www.simiode.org

SIMIODE A SYSTEMIC INITIATIVE FOR MODELING INVESTIGATIONS & OPPORTUNITIES WITH DIFFERENTIAL EQUATIONS

Spread of Oil Slick

Brian Winkel, Director@simiode.org

SIMIODE, Cornwall NY USA

Spread of Oil Slick

We present data on size of an oil slick from photographic data during intermittent fly overs.

1. Data
2. Plots, differencing, and inferences
3. Difference equation and differential equation
4. Solutions, estimate parameters, validate model
5. Discussions

Source:
1-005-S-OilSlick **https://www.simiode.org/resources/196**
1-005-T-OilSlick **https://www.simiode.org/resources/184**

Observation data on oil slick on fly over imagery.

Size of Slick (square miles)	
Initial Observation	10 min. later
1.047	1.139
2.005	2.087
3.348	3.413
5.719	5.765
7.273	7.304
8.410	8.426
9.117	9.127

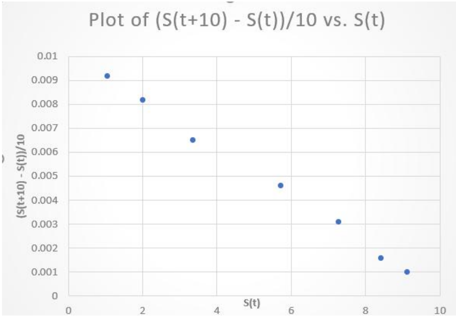
Take 2 minutes to think about how we could build a model for size of oil slick over time.

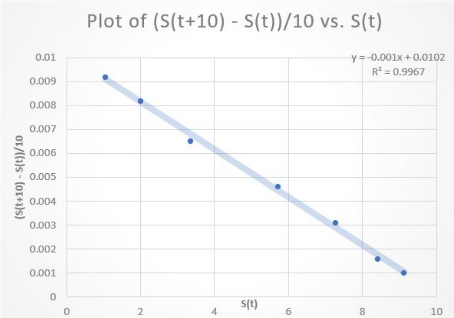
What is missing? How do we proceed?

Size of Slick (square miles)	
Initial Observation	10 min. later
1.047	1.139
2.005	2.087
3.348	3.413
5.719	5.765
7.273	7.304
8.410	8.426
9.117	9.127

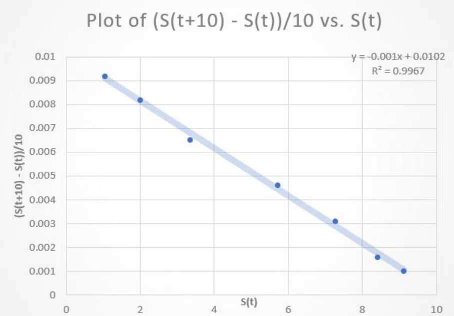
S(t)	S(t+10)	(S(t+10) - S(t))/10
1.047	1.139	0.0092
2.005	2.087	0.0082
3.348	3.413	0.0065
5.719	5.765	0.0046
7.273	7.304	0.0031
8.41	8.426	0.0016
9.117	9.127	0.001

S(t)	S(t+10)	(S(t+10) - S(t))/10
1.047	1.139	0.0092
2.005	2.087	0.0082
3.348	3.413	0.0065
5.719	5.765	0.0046
7.273	7.304	0.0031
8.41	8.426	0.0016
9.117	9.127	0.001



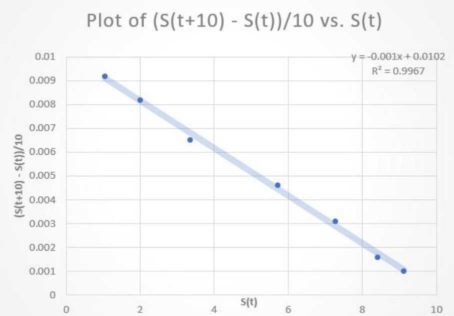


$$\frac{s(t+10)-s(t)}{10} = -0.001s(t) + 0.0102$$



$$\frac{s(t+10)-s(t)}{10} = -0.001s(t) + 0.0102$$

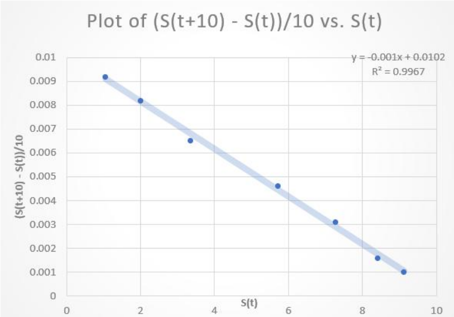
$$\frac{s(t+\Delta t)-s(t)}{\Delta t} = -0.001s(t) + 0.0102$$



$$\frac{s(t+10)-s(t)}{10} = -0.001s(t) + 0.0102$$

$$\frac{s(t+\Delta t)-s(t)}{\Delta t} = -0.001s(t) + 0.0102$$

$$\lim_{\Delta t \to 0} \frac{s(t+\Delta t)-s(t)}{\Delta t} = -0.001s(t) + 0.0102$$



$$\frac{s(t+10)-s(t)}{10} = -0.001s(t) + 0.0102$$

$$\lim_{\Delta t \to 0} \frac{s(t+\Delta t)-s(t)}{\Delta t} = -0.001s(t) + 0.0102$$

$$s'(t) = -0.001s(t) + 0.0102$$

$$s'(t) = -0.001s(t) + 0.0102$$

AHA!!! We have a differential equation model based on a difference equation model.

Remind ourselves of our units!

We seek to solve this differential equation, but we need an initial condition. Where might we get this?

$$s'(t) = -0.001s(t) + 0.0102$$

We seek to solve this differential equation and we do have an initial condition

$$s(0) = 1.047$$

from first observation.

What solution strategies might we employ? One or more?

$$s'(t) = -0.001s(t) \, + \, 0.0102, \quad s(0) = 1.047$$

What solution strategies or techniques do we have for solving this differential equation?

Just look at these two forms and identify a strategy for each.

$$s'(t) = -0.001s(t) + 0.0102, \quad s(0) = 1.047$$

$$s'(t) + 0.001s(t) = 0.0102, \quad s(0) = 1.047$$

OR we could go to

https://www.wolframalpha.com

Syntax:

{s'(t) = -.001 s(t) + .0102,s(0) = 1.047}

Separation of Variables

$$s'(t) = -0.001s(t) + 0.0102, \quad s(0) = 1.047$$

Integrating Factor

$$s'(t) + 0.001s(t) = 0.0102, \quad s(0) = 1.047$$

Separation of Variables

$$s'(t) = -0.001s(t) + 0.0102$$

Separate the variables by dividing both sides by

$-0.001s(t) + 0.0102.$

$$\left(\frac{1}{-0.001s(t) + 0.0102}\right) s'(t) = 1\,.$$

Then what do we do to move on?

We take the antiderivative of both sides with respect to t and do not forget to add the constant of integration C.

$$\left(\frac{1}{-0.001s(t)\ +\ 0.0102}\right) s'(t) = 1\,,$$

$$\int \frac{1}{-0.001s(t)\ +\ 0.0102} s'(t)\, dt = \int 1\, dt + C\,.$$

Identify

$$u = -0.001s(t)\ +\ 0.0102\,.$$

So

$$u' = -0.001s'(t)$$

and make replacements to obtain

$$\frac{1}{-.001}\int \underbrace{\frac{1}{-0.001s(t)\ +\ 0.0102}}_{u}\underbrace{(-0.001s'(t))\,dt}_{du} = \int 1\,dt + C\,.$$

Do we see u and du? Then integrate.

$$\frac{1}{-.001}\int \frac{1}{-0.001s(t)\ +\ 0.0102}(-0.001s'(t))\,dt = \int 1\,dt + C\,,$$

$$\frac{1}{-.001}\ln(|-0.001s(t)\ +\ 0.0102|) = t + C\,.$$

Since $-0.001s(t)\ +\ 0.0102\ > 0$ for our data we have

$$\frac{1}{-.001}\ln(-0.001s(t)\ +\ 0.0102) = t + C\,.$$

We can go after C using initial condition $s(0) = 1.047$.

$$\frac{1}{-.001} \ln(-0.001s(t) + 0.0102) = t + C\,,$$

$$\frac{1}{-.001} \ln(-0.001s(0) + 0.0102) = 0 + C\,.$$

Now using $s(0) = 1.047$ we have

$$\frac{1}{-.001} \ln(-0.001 \cdot 1.047 + 0.0102) = C$$

and finally for C we have

$$4693.97 = C\,.$$

Now substitute $C = 4693.97$ into

$$\frac{1}{-.001} \ln(-0.001s(t) + 0.0102) = t + C$$

$$\frac{1}{-.001} \ln(-0.001s(t) + 0.0102) = t + 4693.97$$

and solve for $s(t)$ we find that

$$s(t) = 10.2 - 9.153e^{-0.001t}\,.$$

There we have it - our solution to the differential equation model

$$s'(t) = -0.001s(t) + 0.0102, \quad s(0) = 1.047\,,$$

namely:

$$s(t) = 10.2 - 9.153e^{-0.001t}\,.$$

STOP and **LOOK** at $s(t)$. What does this tell us about the long term size of the oil slick? Does this make us comfortable or uncomfortable? We have already learned something from our modeling $s(t)$.

Integrating Factor

$$s'(t) + 0.001s(t) = 0.0102, \quad s(0) = 1.047$$

Recall we seek to shape the Left Hand Side of our differential equation as the derivative of a product of two functions.
To do this we create an integrating factor

$$I(t) = e^{\int 0.001\, dt} = e^{0.001t}$$

and multiply both sides of our differential equation by $I(t)$, looking for the derivative of a product using the Product Rule.

$$e^{0.001t}(s'(t) + 0.001s(t)) = e^{0.001t} \cdot 0.0102$$

If we multiply through we see the product $u(t) \cdot v(t)$ with identification of

$$u(t) = e^{0.001t} \text{ and } v(t) = s(t).$$

$$e^{0.001t}(s'(t) + 0.001s(t)) = e^{0.001t} \cdot 0.0102$$

$$e^{0.001t}s'(t) + e^{0.001t}0.001s(t)) = e^{0.001t} \cdot 0.0102$$

$$\underbrace{e^{0.001t}}_{u(t)}\underbrace{s'(t)}_{v'(t)} + \underbrace{s(t)}_{v(t)}\underbrace{e^{0.001t}0.001}_{u'(t)}) = e^{0.001t} \cdot 0.0102$$

OR

$$\frac{d(u(t)v(t))}{dt} = e^{0.001t} \cdot 0.0102 \,.$$

Thus, integrating both sides yields

REMEMBER the constant of integration!

$$\int \frac{d(u(t)v(t))}{dt}\,dt = \int e^{0.001t} \cdot 0.0102\,dt + C\,.$$

From here we see

$$e^{0.001t} \cdot s(t) = u(t) \cdot v(t) = e^{0.001t} \cdot \frac{0.0102}{0.001} + C\,.$$

Let us use our Initial Condition to ascertain C, i.e. $s(0) = 1.047$.

$$1.047 = s(0) = e^{0.001 \cdot 0} \cdot s(0) = e^{0.001 \cdot 0} \cdot \frac{0.0102}{0.001} + C = 10.2 + C\,.$$

Thus we see that $C = 1.047 - 10.2 = -9.153$.

It then follows from

$$e^{0.001t} \cdot s(t) = u(t)v(t) = e^{0.001t} \cdot \frac{0.0102}{0.001} - 9.153$$

that our solution for $s(t)$ is

$$s(t) = 10.2 - 9.153e^{-0.001t}$$

which is exactly what we got with our Separation of Variables technique.

Assignment

1. Write overview of the modeling process to obtain $s(t)$, size of oil slick in square miles, at time t hr, using data.
2. Use your model for $s(t)$ to plot size of oil slick over time.
3. With your model ascertain exact times at which photographs were taken.
4. Plot model over data to affirm or validate your model.
5. Predict long term behavior of oil slick.

CONGRATULATIONS!!!

We went from

- seeing data,
- to realizing a difference equation model,
- to converting to a differential equation model,
- to solving the differential equation,
- to comparing your model with the actual data,
- to filling in observation times of the data,
- to making predictions.

We went from confusing data, to discrete difference equation model using differencing, to linear regression (fitting straight line or TrendLine in Excel), to conversion to differential equation, to solution of the differential equation (in several different ways!!!), to finally answering relevant questions about our oil slick.