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Please show and explain your work. Justification is necessary for credit. Also sign your name here acknowledging you will not use problem solving software (Wolfram Alpha, etc.): Matthew Rende

1. (14 points)

a. (2 pts) Circle all of the following expression which are linear differential equations.

(i) $y''' - 2y'' + 3y = 5$

(ii) $(x^2 - \sin(x)) y^{(5)} - xe^x y + \sin(x^2) = 0$

(iii) $\sqrt{y'} = x \cos(x)y - 3$

(iv) $(y')^2 = y^3 + 2$

* Can't have 'n' be any higher power greater than one *

b. (2 pts) Circle all of the following expression of the form $M(x, y) dx + N(x, y) dy = 0$ such that both $M(x, y)$ and $N(x, y)$ are homogeneous functions of the same degree.

$\frac{\partial}{\partial y} x^2 = x^2$

(i) $x^2 dx + yx dy = 0$

(ii) $(3xy) dx + e^{\left(\frac{x}{y}\right)} dy = 0$

(iii) $\sin(y) dx + (2y + xe^y) dy = 0$

(iv) $(x - 2y) dx + x dy = 0$

c. (4 pts) Circle whether the following differential equation is an ordinary differential equation (ODE) or a partial differential equation (PDE) and state the **order** of the differential equations (you can do this for any PDEs as well).

(i) $\ln(y')y + x^2y = 1$

(ii) $\frac{\partial f}{\partial x} + \frac{\partial^2 f}{\partial y^2} + xy = 0$

(iii) $x^2 \frac{d^3 f}{dx^3} + \frac{df}{dx} = \frac{d^2 f}{dx^2} - \sin(x)$

(iv) $\left(\frac{d}{dx}\right)^4 f(x) + \left(\frac{d^2}{dx^2}\right) f(x) + f'(x) = 0$

ODE PDE Order: 2

ODE PDE Order: 1 least one multivariable function.

ODE PDE Order: 2 least one function of one independent variable.

ODE PDE Order: 4 least one function of one independent variable.

d. (4 pts) Circle all of the following expression which are exact differential equations
(Provide work for partial credit!)

$M = xy$ $N = x^2y$

(i) $xy dx + xy dy = 0$

(ii) $x^2y dx + \frac{1}{3}x^3 dy = 0$

$\frac{\partial}{\partial y} x^2y = x^2$ $\frac{\partial}{\partial x} x^2y = 2xy$

$M = xy^2$ $N = x^2y$

(iii) $xy^2 dx + x^2y dy = 0$

$\frac{\partial}{\partial y} xy^2 = 2xy$, $\frac{\partial}{\partial x} x^2y = 2xy$

(iv) $\sin(xy) dx + xy \cos(x) dy = 0$

COS

No good

e. (2 pts) Suppose $y^2 - x = 3$ is a solution to a differential equation. Is this solution implicit or explicit? Explain your answer in one sentence.

Explicit; for, you can

Solve for y: $y = \pm \sqrt{3+x}$

2. (27 points) Solve the following differential equations using any technique you like that works. However, you must explain WHY you chose that method to receive credit!

a. (9 pts) $y' = \frac{x}{y^2}$

$$\frac{dy}{dx} = \frac{x}{y^2}$$

$$\rightarrow y^2 dy = x dx$$

$$\rightarrow \int y^2 dy = \int x dx$$

$$\rightarrow \frac{1}{3}y^3 = \frac{1}{2}x^2 + C$$

$$\rightarrow y^3 = \frac{3}{2}x^2 + 3C$$

$$\therefore y = \sqrt[3]{\frac{3}{2}x^2 + 3C}$$

\checkmark

Why: separable $\rightarrow \frac{dy}{dx} = y'$

b. (9 pts) $x \frac{dy}{dx} - 2y = x^3 e^x$

why: It looks separable

$$\begin{aligned} x \frac{dy}{dx} - 2y &= x^3 e^x \\ \cancel{x} \frac{dy}{\cancel{x}} - \cancel{2y} &= \cancel{x^2} e^x \\ \rightarrow \frac{dy}{dx} - \frac{2y}{x} &= x^2 e^x \\ \rightarrow \frac{dy}{dx} &= \frac{2y}{x} + x^2 e^x \\ \rightarrow \frac{dy}{dx} &= \frac{1}{x} 2y + x^2 e^x \end{aligned}$$

$$\begin{aligned} x \frac{dy}{dx} - 2y &= x^3 e^x \\ +2y &+2y \\ \rightarrow x \frac{dy}{dx} &= x^3 e^x + 2y \\ -x^3 e^x &-x^3 e^x \\ \rightarrow x - x^3 e^x \frac{dy}{dx} &= 2y \end{aligned}$$

wrong approach

1st-order linear equation
why: it follows form " $\frac{dy}{dx} + P y = Q$ "

$$x \frac{dy}{dx} - 2y = x^3 e^x$$

standard form

$$\rightarrow \frac{dy}{dx} - \frac{2y}{x} = \frac{x^3 e^x}{x}$$

Can't have $x=0$ anymore

$$\rightarrow \frac{dy}{dx} - \frac{2y}{x} = \frac{x^3 e^x}{x}$$

$$\begin{aligned} \rightarrow \frac{dy}{dx} - \frac{2}{x} y &= \frac{x^3 e^x}{x} \\ I \text{-think this is where I'm confusing myself} & \\ \rightarrow \frac{dy}{dx} + \left(-\frac{2}{x} \right) y &= \frac{x^3 e^x}{x} \end{aligned}$$

calc Integrating factor

$$e^{\int P dx} = e^{\int -\frac{2}{x} dx} = e^{-2 \ln|x|}$$

$$\rightarrow e^{-2 \ln|x|} = e^{\ln|x|^{-2}} = x^{-2}$$

Multiply left & Right by Integrating factor

$$x^{-2} \left(\frac{dy}{dx} - \frac{2y}{x} \right) = x^{-2} \left(\frac{x^3 e^x}{x} \right)$$

Product Rule

$$\rightarrow \frac{d}{dx} (x^{-2} y)$$

guide

RCS?

-2

c. (9 pts) M N

$$2xy \, dx + (x^2 + 3y^2) \, dy = 0$$

Exact equation?

Why follows form $M(x,y)dx + N(x,y)dy = 0$

Step 0

$$\frac{\partial}{\partial y}(2xy) = 2x \quad \text{Yes!} \\ \frac{\partial}{\partial x}(x^2 + 3y^2) = 2x$$

Step 1 Anti-deriv w/ respect to "x"

$$F(x,y) = \int (xy) \, dx \\ \rightarrow \frac{1}{2}x^2y \\ \rightarrow \cancel{\frac{1}{2}x^2y} + g(y)$$

Step 2 Take partial deriv of F w/y

$$\frac{\partial}{\partial y}\left(\frac{1}{2}x^2y + g(y)\right) = \cancel{\frac{1}{2}}(x^2y^2) + g'(y)$$

3. (9 points)

a. (8 pts) We note that $y(x) = c_1 e^{3x} + c_2 e^{-3x}$ is a solution to the differential equation $y'' - 9y = 0$. Find a solution to the initial conditions $y(0) = 0$ and $y'(0) = 2$.

From Notes

Ex: (i) Verify that $y = k\cos(2x)$ satisfies the DE $y'' + 4y = 0$.

(ii) Find a solution for the IVP consisting of $y'' + 4y = 0$ and the conditions $y(0) = 3, y'(0) = 0$

Soln]

(i) $y = k\cos(2x), y' = -2k\sin(2x), y'' = -4k\cos(2x)$. Thus, $y'' + 4y = -4k\cos(2x) + 4k\cos(2x) = 0$ ✓

(ii) $y(0) = 3 \quad y'(0) = 0$
 $\rightarrow k\cos(0) = 3 \quad \rightarrow -2k\sin(0) = 0$
 $\rightarrow k = 3 \quad \rightarrow 0 = 0$

Thus, $y = 3\cos(2x)$ works

Blanking out for some reason
But it looks like it checks-out ✓

$$y(0) = 0 \\ \rightarrow c_1 e^{3(0)} + c_2 e^{-3(0)} = 0 \\ \rightarrow c_1(I) + c_2(I) = 0 \\ \rightarrow c = 0$$

3

b. (1 pt) Does the trivial solution also satisfy the initial value problem given above in 3a?
Explain your answer.

Yes

The Trivial Solution

Definition: If the function $y = 0$ is a solution on an interval, we can call this the Trivial Solution.

Step 3 Set partial-deriv of F w/y equal to $N(x,y)$ & solve for $g'(y)$

$$\frac{\frac{1}{2}(x^2y^2) + g'(y)}{x^2y^2} = x^2 + 3y^2$$

$$\rightarrow 2 \cdot \frac{1}{2}x^2 + g'(y) = (y^2 + 3) \cdot 2 \\ \rightarrow g'(y) = 2(y^2 + 3)$$

Step 4 Integrate $g'(y)$

$$g'(y) = 2\left(\frac{1}{2}y^2\right) + 3 \\ \rightarrow g(y) = 2\ln|y^2| + C$$

Step 5 Plug Step 4 in Step 1 anti-derivative w/r to "x"

$$F(x,y) = \frac{1}{2}x^2y + g(y) \\ \rightarrow \frac{1}{2}x^2y + 2\ln|y^2| + C$$

Step 6 Thus we have in form $f(x,y) = C$

$$C = \frac{1}{2}x^2y + 2\ln|y^2| * \text{Implicit Solution}$$

Chain Rule
 $e^x \cdot [X]$

$$y = c_1 e^{3x} + c_2 e^{-3x}$$

$$y' = (c_1 e^{3x} \cdot 3) + (c_2 e^{-3x} \cdot -3)$$

$$y'' =$$

✓ grad. and ?

y ()

Solution to a DE	Particular (one solution)	Parameter	Could it be general	trivial
$y = Ce^{2x}$	✗	✓ - one para	✓ - (?)	✗
$y = 3\cos(4x)$	✓	✗	✗ - (?)	✗
$y = 0$	✓	✗	✗ - (?)	✓