Systems of linear DEs and questions to ask

We will focus on systems of linear differential equations consisting of two differential equations and two unknowns. Such systems can be of the general form

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy,$$

if it is **homogeneous**, or

$$\frac{dx}{dt} = ax + by + f$$
$$\frac{dy}{dt} = cx + dy + g,$$

if it is **nonhomogeneous**. Here, a, b, c, d are constants and x, y, f, g are functions of the independent variable t.

Note: We could consider systems with more equations and unknowns, but we limit ourselves to this "2 by 2" system. Larger systems are tackled similarly, however.

The first thing we would like to do is translate the above system into "matrix form." To do so, we first define what we mean by the 'derivative of a vector of functions.' In particular, we set

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix}.$$

Then, as the video below shows, we can rewrite the system of differential equations above in the form

$$X' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} X,$$

or

$$X' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} X + F,$$

respectively, where we have set $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $F = \begin{pmatrix} f \\ g \end{pmatrix}$.

In this formalism, we can define what a solution to such a system of differential equations is.

Definition

A solution to $X' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} X + F$ is a vector $X = \begin{pmatrix} x \\ y \end{pmatrix}$ which satisfies the equation.

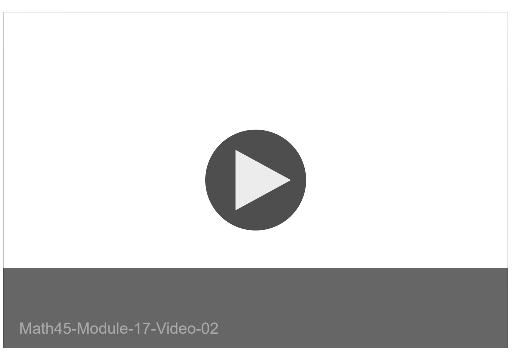
We are now able to ask questions which parallel earlier aspects of the course.

These questions are:

1. How do we verify a given vector is a solution?
2. Does a solution always exist?
3. Is there a notion of a fundamental set of solutions, and what constitutes a fundamental set of solutions and a general solution?
4. How do we solve such systems of differential equations?

In the coming pages of this module, we explore answers to each of these questions.

Discussion, comments, and examples:



WeBWorK module 17 exercises:

- Problem 5

Relevant Wikipedia articles:

- **Systems of linear differential equations** (https://en.wikipedia.org/wiki/System_of_linear_equations)