Exact equations

We are interested in DEs that now have the form $M(x,y)dx + N(x,y)dy = 0$ for multivariable functions $M(x,y)$ and $N(x,y)$. We make two quick remarks:

- We don't tackle all such equations, only ones that satisfy certain properties. These will be known as *exact equations*.
- The dy and dx are known as differentials of equations or infinitesimals. We do not need the specifics of these notions, but we do dwell on them a bit. In particular, we learn to calculate the differential dz of a function $z = f(x,y)$ and see how this fits into the story of exact equations.

Definition:

A differential equation $M(x,y)dx + N(x,y)dy = 0$ is an **exact equation** if $\frac{\partial}{\partial y}M(x,y) = \frac{\partial}{\partial x}N(x,y)$.

Care must be taken!! Note that the partial derivatives are with respect to different variables!

To explain *why* exact equations satisfy the above condition, we first introduce the differential of a 2-variable function. Just as we often introduce the variable y as the image of a single variable function $f(x)$ via $y = f(x)$, we can also introduce z as the image of a 2-variable function $f(x,y)$ via $z = f(x,y)$. In the single variable case, the *differential* of $y = f(x)$ is defined as $dy = f'(x)d = \frac{df}{dx}dx$. In the 2-variable case, we have the following.

Definition:

For a function $z = f(x,y)$ the **differential** of z is $dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$.

Now, suppose z is actually a constant c and not a variable. That is, start with $f(x,y) = c$. The differential in this case is

$$0 = dc = dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

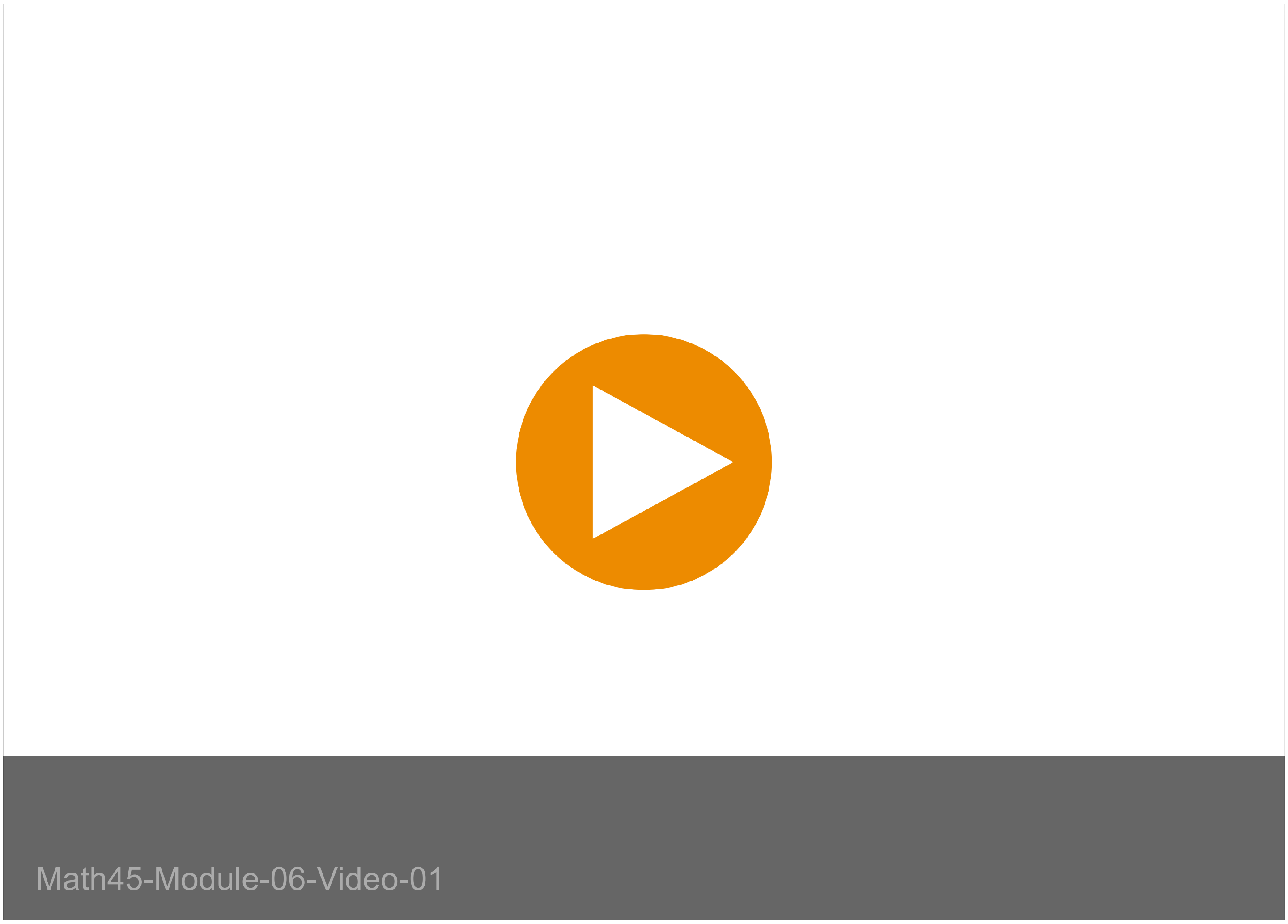
That is,

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0.$$

Thus, for $M(x,y) = \dfrac{\partial f}{\partial x}$ and $N(x,y) = \dfrac{\partial f}{\partial y}$, we obtain $M(x,y)dx + N(x,y)dy = 0$.

The point of this generic discussion is that *given* a specific equation of the form $f(x,y) = c$, we can find a differential equation of the form $M(x,y)dx + N(x,y)dy = 0$ such that $f(x,y) = c$ is a solution to the equation. We look at some examples of this now.

Discussion, comments, and examples:



WeBWorK module 06 exercises:

- Problems 1

Relevant Wikipedia articles:

- **Differential of a function** (https://en.wikipedia.org/wiki/Differential_of_a_function)
- **Infinitesimals** (https://en.wikipedia.org/wiki/Differential_(infinitesimal))