Goal and idea - Module 5

GOAL:

We look at another common "type" of 1st-order ordinary differential equation. Our goal is to learn how to find solutions for those that are called *linear*.

To do so, we:

- Recall what first order linear equations are.
- Discuss how to solve such differential equations.
- Motivate the use of an "integrating factor."

IDEA:

There are a few techniques for solving 1st-order linear DEs. Later in the course we will look at a technique called "variation of parameters." However, for now we focus on a technique which utilizes an "integrating factor."

Motivation for using an integrating factor

Given $\frac{dy}{dx}+Py=Q$. The technique...

Idea

- We want a function F such that
  $$F\left(\frac{dy}{dx}+Py\right)=FQ$$

Taking the differential equation $\frac{dy}{dx}+Py=Q$ and multiply the left and right side by function "F", but what we want it to satisfy condition $F\frac{dy}{dx}+\frac{dF}{dx}y=\frac{d}{dx}(Fy)$

Product Rule

Derivative of "F" with respects to "x"

When you calculate out this derivative $F'y+Fy'$

Satifies

$$F\frac{dy}{dx}+\frac{dF}{dx}y=\frac{d}{dx}(Fy)$$

We are choosing "F" so that we wanna apply it to both sides of our differental equation, but we want it to satisfy the product rule

So if we want to combine $F\left(\frac{dy}{dx}+Py\right)=FQ$ and $F\frac{dy}{dx}+\frac{dF}{dx}y=\frac{d}{dx}(Fy)$ we want a "F" that satisfy the

First Part: $F\left(\frac{dy}{dx}+Py\right)=FQ$

Second Part: $F\frac{dy}{dx}+\frac{dF}{dx}y=\frac{d}{dx}(Fy)$

That is,

$$\frac{dF}{dx}=FP$$

Summary

Looking at the left side of the Second Part: $F\frac{dy}{dx}+\frac{dF}{dx}y=\frac{d}{dx}(Fy)$. The BIG IDEA with this technique of the intergrating factor is that we want to find function "F", multiply it to both the left and right of the equation, and on the left side it takes the whole derivativative "$\frac{dy}{dx}+Py$" and we are able to apply, reverse out, of the product rule "$F(\frac{dy}{dx}+Py)$" on the left hand side and rewrite it more simple "$\frac{d}{dx}(Fy)$"

So what is the function "F" that this would work for?

- Using seprable equation techniques:

$$\frac{dF}{dx} = FP$$

Step 1 | Get all of our "F-stuff" and our "x-stuff" on their own sides

$$\frac{1}{F}dF = Pdx$$

Step 2 | Take the antiderivative of $\frac{1}{F}dF$ and the antiderivative of Pdx

$$\int \frac{1}{F}dF = \int Pdx \longrightarrow \ln|F| = \int Pdx + C$$

Step 3 | Solve for "F"

$$\ln|F| = \int Pdx + C$$

Take "e" of both sides

$$F = e^{\int Pdx + C} \rightarrow F = e^{C}e^{\int Pdx}$$
$$\rightarrow F = Ce^{\int Pdx}$$

- Then multiply $\frac{dy}{dx} + Py = Q$ by F:

$F(\frac{dy}{dx} + Py) = FQ$ gives

$\rightarrow \frac{d}{dx}(Fy) = FQ$, or

$$\rightarrow \frac{d}{dx}[(Ce^{\int Pdx})y] = (Ce^{\int Pdx})Q$$
$$\frac{d}{dx}[(\cancel{C}e^{\int Pdx})y] = (\cancel{C}e^{\int Pdx})Q$$
$$\frac{d}{dx}[(e^{\int Pdx})y] = (e^{\int Pdx})Q$$

The core idea we want some function "F" so that we are able to remanipulate, multiply both sides of the equation, remanipulate the left-hand side back into the product rule, so that we get "$\frac{d}{dx}$" of something.

- Integrating both sides:

$$\frac{d}{dx}[(e^{\int Pdx})y] = (e^{\int Pdx})Q\,dx$$
$$\int \frac{d}{dx}[(e^{\int Pdx})y] = \int (e^{\int Pdx})Q\,dx$$
$$e^{\int Pdx}y = \int e^{\int Pdx}Q\,dx + C$$

So when we go to integrate the left and right side the derivative on left side pops out just "$e^{\int Pdx}$" plus some constant and times "y".

Integrating the right hand side is a little more complicated. We have the integral of "$e^{\int Pdx}$" and we still have the "Q". Integrating all of it we're going to get a constant.

- Solve for "y" *Note: $\frac{1}{e^{\int Pdx}} = e^{-\int Pdx}$*

$$e^{\int Pdx}y = \int e^{\int Pdx}Q\,dx + C$$
$$\rightarrow \frac{e^{\int Pdx}}{e^{\int Pdx}}y = \frac{\int e^{\int Pdx}Q\,dx}{e^{\int Pdx}} + \frac{C}{e^{\int Pdx}}$$
$$\therefore y = e^{-\int Pdx}\int e^{\int Pdx}Q\,dx + Ce^{-\int Pdx}$$

To a certain extent given a first order linear differential equation the only unknowns are the "P" and "Q" ($\frac{dy}{dx} + Py = Q$). You can plug-in "P" and "Q" into the solution

$$y = e^{-\int Pdx}\int e^{\int Pdx}Q\,dx + Ce^{-\int Pdx}$$

Note: If Q=0, then this expression is 0!

*But, don't memorize this!!!
*But, knowing the "Integrating Factor" $F = e^{\int Pdx}$, is useful.

Instead:

(i) Given $\frac{dy}{dx} + P(x)y = Q(x)$,

(ii) Find the integrating factor $e^{\int Pdx}$

(iii) Multiply both sides of $\frac{dy}{dx} + Py = Q$ by $e^{\int Pdx}$ KNOWING that the Left Hand Side becomes $\frac{d}{dx}(e^{\int Pdx}y) = e^{\int Pdx}Q$

(iv) Integrate both sides

(V) Solve for "y"

Ex | Find the general solution for the DE $x\frac{dy}{dx}+2y=3$

Solution |

- 1^{st}-order linear differential equation

Recall

Standard Form Put $\frac{dy}{dx}$ in-front

- Have the highest order derivative in the front.
  - And is normalized meaning you don't have any other coefficient function or anything else in-front of it.

  * Why "3x" was divided *

Everything on one side and zero on the other

$$3x\frac{dy}{dx}+2x+y+4=0$$

equivalent $\Leftrightarrow$

$$\frac{dy}{dx}+\frac{y}{3x}+\frac{2x+4}{3x}=0$$

Step 1 | Rewrite in standard form

$$x\frac{dy}{dx}+2y=3$$

Recall Std. form Put $\frac{dy}{dx}$ in-front

$$\rightarrow \frac{x\frac{dy}{dx}}{x}+\frac{2y}{x}=\frac{3}{x}$$

* In form $\frac{dy}{dx}+Py=Q$ *

$$\rightarrow \frac{dy}{dx}+\frac{2}{x}y=\frac{3}{x}$$

* Can't have $x=0$ anymore *

$$\rightarrow \frac{dy}{dx}+\underbrace{\frac{2}{x}}_{P(x)}y=\underbrace{\frac{3}{x}}_{Q(x)}$$

Step 2 | Calculate Integrating Factor

$$e^{\int P dx}=e^{\int \frac{2}{x}dx}=e^{2\ln|x|}$$

{When you are doing the intergrating factor you do not need to worry about the "+C"}

$$\rightarrow e^{2\ln|x|}=e^{\ln|x^2|}=x^2$$

Step 3 | Multiply left and right by the integrating factor

$$\underbrace{x^2\left(\frac{dy}{dx}+\frac{2}{x}y\right)}_{\text{Product Rule}}=x^2\left(\frac{3}{x}\right)$$

Recall Product Rule $(uv)'=u'v+uv'$

We don't want to actully distribute the "x^2" on the left we'll use the Product Rule

$$\rightarrow \frac{d}{dx}(x^2y)=3x$$

Step 4 | Integrate both sides

$$\frac{d}{dx}(x^2y)=3x$$

$$\rightarrow \int\frac{d}{dx}(x^2y)=\int 3x$$

$$\rightarrow \cancel{\int\frac{d}{dx}}(x^2y)=\int 3x$$

$$\rightarrow x^2y=\int 3dx+C$$

$$\rightarrow x^2y=\frac{3}{2}x^2+C$$

Thus, $y=\frac{3}{2}+\frac{C}{x^2}$ OR $y=\frac{3}{2}+Cx^{-2}$

Notice: Interval of Solutions

* On $(0,\infty)$ or $(-\infty,0)$ *

Transient

Definition: If a term of a solution tends to zero as "x" tends to infinity (i.e. $x\rightarrow\infty$), we call it transient.

Transient Example

In $y=3/2+C/x^2$ the term "C/x^2" is transient

Expectation checklist - Module 5

At the completion of this module, you should:

- know the definitions introduced;
- find and compute the relevant integrating factor;
- know the purpose of the integrating factor;
- if given a 1st-order linear DE, solve it;
    - be able to solve a 1st-order linear IVP; and
- review your integration techniques.

You will be assessed on your understanding of these concepts:

- within the homework,
- on the quizzes, and
- later, on the exam.

Coming up next, we:

- look into a method to solve exact equations.