Reduction of order

WARM-UP: Consider $y''+y=0$. Suppose we know $y_1=\sin(x)$ is a solution.

Q: How many solns do we need for a fund set of solns? A: 2.

Q: Given one soln, can we find the other?

→ Let $u(x)$ be an arbitrary function, and set $y = u(x)\,y_1(x) = u(x)\sin(x) = u\sin(x)$.

Find y' and y''

$$\begin{aligned} y' &= u'\sin(x) + u\cos(x) \\ y'' &= u''\sin(x) + u'\cos(x) + u'\cos(x) - u\sin(x) \\ &= u''\sin(x) + 2u'\cos(x) - u\sin(x) \end{aligned}$$

→ Plug y, y', y'' into the DE

$$y'' + y = 0 \iff u''\sin(x) + 2u'\cos(x) \cancel{- u\sin(x)} \cancel{+ u\sin(x)} = 0$$

$$u'' \sin(x) + 2u' \cos(x) = 0.$$

→ Put this in the form $u'' + P(x)u' + Q(x)u = 0$:

$$u'' + \frac{2\cos(x)}{\sin(x)} u' = 0$$

→ Make substitution u' with w (i.e., $w = u'$):

$$w' + \frac{2\cos(x)}{\sin(x)} w = 0$$

→ Recall how to solve $w' + \frac{2\cos(x)}{\sin(x)} w = 0$!

Find integrating factor:

$$e^{\int P dx} = e^{2\int \frac{\cos(x)}{\sin(x)} dx} = e^{2\ln|u|} = e^{\ln|u^2|}$$

↖ sub $u = \sin(x)$

$$= u^2 = \sin^2(x).$$

Recall:

$$\frac{d}{dx}\left(\sin^2(x) w\right) = \frac{d}{dx}(0)$$

Integrate both sides:

$$\sin^2(x) w = C.$$

solve for w:

$$w = \frac{c}{\sin^2(x)}$$

→ plug back in $u' = w$!

$$u' = \frac{c}{\sin^2(x)}$$

→ separation of variables:

$$\frac{du}{dx} = \frac{c}{\sin^2(x)} \iff du = \frac{c}{\sin^2(x)}\,dx$$

* Integrate:

$$\int du = \int \frac{c}{\sin^2(x)}\,dx$$

$$\iff u = c\int \csc^2(x)\,dx = -c\cot(x) + k$$

$$= -c_1\cot(x) + c_2 = c_1\cot(x) + c_2.$$

→ Plug u into $y = uy_1$:

$$\begin{cases} y = (c_1 \cot(x) + c_2)\sin(x) = c_1\left(\dfrac{\cos(x)}{\sin(x)} + c_2\right)\sin(x) \\ \quad = c_1\cos(x) + c_2\sin(x) \end{cases}$$

→ Take different c_1, c_2 to get 2 lin indep solns.

$$\begin{cases} c_1 = 0,\ c_2 = 1 \Rightarrow \sin(x) \\ c_1 = 1,\ c_2 = 0 \Rightarrow \cos(x) \end{cases}$$

→ Thus, two solns are $y_1 = \sin(x)$, $y_2 = \cos(x)$.

→ General soln for $y'' + y = 0$ is

$$y = c_1 \sin(x) + c_2 \cos(x).$$

this is the method of Reduction of Order.

or could use the formula:

$$y = y_1(x) \int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx.$$

where P is from $y'' + P(x)y' + Q(x)y = 0.$

REDUCTION OF ORDER

* when to use: Given a 2nd-order DE and only 1 soln.
* why to use: Find the other lin indep soln to form a fund set of solns.
* How to use: Two options.
  - → long route above (don't memorize)
  - → the formula

$$y_1(x)\int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx$$