Goal and idea - Module 11

GOAL:

Given a second-order linear DE, we've learned that we need two linear independent solutions to form a fundamental set of solutions for the DE. It is common, however, that in a process of solving the DE, we only find one solution. Luckily, we are able to

- develop a method called "reduction of order" that allows us to bootstrap this solution in a way to find another.

IDEA:

By modifying one solution of a second-order DE and running entering this into the DE we are often able to procure another (linearly independent!) solution.

Warm-up | Consider $y'' + y = 0$.
Suppose we know $y_1 = \sin(x)$ is a solution.

Q: How many solutions do we need for a fundamental set of solutions?

A: $y''+y=0$ is a second order homogeneous linear differential equation, so we need two solutions because the number of solutions is equal to the order of the differential equation.

Q: Given one solution can we find the other?

① Let $u(x)$ be an arbitrary function, and set $y = u(x)y_1(x) = u(x)\sin(x) = u\sin(x)$.

(a) Find y' and y''

*Remember $u\sin(x)$ that "u" is a function of "x" so we have to use the product rule.

Recall The Product Rule
$\boxed{(uv)' = u'v + uv'}$

$$y' = u'\sin(x) + u\cos(x)$$

$$u'\sin(x) + u\cos(x)$$

$$y'' = u''\sin(x) + u'\cos(x) + u'\cos(x) - u\sin(x)$$
$$= u''\sin(x) + 2u'\cos(x) - u\sin(x)$$

(b) Plug-in y, y', y'' into the differential equation

$y = u\sin(x)$
$y' = u'\sin(x) + u\cos(x)$
$y'' = u''\sin(x) + 2u'\cos(x) - u\sin(x)$
$y'' + y = 0$

$$\to u''\sin(x) + 2u'\cos(x) - u\sin(x) + u\sin(x) = 0$$
$$u''\sin(x) + 2u'\cos(x) - \cancel{u\sin(x)} + \cancel{u\sin(x)} = 0$$
$$\to u''\sin(x) + 2u'\cos(x) = 0$$

(c) Put that ($u''\sin(x) + 2u'\cos(x) = 0$) in the form $u'' + P(x)u' + Q(x)u = 0$

$$u'' + \frac{2\cos(x)}{\sin(x)}u'' = 0$$

$P(x) = \frac{2\cos(x)}{\sin(x)}$

$Q(x) = 0$

Almost in form of 1st-order linear equation where we do know to solve, but not quite because it's 2nd-order

But what we can do...

(d) Make subsitution u' with w (ie. $w = u'$)

$$\underbrace{u'' + \frac{2\cos(x)}{\sin(x)}u''}_{w=u'} = 0$$

$$w' + \frac{2\cos(x)}{\sin(x)}w = 0$$

(e) Recall how to solve $w' + \frac{2\cos(x)}{\sin(x)}w = 0$!

Module 05: 1st-Order Linear Equations

(i) Given $\frac{dy}{dx} + P(x)y = Q(x)$,

(ii) Find the integrating factor $e^{\int P dx}$

(iii) Multiply both sides of $\frac{dy}{dx} + Py = Q$ by $e^{\int P dx}$ KNOWING that the Left Hand Side becomes $\frac{d}{dx}(e^{\int P dx}y) = e^{\int P dx}Q$

(iv) Integrate both sides

(V) Solve for "y"

(ii) Find integrating factor

Plus a constant but recall 1st-order linear differential equation when you have "+C" it gets canceled

$$e^{\int P dx} = e^{2\int \frac{\cos(x)}{\sin(x)}dx} = e^{2\ln|u|} = e^{2\ln|u|}$$

Sub $u = \sin(x)$

Recall
$\alpha\ln|x| = \ln|x^\alpha|$
$e^{\ln|x^\alpha|} = x^\alpha$

$$= e^{\ln|u^2|} = u^2 = \sin^2(x)$$

actual integrating factor

(iii) Recall why we care about the integrating factor.
The integrating factor has a property where we take the integrating factor and multiply "y" (in our case "w") then it gives us $w' + \frac{2\cos(x)}{\sin(x)}w = 0$

The left side becomes this

$$\frac{d}{dx}(\sin^2(x)w) = \frac{d}{dx}(0)$$

(iv) Integrate both sides

$$\frac{d}{dx}[\sin^2(x)w] = \frac{d}{dx}[0]$$
$$\to \int\frac{d}{dx}[\sin^2(x)w] = \int\frac{d}{dx}[0]$$
$$\to \sin^2(x)w = C$$

(v) Solve for w

$$w = \frac{C}{\sin^2(x)}$$

(f) Plug back in $u' = w$

$$u' = \frac{C}{\sin^2(x)}$$

(g) Plug back in $u' = w$

$$u' = \frac{c}{\sin^2(x)}$$

(h) Separation of Variables

$$\frac{du}{dx} = \frac{c}{\sin^2(x)} \rightarrow du = \frac{c}{\sin^2(x)}dx$$

*Integrate

$$\int du = \int \frac{c}{\sin^2(x)}dx$$

$$\rightarrow u = c\int \csc^2(x)dx$$

$$= -c\cot(x) + K$$

$$= -C_1\cot(x) + C_2$$

Removed negative because it's a constant $C = \pm\#$

$$= C_1\cot(x) + C_2$$

All we did was solve for 'u'
Recall that we "Let $u(x)$ be an arbitrary function, and set $y = u(x)y_1(x) = u(x)\sin(x) = u\sin(x)$."

(i) Plug u into $y = uy_1$

$$y = (C_1\cot(x) + C_2)\sin(x)$$

$$= C_1\left(\frac{\cos(x)}{\sin(x)} + C_2\right)\sin(x)$$

$$= C_1\cos(x) + C_2\sin(x)$$

(j) Take different c_1, c_2 to get two linearly independent solutions.

$C_1 = 0,\ C_2 = 1 \rightarrow \sin(x)$

$C_1 = 1,\ C_2 = 0 \rightarrow \cos(x)$

(k) Thus, two solutions are

$$y_1 = \sin(x),\ y_2 = \cos(x)$$

(l) General solution for $y'' + y = 0$ is

$$y = C_1\sin(x) + C_2\cos(x)$$

This is the method of Reduction of Order

OR

we could use the formula

$$y_2 = y_1(x)\int \frac{e^{-\int P(x)dx}}{(y_1(x))^2}dx$$

where P is from $y'' + P(x)y' + Q(x)y = 0$

In our case with the differential equation we were given $y'' + 0y' + 1y = 0$

Reduction of Order – Summary

When to use: Given a 2^{nd}-order homogeneous linear differential equation and ONLY ONE solution.

Why to use: Find the other linearly independent solution to form a fundamental set of solutions.

How to use: TWO OPTIONS

(a) Long route above (don't memorize)

(b) The formula

$$y_1(x)\int \frac{e^{-\int P(x)dx}}{(y_1(x))^2}dx$$

Ex Given $y'' + y = 0$, and $y_1 = \cos(x)$, find y_2 with reduction of order (y_2 will be linearly independent)

Soln Need y_2

$$y_2 = y_1\int \frac{e^{-\int P dx}}{(y_1)^2}dx = \cos(x)\int \frac{e^{\int 0 dx}}{\cos^2(x)}dx$$

$$e^{0+c} = e^c = C$$

$$= \cos(x)\int \frac{1}{\cos^2(x)}dx = \cos(x)\int c\sec^2(x)dx$$

$$= C\cos(x)\tan(x) + CK\cos(x) = C\sin(x) + K\cos(x)$$

Answer: $y_2 = \sin(x)$; Gen Soln: $y = C_1\cos(x) + C_2\sin(x)$

Expectation checklist - Module 11

At the completion of this module, you should:

- be able to use one solution of a second-order homogeneous linear differential equation to find another solution to the differential equation that is linearly independent.

This could be done in two ways:

- the method in the video, or
- using the given formula.

You will be assessed on your understanding of these concepts:

- within the homework,
- on the quizzes, and
- later, on the exam.

Coming up next, we:

- Actually solving a certain type or linear differential equation!