

Please show and explain your work. Justification is necessary for credit. Also sign your name here acknowledging you will not use problem solving software (Wolfram Alpha, etc.): Matthew Mendoza

1. (5 points)

- a. (2 pts) Circle all of the following expression which are **homogeneous linear differential equations**.

(i) $y''' - 2y'' + 3y = 5 \quad b(x) \neq 0$

(iii) $\sqrt{y'} - x \cos(x)y = 3 \quad b(x) \neq 0$

(ii) $(x^2 - \sin(x))y^{(5)} - xe^x y' + \sin(x^2)y = 0$

(iv) $y'' = y + 2y'$

Rewritten as $y'' - y' - y = 0$

- b. (2 pts) We have that $y_h = c_1e^x + c_2e^{-x}$ is a general solution for the DE $y'' - y = 0$ and that $y_p = x^3$ is a particular solution to $y'' - y = 6x - x^3$ (you do not need to verify these statements). Provide a general solution to the DE $y'' - y = 6x - x^3$.

$$y = y_h + y_p$$

$$y = C_1 e^x + C_2 e^{-x} + x^3$$

c. (1 pt) Compute the Wronskian of $y_1 = x$ and $y_2 = x^2$.

$$w(f_1, f_2) = \det \begin{pmatrix} f_1 & f_2 \\ f_1' & f_2' \end{pmatrix} \rightarrow (1 \cdot 2x) - (1 \cdot x^2)$$

$$w(x, x^2) = \det \begin{pmatrix} x & x^2 \\ 1 & 2x \end{pmatrix} = -x^2 + 2x$$

2. (10 points) Given that $y_1 = x \sin(x)$ is a solution to the DE $x^2y'' - 2xy' + (x^2 + 2)y = 0$, find a general solution to this differential equation.

$$\begin{aligned} y_2 &= y_1(x) \int \frac{e^{-\int P(x)dx}}{(y_1(x))^2} dx \\ &= x \sin(x) \int \frac{e^{-\int (-2x)dx}}{(x \sin(x))^2} dx \\ &= x \sin(x) \int \frac{e^{+2}}{x^2 \sin^2(x)} dx \end{aligned}$$

=

$$y = x \sin(x) +$$

Recall - Module 10

General Solutions for Nonhomogeneous Linear Differential Equations

Given $(a, b, c$ are constants)

(*) $ay'' + by' + cy = f(x)$

with $f(x) \neq 0$, the general solution is of the form

$$y = y_h + y_p$$

Where

if we are familiar with
the general solution
of the homogeneous piece

This is true for
higher order as well

?? $\rightarrow y_h$ is the general solution of $ay'' + by' + cy = 0$, and

?? $\rightarrow y_p$ is a particular solution of (*)

Reduction of Order - Summary

When to use: Given a 2nd-order homogeneous linear differential equation and **ONLY ONE** solution.

Why to use: Find the other linearly independent solution to form a fundamental set of solutions.

How to use: TWO OPTIONS

@ Long route above (don't memorize)

⑥ The formula

$$y_1(x) \int \frac{e^{-\int P(x)dx}}{(y_1(x))^2} dx$$

Summary
 The most important thing with these problems is (1) Plug-in $y = e^{mx}$ and (2) finding out if $m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ & $m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ are of

Case 1) Two Distinct Roots

Case 2) Repeated Root

Case 3) Complex Conjugate Roots

3. (20 points) Solve the following differential equations. If a technique is asked for, use it.

a. (5 pts) $y'' - 6y' + 9y = 0$

1) Plug-in $y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2e^{mx}$

Skipping a step, instead of plugging-in and refactoring out, we know are able to factor out " e^{mx} "

This gives us

$$e^{mx}(m^2 - 6m + 9) = 0$$

So we need

$$m^2 - 6m + 9 = 0$$

b. (5 pts) $y'' + 4y' + 7y = 0$

$$e^{mx}(m^2 + 4m + 7) = 0$$

$$\rightarrow m^2 + 4m + 7 = 0$$

$$m = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-4 + \sqrt{16 - 28}}{2}$$

$$= \frac{-4 + \sqrt{16 - 28}}{2}$$

$$= \frac{-4 + \sqrt{-12}}{2}$$

$$= \frac{-4 + i\sqrt{12}}{2}$$

c. (10 pts) $y'' - 4y' + 3y = \cos(x) + \sin(x)$ [using the method of undetermined coefficients]

Lets first consider the homogeneous part

General solution for

$$y'' - 4y' + 3y = 0$$

1) Plug-in $y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2e^{mx}$

Skipping a step, instead of plugging-in and refactoring out, we know are able to factor out " e^{mx} "

This gives us

$$e^{mx}(m^2 - 4m + 3) = 0$$

So we need

$$m^2 - 4m + 3 = 0$$

2) Find $m_1 \notin m_2$

$$m = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{4 \pm \sqrt{16 - 12}}{2}$$

$$= \frac{4 \pm \sqrt{4}}{2} = 2+1, 2-1$$

$$M_1 = 3, M_2 = 1$$

$$Y_h = C_1 e^{3x} + C_2 e^x$$

2) Find $m_1 \in m_2$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{6 \pm \sqrt{36 - 36}}{2}$$

$$= \frac{6 \pm 0}{2} = 3$$

case 2 Repeated Root

$$Y = C_1 e^{3x} + X C_2 e^{3x}$$

In summary: To get y_2 all you need to do is multiply y_1 by ' x '.

$$y_2 = e^{-2x} \sin\left(\frac{\sqrt{2}}{2}x\right)$$

$$\therefore Y = C_1 e^{-2x} \cos\left(\frac{\sqrt{2}}{2}x\right) + C_2 e^{-2x} \sin\left(\frac{\sqrt{2}}{2}x\right)$$

* Recall $\Re e^{ix} = \cos(x) + i\sin(x)$
 $e^{ix} = \cos(x) + i\sin(x)$
 $\rightarrow e^{3/2x} (\cos(\frac{\sqrt{2}}{2}x) + i\sin(\frac{\sqrt{2}}{2}x))$
 Focus on the real part
 Euler's Formula Real
 Ignore the complex part Complex

3) Using the formula

(Euler's Formula)

$$y_1 = e^{ax} \cos(bx), y_2 = e^{ax} \sin(bx)$$

We find

$$y_1 = e^{-2x} \cos\left(\frac{\sqrt{2}}{2}x\right)$$

Recall-Module1.0
 General Solutions for Nonhomogeneous Linear Differential Equations
 Given $(a, b, c$ are constants)
 $(*) ay'' + by' + cy = f(x)$
 with $f(x) \neq 0$, the general solution is of the form
 $y = y_h + y_p$
 Where
 $\Rightarrow y_h$ is the general solution of
 $ay'' + by' + cy = 0$, and
 $\Rightarrow y_p$ is a particular solution of $(*)$

Add up all cosine terms
Cosine (LHS = RHS)

$$2A - 4B = 1$$

$$A = \frac{1 + 4B}{2}$$

$$A = \frac{1 + 4(-\frac{1}{10})}{2} \quad \begin{matrix} 2(4B+1) + 2B = 1 \\ 8B + 2 + 2B = 1 \\ 8B + 2 = -1 \end{matrix}$$

Add up all sine terms
Sine (LHS = RHS)

$$4A + 2B = 1$$

$$2(4B+1) + 2B = 1$$

$$8B + 2 + 2B = 1$$

$$8B + 2 = -1$$

$$10B = -1$$

$$B = -\frac{1}{10}$$

$$Y_p = \frac{1 + 4(-\frac{1}{10})}{2} \cos x + -\frac{1}{10} \sin x$$

Case 2) $f(x)$ consists of $\sin(x)$ and/or $\cos(x)$ terms.

Any sines or cosines (not products of them i.e. \sin^2 or \cos^2)

$$y_{p2} = A \cos(x) + B \sin(x)$$

$$y_{p2}' = -A \sin(x) + B \cos(x)$$

$$y_{p2}'' = -A \cos(x) - B \sin(x),$$

$$\text{So } y'' - 4y' + 3y$$

$$\rightarrow (-A \cos(x) - B \sin(x)) - 4(-A \sin(x) + B \cos(x)) + 3(A \cos(x) + B \sin(x))$$

$$\rightarrow (-A \cos(x) - B \sin(x)) + (4A \sin(x) - 4B \cos(x)) + (3A \cos(x) + 3B \sin(x)) = \cos(x) + \sin(x)$$

$$2A \cos(x) - 4B \cos(x) + 4A \sin(x) + 2B \sin(x)$$

Augh running out of time.

4. (5 points) Suppose $y_1 = x$ and $y_2 = x - 1$ are solutions to a 2nd-order homogeneous linear DE on $I = (-\infty, \infty)$. Do y_1 and y_2 form a fundamental set of solutions for this DE on I ? Explain your answer using any math needed.

5. (10 points) Use the method of variation of parameters to find a general solution to the differential equation $x^2y'' - 4xy' + 4y = 3x^3$.

Consider: $a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$
 In particular before we'd have just constants.
 Where $a_2(x)y''$, $a_1(x)y'$, $a_0(x)y$, $g(x)$ are functions

The Method:

Step 1 | First turn
 $a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$
 \downarrow into \downarrow
 $y'' + P(x)y' + Q(x)y = f(x)$ ←
 The idea is to first put it in to this form
 $f(x) = \frac{g(x)}{a_2(x)}$, $P(x) = \frac{a_1(x)}{a_2(x)}$ So you divide everything by whatever function is in y''

Step 2 | Find our homogeneous piece
 $(y_h = c_1y_1 + c_2y_2)$, which is the general solution for $y'' + P(x)y' + Q(x)y = 0$

Step 3 | Then $y_p = u_1y_1 + u_2y_2$, depends on the homogeneous piece (y_1, y_2) where u_1 and u_2 are found by integrating

$u'_1 = \frac{-y_2f(x)}{W}$ and $u'_2 = \frac{y_1f(x)}{W}$,
 and $W(y_1, y_2)$ is the wronskian of y_1, y_2

$$y_p = u_1y_1 + u_2y_2$$