18 – Inner Product

Definition: Inner Product

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ be vectors in $\mathbb{R}^n$. The **inner product** (or **dot product**) of $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \cdots + u_n v_n. \leftarrow \text{result is a number (NOT a vector)}$$

1. Let $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 0 \\ 7 \\ -2 \end{bmatrix}$. Compute $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{u} \cdot \mathbf{u}$.

$\bar{u} \cdot \bar{v} = 3 \cdot 0 + (-1)7 + 5 \cdot (-2) \Rightarrow 0 - 7 - 10 = -17$

$\bar{u} \cdot \bar{v} = (3)^2 + (-1)^2 + (5)^2 \Rightarrow 9 + 1 + 25 = 35$

Theorem

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
3. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$
4. $\mathbf{u} \cdot \mathbf{u} \geq 0$ and $\mathbf{u} \cdot \mathbf{u} = 0 \iff \mathbf{u} = 0$

Definition: Length & Distance

- The **length** (or **norm**) of a vector $\mathbf{v}$ is $||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$. We say $\mathbf{v}$ is **unit vector** if $||\mathbf{v}|| = 1$.

  $\bar{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ $||\bar{v}|| = \sqrt{v_1^2 + \cdots + v_n^2}$

- The **distance** between vectors $\mathbf{u}$ and $\mathbf{v}$ is

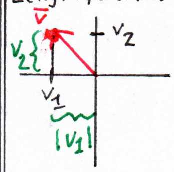
$$\text{dist}(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}|| = \sqrt{(u_1 - v_1)^2 + \cdots + (u_n - v_n)^2}.$$

2. Let $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$.

(a) Compute $\text{dist}(\mathbf{v}, \mathbf{w})$.

$\text{dist}(\bar{v}, \bar{w}) = ||\bar{v} - \bar{w}|| = \left|\left| \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} \right|\right|$

Length & distance



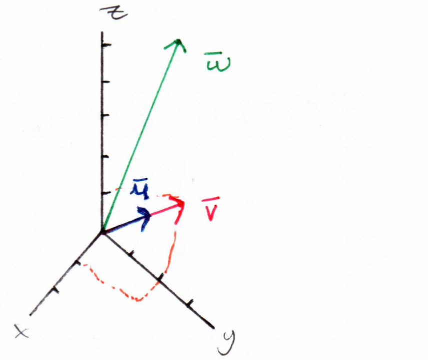
$\bar{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ what is the length of $\bar{v}$? $||\bar{v}||$

$||\bar{v}|| = \sqrt{v_1^2 + v_2^2} = \sqrt{\bar{v} \cdot \bar{v}}$

Angle b/w two vectors

$||\bar{u} - \bar{v}||^2 = ||\bar{u}||^2 + ||$

(b) Find a unit vector $\mathbf{u}$ in the same direction as $\mathbf{v}$. Graph $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.



Theorem

If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then $\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}|| \cdot ||\mathbf{v}|| \cos\theta$.

Definition: Orthogonality

We say vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if $\mathbf{u} \cdot \mathbf{v} = 0$. Equivalently, $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if the angle between them is $90°$.
↑perpendicular

3. Let $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$.

(a) Show that $\mathbf{w} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ is not orthogonal to $\mathbf{v}$. What is the angle between $\mathbf{v}$ and $\mathbf{w}$?

$\bar{v} \cdot \bar{w} = 3 + (-1) + (-1) = 1$; $1 \neq 0$; So NOT orthogonal

side work

$||\bar{v}|| = \sqrt{3^2 + 1^2 + 1^2} = \sqrt{11}$

$||\bar{u}|| = \sqrt{1 + (-1)^2 + (-1)} = \sqrt{3}$

$\bar{v} \cdot \bar{w} = ||\bar{v}|| \, ||\bar{w}|| \cos\theta$

$1 = \sqrt{11} \cdot \sqrt{3} \cos\theta$

$\cos\theta = \frac{1}{\sqrt{33}} \Rightarrow \theta = \arccos\left(\frac{1}{\sqrt{33}}\right) \approx 80°$

(b) Find three different vectors in $\mathbb{R}^3$ that are orthogonal to $\mathbf{v}$. How many other vectors are orthogonal to $\mathbf{v}$?

want $\bar{u}$ such that $\bar{u} \cdot \bar{v} = 0$. write $\bar{u} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$

$\bar{u} \cdot \bar{v} = 3u_1 + u_2 + u_3 = 0$

Possible answers

$\bar{u} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \ldots$

∞ - many answers