The Laplace transform

DEFN/ Let $f(t)$ be a function defined for $t \geq 0$.
The integral

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\,dt$$

is called the Laplace transform of f, so long as the integral exists.

*Recall: $\int_0^{\infty} e^{-st} f(t)\,dt = \lim_{a \to \infty} \int_0^{a} e^{-st} f(t)\,dt$,

and this exists if the limit is a finite number; i.e; converges.

*Note: Generally for a function $f(t)$, we write $\mathcal{L}\{f(t)\} = F(s)$. (Be sure to note domains of $f(t)$ and $F(s)$.)

Ex| compute:

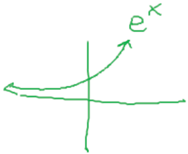
(a) $\mathcal{L}\{3\}$.

soln|

$$\mathcal{L}\{3\} = \int_0^\infty e^{-st}(3)\,dt = \lim_{a\to\infty} 3\int_0^a e^{-st}\,dt$$

$$= 3\lim_{a\to\infty}\left[-\frac{e^{-st}}{s}\Big|_0^a\right]$$

$$= 3 \lim_{a\to\infty}\left[-\frac{e^{-sa}}{s} + \frac{e^{0}}{s}\right]$$



$$= 3\left[-\frac{0}{s}+\frac{1}{s}\right] = \frac{3}{s}.$$ defined on $(0,\infty)$.

(b) $\mathcal{L}\{t\}$.

soln $\mathcal{L}\{t\} = \int_0^\infty e^{-st}(t)\,dt = \lim_{a\to\infty}\int_0^a te^{-st}\,dt$

$u=t \qquad dv = e^{-st}\,dt$

$du=dt \qquad v=-\frac{1}{s}e^{-st}$

$$= \lim_{a\to\infty}\left[-\frac{t}{s}e^{-st}\Big|_0^a + \int_0^a \frac{1}{s}e^{-st}\,dt\right]$$

$$= \lim_{a\to\infty}\left[\left[-\frac{t}{s}e^{-st} - \frac{e^{-st}}{s^2}\right]\Big|_0^a\right]$$

$(s>0)$

$$= \lim_{a\to\infty}\left[\left(-\cancelto{0}{\frac{a}{s}e^{-sa}} - \cancelto{0}{\frac{e^{-sa}}{s^2}}\right) - \left(-\cancelto{0}{\frac{0}{s}} - \frac{1}{s^2}\right)\right]$$

$$= \frac{1}{s^2}, \quad \text{defined on } (0,\infty)$$

* Also works for piecewise functions

Ex1 Find $\mathcal{L}\{f(t)\}$ where $f(t)=\begin{cases}0 & 0\le t\le 2\\ 3 & 2\le t<\infty\end{cases}$.

solu'n

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^2 e^{-st}(0)\,dt + \int_2^\infty e^{-st}(3)\,dt$$

$$= 0 + \lim_{a\to\infty} 3\int_2^a e^{-st}\,dt = 3\lim_{a\to\infty}\left[-\frac{1}{s}e^{-st}\Big|_2^a\right]$$

$$= 3\lim_{a\to\infty}\left[-\frac{1}{s}\cancelto{0}{e^{-sa}} + \frac{e^{-2s}}{s}\right] = \frac{3}{s}e^{-2s},\quad s>0.$$

Thm1 We have

(a) $\mathcal{L}\{c\} = \frac{c}{s}$

(b) $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

(c) $\mathcal{L}\{e^{kt}\} = \frac{1}{s-k}$

(d) $\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}$

(e) $\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}$

for constants c, k and $n = 1, 2, 3, \ldots$

The inverse Laplace transform and linearity

*Idea: Given a function $F(s)$, we want to find $\mathcal{L}^{-1}\{F(s)\}$.
I.e; the function $f(t)$ such that $\mathcal{L}\{f(t)\} = F(s)$.

*From the previous theorem, we have

Thm) we have

(a) $\mathcal{L}^{-1}\left\{\frac{c}{s}\right\} = c$

(b) $\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$

(c) $\mathcal{L}^{-1}\left\{\frac{1}{s-k}\right\} = e^{kt}$

(d) $\mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos(kt)$

(e) $\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin(kt)$

*Note: knowing $\mathcal{L}$ and $\mathcal{L}^{-1}$ of some functions only becomes very useful with the following "linearity" properties.

<u>Thm</u> For any numbers α, β and $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$, we have

(a) $\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{f(t)\}$

(b) $\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha \mathcal{L}^{-1}\{F(s)\} + \beta \mathcal{L}^{-1}\{G(s)\}$.

Ex1 Find

(a) $\mathcal{L}^{-1}\left\{\frac{3s-4}{s^2+9}\right\}$.

soln: $\mathcal{L}^{-1}\left\{\frac{3s-4}{s^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{3s}{s^2+3^2}\right\} + \mathcal{L}^{-1}\left\{\frac{-4}{s^2+3^2}\right\}$

$= 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} - \frac{4}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}$

so in correct form to apply theorem.

$= 3\cos(3t) - \frac{4}{3}\sin(3t)$

(b) $\mathcal{L}\{3t^5 - e^{2t}\}$

<u>soln</u>

$$\mathcal{L}\{3t^5 - e^{2t}\} = 3\mathcal{L}\{t^5\} - \mathcal{L}\{e^{2t}\}$$

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$$= 3\left(\frac{5!}{s^6}\right) - \left(\frac{1}{s-2}\right) = \frac{3\cdot 5!}{s^6} - \frac{1}{s-2}.$$

The Laplace transform of derivatives

GOAL: Find what $\mathcal{L}\{f^{(n)}(t)\}$ is.

→ First, we find

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt = \lim_{a\to\infty} \int_0^a e^{-st} f'(t)\,dt$$

$u = e^{-st}$ $\quad dv = f'(t)\,dt$

$du = -s\,e^{-st}\,dt$ $\quad v = f(t)$

$$= \lim_{a\to\infty} \left[e^{-st} f(t) \Big|_0^a + s\int_0^a e^{-st} f(t)\,dt \right]$$

$$= \lim_{a\to\infty}\left[e^{-st}f(t)\Big|_0^a + s\int_0^a e^{-st}f(t)\,dt\right]$$

$$(s>0) \qquad = \lim_{a\to\infty}\left[\cancelto{0}{e^{-sa}f(a)} - f(0)\right] + s\underbrace{\lim_{a\to\infty}\int_0^a e^{-st}f(t)\,dt}_{\int_0^\infty e^{-st}f(t)\,dt}$$

$$= -f(0) + s\mathcal{L}\{f(t)\}$$

$$= -f(0) + sF(s)$$

That is,

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0).$$

→ Next:

$$\mathcal{L}\{f''(t)\} = \int_0^\infty e^{-st} f''(t)\,dt$$

by parts

$$\overset{\text{by parts}}{=} \lim_{a\to\infty} \left[e^{-st} f'(t) \Big|_0^a \right] + s\int_0^\infty e^{-st} f'(t)\,dt$$

$$= -f'(0) + s\mathcal{L}\{f'(t)\}$$

$$= -f'(0) + s\left[sF(s) - f(0)\right]$$

$$= s^2F(s) - sf(0) - f'(0).$$

Thm] We have

$$\mathcal{L}\{f^{(n)}(t)\} = s^n \underbrace{F(s)}_{\mathcal{L}\{f(t)\}} - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0).$$

*Note: Technically only holds for certain functions. (continuous on $[0,\infty)$, and grow slower than e^{sx}.)

A brief partial fraction decomposition refresher

Goal: separate functions $\frac{p(x)}{q(x)}$ into small pieces.

*sometimes nice to think of the cases that can arise:

(i) Distinct linear terms:

$$\frac{x}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1}$$

$$= \frac{A(x-1) + B(x+3)}{(x+3)(x-1)}$$

$\iff x = (A+B)x + (3B-A)$

$\iff A+B = 1,\ 3B-A = 0$

$\Rightarrow A = 3/4,\ B = 1/4$

$\to$ solve for A, B

$$= \frac{3/4}{x+3} + \frac{1/4}{x-1}$$

(ii) Some repeated linear terms:

$$\frac{x^2-2}{(x-2)(x+1)^3} = \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3}$$

$\rightarrow$ solve for A, B, C, D $\left(A = \frac{2}{27},\ B = \frac{-2}{27},\ C = \frac{7}{9},\ D = \frac{1}{3}\right)$

(iii) Distinct irreducible terms:

$$\frac{x}{(x^2+1)(x^2+2)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2}$$

$\rightarrow$ solve for A, B, C, D $(A=1,\ B=0,\ C=-1,\ D=0)$

(iv) Repeated irreducible terms:

$$\frac{2x-1}{(x^2+x+1)^3} = \frac{Ax+B}{(x^2+x+1)} + \frac{Cx+D}{(x^2+x+1)^2} + \frac{Ex+F}{(x^2+x+1)^3}$$

$\rightarrow$ solve for A, B, C, D, E, F.

(v) Mixing the above:

$$\frac{2x-1}{(x-1)^2(x^2+x+1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{(x^2+x+1)}$$

$\rightarrow$ solve for A, B, C, D, E, F.

Solving IVPs using the Laplace transform

Idea: → Apply $\mathcal{L}$ to the entire IVP, converting it to a function of s (often denoted $Y(s)$).

→ Solve for $Y(s)$

⇝ often use PFD around here to apply $\mathcal{L}^{-1}$!

→ Then use $\mathcal{L}^{-1}$ to transform back into a function of t (often denoted $y(t)$)

Example

Ex) Solve $y'' + y = t$, $y(0) = 0$, $y'(0) = 2$.

soln) → Apply $\mathcal{L}$ to both sides:

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\{t\}$$

$$\Leftrightarrow \mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{t\}$$

$$\Leftrightarrow s^2 Y(s) - s y(0) - y'(0) + Y(s) = \frac{1}{s^2}$$

$\rightarrow$ Solve for $Y(s)$.

$$s^2 Y(s) - sy(0) - y'(0) + Y(s) = \frac{1}{s^2}$$

$$\Leftrightarrow \quad Y(s)(s^2+1) = \frac{1}{s^2} + sy(0) + y'(0)$$

$$\Leftrightarrow \quad Y(s) = \frac{1}{s^2(s^2+1)} + \frac{\cancel{sy(0)}^{\,0} + y'(0)}{s^2+1}$$

$$\Leftrightarrow \quad Y(s) = \frac{1}{s^2(s^2+1)} + \frac{2}{s^2+1}$$

$\Leftrightarrow$ PFD!

$$\frac{1}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}$$

$A = 0$

$B = 1$

$D = -1$

$C = 0$

PFD details:

$$\frac{1}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1} = \frac{As(s^2+1) + B(s^2+1) + (Cs+D)s^2}{s^2(s^2+1)}$$

$$\iff 1 = As^3 + As + Bs^2 + B + Cs^3 + Ds^2$$

$$\iff 1 = (A+C)s^3 + (B+D)s^2 + As + B$$

$$\iff A+C=0,\ B+D=0,\ A=0,\ B=1$$

$$\iff A=0,\ B=1,\ C=0,\ D=-1$$

so $$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$$

Thus,

$$Y(s) = \frac{1}{s^2} - \frac{1}{s^2+1} + \frac{2}{s^2+1}$$

$$\Leftrightarrow \quad Y(s) = \frac{1}{s^2} + \frac{1}{s^2+1}$$

$\rightarrow$ Apply $\mathcal{L}^{-1}$:

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2} + \frac{1}{s^2+1}\right\}$$

$$\Leftrightarrow \quad y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$\Leftrightarrow \quad y(t) = t + \sin(t).$$

Example

Ex] Solve the IVP $y'' + 16y = 32t$, $y(0) = 3$, $y'(0) = -2$.

soln]

$\to$ Apply $\mathcal{L}$:

$$\mathcal{L}\{y'' + 16y\} = \mathcal{L}\{32t\}$$

$$\iff \mathcal{L}\{y''\} + 16\,\mathcal{L}\{y\} = 32\,\mathcal{L}\{t\}.$$

$$\iff s^2 Y(s) - s y(0) - y'(0) + 16 Y(s) = 32\left(\frac{1}{s^2}\right)$$

$$\Longleftrightarrow \quad s^2 Y(s) - s y(0) - y'(0) + 16 Y(s) = \frac{32}{s^2}$$

$$\Longleftrightarrow \quad Y(s)(s^2+16) = \frac{32}{s^2} + s y(0) + y'(0)$$

$$\Longleftrightarrow \quad Y(s)(s^2+16) = \frac{32}{s^2} + 3s - 2$$

$$\Longleftrightarrow \quad Y(s) = \frac{32}{s^2(s^2+16)} + \frac{3s}{s^2+16} - \frac{2}{s^2+16}$$

PFD

$$\Longleftrightarrow \quad Y(s) = \frac{2}{s^2} - \frac{2}{s^2+16} - \frac{2}{s^2+16} + \frac{3s}{s^2+16}$$

$\Leftrightarrow$

$$Y(s) = \frac{2}{s^2} - \frac{4}{s^2+16} + \frac{3s}{s^2+16}$$

$\rightarrow$ Apply $\mathcal{L}^{-1}$:

$$y(t) = 2t - \sin(4t) + 3\cos(4t).$$

Including exponential terms

GOAL: Also solve expressions which include exponential terms, i.e., $\mathcal{L}\{e^{kt}f(t)\}$.

<u>Thm</u>| If $\mathcal{L}\{f(t)\} = F(s)$ and k is a real number, then

$$\mathcal{L}\{e^{kt}f(t)\} = F(s-k)$$

and

$$\mathcal{L}^{-1}\{F(s-k)\} = e^{kt}f(t).$$

Ex| Find

(a) $\mathcal{L}\{e^{-2t} t^4\}$

soln| Recall $\mathcal{L}\{t^4\} = \frac{4!}{s^5} = \frac{24}{s^5} = F(s)$

Thus,

$$\mathcal{L}\{e^{-2t} t^5\} = F(s-(-2)) = F(s+2)$$

$$= \frac{24}{(s+2)^5}.$$

(b) $\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+5}\right\}$

soln|

→ Doesn't fit into forms we know.

→ Try to "translate" into something we know.

[complete the square!]

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+2^2}\right\}$$

to put into sin(kt) form

$$= \frac{1}{2}e^{-t}\sin(2t).$$

Note: can always check by computing $\mathcal{L}\{\frac{1}{2}e^{-t}\sin(2t)\}$.

Ex) Solve $y' - y = 1 + te^{t}$, $y(0) = 0$.

soln)

$\rightarrow$ Apply $\mathcal{L}$:

$$sY(s) - y(0) - Y(s) = \frac{1}{s} + \frac{1}{(s-1)^2}$$

$$\Leftrightarrow Y(s)(s-1) - 0 = \frac{1}{s} + \frac{1}{(s-1)^2}$$

$$\rightarrow \Leftrightarrow Y(s) = \underbrace{\frac{1}{s(s-1)}}_{\text{PFD}} + \frac{1}{(s-1)^3}$$

$$= \frac{A}{s} + \frac{B}{s-1} \qquad A = -1, \quad B = 1$$

$$\Longleftrightarrow$$

$$Y(s) = -\frac{1}{s} + \frac{1}{s-1} + \frac{1}{(s-1)^3}$$

$\rightarrow$ Apply $\mathcal{L}^{-1}$:

$$y(t) = -1 + e^{t} + \frac{1}{2}t^2 e^{t}$$

$$\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$$

$$\Longleftrightarrow$$

$$\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = t^2$$

$$\Longleftrightarrow$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{1}{2}t^2$$