Homogeneous linear DEs with constant coefficients

$\rightarrow y' + \frac{b}{a} y = 0 \iff y' = -\frac{b}{a} y$

* **Order 1** | $ay' + by = 0$, a, b constants.

$\rightarrow$ Could solve as separable equation.

Get: $y = Ce^{-\frac{b}{a}x}$.

$\rightarrow$ OR: consider $y = e^{mx}$ (m a number).

- plug into DE: $y' = me^{mx}$

Gives $ame^{mx} + be^{mx} = 0$

$\Leftrightarrow$ (factoring out e^{mx})

$$\underbrace{e^{mx}}_{\neq 0}\underbrace{(am+b)}_{\text{must have equal } 0!} = 0$$

$\Leftrightarrow$ $am+b=0 \Rightarrow m = -\frac{b}{a}$,

so $y = e^{mx} \Leftrightarrow y = e^{-\frac{b}{a}x}$.

Thus,

→ 1 soln for 1st-order

→ lin indep, so a Fund set of solns.

Thus, $y = Ce^{-b/a\,x}$ is general soln

* **Order 2** | $ay'' + by' + cy = 0$ (a, b, c constants)

→ plug in $y = e^{mx}$;

$y' = me^{mx}$, $y'' = m^2 e^{mx}$

→ This gives

$$am^2 e^{mx} + bme^{mx} + ce^{mx} = 0$$

$$\Leftrightarrow e^{mx}(am^2 + bm + c) = 0$$

So, need $am^2 + bm + c = 0$.

Thus, $m_1 = \dfrac{-b+\sqrt{b^2-4ac}}{2a}$, $m_2 = \dfrac{-b-\sqrt{b^2-4ac}}{2a}$.

But while

$$y_1 = e^{\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)x}, \quad y_2 = e^{\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)x}$$

are solns, there are some issues...

→ Not necessarily different or lin indep

→ Could be complex numbers!

What governs this is b^2-4ac.

CASE II $b^2-4ac>0$:

Then $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ are 2 distinct real numbers.

Thus,

$$y_1 = e^{\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)x}, \quad y_2 = e^{\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)x}$$

are two linearly independent sol'ns

and

$$y = c_1 e^{\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)x} + c_2 e^{\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)x}$$

OR: $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$ $\left(m_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}, \; m_2 = \frac{-b-\sqrt{b^2-4ac}}{2a} \right)$

CASE 2| $b^2 - 4ac = 0$.

Then $y_1 = e^{\left(\frac{-b}{2a}\right)x}$ is a soln.

But need another.... How do we find it?

- A: Reduction of order!

$$y_2 = y_1 \int \frac{e^{-\int P dx}}{(y_1)^2} dx.$$

So, the general solution is:

$$y = c_1 e^{\left(\frac{-b}{2a}\right)x} + c_2 \left(\underbrace{y_1 \int \frac{e^{-\int P dx}}{(y_1)^2} dx}_{c x e^{\frac{-b}{2a}x}} \right).$$

$m = \frac{-b}{2a}$

which becomes:

$$y = c_1 e^{mx} + c_2 x e^{mx}$$

CASE 3 | $b^2 - 4ac < 0$.

Then $\sqrt{b^2-4ac} = \sqrt{(-1)(\underbrace{-b^2+4ac}_{>0})} = \sqrt{-1}\sqrt{-b^2+4ac}$

$$= i\sqrt{4ac-b^2}$$

$$m_1 = \frac{-b}{2a} + \frac{i\sqrt{4ac-b^2}}{2a}, \quad m_2 = \frac{-b}{2a} - \frac{i\sqrt{b^2-4ac}}{2a}$$

$$= \alpha + i\beta \qquad\qquad = \alpha - i\beta$$

↳ This gives

$$y_1 = e^{\left(\frac{-b}{2a} + \frac{i\sqrt{4ac-b^2}}{2a}\right)x}, \quad y_2 = e^{\left(\frac{-b}{2a} - \frac{i\sqrt{4ac-b^2}}{2a}\right)x}$$

complex solns.

* We want real solns.

Euler's Identity: (θ a real number)

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

Thus,

$$y_1 = e^{\left(-\frac{b}{2a}x + i\frac{\sqrt{4ac-b^2}}{2a}x\right)} = e^{-\frac{b}{2a}x}\, e^{i\left(\frac{\sqrt{4ac-b^2}}{2a}x\right)}$$

$$= e^{-\frac{b}{2a}}\left(\underbrace{\cos\left(\frac{\sqrt{4ac-b^2}}{2a}x\right)}_{\text{real}} + \underbrace{\cancel{i\sin\left(\frac{\sqrt{4ac-b^2}}{2a}x\right)}}_{\text{complex}}\right)$$

$$= e^{-\frac{b}{2a}x}\cos\left(\frac{\sqrt{4ac-b^2}}{2a}x\right).$$

Finding y_2 in the same way, gives the same answer y_1.

But, we have reduction of order!

And find: $y_2 = e^{-\frac{b}{2a}x} \sin\left(\frac{\sqrt{4ac-b^2}}{2a}x\right)$

Thus, the general soln is

$$y = c_1 e^{-\frac{b}{2a}x} \cos\left(\frac{\sqrt{4ac-b^2}}{2a}x\right) + c_2 e^{-\frac{b}{2a}x} \sin\left(\frac{\sqrt{4ac-b^2}}{2a}x\right)$$

OR having $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, then

$$y = c_1 e^{\alpha} \cos(\beta x) + c_2 e^{\alpha} \sin(\beta x).$$

Example

Ex) Solve $3y'' - 5y' + 2y = 0$.

soln)

→ Plug in $y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2 e^{mx}$:

→ Get

$$e^{mx}(3m^2 - 5m + 2) = 0,$$

→ so need

$$3m^2 - 5m + 2 = 0.$$

→ so

$$m_1 = \frac{5 + \sqrt{25 - 24}}{6} = \frac{5}{6} + \frac{1}{6} = 1$$

$$m_2 = \frac{5 - 1}{6} = \frac{4}{6} = 2/3$$

→ distinct real numbers (case)

→ thus, the general soln is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

or

$$y = c_1 e^{x} + c_2 e^{\frac{2}{3}x}.$$

Example

Ex| Solve $y'' + 4y' + 4y = 0$.

soln|

$\rightarrow y = e^{mx}$

$\rightarrow$ Gives $e^{mx}(m^2 + 4m + 4) = 0$,

so $m^2 + 4m + 4 = 0$

$\rightarrow$ $m_1 = \dfrac{-4 + \sqrt{16 - 16}}{2} = \dfrac{-4}{2} = -2$.

$m_2 = -2$.

$\rightarrow y_1 = e^{-2x}$

$\rightarrow$ Use reduction of order [could omit in future]

$$\overline{y_2} = y_1 \int \frac{e^{-\int P\,dx}}{(y_1)^2}\,dx = e^{-2x} \int \frac{e^{-\int 4\,dx}}{(e^{-2x})^2}\,dx$$

$$= e^{-2x} \int \frac{e^{-4x}}{e^{-4x}}\,dx = e^{-2x} \int dx$$

$$= e^{-2x}(x + c) = xe^{-2x} + ce^{-2x}$$

consider $y_2 = xe^{-2x}$.

$\rightarrow$ Thus, $y = c_1 e^{-2x} + c_2 x e^{-2x}$ is a gen soln.

Example

Ex1 Solve $y'' - 3y' + 4y = 0$.

soln
$\rightarrow$ plug in $y = e^{mx}$

$\rightarrow$ $e^{mx}(m^2 - 3m + 4) = 0$, so need

$$m^2 - 3m + 4 = 0$$

$\rightarrow$ $m = \dfrac{3 \pm \sqrt{9-16}}{2} = \dfrac{3 \pm i\sqrt{7}}{2}$

$\rightarrow$ $m_1 = \underbrace{\frac{3}{2} + i\frac{\sqrt{7}}{2}}_{\alpha + i\beta}$, $m_2 = \underbrace{\frac{3}{2} - i\frac{\sqrt{7}}{2}}_{\alpha - i\beta}$

$$\boxed{e^{\left(\frac{3}{2}+i\frac{\sqrt{7}}{2}\right)x} = e^{3/2x}e^{i\frac{\sqrt{7}}{2}x} = e^{3/2x}\left(\cos\left(\frac{\sqrt{7}}{2}x\right)+i\sin\left(\frac{\sqrt{7}}{2}x\right)\right)}$$

Euler's Formula

Using the formula:

$$y_1 = e^{\alpha x}\cos(\beta x), \quad y_2 = e^{\alpha x}\sin(\beta x)$$

we find

$$y_1 = e^{\frac{3}{2}x}\cos\left(\frac{\sqrt{7}}{2}x\right), \quad y_2 = e^{\frac{3}{2}x}\sin\left(\frac{\sqrt{7}}{2}x\right).$$

thus, the gen soln is

$$y = c_1 e^{3/2x}\cos\left(\frac{\sqrt{7}}{2}x\right) + c_2 e^{3/2x}\sin\left(\frac{\sqrt{7}}{2}x\right)$$

Higher order

Q: How do we solve?

A: Basically the same.

* Given

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

$(a_n, a_{n-1}, \ldots, a_1, a_0$ are constants$)$

then

→ plug in $y = e^{mx}$:

→ Gives

$$e^{mx}\left(a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0\right) = 0$$

→ So need

$$a_n m^n + \cdots + a_1 m + a_0 = 0.$$

→ solve for m.

* Note: can be hard.

Ex)

$m^{17} - 108m^{15} - 3m^2 + m - 1 = 0$??

situations we can handle:

(1) If n-many distinct real roots, then the gen soln is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}.$$

Ex1 $y''' - 6y'' + 11y' - 6y = 0$

Gives $e^{mx}(\underbrace{m^3 - 6m^2 + 11m - 6}_{= (m-1)(m-2)(m-3)}) = 0$

so $m_1 = 1$, $m_2 = 2$, $m_3 = 3$

Thus,

$$y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}.$$

(2) Repeated roots.

If order n DE, and only 1 real root, but repeated n times, then gen sol^n is

$$y = c_1 e^{mx} + c_2 x e^{mx} + c_3 x^2 e^{mx} + \cdots + c_n x^{n-1} e^{mx}.$$

Ex) $y''' - 6y'' + 12y' + 8$ gives $\underbrace{m^3 - 6m^2 + 12m + 8}_{(m-2)^3} = 0$

so

$$y = c_1 e^{2x} + c_2 x e^{2x} + c_3 x^2 e^{2x}$$

(3) If it contains 2 complex roots only:

Ex1 $\rightsquigarrow$ $(m^2+1)m = 0$

$$\Rightarrow m_1 = 0, \quad \underbrace{m_1 = i}, \quad \underbrace{m_2 = -i}$$

$$\alpha = 0, \quad \beta = 1$$

Then

$$y_1 = e^{0x} = 1, \quad y_2 = e^{0}\cos(x), \quad y_3 = e^{0}\sin(x)$$

Thus,

$$y = c_1 + c_2\cos(x) + c_3\sin(x)$$

(4) combining

Ex) Solve $y''' - 5y'' + 8y' - 4y = 0$.

soln) Get $m^3 - 5m^2 + 8m - 4 = 0$

Note: $m = 1$ is a root, can factor it out

$\Rightarrow \quad (m-1)(m^2 - 4m + 4) = 0$

$\Rightarrow \quad (m-1)(m-2)^2 = 0$

Thus,

$$y = c_1 e^{x} + c_2 e^{2x} + c_3 x e^{2x}$$

is general soln.