<u>Definition</u>:

A **differential equation** is an equation that relates one or more functions and their derivatives.

Ex1

(a) $ax^2+bx+c=0$ is an eqn.

(b) $y''' + y' = 0$ " "

(c) $y^{(3)} + y' = 0$ " "

(d) $\dfrac{x^2y+z}{x-4} = 5$ " "

(e) x^2+y is Not an eqn

(f) $\{1,2,3\}$ is Not an eqn (it's a set)

Ex

(a) $ax^2+bx+c=0$ is Not a DE.

(b) $y''' + y' = 0$ is a DE

(c) $y' = \pi$ is a DE

Q: what is y? (A: $y = \pi x$)

(d) $y'' + y' + y$ is NOT a DE (Not an eqn)

(e) $x'' + x = 0$ is a DE (maybe $x(s) = s^2$)

(f) $\frac{dy}{dx} = 7y$ is a DE

(g) $e^{f'(x)} = f(x)$ is a DE

(h) $\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} = 0$ is a DE (partial derivatives)

Ex

- $y^2 + y = 0$ is NOT a DE (y^2 is $y \cdot y$)
- $f(x) = x^2$ is Not a DE.

Q: Is y in $y'' + y' + y = 0$ a function/variable?
Is x? How about $x'' + x = 0$? ($y' + y = e^x$)

A: We want to find the dependent variables.

Definition:

1. A symbol that represents an input of a function is called an **independent variable**.
2. A symbol that represents an output of a function is called a **dependent variable**.

also

Definition:

A **multivariable function** is a function which consists of more than one independent variable.

Ex1 In the following state the dep and indep variables:

(a) $y = x^2$. dep $= y$, indep $= x$.

(b) $y' = 4$ dep $= y$, indep $= \cancel{x}$ $(y = f(x))$

(c) $f(x) = x^2$ dep $= f$, indep $= x$

(d) $x' + x = y$ dep $= x$, indep $= y$ $(x(y) = ??)$

(e) $\frac{dx}{dy} + x = y$ dep $= x$, indep $= y$

(f) $g(x,y) = \frac{x^2 y}{2}$ dep $= g$, indep $= x, y$.

Ex1 For $g(x,y,z) = x^2 y^3 + z$ we have

$\rightarrow \dfrac{d}{dx} g(x,y,z) = 2xy^3 + 0 = \underline{2xy^3}$

$\rightarrow \dfrac{dg}{dy} = x^2(3y^2) + 0 = \underline{3x^2y^2}$

$\rightarrow \dfrac{dg}{dz} = 0 + \dfrac{d}{dz}(z) = \underline{1}$

Ex| Consider $f(x,y) = \cos(x^2y^3)$

$$\to \frac{df}{dx} = -\sin(x^2y^3)\left(\frac{d}{dx}(x^2y^3)\right)$$

chain rule

$$= -\sin(x^2y^3)(2xy^3) = -2xy^3\sin(x^2y^3).$$

$$\to \frac{df}{dy} = -\sin(x^2y^3)(3x^2y^2) = -3x^2y^2\sin(x^2y^3).$$

Ex| $\frac{d}{dz}\left(\frac{x+y}{z}\right) = (x+y)\frac{d}{dz}\left(\frac{1}{z}\right) = (x+y)\left(-\frac{1}{z^2}\right) = \frac{-(x+y)}{z^2}.$

Ex1 classify as an ODE or PDE in the following:

(a) $\frac{dy}{dx} + e^{x} + 6y + x = 0$ ODE

(b) $\frac{d}{dx}g + \frac{d}{dy}g - xy = 0$ PDE

(c) $\frac{dx}{dt} + \frac{dy}{dt} - x - y = 0$ ODE

Linear: derivatives and function occur as polynomials of degree 1.

Recall: A polynomial in $\overset{y'}{\overset{y}{x}}$ is of the form

$$c_n \underset{(y')^n}{\overset{y^n}{x^n}} + c_{n-1} \underset{(y')^{n-1}}{\overset{y^{n-1}}{x^{n-1}}} + \cdots + c_1 \underset{y'}{\overset{y}{x}} + c_0$$

$c_n, c_{n-1}, \ldots, c_1, c_0$ constants.

$\rightarrow n$ the degree.

So linear DE must only have y or y' or $\ldots y^{(n)}$.

But can't have $(y')^2$, y^2, $(y')^n$, $(y^{(3)})^2$, $\underline{y'' \cdot y'}$.

Ex| Determine if the following are linear or nonlinear:

(a) $yy' = 2y$ NonLin

(b) $y^{(100)} - y'' + y = 0$ Lin

(c) $y^2 - y = 0$ NonLin

(d) $\frac{d^2 f}{dx^2} + f = 0$ Lin

(e) $\left(\frac{df}{dx}\right)^2 + f = 0$ NonLin

(f) $\sqrt{y'} + y = 0$ $\to (y')^{1/2}$ NonLin

(g) $\sin(y^{(5)}) + y = 5$ NonLin

(h) $y' = e^y$ NonLin.

(i) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + \underline{e^x} y = 0$ Lin.

Ex) Find the order of the following DEs:

(a) $y'''' - y''' + y'' - y' + y = 0$ 4th-order DE (linear ODE)

(b) $\frac{d^2}{dx^2} g(x,y) + \frac{d^3}{dy^3} g(x,y) = 0$ 3rd-order PDE

(c) $\frac{d^2 f}{dx^2} + f = 0$ 2nd-order linear ODE

(d) $\left(\frac{df}{dx}\right)^2 + f = 0$ 1st-order nonlinear ODE

(e) $\ln(y'') + y' = 0$ 2nd-order DE

Consider the DE

$$2x + y + 3x\frac{dy}{dx} + 4 = 0.$$

This could also be written

[“multiplying” by dx]

$$2x\,dx + y\,dx + 3x\,dy + 4\,dx = 0$$

Differential form

$$\Leftrightarrow \quad (*) \quad \boxed{(2x + y + 4)\,dx + 3x\,dy = 0}$$

[solve for $\frac{dy}{dx}$]

Normal form

$$\frac{dy}{dx} = -\frac{1}{3x}(2x + y + 4)$$

$$\Leftrightarrow \quad \boxed{(\#) \quad \frac{dy}{dx} = -\frac{(2x + y + 4)}{3x}}$$

OR

[put $\frac{dy}{dx}$ in front]

$$3x\frac{dy}{dx} + 2x + y + 4 = 0$$

standard form

$$\Leftrightarrow \quad \boxed{\frac{dy}{dx} + \frac{y}{3x} + \frac{2x + 4}{3x} = 0}$$

Ex) Express $x^2 dx + y dy = 0$ in normal form.

soln) $\cancel{x^2 \frac{dx}{dx}} + y \frac{dy}{dx} = 0 \iff x^2 + y\frac{dy}{dx} = 0$

$\iff y\frac{dy}{dx} = -x^2 \iff \frac{dy}{dx} = -\frac{x^2}{y}.$

Ex) Is $x^2 y^{(3)} + y'' - y = 3$ in standard form?

soln) No. But $y^{(3)} + \frac{1}{x^2}y'' - \frac{1}{x^2}y - \frac{3}{x^2} = 0.$

We sometimes use $F(x, y, y', \dots, y^{(n)})$ to denote an arbitrary expression involving $x, y, y', y'', \dots, y^{(n)}$.

Ex) $\frac{dy}{dx} = y^3 x + x^2$ or $\frac{dy}{dx} = F(x,y)$, where $F(x,y) = y^3 x + x^2$.

$\Leftrightarrow$

$y' - y^3 x - x^2 = 0$

OR

$F(x, y, y') = 0$ where

$F(x, y, y') = y' - y^3 x - x^2$

Consider DE

$$F(x, y', y'') = 0.$$

OR

$$y^{(5)} = F(x, y, y').$$