Please show and explain your work. Justification is necessary for credit. Also sign your name here acknowledging you will not use problem solving software (Wolfram Alpha, etc.): Matthew Mendez

1. (*14 points*)

a. (*2 pts*) Circle all of the following expression which are **linear** differential equations.

(i) ~~$y''' - 2y''' + 3y = 5$~~

(ii) ~~$(x^2 - \sin(x))\, y^{(5)} - xe^x y + \sin(x^2) = 0$~~

(iii) $\sqrt{y'} = x\cos(x)y - 3$

(iv) ~~$(y')^2 = y^3 + 2$~~

* Can't have 'n' be any higher power greater than one *

b. (*2 pts*) Circle all of the following expression of the form $M(x,y)\,dx + N(x,y)\,dy = 0$ such that **both** $M(x,y)$ and $N(x,y)$ are homogeneous *functions* of the **same** degree.

(i) $x^2\,dx + yx\,dy = 0$ $\rightarrow \frac{dy}{dx} = x^2$

(ii) $(3xy)\,dx + e^{\left(\frac{x}{y}\right)}\,dy = 0$

(iii) $\sin(y)\,dx + (2y + xe^y)\,dy = 0$

(iv) $(x - 2y)\,dx + x\,dy = 0$

c. (*4 pts*) Circle whether the following differential equation is an ordinary differential equation (ODE) or a partial differential equation (PDE) and state the **order** of the differential equations (you can do this for any PDEs as well).

(i) $\ln(y')y + x^2y = 1$ ODE *PDE* *Order:* 2

(ii) $\frac{\partial f}{\boxed{\partial x}} + \frac{\partial^2 f}{\boxed{\partial y^2}} + xy = 0$ ODE *PDE* *Order:* 1 least one multivariable function.

(iii) $x^2\frac{d^3 f}{dx^3} + \frac{df}{dx} = \frac{d^2 f}{dx^2} - \sin(x)$ ODE *PDE* *Order:* 2 least one function of one independent variable.

(iv) $\left(\frac{d}{dx}\right)^4 f(x) + \left(\frac{d^2}{dx^2}\right) f(x) + f'(x) = 0$ ODE *PDE* *Order:* 4 least one function of one independent variable.

d. (*4 pts*) Circle all of the following expression which are **exact** differential equations. (Provide work for partial credit!)

(i) ~~$\overset{M}{xy\,dx} + \overset{N}{xy\,dy} = 0$~~ X

$\frac{\partial}{\partial y} xy = x, \frac{\partial}{\partial x} xy = y$

(ii) $x^2y\,dx + \frac{1}{3}x^3\,dy = 0$

$\frac{\partial}{\partial y} x^2y = x^2 \quad \frac{\partial}{\partial x} x^2$ ✓

(iii) $\overset{M}{xy^2\,dx} + \overset{N}{x^2y\,dy} = 0$

$\frac{\partial}{\partial y} xy^2 = 2xy, \frac{\partial}{\partial x} x^2y = 2xy$ ✓

(iv) ~~$\sin(xy)\,dx + xy\cos(x)\,dy = 0$~~ NO

cos

e. (*2 pts*) Suppose $y^2 - x = 3$ is a solution to a differential equation. Is this solution implicit or explicit? Explain your answer in one sentence.

Explicit; for, you can solve for y: $y = \pm\sqrt{3+x}$

2. (*27 points*) Solve the following differential equations using any technique you like that works. However, **you must explain WHY you chose that method** to receive credit!

a. (*9 pts*) $y' = \frac{x}{y^2}$

$$\frac{dy}{dx} = \frac{x}{y^2}$$

$$\rightarrow y^2 dy = x\,dx$$

$$\rightarrow \int y^2 dy = \int x\,dx$$

$$\rightarrow \frac{1}{3}y^3 = \frac{1}{2}x^2 + C$$

$$\rightarrow y^3 = \frac{3}{2}x^2 + 3C$$

$$\therefore y = \sqrt[3]{\frac{3}{2}x^2 + 3C}$$

Why: seprable $\rightarrow \frac{dy}{dx} = y'$

b. (*9 pts*) $x\frac{dy}{dx} - 2y = x^3 e^x$

~~why: It looks separable~~

$$\cancel{\frac{x\frac{dy}{dx}}{x} - \frac{2y}{x} = \frac{x^3e^x}{x}}$$

$$\cancel{\rightarrow \frac{dy}{dx} - \frac{2y}{x} = x^2 e}$$

$$\cancel{\rightarrow \frac{dy}{dx} = \frac{2y}{x} + x^2 e}$$

$$\cancel{\rightarrow \frac{dy}{dx} = \frac{1}{x}2y + x^2 e}$$

$$\cancel{x\frac{dy}{dx} - 2y = x^3 e^x}$$

$$\cancel{+2y \quad +2y}$$

$$\cancel{\rightarrow x\frac{dy}{dx} = x^3e^x + 2y}$$

$$\cancel{-x^3e^x \quad -x^3e^x}$$

$$\cancel{\rightarrow x - x^3e^x\frac{dy}{dx} = 2y}$$

wrong approach

1st-order linear equation

why: it follows form "$\frac{dy}{dx} + Py = Q$"

$$\frac{x\frac{dy}{dx}}{x} - \frac{2y}{x} = \frac{x^3e^x}{x}$$

standard form

$$\rightarrow \frac{dy}{dx} - \frac{2y}{x} = \frac{x^3e^x}{x}$$

* can't have $x=0$ anymore *

$$\rightarrow \frac{dy}{dx} - \underbrace{\boxed{\frac{2}{x}}}_{P(x)}y = \underbrace{\boxed{\frac{x^3e^x}{x}}}_{Q(x)}$$

$$\rightarrow \frac{dy}{dx} - \underbrace{\boxed{\frac{2}{x}}}_{P(x)}y = \underbrace{\boxed{\frac{x^3e^x}{x}}}_{Q(x)}$$

I think this is where I'm confusing myself

$$\rightarrow \frac{dy}{dx} + \left(\underbrace{\boxed{-\frac{2}{x}}}_{P(x)}y\right) = \underbrace{\boxed{\frac{x^3e^x}{x}}}_{Q(x)}$$

Calc Integrating factor

$$e^{\int P dx} = e^{\int -\frac{2}{x}dx} = e^{-2\ln|x|}$$

$$\rightarrow e^{-2\ln|} = e^{\ln|x^{-2}|} = x^{-2}$$

Multiply left & Right by Integrating factor

$$\underbrace{x^{-2}\left(\frac{dy}{dx} - \frac{2y}{x}\right)}_{\text{Product Rule}} = x^{-2}\left(\frac{x^3e^x}{x}\right)$$

$$\rightarrow \frac{d}{dx}\left(x^{-2}y\right)$$

c. (*9 pts*) $\underbrace{2xy\,dx}_{M} + \underbrace{(x^2+3y^2)}_{N}\,dy = 0$

Exact equation?
why: follows form $M(x,y)dx + N(x,y)dy = 0$

Step 0

$\frac{\partial}{\partial y}(2xy) = 2x$

$\frac{\partial}{\partial x}(x^2+3y^2) = 2x$

Yes ✓

Step I Anti-deriv w/ respect to "x"

$F(x,y) = \int (xy)\,dx$

$\rightarrow \frac{1}{2}x^2y$

$\rightarrow \frac{1}{2}x^2y + g(y)$

Step 2 take partial deriv of F w/ y

$\frac{\partial}{\partial y}\left(\frac{1}{2}x^2y + g(y)\right) = \frac{1}{2}(x^2y^2) + g'(y)$

Step 3 set partial-deriv of F w/ "y" equal to $N(x,y)$ & solve for $g'(y)$

$\frac{\frac{1}{2}(x^2y^2) + g'(y)}{x^2y^2} = \frac{\cancel{x^2}}{\cancel{x^2}y^2} + \frac{3\cancel{y^2}}{x^2\cancel{y^2}}$

$\rightarrow 2 \cdot 1/2 + g'(y) = (y^{-2} + 3)\,2$

$\rightarrow g'(y) = 2(y^{-2} + 3)$

Step 4 Integrate $g'(y)$

$g'(y) = 2\left(\frac{1}{y^2}\right) + 3$

$\rightarrow g(y) = 2\ln|y^2| + C$

Step 5 Plug step 4 in step I anti-derive w/r to "x"

$F(x,y) = \frac{1}{2}x^2y + g(y)$

$\rightarrow \frac{1}{2}x^2y + 2\ln|y^2| + C$

Step 6 Thus we have In form $\boxed{f(x,y) = C}$

$C = \frac{1}{2}x^2y + 2\ln|y^2|$ * Implicit solution

3. (*9 points*)

a. (*8 pts*) We note that $y(x) = c_1e^{3x} + c_2e^{-3x}$ is a solution to the differential equation $y'' - 9y = 0$. Find a solution to the initial conditions $y(0) = 0$ and $y'(0) = 2$.

From Notes

$\boxed{\text{Ex}}$ (i) Verify that $y = K\cos(2x)$ satisfies the DE $y'' = 4y = 0$.

(ii) Find a solution for the IVP consisting of $y'' + 4y = 0$ and the conditions $y(0) = 3$, $y'(0) = 0$

$\boxed{\text{soln}}$

(i) $y = K\cos(2x)$, $y' = -2K\sin(2x)$, $y'' = -4K\cos(2x)$.
Thus, $y'' + 4y = -4K\cos(2x) + 4K\cos(2x) = 0$ ✓

(ii) $y(0) = 3$ $\rightarrow K\cos(0) = 3$ $\rightarrow K = 3$

$y'(0) = 0$ $\rightarrow -2K\sin(0) = 0$ $\rightarrow 0 = 0$

Thus, $y = 3\cos(2x)$ works

Chain Rule $e^x \cdot \boxed{X}$

Blanking out for some reason But it looks like it checks-out ✓

$y = c_1e^{3x} + c_2e^{-3x}$

$y' = (c_1e^{3x} \cdot 3) + (c_2e^{-3x} \cdot -3)$

$y'' =$

$y(0) = 0$

$\rightarrow c_1e^{3(0)} + c_2e^{-3(0)} = 0$

$\rightarrow c_1(1) + c_2(1) = 0$

$\rightarrow c = 0$

✓

b. (*1 pt*) Does the trivial solution also satisfy the initial value problem given above in **3a**? Explain your answer.

Yes

The Trival Solution
Definition: If the function $y = 0$ is a solution on an interval, we can call this the Trival Solution.

	Solution to a DE	Particular (one solution)	Parameter	Could it be general	trival
$y = Ce^{2x}$	✗	✓- one para	✓- (?)		✗
$y = 3\cos(4x)$	✓	✗	✗- (?)		✗
$y = 0$	✓	✗	✗- (?)		✓