Goal and idea - Module 8

GOAL:

We turn the page and consider higher order differential equations. We note, however:

- With first-order DEs, we were able to look at many different types (including nonlinear DEs!).
- With higher order DEs, we restrict our attention only to *linear* differential equations.

In this module, we begin studying higher order linear DEs by

- recalling their definition,
- defining what homogeneous and inhomogeneous linear differential equations are (different homogeneous than before!), and
- learning how to determine whether we are *guaranteed* existence (and uniqueness!) of a solution for a given higher order linear IVP.

IDEA:

We increase our scope of DEs by considering higher order DEs, but restrict or scope by only looking at linear DEs. We make a key classification between such DEs that don't have terms attached to the dependent variable or its derivatives (homogeneous DEs) and those that do (nonhomogeneous DEs). Finally, we raise the question of whether there exists solutions for an IVP (and if so, how many?).

Recall:

[i] Found conditions for when a 1^{st}-order IVP is guaranteed to have a unique solution on an interval
*Note: Differential Equation is not required to be linear.

[ii] Solving some 1^{st}-order differential equations
- Seprable
- Linear
- Exact
- 1^{st}-order homogeneous

Theorem

Suppose we have the initial value problem

The coefficient functions ($a_n(x)$) are all functions of "x" there aren't any "y" terms

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)$$

with initial conditions

as opposed to a boundry value problem (where we plug-in different values)

Plugging-in the same value

$$y(x_0) = y_0,\ y(x_1) = y_1,\ \ldots,\ y(x_{n-1}) = y_{n-1}.$$

If ① $a_n(x), a_{n-1}(x), \ldots, a_1(x), a_0(x)$, and $b(x)$ are continuous on an interval I, ② $a_n(x) \neq 0$ for any x in I, and ③ x_0 is in I, then there exists a unique solution $y(x)$ of the initial value problem on I. (All three conditions)

Ex] Consider the IVP $y'' + \tan(x)y = e^x$ with $y(0)=1$, $y'(0)=0$. Find an interval about $x=0$ where we are guarenteed a unique solution.

- Is this linear?
  - Yes. Even though $\tan(x)$ is not linear in terms of "y" and its derivatives its all linear looking stuff of 2^{nd} order

$$\underbrace{y'' + \tan(x)y = e^x}$$

Nonhomogeneous

$$y'' + \tan(x)y = e^x \neq 0$$

- Our Initial Value Problems
  - $y(0)=1$ & $y'(0)=0$ *2^{nd}-order, so we expect to have two*
    - "Find an interval about $x=0$" makes sense because we know the interval has to enclude zero as one of our three conditions ③ "x_0" is in "I". In short is asking: Find some interval that contains the "x_0"

Where we are guarenteed a unique solution.

Three Required Conditions for a guarenteed unique solution
(i) Coefficent of higest order derivative does not equal zero on I
(ii) All coefficent function are continuous on I
(iii) The "x_0" is in I

Soln]

③ → Need "I" such that zero is in "I".

① → $a_2(x)=1$, $a_1(x)=0$, $a_0(x)=\tan(x)$, $b(x)=e^x$ are all continuous on everything $(-\infty,\infty)$ except $\tan(x)$

Since $\tan(x) = \frac{\sin(x)}{\cos(x)}$ ← equals zero for $x = K + \pi/2$

But $\tan(x)$ is continuous on $(-\pi/2, \pi/2) = I$.

* And zero is in $(-\pi/2, \pi/2)$

② → $a_2(x) = 1 \neq 0$ on $(-\pi/2, \pi/2)$

THUS: $(-\pi/2, \pi/2)$ works!

Recall Linear

Linear: derivatives and functions occur as polynomials of degree one.

Recall: A polynomial in 'x' is of the form $C_n x^n + C_{n-1}x^{n-1} + \ldots + C_1 x + C_0$

y, y' $y^n, (y')^n$ $y^{n-1}, (y')^{n-1}$ y, y'

$C_n, C_{n-1}, \ldots, C_1, C_0$ Constants

→ 'n' is the degree

So linear DE must only have y or y' or $y^{(n)}$

But can't have the derivative squared $(y')^2$, y^2, $(y')^n$, $(y^{(n)})^2$, $y'' \cdot y'$

Definition |

(a) The linear differential equation

$$a_n(x)y^n + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y^1 + a_0(x)y = 0$$

is called "**homogeneous**", while

(b) the linear differential equation

$$a_n(x)y^n + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y^1 + a_0(x)y = b(x),$$

where $b(x) \neq 0$, is called "**nonhomogenous**" OR "**inhomogenous**"

Example | Linear Differential Equations.

Determine which are homogenous and nonhomogenous.

(a) $y' + y - 3 = 0$

Nonhomogeneous: Can move the Constant 3 to the other side

$$y' + y = 3$$

(b) $y' = y$

Homogenous: Can be rewritten as $y' - y = 0$

Expectation checklist - Module 8

At the completion of this module, you should:

- if given a linear initial value problem, be able to determine whether we are guaranteed the existence of a unique solution; and
- know the definitions introduced, and in particular homogeneous and nonhomogeneous linear differential equations.

You will be assessed on your understanding of these concepts:

- within the homework,
- on the quizzes, and
- later, on the exam.

Coming up next, we:

- Fundamental sets of solutions!