Matrices, vectors, and matrix-vector multiplication

DEFN

(a) We call $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ a 2×2 and 3×3 matrix, respectively. Here the letters are entries, which for us are real numbers.

(b) We call $\begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ vectors with entries x, y, z.

(c) We have

$$\rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax+by+cz \\ dx+ey+fz \\ gx+hy+iz \end{pmatrix}$$

Also: Need

(i) Addition of vectors $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$:

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1+b_1 \\ a_2+b_2 \end{pmatrix}.$$

(ii) scalar multiplication of a vector $\begin{pmatrix} a \\ b \end{pmatrix}$ and number or function α:

$$\alpha \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \alpha a \\ \alpha b \end{pmatrix}.$$

Ex)

$\to$ $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} (1)(-1)+(2)(2) \\ (3)(-1)+(4)(2) \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}.$

$\to$ $\begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 8 \\ 10 \end{pmatrix} = \begin{pmatrix} -1+8 \\ 2+10 \end{pmatrix} = \begin{pmatrix} 7 \\ 12 \end{pmatrix}$

$\to$ $e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix}$ or $5 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 15 \end{pmatrix}.$

Systems of linear DEs and questions to ask

* consider a system of linear DE consisting of 2 equations and 2 unknown functions:

$$\frac{dx}{dt} = 2x + 2y$$

$$\frac{dy}{dt} = x + 3y$$

→ we want to convert this to "matrix form!"

- First, we define

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}',$$

or, setting $X = \begin{pmatrix} x \\ y \end{pmatrix}$, we write X'.

- Then,

$$\frac{dx}{dt} = 2x + 2y$$

$$\frac{dy}{dt} = x + 3y$$

becomes

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 2x + 2y \\ x + 3y \end{pmatrix}$$

$\Longleftrightarrow$

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

- Setting $X = \begin{pmatrix} x \\ y \end{pmatrix}$, we have

$$X' = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} X$$

Discussed on the next slide.

DEFN| A solution to $X' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} X + F$ is a vector $X = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ that satisfies the equation. That is, $x(t), y(t)$ satisfying the system.

* We do also have <u>homogeneous</u> systems

$$X' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} X$$

and <u>nonhomogeneous</u> systems

$$X' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} X + F$$

where $F = \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}$. That is,

$$\frac{dx}{dt} = ax + by + f$$

$$\frac{dy}{dt} = cx + dy + g$$

Questions:

(Q1) How do we verify a given vector is a soln?

(Q2) Does a soln always exist?

(Q3) What constitutes a fundamental set of solutions? General solns?

(Q4) solving systems of linear differential equations: How do we do it?

Verifying a vector is a solution

Ex) Verify $y_1 = \begin{pmatrix} 2e^t \\ -e^t \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t = e^t \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

and $y_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} = \begin{pmatrix} e^{4t} \\ e^{4t} \end{pmatrix}$ are solns

to

$$X' = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} X.$$

$\underline{\text{soln}}$ Have $y_1 = \begin{pmatrix} 2e^t \\ -e^t \end{pmatrix}$, $y_2 = \begin{pmatrix} e^{4t} \\ e^{4t} \end{pmatrix}$,

so $y_1' = \frac{d}{dt}\begin{pmatrix} 2e^t \\ -e^t \end{pmatrix} = \begin{pmatrix} \frac{d}{dt}(2e^t) \\ \frac{d}{dt}(-e^t) \end{pmatrix} = \begin{pmatrix} 2e^t \\ -e^t \end{pmatrix}$.

Meanwhile,

$$\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2e^t \\ -e^t \end{pmatrix} = \begin{pmatrix} 4e^t - 2e^t \\ 2e^t - 3e^t \end{pmatrix} = \begin{pmatrix} 2e^t \\ -e^t \end{pmatrix}$$

so $y_1' = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} y_1$ ✓

Also for $y_2 = \begin{pmatrix} e^{4t} \\ e^{4t} \end{pmatrix}$

$$y_2' = \begin{pmatrix} 4e^{4t} \\ 4e^{4t} \end{pmatrix} \text{ and } \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} e^{4t} \\ e^{4t} \end{pmatrix} = \begin{pmatrix} 4e^{4t} \\ 4e^{4t} \end{pmatrix}$$

Thus, y_2 satisfies the system of DEs.

Existence and uniqueness of IVP solutions

Thm1 Consider an IVP of the form

$$X' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} X + \begin{pmatrix} f \\ g \end{pmatrix}$$

with initial condition $X(t_0) = X_0$ (i.e., $\begin{pmatrix} x(t_0) \\ y(t_0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$).

If $f(t)$ and $g(t)$ are continuous on an interval I that contains t_0, then there exists a unique solution to the IVP on I.

Ex1 Are we guaranteed a unique soln for the following?

(a) $x' = 2x + 2y$
$y' = x + 3y$, with $x(5) = 3$, $y(5) = 2$.

soln | can rewrite this system as

$$X' = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} X.$$

Thus $\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, that is, $f = 0$, $g = 0$, which are continuous on $(-\infty, \infty)$. Thus, a unique soln is guaranteed!

(b) $x' = 2x + 2y + \frac{1}{t^2}$

$y' = x + 3y + \sin(t)$, $x(2) = 5$, $y(2) = 3$

soln| This becomes

$$X' = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} X + \begin{pmatrix} 1/t^2 \\ \sin(t) \end{pmatrix}.$$

Since $\sin(t)$ is continuous on $(-\infty, \infty)$, and $1/t^2$ is continuous on $(0, \infty)$, and 2 is in $(0, \infty)$. So there exists a unique soln on $(0, \infty)$.

Fundamental sets of solutions for systems of linear DEs

Thm| If we have an $n \times n$ system of linear DEs; i.e., n-many linear DEs with n-many unknown functions, then

→ n-many
→ solutions
→ that are linearly independent

form a fundamental set of solns for the system.

Q: How do we find if vector functions/solns are linearly independent?

Thm: Given solns $y_1 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$, $y_2 = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$ of a 2×2 system of linear DEs, then they are linearly independent on an interval I if and only if:

$$W(y_1, y_2) = \det\left(\begin{pmatrix} | & | \\ y_1 & y_2 \\ | & | \end{pmatrix}\right) = \det\left(\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}\right) \neq 0$$

on all of I.

* Note: We again call W the Wronskian, though we don't take derivatives here.

Ex| Are the solns $y_1 = \begin{pmatrix} 2e^t \\ -e^t \end{pmatrix}$ and $y_2 = \begin{pmatrix} e^{4t} \\ e^{4t} \end{pmatrix}$ linearly independent?

soln|

$$W(y_1, y_2) = \det\begin{pmatrix} 2e^t & e^{4t} \\ -e^t & e^{4t} \end{pmatrix} = (2e^t)(e^{4t}) - (-e^t)(e^{4t})$$

$$= 2e^{5t} + e^{5t} = 3e^{5t} \neq 0 \text{ on } (-\infty, \infty).$$

Thus, they are lin indep.

AND thus, form a fund set of solns to $X' = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} X$. Thus, $y = c_1 \begin{pmatrix} 2e^t \\ -e^t \end{pmatrix} + c_2 \begin{pmatrix} e^{4t} \\ e^{4t} \end{pmatrix}$ is a general soln!

Solving systems of linear differential equations

Q: How do we solve $x' = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} x$, and similar systems?

Answer:

(i) Find eigenvalues for $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$, call them λ_1, λ_2. [We restrict to 2 real eigenvalues.

(ii) Find eigenvectors for $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$, call them $\vec{v}_1, \vec{v}_2$.

(iii) Then the solns are $y_1 = \vec{v}_1 e^{\lambda_1 t}$, $y_2 = \vec{v}_2 e^{\lambda_2 t}$.

Ex| Solve $X' = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} X$.

soln|

(i) How to find eigenvalues:

solve for λ in

$$\det\left[\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right] = \det\left[\begin{pmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{pmatrix}\right] = 0.$$

$$\iff (2-\lambda)(3-\lambda) - (1)(2) = 0$$

$$\iff \lambda^2 - 5\lambda + 6 - 2 = 0$$

$$\iff \lambda^2 - 5\lambda + 4 = 0$$

$$\iff (\lambda-1)(\lambda-4) = 0$$

$$\iff \underline{\lambda_1 = 1}, \ \underline{\lambda_2 = 4}.$$

eigenvalues!

(ii) How to find eigenvectors:

An eigenvector for the eigenvalue λ_1 is a vector $\vec{v_1}$ that satisfies

$$\begin{pmatrix} 2-\lambda_1 & 2 \\ 1 & 3-\lambda_1 \end{pmatrix} \vec{v_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So for $\lambda_1 = 1$:

$$\begin{pmatrix} 2-1 & 2 \\ 1 & 3-1 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \left.\begin{aligned} a_1 + 2b_1 &= 0 \\ a_1 + 2b_1 &= 0 \end{aligned}\right\} \Rightarrow a_1 = -2b_1$$

Have $a_1 = -2b_1$.

← Find any (at least one nonzero) a_1, b_1 that satisfy this!

Ex1 $b_1 = -1$, $a_1 = 2$.

so $v_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ works! this is our eigenvector!

Need v_2 for $\lambda_2 = 4$:

$$\begin{pmatrix} 2-4 & 2 \\ 1 & 3-4 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\iff \left.\begin{array}{l} -2a_2 + 2b_2 = 0 \\ a_2 - b_2 = 0 \end{array}\right\} \Rightarrow a_2 = b_2.$$

choose. $a_2 = 1 = b_2$.

so, $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ works!

(iii) Have $\lambda_1 = 1$, $\lambda_2 = 4$, $\vec{v}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$,

thus,

$$y_1 = \vec{v}_1 e^{\lambda_1 t} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{t}$$

and

$$y_2 = \vec{v}_2 e^{\lambda_2 t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$$ are solns.

We've seen they are a fund set of solns. Thus, a general soln is

$$y = c_1 \begin{pmatrix} 2e^{t} \\ -e^{t} \end{pmatrix} + c_2 \begin{pmatrix} e^{4t} \\ e^{4t} \end{pmatrix}.$$