Solutions for DEs

Definition:

A function that satisfies a DE for all input of the function is called a **solution** for the DE.

Ex1 Is $y = e^{x^3}$ a solution for $y' = 3x^2 y$?

soln| Note: $y' = 3x^2 e^{x^3}$. Thus

$y' = 3x^2 y$

$\Leftrightarrow 3x^2 e^{x^3} = 3x^2 e^{x^3}$ ✓

yes, satisfies the DE.

We have $y = e^{x^3}$ is a soln.

Note: so are : $y=0$ or $y = 7e^{x^3}$ or $y = 15e^{x^3}$.

Ex1 Find a function y satisfying $\frac{dy}{dx} + y^2 = 0$.

soln Consider $y = \frac{1}{x}$. Then

$$y' = -\frac{1}{x^2} \text{ and } y' + y^2 = 0$$

$$\Leftrightarrow -\frac{1}{x^2} + \frac{1}{x^2} = 0.$$

Q: Is y a soln to the DE?

A: We need to examine the domain of the solns.

Definition:

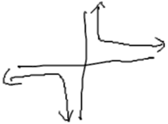
A function defined on an interval that satisfies a DE on the interval is a **solution** for the DE on the interval.

RECALL: $y=\frac{1}{x}$ satisfies $\frac{dy}{dx}+y^2=0$.

However, $y=\frac{1}{x}$ is defined on $(-\infty,0)\cup(0,\infty)$.

thus, $y=\frac{1}{x}$ is a soln for the DE

on $(-\infty,0)$ or $(0,\infty)$.



Definition:

The interval for which a function is a solution for the DE is called the **interval of existence** (or **validity**), or the **domain of the solution**.

Note also: $y=0$ satisfies $\frac{dy}{dx}+y^2=0$.
and the domain of the solution
for $y=0$ is $(-\infty, \infty)$.

Definition:

If the function y=0 is a solution on an interval, we call this the **trivial solution**.

Implicit and Explicit Solutions

Ex) Suppose $y^2 = x^2$. Find $\frac{dy}{dx}$.

soln) Implicit differentiation.

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^2)$$

$$\Leftrightarrow \quad 2y \cdot \left(\frac{dy}{dx}\right) = 2x$$

$$\Leftrightarrow \quad \frac{dy}{dx} = \frac{2x}{2y} = \frac{x}{y}.$$

$\boxed{y = f(x)}$

$y^2 \Leftarrow$

$(x - \text{stuff})^2$

Ex1 consider $\frac{dy}{dx} = \frac{x^2}{1-y^2}$. Verify that

$$y - \frac{y^3}{3} = \frac{x^3}{3} + C \quad (C \text{ a constant})$$ is

a soln for the DE.

soln $\frac{d}{dx}\left(y - \frac{y^3}{3}\right) = \frac{d}{dx}\left(\frac{x^3}{3} + C\right)$

$\Leftrightarrow \frac{dy}{dx} - y^2 \cdot \frac{dy}{dx} = x^2$

$\Leftrightarrow \frac{dy}{dx}(1-y^2) = x^2$

$\Leftrightarrow \frac{dy}{dx} = \frac{x^2}{1-y^2}$.

So $y - \frac{y^3}{3} = \frac{x^3}{3} + C$

is an implicit soln (really its a family)

RECALL For the DE $y' = 3x^2 y$ we had

$y = e^{x^3}$, $y = 0$, $y = 7e^{x^3}$

are all solns.

→ In fact, $y = Ce^{x^3}$ is a soln for any constant C.

→ So $y = Ce^{x^3}$ is a "family" of solns.

→ It is a 1-parameter family of solns.

→ whereas, $y = \pi e^{x^3}$ is a particular soln.

Q: Are all solns for the DE of the form $y = Ce^{x^3}$ for some C?

<u>Definition</u>:

1. A soln for a DE that has no parameters is a **particular** soln.

2. A set of solutions obtained by ranging over n many constants is called an **n-parameter family** of solns.

3. An n-parameter family of solns which gives rise to all solns is called a **general** soln.

Ex1 Consider $y'' - y = 0$. Verify that $y = c_1 e^x + c_2 e^{-x}$ is a 2-parameter family of solns.

Soln:

$$y' = c_1 e^x - c_2 e^{-x}$$

$$y'' = c_1 e^x + c_2 e^{-x}$$

Here, c_1, c_2 are constants.

So

$$y'' - y = 0 \checkmark$$

DONE.

Graphing Solutions

Ex Note $y = \frac{1}{x^2}$ is a soln for $y' = -\frac{2y}{x}$ on the interval $I = (-\infty, 0)$ or on $I = (0, \infty)$.

Graph the soln:

soln Note: the graph of the function $y = \frac{1}{x^2}$ is

For the solution, need interval $(-\infty, 0)$ or $(0, \infty)$.

Ex| Graph the solns for $y' = y$.

soln| $y = e^x$ is a soln. So is $y = 2e^x$, $3e^x$, etc.

In fact, $y = Ce^x$ is a 1-parameter family of solns.

Could graph "all" of them: