

You must show and explain your work! The following formulas may, or may not, be useful.

1. (10 points)

a. (2 pts) Circle all of the following expression which are linear differential equations.

(i) $y''' - y'' + y = 0$

(ii) $(y')^2 = y^2$

(iii) $\ln(y') - y = x \cos(x)$

(iv) $e^x y^{(7)} - (x^2 - 3)y = \sin(x^2)$ → 0, 5

b. (2 pts) Circle all of the following expression which are linear nonhomogeneous differential equations.

(i) $y''' - y'' = y - x^2 \sin(x)$ ~~$y''' - y'' - y = x^2 \sin(x)$~~

(ii) $t^2 \frac{d^3 f}{dt^3} + t \frac{df}{dt} + t^5 = t^5$

(iii) ~~$y^{(3)} - \sec(x)y' - y = e^x$~~

(iv) ~~$x^{(3)} - x^5 y' - 2x^2 = 0$~~

c. (3 pts) Compute $\mathcal{L}\{e^{5t} \sin(3t)\}$.

Laplace Transform	
(a)	$\mathcal{L}\{C\} = \frac{C}{s}$
(b)	$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$
(c)	$\mathcal{L}\{e^{kt}\} = \frac{1}{s-k}$
(d)	$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}$
(e)	$\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}$

~~$\mathcal{L}\{e^{5t}\} \cdot \mathcal{L}\{\sin(3t)\}$~~

~~$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$~~

$\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}$

~~$\frac{5!}{s^5+1}$~~

~~$\frac{3}{s^2+(\sqrt{3})^2}$~~

~~$\frac{5! \cdot 3}{s^6 [s^2 + (\sqrt{3})^2]}$~~

d. (3 pts) Compute $\mathcal{L}^{-1}\left\{\frac{9}{s+2} + \frac{3}{s^2+16}\right\}$.

~~$\mathcal{L}^{-1}\left\{\frac{9}{s+2}\right\}$~~

$+ \quad \mathcal{L}^{-1}\left\{\frac{3}{s^2+16}\right\}$

~~$\rightarrow 9 \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$~~

Does Not Match Known Forms
So multiply by const to force it

~~$\mathcal{L}^{-1}\left\{\frac{1}{s-k}\right\} = e^{kt}$~~ ; ~~$\mathcal{L}^{-1}\left\{\frac{1}{s-(-2)}\right\}$~~

$\rightarrow \mathcal{L}^{-1}\left\{\frac{3}{4} \cdot \frac{4}{s^2+4^2}\right\}$

~~$\rightarrow 9e^{-2t}$~~

Now it matches

$\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin(kt)$

$\rightarrow \frac{3}{4} \sin(4t)$

~~$9e^{-2t} + \frac{3}{4} \sin(4t)$~~

Inverse Laplace Transform

Instead of computing the Inverse Laplace Transform from scratch ...

(a) $\mathcal{L}^{-1}\left\{\frac{C}{s}\right\} = C$

(d) $\mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos(kt)$

(b) $\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$

(e) $\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin(kt)$

(c) $\mathcal{L}^{-1}\left\{\frac{1}{s-k}\right\} = e^{kt}$

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Not separable X

2. (10 points) Solve the following differential equations.

a. (5 pts) $e^y dx + (2y + xe^y) dy = 0$

Step 1 Check if it's exact:

$$e^y dx + (2y + xe^y) dy = 0$$

Partial Der'V

$$(e^y)' = e^y \checkmark$$

$$(2y + xe^y)' \rightarrow (2y)' + (xe^y)' \\ \rightarrow 0 + e^y = e^y \checkmark$$

Find (antiderivative with respect to "x" or "y")

$$F(x, y) = \int (2y + xe^y) dy$$

$$\cancel{\int (2y) dy} + \int (xe^y) dy$$

$$\cancel{y^2} + xe^y + g'(x)$$

$$y^2 + xe^y + g'(x) = e^y$$

$$g'(x) = -y^2 - xe^y + e^y$$

$$0 - e^y + e^y$$

$$g'(x) = 0; \int 0 dx = C_1$$

$$y^2 + xe^y + C_1 = C_2$$

$$\text{So } y^2 + xe^y + C = C_2$$

b. (5 pts) $y' - \frac{3}{x+1}y = (x+1)^4$

Module 05 1st-Order Linear Equations

$$\frac{dy}{dx} + P_y = Q \quad ; \quad P(x) = -\frac{3}{x+1} \quad ; \quad Q(x) = (x+1)^4$$

Step 2 Calculate Integrating Factor

$$e^{\int P dx} = e^{\int -\frac{3}{x+1} dx} = e^{-3 \ln|x+1|} = \frac{1}{(x+1)^3}$$

Step 3 Multiply left and right

by the integrating factor

$$\left(\frac{dy}{dx} - \frac{3}{x+1}\right) \frac{1}{(x+1)^3} = \frac{1}{(x+1)^3} (x+1)^4$$

$$\rightarrow \frac{d}{dx} \left(\frac{1}{x^3 + 3x^2 + 3x + 1} y \right) = x+1$$

Step 4 Integrate both sides

$$\int \frac{d}{dx} \left(\frac{1}{x^3 + 3x^2 + 3x + 1} y \right) dx = \int x+1$$

$$\text{Let } \frac{1}{x^3 + 3x^2 + 3x + 1} = U$$

$$y = \frac{x^2}{2} U + XU + CU$$

$$C$$

2nd-order linear homogeneous

3. (20 points) Solve the following differential equations. Your answers should be written as real general solutions.

a. (5 pts) $y'' - 4y' + 13y = 0$

1) Plug-in $y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2e^{mx}$

Skipping a step, instead of plugging-in and refactoring out, we know are able to factor out " e^{mx} "

This gives us

$$e^{mx}(m^2 - 4m + 13) = 0$$

So we need

$$m^2 - 4m + 13 = 0$$

2) Find $m_1 \notin m_2$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{4 \pm \sqrt{(-4)^2 - [4(1)(13)]}}{2(1)}$$

b. (5 pts) $x^2y'' + 5xy' + 4y = 0$

~~$x^2 - 5x + 4 = 0$~~

2) Find $m_1 \notin m_2$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

~~$5 \pm \sqrt{25 + 16}$~~

~~$m = \frac{5}{2} \pm \frac{\sqrt{41}}{2}$~~

$$\begin{aligned} & \rightarrow \frac{4 \pm \sqrt{16 - 52}}{2} \\ & \rightarrow \frac{4 \pm \sqrt{-36}}{2} \end{aligned}$$

(Case 3) Complex Numbers w/ Conjugate Property

* Recall *
 $e^{i\theta} = \cos(\theta) + i\sin(\theta)$
 $e^{(3/2 + i\pi/2)x} = e^{3/2x} e^{i\pi/2 x}$
 $\rightarrow e^{3/2x} (\cos(\frac{\pi}{2}x) + i\sin(\frac{\pi}{2}x))$
Euler's Formula Real Complex

$$e^{2x} (\cos(\frac{\sqrt{-36}}{2}x) + \sin(\frac{\sqrt{-36}}{2}x))$$

Module 15 Cauchy - Euler

$x^2y'' + 5xy' - 4y = 0$ Need two derivatives

1) Take $y = x^m$

$$y = x^m \quad y' = mx^{m-1} \quad y'' = m(m-1)x^{m-2}$$

$$x^2y'' + 5xy' - 4y = 0$$

$$y = x^m, \quad y' = mx^{m-1}, \quad y'' = m(m-1)x^{m-2}$$

2) Plug into the differential equation

$$\text{This gives: } x^2(m(m-1)x^{m-2}) + 5x(mx^{m-1}) - 4x^m$$

$$x^2(m(m-1)x^{m-2}) + 5x(mx^{m-1}) - 4x^m = 0$$

Factor out " x^m "

$$\rightarrow x^m(m(m-1)) + 5x^m m - 4x^m = 0$$

Simplify

$$\rightarrow x^m(m^2 - m + 5m - 4) = 0$$

$$\rightarrow m^2 + 4m - 4 = 0$$

$$\rightarrow m^2 + 4m - 4 = 0$$

3) Solve for "m"! Quadratic formula

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\rightarrow \frac{4 \pm \sqrt{16 + 16}}{2}$$

$$2 \pm \sqrt{32}$$

$$2 \pm 2\sqrt{2}$$

SO -

Non homogeneous 2nd-order Linear

c. (10 pts) $y'' - 8y' + 15y = 8\cos(x) - 14\sin(x)$

Step 1 Find y_h

$$y'' - 8y' + 15y = 0$$

$$e^{mx} (M^2 - 8M + 15)$$

3) Solve for "m"! Quadratic formula

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$M_1 = \frac{-(-8) + 2}{2}, M_2 = \frac{-(-8) - 2}{2}$$

$$M_1 = 5, M_2 = 3$$

$$y_h = C_1 e^{5t} + C_2 e^{3t}$$

Case 2) $f(x)$ consists of $\sin(x)$ and/or $\cos(x)$ terms.
Any sines or cosines (not products of them i.e. \sin^2 or \cos^2)

Step 1 Then consider $y_p = A\cos(x) + B\sin(x)$

Step 2 Plug-in this y_p (need derivatives)

Step 3 Solve for A, B, C

Rest?

4. (10 points) Use the Laplace Transform to solve the initial value problem $y'' = 2 + 3t$ with initial conditions $y(0) = 0$, $y'(0) = 0$. (You must use the method of Laplace Transform for credit.)

