Goal and idea - Module 10

GOAL:

We look to extend our knowledge of fundamental sets of solutions for homogenenous DEs to that of nonhomogeneous DEs. This involves:

- Learning how the underlying homogeneous DE plays a role;
- As well a particular solution for the nonhomogeneous DE.

IDEA:

By first finding the fundamental set of solutions for the underlying homogeneous and combining this with a particular solution for the nonhomogeneous DE, we are able to construct a fundamental set of solutions for a nonhomogeneous DE.

Consider the linear nonhomogeneous differential equation
$a_n(x)y^{(n)}+\ldots+a_1(x)y'+a_0(x)y=\underset{\text{Not zero}}{\underline{b(x)}}.\ (\bigstar)$

Q: What are the solutions for $(\bigstar)$

Recall $a_n(x)y^{(n)}+\ldots+a_1(x)y'+a_0(x)y=\underset{}{\overset{\text{zero}}{\underline{0}}}$
General solutions for homogeneous linear differential equations boils down to is finding a fundamental set of solutions

A: Theorem

Suppose
(i) $y_P(x)$ is a particular solution for
$a_n(x)y^{(n)}+\ldots+a_1(x)y'+a_0(x)y=b(x)\ (\bigstar)$

↓ AND

(ii) $y_1,\ldots,y_n$ is a fundamental set of solutions for the homogeneous linear differential equation
$a_n(x)y^{(n)}+\ldots+a_1(x)y'+a_0(x)y=0$

→ THEN

The general solution for $(\bigstar)$ is
$y=C_1y_1+C_2y_2+\ldots+C_ny_n+y_P$,
where $C_1,\ldots,C_n$ are arbitrary constants.

Ex Verify $y=C_1\cos(x)+C_2\sin(x)+x\sin(x)+\cos(x)\ln(\cos(x))$ is a general solution to the nonhomogeneous DE $y''+y=\sec(x)$

Soln

To verify if $y=C_1\cos(x)+C_2\sin(x)+x\sin(x)+\cos(x)\ln(\cos(x))$ is a general solution

① We need a particular solution (theorem (i))

What is the particular solution in our differential equation?

Hint: We are expecting what we are given to be a general solution. Some of it the homogeneous fundamental set solution aspect $y=\boxed{C_1y_1+C_2y_2+\ldots+C_ny_n}+y_P$ gets tied with these arbitrary constants and the particular solution does not have an arbitrary constant.

So looking back to our equation we note the arbitrary constants.
$y=C_1\underbrace{\cos(x)}_{y_1}+C_2\underbrace{\sin(x)}_{y_2}+x\sin(x)+\cos(x)\ln(\cos(x))$ tells us that y_1 & y_2 are probably the fundamental set of solution maybe. But what doesn't have a constant is must be our y_P $x\sin(x)+\cos(x)\ln(\cos(x))$
At least we think it is

We still need to check if this is actually a particular solution of our DE

Need to make sure it satisfy this nonhomogeneous DE $y''+y=\sec(x)$

$y_P=x\sin(x)+\cos(x)\ln(\cos(x))$

$y_P'=\cancel{\sin(x)}+x\cos(x)-\sin(x)\ln(\cos(x))-\frac{\cos(x)\cancel{\sin(x)}}{\cancel{\cos(x)}}$ $\quad [\ln(\cos(x))]'=\left[\frac{1}{\cos(x)}\cdot(-\sin(x))\right]$

$y_P''=\cos(x)-x\sin(x)-\cos(x)\ln(\cos(x))-\sin(x)\left[\frac{1}{\cos(x)}\cdot(-\sin(x))\right]$

$=\cos(x)-x\sin(x)-\cos(x)\ln(\cos(x))+\frac{\sin^2(x)}{\cos(x)}$

$y''+y=\left(\cos(x)-x\sin(x)-\cos(x)\ln(\cos(x))+\frac{\sin^2(x)}{\cos(x)}\right)+\left(x\sin(x)+\cos(x)\ln(\cos(x))\right)$

$=\left(\cos(x)\cancel{-x\sin(x)}\cancel{-\cos(x)\ln(\cos(x))}+\frac{\sin^2(x)}{\cos(x)}\right)+\left(\cancel{x\sin(x)}\cancel{+\cos(x)\ln(\cos(x))}\right)$

$=\left(\cos(x)+\frac{\sin^2(x)}{\cos(x)}\right)$

$=\frac{\cos^2(x)+\sin^2(x)}{\cos(x)}=\sec(x).$ ▪

So is a particular solution ✓

② Check: $y_1=\cos(x)$, $y_2=\sin(x)$ a fundamental set of solutions for $y''+y=0$.

- Solutions: Seen before ✓
Take $y_1=\cos(x)$, $y_2=\sin(x)$ individually take two derivatives, plug them in and get zero so they are solutions.
- How many solutions? ✓
Need two because of the second order & we have two.
- Linearly independent? ✓
Yes, $W(\cos(x),\sin(x))=1\neq 0$ on $(-\infty,\infty)$

∴ Thus by theorem
$y=C_1\cos(x)+C_2\sin(x)+x\sin(x)+\cos(x)\ln(\cos(x))$
is a general solution.

Expectation checklist - Module 10

At the completion of this module, you should:

- Understand how to find general solutions for a nonhomogeneous linear DEs by combining
  - the fundamental set of solutions of the underlying homogeneous linear DE and
  - a particular solution of the given nonhomogeneous linear DE.

You will be assessed on your understanding of these concepts:

- within the homework,
- on the quizzes, and
- later, on the exam.

Coming up next, we:

- Reduction of order!