Cauchy-Euler equations

DEFN | A <u>Cauchy-Euler</u> equation has the form $a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_1 x y' + a_0 y = 0$, where $a_0, a_1, \ldots, a_n$ are constants.

Ex | (a) $3x^3 y^{(3)} + 17x^2 y^{(2)} - x y' + 3y = 0$, is a Cauchy-Euler eqn.

(b) $2x y^{(2)} - 3x y' + 5y = 0$ is Not!

Solving Cauchy-Euler equations

To solve such equations:

→ Take $y = x^m$;

→ plug into the DE,

→ solve for m!

I.e., similar to the constant coefficient method (where we used $y = e^{mx}$)

We'll look at 2nd order, and in particular case getting:

- → 2 distinct real m,
- → 1 repeated m, or
- → ~~2 complex (conjugate) m~~

Ex) solve $x^2y'' + 3xy' - 4y = 0$.

soln) $y = x^m$, $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$

This gives

$$x^2\left(m(m-1)x^{m-2}\right)+3x\left(mx^{m-1}\right)-4x^m=0$$

$$\Leftrightarrow x^m\left(m(m-1)\right)+3x^m m-4x^m=0$$

$$\Leftrightarrow x^m\left(m^2-m+3m-4\right)=0$$

$$\Rightarrow m^2-m+3m-4=0$$

$$\Leftrightarrow m^2+2m-4=0$$

$$m_1=\frac{-2+\sqrt{4+16}}{2}=-1+\frac{\sqrt{20}}{2}=-1+\frac{2\sqrt{5}}{2}=-1+\sqrt{5},$$

$m_2=-1-\sqrt{5}$. Thus, $y=c_1x^{-1+\sqrt{5}}+c_2x^{-1-\sqrt{5}}$

Ex) Solve $x^2y'' - 3xy' + 4y = 0$.

soln) $y = x^m$ plugged in gives

$x^m(m^2 - 4m + 4) = 0 \Rightarrow (m-2)^2 = 0$.

so $y_1 = x^2$ works.

$\rightarrow$ Reduction of order: $\left[p = \frac{-3x}{x^2} = -\frac{3}{x}\right]$

$$y_2 = x^2 \int \frac{e^{-\int p}}{(x^2)^2}\,dx = x^2 \int \frac{e^{+\int \frac{3}{x}}}{x^4}\,dx = x^2 \int \frac{e^{3\ln x}}{x^4}\,dx$$

$$= x^2 \int \frac{x^3}{x^4}\,dx = x^2 \int \frac{1}{x}\,dx = x^2 \ln|x|$$

So, $y_2 = x^2 \ln|x|$. Thus, $y = c_1x^2 + c_2x^2\ln|x|$.

* Note: If $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ are complex roots, then have

$$y = x^{\alpha}\left[c_1 \cos(\beta \ln|x|) + c_2 \sin(\beta \ln|x|)\right]$$

is the general soln.

* Note: If have DE of the form

$$a_2 x^2 y'' + a_1 x y' + a_0 y = g(x) \quad [\text{nonhomog}]$$

then would use variation of parameters for y_p.

Example

Ex| Solve $x^2y'' - 3xy' + 4y = 2x^2$

soln|

→ From before we have

$$y_h = c_1 x^2 + c_2 x^2 \ln|x|$$

is the general soln for

$$x^2y'' - 3xy' + 4y = 0.$$

Take $y_1 = x^2$, $y_2 = x^2 \ln|x|$.

→ We use variation of parameters.

→ DE becomes:

$$y'' - \frac{3}{x} y' + \frac{4}{x^2} y = \overbrace{2}^{\to f(x)}$$

→ $W = \det\left(\begin{pmatrix} x^2 & x^2 \ln|x| \\ 2x & 2x\ln|x| + x \end{pmatrix}\right) = (x^2)(2x\ln|x| + x) - (2x)(x^2\ln|x|)$

$= x^3$

→ So

- $u_1' = \frac{-y_2 f(x)}{W} = -\frac{x^2 \ln(x) \cdot 2}{x^3} = -\frac{2\ln(x)}{x}$

- $u_2' = \frac{y_1 f(x)}{W} = \frac{x^2 \cdot 2}{x^3} = \frac{2}{x}$

$\rightarrow$ So [sub $w = \ln(x)$]

- $u_1 = \int u_1' = -2\int \frac{\ln|x|}{x}dx = -2\int w\,dw = -w^2$
  $= -(\ln|x|)^2$

- $u_2 = \int u_2' = \int \frac{2}{x}dx = 2\ln|x|$.

$\rightarrow$ Thus, $y_p = u_1 y_1 + u_2 y_2$
$= -(\ln|x|)^2 x^2 + 2\ln|x|\, x^2 \ln|x|$
$= x^2(\ln|x|)^2$

$\rightarrow$ so,
$y = c_1 x^2 + c_2 x^2 \ln|x| + x^2(\ln|x|)^2$.