terms of y.

EXAMPLE 1.16. Consider the differential equation $y'(2y-e^x) = y^2$. Recall from Example 1.15 that for $y^2 = ye^x$ we have $y' = \frac{ye^x}{2y-e^x}$. Plugging y' into $y'(2y - e^x) = y^2$ and using that $y^2 = ye^x$ we find that the differential equation is satisfied. It may not be obvious, but what we have found is that $y^2 = ye^x$ has an intimate relation to the differential equation. It ends up, a solution does not have to necessarily be expressed explicitly in terms of y. It can be implicitly expressed in terms of y instead.

A solution in which the dependent variable can be expressed in therms of the independent variable is called an **explicit solution**. Meanwhile, a relation $F(x, y) = 0$ is an **implicit solution** to the differential equation if there is at least one function ϕ that satisfies the differential equation and $F(x, y) = 0$.

EXAMPLE 1.17. Continuing our discussion from Examples 1.15 and 1.16 we can take $F(x, y) = y^2 - ye^x$ to fit the definition of an implicit solution just given. However, we still want to see that there is some function ϕ that satisfies *both* $F(x, y) = 0$ and $y'(2y - e^x) = y^2$. Consider the function $y = e^x$. Then $F(x, y) = y^2 - ye^x = (e^x)^2 - e^x e^x = 0$ as desired. Meanwhile, noting that $y' = e^x$ again, we see that $y'(2y - e^x)$ becomes $e^x(2e^x - e^x) = e^x e^x = (e^x)^2$, which is y^2. Thus, the differential equation is also satisfied. It follows that $F(x, y) = y^2 - ye^x$ is an implicit solution to the differential equation $y'(2y - e^x) = y^2$.

1.4. Solution curves

The graph of a solution ϕ if an ordinary differential equation is called a **solution curve**. It is important to note here that the graph of ϕ can be different depending on whether ϕ is graphed as a function or as a solution to a differential equation! Why?[2]

EXAMPLE 1.18. Let us graph the solution $y = \frac{1}{x-1}$ of the ordinary differential equation $y' + y^2 = 0$. However, first we graph $y = \frac{1}{x-1}$ as a function. This can be found in Figure 1.

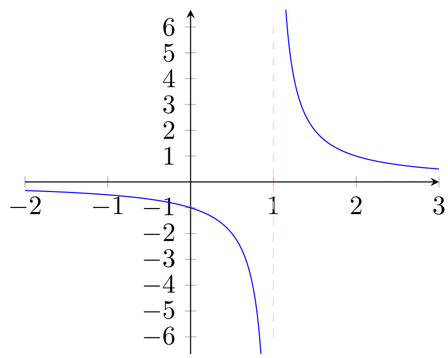

FIGURE 1. The graph of the function $f(x) = \frac{1}{x-1}$.

As a solution, however, we note that $y = \frac{1}{x-1}$ is not defined at $x = 1$. Indeed, technically, we have that $y = \frac{1}{x-1}$ is a solution on $(-\infty, 1)$ and also a (different) solution on $(1, \infty)$. The graphs of both of these solutions can be found in Figure 2.

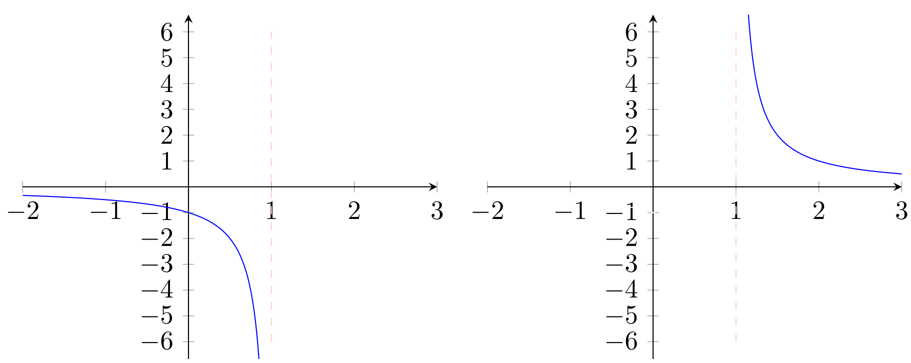

FIGURE 2. The solution $f(x) = \frac{1}{x-1}$ on $(-\infty, 1)$ (left) and the solution $f(x) = \frac{1}{x-1}$ on $(1, \infty)$ (right).

Our last example surrounded the graphing of the particular solution $y = \frac{1}{x-1}$. In relation to the notion of a solution curve we find that Figure 2 provides two

[2]This is because the domain of ϕ when viewed as a solution may be restricted.

separate solution curves. A more complete picture of the graph of a solution curve will incorporate all (or at least more) solution curves.

EXAMPLE 1.19. We have that $y = \frac{1}{x+c}$ is a one-paramter family of solutions for the differential equation $y' + y^2 = 0$ (the case $c = -1$ corresponds to the previous example). Figure 3 contains the graph of the solution curve for the family of solutions $y = \frac{1}{x+c}$ of the differential equation $y' + y^2 = 0$. Technically, since c could be any real number, we don't graph every such solution from the family.

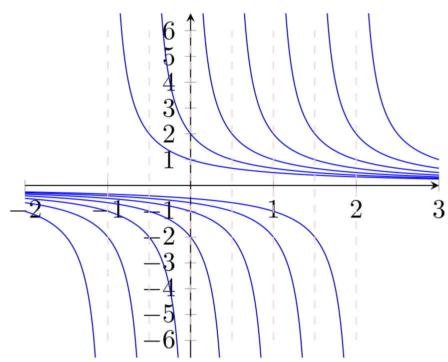

FIGURE 3. A graph of some of the family of solution curves for $f(x) = \frac{1}{x-1}$.

1.5. Initial value problems

Examining the graphs in the previous section, we can see that some of the solutions pass through a point in the xy-plane, while other solutions do not. It is often important to find a particular solution within a family that passes through a desired point.

EXAMPLE 1.20. Recall that $\frac{1}{x+c}$ is a one-parameter family of solutions to the differential equation $y' + y^2 = 0$. Let us find a particular solution which satisfies the condition that $y(7) = 1$. To do so we want a solution so that when $x = 7$ we have $y = 1$. Plugging in $x = 7$ to our family of equations and setting this equal to 1 we have

$$1 = y(7) = \frac{1}{7+c}.$$

Solving for c gives $c = -6$. Thus, the desired particular solution is $y = \frac{1}{x-6}$. Note that geometrically, this solution is the one in the graph of the solution curve which passes through the point $(7, 1)$ in the xy-plane.

We call an nth-order differential equation $F(x, y, y', \ldots, y^{(n)}) = 0$ subject to the conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots \quad y^{(n-1)}(x_{n-1}) = y_{n-1} \tag{1.4}$$

(where $y_0, \ldots, y_{n-1}$ are constants) an nth-order **initial value problem (IVP)** and the conditions 1.4 the **initial conditions**.

EXAMPLE 1.21. Consider the differential equation $y'' + \pi y = 0$. We first verify that $y = k_1 \sin(\sqrt{\pi}x) + k_2 \cos(\sqrt{\pi}x)$ is a one-parameter family of solutions for the differential equation. Noting that $y' = k_1\sqrt{\pi}\cos(\sqrt{\pi}x) - k_2\sqrt{\pi}\sin(\sqrt{\pi}x)$ and $y'' = -k_1\pi\sin(\sqrt{\pi}x) - k_2\pi\cos(\sqrt{\pi}x)$ we find that

$$y''+\pi y = \left(-k_1\pi\sin\left(\sqrt{\pi}x\right) - k_2\pi\cos\left(\sqrt{\pi}x\right)\right)+\pi\left(k_1\sin\left(\sqrt{\pi}x\right) + k_2\cos\left(\sqrt{\pi}x\right)\right) = 0,$$

verifying that y is a family of solutions.

Next, let us find a particular solution to the differential equation $y'' + \pi y = 0$ subject to the conditions

$$y\left(\frac{\sqrt{\pi}}{2}\right) = 2 \quad \text{and} \quad y'\left(\frac{\sqrt{\pi}}{2}\right) = 5.$$

(We note that since $y''+\pi u = 0$ is a second-order differential equation to correspond to the definition above we need two conditions here.) First we note that

$$2 = y\left(\frac{\pi}{2}\right) = k_1\sin\left(\sqrt{\pi}\frac{\sqrt{\pi}}{2}\right) + k_2\cos\left(\sqrt{\pi}\frac{\sqrt{\pi}}{2}\right) = k_1\sin\left(\frac{\pi}{2}\right) + k_2\cos\left(\frac{\pi}{2}\right) = k_1.$$

Meanwhile, using the second initial condition gives

$$\begin{aligned} 5 = y'\left(\frac{\sqrt{\pi}}{2}\right) &= k_1\sqrt{\pi}\cos\left(\sqrt{\pi}\frac{\sqrt{\pi}}{2}\right) - k_2\sqrt{\pi}\sin\left(\sqrt{\pi}\frac{\sqrt{\pi}}{2}\right) \\ &= k_1\sqrt{\pi}\cos\left(\frac{\pi}{2}\right) - k_2\sqrt{\pi}\sin\left(\frac{\pi}{2}\right) = -k_2, \end{aligned}$$

so that $k_2 = -5$. It follows that the desired solution is

$$y = 2\sin\left(\sqrt{\pi}x\right) - 5\cos\left(\sqrt{\pi}x\right).$$

Significant questions arise. Does an ordinary differential equation IVP always have a solution, and how many solutions does an IVP have?

To answer these questions, it is convenient to use partial derivatives. Recall that given a multivariable function $f(x, y)$, the **partial derivative** of f with respect to x (denoted $\frac{\partial f}{\partial x}$) is the derivative of $f(x, y)$ treating x as the variable and y as a constant. Similarly, the partial derivative of f with respect to y (denoted $\frac{\partial f}{\partial y}$) is the derivative of $f(x, y)$ treating y as the variable and x as a constant.

EXAMPLE 1.22. Consider the function $f(x, y) = y^2x^3\sin(x + y^2)$. Treating y as a constant and x as the variable, we find

$$\frac{\partial f}{\partial x} = 3y^2x^2\sin\left(x + y^2\right)+y^2x^3\cos\left(x + y^2\right) = y^2x^2\left(3\sin\left(x + y^2\right) + x\cos\left(x + y^2\right)\right).$$

Meanwhile, treating x as a constant and y as the variable, we have

$$\frac{\partial f}{\partial y} = 2yx^3\sin\left(x + y^2\right) + 2y^3x^3\cos\left(x + y^2\right).$$

The definition of a multivariable function being continuous is studied in multivariable calculus. However, we do need to determine on what regions a multivariable function is continuous for the study of differential equations. Therefore, we point out that multivariable functions consisting of polynomials, exponential, logarithms, trigonometric, and quotient functions are continuous wherever they are defined! Thus, determining where a given function is continuous amounts to determining where the function is defined.

EXAMPLE 1.23. Consider the function$f(x, y) = \ln(y^2 - x)$. Recall that $\ln(z)$ is defined for all $z > 0$. Thus, $\ln(y^2 - x)$ is defined for all (x, y) such that $y^2 - x > 0$. That is, for all (x, y) that satisfy $y^2 > x$. We can graph all such (x, y). First we see that $y^2 > x$ amounts to all (x, y) that satisfy either $y > \sqrt{x}$ or $y < -\sqrt{x}$. Graphing both $y = \sqrt{x}$ and $y = -\sqrt{x}$ we can shade in the regions which $f(x, y) = \ln(y^2 - x)$ is continuous on by shading in everything above $y = \sqrt{x}$ and everything below $y = -\sqrt{x}$. Note that if $x < 0$ then $f(x, y)$ is defined (and thus continuous) for any y.

Our interest in this analysis is sparked from the following theorem which gives us a way to determine not only if a first-order ODE has a solution satisfying an initial value condition, but also tells us such a solution would be unique!

THEOREM 1.24. *Let* $\frac{dy}{dx} = f(x, y)$ *be a first-order ODE and suppose* R *is a rectangular region in the* xy*-plane that contains a point* (x_0, y_0)*. Then the IVP* $\frac{dy}{dx} = f(x, y)$ *with initial condition* $y(x_0) = y_0$ *has a unique solution on some interval about* x_0 *if*

(a) $f(x_0, y_0)$ *and*
(b) $\dfrac{\partial}{\partial y} f(x_0, y_0)$

are both continuous on R*.*

In particular, the previous theorem provides conditions which guarantees both the *existence* and *uniqueness* of a solution to a first order ODE.

EXAMPLE 1.25. Consider the differential equation $\frac{dy}{dx} = \sqrt{yx^2}$. We will attempt to find all points where it is guaranteed that a unique solution to the IVP exists. The points that will accomplish this are the points (x, y) such that (i) $f(x, y)$ and (ii) $\dfrac{\partial}{\partial y} f(x, y)$ are continuous on. Since $\sqrt{w}$ is defined (and continuous) for all $w \geq 0$ we have that $f(x, y)$ is continuous for all (x, y) such that $yx^2 \geq 0$. Since $x^2 \geq 0$ for all x, this amounts to the points (x, y) such that $y \geq 0$ (and any x). Meanwhile, we have

$$\frac{\partial}{\partial y} f(x, y) = \frac{\partial \sqrt{yx^2}}{\partial y} = \frac{x^2}{2\sqrt{yx^2}}.$$

Thus, we have $\dfrac{\partial}{\partial y} f(x, y)$ is continuous on all (x, y) such that $y > 0$ and $x \neq 0$. Indeed, these are essentially the same points in (i), however we must now exclude when $yx^2 = 0$ since we cannot divide by 0.

Therefore, the points that satisfy both parts (a) and (b) of the previous theorem are the points (x, y) such that $y > 0$ and $x \neq 0$. For these points, we are guaranteed a unique solution exists for the IVP. For example, we have that a unique solutions is guaranteed to exist at the point $(1, -4)$. However, it is not guaranteed for the point $(2, 0)$.