Linear combinations of solutions

WARM-UP: We have $y_1(x) = e^x$ and $y_2(x) = e^{-x}$ are both solutions for $y'' - y = 0$. Can you show $2y_1 + 3y_2$ is a soln without taking derivatives?

soln

$$(2y_1 + 3y_2)'' - (2y_1 + 3y_2)$$
$$= (2y_1)'' + (3y_2)'' - 2y_1 - 3y_2$$
$$= (2y_1'' - 2y_1) + (3y_2'' - 3y_2) = 0 + 0 = 0.$$

Thm Suppose $y_1, \dots, y_n$ are solns for a homog linear DE (on I), then $y = c_1 y_1 + \dots + c_n y_n$ is a soln for the DE (on I) for any constants $c_1, c_2, \dots, c_n$.

Notes:

(a) If y is a soln, so is $3y$, $4y$, etc. In fact cy is a soln for any constant c.

(b) A homog linear DE always has $y = 0$ as a soln.

Ex] (i) Consider $y'' + y = 0$. Recall $y_1 = \cos(t)$, $y_2 = \sin(t)$ are solns. Thus so is

$$y = c_1 \cos(t) + c_2 \sin(t).$$

(ii) We could have taken $y_1 = \cos(t)$, $y_2 = 3\cos(t)$ as solns. Then

$$y = c_1 \cos(t) + c_2 \cdot 3\cos(t) = (c_1 + 3c_2)\cos(t)$$

$$= C \cos(t).$$

→ Less interesting....

Linearly independent and dependent functions/solutions

DEFN | Functions $f_1(x), \ldots, f_n(x)$ are:

(a) linearly independent if the only constants $c_1, \ldots, c_n$ such that $c_1 f_1(x) + \cdots + c_n f_n(x) = 0$ are $c_1 = c_2 = \cdots = c_n = 0$.

(b) linearly dependent if they are not linearly independent.

* Note: (a) equivalent to notion that no f_j can be expressed as a linear combination of the other functions.

Ex1

(a) $\sin(x), \cos(x)$] lin indep $c_1\cos(x) + c_2\sin(x) = 0$ $c_1 = 0 = c_2$

(b) e^x, e^{-x}] lin indep

(c) $x, \ln(x)$] lin indep

(d) $x^2+2x+1, x+1, x$ lin indep

$$\underbrace{0}(x^2+2x+1) + \underbrace{0}(x+1) + \underbrace{0}(x) = 0$$

(e) x^2+2x+1, x^2+x+1, x ✓ lin dep

$$\underbrace{1}(x^2+2x+1) + \underbrace{-1}(x^2+x+1) + \underbrace{-1}(x) = 0$$

Fundamental set of solutions

DEFN| Any set $y_1, \ldots, y_n$ of n-many linearly independent solutions for a homogeneous n^{th}-order linear DE $a_n(x) y^{(n)} + \cdots + a_1(x) y' + a_0(x) y = 0$ on an interval I is called a fundamental set of solutions.

* Note: 2 things
  - → number of solns (equal the order)
  - → solns lin indep.

Ex) $y_1 = \cos(t)$, $y_2 = \sin(t)$ is a fundamental set of solutions for $y'' + y = 0$.

→ seen/discussed they are lin indep, and

→ 2 soln and 2nd ord linear homog DE.

Q: Does there always exist a fund set of solns?

A:

Thm) There exists a fundamental set of solns for $a_n(x) y^{(n)} + \cdots + a_1(x) y' + a_0(x) y = 0$ on some interval I.

Q: How do we know if solns are lin indep?

A: Wronskian,

The Wronskian

DEFN: Suppose we can take the first $n-1$ many derivatives of functions $f_1(x), \ldots, f_n(x)$. The Wronskian of $f_1(x), \ldots, f_n(x)$ is the determinant

$$W(f_1, \ldots, f_n) = \det \begin{pmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ f_1'' & f_2'' & \cdots & f_n'' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & & f_n^{(n-1)} \end{pmatrix}$$

Notes:
→ You need to know the $n=2$ case.
→ I would give $n=3$ case in exam.

Notes:

(a) $n=2$:

$$W(f_1, f_2) = \det\left(\begin{pmatrix} f_1 & f_2 \\ f_1' & f_2' \end{pmatrix}\right) = (f_1)(f_2') - (f_1')(f_2)$$

(b) $n=3$:

$$W(f_1, f_2, f_3) = \det\left(\begin{pmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{pmatrix}\right)$$

$$= f_1 \det\left(\begin{pmatrix} f_2' & f_3' \\ f_2'' & f_3'' \end{pmatrix}\right) - f_2 \det\left(\begin{pmatrix} f_1' & f_3' \\ f_1'' & f_3'' \end{pmatrix}\right) + f_3 \det\left(\begin{pmatrix} f_1' & f_2' \\ f_1'' & f_2'' \end{pmatrix}\right)$$

$$= f_1\left(f_2' f_3'' - f_2'' f_3'\right) - f_2\left(f_1' f_3'' - f_1'' f_3'\right) + f_3\left(f_1' f_2'' - f_1'' f_2'\right)$$

Q: How does the Wronskian help?

A:

<u>Thm</u> Suppose $y_1, \dots, y_n$ are solutions to a homog linear DE of nth-order on an interval I. Then $y_1, \dots, y_n$ are linearly independent if and only if:

$$W(y_1, y_2, \dots, y_n) \neq 0$$

for every x in I

<u>Ex</u> (a) $y_1 = e^x,\ y_2 = e^{-x}$: $W(e^x, e^{-x}) = \det\left(\begin{pmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{pmatrix}\right)$

$= -e^x e^{-x} - e^x e^{-x} = -1-1 = -2$

$\neq 0$ on $(-\infty, \infty)$.

$\Rightarrow$ Lin indep on $(-\infty, \infty)$.

(b) $\det\left(\begin{pmatrix} x & 3x \\ 1 & 3 \end{pmatrix}\right) = 3x - 3x = 0$ everywhere $\rightarrow$ Lin dep.

General solutions

Thm If $y_1, \dots, y_n$ form a fundamental set of solns for an nth-order homog linear DE on I, then the general soln of the DE on I is

$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ for arbitrary constants $c_1, c_2, \dots, c_n$.

Ex) Recall $y'' + y = 0$. Note $y_1 = \cos(t)$, $y_2 = \sin(t)$ are solns.

→ How many? Need 2 (2nd order), and have 2!

→ Lin indep?

$$W(\cos(t), \sin(t)) = \det\left(\begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix}\right)$$

$$= \cos(t)\cos(t) - (-\sin(t))\sin(t) = \cos^2(t) + \sin^2(t) = 1$$

$\neq 0$ on $(-\infty, \infty)$ ⇒ So lin indep!

→ So y_1, y_2 fund set of solns.

→ Thus!!!! $\boxed{y = c_1\cos(t) + c_2\sin(t)}$ is a general soln.