Exact Equations

DEFN

(a) A DE $M(x,y)\,dx + N(x,y)\,dy = 0$ is an <u>exact equation</u> if $\dfrac{\partial}{\partial y} M(x,y) = \dfrac{\partial}{\partial x} N(x,y)$.

Ex1 $\quad \underbrace{y}_{M}\,dx + \underbrace{x}_{N}\,dy = 0. \quad \dfrac{\partial}{\partial y}\, y = 1 = \dfrac{\partial}{\partial x}\, x = 1$

an exact eqn.

(b) For a function $z = f(x,y)$ the <u>differential</u> of z is $dz = \dfrac{\partial f}{\partial x}\,dx + \dfrac{\partial f}{\partial y}\,dy$

* Note: the differential of $y=f(x)$ is

$$dy = f'(x)\,dx = \frac{df}{dx}\,dx.$$

Ex) consider an expression $c=f(x,y)$, say

$$\underbrace{e^x + x^2y^2 - \ln|y|} = c.$$

$f(x,y) = e^x + x^2y^2 - \ln|y|$

Then $dz = dc = 0$. And

$$0 = dz = \frac{df}{dx}dx + \frac{df}{dy}dy$$

$$= (e^x + 2xy^2)dx + (2x^2y - \tfrac{1}{y})dy.$$

That is, $e^x + x^2y^2 - \ln|y| = c$ satisfies the DE

$$(e^x + 2xy^2)dx + (2x^2y - 1/y)\,dy = 0.$$

Ex) Find the exact equation solved by

$$\frac{1}{3}x^3 + \frac{1}{2}x^2y - xy^2 + C = 0.$$

soln) Two ways:

(i) using the differential: the expression gives

$$\frac{1}{3}x^3 + \frac{1}{2}x^2y - xy^2 = C.$$

Taking the differential:

$$0 = dz = (x^2 + xy - y^2)\,dx + (\tfrac{1}{2}x^2 - 2xy)\,dy$$

Thus,

$$(x^2 + xy - y^2)\,dx + (\tfrac{1}{2}x^2 - 2xy)\,dy = 0.$$

(ii) We need

$$[\,???\,]\,dx + [\,???\,]\,dy = 0$$

where

$$\frac{d}{dx}\left(\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 y - xy^2\right) \qquad \frac{d}{dy}\left(\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 y - xy^2\right)$$

this gives

$$(x^2 + xy - y^2)\,dx + (\tfrac{1}{2}x^2 - 2xy)\,dy = 0.$$

Exact since:

$$\frac{d}{dy}(x^2 + xy - y^2) = x - 2y = \frac{d}{dx}(\tfrac{1}{2}x^2 - 2xy) \checkmark$$

Solving exact equations

Ex1 Solve $(e^x + 2xy^2)dx + (2x^2y - 1/y)dy = 0.$

solnl

(0) Check if it's exact:

$$\frac{d}{dy}(e^x + 2xy^2) = 4xy$$

$$\frac{d}{dx}(2x^2y - 1/y) = 4xy$$

✓ yes, exact.

(1) Find (antiderivative with respect to x):

$$(\#) \quad F(x,y) = \int (e^x + 2xy^2)\,\underline{\underline{dx}} = e^x + x^2y^2 + \underline{\underline{g(y)}}$$

(2) Take the partial derivative of F (with respect to y):

$$\frac{d}{dy}\left(e^x + x^2y^2 + g(y)\right) = 2x^2y + g'(y)$$

(3) set this equal to $N(x,y) = 2x^2y - 1/y$:

$$2x^2y + g'(y) = 2x^2y - \frac{1}{y}.$$

(4) Solve for $g'(y)$:

$$g'(y) = -\frac{1}{y}.$$

(5) Integrate $g'(y)$:

$$g(y) = -\ln(y) + C$$

(6) Plug this into (#):

$$F(x,y) = e^x + x^2y^2 - \ln|y| + C$$

(7) Thus, we have:

$$\boxed{e^x + x^2y^2 - \ln|y| = C}$$

[$f(x,y) = C$]

is a solution. [Note: can't solve for y, thus implicit]

* Note: could also:
  - → In step (1) take anti-deriv of N with respect to y.
  - → In (2) take partial deriv w.r.t. x
  - → Etc...

<u>Ex</u>| Check if exact, and solve

$$\underbrace{(y^2+1)}\,dx + \underbrace{(2xy)}\,dy = 0.$$

<u>sol</u>|

(0) $\frac{\partial}{\partial y}(y^2+1) = 2y$. $\frac{\partial}{\partial x}(2xy) = 2y$ ✓ Yes, exact.

(1) $F(x,y) = \int 2xy\, dy = xy^2 + g(x).$ (#)

(2) $\frac{d}{dx}\left(xy^2 + g(x)\right) = y^2 + g'(x).$

(3) $y^2 + g'(x) = M(x,y) = y^2 + 1$

(4) $g'(x) = 1$

(5) $g(x) = x + C$

(6) (#) becomes $F(x,y) = xy^2 + x + C.$

(7) Thus, $\boxed{xy^2 + x = C}$ is the soln. [implicit]

$\Rightarrow y^2 = \frac{C-x}{x} \Leftrightarrow y = \pm\sqrt{\frac{C}{x} - 1}$