MATH 45 – Exam Two Review Solutions

Dr. Krauel

---

1. Rework, study, and understand all of the homework and quiz problems.

2. For each of the following differential equations, state whether it is linear or nonlinear. Also, if it is linear state whether it is homogeneous or nonhomogeneous.

(a) Linear and homogeneous.

(b) Nonlinear.

(c) Linear and nonhomogeneous.

(d) Linear and nonhomogeneous.

3. Determine whether a unique solution is guaranteed to exist for the following initial value problems on the given interval I. Explain your answer.

(a) Not guaranteed: Since $x^2 = 0$ when $x = 0$, and 0 is in the given interval.

(b) Guaranteed: Since x^2, x, $\sin(x)$, and x are all continuous and $x^2 \neq 0$ on the given interval.

(c) Not guaranteed: Since $\frac{1}{x-2}$ is not continuous on the given interval.

(d) Not guaranteed: Since 1, $\tan(x)$, and x^2 are all continuous on the interval $\left(0, \frac{\pi}{2}\right)$, but $x_0 = \frac{\pi}{2}$ is not in I.

4. We need to know the initial value condition $y(x_0) = y_0$. $(-\infty, 1)$ or $(1, \infty)$: We note that 1 and $\sin(x)$ are continuous and $1 \neq 0$ on $(-\infty, \infty)$. However, $\frac{1}{1-e^x}$ is only continuous on $(-\infty, 0)$ and $(0, \infty)$. Then it depends on which of these intervals x_0 is in. For example, if $x + 0 = 5$, then the interval would be $(0, \infty)$. While if $x_0 = -3$, we could take $(-\infty, 0)$.

5. $\mathcal{W}(\cos(3t), \sin(3t)) = 3$.

6. Suppose the following functions are solutions to a differential equation. Determine whether they are linearly independent solutions on the given interval.

(a) Linearly independent: Since $\mathcal{W}(e^{2x}, e^{-\frac{3}{2}x}) = -\dfrac{7}{2}e^{\frac{x}{2}} \neq 0$ on $(-\infty, \infty)$.

(b) Linearly dependent: Since $\mathcal{W}(e^x, 3e^{-x+2}) = 0$ on $(0, \infty)$.

(c) Linearly independent: Since $\mathcal{W}(x, xe^x) = x^2e^x \neq 0$ on $(-5, -2)$.

7. Yes: Plug in the functions to verify. Since the order of the differential equation is 3 and we have 3 solutions, these will form a fundamental set if they are linearly independent. Calculating the Wronskian we find $\mathcal{W}(1, \cos(t), \sin(t)) = 1 \neq 0$ on $(-\infty, \infty)$. Therefore, these functions form a fundamental set of solutions on $(-\infty, \infty)$.

8.

(a) $y = c_1e^x + c_2e^{-3x}$: Since $m^2 + 2m - 3 = (m-1)(m+3) = 0$ has two distinct roots.

(b) $y = c_1e^{-2x} + c_2xe^{-2x}$: Since $m^2 + 4m + 4 = (m + 2)^2$. So one solution is $y_1 = e^{-2x}$. The other solution, found by reduction of order is

$$y_2 = e^{-2x}\int \frac{e^{-4x}}{e^{-4x}}\,dx = e^{-2x}\int dx = xe^{-2x}.$$

Thus, the general solution is $y = c_1e^{-2x} + c_2xe^{-2x}$.

(c) $y = c_1e^{3x}\cos(2x) + c_2e^{3x}\sin(2x)$: Since $m^2 - 6m + 13 = 0$ gives $m = 3 \pm 2i$. Thus, the two *complex* solutions are $\overline{y}_1 = e^{(3+2i)x}$ and $\overline{y}_2 = e^{(3-2i)x}$. However, we want the real solutions, which we find by using Euler's formula. This gives, $\overline{y}_1 = e^{3x}e^{2ix} = e^{3x}(\cos(2x)_i\sin(2x))$ and $\overline{y}_2 = e^{3x}e^{-2ix} = e^{3x}(\cos(-2x) + i\sin(-2x))$. The real part of the first solution is $y_1 = e^{3x}\cos(2x)$. Meanwhile, while the real part of the second solution is $e^{3x}\cos(-2x)$, we recall that $\cos(-2x) = \cos(2x)$ so it would simply equal the first equation. As we have seen in class, however, reduction of order will show the second linearly independent solution is $y_2 = e^{3x}\sin(2x)$. Thus, the general solution is $y = c_1e^{3x}\cos(2x) + c_2e^{3x}\sin(2x)$.

(d) $y = c_1e^x + c_2xe^x + c_3e^{-x}$: Since $m^3 - m^2 - m + 1 = (m-1)^2(m+1)$. The term $(m-1)^2$ means there is a repeated root. First we have $y_1 = e^x$. Performing reduction of order gives

$$y_2 = y_1\int \frac{e^{-\int P}}{y_1^2}\,dx.$$

The P here comes from the P in the equation $(m-1)^2 = m^2 - 2m + 1$, that is $P = -2$. Thus,

$$y_2 = y_1\int \frac{e^{-\int P}}{y_1^2}\,dx = e^x\int \frac{e^{2x}}{e^{2x}}\,dx = e^x\int dx = xe^x.$$

Meanwhile, the term $(m + 1)$ gives that $y_3 = e^{-x}$ is also a solution. Thus, the general solution is $y = c_1e^x + c_2xe^x + c_3e^{-x}$.

9. Yes: First, we plug in $y_1 = x$ and $y_2 = xe^x$ to verify these are solutions. Next since the order of the differential equation is order 2 and $\mathcal{W}(x, xe^x) \neq 0$ on $(-\infty, 0)$ and on $(0,\infty)$, we have that the the solution if a general solution on either of these intervals.

10. First we plug in $y_1 = e^{-t}$ and $y_2 = e^{-2t}$ into the homogeneous equation $y''+3y'+2y = 0$ to verify that these are both solutions to this part. Meanwhile, since the order of the differential equation is 2, and $\mathcal{W}(e^{-t}, e^{-2t}) \neq 0$, we have that $y_c = c_1e^{-t} + c_2e^{-2t}$ is the general solution to the homogeneous part. It remains to show that $y_p = \frac{5}{12}e^{2t}$ is a particular solution to the nonhomogeneous equation $y'' + 3y' + 2y = 5e^{2t}$. Plugging in y_p, $y_p' = \frac{5}{6}e^{2t}$, and $y_p'' = \frac{5}{3}e^{2t}$ into the equation to verify it is a solution completes the verification.

11. For the following differential equations we are given that the given y_1 is a solution. Find a second solution y_2.

(a) $y_2 = t^3$: By reduction of order.

(b) $y_2 = \cos(x^2)$: By reduction of order.

12. Solve the following differential equations.

(a) $y = c_1 e^{\frac{2}{3}x} + c_2 e^{-x} - \frac{5}{13}\cos(x) + \frac{1}{13}\sin(x)$: We first find that $y_c = c_1 e^{\frac{2}{3}x} + c_2 e^{-x}$ is the solution to the homogeneous equation $3y''+y'-2y = 0$. The derivatives of $\cos(x)$ is only $\sin(x)$. Thus, the particular solution is of the form $y_p = A\cos(x) + B\sin(x)$. Plugging $y_p' = -A\sin(x) + B\cos(x)$ and $y_p'' = -A\cos(x) - B\sin(x)$ into the nonhomogeneous equation $3y'' + y' - 2y = 2\cos(x)$ gives

$$\begin{aligned}
3y_p''+&y_p' - 2y_p \\
&= 3(-A\cos(x) - B\sin(x)) + (-A\sin(x) + B\cos(x)) - 2(A\cos(x) + B\sin(x)) \\
&= (-3A + B - 2A)\cos(x) + (-3B - A - 2B)\sin(x) \\
&= (-5A + B)\cos(x) + (-5B - A)\sin(x).
\end{aligned}$$

Therefore, we have $(-5A+B)\cos(x)+(-5B-A)\sin(x) = 2\cos(x)$. Thus, $-5A+B = 2$ and $-5B - A = 0$. This gives $A = -\frac{5}{13}$ and $B = \frac{1}{13}$. Thus, $y_p = -\frac{5}{13}\cos(x) + \frac{1}{13}\sin(x)$. The general solution is then $y = c_1 e^{\frac{2}{3}x} + c_2 e^{-x} - \frac{5}{13}\cos(x) + \frac{1}{13}\sin(x)$.

(b) $y = c_1 + c_2 e^{-4x} + \frac{1}{12}x^3 - \frac{1}{16}x^2 + \frac{1}{32}x + \left(-\frac{1}{8}x^2 - \frac{1}{16}x\right)e^{-4x}$: First, we find that $y_c = c_1 + c_2 e^{-4x}$ is the general solution to $y'' + 4y' = 0$. We can look at the particular solution in separate parts for each of the terms x^2 and xe^{-4x}. The x^2 term gives us the particular solution part of $y_{p_1} = Ax^2 + Bx + C$. This duplicates part of our y_c solution since 1 is in both. Therefore, we modify this y_{p_1} and instead take $y_{p_1} = x(Ax^2+Bx+C) = Ax^3+Bx^2+Cx$. Plugging in y_{p_1} into the equation $y''+4y' = x^2$ gives $6Ax+2B+4(3Ax^2+2Bx+C) = x^2$. This in turn gives $12Ax^2+(8B+6A)x+(4C+2B) = x^2$. Thus, $A = \frac{1}{12}$, $B = -\frac{1}{16}$, and $C = \frac{1}{32}$. Thus,

$$y_{p_1} = \frac{1}{12}x^3 - \frac{1}{16}x^2 + \frac{1}{32}x.$$

Now we move on to the other part. If we considered $y_{p_2} = Axe^{-4x} + Be^{-4x}$, then we would have an overlap with the e^{-4x} in the y_c part. So instead we take $y_{p_2} = x(Axe^{-4x} + Be^{-4x}) = Ax^2e^{-4x} + Bxe^{-4x}$. Plugging this into $y'' + 4y' = xe^{-4x}$ gives $(2A - 4B)e^{-4x} - 8Axe^{-4x} = xe^{-4x}$. Thus $A = -\frac{1}{8}$ and $B = -\frac{1}{16}$. Thus,

$$y_{p_2} = \left(-\frac{1}{8}x^2 - \frac{1}{16}x\right)e^{-4x}.$$

Thus, the general solution is

$$y = c_1 + c_2 e^{-4x} + \frac{1}{12}x^3 - \frac{1}{16}x^2 + \frac{1}{32}x + \left(-\frac{1}{8}x^2 - \frac{1}{16}x\right)e^{-4x}.$$

13.

(a) $y = c_1\cos(3t) + c_2\sin(3t) - \frac{1}{9}\cos(3t)\ln|\sec(3t) + \tan(3t)|$: We need $y = y_c + y_p$ and we first find y_c. Solving the homogeneous equation $y'' + 9y = 0$ via the equation $m^2 + 9 = 0$ gives (after finding the complex solutions) $y_1 = \cos(3t)$ and $y_2 = \sin(3t)$.

Thus, $y_c = c_1\cos(3t) + c_2\sin(3t)$. To apply the method of variation of parameters we also note that $f(x) = \tan(3t)$ and

$$W = W(y_1, y_2) = \det\left(\begin{pmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{pmatrix}\right) = 3.$$

Thus,

$$u_1' = \frac{-y_2 f(x)}{W} = \frac{-\sin(3t)\tan(3t)}{3} = -\frac{1}{3}\frac{\sin^2(3t)}{\cos(3t)}$$

and

$$u_2' = \frac{y_1 f(x)}{W} = \frac{\cos(3t)\tan(3t)}{3} = \frac{1}{3}\sin(3t).$$

Therefore,

$$\begin{aligned} u_1 &= \int u_1'\,dt = -\frac{1}{3}\int \frac{\sin^2(3t)}{\cos(3t)}\,dt = -\frac{1}{3}\int \frac{1-\cos^2(3t)}{\cos(3t)}\,dt \\ &= -\frac{1}{3}\int \sec(3t)\,dt + \frac{1}{3}\int \cos(3t)\,dt = -\frac{1}{9}\ln|\sec(3t)+\tan(3t)| + \frac{1}{9}\sin(3t) \end{aligned}$$

and

$$u_2 = \int u_2'\,dt = \frac{1}{3}\int \sin(3t)\,dt = -\frac{1}{9}\cos(3t)$$

Using that $y_p = u_1 y_1 + u_2 y_2$ we find

$$\begin{aligned} y_p &= -\frac{1}{9}\cos(3t)\ln|\sec(3t)+\tan(3t)| + \frac{1}{9}\sin(3t)\cos(3t) - \frac{1}{9}\cos(3t)\sin(3t) \\ &= -\frac{1}{9}\cos(3t)\ln|\sec(3t)+\tan(3t)| \end{aligned}$$

and thus

$$y = y + c + y_p = c_1\cos(3t) + c_2\sin(3t) - \frac{1}{9}\cos(3t)\ln|\sec(3t)+\tan(3t)|.$$

(b) $y = c_1\cos(3t) + c_2\sin(3t) - \frac{1}{9}\cos(3t)\ln|\sec(3t)+\tan(3t)|$: We need $y = y_c + y_p$ and we first find y_c. Solving the homogeneous equation $y'' - y = 0$ via the equation $m^2 - 1 = 0$ gives (after finding the two real roots) $y_1 = e^t$ and $y_2 = e^{-t}$. Thus, $y_c = c_1 e^t + c_2 e^{-t}$. To apply the method of variation of parameters we also not that $f(x) = t + 3$ and

$$W = W(y_1, y_2) = \det\left(\begin{pmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix}\right) = -2.$$

Thus,

$$u_1' = \frac{-y_2 f(x)}{W} = \frac{-e^{-t}(t+3)}{-2} = \frac{1}{2}(t+3)e^{-t}$$

and

$$u_2' = \frac{y_1 f(x)}{W} = \frac{e^t(t+3)}{-2} = -\frac{1}{2}(t+3)e^t.$$

Therefore, (using integration by parts)

$$u_1 = \int u_1'\,dt = \frac{1}{2}\int (t+3)e^{-t}\,dt = \frac{1}{2}\left(-(t+3)e^{-t} + \int e^{-t}\,dt\right)$$
$$= -\frac{1}{2}e^{-t}\,(t+4)$$

and

$$u_2 = \int u_2'\,dt = -\frac{1}{2}\int (t+3)e^{t}\,dt = -\frac{1}{2}e^{t}\,(t+2)$$

Using that $y_p = u_1y_1 + u_2y_2$ we find

$$y_p = -\frac{1}{2}e^{-t}\,(t+4)\,e^{t} - \frac{1}{2}e^{t}\,(t+2)\,e^{-t} = -\frac{1}{2}\,(t+4+t+2) = -\frac{1}{2}\,(2t+6) = -(t+3).$$

and thus

$$y = y + c + y_p = c_1e^t + c_2e^{-t} - (t+3).$$

14.

(a) $y = c_1x^{-5} + c_2x^6$: We note this is a homogeneous Cauchy-Euler equation. Thus, substituting $y = x^m$ gives that we must have $m^2 - m - 30 = 0$ which means $m = 6, -5$. Therefore, we have $y_1 = x^{-5}$ and $y_2 = x^6$ are solutions. Since these are linearly independent and the order of the DE is two, we have the general solution is $y = c_1x^{-5} + c_2x^6$.

(b) $y{=}c_1x^3{+}c_2x^3\ln(x)$: We note this is a homogeneous Cauchy-Euler equation. We want to find a general solution. The equation $m(m-1)-5m+9 = (m-3)^2 = 0$ means we have a repeated root. We have $y_1 = x^3$ while reduction of order shows that $y_2 = x^3\ln(x)$. Thus, $y{=}c_1x^3 + c_2x^3\ln(x)$.

15. Explain your answers to the following questions.

(a) No: They are linearly dependent.

(b) No: They are linearly dependent since $\sin(-5x) = -\sin(5x)$.

(c) Yes: But they would not be linearly independent.

(d) No: Since this would imply they are linearly dependent.