Goal and idea - Module 13

GOAL:

In the last module, we learned how to solve *homogeneous* linear differential equations with constant coefficients. We turn to finding solutions for such *nonhomogeneous* differential equations. However, we won't be able to solve all such equations. In this module, we

- recall nonhomogeneous linear differential equations with constant coefficients; and
- learn to solve such differential equations in the case the nonhomogeneous portion consists of
  - polynomials,
  - $\sin(x), \cos(x)$, and
  - terms of $x^k e^{mx}$.

IDEA:

We take the needed particular solution to be a linear combination of the terms above, plug into the differential equation, and find the appropriate constants.

Step 1 | Find Y_h

Then we consider three cases where we can find Y_p:

Case 1) f(x) is a polynomial

Since order 2 DE

Step 1 | Then take $y_p = Ax^2 + Bx + C$, Undetermined cofficients
where A, B, C are constants

Step 2 | Plug-in this y_p (need derivatives)

Step 3 | Solve for A, B, C

The idea is we are taking a guess as we are using the fact that f(x) is of a polynomial then the particular solution is probably itself going to be some polynomial type situation. In which case you can plug it in and just figure out what are the coefficients of that polynomial.

If your y_h contains some polynomial terms (say y_h has an x^2) then you can actually omit the x^2 in Ax^2+Bx+C and just consider the remaining aspects, but it wouldn't hurt you to include it either.

Example | Solve $Y''+3Y'+2Y=2x^2+2x-6$

When we are given homogeneous we notice a polynomial, So undermined coefficents might be a great approach.

Solution

1 | Take $y_p = Ax^2+Bx+C$

2 | Then we need our derivatives because we need to plug this in

$$y_p = Ax^2+Bx+C$$
$$y_p' = 2Ax+B$$
$$y_p'' = 2A$$

Lets first consider the homogeneous part
General solution for $Y''+3Y'+2Y=0$
1) Plug-in $y=e^{mx}$
2) Find m
is
$$Y_h = C_1e^{-x}+C_2e^{-2x}$$

Gives:
$$Y''+3Y'+2Y=2x^2+2x-6$$
$$\rightarrow 2A+3(2Ax+B)+2(Ax^2+Bx+C)=2x^2+2x-6$$

The idea is that now we have a polynomial on both the left and right side, and we need to equate the coefficents allowing us to solve for A, B, and C.

So what ends up happening here is that we can regroup all "x^2" terms, "x" terms, and then all of the constants together.

$$2A+3(2Ax+B)+2(Ax^2+Bx+C) = 2x^2+2x-6$$
$$\rightarrow 2Ax^2+(6A+2B)x+(2A+3B+2C) = 2x^2+2x-6$$

3 | Solve for A, B, C : then what we need
x^2(LHS=RHS), x(LHS=RHS), constants(LHS=RHS)

$$2Ax^2+(6A+2B)x+(2A+3B+2C) = 2x^2+2x-6$$
$$\rightarrow 2A=2,\quad 6A+2B=2,\quad 2A+3B+2C=-6$$

Thus, $A=1$, $B=-2$, $C=-1$
$$y_p = Ax^2+Bx+C = x^2-2x-1$$

So the general solution is...
$$y = C_1e^{-x}+C_2e^{-2x}+x^2-2x-1$$

Recall - Module10

General Solutions for Nonhomogeneous Linear Differential Equations

Given (a, b, c are constants)

$$(*)\quad ay''+by'+cy=f(x)$$

with $f(x)\neq 0$, the general solution is of the form

$$y = y_h + y_p$$

Where

🙂 → y_h is the general solution of $ay''+by'+cy=0$, and

We are familiar with finding the general solution of the homogeneous piece

This is true for higer order as well

??? → y_p is a particular solution of (*)

Case 2) f(x) consists of sin(x) and/or cos(x) terms.

Any sines or cosines (not products of them i.e. $\sin^2$ or $\cos^2$)

Step 1 | Then consider $Y_p = A\cos(x) + B\sin(x)$

Step 2 | Plug-in this y_p (need derivatives)

Step 3 | Solve for A, B, C

The idea is when f(x) has some particular form what are we guessing? We don't know how much they are going to occur, but the reason for this is that the derivative of cosine or sine is some cosine or sine and then its derivative are cosines or sines (its plus or minus cosines and sines).

So really the question is: If you have a cosine or sine ocurring in that f(x) term it means its derivatives of themselves, but how many of them do you need? And that's where the A&B come in.

Case 3) f(x) has a x^ke^{mx} form

K&m are integers, but k is a non-negative integer

Step 1 | Then take all derivatives of x^ke^{mx}

Take the derivative, take the derivative of what you just gotten, the derivative of that... If it's a 2^{nd}-order differential equation then you'd want to take that many derivatives and see what pops up in this process. At some point there's no new terms being introduced. and then include (with unknown constants) in the Y_p.

Step 2 | Plug-in this y_p (need derivatives)

Step 3 | Solve for A, B, C

Case 3) f(x) has a $x^k e^{mx}$ form

k & m are integers, but k is a non-negative integer

Step 1 Then take all derivatives of $x^k e^{mx}$, and include (with unknown constants) in the y_p.

Step 2 Plug-in this y_p (need derivatives)

Step 3 Solve for A, B, C

Example If $f(x) = x^2 e^{2x}$, then derivates include:

Step 1 Then take all derivatives of $x^k e^{mx}$, and include (with unknown constants) in the y_p

$$f(x) = x^2 e^{2x}$$

Recall Product Rule

$$\boxed{(uv)' = u'v + uv'}$$

$$\rightarrow \underbrace{2xe^{2x},\ 2x^2e^{2x}}_{}, \ 2e^{2x}$$

When finding the derivatives we don't care about the constants

Technically, we are looking for linearly independent terms

So...The y_p we would consider is

$$y_p = Ax^2e^{2x} + Bxe^{2x} + Ce^{2x}$$

Notice: You take the first term and you take the decending ones as you the powers of x lower & lower

Note: We create/write out the actual honest derivatives we made all the various terms that occur as we take the derivatives, the derivatives of the derivatives, and so on..

The nice thing is when you've done it a couple of times with $x^k e^{mx}$ terms you'll realize what those terms always are: $e^{2x}, xe^{2x}, x^2e^{2x}$, and all the way up to x^k

Case 4) Combine all of the above

Ex Solve $y'' + 3y' + 2y = 2x^2 + 2x - 6 + 2\cos(x) + 3xe^{2x}$

$$\underbrace{y'' + 3y' + 2y}_{\text{Regular Homogeneous *Not interesting*}} = \underbrace{\underbrace{2x^2 + 2x - 6}_{\text{Case 1 Polynomial}} + \underbrace{2\cos(x)}_{\text{Case 2 Sine or cosine Present}} + \underbrace{3xe^{2x}}_{\text{Case 3 } x^k e^{mx} \text{ form}}}_{\text{* Is interesting *}}$$

Soln

Lets first consider the homogeneous part

General solution for

$$y'' + 3y' + 2y = 0$$

1) Plug-in $y = e^{mx}$
2) Find m

is

$$y_h = C_1 e^{-x} + C_2 e^{-2x}$$

Our y_p will have the form:

$$y_p = y_{p_1} + y_{p_2} + y_{p_3}$$

$$y_{p_1} = Ax^2 + Bx + C$$

$$y_{p_2} = A\cos(x) + B\sin(x)$$

$$y_{p_3} = Axe^{2x} + Be^{2x}$$

Then can compute eachone independently (then add)

y_{p_1}

Independently: We plug-in the left and then comparing it to the polynomial peice.

$$y_{p_1} = C_1e^{-x} + C_2e^{-2x} + x^2 - 2x - 1$$

y_{p_2}

$$y'' + 3y' + 2y = 2x^2 + 2x - 6 + 2\cos(x) + 3xe^{2x}$$

$y_{p_2} = A\cos(x) + B\sin(x)$

$y_{p_2}' = -A\sin(x) + B\cos(x)$

$y_{p_2}'' = -A\cos(x) - B\sin(x)$,

So $y'' + 3y' + 2y$

$\rightarrow -A\cos(x) - B\sin(x) + 3(-A\sin(x) + B\cos(x)) + 2(A\cos(x) + B\sin(x)) = 2\cos(x)$

Cosine (LHS = RHS) (Add up all cosine terms), Sine (LHS = RHS) (Add up all sine terms)

$\rightarrow A + 3B = 2$, $-3A + B = 0$

$\rightarrow A = 1/5$, $B = 3/5$

Therefore - $y_{p_2} = \frac{1}{5}\cos(x) + \frac{3}{5}\sin$

y_{p_3}

$y_{p_3} = Axe^{2x} + Be^{2x}$

$y_{p_3}' = 2Axe^{2x} + (2B + A)e^{2x}$

$y_{p_3}'' = 4Axe^{2x} + (4A + 4B)e^{2x}$

Plugging in gives

$$\left.\begin{array}{l} 12A = 3 \\ 7A + 12B = 0 \end{array}\right\} \begin{array}{l} A = 1/4 \\ B = 7/48 \end{array}$$

So, $y_{p_3} = \frac{1}{4}xe^{2x} - \frac{7}{48}e^{2x}$

Thus the general solution is

$$\boxed{y = C_1e^{-x} + C_2e^{-2x} + x^2 - 2x - 1 + \frac{1}{5}\cos(x) + \frac{3}{5}\sin + \frac{1}{4}xe^{2x} - \frac{7}{48}e^{2x}}$$

Expectation checklist - Module 13

At the completion of this module, you should:

- Be able to find the general solution for 2nd-order homogeneous linear differential equations with constant coefficients such that the nonhomogeneous term consists of:
  - a polynomial of degree k;
  - a sum of $\sin(x)$ and/or $\cos(x)$;
  - of the form $x^k e^{mx}$; or
  - a combination of the above terms.

Coming up next, we:

- Solve some nonhomogeneous linear differential equations of a different form, and with a different method, than those seen here.