Goal and idea - Module 3

GOAL:

Introduce initial value problems and develop some criteria to help determine the existence and uniqueness of solutions for such problems.

To do so, we:

- Define what an initial value problem is, including initial conditions.
- Develop a way to find whether (certain) initial value problems have a unique solution.
- Briefly discuss what a boundary value problem is.

IDEA:

As we have previously seen, it is common that a DE has infinite many solutions. However, in graphing the solutions we see that while they all look similar they are each shifted. In particular, given a specific point in the graph, it appears that there is only one solution which would pass through this point. This is the geometric motivation of initial value problems. In other words, an initial value problem asks whether there is a solution for a DE which given an input, has a desired output. We would like to study such problems, and in particular, develop some criteria to help determine the existence and uniqueness of such solutions.

Time: Videos run 42:36 minutes.

Module 03 SUBJECT:

Initial Value Problems (IVPs)

DATE: 2020/09/20 PAGE NO: 01/01

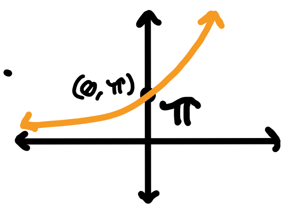
Question: Find a solution for $y' = 3x^2y$ such that $\underbrace{y(0)=\pi}_{\text{*The initial condition*}}$.

$y(0)=e^{0^3}=1\neq\pi$, $y(0)=0\neq\pi$, $y(0)=7e^{0^3}=7\neq\pi$

Note

We'd seen $y=e^{x^3}$, $y=0$, $y=7e^{x^3}$ are solutions to the DE;
but, also $y=Ce^{x^3}$ is a solution for all C.

Soln Note $y(0)=Ce^{0}=C$ (for $y=Ce^{x^3}$). So for $y(0)=\pi$, $C=\pi$.
Thus, $y=\pi e^{x^3}$ satisfies the DE and condition.



Definition

We call an n-th order DE $F(x,y,y',\dots y^{(n)})=0$
subject to the conditions $y(x_0)=y_0, y'(x)=y_1,\dots,y^{(n-1)}(x_0)=y_{n-1}$
(where $y_0, y_1,\dots,y_{n-1}$ are constants) an n-th order
inital value problem (IVP) and the relations the initial conditions.

Ex $y=\dfrac{1}{K+e^{-x}}$ is a 1-parameter family of solutions for $y'=e^{-x}y$. Find a solution to the IVP with initial condition $y(-1)=2$.

Soln

Assuming $y=\frac{1}{K+e^{-x}}$ is a solution, so
what we are interested in is solving for K
* (What is the percise constant 'K' will hold for) *
when we plug-in -1, on one hand we want 2,
and the other we take our function and plug-in -1.

$2 = y(-1) = \frac{1}{K+e^{-(-1)}}$

$\to 2 = \frac{1}{K+e^1}$

$\to (K+e)2 = 1$

$\to 2K+2e = 1$

$\to 2K = 1-2e$

$\to K = \frac{1-2e}{2}$

Thus, $y = \dfrac{1}{\frac{1-2e}{K}+e^{-x}}$

Ex (i) Verify that $y=K\cos(2x)$ satisfies the DE $y''=4y=0$.

(ii) Find a solution for the IVP consisting of $y''+4y=0$ and the conditions $y(0)=3$, $y'(0)=0$

Soln

(i) $y=K\cos(2x)$, $y'=-2K\sin(2x)$, $y''=-4K\cos(2x)$.
Thus, $y''+4y=-4K\cos(2x)+4K\cos(2x)=0$ ✓

(ii)

$y(0)=3$	$y'(0)=0$
$\to K\cos(0)=3$	$\to -2K\sin(0)=0$
$\to K=3$	$\to 0=0$

Thus, $y=3\cos(2x)$ works

Questions

Existance: Is there always a solution to an IVP?

Uniqueness: How many solutions are there?

Module 03

SUBJECT: Existance and Uniqueness of Solutions DATE: 2020/09/20 PAGE NO: 01/01

Theorem

Suppose we have a DE of form $\boxed{\frac{dy}{dx}=f(x,y)}$, where $f(x,y)$ is expressing the rest of the DE as a multivarable function. Additionly, consider a rectangular region R in the xy-plane given by $a \le x \le b$ and $c \le y \le d$ for numbers a,b,c,d which contains a point (x_0, y_0). If

*· $f(x,y)$ and

*· $\frac{\partial}{\partial y} f(x,y)$

are both continuous on R, then on some interval around x_0 there exists a unique solution to the DE, $\frac{dy}{dx}=f(x,y)$

* A multivarable function $f(x,y)$ is continuous where it is defined (the domain of the function).

* Note: Theorem does not say when a solution does not exist. So if something fails this theorem its inconclusive. It does not give you information. Can't say 'yes' or 'no'.

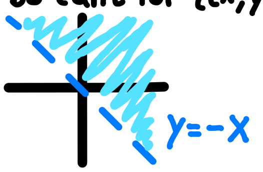
Ex

(i) $\sqrt{x+y}$

* Need (for continuous) $x+y \ge 0$

$\rightarrow y \ge -x$

So can't for $\{(x,y) \mid y \ge -x\}$



SUB EX| Points

· $(1,2)$ ✓

· $(-1,-1)$ ✗

(ii) $\frac{x}{x^2+y^2}$ is continuous everywhere except at $(0,0)$.

EX Suppose we had $\frac{dy}{dx}=\sqrt{xy}$. Describe the points (x_0, y_0) where the IVP $y(x_0)=y_0$ would guarantee to have a unique solution.

Soln A|

1) $f(x,y)=\sqrt{xy}$

Partial Derivative

2) $\frac{df}{dy}=\frac{x}{2\sqrt{xy}}$

S

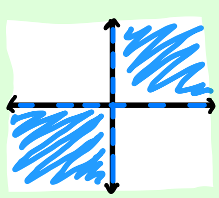
what are the values of x & y you are allowed to plug-in?

1) $xy \ge 0$

2) $xy \ne 0$

· Need $xy > 0$

· Need $x>0$ and $y>0$ OR $x<0$ and $y<0$



Soln B| Graph it as a set

That is, $\{(x,y) \mid x,y>0 \text{ OR } x,y<0\}$

Ex| Consider $y'=\sqrt{y^2-9}$. Does this DE have a unique solution at $(-1,5)$?

Soln| $f(x,y)=\sqrt{y^2-9}$ is continuous for $y^2-9 \ge 0$

Is already $\frac{dy}{dx}$ in form

$f(x,y)$

$y'=\sqrt{y^2-9}$

$\rightarrow y^2 \ge 9$ OR $y \ge 3$ OR $y \le -3$

We need these two conditions: $y \le -3$ OR $3 \le y$

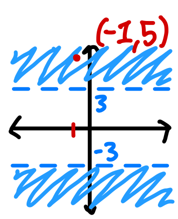
The second peice we need is the partial derivative of 'y' with respects to our $f(x,y)$. (AKA: Partial derivative with respects to 'y' of $\sqrt{y^2-9}$)

$$\frac{d}{dy} f(x,y)=\frac{d}{dy}\sqrt{y^2-9}=\frac{d}{dy}(y^2-9)^{1/2}$$

$$\rightarrow \frac{1}{2}(y^2-9)^{-1/2}\cdot(2y)=\frac{y}{\sqrt{y^2-9}}$$

Defined for $y^2-9>0 \Longleftrightarrow y^2>9$

$\rightarrow$ Need $y<-3$ or $y>3$



Thus a unique solution exists at $(-1,5)$. If at $(-1,1)$ we can only say it's not guarantee. We don't know for $(-1,1)$ it's inconclusive.

Definition: Boundry Value Problem

We call an Ordinary Differential Equation (ODE) $F(x, y, y', \ldots, y^{(n)}) = 0$ subject to the conditions

$y(x_0) = y_0,\ y'(x_1) = y_1, \ldots, y^{(n-1)}(x_{n-1}) = y_{n-1}$ *

A Boundry Value Problem (BVP) and * the boundry conditions.

Initial Value Problem (IVP) VS Boundry Value Problem (BVP)

Initial Value Problem (IVP)

$y(x_0) = y_0,\ y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}$

- IVP we'd have $y(x_0)$ and at the derivative we'd have (x_0) again in $y'(x_0)$.
- We'd only have the output change $y(x_0) = y_0,\ y(x_0) = y_1,\ y(x_0) = y_2$

Boundry Value Problem (BVP)

$y(x_0) = y_0,\ y'(x_1) = y_1, \ldots, y^{(n-1)}(x_{n-1}) = y_{n-1}$

- BVP changes not only the outputs, but also the inputs $y(x_0) = y_0,\ y'(x_1) = y_1,\ y''(x_2) = y_2$

Expectation checklist - Module 3

At the completion of this module, you should:

- know the definitions introduced;
- Determine if a given function is a solution to an IVP;
- if given a first-order IVP determine if
  - there is a unique solution for the IVP, or
  - there is not enough information to conclude such a result;
- if given a first-order IVP be able to provide a region (both graphically and with mathematical notation) where you are guaranteed the existence of a unique solution for the IVP;
- find whether a multivariable function is continuous under the conventions in this class; and
- have an idea of what boundary value problems are.

You will be assessed on your understanding of these concepts:

- within the homework,
- on the quizzes, and
- later, on the exam.

Coming up next, we:

- begin a probe on *finding* solutions for first-order ODEs, and in particular methods surrounding
  - separable equations
  - first-order linear equations
  - exact equations
  - and first-order homogeneous equations.