Variation of parameters

Consider $a_2(x)\,y'' + a_1(x)\,y' + a_0(x)\,y = g(x)$,

where $a_2(x), a_1(x), a_0(x), g(x)$ are functions.

Method:

(i) First turn

$$a_2(x)\,y'' + a_1(x)\,y' + a_0(x)\,y = g(x)$$

into $y'' + P(x)\,y' + Q(x)\,y = f(x)$.

(ii) Find $y_h = c_1 y_1 + c_2 y_2$, which is the general soln for $y'' + P(x) y' + Q(x) y = 0$.

(iii) Then

$$y_p = u_1 y_1 + u_2 y_2,$$

where u_1 and u_2 are found by integrating

$$u_1' = \frac{-y_2 f(x)}{W} \quad \text{and} \quad u_2' = \frac{y_1 f(x)}{W},$$

and $W = W(y_1, y_2)$ is the Wronskian of y_1, y_2.

Ex| solve $y'' + y = \csc(x)$.

soln|

$\rightarrow$ Need y_h. Here $y'' + y = 0$, plug in $y = e^{mx}$:

$$m^2 + 1 = 0 \Rightarrow m_1 = i, \; m_2 = -i,$$

$$\text{Thus, } y_h = c_1 \underbrace{\cos(x)}_{y_1} + c_2 \underbrace{\sin(x)}_{y_2}$$

$\rightarrow$ Need:

$$W = \det\left(\begin{pmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{pmatrix}\right) = 1.$$

Thus,

$$u_1' = \frac{-y_2 f(x)}{W} = \frac{-\sin(x)\ \csc(x)}{1} = \frac{-\sin(x)}{\sin(x)} = -1$$

Therefore,

$$u_1 = \int u_1' = -\int 1 dx = -x \quad [\text{No } + c \text{ here}]$$

Also,

$$u_2' = \frac{y_1 f(x)}{W} = \frac{\cos(x)\csc(x)}{1} = \frac{\cos(x)}{\sin(x)},$$

And $u_2 = \int u_2' = \int \frac{\cos(x)}{\sin(x)} dx = \ln|\sin(x)|$

[u-sub $u = \sin(x)$]

Thus,

$$y_p = u_1 y_1 + u_2 y_2$$

$$= -x\cos(x) + \ln|\cos(x)|\sin(x)$$

$$= -x\cos(x) + \sin(x)\ln|\cos(x)|$$

Thus,

$$y = c_1\cos(x) + c_2\sin(x) - x\cos(x) + \sin(x)\ln|\cos(x)|.$$