Goal and idea - Module 2

GOAL:

Solutions to differential equations are, at their core, functions. However, it's not just the function (or functions!?), it is an appropriate domain of the function(s). Our goal is to study what it means to be a solution of a differential equation.

To do so, we:

- Define what solutions to DEs are, including their interval of existence.
- Study the difference between implicit and explicit solutions.
- Introduce different forms of solutions, including
  - particular solutions,
  - n-parameter families of solutions, and
  - general solutions.
- Finally, we examine how we can graph solutions, and introduce the notion of solution curves.

IDEA:

Whether a function is a solution to a DE or not depends on more than just how its derivatives relate. It also depends on where the function is being defined (i.e., the domain). Exploring this idea is much of this module. The remaining aspects focuses on how we can succinctly describe and write more than one solution at once, and in introducing terminology to distinguish whether a family of solutions exhausts *all* solutions for a DE.

Time: Videos run 44:06 minutes.

Module 02
SUBJECT: **Solutions, domain of solution, trivial solution** DATE: 2020/09/08 PAGE NO: 01/01

Solutions for DEs

Definition: A function that satisfies a DE for all input of the function is called a **Solution** for the DE.

Ex | Is $y=e^{x^3}$ a solution for $y'=3x^2y$?

What we know about $y'=3x^2y$
- Is first order
- Appears to be linear
- Is written in normal form

We are more interested if $y=e^{x^3}$ is an actual solution.
Soln.

Step 1 | We need to verify it by finding out its derivative.

chain rule

$$y=e^{x^3} \rightarrow y'=[e^{x^3}]'\cdot[x^3]'\cdot[3]'$$
$$\rightarrow y'=e^{x^3}\cdot 3x^2\cdot 1$$
$$\rightarrow y'=3x^2e^{x^3}$$

Step 2 | Now that we have the derivative thus our DE we are interested in we need to see if it satisfies.

Derivative — Differental

$y'=3x^2e^{x^3}$. Thus $y'=3x^2y$

equivalent $\Leftrightarrow$

$3x^2e^{x^3}=3x^2y$ $y=e^{x^3}$

$\rightarrow 3x^2e^{x^3}=3x^2e^{x^3}$

Yes $y=e^{x^3}$ satisfies the differental

Step 3 | Double check. How about for all input? What are the possible inputs of 'y'?

You can plug-in any real number into 'x' is there an issue? No. There isn't any issue. You can plug-in any real number into 'x' of $y=e^{x^3}$ you'll get 'e' to some number that is still a "nice number." The same goes when you plug-in any real number into 'x' of $y'=3x^2y$ or $y'=3x^2e^{x^3}$ there is no real issue.

So...
We have $y=e^{x^3}$ is a solution

Ex | Find a function 'y' satisfying $\frac{dy}{dx}+y^2=0$

Soln | Consider $y=\frac{1}{x}$.
Then $y'=-\frac{1}{x^2}$ and $y'+y^2=0$
$$\Longleftrightarrow -\frac{1}{x^2}+\frac{1}{x^2}=0$$

Question: Is 'y' a solution to the DE?

It goes back to the definition

Solution for the DE
A function that satisfies a DE for all input of the function

'y' certainly satisfies the DE, but does it satifies the DE for all inputs of the function?

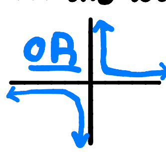
In ways you can argue YES it does because we already excluded the input $y=\frac{1}{x}$ of zero for instance.

Answer: We need to examine the domain of the solution

Solution for the DE on the Interval

Definition: A function defined on an interval that satisfies a DE on the interval is a **Solution** for the DE on the interval.

Recall: $y=\frac{1}{x}$ satisfies $\frac{dy}{dx}+y^2=0$; however, $y=\frac{1}{x}$ is defined on $(-\infty,0)\cup(0,\infty)$. Thus, $y=\frac{1}{x}$ is a solution for the DE on $(-\infty,0)$ **OR** $(0,\infty)$.



Two very different functions.
It's discontinuous, there's a gap, at zero.
We only want to consider the continuous functions.

Interval of Existance/Validity or Domain of the Solution

Definition: The interval for which a function is a solution for the DE is called the **interval of existance** (or **Validity**), or the **domain of the solution**

Note also

$y=0$ satisfies $\frac{dy}{dx}+y^2=0$
and the domain of the solution for $y=0$ is $(-\infty,\infty)$.

The Trival Solution

Definition: If the function $y=0$ is a solution on an interval, we can call this the Trival Solution.

Solution to a DE	Particular (one solution)	Parameter	Could it be general	trival
$y=Ce^{2x}$	✗	✓-one para	✓-(?)	✗
$y=3\cos(4x)$	✓	✗	✗-(?)	✗
$y=0$	✓	✗	✗-(?)	✓

A Note on Domain & Range of Functions and Solutions 2020 09 19 01/01

A note on domain and range of functions and solutions

We take a brief moment to review what functions we are interested in this course and to recall their domains.

Note, however, that the domain of a function is not necessarily the domain of the solution of a DE!!

Below is a table with some common functions we will encounter and their domains.

Polynomials (x^2, x^3, 1, $2+3x^5$, etc.)	$(-\infty, \infty)$
$\frac{1}{x-c}$, $\frac{1}{(x-c)^2}$, etc. (c a constant)	$(-\infty, c) \cup (c, \infty)$, i.e., all numbers but c
$\sqrt{x}$, $\sqrt[4]{x}$, $\sqrt[6]{x}$, etc. (even roots)	$[0, \infty)$
$\ln(x)$	$(0, \infty)$
e^x	$(-\infty, \infty)$
$\sin(x), \cos(x)$	$(-\infty, \infty)$
$\tan(x)$	All x except $x = \frac{\pi}{2}k$ for integers k
$\sec(x)$	All x except $x = \frac{\pi}{2}k$ for integers k
$\csc(x)$	All x except $x = \pi k$ for integers k
$\cot(x)$	All x except $x = \pi k$ for integers k

Being able to extend these properties will also be crucial. For example, using the fourth line of the table to deduce the domain of $\ln(x+5)$ is $(-5, \infty)$.

While the domain of a solution to a DE may be different that the domain of the underlying function, it can only be more restrictive. That is, to determine the domain of a solution, we can first consider the domain of the underlying function, and then possibly remove points.

WeBWorK module 02 exercises:

- Problems 4,5

Relevant Wikipedia articles:

- [Domain of a function (https://en.wikipedia.org/wiki/Domain_of_a_function#Natural_domain)](https://en.wikipedia.org/wiki/Domain_of_a_function#Natural_domain)

Can Solve for y, $y=$something on its own

Ex| Suppose y^2+x^2. Find $\frac{dy}{dx}$.

Soln| Implicit Differentation

Idea: Take the derivative with respects to 'x' on both sides of the equation. Treating 'y' as some function of 'x' (ie. $y=f(x)$),

$$\frac{d}{dx}(y^2)=\frac{d}{dx}(x^2)$$

$$\rightarrow 2y\left(\frac{dy}{dx}\right)=2x$$

$$\rightarrow \frac{dy}{dx}=\frac{2x}{2y} \therefore \frac{x}{y}$$

Ex| Consider $\frac{dy}{dx}=\frac{x^2}{1-y^2}$. Verify that $y-\frac{y^3}{3}=\frac{x^3}{3}+C$ (C is a constant) is a solution for the DE.

soln| Implicit Differentation

$$y-\frac{y^3}{3}=\frac{x^3}{3}+C$$

* Take the derivative of both sides *

$$\rightarrow \frac{d}{dx}\left(y-\frac{y^3}{3}\right)=\frac{d}{dx}\left(\frac{x^3}{3}+C\right)$$

$$\rightarrow \frac{dy}{dx}-\frac{3y^2}{3}\left(\frac{dy}{dx}\right)=\frac{3x^2}{3}$$

$$\rightarrow \frac{dy}{dx}-y^2\left(\frac{dy}{dx}\right)=x^2$$

* Solve for $\frac{dy}{dx}$ *

$$\rightarrow \frac{dy}{dx}(1-y^2)=x^2$$

$$\therefore \frac{dy}{dx}=\frac{x^2}{1-y^2}$$

As a first attempt you might try to isolate 'y' and solve for it

$$y-\frac{y^3}{3} \rightsquigarrow y\left(1-\frac{y^2}{3}\right)$$

You'll find you can't isolate 'y'. You can't solve explicitly for 'y' and then find its derivative.

So did we actully verify that it's a solution?

Yes in a sence that the whole expresson

$$y-\frac{y^3}{3}=\frac{x^3}{3}+C$$

satisfies it.

When you differentiate it you get back your original differental equation.

$$\frac{dy}{dx}=\frac{x^2}{1-y^2}$$

So, $y-\frac{y^3}{3}=\frac{x^3}{3}+C$ is an implicit solution (a family)

Recall| For the DE $y'=3x^2y$ we had $y=e^{x^3}$, $y=0$, $y=7e^{x^3}$ are all solutions

→ In fact, $y=Ce^{x^3}$ is a solution for any constant C.

→ So $y=Ce^{x^3}$ is a "family" of solutions.

→ It is a 1-parameter family of solutions.

→ Whereas, $y=\pi e^{x^3}$ is a particular solution.

Question: Are all solutions for the DE of the form $y=Ce^{x^3}$ for some C?

Definition

1. A solution for a DE that has no parameters is a **particular solution.**
2. A set of solutions obtained by ranging over 'n' many constants is called an **n-parameter family** of solutions.
3. An n-parameter family of solutions which gives rise to all solutions is called a **general** solution.

Ex| Consider $y''-y=0$. Verify that $y=c_1e^x+c_2e^{-x}$ is a 2-parameter family of solutions.

Soln| $y'=[c_1e^x]'+[c_2e^{-x}]'$

→ $=c_1e^x+c_2e^{-x}\cdot[-x^1]$

→ $=c_1e^x+c_2e^{-x}\cdot(-1x^0)$

$y'=c_1e^x-c_2e^{-x}$

$y''=[c_1e^x]''+[c_2e^{-x}]''$

$y'=[c_1e^x]'-[c_2e^{-x}]'$

→ $=c_1e^x-[c_2e^{-x}]'$

→ $=c_1e^x-c_2e^{-x}\cdot[-x^1]$

→ $=c_1e^x-c_2e^{-x}(-1x^0)$

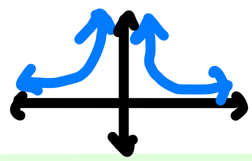
$y''=c_1e^x+c_2e^{-x}$

Here, c_1 and c_2 are constants

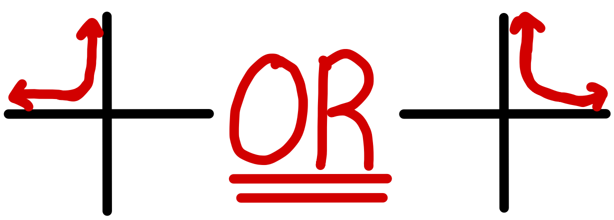
So, $y''-y=0$ ✓

Ex | Note $y=\frac{1}{x^2}$ is a solution for $y'=-\frac{2y}{x}$.
Graph the solution:

Soln | Note: The graph of the function

$y=\frac{1}{x^2}$ is 

For the solution; need interval $(-\infty,0)$ OR $(0,\infty)$



Ex | Graph the solutions for $y'=y$

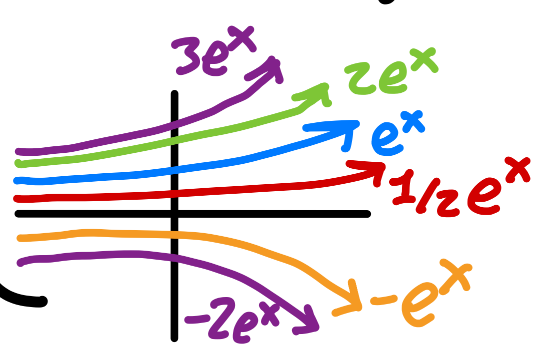
Soln | $y=e^x$ is a solution so is...

→ $y=2e^x$

→ $y=3e^x$

→ $y=9000e^x$

- In fact, $y=Ce^x$ is a 1-parameter family of solutions.
- You could graph "all" of them.



Expectation checklist - Module 2

At the completion of this module, you should:

- know the definitions introduced,
- if given a function and DE be able to
  - determine if the function is a solution to the DE and
  - if so determine what the domain of the solution is,
- be able to determine if the solution set of a DE contains the trivial solution,
- implicitly differentiate a given relation,
- determine (via implicit differentiation) whether a given relation is an (implicit) solution for a given DE,
- check whether a given function is a particular solution for a DE,
- check whether a given family of functions is an n-parameter family of solutions for a DE, and
- be able to graph solutions, taking their domain into account.

You will be assessed on your understanding of these concepts:

- within the homework,
- on the quizzes, and
- later, on the exam.

Coming up next, we:

- introduce initial value problems,
- learn one way to determine whether solutions exist and are unique, and
- make a comment on boundary value problems.