Goal and idea - Module 7

GOAL:

For the final 1st-order DE in which we study, we turn to 1st-order homogeneous DEs and the technique of 'substituion.'

To do so, we:

- Define what 1st-order homogeneous equations are, and determine whether a DE is such an equation.
- Discuss how to solve such differential equations.
- Briefly discuss the unfortunate fact that we see another type of DE called 'homogeneous' later in the class that is very different from this current case.

IDEA:

By substituting variables, we are able to alter certain DEs (1st-order homogeneous DEs) into a separable equation which we can solve. We then re-substitute our variables to find a solution for the original equation.

Definition 1^{st}–Order Homogeneous Differential Equation

[a] A differential equation $\frac{dy}{dx} = f(x,y)$ is a <u>1^{st}–Order Homogeneous Differential Equation</u> if $f(x,y)$ is a homogeneous function of degree zero.

If you have this Differential Equation it does not have to be expressed in this normal form

$$\frac{dy}{dx} = f(x,y)$$

, but the idea would be

If you manipulate it into this form
- If you don't know if it's separable
- If you don't know if it's exact

If that is a homogeneous function of degree zero

THEN it is a 1^{st}-Order Homogeneous DE

<u>Example</u> If we had . . .

$$\frac{dy}{dx} = \frac{xy^2 + x^3\cos\left(\frac{2x}{3y}\right)}{5y^2x + y^3}$$

Then

$$\frac{(tx)(ty)^2 + (tx)^3\cos\left(\frac{2(tx)}{3(ty)}\right)}{5(ty)^2(tx) + (ty)^3}$$

$$\rightarrow \frac{(xt^{1+2}y^2) + (t^3x^3\cos\left(\frac{2\cancel{t}x}{3\cancel{t}y}\right))}{t^{2+1}(5y^2x) + t^3y^3}$$

$$\frac{1\cancel{t^3}(xy^2 + x^3\cos\left(\frac{2x}{3y}\right))}{1\cancel{t^3}(5xy^2 + y^3)}$$

$$\rightarrow \frac{(xy^2 + x^3\cos\left(\frac{2x}{3y}\right))}{(5xy^2 + y^3)} = t^0F(x,y) \text{ or simply } F(x,y)$$

This calculation seeing that it showed degree zero would tell us the whole DE would be a 1^{st}–Order Homogeneous Differential Equation

[b] A differential equation in exact form $M(x,y)dx + N(x,y)dy = 0$ <u>1^{st}–Order Homogeneous Differential Equation</u> if

✳ Rearranging the expression gives ✳

$$M(x,y)dx + N(x,y)dy = 0$$

$$\rightarrow \frac{M(x,y)dx}{dx} + \frac{N(x,y)dy}{dx} = 0$$

$$\rightarrow M(x,y) + N(x,y)\frac{dy}{dx} = 0 \quad M(x,y)$$

$$\rightarrow \frac{N(x,y)\frac{dy}{dx}}{N(x,y)} = \frac{M(x,y)}{N(x,y)}$$

$$\rightarrow \frac{dy}{dx} = \frac{M(x,y)}{N(x,y)}$$

$M(x,y)$ and $N(x,y)$ are Homogeneous functions of the same degree.

Definition [B]: If you are given a differential equation in differential form $(M(x,y)dx + N(x,y)dy = 0)$ then you could ask Is it exact or is it homogeneous?

Definition Homogeneous

A function $f(x,y)$ is called **homogeneous** of degree α (alpha), α is a real number, if for $t = 0$ we have $f(tx,ty) = t^{\alpha}f(x,y)$

<u>Ex</u> Consider $f(x,y) = \sqrt{x+y}$

The Idea: We want to factor and modify this equation $(f(tx,ty) = \sqrt{tx+ty})$, So we get back our original function $(\sqrt{x+y})$

<u>Soln</u>

Then $f(tx,ty) = \sqrt{tx+ty}$

$$\rightarrow \sqrt{tx+ty} = \sqrt{t(x+y)} = \sqrt{t}\sqrt{x+y}$$

$$\rightarrow \sqrt{t}\sqrt{x+y} = t^{1/2}\sqrt{x+y} = t^{1/2}f(x,y)$$

So $f(x,y)$ is homogeneous of degree $1/2$.

<u>Ex</u> Show $F(x,y) = \frac{xy^2 + x^3\cos\left(\frac{2x}{3y}\right)}{5y^2x + y^3}$ is homogeneous of degree zero.

<u>Soln</u>

$$F(tx,ty) = \frac{(tx)(ty)^2 + (tx)^3\cos\left(\frac{2(tx)}{3(ty)}\right)}{5(ty)^2(tx) + (ty)^3}$$

$$\rightarrow \frac{(xt^{1+2}y^2) + (t^3x^3\cos\left(\frac{2\cancel{t}x}{3\cancel{t}y}\right))}{t^{2+1}(5y^2x) + t^3y^3}$$

Recall $t^0 = 1$

$$\rightarrow \frac{1\cancel{t^3}(xy^2 + x^3\cos\left(\frac{2x}{3y}\right))}{1\cancel{t^3}(5xy^2 + y^3)}$$

$$\rightarrow \frac{(xy^2 + x^3\cos\left(\frac{2x}{3y}\right))}{(5xy^2 + y^3)} = t^0F(x,y) \text{ or simply } F(x,y)$$

To solve such a DE

Step 1 | Subsitute $y=ux$

Step 2 | Replace "$\frac{dy}{dx}$" with "$x\frac{du}{dx}+u$"

→ "u" is a function of "x"

→ $\frac{dy}{dx} = \frac{d}{dx}(ux) = u'x + ux' = \frac{du}{dx}\cdot x + u = x\frac{du}{dx} + u$

Step 3 | Solve the separable equation

* Problem the separable equation is now in terms of "x's" and "u's"

Step 4 | Plug back-in "$y=ux$", that is, "$u=\frac{y}{x}$"

Ex | Solve $\frac{dy}{dx} = \frac{-3y}{3x-7y}$

Soln |

Check:

$$\frac{-3(ty)}{3(tx)-7(ty)} = \frac{t(-3y)}{t(3x-7y)} = \frac{-3y}{3x-7y}$$

So it's a 1^{st}-order homogeneous differential equation

Step 1&2 | We have $y=ux$. $\frac{dy}{dx} = x\frac{du}{dx}+u$, this gives:

$$x\frac{du}{dx} + u = \frac{-3ux}{3x-7ux}$$

$$\rightarrow x\frac{du}{dx} = \frac{-3ux}{3x-7ux} - u$$

$$\rightarrow \frac{du}{dx} = \frac{-3ux}{3x-7ux} - \frac{u}{x}$$

$$\rightarrow \frac{du}{dx} = -\frac{3ux-u(3x-7ux)}{3x^2-7ux^2}$$

$$\rightarrow \frac{du}{dx} = \frac{-3u-u(3x-7x)}{3x-7ux}$$

$$\rightarrow \frac{du}{dx} = \frac{-6u+7u^2}{x(3x-7u)}$$

$$\rightarrow \frac{3-7u}{7u^2-6u}du = \frac{1}{x}dx$$

we want $\underbrace{\int \frac{3-7u}{7u^2-6u}du}$ $= \int \frac{1}{x}dx$

$w = 7u^2-6u$

$dw = (14u-6) \rightarrow du = \frac{dw}{14u-6}$

$$= \int \frac{3-7u}{w}\frac{dw}{14u-6} = -\frac{1}{2}\int \frac{1}{w}dw$$

$$= -\frac{1}{2}\ln|w| + C_1 = -\frac{1}{2}\ln|7u^2-6u| + C_1$$

$$\rightarrow -\frac{1}{2}\ln|7u^2-6u| + C_1 = \ln|x| + C_2$$

$$\rightarrow -\frac{1}{2}\ln|7u^2-6u| = \ln|x| + C$$

Step 3 | Solve the separable equation

$$-\frac{1}{2}\ln|7u^2-6u| = \ln|x| + C$$

$$\rightarrow \ln|7u^2-6u| = -2\ln|x| + C$$

Recall: $\alpha\ln|x| = \ln|x^{\alpha}|$

$$\rightarrow e^{\ln|7u^2-6u|} = e^{-2\ln|x|+C}$$

$e^{\ln|x^{-2}|}$, $\underline{\underline{e^{C}}} = \underline{\underline{C}}$

$$\rightarrow 7u^2-6u = Cx^{-2}$$

Step 4 | Plug Back-in "$u=\frac{y}{x}$"

$$\frac{7y^2}{x^2} - \frac{6y}{x}x = Cx^{-2}$$

→ $7y^2-6yx = C$
Is an implicit solution

Expectation checklist - Module 7

At the completion of this module, you should:

- know the definitions introduced, and in particular
  - homogeneous function of degree α, and
  - 1st-order homogeneous differential equation;
- be able to determine if a function is homogeneous and of what degree;
- if given a 1st-order homogeneous differential equation, solve it using a substitution;
- be able to do this also for 1st order homogeneous differential equations expressed in differential form.

You will be assessed on your understanding of these concepts:

- within the homework,
- on the quizzes, and
- later, on the exam.

Coming up next, we:

- Higher order differential equations!