General solutions for nonhomogeneous linear DEs

Consider the linear nonhomogeneous DE

$$a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = b(x). \qquad (\triangle)$$

Q: What are the sol'ns for $(\triangle)$

Thm suppose

(i) $y_p(x)$ is a particular soln for

$$a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = b(x) \qquad (*)$$

and

(ii) $y_1, \ldots, y_n$ is a fundamental set of solns for the homogeneous linear DE

$$a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = 0.$$

Then the general soln for $(*)$ is

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n + y_p,$$

where $c_1, \ldots, c_n$ are arbitrary constants.

Ex| Verify $y = \underline{c_1}\overset{y_1}{\cos(x)} + \underline{c_2}\overset{y_2}{\sin(x)} + \overbrace{x\sin(x) + \cos(x)\ln(\cos(x))}^{y_p}$ is a general solution to the nonhomog DE

$$y'' + y = \sec(x).$$

$\left[\frac{1}{\cos(x)} \cdot (-\sin(x))\right]$

soln| Check:

$$\rightarrow y_p = x\sin(x) + \cos(x)\ln(\cos(x))$$

$$y_p' = \cancel{\sin(x)} + x\cos(x) - \sin(x)\ln(\cos(x)) - \frac{\cancel{\cos(x)}\sin(x)}{\cancel{\cos(x)}}$$

$$y_p'' = \cos(x) - x\sin(x) - \cos(x)\ln(\cos(x)) - \sin(x)\left[\frac{1}{\cos(x)}(-\sin(x))\right]$$

$$= \cos(x) - x\sin(x) - \cos(x)\ln(\cos(x)) + \frac{\sin^2(x)}{\cos(x)}.$$

$$y_p'' + y_p = \cos(x) \cancel{- x\sin(x)} \cancel{- \cos(x)\ln(\cos(x))} + \frac{\sin^2(x)}{\cos(x)}$$

$$+ \cancel{x\sin(x)} + \cancel{\cos(x)\ln(\cos(x))}$$

$$y_p'' + y_p = \cos(x) + \frac{\sin^2(x)}{\cos(x)} = \frac{\cos^2(x) + \sin^2(x)}{\cos(x)} = \frac{1}{\cos(x)}$$

$= \sec(x)$. So particular soln. ✓

→ Check: $y_1 = \cos(x)$, $y_2 = \sin(x)$ a Fund set of solns:

For $y'' + y = 0$.

- solns. seen before ✓
- How many? Need 2, have 2 ✓
- ✓ Lin indep? Yes, $W(\cos(x), \sin(x)) = 1 \neq 0$ on $(-\infty, \infty)$. ✓

→ Thus, by thm,

$$y = c_1 \cos(x) + c_2 \sin(x) + x\sin(x) + \cos(x)\ln(\cos(x))$$

is a gen soln.