Existence and uniqueness of solutions

Questions arise:

- Is there always a solution for an IVP?
- How many solutions can there be?

We develop a criteria, which when satisfied, shows the existence of a unique solution for first order IVPs.

Some important notes:

1. This criteria can only be applied to first order IVPs (later in the course we do consider criteria for higher order).
2. If an IVP does not satisfy the criteria, then *we can NOT determine* that there is not a unique solution. That is, this criteria is sufficient to establish a unique solution, but not necessary.

Theorem:

Suppose we have a DE of the form $\frac{dy}{dx} = f(x, y)$, where $f(x, y)$ is expressing the rest of the DE as a multivariable function. Additionally, consider a rectangular region R in the xy-plane given by $a \leq x \leq b$ and $c \leq y \leq d$ for numbers a, b, c, d which contains a point $(x_0 y_0)$. If

- $f(x, y)$ and
- $\frac{\partial}{\partial y} f(x, y)$

are both continuous on R, then on some interval around x_0 there exists a unique solution to the DE $\frac{dy}{dx} = f(x, y)$.

Notes:

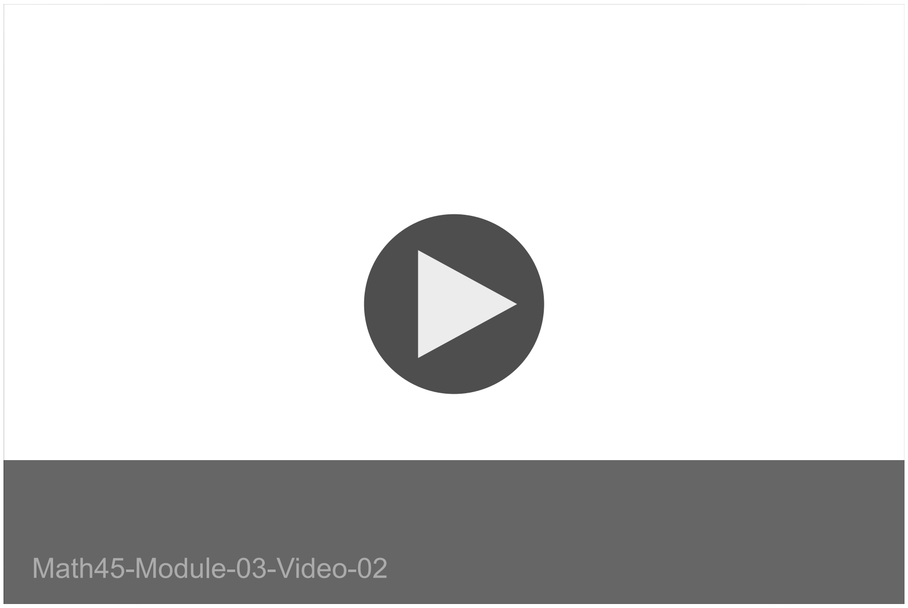
- This gives both the existence and uniqueness for a solution on "some" interval. It does not tell us what interval, or even what the solution is.
- The DE is given in normal form.
- To apply this theorem, we need to determine whether the two functions are continuous. Since these are multivariable functions, this may not be familiar.

To address the last bullet point, in this class we take the following convention.

We are okay to say that a multivariable function $f(x, y)$ is continuous wherever it is defined.

In other words, we reduce the notion of where it is continuous to that of the domain of the function.

Discussion, comments, and examples:



WeBWorK module 03 exercises:

- Problems 3,4

Relevant Wikipedia articles:

- **Existence and uniqueness results for ODEs. (https://en.wikipedia.org/wiki/Ordinary_differential_equation#Existence_and_uniqueness_of_solutions)**