The inverse Laplace transform and linearity

Here we introduce the inverse Laplace transform.

Suppose we are given a function of s, say $F(s)$. We want to ask the question, is this function $F(s)$ a Laplace transform of a function $f(t)$, and if so, what is the $f(t)$? In other words, given $F(s)$, we want to be able to find the $f(t)$ such that $\mathcal{L}\{f(t)\} = F(s)$.

Definition

Given a function $F(s)$, we call the function $f(t)$ such that $\mathcal{L}\{f(t)\} = F(s)$ the **inverse Laplace transform**, and denote it by $\mathcal{L}^{-1}\{F(s)\}$.

While there is an integral formula for the inverse Laplace transform, it is quite difficult to use in practice. Instead, it is typically much more manageable to compute the inverse Laplace transform by referring to the following theorem, and using linearity.

Theorem

We have

(a) $\mathcal{L}^{-1}\left\{\dfrac{C}{s}\right\} = C$

(b) $\mathcal{L}^{-1}\left\{\dfrac{n!}{s^{n+1}}\right\} = t^n$

(c) $\mathcal{L}^{-1}\left\{\dfrac{1}{s-k}\right\} = e^{kt}$

(d) $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+k^2}\right\} = \cos(kt)$

(e) $\mathcal{L}^{-1}\left\{\dfrac{k}{s^2+k^2}\right\} = \sin(kt)$

where C is an arbitrary constant, k is a given number, and n is any nonzero integer.

Note: Compare this table with the one for the Laplace transform and feel comfortable with how they relate.

Knowing the inverse Laplace transform on the functions given above, we can do a lot so long as we use the following linearity properties.

Theorem

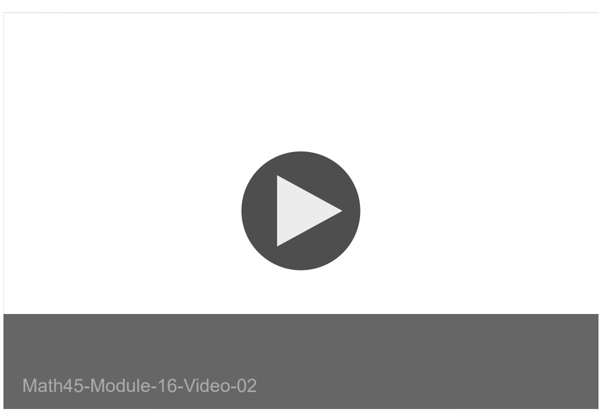
We have for any numbers α and β and known Laplace transforms $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$ that the following linearity properties hold:

(a) $\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha\mathcal{L}\{f(t)\} + \beta\mathcal{L}\{g(t)\}$, and

(b) $\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha\mathcal{L}^{-1}\{F(s)\} + \beta\mathcal{L}^{-1}\{G(s)\}$.

We examine how we can use the theorems so far seen to compute the Laplace transform, and the inverse Laplace transform, for many functions.

Discussion, comments, and examples:



WeBWorK module 16 exercises:

- Problems 4, 5, 6, 7

Relevant Wikipedia articles:

- [The inverse Laplace transform](https://en.wikipedia.org/wiki/Laplace_transform#Inverse_Laplace_transform) (https://en.wikipedia.org/wiki/Laplace_transform#Inverse_Laplace_transform)