Goal and idea - Module 15

GOAL:

To this point, we can really only solve linear differentiation equations with constant coefficients (though they could now be homogeneous or nonhomogeneous). Here we examine another type of linear ODE that we can solve. We will

- introduce Cauchy-Euler equations; and
- learn to solve such differential equations.

IDEA:

In the constant coefficient case we input the function e^{mx} and solved for m. Here we consider x^m.

Cauchy - Euler Equations

A Cauchy - Euler equation has the form

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{n-1} + \ldots + a_1 x y' + a_0 y = 0$$

where $a_0, a_1, \ldots, a_n$ are constants

Note
- Is a linear differential equation
- Can be n^{th}-order

Key Take Away

The presents of the n^{th} derivative we have x^n on the $n-1$ derivative we have x^{n-1}, so whatever that order of the derivative is we have the exact order powered polynomial right in-front of it.

Example

(a) $3x^3y^{(3)} + \pi x^2 y^{(2)} - xy' + 3y = 0$

$$3x^3y^{(3)} + \pi x^2 y^{(2)} - xy' + 3y = 0$$

Is a Cauchy - Euler equation

(b) $2xy^{(2)} - 3xy' + 5y = 0$

OFF By ONE $2\cancel{x}y^{(2)} - 3xy' + 5y = 0$

Is not a Cauchy - Euler equation

To Solve Such Equations

1) Take $y = x^m$
2) Plug into the differential equation
3) Solve for "m"!

I.e, simular to the constant coefficent method (Module12: where we used $y = e^{mx}$)

Here we take the derivative of "x^m" we get mx^{m-1} and then gather up some other stuff as you go

We'll look at 2nd-Order, and in particular case getting:

→ Two distinct real "m" *focus*

→ One repeated "m" *focus*

→ Two complex (conjugate)

Works for higer order as well. Higer the order the harder it gets to just find the roots of the polynomial that comes out

Ex| Solve $x^2y'' + 3xy' - 4y = 0$

Solution

$x^2y'' + 3xy' - 4y = 0$ Need two derivatives

1) Take $y = x^m$

$y = x^m$, $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$

Big Idea: This is why we need the "x^2" next to the y'' is when we plug-in $y''=m(m-1)x^{m-2}$ we add "x^{m-2}" when we multiply that by "x^2" we add exponents $x^2y'' = m(m-1)x^{m-2} \cdot x^2$, (x^{m-2+2}) it gives us back x^m and is simular for the first derivative.

$x^2y'' + 3xy' - 4y = 0$

$y = x^m$, $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$

2) Plug into the differential equation

This gives: $x^2(m(m-1)x^{m-2}) + 3x(mx^{m-1}) - 4x^m$

$x^2(m(m-1)x^{m-2}) + 3x(mx^{m-1}) - 4x^m = 0$

Factor out "x^m"

$\rightarrow x^m(m(m-1)) + 3x^m m - 4x^m = 0$

Simplify

$\rightarrow x^m(m^2 - m + 3m - 4) = 0$

So long "x" isn't zero we require the parenthetical piece if "x" equals zero and it works then "y" equals zero and we'd have the trival solution.

$\rightarrow m^2 - m + 3m - 4 = 0$

$\rightarrow m^2 + 2m - 4 = 0$

3) Solve for "m"! Quadratic formula

$$\rightarrow \begin{cases} m_1 = \frac{-2+\sqrt{4+16}}{2} = -1 + \frac{\sqrt{20}}{2} = -1 + \frac{2\sqrt{5}}{2} = -1 + \sqrt{5} \\ m_2 = -1 - \sqrt{5} \end{cases}$$

Two distinct real roots "m"

Thus, the general solution is

$$y = C_1 x^{-1+\sqrt{5}} + C_2 x^{-1-\sqrt{5}}$$

What happens when you have repeated roots?

Ex| Solve $x^2y'' - 3xy' + 4y = 0$

Solution

$y = x^m$ Plugged-in gives

$x^m(m^2 - 4m + 4) = 0$

$\rightarrow (m-2)^2 = 0$

Standard Form

- Have the highest order derivative in the front.
  - And is normalized meaning you don't have any other coefficient function or anything else in-front of it.

So $y_1 = x^2$ works ✓

How do we find the other one?

Technically by Reduction of Order

$$y_2 = x^2 \int \frac{e^{-\int P}}{(x^2)^2} dx$$

Reduction of Order formula

$$y_2 = y_1(x) \int \frac{e^{-\int P(x)dx}}{(y_1(x))^2} dx$$

To calculate "P" we need to look back to $x^2y'' - 3xy' + 4y = 0$ our "P" isn't $-3x$ it's actually

$P = \frac{-3x}{x^2} = \frac{-3}{x}$ to get "P" we need the DE in Standard Form. We need the leading term to have a coefficient of one.

$$y_2 = x^2 \int \frac{e^{-\int P}}{(x^2)^2} dx$$

$$\rightarrow x^2 \int \frac{e^{-\int \frac{-3}{x}}}{x^4} dx$$

$$\rightarrow x^2 \int \frac{e^{3\ln|x|}}{x^4} dx$$

$$\rightarrow x^2 \int \frac{\cancel{e}^{\cancel{\ln}|x^3|}}{x^4} dx$$

$$\rightarrow x^2 \int \frac{x^3}{x^4} dx \rightarrow x^2 \int \frac{1}{x} dx \rightarrow x^2 \ln|x|$$

So, $y_2 = x^2 \ln|x|$ ✓

What we learn is whatever we get from y_1 Reduction of Order will add on a natural log of x

Thus, the general solution is

$$y = C_1 x^2 + C_2 x^2 \ln|x|$$

*Note1: If $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ are complex roots, then have $y = x^{\alpha}[C_1\cos(\beta\ln|x|) + C_2\sin(\beta\ln|x|)]$ is the general solution.

*Note2: If have differential equation of the form $a_2x^2y'' + a_1xy' + a_0y = g(x)$ (Non-homogeneous) then we would use Variation of Parameters for y_P.

Expectation checklist - Module 15

At the completion of this module, you should:

- Be able to find the general solution for 2nd-order Cauchy-Euler equations which stem from
  - two distinct real roots, or
  - one repeated root;
- Be able to solve some nonhomogeneous differential equations whose underlying homogeneous equation is a Cauchy-Euler equation via the utilization of the methods of this section along with variation of parameter.
- Know where reduction of order is utilized, even if we omit its steps.

Coming up next, we:

- Study the Laplace Transform!