Goal and idea - Module 16

GOAL:

We develop a method in which we transform differential equations into algebraic expressions which we are able to solve. However, we then also need to transform these answers back to proper solutions for the differential equation. We will

- define the Laplace transform of functions and learn how to calculate it;
- discuss the inverse of the Laplace transform and how to compute it;
- determine the Laplace transform of derivatives of a function;
- recall partial fraction decomposition, which is often needed in this process;
- solve IVPs utilizing the theory of the Laplace transform; and
- expand our use of this process by learning how to translate the functions we evaluate by exponential terms.

IDEA:

We learn about the Laplace transform defined on a number of common functions (polynomials, exponential functions, trig, etc.) and then note how the linearity of the Laplace transform allows us to compute linear combinations of these functions. In general, we use the Laplace transform to transform the differential equation into a simpler equation we can solve. We then use the inverse of the Laplace transform to map these solutions back into the appropriate form.

Defn Laplace Transform
Let f(t) be at a function defined for $t \geq 0$.
The integral $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$
is called the Laplace Transform of "f," so long as the integral exists.

1) $\int^\infty$ ← Is an improper integral
2) Although we are taking the antiderivative with respects to "dt," but end of the day what we are viewing this as is a function of "s"
$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$

Big Idea: We are taking functons of "t" and transforming them into functions of "s" where hopefully a lot of the manipulations we do are easier in "s-land"; then, introduce an inverse laplace to be able to map what we've done in "s" back into "t."

Recall Improper Integral
$$\int_0^\infty e^{-st} f(t)\,dt = \lim_{a\to\infty} \int_0^a e^{-st} f(t)\,dt$$
and this exists if the limit is a finite number ie. it converges

Note
Generally for a function f(t), we write $\mathcal{L}\{f(t)\} = F(s)$. Doesn't mean antiderivative in this context
(Be sure to note domains of f(t) and F(s))

Example (1) Compute $\mathcal{L}\{3\}$

Soln

1) Know the definition
$$\mathcal{L}\{3\} = \int_0^\infty e^{-st}(3)\,dt$$

2) Find the Improper Integral
$$\mathcal{L}\{3\} = \int_0^\infty e^{-st}(3)\,dt$$

Integral property Pull out constants
$$\to \lim_{a\to\infty} 3\int_0^\infty e^{-st}\,dt$$

Limit property Pull out constants
$$\to 3\lim_{a\to\infty}\left[-\frac{e^{-st}}{s}\Big|_0^a\right]$$

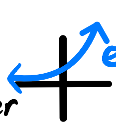
Find the antiderivative w/ varable "t"

Definate integral

* Recall *
The Fundamental Theorem of Calculus
If "f" is continuous on [a,b] and "F" is an antiderivative of "f" then, $\int_a^b f(x)\,dx = F(b) - F(a)$

$$\to 3\lim_{a\to\infty}\left[\frac{e^{-sa}}{s} - \left(-\frac{e^{-s0}}{s}\right)\right]$$
$$\to 3\lim_{a\to\infty}\left[-\frac{e^{-sa}}{s} + \frac{e^{0}}{s}\right]$$

Simplify and take the limits



$$\to 3\lim_{a\to\infty}\left[-\frac{0}{s} + \frac{1}{s}\right]$$
$$\to 3\left[-\frac{0}{s} + \frac{1}{s}\right]$$

∴ 3/s ← Our F(s) defined on $(0, \infty)$

Summary: This is the general formula. The Laplace Transform of any constant (c/s)

Example (2) $\mathcal{L}\{t\}$

Soln

1) Know the definition
$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}(t)\,dt$$

2) Find the Improper Integral
$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}(t)\,dt$$

Integral property Pull out constants
$$\to \lim_{a\to\infty}\int_0^a (t)e^{-st}\,dt$$

Integration by parts

Recall – Integration By Parts
$$\int u\,dv = uv - \int v\,du$$

$$\lim_{a\to\infty}\int_0^a \underbrace{(t)}_{u}\,\underbrace{e^{-st}\,dt}_{dv}$$

$u = t$, $du = dt$; $dv = e^{-st}\,dt$, $v = -\frac{1}{s}e^{-st}$

$$\to \lim_{a\to\infty}\left[t\cdot -\frac{1}{s}e^{-st} - \left(\int -\frac{1}{s}e^{-st}\,dt\right)\right]$$
$$\to \lim_{a\to\infty}\left[-\frac{t}{s}e^{-st} + \int \frac{1}{s}e^{-st}\,dt\right]$$

Include our bounds & integral
$$\to \lim_{a\to\infty}\left[-\frac{t}{s}e^{-st}\Big|_0^a + \int_0^a \frac{1}{s}e^{-st}\,dt\right]$$
$$\to \lim_{a\to\infty}\left[\left[-\frac{t}{s}e^{-st} - \frac{e^{-st}}{s^2}\right]\Big|_0^a\right]$$
$$\to \lim_{a\to\infty}\left[\left(-\frac{a}{s}e^{-sa} - \frac{e^{-sa}}{s^2}\right) - \left(-\frac{0}{s}e^{-s0} - \frac{e^{-s0}}{s^2}\right)\right]$$
$$\to \lim_{a\to\infty}\left[\left(-\frac{a}{s}e^{-sa} - \frac{e^{-sa}}{s^2}\right) - \left(-\frac{0}{s}e^{0} - \frac{e^{0}}{s^2}\right)\right]$$
$$\to \lim_{a\to\infty}\left[\left(-\frac{a}{s}e^{-sa} - \frac{e^{-sa}}{s^2}\right) - \left(-\frac{0}{s} - \frac{1}{s^2}\right)\right]$$

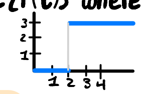
making the assumption that $s > 0$
$$\to \lim_{a\to\infty}\left[\left(-\cancel{\frac{a}{s}e^{-sa}}^{0} - \cancel{\frac{e^{-sa}}{s^2}}^{0}\right) - \left(-\cancel{\frac{0}{s}}^{0} - \frac{1}{s^2}\right)\right]$$
otherwise it would be divergent. It would be going to infinity.

$\therefore -\frac{1}{s^2}$, defined on $(0, \infty)$

*Also works for Piecewise functions

Example (3) Find $\mathcal{L}\{f(t)\}$ where

$f(t) = \begin{cases} 0 & 0 \leq t \leq 2 \\ 3 & 2 \leq t \leq \infty \end{cases}$



Soln

1) Know the definition
$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}(t)\,dt$$

Because you can break-up integrals over addition
$$\to \int_0^2 e^{-st}(0)\,dt + \int_2^\infty e^{-st}(3)\,dt$$

compute...
$$\to \int_0^2 e^{-st}(0)\,dt + \int_2^\infty e^{-st}(3)\,dt$$

The antiderivative of zero is a constant so you'd put a "C" but remember you have the bounds from (0,2) the bounds would be C – C, so you still get zero
$$\to (0) + \lim_{a\to\infty} 3\int_2^a e^{-st}\,dt$$
$$\to 3\lim_{a\to\infty}\left[-\frac{1}{s}e^{-st}\Big|_2^a\right]$$
$$\to 3\lim_{a\to\infty}\left[\left(-\frac{1}{s}e^{-sa}\right) - \left(-\frac{1}{s}e^{-2t}\right)\right]$$
$$\to 3\lim_{a\to\infty}\left[-\frac{1}{s}e^{-sa} + \frac{e^{-2s}}{s}\right]$$
$$\to 3\lim_{a\to\infty}\left[-\cancel{\frac{1}{s}e^{-sa}}^{0} + \frac{e^{-2s}}{s}\right]$$

$\therefore 3\frac{e^{-2s}}{s}$, Domain: $s > 0$

Theorem
For constants C, K and n = 1, 2, 3, ...

(A) $\mathcal{L}\{C\} = \frac{C}{s}$

(B) $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

(C) $\mathcal{L}\{e^{Kt}\} = \frac{1}{s-K}$

(D) $\mathcal{L}\{\cos(Kt)\} = \frac{s}{s^2+K^2}$

(E) $\mathcal{L}\{\sin(Kt)\} = \frac{K}{s^2+K^2}$

* Idea: Given a function $F(s)$, we want to find $\mathcal{L}^{-1}\{F(s)\}$.

I.e. the function $f(t)$ such that $\mathcal{L}\{f(t)\} = F(s)$.

* From the previous theorem, we have

Laplace Transform

(A) $\mathcal{L}\{C\} = \frac{C}{s}$

(B) $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

(C) $\mathcal{L}\{e^{kt}\} = \frac{1}{s-k}$

(D) $\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}$

(E) $\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}$

$\rightarrow$

Inverse Laplace Transform

Instead of computing the Inverse Laplace Transform from scratch ...

(a) $\mathcal{L}^{-1}\{\frac{C}{s}\} = C$

(b) $\mathcal{L}^{-1}\{\frac{n!}{s^{n+1}}\} = t^n$

(c) $\mathcal{L}^{-1}\{\frac{1}{s-k}\} = e^{kt}$

(d) $\mathcal{L}^{-1}\{\frac{s}{s^2+k^2}\} = \cos(kt)$

(e) $\mathcal{L}^{-1}\{\frac{k}{s^2+k^2}\} = \sin(kt)$

Note: Knowing $\mathcal{L}$ and $\mathcal{L}^{-1}$ of some functions only becomes very useful with the following "linearity" properties.

THEOREM

For any numbers α, β (are real numbers) and $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$, we have

Inverse Laplace Transform

(i) $\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha\mathcal{L}^{-1}\{F(s)\} + \beta\mathcal{L}^{-1}\{G(s)\}$

Laplace Transform

We are able to reduce it to things we know

(ii) $\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha\mathcal{L}\{f(t)\} + \beta\mathcal{L}\{g(t)\} = \alpha F(s) + \beta G(s)$

Example (i) — Inverse Laplace Transform

Find $\mathcal{L}^{-1}\left\{\frac{3s-4}{s^2+9}\right\}$

Soln

$\mathcal{L}^{-1}\left\{\frac{3s-4}{s^2+9}\right\}$ Looks nothing of the form we've seen before, but (based on their denominator) looks like:

(d) $\mathcal{L}^{-1}\{\frac{s}{s^2+k^2}\} = \cos(kt)$ & (e) $\mathcal{L}^{-1}\{\frac{k}{s^2+k^2}\} = \sin(kt)$

Breaking it over the numerator because we need single terms on top

$\rightarrow \mathcal{L}^{-1}\left\{\frac{3s}{s^2+3^2}\right\} + \mathcal{L}^{-1}\left\{\frac{-4}{s^2+3^2}\right\}$

Now we are starting to see the base forms of (d) & (e)

$\rightarrow 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} - \frac{4}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}$

So in correct form to apply theorem

$\therefore 3\cos(3t) - \frac{4}{3}\sin(3t)$

Example (ii) — Laplace Transform

$\mathcal{L}\{3t^5 - e^{2t}\}$

Soln

Using the linearity property we can pull out a constant

$\rightarrow \mathcal{L}\{3t^5 - e^{2t}\} = 3\mathcal{L}\{t^5\} - \mathcal{L}\{e^{2t}\}$

Reduce it to things we know

Based on their forms they look like Laplace Transforms theorems: (b) $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ & (c) $\mathcal{L}\{e^{kt}\} = \frac{1}{s-k}$

Apply theorems...

$\rightarrow 3\left(\frac{5!}{s^{5+1}}\right) - \left(\frac{1}{s-2}\right)$

$\rightarrow \frac{3 \cdot 5!}{s^6} - \frac{1}{s-2}$

$\therefore \frac{120}{s^6} - \frac{1}{s-2}$

←Goal

Module 16 SUBJECT: The Laplace Transform of Derivatives DATE: 2020/12/09 PAGE NO: 01/01

Goal: Find what $\mathcal{L}\{f^{(n)}(t)\}$ is

* Keep in mind this is for a general function, so we never really plug-in the exact function; but, the idea is we want to figure out how we can rewrite the Laplace Transform for the derivatives of a general function and place it in terms of the Laplace Transform of the original function

1 First, we find

Integration by Parts

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt = \lim_{a\to\infty}\int_0^a e^{-st} f'(t)\,dt$$

Recall – Integration By Parts: $\int u\,dv = uv - \int v\,du$

$u = e^{-st}$ $\quad dv = f'(t)dt$

$du = -se^{-st}dt$ $\quad v = f(t)$

$$\rightarrow \lim_{a\to\infty}\left[e^{-st}f(t)\Big|_0^a - \int_0^a f(t)\,(-se^{-st})\,dt\right]$$

$$\rightarrow \lim_{a\to\infty}\left[e^{-st}f(t)\Big|_0^a + s\int_0^a f(t)e^{-st}\,dt\right]$$

$$\rightarrow \lim_{a\to\infty}\left[e^{-sa}f(a) - e^{-s0}f(0)\right] + s\lim_{a\to\infty}\int_0^a f(t)e^{-st}\,dt$$

$(s>0)$ ↘ ↙ *$f(a) <$ exponential growth* e^x

$$\rightarrow \lim_{a\to\infty}\left[e^{-sa}f(a) - f(0)\right] + s\lim_{a\to\infty}\int_0^a f(t)e^{-st}\,dt$$

$$\rightarrow \lim_{a\to\infty}\left[\cancel{e^{-sa}f(a)}^{\,0} - f(0)\right] + s\lim_{a\to\infty}\int_0^a f(t)e^{-st}\,dt$$

{As you can see we are reducing it back into our original function – there's no derivatives in this term

$$\rightarrow -f(0) + s\lim_{a\to\infty}\int_0^a f(t)e^{-st}\,dt \rightarrow \int_0^\infty e^{-st}f(t)\,dt$$

The limit rewriten equals to an Improper Integral and is our Laplace Transformation

$$\rightarrow -f(0) + s\,\mathcal{L}\{f(s)\}$$

$$\rightarrow -f(0) + sF(s)$$

That is,

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

2 Next:

Integration by Parts

$$\mathcal{L}\{f''(t)\} = \int_0^\infty e^{-st} f''(t)\,dt$$

Recall – Integration By Parts: $\int u\,dv = uv - \int v\,du$

$u = e^{-st}$ $\quad dv = f''(t)dt$

$du = -se^{-st}dt$ $\quad v = f'(t)$

$$\rightarrow \lim_{a\to\infty}\left[e^{-st}f'(t)\Big|_0^a\right] - \int_0^\infty f'(t)\,(-se^{-st})\,dt$$

$$\rightarrow \lim_{a\to\infty}\left[e^{-st}f'(t)\Big|_0^a\right] + s\int_0^\infty f'(t)e^{-st}\,dt$$

$$\rightarrow \lim_{a\to\infty}\left[e^{-sa}f'(a) - e^{-s0}f'(0)\right] + s\int_0^\infty f'(t)e^{-st}\,dt$$

$$\rightarrow \lim_{a\to\infty}\left[e^{-sa}f'(a) - f'(0)\right] + s\int_0^\infty f'(t)e^{-st}\,dt$$

$$\rightarrow \lim_{a\to\infty}\left[\cancel{e^{-sa}f'(a)}^{\,0} - f'(0)\right] + \underbrace{s\int_0^\infty f'(t)e^{-st}\,dt}_{\text{Laplace Transformation of the derivative}}$$

$$\rightarrow -f'(0) + s\,\mathcal{L}\{f'(t)\}$$

$$\rightarrow -f'(0) + s\left[F(s) - f(0)\right]$$

Once you do one derivative and as higher derivatives continue you do one application of Integration by Parts and it reduces it to the previous case via recursion

That is,

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

$\mathcal{L}\{f''(t)\}$ $\qquad \mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$

So what we learn is that we're able to rewrite the Laplace Transform of derivatives in terms of the original Laplace Transformation; and then, minus terms of the original function plugged in at zero and it's derivatives at zero, and also "s" and stuff...

Theorem We have

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\underbrace{F(s)}_{\mathcal{L}\{f(t)\}} - s^{n-1}f(0) - s^{n-2}f'(0) - (\cdots) - sf^{(n-2)}(0) - sf^{(n-1)}(0)$$

* Note: Technically only holds for certain functions (Continuous on $[0,\infty)$, and grow slower than e^{st})

Goal: Separate functions $\frac{P(x)}{q(x)}$ into small pieces.

Key is to look at the denominator

* Sometimes nice to think of the cases that can arise

(i) Distinct linear terms:

Linear Polynomials

Polynomials of degree one

$$\frac{x}{\underbrace{(x+3)(x-1)}_{x^2+2x-3}} = \frac{A}{(x+3)} + \frac{B}{(x-1)}$$

Might have seen this on the denominator, but you would factor it apart into linear terms

Then, solve for A, B

$$\frac{A}{(x+3)} + \frac{B}{(x-1)}$$

$$\rightarrow \frac{A(x-1)+B(x+3)}{(x+3)(x-1)}$$

The denominators are the same, so it comes down to the numerators

$$\rightarrow x = (A+B)x + (3B-A)$$

On the left we have the numerator "x"

On the right grouped based on the Polynomial x-terms

Then equate the coefficents

$$\rightarrow A+B = 1, \quad 3B-A = 0$$

As a result,

$$A = 3/4, \quad B = 1/4$$

In this particular case we end up learning Because the starting expression is hard to deal with, for the denominator is "clunky"

$$\frac{x}{(x+3)(x-1)}$$

After finding out what **A** and **B** are is that we are actually able to rewrite it as...

$$\frac{x}{(x+3)(x-1)} = \frac{3/4}{(x+3)} + \frac{1/4}{(x-1)}$$

So the reason why you first learn about Partial Fraction Decomposition may have been in computing integrals because it wouldn't know how to take the integral of the left (starting expression), but might know how to take the integral of rewritten version independently. So instead of one integral you'd have to do two, but they are much easier.

(ii) Some repeated linear terms:

$$\frac{x^2-2}{(x-2)(x+1)^3}$$

We can have linear terms on the bottom, but one or both of them have higher powers

In this case where ever we have a linear term we still have it as the denominator with a variable numerator; and then, keep tacking on different powers until you exhaust that particular linear term.

$$\frac{x^2-2}{(x-2)(x+1)^3} = \frac{A}{(x-2)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3}$$

Then, solve for A, B, C, and D

$$A = \frac{2}{27} \quad B = \frac{-2}{27} \quad C = \frac{7}{9} \quad D = \frac{1}{3}$$

(iii) Distinct irreducible terms:

$$\frac{x}{(x^2+1)(x^2+2)}$$

The denominator cannot factor into linear terms, no because it would have complex root, so it's an irreducible polynomial

In this case if you have an irreducible term instead of putting an **A** on the top you need to put an expression in the form **Ax + B** and is done for each of the linear terms with different lettered variables **(Cx + D)** , **(Ex + F)** , **(Gx + H)** , **(Ix + J)** , . . .

$$\frac{x}{(x^2+1)(x^2+2)} = \frac{(Ax+B)}{(x^2+1)} + \frac{(Cx+D)}{(x^2+2)}$$

Then, solve for A, B, C, and D

$$A = 1 \quad B = 0 \quad C = -1 \quad D = 0$$

(iv) Repeated irreducible terms:

$$\frac{2x-1}{(x^2+x+1)^3}$$

In this case just as before instead of putting an **A** for the numerator we need to put an expression in the form **(Ax + B)** and for each linear terms with different lettered variables

$$\frac{2x-1}{(x^2+x+1)^3} = \frac{(Ax+B)}{(x^2+x+1)} + \frac{(Cx+D)}{(x^2+x+1)^2} + \frac{(Ex+F)}{(x^2+x+1)^3}$$

Then, solve for A, B, C, D, and F

(V) Mixing of any previous cases...

$$\frac{2x-1}{(x^2+1)^2(x^2+x+1)^2} = \frac{A}{(x^2+1)} + \frac{B}{(x^2+1)^2} + \frac{(Cx+D)}{(x^2+x+1)^2} + \frac{(Ex+F)}{(x^2+x+1)^3}$$

Then, solve for A, B, C, D, and F

Theory and set-up

The Big Idea:

1| Apply Laplace Transform($\mathcal{L}$) to the entire IVP, coverting it to a function of "s" (often denoted $Y(s)$)

2| Solve for $Y(s)$.

2.5| → Often use Partial Fraction Decomposition (PFD) around here to apply inverse Laplace Transform $\mathcal{L}^{-1}$.

3| Then use the inverse Laplace Transform $\mathcal{L}^{-1}$ to transform back into a function of "t" (often denoted $y(t)$).

Example 1 | Solve $y''+y=t,\ y(0)=0,\ y'(0)=2$

Solution

1| Apply $\mathcal{L}$ to both sides

$$y''+y=t$$

$$\to \mathcal{L}\{y''+y\} = \mathcal{L}\{t\}$$

Linearity Properties w/ transformation

$$\to \mathcal{L}\{y''\}+\mathcal{L}\{y\} = \mathcal{L}\{t\}$$

What we want to solve for

$$\to s^2Y(s)-sy(0)-y'(0) + Y(s) = \frac{1}{s^2}$$

2| Solve for $Y(s)$

$$s^2Y(s)-sy(0)-y'(0)+Y(s) = \frac{1}{s^2}$$

Factor out $Y(s)$

$$\to Y(s)(s^2+1) = \frac{1}{s^2}+sy(0)+y'(0)$$

$$\to Y(s) = \frac{1}{s^2(s^2+1)} + \frac{sy(0)+y'(0)}{s^2+1}$$

$$\to Y(s) = \frac{1}{s^2(s^2+1)} + \frac{s\cancel{y(0)}^{0}+\cancel{y'(0)}^{2}}{s^2+1}$$

$$\to Y(s) = \frac{1}{s^2(s^2+1)} + \frac{2}{s^2+1}$$

Thus,

$$Y(s) = \frac{1}{s^2} - \frac{1}{s^2+1} + \frac{2}{s^2+1}$$

$$\to Y(s) = \frac{1}{s^2} + \frac{1}{s^2+1}$$

3| Apply inverse Laplace Transform $\mathcal{L}^{-1}$

$$Y(s) = \frac{1}{s^2} + \frac{1}{s^2+1}$$

Apply to both sides

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2} + \frac{1}{s^2+1}\right\}$$

L-Notation & A- Linearity Property

$$\to y(t) = \underset{(b)}{\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}} + \underset{(e)}{\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}}$$

Lucky! $\mathcal{L}^{-1}$ is exactly of theorem form

$\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}=t^n$ $\quad \mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\}=\sin(kt)$

Therefore,

$$y(t) = t + \sin(t)$$

What we would like to do is apply the inverse Laplace to transform both sides where the left would become y(t) [our solution], but we already applied our initial Conditions (that's what we turned into our "plugged-in" numbers.
The right side we can't apply the Laplace transforms to the first piece; however, the second piece Is not so bad, but the first piece we need to do a bit more...

Partial Fraction Decomposition (PFD)

$$\frac{1}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cx+D}{s^2+1}$$

$A=0 \quad B=1$

$C=-1 \quad D=0$

Recall

Inverse Laplace Transform forms

(a) $\mathcal{L}^{-1}\left\{\frac{c}{s}\right\}=C$

(b) $\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}=t^n$

(c) $\mathcal{L}^{-1}\left\{\frac{1}{s-k}\right\}=e^{kt}$

(d) $\mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\}=\cos(kt)$

(e) $\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\}=\sin(kt)$

Example 2 | Solve the IVP $y''+16y=32t,\ y(0)=3,\ y'(0)=-2$

Solution

1| Apply $\mathcal{L}$ to both sides

$$y''+16y = 32t$$

$$\to \mathcal{L}\{y''+16y\} = \mathcal{L}\{32t\}$$

Linearity Property

$$\to \mathcal{L}\{y''\}+\mathcal{L}\{16y\} = \mathcal{L}\{32t\}$$

$$\to \mathcal{L}\{y''\}+\mathcal{L}16\{y\} = \mathcal{L}32\{t\}$$

$$\to s^2Y(s)-sy(0)-sy'(0) + 16Y(s) = 32\left(\frac{1}{s^2}\right)$$

2| Solve for $Y(s)$.

$$s^2Y(s)-sy(0)-sy'(0)+16Y(s) = \frac{32}{s^2}$$

Factor out $Y(s)$

$$\to Y(s)(s^2+16) = \frac{32}{s^2}+sy(0)+sy'(0)$$

$$\to Y(s)(s^2+16) = \frac{32}{s^2}+3s+(-2)$$

$$\to Y(s) = \frac{32}{s^2(s^2+16)} + \frac{3s}{(s^2+16)} - \frac{2}{(s^2+16)}$$

Partial Fraction Decomposition

So we can notice the double $\frac{2}{(s^2+16)}$

swaped

$$\to Y(s) = \frac{2}{s^2} - \frac{2}{(s^2+16)} - \frac{2}{(s^2+16)} + \frac{3s}{(s^2+16)}$$

$$\to Y(s) = \frac{2}{s^2} - \frac{4}{(s^2+16)} + \frac{3s}{(s^2+16)}$$

3| Apply inverse Laplace Transform $\mathcal{L}^{-1}$

Recall Inverse Laplace Transform forms

(a) $\mathcal{L}^{-1}\left\{\frac{c}{s}\right\}=C$ (b) $\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}=t^n$ (c) $\mathcal{L}^{-1}\left\{\frac{1}{s-k}\right\}=e^{kt}$

(d) $\mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\}=\cos(kt)$ (e) $\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\}=\sin(kt)$

$$Y(s) = \frac{2}{(b)\,s^2} - \frac{4}{(e)\,(s^2+16)} + \frac{3s}{(d)\,(s^2+16)}$$

$$2\frac{1}{s^2} \qquad 3\frac{s}{s^2+16}$$

$$\therefore y(t) = 2t - \sin(4t) + 3\cos(4t)$$

Goal: Also solve expressions which include exponential terms (i.e. $\mathcal{L}\{e^{kt}f(t)\}$)

Theorem

If $\mathcal{L}\{f(t)\}=F(s)$ and "k" is a real number,
THEN
$$\mathcal{L}\{e^{kt}f(t)\}=F(s-k)$$
AND
$$\mathcal{L}^{-1}\{F(s-k)\}=e^{kt}f(t)$$

Example

(a) Find $\mathcal{L}\{e^{-2t}t^4\}$

Solution

* Recall $\mathcal{L}\{t^4\}=\frac{4!}{s^5}=\frac{24}{s^5}=F(s)$

Thus,
$$\mathcal{L}\{e^{-2t}t^5\}=F(s-(-2))=F(s+2)$$

$\therefore \frac{24}{(s+2)^5}$

(b) Find $\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+5}\right\}$

Solution

Doesn't fit into forms we know

* Try to "translate" into something we know [Complete the square!]

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\right\}$$

$$\rightarrow \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+2^2}\right\}$$ To put into $\sin(kt)$ form

Remember it's shifted $(s-k)$

$\therefore \frac{1}{2}e^{-t}\sin(2t)$

* Note: Can always check by computing $\mathcal{L}\{\frac{1}{2}e^{-t}\sin(2t)\}$

(c) IVP - Solve
$$y'-y=1+te^t,\ y(0)=0$$

Solution

1 Apply $\mathcal{L}$ to both sides

$$\mathcal{L}\{y'-y\}=\mathcal{L}\{1+te^t\}$$

$$\rightarrow sY(s)-y(0)-Y(s)=\frac{1}{s}+\frac{1}{(s-1)^2}$$

2 Solve for $Y(s)$

$$sY(s)-y(0)-Y(s)=\frac{1}{s}+\frac{1}{(s-1)^2}$$

Factor out $Y(s)$

$$\rightarrow Y(s)(s-1)-0=\frac{1}{s}+\frac{1}{(s-1)^2}$$

Simplify

$$\rightarrow Y(s)=\frac{1}{s(s-1)}+\frac{1}{(s-1)^3}$$

Partial Fraction Decomposition

Partial Fraction Decomposition
$\frac{A}{s}+\frac{B}{s-1}\ \therefore\ A=-1,\ B=1$

$$\rightarrow Y(s)=-\frac{1}{s}+\frac{1}{s-1}+\frac{1}{(s-1)^3}$$

3 Apply inverse Laplace Transform $\mathcal{L}^{-1}$

$$Y(s)=-\frac{1}{s}+\frac{1}{s-1}+\frac{1}{(s-1)^3}$$

$$\rightarrow \mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{-\frac{1}{s}+\frac{1}{s-1}+\frac{1}{(s-1)^3}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}=t^n \rightarrow \mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\}=t^2 \rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}=\frac{1}{2}t^2$$

$$\therefore y(t)=-1+e^t+\frac{1}{2}t^2e^t$$

Expectation checklist - Module 16

At the completion of this module, you should:

- be able to compute the Laplace transform of a given function;
- be able to compute the inverse Laplace transform of a given function;
- know the Laplace transform and inverse Laplace transform for the functions given in the tables of this module;
- be able to use the tables and the linearity properties to compute the Laplace transform and inverse Laplace transform for many functions;
- apply partial fraction decomposition to decompose functions so that we can apply the inverse Laplace transform;
- compute the Laplace transform of derivatives of a function;
- apply the theory described here to solve IVPs;
- include "frequency shifting" into the theory describe above.

Coming up next, we:

- What if we need to solve for functions that satisfy more than one differential equation at a time? Well, that would be systems of differential equations, which is coming up next!