Undetermined coefficients

RECALL Given (a, b, c constants)

$$a y'' + by' + cy = f(x) \qquad (*)$$

with $f(x) \neq 0$, the general soln is of the form

$$y = y_h + y_p$$

where

☆ → y_h is the general soln of

$$ay'' + by' + cy = 0,$$ and

? → y_p is a particular soln of $(*)$.

* First, find y_h.

Then we consider 3 cases where we can find y_p:

$\boxed{1}$ $f(x)$ is a polynomial.

since order 2 DE

→ Then take $y_p = Ax^2 + Bx + C$, where A, B, C are constants.

→ Plug in this y_p (Need derivatives)

→ Solve for A, B, C.

Thus, $A = 1$, $B = -2$, $C = -1$, so

$$y_p = x^2 - 2x - 1,$$ and

the gen soln is

$$y = c_1 e^{-x} + c_2 e^{-2x} + x^2 - 2x - 1.$$

$\boxed{2}$ $f(x)$ consists of $\sin(x)$ and $\cos(x)$ terms.

→ Then consider $y_p = A\cos(x) + B\sin(x)$.

→ Plug in

→ solve for A, B.

$\boxed{3}$ $f(x)$ has a $x^k e^{mx}$ form.

→ Then take all derivatives of $x^k e^{mx}$, and include (with unknown constants) in the y_p.

→ plug in

→ solve for the constants.

Ex) ~~If~~ $f(x) = x^2 e^{2x}$, then derivates include:

$2xe^{2x}$, $2x^2e^{2x}$, $2e^{2x}$, so....

The y_p we would consider is

$$y_p = Ax^2e^{2x} + Bxe^{2x} + Ce^{2x}$$

$\boxed{4}$ Combine all of the above.

Ex1 solve $y'' + 3y' + 2y = 2x^2 + 2x - 6 + 2\cos(x) + 3xe^{2x}$.

soln

$y_h = c_1 e^{-x} + c_2 e^{-2x}$

our y_p will have the form:

$y_p = y_{p_1} + y_{p_2} + y_{p_3}$, where

$y_{p_1} = Ax^2 + Bx + C$,

$y_{p_2} = A\cos(x) + B\sin(x)$, and

$y_{p_3} = Axe^{2x} + Be^{2x}$.

Then can compute each one independently
(then add)

Ex1
$\rightarrow y_{p_1} = x^2 - 2x - 1$.

$\rightarrow y_{p_2} = A\cos(x) + B\sin(x)$

$y_{p_2}' = -A\sin(x) + B\cos(x)$

$y_{p_2}'' = -A\cos(x) - B\sin(x)$, so

$-A\cos(x) - B\sin(x) + 3(-A\sin(x) + B\cos(x))$
$+ 2(A\cos(x) + B\sin(x)) = 2\cos(x)$.

$\Rightarrow \quad A + 3B = 2, \quad -3A + B = 0$

$\Rightarrow \quad A = 1/5, \quad B = 3/5$

so $\quad y_{p_2} = \frac{1}{5}\cos(x) + \frac{3}{5}\sin(x).$

$\rightarrow \quad y_{p_3} = Axe^{2x} + Be^{2x}$

$y_{p_3}' = 2Axe^{2x} + (2B + A)e^{2x}$

$y_{p_3}'' = 4Axe^{2x} + (4A + 4B)e^{2x}$

plugging in gives

$$\left.\begin{aligned} 12A &= 3 \\ 7A + 12B &= 0 \end{aligned}\right\} \quad \begin{aligned} A &= \frac{1}{4} \\ B &= -\frac{7}{48} \end{aligned}$$

so $\quad y_{p_3} = \frac{1}{4}xe^{2x} - \frac{7}{48}e^{2x}$

Thus, the general soln is

$$y = y_h + y_p$$

$$= c_1 e^{-x} + c_2 e^{-2x} + x^2 - 2x - 1$$

$$+ \frac{1}{5}\cos(x) + \frac{3}{5}\sin(x)$$

$$+ \frac{1}{4} x e^{2x} - \frac{7}{48} e^{2x}.$$