32.2-4

Alice has a copy of a long n-bit file $A = \langle a_{n-1}, a_{n-2}, \ldots, a_0 \rangle$, and Bob similarly has an n-bit file $B = \langle b_{n-1}, b_{n-2}, \ldots, b_0 \rangle$. Alice and Bob wish to know if their files are identical. To avoid transmitting all of A or B, they use the following fast probabilistic check. Together, they select a prime $q > 1000n$ and randomly select an integer x from $\{0, 1, \ldots, q-1\}$. Then, Alice evaluates

$$A(x) = \left(\sum_{i=0}^{n-1} a_i x^i \right) \bmod q$$

and Bob similarly evaluates $B(x)$. Prove that if $A \neq B$, there is at most one chance in 1000 that $A(x) = B(x)$, whereas if the two files are the same, $A(x)$ is necessarily the same as $B(x)$. (*Hint:* See Exercise 31.4-4.)

32.3 String matching with finite automata

Many string-matching algorithms build a finite automaton—a simple machine for processing information—that scans the text string T for all occurrences of the pattern P. This section presents a method for building such an automaton. These string-matching automata are very efficient: they examine each text character *exactly once*, taking constant time per text character. The matching time used—after preprocessing the pattern to build the automaton—is therefore $\Theta(n)$. The time to build the automaton, however, can be large if Σ is large. Section 32.4 describes a clever way around this problem.

We begin this section with the definition of a finite automaton. We then examine a special string-matching automaton and show how to use it to find occurrences of a pattern in a text. Finally, we shall show how to construct the string-matching automaton for a given input pattern.

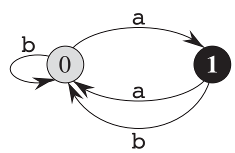
Finite automata

A ***finite automaton*** M, illustrated in Figure 32.6, is a 5-tuple $(Q, q_0, A, \Sigma, \delta)$, where

- Q is a finite set of ***states***,
- $q_0 \in Q$ is the ***start state***,
- $A \subseteq Q$ is a distinguished set of ***accepting states***,
- Σ is a finite ***input alphabet***,
- δ is a function from $Q \times \Sigma$ into Q, called the ***transition function*** of M.

state	input a	input b
0	1	0
1	0	0

(a)



(b)

Figure 32.6 A simple two-state finite automaton with state set $Q = \{0, 1\}$, start state $q_0 = 0$, and input alphabet $\Sigma = \{\texttt{a}, \texttt{b}\}$. **(a)** A tabular representation of the transition function δ. **(b)** An equivalent state-transition diagram. State 1, shown blackend, is the only accepting state. Directed edges represent transitions. For example, the edge from state 1 to state 0 labeled b indicates that $\delta(1, \texttt{b}) = 0$. This automaton accepts those strings that end in an odd number of a's. More precisely, it accepts a string x if and only if $x = yz$, where $y = \varepsilon$ or y ends with a b, and $z = \texttt{a}^k$, where k is odd. For example, on input abaaa, including the start state, this automaton enters the sequence of states $\langle 0, 1, 0, 1, 0, 1 \rangle$, and so it accepts this input. For input abbaa, it enters the sequence of states $\langle 0, 1, 0, 0, 1, 0 \rangle$, and so it rejects this input.

The finite automaton begins in state q_0 and reads the characters of its input string one at a time. If the automaton is in state q and reads input character a, it moves ("makes a transition") from state q to state $\delta(q, a)$. Whenever its current state q is a member of A, the machine M has ***accepted*** the string read so far. An input that is not accepted is ***rejected***.

A finite automaton M induces a function ϕ, called the ***final-state function***, from Σ^* to Q such that $\phi(w)$ is the state M ends up in after scanning the string w. Thus, M accepts a string w if and only if $\phi(w) \in A$. We define the function ϕ recursively, using the transition function:

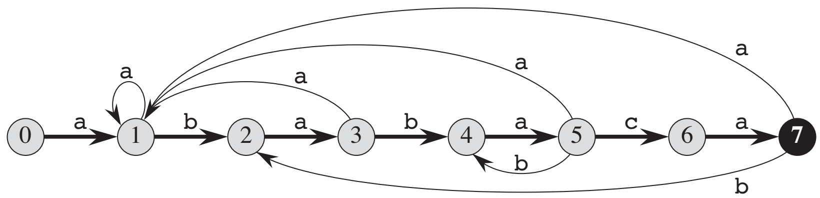
$$
\begin{aligned}
\phi(\varepsilon) &= q_0 , \\
\phi(wa) &= \delta(\phi(w), a) \quad \text{for } w \in \Sigma^*, a \in \Sigma .
\end{aligned}
$$

String-matching automata

For a given pattern P, we construct a string-matching automaton in a preprocessing step before using it to search the text string. Figure 32.7 illustrates how we construct the automaton for the pattern $P = \texttt{ababaca}$. From now on, we shall assume that P is a given fixed pattern string; for brevity, we shall not indicate the dependence upon P in our notation.

In order to specify the string-matching automaton corresponding to a given pattern $P[1 \,..\, m]$, we first define an auxiliary function σ, called the ***suffix function*** corresponding to P. The function σ maps Σ^* to $\{0, 1, \ldots, m\}$ such that $\sigma(x)$ is the length of the longest prefix of P that is also a suffix of x:

$$\sigma(x) = \max\{k : P_k \sqsupset x\} . \tag{32.3}$$



(a)

state	input a	input b	input c	P
0	1	0	0	a
1	1	2	0	b
2	3	0	0	a
3	1	4	0	b
4	5	0	0	a
5	1	4	6	c
6	7	0	0	a
7	1	2	0	

(b)

i	—	1	2	3	4	5	6	7	8	9	10	11
$T[i]$	—	a	b	a	b	a	b	a	c	a	b	a
state $\phi(T_i)$	0	1	2	3	4	5	4	5	6	**7**	2	3

(c)

Figure 32.7 **(a)** A state-transition diagram for the string-matching automaton that accepts all strings ending in the string `ababaca`. State 0 is the start state, and state 7 (shown blackened) is the only accepting state. A directed edge from state i to state j labeled a represents $\delta(i, a) = j$. The right-going edges forming the "spine" of the automaton, shown heavy in the figure, correspond to successful matches between pattern and input characters. The left-going edges correspond to failing matches. Some edges corresponding to failing matches are omitted; by convention, if a state i has no outgoing edge labeled a for some $a \in \Sigma$, then $\delta(i, a) = 0$. **(b)** The corresponding transition function δ, and the pattern string $P =$ `ababaca`. The entries corresponding to successful matches between pattern and input characters are shown shaded. **(c)** The operation of the automaton on the text $T =$ `abababacaba`. Under each text character $T[i]$ appears the state $\phi(T_i)$ that the automaton is in after processing the prefix T_i. The automaton finds one occurrence of the pattern, ending in position 9.

The suffix function σ is well defined since the empty string $P_0 = \varepsilon$ is a suffix of every string. As examples, for the pattern $P =$ `ab`, we have $\sigma(\varepsilon) = 0$, $\sigma($`ccaca`$) = 1$, and $\sigma($`ccab`$) = 2$. For a pattern P of length m, we have $\sigma(x) = m$ if and only if $P \sqsupset x$. From the definition of the suffix function, $x \sqsupset y$ implies $\sigma(x) \leq \sigma(y)$.

We define the string-matching automaton that corresponds to a given pattern $P[1 \mathinner{.\,.} m]$ as follows:

- The state set Q is $\{0, 1, \ldots, m\}$. The start state q_0 is state 0, and state m is the only accepting state.
- The transition function δ is defined by the following equation, for any state q and character a:

$$\delta(q, a) = \sigma(P_q a) . \tag{32.4}$$

We define $\delta(q, a) = \sigma(P_q a)$ because we want to keep track of the longest prefix of the pattern P that has matched the text string T so far. We consider the most recently read characters of T. In order for a substring of T—let's say the substring ending at $T[i]$—to match some prefix P_j of P, this prefix P_j must be a suffix of T_i. Suppose that $q = \phi(T_i)$, so that after reading T_i, the automaton is in state q. We design the transition function δ so that this state number, q, tells us the length of the longest prefix of P that matches a suffix of T_i. That is, in state q, $P_q \sqsupset T_i$ and $q = \sigma(T_i)$. (Whenever $q = m$, all m characters of P match a suffix of T_i, and so we have found a match.) Thus, since $\phi(T_i)$ and $\sigma(T_i)$ both equal q, we shall see (in Theorem 32.4, below) that the automaton maintains the following invariant:

$$\phi(T_i) = \sigma(T_i) . \tag{32.5}$$

If the automaton is in state q and reads the next character $T[i+1] = a$, then we want the transition to lead to the state corresponding to the longest prefix of P that is a suffix of $T_i a$, and that state is $\sigma(T_i a)$. Because P_q is the longest prefix of P that is a suffix of T_i, the longest prefix of P that is a suffix of $T_i a$ is not only $\sigma(T_i a)$, but also $\sigma(P_q a)$. (Lemma 32.3, on page 1000, proves that $\sigma(T_i a) = \sigma(P_q a)$.) Thus, when the automaton is in state q, we want the transition function on character a to take the automaton to state $\sigma(P_q a)$.

There are two cases to consider. In the first case, $a = P[q+1]$, so that the character a continues to match the pattern; in this case, because $\delta(q, a) = q+1$, the transition continues to go along the "spine" of the automaton (the heavy edges in Figure 32.7). In the second case, $a \neq P[q+1]$, so that a does not continue to match the pattern. Here, we must find a smaller prefix of P that is also a suffix of T_i. Because the preprocessing step matches the pattern against itself when creating the string-matching automaton, the transition function quickly identifies the longest such smaller prefix of P.

Let's look at an example. The string-matching automaton of Figure 32.7 has $\delta(5, \texttt{c}) = 6$, illustrating the first case, in which the match continues. To illustrate the second case, observe that the automaton of Figure 32.7 has $\delta(5, \texttt{b}) = 4$. We make this transition because if the automaton reads a `b` in state $q = 5$, then $P_q\texttt{b} = \texttt{ababab}$, and the longest prefix of P that is also a suffix of `ababab` is $P_4 = \texttt{abab}$.

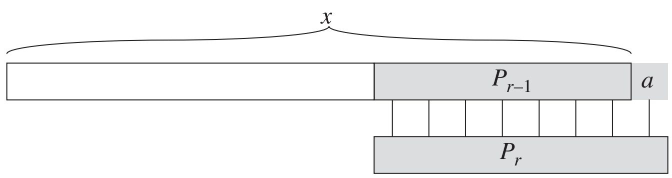


Figure 32.8 An illustration for the proof of Lemma 32.2. The figure shows that $r \le \sigma(x) + 1$, where $r = \sigma(xa)$.

To clarify the operation of a string-matching automaton, we now give a simple, efficient program for simulating the behavior of such an automaton (represented by its transition function δ) in finding occurrences of a pattern P of length m in an input text $T[1 \mathinner{..} n]$. As for any string-matching automaton for a pattern of length m, the state set Q is $\{0, 1, \ldots, m\}$, the start state is 0, and the only accepting state is state m.

```
FINITE-AUTOMATON-MATCHER(T, δ, m)
n = T.length
q = 0
for i = 1 to n
    q = δ(q, T[i])
    if q == m
        print "Pattern occurs with shift" i − m
```

From the simple loop structure of FINITE-AUTOMATON-MATCHER, we can easily see that its matching time on a text string of length n is $\Theta(n)$. This matching time, however, does not include the preprocessing time required to compute the transition function δ. We address this problem later, after first proving that the procedure FINITE-AUTOMATON-MATCHER operates correctly.

Consider how the automaton operates on an input text $T[1 \mathinner{..} n]$. We shall prove that the automaton is in state $\sigma(T_i)$ after scanning character $T[i]$. Since $\sigma(T_i) = m$ if and only if $P \sqsupset T_i$, the machine is in the accepting state m if and only if it has just scanned the pattern P. To prove this result, we make use of the following two lemmas about the suffix function σ.

Lemma 32.2 (Suffix-function inequality)
For any string x and character a, we have $\sigma(xa) \le \sigma(x) + 1$.

Proof Referring to Figure 32.8, let $r = \sigma(xa)$. If $r = 0$, then the conclusion $\sigma(xa) = r \le \sigma(x) + 1$ is trivially satisfied, by the nonnegativity of $\sigma(x)$. Now assume that $r > 0$. Then, $P_r \sqsupset xa$, by the definition of σ. Thus, $P_{r-1} \sqsupset x$, by

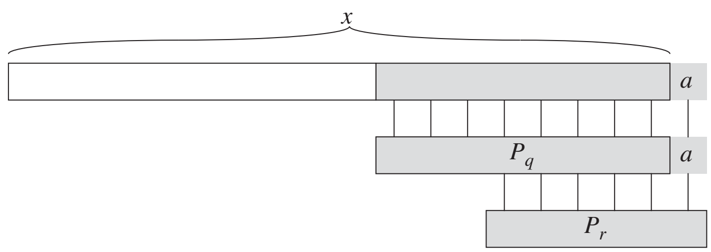


Figure 32.9 An illustration for the proof of Lemma 32.3. The figure shows that $r = \sigma(P_q a)$, where $q = \sigma(x)$ and $r = \sigma(xa)$.

dropping the a from the end of P_r and from the end of xa. Therefore, $r-1 \leq \sigma(x)$, since $\sigma(x)$ is the largest k such that $P_k \sqsupset x$, and thus $\sigma(xa) = r \leq \sigma(x) + 1$. ∎

Lemma 32.3 (Suffix-function recursion lemma)
For any string x and character a, if $q = \sigma(x)$, then $\sigma(xa) = \sigma(P_q a)$.

Proof From the definition of σ, we have $P_q \sqsupset x$. As Figure 32.9 shows, we also have $P_q a \sqsupset xa$. If we let $r = \sigma(xa)$, then $P_r \sqsupset xa$ and, by Lemma 32.2, $r \leq q + 1$. Thus, we have $|P_r| = r \leq q + 1 = |P_q a|$. Since $P_q a \sqsupset xa$, $P_r \sqsupset xa$, and $|P_r| \leq |P_q a|$, Lemma 32.1 implies that $P_r \sqsupset P_q a$. Therefore, $r \leq \sigma(P_q a)$, that is, $\sigma(xa) \leq \sigma(P_q a)$. But we also have $\sigma(P_q a) \leq \sigma(xa)$, since $P_q a \sqsupset xa$. Thus, $\sigma(xa) = \sigma(P_q a)$. ∎

We are now ready to prove our main theorem characterizing the behavior of a string-matching automaton on a given input text. As noted above, this theorem shows that the automaton is merely keeping track, at each step, of the longest prefix of the pattern that is a suffix of what has been read so far. In other words, the automaton maintains the invariant (32.5).

Theorem 32.4
If ϕ is the final-state function of a string-matching automaton for a given pattern P and $T[1 \,..\, n]$ is an input text for the automaton, then

$$\phi(T_i) = \sigma(T_i)$$

for $i = 0, 1, \ldots, n$.

Proof The proof is by induction on i. For $i = 0$, the theorem is trivially true, since $T_0 = \varepsilon$. Thus, $\phi(T_0) = 0 = \sigma(T_0)$.

Now, we assume that $\phi(T_i) = \sigma(T_i)$ and prove that $\phi(T_{i+1}) = \sigma(T_{i+1})$. Let q denote $\phi(T_i)$, and let a denote $T[i+1]$. Then,

$$\begin{aligned}
\phi(T_{i+1}) &= \phi(T_i a) && \text{(by the definitions of } T_{i+1} \text{ and } a) \\
&= \delta(\phi(T_i), a) && \text{(by the definition of } \phi) \\
&= \delta(q, a) && \text{(by the definition of } q) \\
&= \sigma(P_q a) && \text{(by the definition (32.4) of } \delta) \\
&= \sigma(T_i a) && \text{(by Lemma 32.3 and induction)} \\
&= \sigma(T_{i+1}) && \text{(by the definition of } T_{i+1}) \ . \qquad \blacksquare
\end{aligned}$$

By Theorem 32.4, if the machine enters state q on line 4, then q is the largest value such that $P_q \sqsupset T_i$. Thus, we have $q = m$ on line 5 if and only if the machine has just scanned an occurrence of the pattern P. We conclude that FINITE-AUTOMATON-MATCHER operates correctly.

Computing the transition function

The following procedure computes the transition function δ from a given pattern $P[1..m]$.

```
COMPUTE-TRANSITION-FUNCTION(P, Σ)
1  m = P.length
2  for q = 0 to m
3      for each character a ∈ Σ
4          k = min(m + 1, q + 2)
5          repeat
6              k = k − 1
7          until P_k ⊐ P_q a
8          δ(q, a) = k
9  return δ
```

This procedure computes $\delta(q, a)$ in a straightforward manner according to its definition in equation (32.4). The nested loops beginning on lines 2 and 3 consider all states q and all characters a, and lines 4–8 set $\delta(q, a)$ to be the largest k such that $P_k \sqsupset P_q a$. The code starts with the largest conceivable value of k, which is $\min(m, q+1)$. It then decreases k until $P_k \sqsupset P_q a$, which must eventually occur, since $P_0 = \varepsilon$ is a suffix of every string.

The running time of COMPUTE-TRANSITION-FUNCTION is $O(m^3 |\Sigma|)$, because the outer loops contribute a factor of $m |\Sigma|$, the inner **repeat** loop can run at most $m+1$ times, and the test $P_k \sqsupset P_q a$ on line 7 can require comparing up to m characters. Much faster procedures exist; by utilizing some cleverly computed information about the pattern P (see Exercise 32.4-8), we can improve the time required to compute δ from P to $O(m\,|\Sigma|)$. With this improved procedure for computing δ, we can find all occurrences of a length-m pattern in a length-n text over an alphabet Σ with $O(m\,|\Sigma|)$ preprocessing time and $\Theta(n)$ matching time.

Exercises

32.3-1
Construct the string-matching automaton for the pattern $P = \texttt{aabab}$ and illustrate its operation on the text string $T = \texttt{aaababaabaababaab}$.

32.3-2
Draw a state-transition diagram for a string-matching automaton for the pattern $\texttt{ababbabbababbababbabb}$ over the alphabet $\Sigma = \{\texttt{a}, \texttt{b}\}$.

32.3-3
We call a pattern P ***nonoverlappable*** if $P_k \sqsupset P_q$ implies $k = 0$ or $k = q$. Describe the state-transition diagram of the string-matching automaton for a nonoverlappable pattern.

32.3-4 ★
Given two patterns P and P', describe how to construct a finite automaton that determines all occurrences of *either* pattern. Try to minimize the number of states in your automaton.

32.3-5
Given a pattern P containing gap characters (see Exercise 32.1-4), show how to build a finite automaton that can find an occurrence of P in a text T in $O(n)$ matching time, where $n = |T|$.

★ 32.4 The Knuth-Morris-Pratt algorithm

We now present a linear-time string-matching algorithm due to Knuth, Morris, and Pratt. This algorithm avoids computing the transition function δ altogether, and its matching time is $\Theta(n)$ using just an auxiliary function π, which we precompute from the pattern in time $\Theta(m)$ and store in an array $\pi[1\,..\,m]$. The array π allows us to compute the transition function δ efficiently (in an amortized sense) "on the fly" as needed. Loosely speaking, for any state $q = 0, 1, \ldots, m$ and any character