Matthew Mendoza **Math-45-Krauel-F20**

Assignment Math45-Homework-WEEK-08 due 10/24/2020 at 11:59pm PDT

Are $f_1(x) = x$, $f_2(x) = x^2$, and $f_3(x) = 3x - 8x^2$ linearly independent or linearly dependent on $(-\infty, \infty)$?

- A. Linearly dependent
- B. Linearly independent

Solution:
SOLUTION:
The easiest way solve this problem is to hopefully notice that

$$(-3)f_1(x) + (8)f_2(x) + (1)f_3(x) = 0.$$

Thus, the functions are linearly dependent. (Other constants would also work.)

However, another attempt may have gone about finding the Wronskian $W(f_1, f_2, f_3)$ of the functions f_1, f_2, and f_3. If so, we would find

$$\begin{aligned}
W(f_1, f_2, f_3) &= \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix} = \begin{vmatrix} x & x^2 & 3x - 8x^2 \\ 1 & 2x & 3 - 16x \\ 0 & 2 & -16 \end{vmatrix} \\
&= x \begin{vmatrix} 2x & 3 - 16x \\ 2 & -16 \end{vmatrix} - x^2 \begin{vmatrix} 1 & 3 - 16x \\ 0 & -16 \end{vmatrix} + \left(3x - 8x^2\right) \begin{vmatrix} 1 & 2x \\ 0 & 2 \end{vmatrix} \\
&= x((2x)(-16) - (2)(3 - 16x)) - x^2((1)(-16) - (0)(3 - 16x)) + \left(3x - 8x^2\right)((1)(2) - (0)(2)) \\
&= -6x + 16x^2 + \left(3x - 8x^2\right)(2) \\
&= -6x + 16x^2 + 6x - 16x^2 = 0
\end{aligned}$$

for all x on $(-\infty, \infty)$. However, since we are given that these functions are only functions (and not solutions to a linear differential equation), this does NOT imply that the functions are linearly dependent (though it does suggest they might be). Thus, this method is inconclusive.

Had we not been able to find constants to show linear dependence, we could instead proceed with the definition of linear independence/dependence. Namely, the given functions are linearly independent if the only constants c_1, c_2, and c_3 such that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$$

are $c_1 = c_2 = c_3 = 0$. Otherwise, they are linearly dependent. Plugging in the functions the display above becomes

$$c_1 x + c_2 x^2 + c_3 \left(3x - 8x^2\right) = 0,$$

or

$$(c_1 + 3c_3)x + (c_2 - 8c_3)x^2 = 0.$$

This holds if $c_1 + 3c_3 = 0$ and $c_2 - 8c_3 = 0$. That is, if $c_1 = -3c_3$ and $c_2 = 8c_3$. Therefore, if we take c_3 to be any nonzero number, we find that there are c_1, c_2, and c_3, at least one of which that is nonzero, which satisfies the equation above (for example if $c_3 = 1$ we have $c_1 = -3$ and $c_2 = 8$). Thus, the functions are linearly dependent. Thus, the correct answer is A.

Correct Answers:

- `A`

Are $f_1(x) = 3$, $f_2(x) = \sin^2(x)$, and $f_3(x) = \cos^2(x)$ linearly independent or linearly dependent on $(-\infty, \infty)$?

- A. Linearly dependent
- B. Linearly independent

Solution:
SOLUTION:
The easiest way to solve this problem is if we happen to notice that

$$f_1(x) + (3)f_2(x) + (3)f_3(x) = 0.$$

Thus, the functions are linearly dependent.

Alternatively, we could attempt to compute the Wronskian $W(f_1, f_2, f_3)$. However, we would find that $W(f_1, f_2, f_3) = 0$. Since these functions are not known to be solutions to a linear differential equation, this is inconclusive and we must proceed by the definition of linear independence/dependence. (Had we known these functions are solutions to a linear differential equation, we would be done!)

Meanwhile, we could also use the defnition of linear independence/dependence. The given functions are linearly independent if the only constants c_1, c_2, and c_3 such that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$$

are $c_1 = c_2 = c_3 = 0$. Otherwise, they are linearly dependent. Plugging in the functions the display above becomes

$$3c_1 + c_2 \sin^2(x) + c_3 \cos^2(x) = 0.$$

If $c_2 = c_3$ then we have

$$3c_1 + c_2\left(\sin^2(x) + \cos^2(x)\right) = 0$$

or

$$3c_1 + c_2 = 0.$$

This holds for $c_2 = -3c_1$. For example, $c_1 = 1$, $c_2 = c_3 = -3$. Since we have a nonzero constant that satisfies this idendity, we conclude that these functions are linearly dependent. Thus, the correct answer is A.

Correct Answers:

- `A`

Are $f_1(x) = e^{-2x}$ and $f_2(x) = e^{3x}$ solutions to the differential equation $y'' - y' - 6y = 0$ on the interval $(-\infty, \infty)$?

- A. No
- B. Yes

Are $f_1(x) = e^{-2x}$ and $f_2(x) = e^{3x}$ linearly independent or linearly dependent on $(-\infty, \infty)$?

- A. Linearly independent
- B. Linearly dependent

Do $f_1(x) = e^{-2x}$ and $f_2(x) = e^{3x}$ form a fundamental set of solutions of the differential equation $y'' - y' - 6y = 0$ on the interval $(-\infty, \infty)$?

- A. Yes
- B. No

Solution:

SOLUTION:

For the first part, we verify that the indeed the functions satisfy the differential equation. Additionally, they are defined over the interval $(-\infty, \infty)$ and are thus solutions to the differential equation.

Since there are two solutions to a second order linear differential equation, we have that they are linearly independent if and only if the Wronskian $W(f_1, f_2) \neq 0$ on $(-\infty, \infty)$. We find

$$\begin{aligned} W(f_1, f_2) &= \begin{vmatrix} e^{-2x} & e^{3x} \\ -2e^{-2x} & 3e^{3x} \end{vmatrix} = \left(e^{2x}\right)\left(3e^{3x}\right) - \left(-2e^{-2x}\right)\left(e^{3x}\right) \\ &= 3e^{(3-2)x} + 2e^{(3-2)x} = (3+2)e^{(3-2)x} \\ &= 5e^{x} \end{aligned}$$

which is nonzero for any x in $(-\infty, \infty)$. Thus, the functions are linearly independent.

Since by the first two parts of the problem we have that there are two functions for a second order linear differential equation, that the functions satisfy the differential equation, and that they are linearly independent, we have that the functions form a fundamental set of solutions for the differential equation.

Thus, the correct answers are B, A, and A.

Correct Answers:

- `B`
- `A`
- `A`

Consider $y = c_1 e^{4x} + c_2 e^{5x} + 3e^{x}$ and the differential equation $y'' - 9y' + 20y = 36e^{x}$. Which of the following best describes y as a solution to this differential equation on the interval $(-\infty, \infty)$?

- A. y is a two-parameter family of solutions, but not general
- B. y is a general solution
- C. y is not a solution
- D. y is a general solution, but not linearly independent

Solution:

SOLUTION:

For starters, one can verify that $36e^{x}$ is a particular solution of the differential equation. Next, it can be shown that e^{4x} and e^{5x} are linearly independent and solutions to the associated homogeneous differential equation

$$y'' - 9y' + 20y = 0.$$

Thus, e^{4x} and e^{5x} form a fundamental set of solutions to this associated homogeneous differential equation. It follows that y is a general solution. Thus, the correct answers are B.

Correct Answers:

- `B`

Are $f_1(x) = x$, $f_2(x) = x - 1$, and $f_3(x) = x + 4$ linearly independent or linearly dependent on $(-\infty, \infty)$?

- A. Linearly independent

- B. Linearly dependent

Solution:
SOLUTION:
The easiest way to solve this problem, is to happen to notice that the functions are linearly dependent and find constants c_1, c_2, and c_3 such that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0.$$

In particular,

$$(-5)x + (4)(x-1) + (1)(x+4) = 0.$$

Thus, the functions are linearly dependent.

We may have first computed the Wronskian $W(f_1, f_2, f_3)$ of the functions f_1, f_2, and f_3. If so, we would find

$$\begin{aligned} W(f_1,f_2,f_3) &= \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix} = \begin{vmatrix} x & x^2 & 1x-4x^2 \\ 1 & 2x & 1-8x \\ 0 & 2 & -8 \end{vmatrix} \\ &= x\begin{vmatrix} 2x & 1-8x \\ 2 & -8 \end{vmatrix} - x^2 \begin{vmatrix} 1 & 1-8x \\ 0 & -8 \end{vmatrix} + \left(1x-4x^2\right) \begin{vmatrix} 1 & 2x \\ 0 & 2 \end{vmatrix} \\ &= x((2x)(-8)-(2)(1-8x)) \\ &\quad - x^2((1)(-8)-(0)(1-8x)) \\ &\quad + \left(1x-4x^2\right)((1)(2)-(0)(2)) \\ &= -2x + 8x^2 + \left(1x-4x^2\right)(2) \\ &= -2x+8x^2+2x-8x^2 = 0 \end{aligned}$$

for all x on $(-\infty, \infty)$. However, since we are given that these functions are only functions (and not solutions to a linear differential equation), this does NOT imply that the functions are linearly dependent (though it does suggest they might be).

If we couldn't find the constants to show linear dependence above, then we could proceed directly by the definition. The given functions are linearly independent if the only constants c_1, c_2, and c_3 such that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$$

are $c_1 = c_2 = c_3 = 0$. Otherwise, they are linearly dependent. Plugging in the functions the display above becomes

$$c_1 x + c_2 x^2 + c_3\left(1x - 4x^2\right) = 0,$$

or

$$(c_1 + 1c_3)x + (c_2 - 4c_3)x^2 = 0.$$

This holds if $c_1 + 1c_3 = 0$ and $c_2 - 4c_3 = 0$. That is, if $c_1 = -1c_3$ and $c_2 = 4c_3$. Therefore, if we take c_3 to be any nonzero number, we find that there are c_1, c_2, and c_3, at least one of which that is nonzero, which satisfies the equation above (for example if $c_3 = 1$ we have $c_1 = -1$ and $c_2 = 4$). Thus, the functions are linearly dependent. Thus, the correct answer is B.

Correct Answers:

- B

Are $f_1(x) = e^x\cos(5x)$ and $f_2(x) = e^x\sin(5x)$ solutions to the differential equation $y'' - 2y' + 26y = 0$ on the interval $(-\infty, \infty)$?

- A. No
- B. Yes

Are $f_1(x) = e^x\cos(5x)$ and $f_2(x) = e^x\sin(5x)$ linearly independent or linearly dependent on $(-\infty, \infty)$?

- A. Linearly dependent
- B. Linearly independent

Do $f_1(x) = e^x\cos(5x)$ and $f_2(x) = e^x\sin(5x)$ form a fundamental set of solutions of the differential equation $y'' - 2y' + 26y = 0$ on the interval $(-\infty, \infty)$?

- A. No
- B. Yes

Solution:
SOLUTION:
For the first part, we verify that the indeed the functions satisfy the differential equation. Additionally, they are defined over the interval $(-\infty, \infty)$ and are thus solutions to the differential equation.

Since there are two solutions to a second order linear differential equation, we have that they are linearly independent if and only if the Wronskian $W(f_1, f_2) \neq 0$ on $(-\infty, \infty)$. We find

$$\begin{aligned} W(f_1,f_2) &= \begin{vmatrix} e^{-5x} & e^{6x} \\ -5e^{-5x} & 6e^{6x} \end{vmatrix} = \left(e^{5x}\right)\left(6e^{6x}\right) - \left(-5e^{-5x}\right)\left(e^{6x}\right) \\ &= 6e^{(6-5)x} + 5e^{(6-5)x} = (6+5)e^{(6-5)x} \\ &= e^x \end{aligned}$$

which is nonzero for any x in $(-\infty, \infty)$. Thus, the functions are linearly independent.

Since by the first two parts of the problem we have that there are two functions for a second order linear differential equation, that the functions satisfy the differential equation, and that they are linearly independent, we have that the functions form a fundamental set of solutions for the differential equation.

Thus, the correct answers are B, B, and B.

Correct Answers:

- B
- B
- B

It can be verified that $y_1 = e^{2x}$ and $y_2 = xe^{2x}$ form a fundamental set of solutions of the differential equation $y'' - 4y' + 4y = 0$. It can also be verified that $y_p = x^2e^{2x} + x - 2$ is a particular solution to the differential equation $y'' - 4y' + 4y = 2e^{2x} + 4x - 12$. Using this information, which of the following is the general solution to the differential equation $y'' - 4y' + 4y = 2e^{2x} + 4x - 12$?

- A. $y = c_1e^{2x} + c_2xe^{2x} + x^2e^{2x} + x - 2$
- B. $y = c_1e^{2x} + c_2xe^{2x} + c_3x^2e^{2x} + c_4x - c_5$
- C. $y = c_1e^{2x} + c_2xe^{2x} + c_3\left(x^2e^{2x} + x - 2\right)$
- D. $y = e^{2x} + xe^{2x} + x^2e^{2x} + x - 2$

Solution:

SOLUTION:

Since $y_1 = e^{2x}$ and $y_2 = xe^{2x}$ form a fundamental set of solutions of the differential equation $y'' - 4y' + 4y = 0$ and $y_p = x^2e^{2x} + x - 2$ is a particular solution to the differential equation $y'' - 4y' + 4y = 2e^{2x} + 4x - 12$ we have that $y = c_1e^{2x} + c_2xe^{2x} + x^2e^{2x} + x - 2$ is the general solution for $y'' - 4y' + 4y = 2e^{2x} + 4x - 12$.

Thus, the correct answers are A.

Correct Answers:

- A

Are the functions $f_1(x) = e^{x+3}$ and $f_2(x) = e^{x-4}$ linearly dependent or independent?

- A. Linearly dependent
- B. Linearly independent

Which of the following best describes the correct choice for part (a)? (Careful!!)

- A. Since the functions are scalar multiples of each other. That is, $f_1 = cf_2$ for some constant c.
- B. Since the Wronskian equals zero for at least one x on $(-\infty, \infty)$.
- C. Since the Wronskian never equals zero on $(-\infty, \infty)$.
- D. Since the only solution to $c_1f_1 + c_2f_2 = 0$ is $c_1 = c_2 = 0$.

Solution:

SOLUTION:

Note that we can write $f_1(x) = e^{x+3} = e^3e^x$ and $f_2(x) = e^{x-4} = e^{-4}e^x$. Thus, we see that these two functions are both multiples of e^x. For this reason, they are linearly dependent.

To explain further, we find a constant c such that $f_1(x) = cf_2(x)$. That is, a c such that $e^3e^x = ce^{-4}e^x$. Since $e^x \neq 0$ for any x, we divide both side of this last equation by e^x to find $e^3 = ce^{-4}$, or $c = e^3e^4 = e^{3+4} = e^7$. Since $f_1(x) = cf_2(x)$ for some constant c, the functions are linearly dependent.

Thus, the correct answers are A and A.

Correct Answers:

- A
- A

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9. (1 point) The function $y_1(x) = e^{7x}$ is a solution to the differential equation $y'' - 14y' + 49y = 0$. Use reduction of order to find another solution y_2 to this differential equation.

$y_2 =$ ________ help (formulas)

Solution:

SOLUTION:

We will solve this problem in two ways. The first way will go through the process of reduction of order. The second way will simply compute y_2 using the general solution formula which can be derived from the reduction of order method.

Method 1: Performing the steps of reduction of order

Note that $y_1 = e^{7x}$, $y_1' = 7e^{7x}$, and $y_1'' = 49e^{7x}$. We take $y = u(x)y_1(x)$, in which case the product rule gives

$$y' = u'y_1 + uy_1' = u'e^{7x} + 7ue^{7x} = \left(u' + 7u\right)e^{7x}$$

and

$$y'' = \left(u' + 7u\right)' e^{7x} + 7\left(u' + 7u\right) e^{7x}$$
$$= \left(u'' + 14u' + 49u\right) e^{7x}.$$

Thus,

$$0 = y'' - 14y' + 49y$$
$$= \left(u'' + 14u' + 49u\right) e^{7x} - 14\left(u' + 7u\right) e^{7x} + 49e^{7x}$$
$$= u'' e^{7x}.$$

Since $e^{7x} \neq 0$ for every x in $(-\infty, \infty)$ we have that $u'' = 0$ for all x on $(-\infty, \infty)$. We make the substitution $w = y'$ so that we have

$$w' = 0$$

for all x in $(-\infty, \infty)$, which is a first order linear differential equation. It is also separable, and we could write it as $\frac{dw}{dx} = 0$, or $dw = 0\,dx$ and integrate both sides to find $w = c_1$, where c_1 is an arbitrary constant. (We could have solved the differential equation $w' = 0$ other ways as well.) Since $u' = w = c_1$, we integrate again to find

$$u = c_1 x + c_2,$$

where c_1 and c_2 are both arbtrary constants.

Thus,

$$y(x) = u(x)y_1(x) = (c_1 x + c_2) e^{7x}$$
$$= c_1 x e^{7x} + c_2 e^{7x}$$

is a two-parameter family of solutions to the differential equation.

Note that this family of solutions contains the initial solution $y_1(x) = e^{7x}$, which arises if we take $c_1 = 0$ and $c_2 = 1$. Thus, the desired second solution y_2 is the other piece. We therefore take $y_2 = xe^{7x}$ (we could have taken any nonzero multiple of this).

Method 2: Using the formula.

Here we use the formula

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1(x)^2}\,dx$$

which produces the desired solution y_2 given a differential equation in the form $y'' + P(x)y' + Q(x)y = 0$ and one solution y_1. We note that in our case we have $P(x) = -14$ and $y_1(x) = e^{7x}$. Thus,

$$y_2 = e^{7x} \int \frac{e^{14\int dx}}{e^{14x}}\,dx$$
$$= e^{7x} \int \frac{e^{14x}}{e^{14x}}\,dx$$
$$= e^{7x} \int dx$$
$$= xe^{7x}.$$

Note that we omit the "plus c" with this integral (if we include it, it will produce a multiple of the solution y_1).

Comments:

Regardless of the method taken above, note that the Wronskian $W(e^{7x}, xe^{7x}) \neq 0$ for any x in $(-\infty, \infty)$ and since y_1 and y_2 are two linearly independent solutions to the second order homogeneous linear differential equation $y'' - 14y' + 49y = 0$ we have they form a fundamental set of solutions to the differential equation, and thus

$$y = c_1 xe^{7x} + c_2 e^{7x}$$

is the general solution to this differential equation. However, the solution to the problem is simply $y_2 = xe^{7x}$ (or any nonzero multiple of this).

Correct Answers:

- `x*e^(7*x)`

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10. (1 point) The function $y_1(x) = \cos(5x)$ is a solution to the differential equation $y'' + 25y = 0$. Use reduction of order to find another solution y_2 to this differential equation.

$y_2 =$ __________ help (formulas)

Solution:

SOLUTION:

We will solve this problem in two ways. The first way will go through the process of reduction of order. The second way will simply compute y_2 using the general solution formula which can be derived from the reduction of order method.

Method 1: Performing the steps of reduction of order

Note that $y_1 = \cos(5x)$, $y_1' = -5\sin(5x)$, and $y_1'' = -25\cos(5x)$. We take $y = u(x)y_1(x)$, in which case the product rule gives

$$y' = u'y_1 + uy_1' = u'\cos(5x) - 5u\sin(5x)$$

and

$$y'' = u''\cos(5x) - 5u'\sin(5x) - 5u'\sin(5x) - 25u\cos(5x)$$
$$= \left(u'' - 25u\right)\cos(5x) - 10u'\sin(5x).$$

Thus,

$$0 = y'' + 25y$$
$$= \left(u'' - 25u\right)\cos(5x) - 10u'\sin(5x) + 25u\cos(5x)$$
$$= u''\cos(5x) - 10u'\sin(5x).$$

We make the substitution $w = u'$ so that $w' = u''$ and we have

$$\cos(5x)w' - 10\sin(5x)w = 0$$

which is a first order linear differential equation. We solve it using the methods developed for such differential equations before. Namely, we rewrite it as

$$w' - 10\frac{\sin(5x)}{\cos(5x)}w = 0$$

and note that the integrating factor is

$$\begin{aligned} e^{\int P(x)\,dx} &= e^{-10\int \frac{\sin(5x)}{\cos(5x)}\,dx} \\ &= e^{2\ln(\cos(5x))} \\ &= e^{\ln(\cos^2(5x))} \\ &= \cos^2(5x). \end{aligned}$$

Thus, multiplying both sides of $w' - 10\frac{\sin(5x)}{\cos(5x)}w = 0$ by $\cos^2(5x)$ and recalling the product rule on the left we find this becomes

$$\frac{d}{dx}\left(\cos^2(5x)w\right) = 0.$$

Integrating both sides gives

$$\cos^2(5x)w = c_1,$$

where c_1 is an arbitrary constant. Solving for w gives $w = c_1\cos^{-2}(5x) = c_1\sec^2(5x)$. Since $u' = w = c_1\sec^2(5x)$ we integrate to find

$$u = \frac{c_1}{5}\tan(5x) + c_2 = c_1\tan(5x) + c_2,$$

where c_1 and c_2 are arbitrary constants.

Thus,

$$\begin{aligned} y(x) = u(x)y_1(x) &= (c_1\tan(5x) + c_2)\cos(5x) \\ &= c_1\sin(5x) + c_2\cos(5x) \end{aligned}$$

is a two-parameter family of solutions to the differential equation.

Note that this family of solutions contains the initial solution $y_1(x) = \cos(5x)$, which arises if we take $c_1 = 0$ and $c_2 = 1$. Thus, the desired second solution y_2 is the other piece. We therefore take $y_2 = \sin(5x)$ (we could have taken any nonzero multiple of this).

Method 2: Using the formula.

Here we use the formula

$$y_2(x) = y_1(x)\int \frac{e^{-\int P(x)\,dx}}{y_1(x)^2}\,dx$$

which produces the desired solution y_2 given a differential equation in the form $y'' + P(x)y' + Q(x)y = 0$ and one solution y_1. We note that in our case we have $P(x) = 0$ and $y_1(x) = \cos(5x)$.

Thus,

$$\begin{aligned} y_2 &= \cos(5x)\int \frac{e^{\int 0\,dx}}{\cos^2(5x)}\,dx \\ &= \cos(5x)\int \frac{e^0}{\cos^2(5x)}\,dx \\ &= \cos(5x)\int \sec^2(5x)\,dx \\ &= \frac{1}{5}\cos(5x)\tan(5x) \\ &= \frac{1}{5}\sin(5x). \end{aligned}$$

Note that we omit the "plus c" with this integral (if we include it, it will produce a multiple of the solution y_1). We could take our solution y_2 to be $\frac{1}{5}\sin(5x)$. However, since we can also choose any nonzero multiple, we take $y_2 = \sin(5x)$.

Comments:

Regardless of the method taken above, note that the Wronskian $W(\cos(5x), \sin(5x)) = 1 \neq 0$ for any x in $(-\infty, \infty)$ and since y_1 and y_2 are two linearly independent solutions to the second order homogeneous linear differential equation $y'' + 25y = 0$ we have they form a fundamental set of solutions to the differential equation, and thus

$$y = c_1\cos(5x) + c_2\sin(5x)$$

is the general solution to this differential equation. However, the solution to the problem is simply $y_2 = \sin(5x)$ (or any nonzero multiple of this).

Correct Answers:

- `sin(5*x)`

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11. (1 point) The function $y_1(x) = e^{\frac{2}{3}x}$ is a solution to the differential equation $9y'' - 12y' + 4y = 0$. Use reduction of order to find another solution y_2 to this differential equation.

$y_2 =$ __________ help (formulas)

Solution:

SOLUTION:

We will solve this problem in two ways. The first way will go through the process of reduction of order. The second way will simply compute y_2 using the general solution formula which can be derived from the reduction of order method.

Method 1: Performing the steps of reduction of order

Note that $y_1 = e^{\frac{2}{3}x}$, $y_1' = \frac{2}{3}e^{\frac{2}{3}x}$, and $y_1'' = \frac{4}{9}e^{\frac{2}{3}x}$. We take $y = u(x)y_1(x)$, in which case the product rule gives

$$y' = u'y_1 + uy_1' = u'e^{\frac{2}{3}x} + \frac{2}{3}ue^{\frac{2}{3}x} = \left(u' + \frac{2}{3}u\right)e^{\frac{2}{3}x}$$

and

$$y'' = \left(u' + \frac{2}{3}u\right)' e^{\frac{2}{3}x} + \frac{2}{3}\left(u' + \frac{2}{3}u\right) e^{\frac{2}{3}x}$$
$$= \left(u'' + \frac{4}{3}u' + \frac{4}{9}u\right) e^{\frac{2}{3}x}.$$

Thus,

$$0 = 9y'' - 12y' + 4y$$
$$= 9\left(u'' + \frac{4}{3}u' + \frac{4}{9}u\right) e^{\frac{2}{3}x} - 12\left(u' + \frac{2}{3}u\right) e^{\frac{2}{3}x} + 4ue^{\frac{2}{3}x}$$
$$= 16u''e^{\frac{2}{3}x}.$$

Since $16e^{\frac{2}{3}x} \neq 0$ for every x in $(-\infty,\infty)$ we have that $u'' = 0$ for all x on $(-\infty,\infty)$. We make the substitution $w = y'$ so that we have

$$w' = 0$$

for all x in $(-\infty,\infty)$, which is a first order linear differential equation. It is also separable, and we could write it as $\frac{dw}{dx} = 0$, or $dw = 0\,dx$ and integrate both sides to find $w = c_1$, where c_1 is an arbitrary constant. (We could have solved the differential equation $w' = 0$ other ways as well.) Since $u' = w = c_1$, we integrate again to find

$$u = c_1x + c_2,$$

where c_1 and c_2 are both arbtrary constants.

Thus,

$$y(x) = u(x)y_1(x) = (c_1x + c_2)\, e^{\frac{2}{3}x}$$
$$= c_1xe^{\frac{2}{3}x} + c_2e^{\frac{2}{3}x}$$

is a two-parameter family of solutions to the differential equation.

Note that this family of solutions contains the initial solution $y_1(x) = e^{\frac{2}{3}x}$, which arises if we take $c_1 = 0$ and $c_2 = 1$. Thus, the desired second solution y_2 is the other piece. We therefore take $y_2 = xe^{\frac{2}{3}x}$ (we could have taken any nonzero multiple of this).

Method 2: Using the formula.

Here we use the formula

$$y_2(x) = y_1(x)\int \frac{e^{-\int P(x)\,dx}}{y_1(x)^2}\,dx$$

which produces the desired solution y_2 given a differential equation in the form $y'' + P(x)y' + Q(x)y = 0$ and one solution y_1. We note that in our case we have our differential equation becomes $y'' - \frac{12}{9}y' + \frac{4}{9}y = 0$ so that $P(x) = -\frac{12}{9}$ and $y_1(x) = e^{\frac{2}{3}x}$. Thus,

$$y_2 = e^{\frac{2}{3}x}\int \frac{e^{\frac{12}{9}\int dx}}{e^{\frac{4}{3}x}}\,dx$$
$$= e^{\frac{2}{3}x}\int \frac{e^{\frac{4}{3}x}}{e^{\frac{4}{3}x}}\,dx$$
$$= e^{\frac{2}{3}x}\int dx$$
$$= xe^{\frac{2}{3}x}.$$

Note that we omit the "plus c" with this integral (if we include it, it will produce a multiple of the solution y_1).

Comments:

Regardless of the method taken above, note that the Wronskian $W(e^{\frac{2}{3}x}, xe^{\frac{2}{3}x}) \neq 0$ for any x in $(-\infty,\infty)$ and since y_1 and y_2 are two linearly independent solutions to the second order homogeneous linear differential equation $9y'' - 12y' + 4y = 0$ we have they form a fundamental set of solutions to the differential equation, and thus

$$y = c_1xe^{\frac{2}{3}x} + c_2e^{\frac{2}{3}x}$$

is the general solution to this differential equation. However, the solution to the problem is simply $y_2 = xe^{\frac{2}{3}x}$ (or any nonzero multiple of this).

Correct Answers:

- `x*e^(2*x/3)`