Goal and idea - Module 4

GOAL:

We have turned a page, and began to learn how to solve many 1st-order differential equations. We begin with a class of 1st-order differential equations called separable equations. Our goal is to learn how to find solutions for such differential equations.

To do so, we:

- Define what separable equations are.
- Discuss how to solve such differential equations.
- Briefly review and discuss antiderivatives.

IDEA:

We will find that separable equations are differential equations, in which we can separate the variables in such a way that we are able to solve the differential equation by integrating. The advantage is that solving such differential equations is as simple as integrating! The disadvantage is, we must recall all of our integration techniques!

- Actually solving differential equations
- Before we were
  - Given a solution and verifying

Separable Equations

Separable Equations

A 1st-order DE which can be expressed in the form $\frac{dy}{dx}=g(x)h(y)$ is called "separable"

Note

If a differential equation is separable, we can put all "x-stuff" on one side of the equation, and "y-stuff" on the other.

* Interchangeable: $\frac{dy}{dx}=y'$ *

(a) $\frac{dy}{dx}=xe^y$

This is "nice" because it is writen in the same form of the definition $\frac{dy}{dx}=g(x)h(x)$, so there is no need to modify.

$\frac{dy}{dx}=\underbrace{x}_{g(x)}\cdot\underbrace{e^y}_{h(y)}$ is separable $\underbrace{\frac{1}{e^y}dy}_{\text{all y-stuff on one side}}=\underbrace{xdx}_{\text{all x-stuff on one side}}$

(b) $e^x\frac{dy}{dx}-y=0$

Manipulating this equation

$e^x\frac{dy}{dx}-y=0$

$\rightarrow e^x\frac{dy}{dx}=y$

$\rightarrow \frac{dy}{dx}=\underbrace{y}_{h(y)}\underbrace{e^{-x}}_{g(x)}$

is separable

$\underbrace{\frac{1}{y}dy}_{\text{all y-stuff on one side}}=\underbrace{e^{-x}dx}_{\text{all x-stuff on one side}}$

(c) $\frac{dy}{dx}=x+y+\cos(x)$

No, you can not separate. When you try to

$\frac{dy}{dx}=x+y+\cos(x)$

$\rightarrow \frac{dy}{dx}=(x+\cos(x))+y$

Although you try to separate the 'x' and 'y' it is hard to prove or guarantee that it is not separable.

Question:

Can you isolate the "x" terms. Is it possible to gather all of the "x" stuff on the left and all of the "y" stuff on the right?

Answer:

No, because when you multiply both sides by dx you are going to get $(x+\cos(x))dx + ydx$ and you won't be able to separate.

Not a separable equation.

Another example of this is : $y'=e^{xy}$ ← Not Separable

Concider the following steps:

Given $\frac{dy}{dx} = g(x)h(y)$

Step 1 Treating dy, dx as "y" and "x" terms put all "y-stuff" on the left and "x-stuff" on the right.

$$* \quad \frac{1}{h(y)}dy = g(x)dx$$

Step 2 Integrate both sides of $\frac{1}{h(y)}dy = g(x)dx$:

$\rightarrow \int \frac{1}{h(y)}dy = \int g(x)dx$

$\rightarrow H(y) + C_1 = G(x) + C_2$

*Note: Here $H(y)$ is an antiderivative of $\frac{1}{h(y)}$

*C_1, C_2 are arbitrary constants

$\rightarrow$ Need only one constant:

$H(y) = G(x) + C$ (C is a constant)

Step 3 Solve for "y" (if possible)

Example 1 Solve $\frac{dy}{dx} = 7y$

For starters ask: Is this seprable?

Is this even a seprable equation

- We see a function of 'y': $7y$
- What about the function of 'x'?
  - 7 is just a constant it's a function of 'x'.

$\frac{dy}{dx} = 7y$ ← still has a $g(x)$ Say $g(x) = 1$.
A constant eventhough it is a function of 'x' it does not have to have a 'x'

Soln

Step#1 Get all of our x-stuff on one side and our y-stuff on another

$$\frac{dy}{dx} = 7y$$

$$\rightarrow \left(\frac{1}{y}\right)\frac{dy}{dx} = 7\cancel{y}\cancel{\left(\frac{1}{y}\right)}$$

$$\rightarrow \cancel{dx}\left(\frac{1}{y}\right)\frac{dy}{\cancel{dx}} = 7\,dx$$

$$\rightarrow \left(\frac{1}{y}\right)dy = 7\,dx$$

Step#2 Integrate the separated equation

Find the antiderivative

$$\left(\frac{1}{y}\right)dy = 7\,dx$$

$$\rightarrow \int \frac{1}{y}dy = \int 7\,dx$$

$$\rightarrow \ln|y| = 7x + C$$ ← *Need some constant for the whole equation

$\underbrace{\ln|y| + C_1}$ $\underbrace{7x + C_2}$ Combined

Recall: $\alpha \ln|x| = \ln|x^{\alpha}|$

Step#3 Solve for 'y'

$$\ln|y| = 7x + C$$

The 'y' is inside a natural log so, we need to take 'e' of both sides of the equation to isolate the 'y'

$$\rightarrow e^{\ln|y|} = e^{7x + C}$$ ← Arbitrary Constant

$$\rightarrow y = e^{7x}\boxed{e^{C}}$$ ← Arbitrary Constant

$$\rightarrow y = e^{7x}C \rightarrow \boxed{y = Ce^{7x}}$$

Example 2

Solve $e^x y \frac{dy}{dx} = e^{-y} + e^{-2x-y}$ generally, and also find a soln to the Initial Value Problem (IVP) with $y(0)=0$.

soln | Step #1 Separate the varables all "y-stuff" and "x-stuff" on opposite sides.

$$e^x y \frac{dy}{dx} = e^{-y} + e^{-2x-y}$$

$$\rightarrow \frac{dy}{dx} = \frac{e^{-y} + e^{-2xy}}{e^x y}$$

$$\rightarrow \frac{dy}{dx} = \frac{e^{-y} + e^{-2x}e^{y}}{e^x y}$$

$$\rightarrow \frac{dy}{dx} = \frac{e^{-y}(1+e^{-2x})}{e^x y}$$

$$\rightarrow \frac{dy}{dx} = \frac{e^{-y}}{y} \cdot \frac{(1+e^{-2x})}{e^x}$$

$$\rightarrow \left(\frac{\cancel{dx}}{1}\right)\cdot\left(\frac{y}{e^{-y}}\right)\frac{dy}{\cancel{dx}} = \frac{\cancel{e^{-y}}}{\cancel{y}} \cdot \frac{(1+e^{-2x})}{e^x}\left(\frac{dx}{1}\right)\cdot\left(\frac{\cancel{y}}{\cancel{e^{-y}}}\right)$$

$$\rightarrow \underbrace{\left(\frac{y}{e^{-y}}\right)}dy = \underbrace{\frac{(1+e^{-2x})}{e^x}}\left(\frac{dx}{1}\right)$$

$$\left(\frac{y}{e^{-y}}\right) \rightarrow \left(\frac{y}{\frac{1}{e^y}}\right) \qquad \frac{1}{e^x} + \frac{e^{-2x}}{e^x} \left\{ e^{-2x} \cdot e^{-x} \right.$$

$$y e^y dy = (e^{-x} + e^{-3x})dx$$

Step #2 Integrate *Find the antiderivatives*

$$y e^y dy = (e^{-x} + e^{-3x})dx$$

$$\rightarrow \int \underbrace{y e^y dy} = \int \underbrace{(e^{-x} + e^{-3x})}dx$$

Recall - Integration By Parts

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Recall - Integral Subsitution

$$\boxed{\int f(g(x))\cdot g'(x)dx = \int f(u)du,\ u=g(x)}$$

Integration by parts

$u = y$ $\quad dv = e^y dy$

$du = 1\cdot dy$ $\quad v = e^y$

$uv - \int v\,du$

$\rightarrow ye^y - \int e^y dy$

U-subsitution: Integral Subsitution

$u = -x$

$\rightarrow \frac{d}{dx}(-x)$

Take out constant

$\rightarrow -1\frac{d}{dx}(x)$

$\rightarrow -1\cdot 1 = -1$

$du = -1dx$

$dx = -\frac{1}{1}$

$u = -3x$

$\rightarrow \frac{d}{dx}(-3x)$

Take out constant

$\rightarrow -3\frac{d}{dx}(x)$

$\rightarrow -3\cdot 1 = -3$

$du = -3dx$

$dx = (-\frac{1}{3})du$

$\rightarrow \int -\frac{1}{3}e^u du$

Take out constant

$\rightarrow -\frac{1}{3}\cdot\int e^u du$

$\rightarrow -1/3 e^u$

Subsitute back

$u = -3x$

$\rightarrow -\frac{1}{3}e^{-3x}$

$$\rightarrow ye^y - \underbrace{\int e^y dy}_{\text{Antiderivative}(e^y) = e^y} = -e^{-x} - \frac{1}{3}e^{-3x} + C$$

$$\rightarrow ye^y - e^y = -e^{-x} - \frac{1}{3}e^{-3x} + C$$

$$\rightarrow e^y(y-1) = -e^{-x}(1 + \tfrac{1}{3}e^{-2x}) + C$$

Step #3 Solve for 'y' if you can

In this case, this example, we can not explicitly solve for 'y'!

$$e^y(y-1) = -e^{-x}(1+\tfrac{1}{3}e^{-2x}) + C \leftarrow \text{An Implicit solution}$$

Now, the Initial Value Problem (IVP) $y(0)=0$. (I.E. $x=0, y=0$)

This gives $e^0(0-1) = -e^0(1+\frac{1}{3}e^0) + C$

$\rightarrow -1 = -(1+\frac{1}{3}) + C$

$\rightarrow -1 = -\frac{4}{3} + C$

$\rightarrow C = -1 + \frac{4}{3} = \frac{1}{3}$

$\therefore C = 1/3$. Thus, the Initial Value Problem (IVP) has the particular solution

$$e^y(y-1) = -e^{-x}(1+\tfrac{1}{3}e^{-2x}) + \tfrac{1}{3}$$

Module 04

A note on antiderivatives

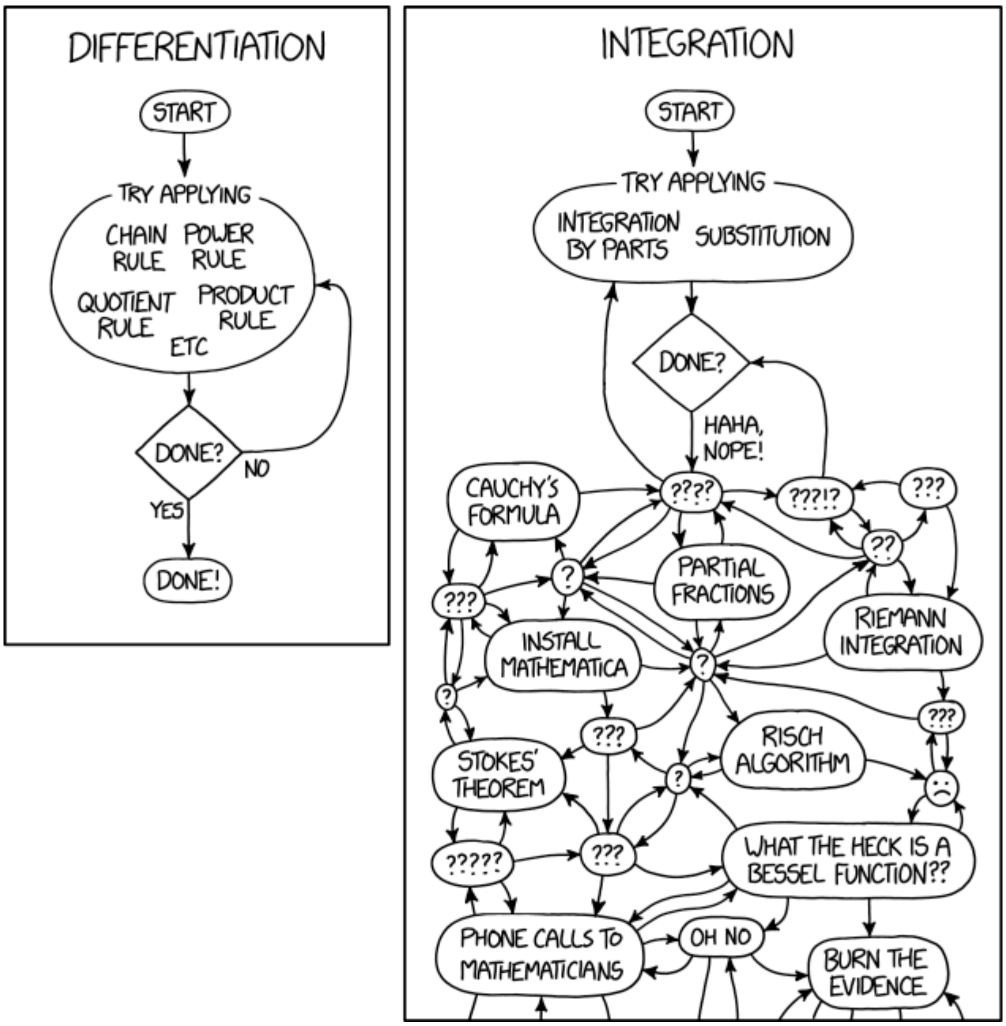
At this point, you might have realized that integrating will be a tool we use THROUGHOUT this course. Techniques which will be utilized often are:

- u-substitution
- integration by parts
- partial fraction decomposition.

However, trig-sub, etc., will also be used at times. We do not do an exhaustive review in this course, though we will try to work out some examples in detail.

Please consider consulting the Calculus Textbook linked on Canvas for more information on a particular technique.

Also, enjoy this **xkcd comic** (https://xkcd.com/2117/) :



WeBWorK module 04 exercises:

- Problems 4, 5

Relevant Wikipedia articles:

- **Some integration techniques** (https://en.wikipedia.org/wiki/Integral#Computation)

Expectation checklist - Module 4

At the completion of this module, you should:

- know the definitions introduced;
- determine if a given DE is separable;
- if given a separable DE, solve it;
  - note the solution may be explicit or implicit;
  - be able to solve a separable IVP; and
- review your integration techniques.

You will be assessed on your understanding of these concepts:

- within the homework,
- on the quizzes, and
- later, on the exam.

Coming up next, we:

- look into a method to solve linear 1st order differential equations.