1st-order homogeneous DEs

We note right away, that very shortly we will encounter yet another (different) definition of *homogeneous* in this class. In an attempt to keep the matters separate, we refer to the homogeneous notion at hand as being a 1st-order homogeneous differential equation (as opposed to later a homogeneous differential equation). First we look at what it means for a *function* to be homogeneous.

Definition:

A function $f(x, y)$ is called **homogeneous (of degree α)** if for some $t \neq 0$ we have $f(tx, ty) = t^{\alpha} f(x, y)$.

Here, α is a real number. For example, since for $f(x, y) = x^2 y^2$ we have

$$f(tx, ty) = (tx)^2 (ty)^2 = t^2 x^2 y^2 = t^2 f(x, y),$$

we have $f(x, y) = x^2 y^2$ is a homogeneous function of degree 2. Note that $f(x, y) = x^2 y^3$ is not homogeneous. Now, we can return to differential equations. We break it into two (equivalent) notions.

Definition:

1. A differential equation $\dfrac{dy}{dx} = f(x, y)$ is a **1st-order homogeneous differential equation** if $f(x, y)$ is a homogeneous function of degree 0.
2. A differential equation $M(x, y)dx + N(x, y)dy = 0$ is a **1st-order homogeneous differential equation** if $M(x, y)$ and $N(x, y)$ are both homogeneous functions of the same degree.

These notions are the same, after rearranging. Indeed, if $M(x, y)dx + N(x, y)dy = 0$ is a 1st-order homogeneous equation, then $M(x, y)$ and $N(x, y)$ are both homogeneous functions of the same degree, say α. Dividing both sides of $M(x, y)dx + N(x, y)dy = 0$ by dx and rearranging the expression gives

$$\frac{dy}{dx} = \frac{M(x, y)}{N(x, y)}.$$

Thus, if $M(x, y)$ and $N(x, y)$ are both homogeneous functions of the same degree, it means $f(x, y) = \dfrac{M(x, y)}{N(x, y)}$ is a homogeneous equation of degree 0, as in the part 1 of the definition.

We will find that given a differential equation as in part 2 of the definition, it will be convenient to alter it into the form seen in part 1.

Discussion, comments, and examples:

WeBWorK module 07 exercises:

- Problems 1, 2, 3

Relevant Wikipedia articles:

- **Homogeneous functions (https://en.wikipedia.org/wiki/Homogeneous_function)**