Finances -- Separable equation

Note: The following is an approximation, see below for more comments on other ways to go about planning for retirement

It's time to plan for retirement!

Question: How much money do we want in retirement?

Let's not plan on social security, additional income, or any possible pension. Instead, we will look to find how much money we will need to have invested.

Here are our assumptions and variables:

- Let $M = M(t)$ denote our amount of money, which is a function of time t.
- We will assume that we have our money invested in a well diversified collection of stocks and bonds, which is expected to return an *average* of 7% annually.
- We will say that the interest is compounded *continuously* for this example.
- Let's assume we want to live on $105,000 a year during your golden years (why not, right?).
- We will take $t = 0$ to be the time we start our retirement. So $M(0)$ is the initial amount of money we have set aside for retirement.

Then a model representing the rate of change of our money can be created as

$$\frac{dM}{dt} = .07M - 105000.$$

The derivative is our rate of change of money, and we say this is equal to the 7% growth of our money, minus the $1005000 we will be spending. This isn't how much money we have, it is the rate of change we expect in our retirement years.

The question remains: how much money do we need? That is, what is $M(0)$?

Let's first solve the differential equation so that we have our function M. Separating variables we rewrite the equation above as

$$\frac{1}{.07M - 105000} dM = dt.$$

Integrating both sides (using a u-substitution on the left) gives

$$\frac{1}{0.07} \ln|0.7M - 105000| = t + C.$$

Multiplying both sides by 0.07, expontiation both sides, and manipulating the constant we get $0.07M = Ce^{0.07t} + 105000$. Solving for M and relabeling the constant again we find

$$M(t) = Ce^{0.07t} + 1500000.$$

Let's return now to our question again: how much money do we need?

That is, what should our $M(0)$ be? Well, it depends on two additional assumptions we need to make:

- How long do we plan to live?
- How much money do we want to leave behind when we die?

We will show how we can answer the question making two different sets of assumptions here.

Scenario 1: Not worry about how long we live, and plan on leaving our entire retirement account behind.

In other words, we are asking what $M(0)$ needs to be so that withdrawing $105,000 a year won't change the value. That is, we want $M(t)$ to be constant and never change. The only way this can happen in the formula $M(t) = Ce^{0.07t} + 1500000$ is if $C = 0$, as otherwise $e^{0.07t} \to \infty$ as $t \to \infty$. Thus, we find that $M(t) = 1500000$ for all t. Thus, we need $M(0) = 1500000$, or $1,500,000!

In hindsight, you could argue this without using differential equations. You could just ask what M is needed so that $0.07M = 105000$.

Scenario 2: Plan on living 30 year, and don't leave a thing. Yolo.

We want to find $M(0)$, and we require that $M(30) = 0$. This latter piece is our initial condition, and we use it in our formula above to solve for C. Indeed, we have $0 = Ce^{(0.07)(30)} + 1500000$, which gives

$$C = \frac{-1500000}{e^{2.1}} \approx -183685$$

Thus,

$$M(t) = -183685e^{0.07t} + 1500000$$

in this scenario. Thus, $M(0) = 1500000 - 183685 = 1316315$. That is, we should start with $1,316,315!

Remaining questions?

You should have some! Here are some big ones I have, along with some comments.

1. Can we safely assume that we would return 7%?
2. What about inflation?
3. What's the deal with a withdrawal rate?

Let's look at question 1 first. The total annual return (including dividends) fot eh S&P 500 is 14.69% ([according to Wikipedia](https://en.wikipedia.org/wiki/S%26P_500_Index) (https://en.wikipedia.org/wiki/S%26P_500_Index)). People say the historical average return of the stock market is 10%. But past performance does not mean future rates will continue. Many annalists expect much lower returns moving forward. Also, differences in tax codes can greatly effect what this means in terms of "making money." On the other hand, there are many reasons to believe that a 5-10% annual return is probable.

For the second question, the answer is: inflation is a serious thing! In fact, if you were surprised you "only" needed around 1.4 million dollars for a comfortable retirement (and that's not including social security, or anything else!), then it may be because we did not account at all for inflation or taxes. Both of these will eat significantly away at things. This is one reason that something like 1 3-4% rule comes about, and thus the need for more money. See the next answer for more information on that.

The third question is interesting. One can ask what withdrawal rate to make on your first year of retirement (the future years will be readjusted based on inflation). Using simulations, Monte Carlo simulations, stochastic process, and other tools, researchers have simulated what rate would have worked historically over every 30-year interval, as well as many future predictions. The answer? One common answer is the 4% rule, which states you should try to have enough saved so that you are happy with withdrawing 4% the first year. This model works in most cases. In the worst 30-year interval the max withdrawal rate was 3.5%. To determine how much money we would want in retirement using a 3.5% withdrawal rate, we would need $0.035M = 105000$, or $3,000,000!

To read more about withdrawal rates, see

- **the Trinity Study** (https://en.wikipedia.org/wiki/Trinity_study)
- **Retirement spend-down** (https://en.wikipedia.org/wiki/Retirement_spend-down)

Additional relevant Wikipedia article

- **Compound interest** (https://en.wikipedia.org/wiki/Compound_interest)