MATH 45 – Module 16–17 Review Solutions

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1. Rework, study, and understand all of the homework and quiz problems.

2. See the review solutions for these reviews.

3. $-\dfrac{3}{s^2+9}$: Taking $u = \sin(3t)$ and $dv = e^{-st}\,dt$ so that $du = 3\cos(3t)$ and $v = -\frac{1}{s}e^{-st}$, integration by parts gives $\mathcal{L}(\sin(3t))$ equals

$$\begin{aligned}
\int_0^\infty e^{-st}\sin(3t)\,dt &= \lim_{b\to\infty}\int_0^b \sin(3t)e^{-st}\,dt \\
&= \lim_{b\to\infty}\left(-\frac{\sin(3t)e^{-st}}{s}\Bigg|_0^b + \frac{3}{s}\int_0^b \cos(3t)e^{-st}\,dt\right) \\
&= \lim_{b\to\infty}\left(-\frac{\sin(3b)e^{-sb}}{s} + \frac{\sin(0)e^{-0}}{s} + \frac{3}{s}\int_0^b \cos(3t)e^{-st}\,dt\right) \\
&= -0 + 0 + \lim_{b\to\infty}\frac{3}{s}\int_0^b \cos(3t)e^{-st}\,dt.
\end{aligned}$$

Applying integration by parts again with $u = \cos(3t)$ and $dv = e^{-st}\,dt$ so that $du = -3\sin(3t)$ and $v = -\frac{1}{s}e^{-st}$ we find $\mathcal{L}(\sin(3t))$ equals

$$\begin{aligned}
\int_0^\infty e^{-st}\sin(3t)\,dt = \lim_{b\to\infty}\frac{3}{s}\int_0^b \cos(3t)e^{-st}\,dt &= \lim_{b\to\infty}\frac{3}{s}\left(-\frac{\cos(3t)e^{-st}}{s}\Bigg|_0^b - \frac{3}{s}\int_0^b \sin(3t)e^{-st}\,dt\right) \\
&= 0 - \frac{3}{s^2} - \lim_{b\to\infty}\frac{9}{s^2}\int_0^b \sin(3t)e^{-st}\,dt \\
&= -\frac{3}{s^2} - \frac{9}{s^2}\int_0^\infty \sin(3t)e^{-st}\,dt.
\end{aligned}$$

Rearranging gives

$$\left(1 + \frac{9}{s^2}\right)\int_0^\infty e^{-st}\sin(3t)\,dt = -\frac{3}{s^2}.$$

Solving for $\int_0^\infty \sin(3t)e^{-st}\,dt$ gives

$$\int_0^\infty \sin(3t)e^{-st}\,dt = -\frac{3}{s^2\left(1+\frac{9}{s^2}\right)} = -\frac{3}{s^2+9}.$$

Note that replacing 3 in the calculation above with the arbitrary constant k would give the general formula.

4.

(a) $\frac{6}{s^2+9}$: We have

$$\mathcal{L}(2\sin(3t)) = 2\mathcal{L}(\sin(3t)) = 2\left(\frac{3}{s^2+3^2}\right) = \frac{6}{s^2+9}.$$

(b) $\frac{7!}{3s^8}+\frac{e^3}{2s-10}$: We have

$$\mathcal{L}\left(\frac{t^7}{3}+\frac{e^{5t+3}}{2}\right)=\mathcal{L}\left(\frac{t^7}{3}\right)+\mathcal{L}\left(\frac{e^{5t}e^3}{2}\right)=\frac{1}{3}\mathcal{L}\left(t^7\right)+\frac{e^3}{2}\mathcal{L}\left(e^{5t}\right)$$
$$=\frac{1}{3}\frac{7!}{s^{7+1}}+\frac{e^3}{2}\frac{1}{s-5}=\frac{7!}{3s^8}+\frac{e^3}{2s-10}.$$

(c) $\frac{3}{2}\cos\left(\sqrt{\frac{5}{2}}t\right)$: Noting it resembles the formula for $\cos(kt)$ we find

$$\mathcal{L}^{-1}\left(\frac{3s}{2s^2+5}\right)=\mathcal{L}^{-1}\left(\frac{3s}{2\left(s^2+\frac{5}{2}\right)}\right)=\frac{3}{2}\mathcal{L}^{-1}\left(\frac{s}{s^2+\left(\frac{\sqrt{5}}{\sqrt{2}}\right)^2}\right)=\frac{3}{2}\cos\left(\sqrt{\frac{5}{2}}t\right).$$

(d) $\frac{1}{2}t^3-\frac{1}{3}$: We have

$$\mathcal{L}^{-1}\left(\frac{3}{s^4}-\frac{1}{3s}\right)=\mathcal{L}^{-1}\left(\frac{3}{s^4}\right)-\mathcal{L}^{-1}\left(\frac{1}{3s}\right)=\frac{1}{2}\mathcal{L}^{-1}\left(\frac{6}{s^{3+1}}\right)-\frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{s}\right)=\frac{1}{2}t^3-\frac{1}{3}.$$

5.

(a) $\frac{7!}{3(s-2)^8}+\frac{e^3}{2(s-2)}$: From part (b) of the previous exercise we saw $\mathcal{L}\left(\frac{t^7}{3}\right)=\frac{7!}{3s^8}$. Using that $\mathcal{L}(e^{kt}f(t))=F(s-k)$ we find

$$\mathcal{L}\left(e^{2t}\frac{t^7}{3}+e^{-3t}\frac{e^{5t+3}}{2}\right)=\frac{1}{3}\mathcal{L}\left(e^{2t}t^7\right)+\frac{e^3}{2}\mathcal{L}\left(e^{2t}\right)=\frac{7!}{3(s-2)^8}+\frac{e^3}{2(s-2)}.$$

(b) $\frac{2}{3}e^{2t}\sin(3t)$: Completing the square and using that $\mathcal{L}(e^{kt}f(t))=F(s-k)$ we find

$$\mathcal{L}^{-1}\left(\frac{2}{s^2-4s+13}\right)=\frac{2}{3}\mathcal{L}^{-1}\left(\frac{3}{(s-2)^2+9}\right)=\frac{2}{3}e^{2t}\sin(3t).$$

6.

(a) $2e^{3t}$: Using the Laplace transform we find $\mathcal{L}(y'-2y)=\mathcal{L}(2e^{3t})$ which becomes $\mathcal{L}(y')-2\mathcal{L}(y)=2\mathcal{L}(e^{3t})$. This in turn becomes $sY(s)-y(0)-2Y(s)=\frac{2}{s-3}$, or $sY(2)-2-2Y(s)=\frac{2}{s-3}$. Solving for $Y(s)$ we have

$$Y(s)=\frac{2}{(s-3)(s-2)}+\frac{2}{s-2}.$$

Applying the inverse Laplace we have

$$y=\mathcal{L}^{-1}\left(\frac{2}{(s-3)(s-2)}\right)+\mathcal{L}^{-1}\left(\frac{2}{s-2}\right).$$

By partial fraction decomposition (actually we could simplify and avoid PFD, but we do it here for practice) we find

$$\frac{2}{(s-3)(s-2)} = \frac{2}{s-3} - \frac{2}{s-2}.$$

Thus,

$$y = \mathcal{L}^{-1}\left(\frac{2}{s-3}\right) - \mathcal{L}^{-1}\left(\frac{2}{s-2}\right) + \mathcal{L}^{-1}\left(\frac{2}{s-2}\right) = \mathcal{L}^{-1}\left(\frac{2}{s-3}\right) = 2e^{3t}.$$

(b) $3e^{-2t} - 2e^{-3t}$: Using the Laplace transform we find $\mathcal{L}(y'' + 5y' + 6y) = \mathcal{L}(0)$ which becomes $\mathcal{L}(y'') + 5\mathcal{L}(y') + 6\mathcal{L}(y) = 0$. This in turn becomes $s^2Y(s) - sy(0) - y'(0) + 5(sY(s) - y(0)) + 6Y(s) = 0$, or $s^2Y(s) - s - 0 + 5(sY(s) - 1) + 6Y(s) = 0$. Solving for $Y(s)$ we have

$$Y(s) = \frac{s+5}{s^2+5s+6}.$$

Applying the inverse Laplace we have

$$y = \mathcal{L}^{-1}\left(\frac{s+5}{s^2+5s+6}\right).$$

By partial fraction decomposition we find

$$y = \mathcal{L}^{-1}\left(\frac{3}{s+2} - \frac{2}{s+3}\right).$$

Thus,

$$y = \mathcal{L}^{-1}\left(\frac{3}{s+2}\right) - \mathcal{L}^{-1}\left(\frac{2}{s+3}\right) = 3e^{-2t} - 2e^{-3t}.$$

7.

(a) We have

$$\mathbf{X}' = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \mathbf{X}.$$

(b) We have

$$\frac{dy}{dx} = 2x + 2y$$
$$\frac{dy}{dy} = 3x + y.$$

8.

(a) We have $y' = 5\binom{1}{1}e^{5t}$ and $\left(\begin{smallmatrix} 1 & 4 \\ 2 & 3 \end{smallmatrix}\right) \begin{pmatrix} e^{5t} \\ e^{5t} \end{pmatrix} = \begin{pmatrix} e^{5t} + 4e^{5t} \\ 2e^{5t} + 3e^{5t} \end{pmatrix} = \begin{pmatrix} 5e^{5t} \\ 5e^{5t} \end{pmatrix}.$

(b) The differential equation
$$\mathbf{X}' = \begin{pmatrix} 2 & 2 \\ 3 & 1 \end{pmatrix} \mathbf{X}.$$
with $y = \begin{pmatrix} -\frac{2}{3} \\ 1 \end{pmatrix} e^{-t}$. We have $y' = \begin{pmatrix} \frac{2}{3} \\ -1 \end{pmatrix} e^{-t}$ and $\left(\begin{smallmatrix} 2 & 2 \\ 3 & 1 \end{smallmatrix}\right) \begin{pmatrix} -\frac{2}{3}e^{-t} \\ e^{-t} \end{pmatrix} = \begin{pmatrix} -\frac{4}{3}e^{-t} + 2e^{-t} \\ -2e^{-t} + e^{-t} \end{pmatrix} = \begin{pmatrix} \frac{2}{3}e^{-t} \\ -e^{-t} \end{pmatrix}$.

9.

(a) Linearly dependent: Since
$$W(y_1, y_2) = \det\left(\begin{pmatrix} e^{5t} & 2e^{-3t} \\ e^{5t} & 2e^{-3t} \end{pmatrix}\right) = 0.$$

(b) Linearly independent: Since
$$W(y_1, y_2) = \det\left(\begin{pmatrix} e^{4t} & -\frac{2}{3}e^{-t} \\ e^{4t} & e^{-t} \end{pmatrix}\right) = e^{3t} + \frac{2}{3}e^{3t} = \frac{5}{3}e^{3t} \neq 0$$
on $(-\infty, \infty)$.

10. Yes, they do. This follows from the fact it is a system of two equations with two variables and
$$W(y_1, y_2) = \det\left(\begin{pmatrix} e^{5t} & -2e^{-t} \\ e^{5t} & e^{-t} \end{pmatrix}\right) = e^{4t} + 2e^{4t} = 3e^{4t} \neq 0$$
on $(-\infty, \infty)$.

11. By Exercise 10(b) we know $y_2 = \begin{pmatrix} -\frac{2}{3} \\ 1 \end{pmatrix} e^{-t}$ is a solution to the system of DEs. It can similarly be shown that $y_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$ is a solution. By Exercise 11(b) these are linearly dependent. Since the system consists of two equations with two unknowns, we have that y_1 and y_2 form a fundamental set of solutions and thus $y = c_1 y_1 + c_2 y_2$ is the general solution.

12.

(a) $y = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-t}$: In matrix form this is
$$\mathbf{X}' = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \mathbf{X}.$$

Solving for λ in the equation
$$\begin{pmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda) - 6 = \lambda^2 - 4\lambda - 5 = 0$$

we find $\lambda = 5$ and $\lambda = -1$. In the case $\lambda = 5$ we solve the equation

$$\begin{pmatrix} 1-5 & 4 \\ 2 & 3-5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives $-4x = 4y$ and $2x = -2y$. Thus, $x = -y$. We could take any (nonzero) values that satisfy this, so we might as well take $x = 1$ and $y = 1$. Thus the first solution is

$$y_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t}.$$

In the case $\lambda = -1$ we solve the equation

$$\begin{pmatrix} 1+1 & 4 \\ 2 & 3+1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives $2x = -4y$ and $2x = -4y$. Thus, $x = -2y$. We could take any (nonzero) values that satisfy this, so we might as well take $y = 1$ which gives $x = -2$. Thus the second solution is

$$y_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-t}.$$

The fact that these are linearly independent and form a fundamental set of solutions should also be shown. However this follows from Exercise 12 above.

(b) $y = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} -\frac{2}{3} \\ 1 \end{pmatrix} e^{-t}$: Solving for λ in the equation

$$\begin{pmatrix} 2-\lambda & 2 \\ 3 & 1-\lambda \end{pmatrix} = (2-\lambda)(1-\lambda) - 6 = \lambda^2 - 3\lambda - 4 = 0$$

we find $\lambda = 4$ and $\lambda = -1$. In the case $\lambda = 4$ we solve the equation

$$\begin{pmatrix} 2-4 & 2 \\ 3 & 1-4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives $2x = 2y$ and $3x = 3y$. Thus, $x = -y$. We could take any (nonzero) values that satisfy this, so we might as well take $x = 1$ and $y = 1$. Thus the first solution is

$$y_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}.$$

In the case $\lambda = -1$ we solve the equation

$$\begin{pmatrix} 2+1 & 2 \\ 3 & 1+1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives $3x = -2y$ and $3x = -2y$. Thus, $x = -\frac{2}{3}y$. We could take any (nonzero) values that satisfy this, so we might as well take $y = 1$ which gives $x = -\frac{2}{3}$. Thus the second solution is

$$y_2 = \begin{pmatrix} -\frac{2}{3} \\ 1 \end{pmatrix} e^{-t}.$$

That these form a fundamental set of solutions follows from Exercise 12.