MATH 45 – Exam One Review Solutions

Dr. Krauel

---

1. Rework, study, and understand all of the homework and quiz problems.

2.

(a) No.

(b) Yes.

(c) No.

(d) Yes.

3.

(a) ODE, linear, separable, 1st order.

(b) ODE, linear, separable, 1st order.

(c) PDE, linear, not separable, 2nd order.

(d) ODE, linear, 2nd order.

(e) ODE, linear, 3rd order.

(f) ODE, non-linear, separable, 1st order.

(g) ODE, non-linear, 1st order.

(h) ODE, linear, 1st order.

(i) ODE, linear, separable, 1st order.

(j) PDE, linear.

4.

(a) No.

(b) Yes.

(c) No.

(d) Yes.

5. $y(x) = 1$ is a solution. $y(x) = 1 + x^2$ is not a solution. $y(x) = 1 - x^2$ is not a solution (however, it can be shown that this one is a solution to the differential equation $(y')^2+4y-4 = 0$.

6.

(a) Using that $y' = -Ce^{-x}$ we find that $y' + y = 0$. Plugging in $x = 0$ gives $3 = y(0) = C$, so $C = 3$. Thus, $y = 3e^{-x}$ is the solution to the IVP.

(b) Using that $y' = -2xCe^{-x^2}$ we find that $y' + 2xy = 0$. Plugging in $x = 0$ gives $-1 = y(0) = C$, so $C = -1$. Thus, $y = -e^{-x^2}$ is the solution to the IVP.

(c) We have $y' = -Ce^{-x} + 1$ and $x - y = x - Ce^{-x} - x + 1$. Plugging in $x = 0$ gives $1 = y(0) = C - 1$ so that $C = 2$. Thus, $y = 2e^{-x} + x - 1$ is the solution to the IVP.

7. $f(t) = \frac{1}{60}e^{-4t} + \frac{7}{6}e^{3t} - \frac{1}{10}e^{t} - \frac{1}{6}e^{2t} + \frac{1}{12}$: For starters, we have $f(t) = c_1e^{-4t} + c_2e^{3t} - \frac{1}{10}e^{t} - \frac{1}{6}e^{2t} + \frac{1}{12}$ and $f(0) = 1$. Thus,

$$c_1 + c_2 = 1 + \frac{1}{10} + \frac{1}{6} - \frac{1}{12} = \frac{71}{60}.$$

On the other hand, we have $f'(t) = -4c_1e^{-4t} + 3c_2e^{3t} - \frac{1}{10}e^{t} - \frac{1}{3}e^{2t}$ and $f'(0) = 3$. Thus,

$$-4c_1 + 3c_2 - \frac{1}{10} - \frac{1}{3} = 3,$$

or

$$-4c_1 + 3c_2 = \frac{206}{60}.$$

Then solving

$$c_1 + c_2 = \frac{71}{60} \qquad -4c_1 + 3c_2 \qquad\qquad = \frac{103}{50}$$

gives $c_1 = \frac{1}{60}$ and $c_2 = \frac{7}{6}$. Thus, the answer is $f(t) = \frac{1}{60}e^{-4t} + \frac{7}{6}e^{3t} - \frac{1}{10}e^{t} - \frac{1}{6}e^{2t} + \frac{1}{12}$.

8.

(a) $y = -\frac{1}{\cos(x)+C}$: We can rewrite this giving $\frac{dy}{dx} = -y^2\sin(x)$. Separating variables we get $\frac{dy}{y^2} = -\sin(x)\,dx$. Integrating gives $-\frac{1}{y} = \cos(x) + C$. Solving for y we find $y = -\frac{1}{\cos(x)+C}$.

(b) $y = \pm\sqrt{\frac{2}{3}\ln|1+x^3| + C}$: Separating variables we get $y\,dy = \frac{x^2}{1+x^3}\,dx$. Integrating gives $\frac{y^2}{2} = \frac{1}{3}\ln|1+x^3| + C$. This is the solution in implicit form. Solving for y we find $y = \pm\sqrt{\frac{2}{3}\ln|1+x^3| + C}$, which is the explicit form of the solution.

9. Find the solution to the given initial value problem.

(a) $y = \frac{-1}{x-x^2+6}$: Separating variables we get $\frac{dy}{y^2} = (1-2x)\,dx$. Integrating gives $-\frac{1}{y} = x - x^2 + C$. Solving for y we find $y = -\frac{1}{x-x^2+C}$. Next, we plug in the initial conditions to find $-\frac{1}{6} = y(0) = \frac{-1}{0-0^2+C} = -\frac{1}{C}$, so that $C = 6$.

(b) $\frac{dy}{dt} = e^{t+y}$, $y(0) = 0$. First we separate the variable to get $e^{-y}\,dy = e^t dt$. Integrating both sides we find $-e^{-y} = e^t + C$. Solving for y we get $e^{-y} = -e^t + C$ or $-y = \ln(-e^t + C)$, so that $y = -\ln(C - e^t)$. Now we use the initial condition to solve for C. In fact, we could have done this at any state, and it is easier to use the earlier step where we had $e^{-y} = -e^t + C$. Taking $t = 0$ and $y = 0$ here we find $e^0 = -e^0 + C$ so that $C = 2$. Thus, the solution to the IVP is $y = -\ln(2 - e^t)$.

10. Separating the equation we find $y\,dy = x^2\,dx$. Integrating gives $\frac{y^2}{2} = \frac{1}{3}x^3 + C$. This is the solution in implicit form. Solving for y gives the explicit solutions $y = \pm\sqrt{\frac{2}{3}x^3 + C}$.

11. $y^2 - 5y = x^3 - e^x - 3$: Separating the equation we find $(2y-5)\,dy = (3x^2 - e^x)\,dx$. Integrating gives $y^2 - 5y = x^3 - e^x + C$. Taking $x = 0$ and $y = 1$ gives, $1 - 5 = 0^3 - e^0 + C$, so that $C = -3$. Thus the implicit form of the solution to the IVP is $y^2 - 5y = x^3 - e^x - 3$. We include an attempt to solve for y explicitly to serve as an example, in case one wanted an explicit solution (where we would then need to restrict the domain). We find by completing the square that

$$\left(y - \frac{5}{2}\right)^2 - \frac{25}{2} = x^3 - e^x - \frac{6}{2},$$

or

$$y = \pm\sqrt{x^3 - e^x + \frac{19}{2}} + \frac{5}{2}.$$

12.

(a) We have $f(x,y) = x\ln(y)$ and $\frac{\partial}{\partial y} f(x,y) = \frac{x}{y}$. The function $x\ln(y)$ is continuous so long as $y > 0$ and for all x. Meanwhile $\frac{x}{y}$ is continuous for all x and $y \neq 0$. Thus, both functions are continuous about the point $(1,1)$ showing that there is a unique solution at this point. However, since the functions are not continuous at $y = 0$ we are not guaranteed by the theorem that there is a unique solution at $(1,0)$.

(b) We have $f(x,y) = \frac{x-1}{y}$ and $\frac{\partial}{\partial y} f(x,y) = -\frac{(x-1)}{y^2}$. The function $\frac{x-1}{y}$ is continuous so long as $y \neq 0$ and for all x. Meanwhile $-\frac{(x-1)}{y^2}$ is continuous for all x and $y \neq 0$. Thus, both functions are continuous about the point $(0,1)$ showing that there is a unique solution at this point. However, since the functions are not continuous at $y = 0$ we are not guaranteed by the theorem that there is a unique solution at $(1,0)$.

13. We begin by rewriting the differential equation as

$$y\cos(y)\,dy = \frac{e^x}{1+e^x}\,dx.$$

Integrating both sides we have

$$\int y\cos(y)\,dy = \int \frac{e^x}{1+e^x}\,dx. \tag{1}$$

We will do each of these integrals independently. For the integral on the left, we note that integration by parts works. Taking $u = y$ and $dv = \cos(y)\,dy$, we have $du = dy$ and $v = \sin(y)$. Then integration by parts gives

$$\begin{aligned}\int y\cos(y)\,dy = uv - \int v\,du &= y\sin(y) - \int \sin(y)\,dy \\ &= y\sin(y) + \cos(y) + C.\end{aligned}$$

We turn our attention to the integral on the right side above. Using the u-substitution $u = 1 + e^x$ we have $du = e^x\,dx$ so that

$$\int \frac{e^x}{1+e^x}\,dx = \int \frac{1}{u}\,du = \ln(u) + C = \ln(1+e^x) + C.$$

Therefore, Equation (1) above becomes (combining the C values to one new C)

$$y\sin(y) + \cos(y) = \ln(1+e^x) + C.$$

We cannot solve for y here. Thus, the solution must be left as an *implicit* solution. This is a 1-parameter family of solutions for the differential equation.

We now want to find a solution to the initial value problem. Taking $x = 0$ and $y = \frac{\pi}{2}$ we have

$$\frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right) = \ln\left(1 + e^0\right) + C,$$

which becomes $\frac{\pi}{2} = \ln(2) + C$. Solving for C gives $C = \frac{\pi}{2} - \ln(2)$. Thus, a solution for the IVP is $y\sin(y) + \cos(y) = \ln(1+e^x) + \left(\frac{\pi}{2} - \ln(2)\right)$.

14. As a separable equation we rewrite the differential equation as $\frac{dy}{dx} = x(1-y)$ or $\frac{1}{1-y}\,dy = x\,dx$. Integrating both sides gives $-\ln(1-y) = \frac{1}{2}x^2 + C$. Solving for y we find $\ln(1-y) = -\frac{1}{2}x^2 + C$, so that $1 - y = e^{-\frac{1}{2}x^2+C}$. Thus,

$$y = 1 - e^{-\frac{1}{2}x^2+C} = 1 - Ce^{-\frac{1}{2}x^2}.$$

(Note we kept using C to denote a different constant at each step.)

Now we solve this differential equation using the theory of linear equations. First, we place it into the form

$$\frac{dy}{dx} + xy = x. \tag{2}$$

We want to multiply this equation by the 'integrating factor' which will allow us to nicely rewrite the left side of 2. The integrating factor is given by $e^{\int x\,dx} = e^{\frac{1}{2}x^2}$. (Recall this is from $e^{\int P(x)\,dx}$, where $P(x)$ is the coefficient of y in the equation 2.) Multiplying 2 by the integrating factor gives

$$e^{\frac{1}{2}x^2}\left(\frac{dy}{dx} + xy\right) = e^{\frac{1}{2}x^2}(x).$$

The whole point in this integrating factor function is that the left side of this last equation becomes $e^{\frac{1}{2}x^2}\left(\frac{dy}{dx} + xy\right) = \frac{d}{dx}\left(e^{\frac{1}{2}x^2}y\right)$. Thus, the previous display becomes

$$\frac{d}{dx}\left(e^{\frac{1}{2}x^2}y\right) = e^{\frac{1}{2}x^2}(x).$$

We are now able to integrate both sides to find

$$e^{\frac{1}{2}x^2}y = \int xe^{\frac{1}{2}x^2}\,dx.$$

That is, (using for example a u-sub with $u = \frac{1}{2}x^2$)

$$e^{\frac{1}{2}x^2}y = e^{\frac{1}{2}x^2} + C.$$

Thus,

$$y = \frac{e^{\frac{1}{2}x^2}}{e^{\frac{1}{2}x^2}} + \frac{C}{e^{\frac{1}{2}x^2}} = 1 + Ce^{-\frac{1}{2}x^2}.$$

Note that this constant is arbitrary, so we could easily replace it with $-C$ to get the same expression we got in the previous method (or turn that one to $+C$).

15. $y = t + \frac{C}{t}$: We begin by putting it in the form

$$\frac{dy}{dt} + \frac{1}{t}y = 2.$$

Our integrating factor is (remembering we don't need the constant from integrating here)

$$e^{\int \frac{1}{t}\,dt} = e^{\ln(t)} = t.$$

Multiplying our equation through with the integrating factor gives

$$e^{\int \frac{1}{t}\,dt}\left(\frac{dy}{dt}+\frac{1}{t}y\right)=e^{\int \frac{1}{t}\,dt}(2)$$

or

$$t\left(\frac{dy}{dt}+\frac{1}{t}y\right)=2t,$$

which can be written as

$$t\left(\frac{dy}{dt}\right)+(1)y=2t.$$

Knowing the left side is equal to $\frac{d}{dt}(ty)$ we have

$$\frac{d}{dt}(ty)=2t.$$

Integrating both sides gives $ty=t^2+C$. Thus, $y=t+\frac{C}{t}$ is the solution.

16.

(a) Exact: We note that $\frac{d}{dy}(-4xy^2+y)=-8xy+1=\frac{d}{dx}(-4x^2y+x)$.

(b) Exact: We first rewrite this as $(4e^x\sin(y)-3y)\,dx+(-3x+4e^x\cos(y))\,dy=0$. Next note that $\frac{d}{dy}(4e^x\sin(y)-3y)=4e^x\cos(y)-3=\frac{d}{dx}(-3x+4e^x\cos(y))$.

(c) Not exact: We have $\frac{d}{dy}(y^2)=2y\neq 2x=\frac{d}{dx}(x^2)$.

17.

(a) $C=-2x^2y^2+xy+k$: We know from above this is an exact equation. We could integrate either $M(x,y)$ with respect to x or $N(x,y)$ with respect to y. We do the former and find

$$f(x,y)=\int(-4xy^2+y)\,dx=-2x^2y^2+xy+g(y),$$

where $g(y)$ is some function of y. For $f(x,y)$ to be the solution to the differential equation, we must have $\frac{d}{dy}f(x,y)=-4x^2y+x$ (since exact equations satisfy $\frac{d}{dx}f(x,y)=M(x,y)$ and $\frac{d}{dy}f(x,y)=N(x,y)$). from our calculation above we find

$$\frac{d}{dy}\left(-2x^2y^2+xy+g(y)\right)=-4x^2y+x+g'(y).$$

Therefore, for $-4x^2y+x+g'(y)$ to equal $N(x,y)$ we must have $g'(y)=0$. This implies that $g(y)$ is a constant k. So from above we have $f(x,y)=-2x^2y^2+xy+k$. Technically, however, the solution to such a differential equation is this expression set equal to a constant. That is, $C=f(x,y)$, or $C=-2x^2y^2+xy+k$. Combining the constants, we write $C=-2x^2y^2+xy$. (One can then apply implicit differentiation to confirm this satisfies the differential equation).

(b) We know from above this is an exact equation. We could integrate either $M(x,y)$ with respect to x or $N(x,y)$ with respect to y. We do the latter and find

$$f(x,y) = \int (-3x + 4e^x \cos(y))\, dy = -3xy + 4e^x \sin(y) + g(x),$$

where $g(x)$ is some function of x. For $f(x,y)$ to be the solution to the differential equation, we must have $\frac{d}{dx} f(x,y) = 4e^x \sin(y) - 3y$ (since exact equations satisfy $\frac{d}{dx} f(x,y) = M(x,y)$ and $\frac{d}{dy} f(x,y) = N(x,y)$). from our calculation above we find

$$\frac{d}{dx}\left(-3xy + 4e^x \sin(y) + g(x)\right) = -3y + 4e^x \sin(y) + g'(x).$$

Therefore, for $-3y + 4e^x \sin(y) + g'(x)$ to equal $M(x,y)$ we must have $g'(x) = 0$. This implies that $g(x)$ is a constant k. So from above we have $f(x,y) = -3xy + 4e^x \sin(y) + k$. Technically, however, the solution to such a differential equation is this expression set equal to a constant. That is, $C = f(x,y)$, or $C = -3xy + 4e^x \sin(y) + k$. Combining the constants, we write $C = -3xy + 4e^x \sin(y)$. (One can then apply implicit differentiation to confirm this satisfies the differential equation).

(c) Not exact.

18.

(a) Plugging in tx for x and ty for y we find $(tx + ty) = t(x+y)$ and $(tx) = t(x)$. Thus, both $M(x,y)$ and $N(x,y)$ are homogeneous functions of degree 1.

(b) Plugging in tx for x and ty for y we find $(ty)^2 = t^2y^2$ and $(tx)^2(ty) = t^3(x^2y)$. Thus, $M(x,y)$ and $N(x,y)$ are homogeneous functions of degree 2 and 3, respectively. Since the degrees are different, the differential equation is not homogeneous.

(c) Plugging in tx for x and ty for y we find $(ty) = t(y)$ so that $M(x,y)$ is a homogeneous function of degree 1. That $N(x,y)$ is a homogeneous function is much more unclear. However, if we note that

$$\ln(x) - \ln(y) = \ln\left(\frac{x}{y}\right)$$

then we find

$$\begin{aligned}(tx)(\ln(tx) - \ln(ty) - 1) &= (tx)\left(\ln\left(\frac{tx}{ty}\right) - 1\right) = tx\left(\ln\left(\frac{x}{y}\right) - 1\right)\\ &= t\left(x(\ln(x) - \ln(y) - 1)\right)\end{aligned}$$

showing $N(x,y)$ is a homogeneous function of degree 1. Thus, both $M(x,y)$ and $N(x,y)$ are homogeneous functions of degree 1.

(d) Neither e^x or $-e^y$ are homogeneous functions since $e^{tx} \neq te^x$ and $-e^{ty} \neq -te^y$.

19.

(a) $y = \frac{Cx^3 - x}{2}$: We rewrite this to be $(x+y) + x\frac{dy}{dx} = 0$. Let $y = ux$ or $x = vy$. One can usually use either, however, it is typically easier to use $y = ux$ if $N(x, y)$ is simpler and $y = vx$ is $M(x, y)$ is Simpler. In this problem, since $N(x, y)$ is simpler than $M(x, y)$ we use $y = ux$. The product rule shows that $\frac{dy}{dx} = x\frac{du}{dx} + u$. Plugging this into the differential equation we have

$$(x + ux) + x\left(x\frac{du}{dx} + u\right) = 0.$$

We simplify this to the form $x^2\frac{du}{dx} = -x(1+2u)$. Separating variables gives $\frac{1}{1+2u}\,du = -\frac{1}{x}\,dx$. Integrating gives $\frac{1}{2}\ln(1+2u) = -\ln(x)+c$ which can be rewritten as $\ln(\sqrt{1+2u}) = \ln(x^{-1}) + c$. Exponentiating gives $\sqrt{1+2u} = e^{\ln(x^{-1})+c} = Cx^{-1} = \frac{C}{x}$. Thus, $1 + 2u = Cx^{-2}$. Using that $y = ux$ we substitute in $u = \frac{y}{x}$ to find $1 + 2\frac{y}{x} = Cx^{-2}$. Solving for y gives $y = \frac{C}{x} - \frac{x}{2}$ (where we redefined C yet again).

(b) Not of same degree. However, one could rewrite this as $x^2y\,dy = -x^2\,dx$ which is $y\,dy = -dx$. Thus, separation of variables shows that $\frac{1}{2}y^2 = -x + c$ is an implicit solution.

(c) $y\ln(\frac{x}{y}) = -e$: Let $y = ux$ or $x = vy$. In this case, since $M(x, y)$ is simpler than $N(x, y)$ we use $x = vy$. In this case we can use the product rule to find that $\frac{dx}{dy} = y\frac{dv}{dx} + v$. Then rewriting the differential equation as (where we also use $\ln(x) - \ln(y) = \ln(\frac{x}{y})$)

$$y\frac{dx}{dy} + x\left(\ln\left(\frac{x}{y}\right) - 1\right) = 0$$

and plugging in $x = vy$ we get

$$y\left(y\frac{dv}{dy} + v\right) + yv\left(\ln(v) - 1\right) = 0.$$

Simplifying this becomes $y^2\frac{dv}{dy} + yv\ln(v) = 0$. Separating variables we find $\frac{1}{v\ln(v)}\,dv = -\frac{1}{y}\ ,dy$. Integrating gives $\ln(\ln(v)) = -\ln(y) + c = \ln(y^{-1}) + c$. Thus, $\ln(v) = e^{\ln(y^{-1})+c} = Ce^{\ln(y^{-1})} = Cy^{-1} = \frac{C}{y}$. Therefore, $v = e^{\frac{C}{y}}$. Using $x = vy$ so that $v = \frac{x}{y}$ this becomes $\frac{x}{y} = e^{\frac{C}{y}}$. We could also write this as $y\ln(\frac{x}{y}) = C$. In either case, taking $x = 1$ and $y = e$ from the initial conditions gives $e\ln(\frac{1}{e}) = C$. That is $C = e(\ln(1) - \ln(e)) = e(0-1) = -e$. Therefore, $y\ln(\frac{x}{y}) = -e$ is our final solution.

(d) Not homogeneous. However, separation of variables leads to $e^y = e^x + c$ so that $y = \ln(e^x + c)$.

20. Explain your answer to the following questions.

(a) See the definitions.

(b) No. Only $y = 0$ can be the trivial solution (if it is a solution, that is).

(c) Suppose $f(x) = \sqrt{x^2 - 1}$ satisfies a differential equation. Can we say $f(x) = \sqrt{x^2 - 1}$ is a solution to the differential equation? We need to specify an interval. The function $f(x)$ is only defined on $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$. To be a solution to a differential equation, we must be able to take its derivative (since a differential equation has derivatives of this function). A differentiable function must be defined (and continuous on an interval about) such places. Thus, this function is only a solution on the intervals $(-\infty, -1)$ or $(-1, 1)$ or $(1, \infty)$ (and not all at once on their union).