You must show and explain your work! The following formulas may, or may not, be useful.

1. (*10 points*)

a. (*2 pts*) Circle all of the following expression which are **linear** differential equations.

(i) $y''' - y'' + y = 0$

(ii) $\cancel{(y')^3 = y^2}$

(iii) $\ln(y') - y = x\cos(x)$

(iv) $e^x y^{(7)} - \left(x^2 - 3\right) y = \sin\left(x^2\right)$

b. (*2 pts*) Circle all of the following expression which are linear **nonhomogeneous** differential equations.

(i) $y''' - y'' = y - x^2\sin(x)$

Rewrite $y''' - y'' - y = x^2 \sin x$

(ii) $t^2 \dfrac{d^3 f}{dt^3} + t\dfrac{df}{dt} + t^5 = t^5$

(iii) $\cancel{y^{(3)}} - \sec(x)y' - y = e^x$

(iv) $\cancel{y^{(3)}} - x^5 y' - 2x^2 = 0$

c. (*3 pts*) Compute $\mathcal{L}\left\{e^{5t}\sin(3t)\right\}$.

Laplace Transform

(a) $\mathcal{L}\{C\} = \frac{C}{s}$

(b) $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

(c) $\mathcal{L}\{e^{kt}\} = \frac{1}{s-k}$

(d) $\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}$

(e) $\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}$

$$\mathcal{L}\{e^{5t}\}\,\mathcal{L}\{\sin(3t)\}$$

$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ $\qquad$ $\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}$

$$\frac{5!}{s^{5+1}} \qquad \frac{3}{s^2+(\sqrt{3})^2}$$

$$\frac{5!}{s^6}$$

$$\therefore \frac{5!\cdot 3}{s^6\left[s^2+(\sqrt{3})^2\right]}$$

d. (*3 pts*) Compute $\mathcal{L}^{-1}\left\{\frac{9}{s+2} + \frac{3}{s^2+16}\right\}$.

$$\mathcal{L}^{-1}\left\{\frac{9}{s+2}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{s^2+16}\right\}$$

$\rightarrow 9\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$

Does Not match known forms So multiply by const to force it

$\mathcal{L}^{-1}\left\{\frac{1}{s-k}\right\} = e^{kt}$; $\mathcal{L}^{-1}\left\{\frac{1}{s-(-2)}\right\}$ $\qquad$ $\rightarrow \mathcal{L}^{-1}\left\{\frac{3}{4}\cdot\frac{4}{s^2+4^2}\right\}$

$\rightarrow 9e^{-2t}$

Now it matches

$\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin(kt)$

$\rightarrow \frac{3}{4}\sin(4t)$

Inverse Laplace Transform

Instead of computing the Inverse Laplace Transform from scratch ...

(a) $\mathcal{L}^{-1}\left\{\frac{C}{s}\right\} = C$

b) $\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$

(c) $\mathcal{L}^{-1}\left\{\frac{1}{s-k}\right\} = e^{kt}$

(d) $\mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos(kt)$

(e) $\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin(kt)$

$$\therefore 9e^{-2t} + \frac{3}{4}\sin(4t)$$

Not separable ✗

Exact Equation

Definition

(a) A differential equation $M(x,y)dx + N(x,y)dy = 0$ is an **exact equation** if $\frac{\partial}{\partial y}M(x,y) = \frac{\partial}{\partial x}N(x,y)$

2. (*10 points*) Solve the following differential equations.

a. (*5 pts*) $\underbrace{e^y}_{M}\,dx + \underbrace{(2y + xe^y)}_{N}\,dy = 0$

Step 0 | Check if it's exact:

$e^y dx + (2y + xe^y)dy = 0$

Partial Deriv

$(e^y)' = e^y$ ✓

$(2y + xe^y)' \to (2y)' + (xe^y)'$

$\to 0 + e^y = e^y$ ✓

Find (antiderivative with respect to "x" or "y")

$F(x,y) = \int(2y + xe^y)dy$

$\int(2y) + \int(xe^y)$

$y^2 + xe^y + g'(x)$

$y^2 + xe^y + g'(x) = e^y$

$g'(x) = -y^2 - xe^y + e^y$

$0 - e^y + e^y$

$g'(x) = 0;\ \int 0\,dx = C_1$

$y^2 + xe^y + C_1 = C_2$

$\therefore y^2 + xe^y + C$

b. (*5 pts*) $y' - \frac{3}{x+1}y = (x+1)^4$

Module 05

1^{st}-Order Linear Equations

$\frac{dy}{dx} + Py = Q.\ ;\ P(x) = -\frac{3}{x+1}\ ;\ q(x) = (x+1)^4$

Step 2 | Calculate Integrating Factor

$e^{\int P dx} = e^{\int -\frac{3}{x+1}dx} = e^{-3\ln|x+1|} = \frac{1}{(x+1)^3}$

Step 3 | Multiply left and right by the integrating factor

$\left(\frac{dy}{dx} - \frac{3}{x+1}\right)\frac{1}{(x+1)^3} = \frac{1}{(x+1)^3}(x+1)^4$

$\to \frac{d}{dx}\left(\frac{1}{x^3+3x^2+3x+1}y\right) = x+1$

Step 4 | Integrate both sides

$\cancel{\int \frac{d}{dx}}\left(\frac{1}{x^3+3x^2+3x+1}y\right) = \cancel{\int x+1}$

$\frac{x^2}{2} + x$

$\therefore$ Let $\frac{1}{x^3+3x^2+3x+1} = U$

$\therefore y = \frac{x^2}{2}U + xU + CU$

2nd-order linear homog.

3. (*20 points*) Solve the following differential equations. Your answers should be written as real general solutions.

a. (*5 pts*) $y'' - 4y' + 13y = 0$

1) Plug-in $y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2e^{mx}$

Skipping a step, instead of plugging-in and refactoring out, we know are able to factor out "e^{mx}"

This gives us

$$e^{mx}(m^2 - 4m + 13) = 0$$

So we need

$$m^2 - 4m + 13 = 0$$

2) Find m_1 & m_2

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{4 \pm \sqrt{(-4)^2 - [4(1)(13)]}}{2(1)}$$

$$\rightarrow \frac{4 \pm \sqrt{16 - 52}}{2}$$

$$\rightarrow \frac{4 \pm \sqrt{-36}}{2}$$

Case 3) Complex Numbers w/Conjugate Property

* Recall *

$e^{i\theta} = \cos(\theta) + i\sin(\theta)$

$$e^{(\underbrace{3/2}_{\alpha} + \underbrace{\frac{i\sqrt{7}}{2}}_{i\beta})x} = e^{3/2x}e^{\frac{i\sqrt{7}}{2}x}$$

Focus on the real — Ignore the complex

$$\rightarrow e^{3/2x}\left(\underbrace{\cos\left(\frac{\sqrt{7}}{2}x\right)}_{\text{Real}} + \underbrace{i\sin\left(\frac{\sqrt{7}}{2}x\right)}_{\text{Complex}}\right)$$

Euler's Formula

$$e^{2x}\left(\cos\frac{\sqrt{-36}}{2}x + \sin\frac{\sqrt{-36}}{2}x\right)$$

b. (*5 pts*) $x^2y'' + 5xy' + 4y = 0$

Module15 Cauchy-Euler

$$\cancel{x^2 - 5x + 4 = 0}$$

~~2) Find m_1 & m_2~~

$$\cancel{m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

$$\cancel{\frac{5 \pm \sqrt{25 + 16}}{2}}$$

$$\cancel{M = \frac{5}{2} \pm \frac{\sqrt{41}}{2}}$$

$x^2y'' + 5xy' - 4y = 0$ Need two derivatives

1) Take $y = x^m$

$y = x^m$ $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$

$x^2y'' + 5xy' - 4y = 0$

$y = x^m$, $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$

2) Plug into the differential equation

This gives: $x^2(m(m-1)x^{m-2}) + 5x(mx^{m-1}) - 4x^m$

$$x^2(m(m-1)x^{m-2}) + 5x(mx^{m-1}) - 4x^m = 0$$

Factor out "x^m"

$$\rightarrow x^m(m(m-1)) + 5x^m m - 4x^m = 0$$

Simplify

$$\rightarrow x^m(m^2 - m + 5m - 4) = 0$$

$$\rightarrow m^2 - m + 5m - 4 = 0$$

$$\rightarrow m^2 + 4m - 4 = 0$$

3) Solve for "m"! Quadratic formula

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{4 \pm \sqrt{16 + 16}}{2}$$

$$2 \pm \sqrt{32}$$

$$\frac{2 \pm 2\sqrt{2}}{2}$$

c. (*10 pts*) $y'' - 8y' + 15y = 8\cos(x) - 14\sin(x)$

Non homog
2nd-order
Linear

Step 1 | Find y_h

$$y'' - 8y' + 15y = 0$$

$$e^{mx}(m^2 - 8m + 15)$$

3) Solve for "m"! Quadratic formula

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m_1 = \frac{-(-8)+2}{2}, \quad m_2 = \frac{-(-8)-2}{2}$$

$$m_1 = 5, \quad m_2 = 3$$

$$y_h = C_1 e^{5t} + C_2 e^{3t}$$

Case 2) f(x) consists of sin(x) and/or cos(x) terms.

Any sines or cosines (not products of them i.e. $\sin^2$ or $\cos^2$)

Step 1 | Then consider $y_p = A\cos(x) + B\sin(x)$

$$8A + (-14)B$$

Step 2 | Plug-in this y_p (need derivatives)

Step 3 | Solve for A, B, C

4. (*10 points*) Use the Laplace Transform to solve the initial value problem $y'' = 2 + 3t$ with initial conditions $y(0) = 0$, $y'(0) = 0$. (You must use the method of Laplace Transform for credit.)