# The Wronskian

The main hurdle for us in finding whether a set of functions is a fundamental set of solutions, is determining whether the solutions are linearly independent. To tackle this, we employ the Wronskian.

However, this depends on finding the determinant of matrices, which we may have no experience with. We will only ever be concerned with determinants of $2 \times 2$ and $3 \times 3$ matrices (and really, for quizzes and exams only $2 \times 2$). Thus, we provide the formulas for these.

**Definition**

- Suppose $a, b, c, d$ (which could be functions) are entries of a $2 \times 2$ matrix $A$. The **determinant** of $A$, denoted $\det(A)$, is given by

$$\det(A) = \det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = ad - bc.$$

- Suppose $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}$ (which could be functions) are entries of a $3 \times 3$ matrix $A$. The **determinant** of $A$, denoted $\det(A)$, is given by

$$\det(A) = \det\left(\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\right)$$

$$= a_{11}\left(a_{22}a_{33} - a_{32}a_{23}\right) = a_{21}\left(a_{21}a_{33} - a_{31}a_{23}\right) + a_{13}\left(a_{21}a_{32} - a_{31}a_{22}\right).$$

In the video below, we provide some useful comments about these formulas.

However, we are not able to define the Wronskian.

**Definition**

Suppose we can take at least $n-1$ derivatives of functions $y_1, y_2, \ldots, y_n$. The **Wronskian** of $y_1, y_2, \ldots, y_n$is the determinant

$$W\left(y_1, \ldots, y_n\right) = \det\left(\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}\right).$$

Again, we don't worry about such big matrices, and in computation will restrict ourselves to the $n = 2, 3$ cases typically.

Now, we get in to a bit of a subtlety using the Wronskian. In particular, there are two subtly different results one could look at concerning the Wronskian and linear independence. In particular,

- the linear independence of (just) functions, and
- the linear independent of solutions to a homogeneous linear differential equation.

It ends up that the result is stronger for solutions to linear differential equations. We discuss this more in the video below, but only list the result for *solutions* here.

**Theorem**

Let $y_1, y_2, \ldots, y_n$ be solutions to a homogeneous linear $n$th-order differential equation on an interval $I$. Then the solutions $y_1, y_2, \ldots, y_n$ are linearly independent if and only if $W(y_1, y_2, \ldots, y_n) \neq 0$ for every $x$ in $I$.

Just keep in mind. The result above (and in particular the 'if and only if' part) can be used so long as you know the functions are solutions to a homogeneous linear DE. If you only have functions (not solutions to a linear DE), then care must be taken.

## Discussion, comments, and examples:


## WeBWorK module 09 exercises:

- Problems 4, 5

## Relevant Wikipedia articles:

- **Determinant of matrices** **(https://en.wikipedia.org/wiki/Determinant)**
- **Wronskian** **(https://en.wikipedia.org/wiki/Wronskian)**
- **The Wronskian and linear independence** **(https://en.wikipedia.org/wiki/Wronskian#The_Wronskian_and_linear_independence)**