Separable Equations

DEF'N A 1^{st}-order DE which can be expressed in the form

$$\frac{dy}{dx} = g(x)\,h(y)$$

is called <u>separable</u>.

* Note: If a DE is separable, we can put all "x-stuff" on one side of the equation, and "y-stuff" on the other.

$g(x) \cdot h(y)$

Ex (a) $\frac{dy}{dx} = \underbrace{x}\,\underbrace{e^y}$ IS separable.

$\left[\frac{1}{e^y}\,dy = x\,dx \right]$

(b) $e^x \frac{dy}{dx} - y = 0 \iff e^x \frac{dy}{dx} = y \iff \frac{dy}{dx} = y e^{-x}$ ✓

$\iff \frac{1}{y}\,dy = e^{-x}\,dx.$ ✓

IS separable

(c) $\frac{dy}{dx} = x + y + \cos(x)$

$\iff \frac{dy}{dx} = (x + \cos(x)) + y.$ Can't separate.

Not a separable equation.

$y = e^{xy}$

Solving Separable Equations

Consider the following steps:

Given $\frac{dy}{dx} = g(x)\,h(y)$.

(1) Treating dy, dx as y and x terms, put all "y-stuff" on the left and "x-stuff" on the right:

$$\star \quad \frac{1}{h(y)}\,dy = g(x)\,dx$$

(2) Integrate both sides of $\frac{1}{h(y)} dy = g(x)\, dx$:

$\to \int \frac{1}{h(y)} dy = \int g(x) dx.$

$\to \quad H(y) + C_1 = G(x) + C_2$

* Note: Here $H(y)$ is anti-deriv. of $\frac{1}{h(y)}$.
* C_1, C_2 are arbitrary constants.

$\to$ Need only one constant:

$H(y) = G(x) + C$ (C a constant).

(3) Solve for y (if possible).

Ex1 Solve $\frac{dy}{dx} = 7y$.

solu

(1) $\frac{1}{y}\,dy = 7\,dx$

(2) $\int \frac{1}{y}\,dy = \int 7\,dx \iff \ln|y| = 7x + C$.

arbitrary constant.

(3) $e^{\ln|y|} = e^{7x+C} \iff y = e^{7x}e^{C}$

$\iff y = \underbrace{e^{C}}\,e^{7x} \iff \boxed{y = \underbrace{C}\,e^{7x}}$ C a constant.

Ex| solve $e^x y \frac{dy}{dx} = e^{-y} + e^{-2x-y}$ generally, and also find a soln to the IVP with $y(0)=0$.

soln|

(1) $$\frac{dy}{dx} = \frac{e^{-y} + e^{-2x}e^{-y}}{e^x y} \iff \frac{dy}{dx} = \underbrace{\frac{e^{-y}}{y}} \cdot \underbrace{\frac{(1+e^{-2x})}{e^x}}$$

$$\iff \frac{y}{e^{-y}}\,dy = \frac{(1+e^{-2x})}{e^x}\,dx \qquad \frac{1}{e^x} + \frac{e^{-2x}}{e^x} \qquad e^{-2x}\cdot e^{-x}$$

$$\iff \underline{y e^{y}\,dy = \left(e^{-x} + e^{-3x}\right)dx}$$

$$ye^y\,dy = (e^{-x} + e^{-3x})\,dx$$

(2) $$\int ye^y\,dy = \int (e^{-x} + e^{-3x})\,dx$$

$\downarrow$ By parts.

$\int u\,dv = uv - \int v\,du$.

$u = y \quad dv = e^y\,dy$

$du = 1 \cdot dy \quad v = e^y$

$u = -x, \quad du = -1\,dx$

$u = -3x \quad du = -3\,dx$

$$\Leftrightarrow \quad ye^y - \int e^y\,dy = -e^{-x} - \frac{1}{3}e^{-3x} + C$$

$$\Leftrightarrow \quad ye^y - e^y = -e^{-x} - \frac{1}{3}e^{-3x} + C.$$

$\Leftrightarrow$ $\boxed{e^y(y-1) = -e^{-x}(1+\tfrac{1}{3}e^{-2x}) + C}$

(3) Can't explicitly solve for y!

An implicit soln!

Now, the IVP $y(0)=0$. I.e., $x=0$, $y=0$.

This gives

$$e^0(0-1) = -e^0(1+\tfrac{1}{3}e^0) + C$$

$\Leftrightarrow$ $-1 = -(1+\tfrac{1}{3}) + C$ $\Leftrightarrow$ $-1 = -\tfrac{4}{3} + C$

$\Leftrightarrow$ $C = \tfrac{1}{3}$. Thus, the IVP has the soln

$$\boxed{e^y(y-1) = -e^{-x}(1+\tfrac{1}{3}e^{-2x}) + \tfrac{1}{3}}$$