Goal and idea - Module 1

GOAL:

The ultimate goal of this module is to develop terminology to discuss what differential equations are and what characterizes them.

To do so, we need to:

- Define what differential equations (DEs) are.
- Recognize what dependent and independent variables are.
- Introduce some ways in which we can characterize DEs. While there are many ways to do so in the world, we focus on classifying based on
  - ordinary or partial,
  - linear or non-linear, and
  - order.
- Finally, we examine different ways in which DEs can be written, or expressed. Many of these 'forms' have their own name.

IDEA:

We need to develop a lot of notation, which is cumbersome. Meanwhile, the scope of DEs is too large for us to master at the moment and we need to limit what we will (and can) study. In an attempt to accomplish both of these, we introduce notation, ways in which DEs can be written, and ultimately some characteristics of DEs which we will use to limit the scope of our study.

Time: Videos run 67:26 minutes.

DEs and variables

Definition

A differental equation is an equation that relates one or more functions and their derivatives.

Definition

1. A symbol that represents an input of a function is called an independent variable
2. A symbol that represents an output of a function is called an dependent variable

Example

$y = x^2$ ← independent variable (input)

dependent variable (output)

Definition

A multivariable function is a function which consists of more than one independent variable.

Example | In the following state the dependent and independent variables

(a) $y = x^2$ dep $= y$ indep $= x$

(b) $y' = 4$ dep $= y$ indep $= x$ ($y = f(x)$)

(c) $f(x) = x^2$ dep $= f$ indep $= x$

(d) $x' + x = y$ dep $= x$ indep $= y$

(e) $\frac{dx}{dy} + x = y$ dep $= x$ indep $= y$

(f) $g(x,y) = \frac{x^2 y}{2}$ dep $= g$ indep $= x, y$

Ex For $g(x,y,z) = x^2y^3+z$ we have

$\rightarrow \frac{d}{dx} g(x,y,z) = 2xy^3+0 = \boxed{2xy^3}$

$\rightarrow \frac{dg}{dy} = x^2(3y^2)+0 = \boxed{3x^2y^2}$

$\rightarrow \frac{dg}{dz} = 0+\frac{d}{dz}(z) = \boxed{1}$

Ex Consider $f(x,y) = \cos(x^2y^3)$

Recall Chain Rule
$$[f(g(x))]' = f'(g(x))\,g'(x)$$

$\rightarrow \frac{df}{dx} = -\sin(x^2y^3)\left(\frac{d}{dx}(x^2y^3)\right)$

Derivative of f w/ respects to x

Chain Rule

$= -\sin(x^2y^3)(2xy^3) = \boxed{-2xy^3\sin(x^2y^3)}$

$\rightarrow \frac{df}{dy} = -\sin(x^2y^3)(3y^2x^2) = \boxed{-3x^2y^2\sin(3y^2x^2)}$

Derivative of f with respects to 'y'

Ex $\frac{d}{dz}\left(\frac{x+y}{z}\right)$

The derivative with respects to 'z'

An application of the Quotient Rule

$\rightarrow \frac{d}{dz}\left(\frac{x+y}{z}\right) = (x+y)\frac{d}{dz}\left(\frac{1}{z}\right) = (x+y)\left(-\frac{1}{z^2}\right) = \boxed{-\frac{(x+y)}{z^2}}$ ∎

Because 'x+y' is not 'z' it can be treated as a constant and so we can pull constants out of the derivative

ODEs and PDEs

We turn to ways in which we can classify DEs.
The first way is based on whether it includes a single-variable function and its derivatives, or a multivariable functions and its partial derivatives

Definition

- An ordinary differential equation (ODE) is a DE consisting of derivatives of at least one function of one independent variable.
- A partial differential equation (PDE) is a DE consisting of partial derivatives of at least one multivariable function.

Ex| Classify as an ODE or PDE in the following:

'y' is dependent on 'x'

(a) $\frac{dy}{dx} + e^x + 6y + x = 0$ ODE

Partial Derivatives 'g' is a function of 'x' 'g' is also a function of 'y'

(b) $\frac{d}{dx}g + \frac{d}{dy}g - xy = 0$ PDE - Multivariable function, and derivatives with respects to both 'x' & 'y'

(c) $\frac{dx}{dt} + \frac{dy}{dt} - x - y = 0$ ODE - Least one function of one independent variable.

Both 'x' and 'y' is a function of 't'

$\frac{dx}{dt}$ & $\frac{dy}{dt}$ $\frac{\text{Dependent}}{\text{Independent}}$

Linear: derivatives and functions occur as polynomials of degree one.

Recall: A polynomial in 'x' is of the form $C_n x^n + C_{n-1} x^{n-1} + \ldots + C_1 x + C_0$

(y, y' ... $y^n, (y')^n$ ... $y^{n-1}, (y')^{n-1}$ ... y, y')

$C_n, C_{n-1}, \ldots, C_1, C_0$ constants

→ 'n' is the degree

So linear DE must only have y or y' or $y^{(n)}$

But can't have the derivative squared $(y')^2$, y^2, $(y')^n$, $(y^{(3)})^2$, $y'' \cdot y'$

* Can't have 'n' be any higher power greater than one *

Ex Determine if the following are linear or non-linear.

(a) $yy' = 2y$ — **Non-Linear:** There are some terms that are not degree one polynomial terms

(b) $y^{(100)} - y'' + y = 0$ — **Linear:** The 100 in parentheses means the 100th derivative. It isn't the 100th derivative squared, so it's a still a degree one polynomial term.

(c) $y^2 - y = 0$ — **Non-Linear:** The two isn't in parentheses, so it is of degree two polynomial.

(d) $\frac{d^2 f}{dx^2} + f = 0$ — **Linear:** Also applies to partial DE. * It's okay to have higher-order derivatives *
$\frac{d^2 f}{dx^2}$ the 2 denotes the second derivative of 'f'

(e) $\left(\frac{df}{dx}\right)^2 + f = 0$ — **Non-Linear:** The difference between (d) and (e) is that in (e) the derivative, as a whole, is being squared. It is a polynomial of degree two term.

(f) $\sqrt{y''} + y = 0$ — **Non-Linear:** $\sqrt{y''} \to (y'')^{1/2}$. y prime is not a polynomial of degree one, so it's non-linear

(g) $\sin(y^{(5)}) + y = 5$ — **Non-Linear:** Sine of 'y' to the fifth derivative of 'y' in sine. Recall from Calculus there is a power series expansion of sine and is an infinite polynomial that has higher degrees. When y-prime in another function like this: it is not in linear form (meaning not polynomial of degree one) is therefore non-linear.

(h) $y' = e^y$ — **Non-Linear:** The y-prime is fine and is in polynomial degree one (i.e. Linear form), but e^y the y is not in linear form. The y is up in the exponential space.

(i) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + e^x y = 0$ — **Linear:** The non-linear e^x is with x. You'll only need to care about the dependent variable. You don't have to care with what's happening with the independent variable. What you need to focus on is 'y' and its derivatives and looking to see if it's degree one terms.

$\frac{dx}{dt}$ & $\frac{dy}{dt}$: Dependent / Independent

$$\underbrace{\frac{d^2 y}{dx^2}}_{\text{degree one?}} + \underbrace{\frac{dy}{dx}}_{\text{degree one?}} + e^x \underbrace{y}_{\text{degree one?}} = 0$$

Degree one term of the second derivative

Order of a linear differental equation is simply the higest derivative that occurs in that differential equation.

Definition:
A <u>*n-th order*</u> DE is a DE whose highest derivative has order <u>*n*</u>

<u>Ex</u>| Find the order of the following DE:

(a) $y'''' - y''' + y'' - y' + y = 0$ . . 4^{th} order DE (linear ODE)

(b) $\frac{d^2}{dx^2} g(x,y) + \frac{d^3}{dy^3} g(x,y) = 0$ . . . . . 3^{rd} order PDE

(c) $\frac{d^2 f}{dx^2} + f = 0$ . . . . 2^{nd} order linear ODE

(d) $\left(\frac{df}{dx}\right)^2 + f = 0$ . . . . 1^{st} order non-linear ODE

(e) $\ln(y'') + y' = 0$ . . . . 2^{nd} order DE non-linear

Module 01 SUBJECT:

Expressions (forms) of Differental Equations

DATE: 2020/09/07 PAGE NO: 01/01

There are a number of ways writing a DE, so consider the DE $2x+y+3x\frac{dy}{dx}+4=0$

This could also be written in these three common ways

1 "Multiplying" by dx

Differential Form

- We don't deal with this form too much
- Nice that it does not have a $\frac{dy}{dx}$

$$2x\,dx+y\,dx+3x\,dy+4\,dx=0$$

equivalent $\Leftrightarrow$

$$(2x+y+4)dx+3x\,dy=0$$

2 Solve for $\frac{dy}{dx}$

Normal Form

Bring everything on the right-hand side and divide by $3x$

$$\frac{dy}{dx}=-\frac{1}{3x}(2x+y+4)$$

equivalent $\Leftrightarrow$

$$\frac{dy}{dx}=-\frac{(2x+y+4)}{3x}$$

3 Put $\frac{dy}{dx}$ in-front

Standard Form

- Have the highest order derivative in the front.
  - And is normalized meaning you don't have any other coefficient function or anything else in-front of it.

  * Why "3x" was divided *

Everything on one side and zero on the other

$$3x\frac{dy}{dx}+2x+y+4=0$$

equivalent $\Leftrightarrow$

$$\frac{dy}{dx}+\frac{y}{3x}+\frac{2x+4}{3x}=0$$

Ex Express $x^2dx+y\,dy=0$ in Normal Form

$x^2dx+y\,dy=0$ is in Differental Form

To have Normal Form

- We want $\frac{dy}{dx}$ on one side and equals everything else
  - Divide both sides by dx

solution

$$\frac{x^2dx}{dx}+\frac{y\,dy}{dx}=\frac{0}{dx}$$

$$\rightarrow x^2\frac{dx}{dx}+y\frac{dy}{dx}=0$$

$$\rightarrow x^2\cancel{\frac{dx}{dx}}^{(1)}+y\frac{dy}{dx}=0$$

$$\rightarrow x^2+y\frac{dy}{dx}=0$$

$$-x^2 \qquad -x^2$$

$$\rightarrow \frac{y\frac{dy}{dx}}{y}=\frac{-x^2}{y}$$

$$\therefore \frac{dy}{dx}=-\frac{x^2}{y}$$

Ex Is $x^2y^{(3)}+y''-y=3$ in Standard Form?

What does it mean to be in standard form?

In Standard Form you want the highest derivative on its own out in-front. Everything on one side and zero on the other.

solution

No, but $y^{(3)}+\frac{1}{x^2}y''-\frac{1}{x^2}y-\frac{3}{x^2}=0$ is.

A Note on Notation

We sometimes use $F(x,y,y',\dots,y^{(n)})$ to denote an arbitrary expresson involving $x,y,y',y'',\dots,y^{(n)}$

Ex $\frac{dy}{dx}=y^3x+x^2$ or $\frac{dy}{dx}=F(x,y)$, OR (write all of this as)

$y'-y^3x-x^2=0$

where $F(x,y)=y^3x+x^2$

$F(x,y,y')=0$

where $F(x,y,y')=y'-y^3x-x^2$

The reason for that notation consider DE

$F(x,y',y'')=0$ * You can quickly look at it and see that it's a second order DE *

OR

$y^{(5)}=F(x,y,y')$

Expectation checklist - Module 1

At the completion of this module, you should:

- know the definitions introduced,
- be able to determine what are dependent and independent variables in a (differential) equation,
- be able to compute basic partial derivatives of multivariable functions,
- determine whether a DE is an ODE or PDE and provide examples of each,
- determine whether a DE is linear or nonlinear and provide examples of each,
- determine the order of a DE and provide examples of DEs of any order,
- determine whether a given DE is written in normal, standard, or differential form, and
- be able to manipulate a given DE into normal, standard, or differential form if possible.

You will be assessed on your understanding of these concepts:

- within the homework,
- on the quizzes, and
- later, on the exam.

Coming up next, we:

- turn to developing terminology surrounding solutions to DEs,
- discuss the importance of studying the appropriate domains for which a function is a solution to a DE,
- determine whether a given function is a solution to a DE, and
- visualize solutions by graphing them.

Week 2

Graded Materials:

- **Wednesday (11:59pm)**
  - Module 02 Exercises.
- **Friday (11:59pm)**
  - Homework WEEK 02.
- **Wednesday Class Time: In-class Quiz**
  - Class is the time listed on the syllabus, and the quiz will take place once we have discussed math.
  - Please have a camera ready which can be turned on during the quiz.
  - Please have some sort of photo or pdf making device handy (you will need to scan or take a photo of your quiz).
  - The quiz is **not** open notes.
  - Quiz material: You will be expected to
    - Determine whether given differential equations are ODEs or PDEs and linear or non-linear.
    - Determine whether given function is a solution to a differential equation.
    - Verify a given function is a solution to a differential equation.
    - State whether given solutions are 1 or 2-parameter families of solutions, general solutions, trivial solutions, or particular solutions.
    - Determine the domain of a solution (interval of validity)
  - Please see the "Expectation Checklist" for Module 2.

Material to Master:

- Module 2.