29.5/50

Please show and explain your work. Justification is necessary for credit. Also sign your name here acknowledging you will not use problem solving software (Wolfram Alpha, etc.): Matthew Mendoza

1. (*5 points*)

4.5

a. (*2 pts*) Circle all of the following expression which are **homogeneous linear differential equations**.

(i) ~~$y''' - 2y''' + 3y = 5$~~ $b(x) \neq 0$

(ii) $\left(x^2 - \sin(x)\right) y^{(5)} - xe^x y' + \sin\left(x^2\right) y = 0$

(iii) ~~$\sqrt{y'} = x\cos(x)y - 3$~~ $b(x) \neq 0$

(iv) $y'' = y + 2y'$ Rewritten as $y'' - 2y' - y = 0$

Nice!

b. (*2 pts*) We have that $y_h = c_1e^x + c_2e^{-x}$ is a general solution for the DE $y'' - y = 0$ and that $y_p = x^3$ is a particular solution to $y'' - y = 6x - x^3$ (you do not need to verify these statements). Provide a general solution to the DE $y'' - y = 6x - x^3$.

$$y = y_h + y_p$$

$$y = C_1e^x + C_2e^{-x} + x^3 \longleftarrow$$

Recall – Module 10

General Solutions for Nonhomogeneous Linear Differential Equations

Given (a, b, c are constants)

(*) $ay'' + by' + cy = f(x)$

with $f(x) \neq 0$, the general solution is of the form

$y = y_h + y_p$

Where

🙂 $\rightarrow y_h$ is the general solution of $ay'' + by' + cy = 0$, and

We are familiar with finding the general solution of the homogeneous piece

This is true for higer order as well

??? $\rightarrow y_p$ is a particular solution of (*)

c. (*1 pt*) Compute the Wronskian of $y_1 = x$ and $y_2 = x^2$.

$$w(f_1, f_2) = \det\begin{pmatrix} f_1 & f_2 \\ f_1' & f_2' \end{pmatrix}$$

$$w(x, x^2) = \det\begin{pmatrix} x & x^2 \\ 1 & 2x \end{pmatrix} \rightarrow (1 \cdot 2x) - (1 \cdot x^2)$$

$$= -x^2 + 2x$$

2

2. (*10 points*) Given that $y_1 = x\sin(x)$ is a solution to the DE $x^2y'' - 2xy' + (x^2 + 2)y = 0$, find a general solution to this differential equation.

1.5

$$y_2 = y_1(x)\int \frac{e^{-\int P(x)dx}}{(y_1(x))^2}dx$$

$$= x\sin(x)\int \frac{e^{-\int(-2x)dx}}{(x\sin(x))^2}dx$$

$$= x\sin(x)\int \frac{e^{+2}}{x^2\sin^2(x)}dx$$

$$=$$

$$y = x\sin(x) +$$

Reduction of Order – Summary

When to use: Given a 2^{nd}-order homogeneous linear differential equation and ONLY ONE solution.

Why to use: Find the other linearly independent solution to form a fundamental set of solutions.

How to use: TWO OPTIONS

ⓐ Long route above (don't memorize)

ⓑ The formula

$$y_1(x)\int \frac{e^{-\int P(x)dx}}{(y_1(x))^2}dx$$

10

Summary

The most important thing with these problems is (1) Plug-in $y=e^{mx}$ and (2) Finding out if $m_1=\frac{-b+\sqrt{b^2-4ac}}{2a}$ & $m_2=\frac{-b+\sqrt{b^2-4ac}}{2a}$ are of

Case 1) Two Distinct Roots

Case 2) Repeated Root

Case 3) Complex Conjugate Roots

3. (*20 points*) Solve the following differential equations. If a technique is asked for, use it.

a. (*5 pts*) $y'' - 6y' + 9y = 0$

1) Plug-in $y=e^{mx}$, $y'=me^{mx}$, $y''=m^2e^{mx}$

Skipping a step, instead of plugging-in and refactoring out, we know are able to factor out "e^{mx}"

This gives us

$$e^{mx}(m^2-6m+9)=0$$

So we need

$$m^2-6m+9=0$$

2) Find m_1 & m_2

$$m=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$=\frac{+6\pm\sqrt{6^2-4\cdot 9}}{2}$$
$$=\frac{6\pm\sqrt{36-36}}{2}$$
$$=\frac{6\pm 0}{2}=3$$

Case 2 Repeated Root

$$y=C_1e^{3x}+xC_2e^{3x}$$

In summary: To get y_2 all you need to do is multiply y_1 by 'x'.

b. (*5 pts*) $y'' + 4y' + 7y = 0$

$$e^{mx}(m^2+4m+7)=0$$
$$\rightarrow m^2+4m+7=0$$
$$m=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$=\frac{-4\pm\sqrt{4^2-4\cdot 7}}{2}$$
$$=\frac{-4\pm\sqrt{16-28}}{2}$$
$$=\frac{-4\pm\sqrt{-12}}{2}$$
$$=\frac{-4\pm i\sqrt{12}}{2}=-2\pm\frac{i\sqrt{12}}{2}$$

∗ Recall ∗

$e^{i\theta}=\cos(\theta)+i\sin(\theta)$

$$e^{(\underbrace{3/2}_{\alpha}+\underbrace{\frac{i\sqrt{7}}{2}}_{i\beta})x}=e^{3/2x}e^{\frac{i\sqrt{7}}{2}x}$$

$$\rightarrow e^{3/2x}(\underbrace{\cos(\tfrac{\sqrt{7}}{2}x)}_{\text{Real}}+\underbrace{i\sin(\tfrac{\sqrt{7}}{2}x)}_{\text{Complex}})$$

Euler's Formula — Focus on the real — Ignore the complex

3) Using the formula (Euler's Formula)

$$y_1=e^{\alpha x}\cos(\beta x),\quad y_2=e^{\alpha x}\sin(\beta x)$$

We find

$$y_1=e^{-2x}\cos(\tfrac{\sqrt{12}}{2}x)$$
$$y_2=e^{-2x}\sin(\tfrac{\sqrt{12}}{2}x)$$

$$\therefore\ y=C_1e^{-2x}\cos(\tfrac{\sqrt{12}}{2}x)+C_2e^{-2x}\sin(\tfrac{\sqrt{12}}{2}x)$$

ok

c. (*10 pts*) $y'' - 4y' + 3y = \cos(x) + \sin(x)$ [using the method of undetermined coefficients]

Lets first consider the homogeneous part

General solution for

$$y''-4y'+3y=0$$

1) Plug-in $y=e^{mx}$, $y'=me^{mx}$, $y''=m^2e^{mx}$

Skipping a step, instead of plugging-in and refactoring out, we know are able to factor out "e^{mx}"

This gives us

$$e^{mx}(m^2-4m+3m)=0$$

So we need

$$m^2-4m+3=0$$

2) Find m_1 & m_2

$$m=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$=\frac{4\pm\sqrt{16-12}}{2}$$
$$=\frac{4\pm\sqrt{4}}{2}=2+1,\ 2-1$$
$$m_1=3,\ m_2=1$$

$$y_h=C_1e^{3x}+C_2e^{x}$$

Recall-Module10

General Solutions for Nonhomogeneous Linear Differential Equations

Given (a,b,c are constants)

(∗) $ay''+by'+cy=f(x)$

with $f(x)\neq 0$, the general solution is of the form

$$y=y_h+y_p$$

Where

🙂 → y_h is the general solution of $ay''+by'+cy=0$, and

We are familiar with finding the general solution of the homogeneous piece

This is true for higer order as well

??? → y_p is a particular solution of (∗)

Case 2) f(x) consists of sin(x) and/or cos(x) terms.

Any sines or cosines (not products of them i.e. $\sin^2$ or $\cos^2$)

$$y_{p_2}=A\cos(x)+B\sin(x)$$
$$y_{p_2}'=-A\sin(x)+B\cos(x)$$
$$y_{p_2}''=-A\cos(x)-B\sin(x),$$

So $y''-4y'+3y$

$$\rightarrow(-A\cos(x)-B\sin(x))-4(-A\sin(x)+B\cos(x))+3(A\cos(x)+B\sin(x))$$
$$\rightarrow(\underline{-A\cos(x)}-\underline{B\sin(x)})+(\underline{4A\sin(x)}-\underline{4B\cos(x)})+(\underline{3A\cos(x)}+\underline{3B\sin(x)})=\cos(x)+\sin(x)$$
$$2A\cos(x)-4B\cos(x)+4A\sin(x)+2B\sin(x)$$

Cosine (LHS = RHS) — Add up all cosine terms

$$2A-4B=1$$
$$A=\frac{1+4B}{2}$$
$$A=\frac{1+4(-1/10)}{2}$$

Sine (LHS = RHS) — Add up all sine terms

$$4A+2B=1$$
$$2(4B+1)+2B=1$$
$$8B+2+2B=1$$
$$-2\qquad -2$$
$$8B+2B=-1$$
$$10B=-1$$
$$B=-1/10$$

$$y_p=\frac{1+4(-1/10)}{2}\cos+-1/10\sin$$

10/10

Nice

augh running out of time...

4. (*5 points*) Suppose $y_1 = x$ and $y_2 = x - 1$ are solutions to a 2nd-order homogeneous linear DE on $I = (-\infty, \infty)$. Do y_1 and y_2 form a fundamental set of solutions for this DE on I? Explain your answer using any math needed.

0/5

5. (*10 points*) Use the method of variation of parameters to find a general solution to the differential equation $x^2y'' - 4xy' + 4y = 3x^3$.

Consider: $a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$

In particular before we'd have just constants. Where $a_2(x)y''$, $a_1(x)y'$, $a_0(x)y$, $g(x)$ are functions

The Method:

Step 1 First turn

$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$

↓ into ↓

$y'' + P(x)y' + Q(x) = f(x)$

The idea is to first put it in to this form

$f(x) = \frac{g(x)}{a_2(x)}$, $P(x) = \frac{a_1(x)}{a_2(x)}$ So you divide everything by whatever function is in y''

Step 2 Find our homogeneous peice ($y_h = c_1y_1 + c_2y_2$), which is the general solution for $y'' + P(x)y' + Q(x)y = 0$

Step 3 Then $y_p = u_1y_1 + u_2y_2$, depends on the homogeneous peice (y_1 & y_2) where u_1 and u_2 are found by integrating

$u_1' = \frac{-y_2 f(x)}{W}$ and $u_2' = \frac{y_1 f(x)}{W}$,

and $W(y_1, y_2)$ is the wronskian of y_1, y_2

$$y_p = u_1y_1 + u_2y_2$$

So 2 But right

2

3/10