Existence and uniqueness of solutions for IVPs

Recall that an nth-order linear differential equation is of the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)$$

for functions $a_0(x), a_1(x), \ldots, a_n(x)$, and $b(x)$. This can also be expressed as

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = b(x).$$

In Module 3 we defined what it means to be an *initial value problem (IVP)*, and we also practiced a method to determine whether a 1st-order IVP was guaranteed to have a unique solution (remember the partial derivatives with respect to y?). That method was only valid for 1st-order IVPs. The following works for higher order *linear* IVPs.

Theorem

Suppose we have the initial value problem

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)$$

with initial conditions

$$y(x_0) = y_0, y(x_1) = y_1, \ldots, y(x_{n-1}) = y_{n-1}.$$

If $a_n(x), a_{n-1}(x), \ldots, a_1(x), a_0(x)$, and $b(x)$ are continuous on an interval I, $a_n(x) \neq 0$ for any x in I, and x_0 is in I, then there exists a unique solution $y(x)$ of the initial value problem on I.

Notes:

- The theorem for 1st-order IVPs, while restricted to only 1st-order, had the advantage of being able to be applied to nonlinear IVPs.
- The theorem above can be applied to higher order IVPs, but only those that are linear.
- Be sure to check *all* of the conditions of this theorem when using it!

Discussion, comments, and examples:

[(https://csus.instructure.com/courses/66043/external_tools/retrieve?url=https%3A%2F%2Fcsus.mediasite.com%2FMediasite%2FLti%2FHome%2FLaunch%3FmediasiteId%3D484acb714e524c3ba40383bfab2e3d5e1d)](https://csus.instructure.com/courses/66043/external_tools/retrieve?url=https%3A%2F%2Fcsus.mediasite.com%2FMediasite%2FLti%2FHome%2FLaunch%3FmediasiteId%3D484acb714e524c3ba40383bfab2e3d5e1d)

WeBWorK module 08 exercises:

- Problems 1, 2, 3

References:

- **Theorem 9.11 in this text.** (https://digitalcommons.trinity.edu/cgi/viewcontent.cgi?article=1008&context=mono) (However, it is rephrased in a very different way.)
- See Zill's textbook as well.