1st-order homogeneous differential equations

DEFN] A function $f(x,y)$ is called <u>homogeneous</u> of degree α (α a real number) if for $t \neq 0$ we have

$$f(tx, ty) = t^{\alpha} f(x,y).$$

Ex] Consider $f(x,y) = \sqrt{x+y}$.

Then $f(tx,ty) = \sqrt{tx+ty} = \sqrt{t(x+y)} = \sqrt{t}\,\sqrt{x+y}$

$$= t^{1/2}\sqrt{x+y} = t^{1/2} f(x,y).$$

So $f(x,y)$ is homog of degree $1/2$.

Ex| show $F(x,y) = \dfrac{xy^2 + x^3\cos\left(\frac{2x}{3y}\right)}{5y^2x + y^3}$ is homogeneous of degree 0.

soln|

$$F(tx, ty) = \frac{(tx)(ty)^2 + (tx)^3\cos\left(\frac{2(tx)}{3(ty)}\right)}{5(ty)^2(tx) + (ty)^3}$$

$$= \frac{t^3\left(xy^2 + x^3\cos\left(\frac{2x}{3y}\right)\right.}{t^3\left(y^2x + y^3\right)} = F(x,y).$$

DEFN|

(a) A DE $\frac{dy}{dx} = f(x,y)$ is a <u>1st-order homogeneous differential equation</u> if $f(x,y)$ is a homogeneous function of degree 0.

(b) A DE $M(x,y)dx + N(x,y)dy = 0$ is a <u>1^{st}-order homogeneous differential equation</u> if $M(x,y)$ and $N(x,y)$ are homogeneous functions of the same degree.

Solving 1st-order homogeneous differential equations

To solve such a DE:

(i) substitute $y = ux$.

(ii) Replace $\frac{dy}{dx}$ with $x\frac{du}{dx} + u$,

→ u is a function of x:

→ $\frac{dy}{dx} = \frac{d}{dx}(ux) = u'x + ux' = \frac{du}{dx}\cdot x + u$

$= x\frac{du}{dx} + u.$

(iii) solve the separable equation.

(iv) plug back in $y = ux$, that is, $u = \frac{y}{x}$.

Ex1 Solve $\frac{dy}{dx} = \frac{-3y}{3x-7y}$.

soln/ check: $\frac{-3(ty)}{3(tx)-7(ty)} = \frac{t(-3y)}{t(3x-7y)} = \frac{-3y}{3x-7y}$,

so it's a 1st-order homog DE.

(i)+(ii) we have $y = ux$, $\frac{dy}{dx} = x\frac{du}{dx} + u$, this gives:

$$x\frac{du}{dx} + u = \frac{-3ux}{3x-7ux}$$

$$\Leftrightarrow \quad x\frac{du}{dx} = \frac{-3ux}{3x-7ux} - u$$

$$\frac{du}{dx} = \frac{-3ux}{x(3x-7ux)} - \frac{u}{x}$$

$$\Leftrightarrow \quad \frac{du}{dx} = \frac{-3u}{3x-7ux} - \frac{u}{x}$$

$$\Leftrightarrow \quad \frac{du}{dx} = \frac{-3ux - u(3x-7ux)}{3x^2-7ux^2}$$

$$\Leftrightarrow \quad \frac{du}{dx} = \frac{-3u - u(3-7u)}{3x-7ux}$$

$$\Leftrightarrow \quad \frac{du}{dx} = \frac{-6u+7u^2}{x(3-7u)} \quad \Leftrightarrow \quad \frac{3-7u}{7u^2-6u}\,du = \frac{1}{x}\,dx$$

(iii) want $\displaystyle\int \frac{3-7u}{7u^2-6u}\,du = \int \frac{1}{x}\,dx$.

$w = 7u^2-6u, \quad dw = (14u-6)\,du \iff du = \dfrac{dw}{14u-6}$

$$= \int \frac{3-7u}{w}\,\frac{dw}{14u-6} = -\frac{1}{2}\int \frac{1}{w}\,dw$$

$$= -\frac{1}{2}\ln|w| + c_1 = -\frac{1}{2}\ln|7u^2-6u| + c_1$$

$\iff$

$$-\frac{1}{2}\ln|7u^2-6u| + c_1 = \ln|x| + c_2$$

$\iff$

$$-\frac{1}{2}\ln|7u^2-6u| = \ln|x| + C.$$

$$-\tfrac{1}{2}\ln|7u^2-6u| = \ln|x| + C$$

$\alpha \ln|x| = \ln|x^\alpha|$

$$\Leftrightarrow \quad \ln|7u^2-6u| = -2\ln|x| + C$$

$$\Rightarrow \quad e^{\ln|7u^2-6u|} = e^{\ln|x^{-2}| + C} = e^{\ln|x^{-2}|}\underbrace{e^{C}}_{C}$$

$$\Leftrightarrow \quad 7u^2-6u = Cx^{-2}$$

(iv) plug in $u=\frac{y}{x}$:

$$\frac{7y^2}{x^2} - \frac{6y}{x} = Cx^{-2} \Leftrightarrow \boxed{7y^2 - 6yx = C}$$

(an implicit soln).

Ex| Consider $(y^2+yx)dx + x^2 dy = 0$.

(a) Is this DE a 1^{st}-order homogeneous DE?

(b) Solve this DE.

(c) Solve the IVP subject to $y(1)=2$.

soln|

(a) Since $M(tx,ty) = (ty)^2 + (ty)(tx) = t^2y^2 + t^2yx$
$= t^2(y^2+yx) = t^2 M(x,y)$

and

$N(tx,ty) = (tx)^2 = t^2x^2 = t^2 N(x,y)$ are homog of degree 2, this IS a homog DE.

(b) $$(y^2+yx)dx + x^2 dy = 0.$$

$$\Longleftrightarrow \quad (y^2+yx) + x^2\frac{dy}{dx} = 0.$$

(i)+(ii) $y = ux$ and $\frac{dy}{dx} = x\frac{du}{dx} + u$:

$$(ux)^2 + (ux)x + x^2\left(x\frac{du}{dx} + u\right) = 0$$

$$\Longleftrightarrow \quad u^2x^2 + ux^2 + x^3\frac{du}{dx} + ux^2 = 0$$

$$\Longleftrightarrow \quad x^3\frac{du}{dx} = -2ux^2 - u^2x^2$$

$$\Longleftrightarrow \quad \frac{du}{dx} = \frac{-2u-u^2}{x} \quad \Longleftrightarrow \quad \frac{-1}{2u+u^2}du = \frac{1}{x}dx$$

(iii) $$\underbrace{\int \frac{1}{2u+u^2}\,du}_{\#1} = \underbrace{\int \frac{1}{x}\,dx}_{\#2}$$

$\boxed{\#2}$ $\int \frac{1}{x}\,dx = \ln|x| + C$

$\boxed{\#1}$

$$= -\int \frac{1}{u(2+u)}\,du = -\int \left[\frac{1/2}{u} + \frac{-1/2}{2+u}\right] du = -\frac{1}{2}\ln|u| + \frac{1}{2}\ln|2+u|$$

PARTIAL FRACTION DECOMP.

$$\frac{1}{u(2+u)} = \frac{A}{u} + \frac{B}{2+u} = \frac{A(2+u) + Bu}{u(2+u)}$$

$$\Leftrightarrow \quad 1 = 2A + Au + Bu$$

$$\Leftrightarrow \quad 1 = 2A + u(A+B)$$

$A+B=0$

$2A=1$

$\Rightarrow A = 1/2,\ B = -1/2.$

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$$-\int \frac{1}{2u+u^2}\,du = -\int \frac{1}{u(2+u)}\,du = -\int \left[\frac{\frac{1}{2}}{u} - \frac{\frac{1}{2}}{2+u}\right] du$$

$$= -\tfrac{1}{2}\ln|u| + \tfrac{1}{2}\ln|2+u| + C$$

so

$$\int \frac{-1}{2u+u^2}\,du = \int \frac{1}{x}\,dx \quad \text{becomes}$$

$$-\tfrac{1}{2}\ln|u| + \tfrac{1}{2}\ln|2+u| = \ln|x| + C$$

$$\Longleftrightarrow \ln|u^{-1/2}| + \ln|(2+u)^{1/2}| = \ln|x| + C$$

$$\Longleftrightarrow e^{\ln|u^{-1/2}\cdot(2+u)^{1/2}|} = e^{\ln|x| + C} = e^{\ln|x|}\underbrace{e^{C}}_{c}$$

$$\Longrightarrow u^{-1/2}(2+u)^{1/2} = cx$$

$$u^{-1/2}(2+u)^{1/2} = Cx$$

$$\Leftrightarrow \frac{(2+u)^{1/2}}{u^{1/2}} = Cx \Leftrightarrow \frac{\sqrt{2+u}}{\sqrt{u}} = Cx$$

$$\Leftrightarrow \sqrt{\frac{2+u}{u}} = Cx \Leftrightarrow \frac{2+u}{u} = Cx^2$$

(iv) $u = \frac{y}{x}$:
$$\Leftrightarrow \frac{2+\frac{y}{x}}{\frac{y}{x}} = Cx^2$$

$$\Leftrightarrow 2+\frac{y}{x} = Cx^2 \cdot \frac{y}{x} \Leftrightarrow 2+\frac{y}{x} = Cyx$$

$$\Leftrightarrow 2x+y = Cyx^2 \Leftrightarrow y - Cyx^2 = -2x$$

$$\Leftrightarrow y(1-Cx^2) = -2x \Leftrightarrow \boxed{y = \frac{-2x}{1-Cx^2}}$$

(c) IVP with $y(1)=2$.

→ Have: $y=\dfrac{-2x}{1-cx^2}$ is a soln,

and $x=1$, $y=2$ gives

$$2=\frac{-2}{1-c} \iff 2(1-c)=-2$$

$$\iff 2-2c=-2$$

$$\iff -2c=-4 \iff c=2.$$

Thus, $y=\dfrac{-2x}{1-2x^2}$ is the soln.