Goal and idea - Module 12

GOAL:

We begin solving higher order differential equations! As mentioned before, we limit ourselves to linear differential equations. However, we still make additional limitations. In particular, we focus on such differential equations with *constant coefficients*. We will

- introduce linear differential equations with constant coefficients;
- learn to solve such differential equations.

IDEA:

We begin by looking at order 2 linear differential equations, and from this we discuss how we could go about the higher order case.

Order 1 | $ay'+by=0$, a, b constants

Could solve as seprable equation

- Get: $y=Ce^{-\frac{b}{a}x}$

$ay'+by=0$ because of constants it could have been rewritten as $y'+\frac{b}{a}y=0$ OR $y'=-\frac{b}{a}y$

OR

Consider $y=e^{mx}$ ("m" is a number)

Plug into DE; $y'=me^{mx}$

Gives $ame^{mx}+be^{mx}=0$

Factoring out e^{mx}

$\rightarrow \underbrace{e^{mx}}_{\neq 0}\underbrace{(am+b)}_{\text{must have equal zero!}}=0$

$\rightarrow am+b=0$

Solve for "m"

$\rightarrow m=-b/a$

So

$\rightarrow y=e^{mx}$

Is actually/becomes

$\rightarrow y=e^{-b/a\,x}$

Thus

- One solution for 1st-order
- Is linearly independent, so is a fundamental set solution

As a result

$y=Ce^{-b/a\,x}$ is a general solution

Order 2 | $ay''+by'+cy=0$ (a,b,c constants)

constant coeffcients

- Plug-in $y=e^{mx}$; $y'=me^{mx}$; $y''=m^2e^{mx}$
- This gives: $am^2e^{mx}+bme^{mx}+ce^{mx}=0$

Factoring out e^{mx}

$\rightarrow e^{mx}(am^2+bm+c)=0$

So, we need

$\rightarrow am^2+bm+c=0$

Because it's a degree 2 polynomial thus we have the Quadratic Formula: $m=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

But While

$$y_{1,2}=e^{\left(\frac{-b\pm\sqrt{b^2-4ac}}{2a}\right)x}$$

are solutions, there are some issues...

→ Not necessarily different or linearly independent

→ Could be complex numbers!

- $\sqrt{b^2-4ac}$ ← could be a negative number

then you'd have "e" to some complex number

what governs this is b^2-4ac.

Case 1 | $b^2-4ac>0$ Greater than zero

Then $\frac{-b\pm\sqrt{b^2-4ac}}{2(a)}$ are two distinct real numbers

Thus, $y_1=e^{\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)x}$, $y_2=e^{\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)x}$

are two linearly independent solutions, And our general solution is going to be of this form

$$y=C_1e^{\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)x}+C_2e^{\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)x}$$

So when we solve our quadratic formula we had before ($am^2+bm+c=0$) then case 1 when we get two distinct real numbers popping out as the solutions for that polynomial then we are immediatly done once you know how to write the general solution.

$\left(m_1=\frac{-b+\sqrt{b^2-4ac}}{2a} \;\&\; m_2=\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$

OR: $y=C_1e^{m_1x}+C_2e^{m_2x}$

Case 2 | $b^2-4ac=0$ Equal to zero

Real Numbers Repeated

Then $y_1=e^{\left(-\frac{b}{2a}\right)}$ is a solution. Why?

Because we took our e^{mx} and we solved for "m", we figured out that "m" works, forces this to be a solution of our differential equation.

BUT WE NEED ANOTHER... How do we find it?

Answer: Reduction of Order!

We can do this generically

$$y_2=y_1\int\frac{e^{-\int Pdx}}{(y_1)^2}dx$$

so the general solution is:

① In an abstract form

$$y=C_1e^{\left(-\frac{b}{2a}\right)x}+C_2\left(y_1\int\frac{e^{-\int Pdx}}{(y_1)^2}dx\right)$$

② Which becomes

$$\underbrace{y=C_1e^{\left(-\frac{b}{2a}\right)x}+C_2\left(y_1\int\frac{e^{-\int Pdx}}{(y_1)^2}dx\right)}$$

$$y=C_1e^{mx}+C_2xe^{mx}$$

* Note *
$e^{\left(-\frac{b}{2a}\right)x}$ is rewritten as e^{mx}; for, $m=-\frac{b}{2a}$.
And $y_1\int\frac{e^{-\int Pdx}}{(y_1)^2}dx$ becomes $Cxe^{-\frac{b}{2a}x}$

Case 3 | $b^2-4ac<0$ Less than zero

If less than zero then we could rewrite as (factoring out a −1)

Complex Numbers w/Conjugate Property

Then

$$\sqrt{b^2-4ac}=\sqrt{(-1)(\underbrace{-b^2+4ac}_{\text{This term has to be positive }(>0)})}=\sqrt{-1}\,\sqrt{-b^2+4ac}$$

$$=i\sqrt{-b^2+4ac}$$

$\left(m_1=\frac{-b}{2a}+\frac{i\sqrt{b^2-4ac}}{2a} \;\&\; m_2=\frac{-b}{2a}-\frac{i\sqrt{b^2-4ac}}{2a}\right)$

The problem is this gives

$y_1=\left(\frac{-b}{2a}+\frac{i\sqrt{b^2-4ac}}{2a}\right)x \;\&\; y_2=\left(\frac{-b}{2a}-\frac{i\sqrt{b^2-4ac}}{2a}\right)x$

Complex Solutions!

* WE WANT REAL SOLUTIONS *

Euler's Identity: (θ is a real number)

$$e^{i\theta}=\cos(\theta)+i\sin(\theta)$$

Thus,

$$y_1=\left(\frac{-b}{2a}+\frac{i\sqrt{b^2-4ac}}{2a}\right)x$$

$$y_1=e^{-\frac{b}{2a}x}\,e^{i\left(\frac{\sqrt{b^2-4ac}}{2a}\right)x}$$

Now using Euler's Identity → Focus on the real, Ignore the complex

$$y_1=e^{-\frac{b}{2a}x}\Big(\underbrace{\cos\left(\frac{\sqrt{b^2-4ac}}{2a}x\right)}_{\text{Real}}+\underbrace{i\sin\left(\frac{\sqrt{b^2-4ac}}{2a}x\right)}_{\text{Complex}}\Big)$$

Case 3 | $b^2-4ac < 0$ Less than zero

If less than zero then we could rewrite as (factoring out a -1)

Then $\sqrt{b^2-4ac} = \sqrt{(-1)(\underbrace{-b^2+4ac}_{\text{This term has to be positive } (>0)})} = \sqrt{-1}\,\sqrt{-b^2+4ac} = i\sqrt{-b^2+4ac}$

$$\left(m_1 = \underbrace{\frac{-b}{2a} + \frac{i\sqrt{b^2-4ac}}{2a}}_{=\alpha+i\beta} \;\&\; m_2 = \underbrace{\frac{-b}{2a} - \frac{i\sqrt{b^2-4ac}}{2a}}_{=\alpha-i\beta}\right)$$

The problem is this gives

$$y_1 = \left(\frac{-b}{2a} + \frac{i\sqrt{b^2-4ac}}{2a}\right)x \;\&\; y_2 = \left(\frac{-b}{2a} - \frac{i\sqrt{b^2-4ac}}{2a}\right)x$$

Complex Solutions!

* WE WANT REAL SOLUTIONS *

Euler's Identity: (Θ is a real number)

$$\boxed{e^{i\Theta} = \cos(\Theta) + i\sin(\Theta)}$$

Thus,

$$y_1 = \left(\frac{-b}{2a} + \frac{i\sqrt{b^2-4ac}}{2a}\right)x$$

$$y_1 = e^{-\frac{b}{2a}x}\, e^{i\left(\frac{\sqrt{b^2-4ac}}{2a}\right)x}$$

Now using Euler's Identity

Focus on the real | Ignore the complex

$$y_1 = e^{-\frac{b}{2a}x}\left(\underbrace{\cos\left(\frac{\sqrt{b^2-4ac}}{2a}x\right)}_{\text{Real}} + \underbrace{i\sin\left(\frac{\sqrt{b^2-4ac}}{2a}x\right)}_{\text{Complex}}\right)$$

So...

$$y_1 = e^{-\frac{b}{2a}x}\left(\cos\left(\frac{\sqrt{b^2-4ac}}{2a}x\right)\right)$$

Now we can solve for one real solution.

The problem is finding y_2 in the same way gives the same answer y_1, but we have Reduction of Order!

And find:

$$y_2 = e^{-\frac{b}{2a}x}\left(\sin\left(\frac{\sqrt{b^2-4ac}}{2a}x\right)\right)$$

Thus, the general solution for the complex case is

$$y = C_1 e^{-\frac{b}{2a}x}\left(\cos\left(\frac{\sqrt{b^2-4ac}}{2a}x\right)\right) + C_2 e^{-\frac{b}{2a}x}\left(\sin\left(\frac{\sqrt{b^2-4ac}}{2a}x\right)\right)$$

OR having $m_1 = \alpha + i\beta$ & $m_2 = \alpha - i\beta$ then

$$y = C_1 e^{\alpha}\cos(\beta x) + C_2 e^{\alpha}\sin(\beta x)$$

Ex| Solve $3y''-5y'+2y=0$

Solution|

1) Plug-in $y=e^{mx}$, $y'=me^{mx}$, $y''=m^2e^{mx}$

Skipping a step, instead of plugging-in and refactoring out, we know are able to factor out "e^{mx}"

This gives us

$$e^{mx}(3m^2-5m+2)=0$$

So we need

$$3m^2-5m+2=0$$

2) Find m_1 & m_2

$$m_1=\frac{-b+\sqrt{b^2-4ac}}{2a} \quad \& \quad m_2=\frac{-b+\sqrt{b^2-4ac}}{2a}$$

$$m_1=\frac{-(-5)+\sqrt{-5^2-4(3)(2)}}{2(3)} \quad \& \quad m_2=\frac{-(-5)-\sqrt{-5^2-4(3)(2)}}{2(3)}$$

$$m_1=\frac{5+\sqrt{25-24}}{6} \quad \& \quad m_2=\frac{5-\sqrt{25-24}}{6}$$

$$m_1=\frac{5}{6}+\sqrt{\frac{1}{6}} \quad \& \quad m_2=\frac{5}{6}-\sqrt{\frac{1}{6}}$$

$$m_1=\frac{5}{6}+\frac{1}{6} \quad \& \quad m_2=\frac{5}{6}-\frac{1}{6}$$

$$m_1=1 \quad \& \quad m_2=\frac{4}{6}=\frac{2}{3}$$

3) What are they (m_1 & m_2)?

$m_1=1$ & $m_2=\frac{2}{3}$ are **Two Distinct Real Numbers**

→ Had m_1 & m_2 both equal 1
it would have been case two

→ Had m_1 & m_2 have any imaginary
"i" complex numbers we'd have case three

Thus, the general solution is

$$y=C_1e^{m_1x}+C_2xe^{m_2x}$$

OR

$$y=C_1e^{1x}+C_2xe^{\frac{2}{3}x}$$

Ex Solve $y''+4y'+4y=0$

Solution

1) Plug-in $y=e^{mx}$, $y'=me^{mx}$, $y''=m^2e^{mx}$

Skipping a step, instead of plugging-in and refactoring out, we know are able to factor out "e^{mx}"

This gives us

$$e^{mx}(m^2+4m+4)=0$$

So we need

$$m^2+4m+4=0 \leftarrow \boxed{(m+2)^2=0}$$

Or Note

2) Find m_1 & m_2

$$m_1=\frac{-b+\sqrt{b^2-4ac}}{2a} \quad \& \quad m_2=\frac{-b+\sqrt{b^2-4ac}}{2a}$$

$$m_1=\frac{-(4)+\sqrt{4^2-4(1)(4)}}{2(1)} \quad \& \quad m_2=\frac{-(4)-\sqrt{^2-4(1)(4)}}{2(1)}$$

$$m_1=\frac{-4+\sqrt{16-16}}{2} \quad \& \quad m_2=\frac{-4-\sqrt{16-16}}{2}$$

$$m_1=\frac{-4}{2}=-2 \quad \& \quad m_2=\frac{-4}{2}=-2$$

3) What are they (m_1 & m_2)?

m_1 & $m_2=-2$ are Real Numbers Repeated

It ends up these three cases we note are the only three case that are possible

- Case 1) Two Distinct Real Numbers
- Case 2) Real Number Repeated
- Case 3) Complex Numbers w/Conjugate Property

We only have one solution at the moment $\rightarrow y_1=e^{-2x}$, and we want a second one (y_2)

Typically we can skip this step (once we know how it works), but for now...

Skip

4) Reduction of Order [Could omit in future]

$$\bar{y}_2=y_1(x)\int\frac{e^{-\int P(x)dx}}{(y_1(x))^2}dx$$

Find $P(x)$

Looking back at the DE: $y''+4y'+4y=0$

Since 1 is the Coefficent

we need it in the form

$y''+Py'+Qy=0$

$y''+4y'+4y=0$

$\therefore P(x)$ in our case is 4

$$\bar{y}_2=y_1\int\frac{e^{-\int Pdx}}{(y_1)^2}dx$$

$\bar{y}_2$ because whenever we calculate this we will get stuff we don't need

$$=e^{-2x}\int\frac{e^{-\int 4dx}}{(e^{-2x})^2}dx$$

$$=e^{-2x}\int\frac{e^{-4x}}{e^{-4x}}dx=e^{-2x}\int\frac{\cancel{e^{-4x}}}{\cancel{e^{-4x}}}dx$$

$$=e^{-2x}\int 1\,dx \quad =e^{-2x}(x+C)$$

$$=xe^{-2x}+Ce^{-2x}$$

Looking back y_1 is already contained here

At the end of the day we only care about linear independent functions.

We are looking for Fundamental Set, and so we would consider everything that's going to be linearly independent or in this case whatever doesn't have what we already have

Recall

Fundamental Set of Solutions: Two Things
- $\rightarrow$ Number of solutions (equal to the order) of the differential equation.
- $\rightarrow$ The solutions need to be linearly independent.

Consider $y_2=xe^{-2x}$

Thus,

$y=C_1e^{-2x}+C_2xe^{-2x}$ is a general solution

In summary: To get y_2 all you need to do is multiply y_1 by 'x'.

Ex| Solve $y'' - 3y' + 4y = 0$

Solution|

1) Plug-in $y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2 e^{mx}$

Skipping a step, instead of plugging-in and refactoring out, we know we are able to factor out "e^{mx}"

This gives us

$$e^{mx}(m^2 - 3m + 4) = 0$$

So we need

$$m^2 - 3m + 4 = 0$$

2) Find m_1 & m_2

$$m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \& \quad m_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$m_1 = \frac{-(3) + \sqrt{(3)^2 - 4(1)(4)}}{2(1)} \quad \& \quad m_2 = \frac{-(3) - \sqrt{(3)^2 - 4(1)(4)}}{2(1)}$$

$$m_1 = \frac{-3 + \sqrt{9 - 16}}{2} \quad \& \quad m_2 = \frac{-3 - \sqrt{9 - 16}}{2}$$

$$m_1 = \frac{-3 + \sqrt{-7}}{2} \quad \& \quad m_2 = \frac{-3 - \sqrt{-7}}{2}$$

$$m_1 = \frac{-3 + i\sqrt{7}}{2} \quad \& \quad m_2 = \frac{-3 - i\sqrt{7}}{2}$$

$$m_1 = \underbrace{\frac{-3}{2} + \frac{i\sqrt{7}}{2}}_{\alpha + i\beta} \quad \& \quad m_2 = \underbrace{\frac{-3}{2} - \frac{i\sqrt{7}}{2}}_{\alpha - i\beta}$$

This is always the case where they are conjugates (one is plus & the other is minus)

⁎ Recall ⁎

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$e^{(\underset{\alpha}{3/2} + \underset{i\beta}{\frac{i\sqrt{7}}{2}})x} = e^{3/2x} e^{\frac{i\sqrt{7}}{2}x}$$

$$\rightarrow e^{3/2x}\Big(\underbrace{\cos\left(\tfrac{\sqrt{7}}{2}x\right)}_{\text{Real}} + \underbrace{i\sin\left(\tfrac{\sqrt{7}}{2}x\right)}_{\text{Complex}}\Big)$$

Focus on the real — Ignore the complex

Euler's Formula

3) Using the formula

(Euler's Formula)

$$y_1 = e^{\alpha x}\cos(\beta x), \quad y_2 = e^{\alpha x}\sin(\beta x)$$

We find

$$y_1 = e^{3/2x}\cos\left(\tfrac{\sqrt{7}}{2}x\right), \quad y_2 = e^{3/2x}\sin\left(\tfrac{\sqrt{7}}{2}x\right)$$

Thus, the general solution is

$$y = C_1 e^{3/2x}\cos\left(\tfrac{\sqrt{7}}{2}x\right) + C_2 e^{3/2x}\sin\left(\tfrac{\sqrt{7}}{2}x\right)$$

Summary

The most important thing with these problems is (1) Plug-in $y = e^{mx}$ and (2) Finding out if $m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ & $m_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ are of

Case 1) Two Distinct Roots

Case 2) Repeated Root

Case 3) Complex Conjugate Roots

In the case of complex roots (case 3) you identify, you have to work it out and figure out, what are your Alpha (α) and Beta (β)

$\alpha + i\beta$ $\qquad$ $\alpha - i\beta$

but then it's okay to then move forward and realize

$$y_1 = e^{\alpha x}\cos(\beta x), \quad y_2 = e^{\alpha x}\sin(\beta x)$$

are the solutions showing you where Alpha and Beta go (what the solution looks like).

All examples of Order Two Homogeneous Linear DEs with Constant Coefficents follows the same steps (1) Plug-in $y = e^{mx}$ and (2) Find m. It just so happens that when "m" is a complex number a cosine and sine "magically appears" using the Euler's Formula.

Find $\rightarrow e^{mx}$ Don't be afraid for any Order 2 homog. linear DEs use the Quadratic Formula.

Higher Order

Question: How do we solve?
Answer: Basically the same.

* Given *

Homogeneous Linear Differential Equation

Any order DE
Not just order one two...

Some constant "an" times an n^{th}-order derivative

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$$

($a_n, a_{n-1}, \ldots, a_1, a_0$ are constants)

Then (just as before)

1) Plug-in $y=e^{mx}$

Skipping a step, instead of plugging-in and refactoring out, we know we are able to factor out "e^{mx}"

This gives us

$$e^{mx}(a_n m^n + a_{n-1} m^{n-1} + \ldots + a_1 m + a_0) = 0$$

2) Find m

* Note: can be hard! *
Example
$$m^{17} - 108m^{15} - 3m^2 + m - 1 = 0$$

Unlike 2^{nd}-Order DEs we are guaranteed two Ms because of the Quadratic Formula

Finding roots of polynomials is our limitation

Because of that if you able to find the "ms" we just plug it back into e^{mx} and we get what we need, but because it's so difficult we are going to limit ourselves to just a couple of situations that are worth knowing about.

Situations we can handle:

① If n-many distinct real roots; then, the general solution is $y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \ldots + C_n e^{m_n x}$

Example: $y''' - 6y'' + 11y' - 6y = 0$

Gives
$$e^{mx}(m^3 - 6m^2 + 11m - 6) = 0$$
$$= (m-1)(m-2)(m-3)$$

So $m_1 = 1$, $m_2 = 2$, $m_3 = 3$

Thus,
$$y = C_1 e^{1x} + C_2 e^{2x} + C_3 e^{3x}$$

Situations we can handle - Continued

② If order n DE, and only one real root, but repeated n-times; then, the general solution is

$$y = C_1 e^{mx} + C_2 x e^{mx} + C_3 x^2 e^{mx} + \ldots + C_n x^{n-1} e^{mx}$$

Example: $y''' - 6y'' + 12y' - 8 = 0$

Gives
$$e^{mx}(m^3 - 6m^2 + 12m - 8) = 0$$
$$= (m-2)(m-2)(m-2)$$
OR
$$(m-2)^3$$

One root repeated

So
$$y = C_1 e^{2x} + C_2 x e^{2x} + C_3 x^3 e^{2x}$$

③ If it contains two complex roots only (But has some other stuff as well)

Example: $y''' + y' = 0$

Gives
$$e^{mx}(m^3 + m) = 0$$
$$(m^2+1)m$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-0 \pm \sqrt{0^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{\pm\sqrt{-4}}{2} = \frac{\pm -i\sqrt{4}}{2}$$

$$m_1 = \underbrace{\frac{+-i\sqrt{4}}{2}}_{\alpha+i\beta}, \quad m_2 = \underbrace{\frac{--i\sqrt{4}}{2}}_{\alpha-i\beta}$$

$$m_1 = \frac{+-i\cancel{2}}{\cancel{2}}, \quad m_2 = \frac{--i\cancel{2}}{\cancel{2}}$$

$$m_1 = -i, \quad m_2 = i$$

$$m_1 = 0$$

We have three roots, but two of them are complex conjugates

(Euler's Formula) $y_1 = e^{\alpha x}\cos(\beta x)$, $y_2 = e^{\alpha x}\sin(\beta x)$

$m_1 = 0$, $m_1 = -i$, $m_2 = i$
$\alpha = 0$, $\beta = i$

Then
$$y_1 = e^{0x} = 1, \quad y_2 = e^{0}\cos(x), \quad y_3 = e^{0}\sin(x)$$

Thus
$$y = C_1 + C_2\cos(x) + C_3\sin(x)$$

④ Combining

Example: Solve $y''' - 5y'' + 8y' - 4y = 0$

Get $m^3 - 5m^2 + 8m - 4 = 0$

Try plugging in things and guess

* Note: m=1 is a root, can factor it out

$\rightarrow (m-1)(m^2 - 4m + 4) = 0$

$\rightarrow (m-1)(m-2)^2 = 0$

Thus
$$y = C_1 e^{x} + C_2 e^{2x} + C_3 x e^{2x}$$
is a general solution

Expectation checklist - Module 12

At the completion of this module, you should:

- Be able to find the general solution for *any* 2nd-order homogeneous linear differential equations with constant coefficients;
- Understand how the following three cases arise in order 2, and how to write the corresponding solutions:
    - two distinct real roots,
    - one repeated root, and
    - complex conjugate roots;
- Be able to solve most higher order homogeneous linear differential equations so long as roots are able to be found, and in particular, those with distinct real roots; and
- Know where reduction of order is utilized, even if we omit its steps.

Coming up next, we:

- Solve some homogeneous linear differential equations!