Motivation for using an integrating factor

Given $\frac{dy}{dx} + Py = Q$. The technique...

- We want a function F such that

$$F\left(\frac{dy}{dx} + Py\right) = FQ$$

satisfies

the product rule.

$$F\frac{dy}{dx} + \frac{dF}{dx}y = \frac{d}{dx}(Fy) \qquad (\#)$$

That is, $\dfrac{dF}{dx} = FP$

- using separable equation techniques:

$$\frac{dF}{dx} = FP$$

(i) $\frac{1}{F} dF = P\,dx$

(ii) $\int \frac{1}{F} dF = \int P\,dx \iff \ln|F| = \int P\,dx + C$

(iii) $F = e^{\int P\,dx + C} \iff F = e^{C} e^{\int P\,dx}$

$\iff F = C e^{\int P\,dx}$.

- Then multiplying $\frac{dy}{dx} + Py = Q$ by F:

$$F\left(\frac{dy}{dx}+Py\right) = FQ \quad \text{gives}$$

$$\Leftrightarrow \quad \frac{d}{dx}(Fy) = FQ, \quad \text{or}$$

$$\Leftrightarrow \quad \frac{d}{dx}\left(\left(\cancel{c}e^{\int P\,dx}\right)y\right) = \left(\cancel{c}e^{\int P\,dx}\right)Q.$$

- Integrating both sides:

$$e^{\int P\,dx}\,y = \int e^{\int P\,dx}\,Q\,dx + C$$

• solve for y:

$$\left[\frac{1}{e^{\int p\,dx}} = e^{-\int p\,dx}\right]$$

$$y = e^{-\int p\,dx} \int e^{\int p\,dx}\, Q\, dx + C e^{-\int p\,dx}.$$

* But, don't memorize this!!
* But, knowing the "integrating factor" $\boxed{F = e^{\int p\,dx}}$, is useful.

- Instead:

(i) Given $\frac{dy}{dx} + P(x)\, y = Q(x)$, (ii) find the integrating factor $e^{\int P\,dx}$.

(iii) Multiply both sides of $\frac{dy}{dx} + Py = Q$ by $e^{\int P\,dx}$ KNOWING that the Left Hand side becomes

$$\frac{d}{dx}\left(e^{\int P\,dx}\, y\right) = e^{\int P\,dx}\, Q$$

(iv) Integrate both sides and solve for y.

Ex] Find the general soln for the DE

$$x\frac{dy}{dx} + 2y = 3.$$

soln]

(i) $$\frac{dy}{dx} + \underbrace{\frac{2}{x}}_{P(x)}y = \underbrace{\frac{3}{x}}_{Q(x)}$$

(ii) Integrating factor:

$$e^{\int P dx} = e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = e^{\ln|x^2|} = x^2.$$

(iii) $x^2\left(\frac{dy}{dx} + \frac{2}{x}y\right) = x^2\frac{3}{x}$

$\Leftrightarrow \underbrace{\qquad\qquad}_{\#}$

$$\frac{d}{dx}\left(x^2 y\right) = 3x$$

(iv) Integrate:

$x^2 y = \int 3x\,dx + c \Leftrightarrow x^2 y = \frac{3}{2}x^2 + c$

Thus, $\boxed{y = \frac{3}{2} + \frac{c}{x^2}}$ or $\boxed{y = \frac{3}{2} + cx^{-2}}$

(on $(0,\infty)$ or $(-\infty,0)$)

DEFN If a term of a solution tends to 0 as x tends to infinity (ie; $x \to \infty$), we call it <u>transient</u>.

Ex In $y = \frac{3}{2} + \frac{c}{x^2}$ the term $\frac{c}{x^2}$ is transient.