1.1 Systems of Linear Equations

Related :

Example 01

Example

Consider the following system

$$\begin{aligned} x + 2y &= 1 \\ 3x + 4y &= -1 \end{aligned}$$

(a) Is $(-3, 1)$ a solution to the system?

❌ $(-3, 1)$ is NOT a solution because it is not a solution to equation #2: $3x + 4y = -1$

$(-3, 1)$ is not a solution to $3x + 4y = -1$

$$\begin{aligned} 3x + 4y &= -1 \\ 3\left(3\right) + 4\left(1\right) &= -1 \\ 9 + 4 &= -1 \\ 13 &\stackrel{?}{=} 1 \\ 13 &\neq 1 \end{aligned}$$

What about $(-3, 2)$?

✅ $(-3, 2)$ is a solution to the system

$(-3, 2)$ is a solution to $x + 2y = 1$

$$\begin{aligned} x + 2y &= 1 \\ (-3) + 2\left(2\right) &= 1 \\ (-3) + 4 &= 1 \\ 1 &\stackrel{?}{=} 1 \\ 1 &= 1 \checkmark \end{aligned}$$

$(-3, 2)$ is a solution to $3x + 4y = -1$

$$
\begin{aligned}
3x+4y&=-1\\
3\left(-3\right)+4\left(2\right)&=-1\\
-9+8&=-1\\
-1&\overset{?}{=}-1\\
-1&=-1\checkmark
\end{aligned}
$$

(b) Find <u>all</u> solutions to the system

Use elimination:

> **Symbol Tilde:** $\sim$
>
> The tilde is read as "equivalent": same solution set
>
> > **Example**
> >
> > $$\underset{-3r_1+r_2\to r_2}{\sim}$$
>
> "r" means `row`, so r_1 is `row 1`

$$
\begin{aligned}
x+2y&=1\\
3x+4y&=-1
\end{aligned}
\quad \underset{-3r_1+r_2\to r_2}{\sim}
$$

> **Scratch Work**
>
> $$
> \begin{array}{lr}
> -3r_1 & 3x-6y=-3\\
> +r_2 & 3x+4y=-1\\
> \hline
> & 0-2y=-4
> \end{array}
> $$

$$
\begin{aligned}
x+2y&=1\\
2y&=-4
\end{aligned}
\quad \underset{-\frac{1}{2}r_2\to r_2}{\sim}
$$

> **Solve This**
>
> $$
> \begin{aligned}
> x+2y&=1\\
> y&=2
> \end{aligned}
> $$
>
> plug-and-chug using: $y=2\quad\checkmark$
>
> $$
> \begin{aligned}
> x+2\left(2\right)&=1\\
> \Rightarrow x&=-3
> \end{aligned}
> $$

✓ **Only one solution:**

$$\begin{aligned} x &= -3 \\ y &= 2 \end{aligned} \quad OR \quad (-3, 2)$$

(c) Interpret the system geometrically

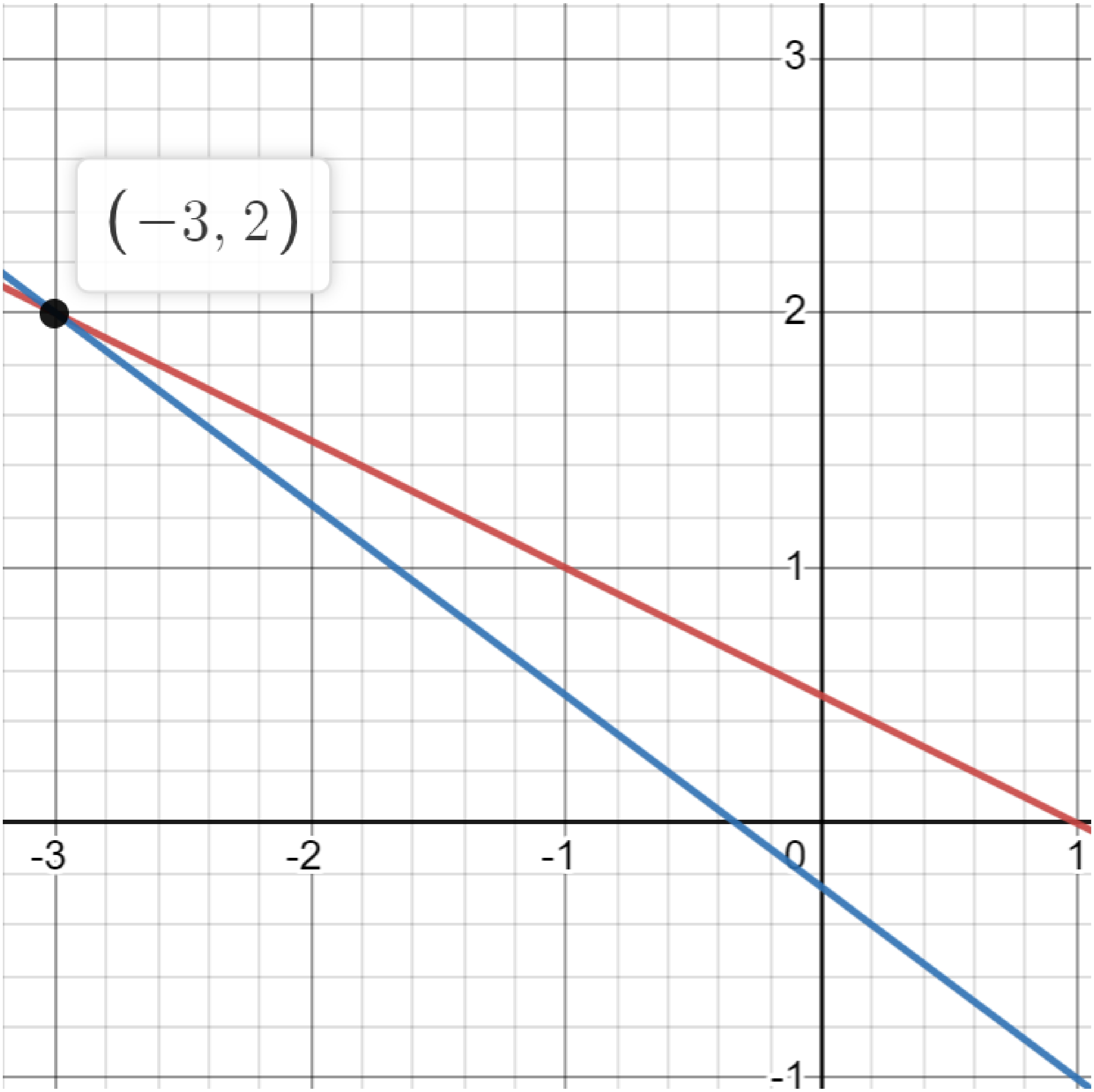
- Solutions to $x + 2y = 1$ are the points on the line it describes
- Similarly for $3x + 4 = -1$
- Solution(s) are the points on that line

Put into the standard from we get the following

$$x + 2y = 1$$
$$\Rightarrow \ y = -\frac{1}{2}x + \frac{1}{2}$$

$$3x + 4y = 1$$
$$\Rightarrow y = -\frac{1}{2}x + \frac{1}{2}$$

When graphed we get the following



When manipulating linear systems, we do not want to change the solution sets.

What are the allowable operations?

Elementary Row Operations

1. **Replacement**: Replace a row by itself plus any multiple of another row
   $r_i + cr_j \to r_i$
2. **Interchange**: Swap any two rows $r_i \longleftrightarrow r_j$
3. **Scaling**: Multiply any row by a non-zero number $cr_i \to r_i$

If one system can be transformed into another by a series of row operations then we say: the systems are **row equivalent**.

> **Theorem**
>
> If two systems are row equivalent, then they have the same solution sets

Example Matrix Notation (by example)

Convert the system to augmented matrix form and then solve.

> **Example**
>
> $$\begin{aligned} x_1 - 3x_2 &= 5 \\ -x_1 + x_2 + 5x_3 &= 2 \\ x_2 + x_3 &= 0 \end{aligned}$$

Step 1) Rewrite to align

$$\begin{array}{rrrl} x_1 - & 3x_2 & & = 5 \\ -x_1 + & x_2 + & 5x_3 & = 2 \\ & x_2 + & x_3 & = 0 \end{array}$$

Step 2) Write in matrix form

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ -1 & 1 & 5 & 2 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

Step 3) $r_1 + r_2 \to r_2$ - take `row 1` and add `row 2`; then, take the sum to replace `row 2`

> **Replacement**: Replace a row by itself plus any multiple of another row

$$\left[\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ -1 & 1 & 5 & 2 \\ 0 & 1 & 1 & 0 \end{array}\right] \underset{r_1+r_2\to r_2}{\sim} \left[\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ 0 & -2 & 5 & 7 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

Step 4) $r_2 \longleftrightarrow r_3$ - swap `row 2` with `row 3`

> **Interchange**: Swap any two rows

$$\left[\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ 0 & -2 & 5 & 7 \\ 0 & 1 & 1 & 0 \end{array}\right] \underset{r_2 \longleftrightarrow r_3}{\sim} \left[\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & 5 & 7 \end{array}\right]$$

Step 5) $2r_2 + r_3 \rightarrow r_3$ - double `row 2` and add `row 3`; then, take the result to replace row 3

> **Replacement**: Replace a row by itself plus any multiple of another row

$$\left[\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & 5 & 7 \end{array}\right] \underset{2r_2+r_3 \rightarrow r_3}{\sim} \left[\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 7 & 7 \end{array}\right]$$

Step 6) $\frac{1}{7}r_3 \ \rightarrow \ r_3$ - scale down `row 3` by $\frac{1}{7}$ (i.e. divide `row 3` by 7)

> **Scaling**: Multiply any row by a non-zero number

$$\left[\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 7 & 7 \end{array}\right] \underset{\frac{1}{7}r_3 \rightarrow r_3}{\sim} \left[\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

This is to say that the original system is equitant to

$$\begin{array}{c} \\ Original \end{array} \quad \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ -1 & 1 & 5 & 2 \\ 0 & 1 & 1 & 0 \end{array}\right] \end{array} \quad \underset{is\ equivalent\ to}{\sim} \quad \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \end{array} \quad Modifie$$

Modified

$$\begin{array}{lr} Equation\ 1 & x_1 - 3x_2 = 5 \\ Equation\ 2 & x_2 + x_3 = 0 \\ Equation\ 3 & x_3 = 1 \end{array}$$

Step 7) Finish solving

$$Equation \, ① \, x_1 - 3\,(-1) = 5$$
$$\Rightarrow x_1 = 2$$

$$Equation \, ② \, 2x + 1 = 0$$
$$\Rightarrow x_2 = -1$$

$$Equation \, ③ \, x_3 = 1$$

> ✓ **Done**
>
> $$\begin{aligned} x_1 &= 2 \\ x_2 &= -1 \quad OR \quad (2, -1, 1) \\ x_3 &= 1 \end{aligned}$$

Example: Determine if the follow is consistent

`consistent`: There is at least one solution

Using the elimination method

> **Example**
>
> $$\begin{aligned} x + y &= 1 \\ x + 2y &= 2 \\ -x + 3y &= 7 \end{aligned}$$

Step 1) Write in matrix form

> Reduce: make zeros

$$\begin{array}{ccc} x & y & = \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 2 \\ -1 & 3 & 7 \end{array}\right]$$

Step 2) $-r_1 + r_2 \rightarrow r_2$ make `row 1` negative of the same values and take the sum of `row 1` and `row 2`; after, $r_1 + r_3 \rightarrow r_3$ add `row 1` and `row 3`, and use the sum to replace `row 3`

> **Replacement**: Replace a row by itself plus any multiple of another row

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 2 \\ -1 & 3 & 7 \end{array}\right] \underset{\substack{-r_1 + r_2 \to r_2 \\ r_1 + r_3 \to r_3}}{\sim} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 4 & 8 \end{array}\right]$$

Step 3) $-4r_2 + r_3 \to r_3$ multiply `row 2` by -4 then add `row 2` with `row 3`; after, take the sum and replace it with `row 3`

> **Replacement**: Replace a row by itself plus any multiple of another row

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 4 & 8 \end{array}\right] \underset{-4r_2 + r_3 \to r_3}{\sim} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 1 \\ \boxed{0} & \boxed{4} & \boxed{8} \end{array}\right]$$

All together:

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 2 \\ -1 & 3 & 7 \end{array}\right] \underset{\substack{-r_1 + r_2 \to r_2 \\ r_1 + r_3 \to r_3}}{\sim} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 4 & 8 \end{array}\right] \underset{-4r_2 + r_3 \to r_3}{\sim} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 1 \\ \boxed{0} & \boxed{4} & \boxed{8} \end{array}\right]$$

✓ **Done**

$$\begin{aligned} x + y &= 1 \\ y &= 1 \\ 0 &= 4 \end{aligned}$$

$$\text{No Solutions} \therefore \boxed{\text{Inconsistent}}$$

$$\left.\begin{aligned} x + y &= 1 \\ y &= 1 \\ 0 &= 4 \end{aligned}\right\} \text{Geometrically 3 lines in } \mathbb{R}^2$$

Example: Find all solutions

Example

$$\begin{aligned} 2x - y + 3z &= 4 \\ 2x + 3y - 5z &= 0 \end{aligned}$$

Step 1) Write in matrix form

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 4 \\ 2 & 3 & -5 & 0 \end{array}\right]$$

Step 2) $-r_1 + r_2 \rightarrow r_2$ multiply `row 1` with -1, add `row 2`, and replace the sum with `row 2`

> **Replacement**: Replace a row by itself plus any multiple of another row

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 4 \\ 2 & 3 & -5 & 0 \end{array}\right] \underset{-r_1 + r_2 \rightarrow r_2}{\sim} \left[\begin{array}{ccc|c} 2 & -1 & 3 & 4 \\ 0 & 4 & -8 & -4 \end{array}\right]$$

Step 3) $\frac{1}{4} r_2 \rightarrow r_2$ replace `row 2` with a $\frac{1}{4}$ scale-down version of `row 2`

> **Scaling**: Multiply any row by a non-zero number

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 4 \\ 0 & 4 & -8 & -4 \end{array}\right] \underset{\frac{1}{4} r_2 \rightarrow r_2}{\sim} \left[\begin{array}{ccc|c} 2 & -1 & 3 & 4 \\ 0 & 1 & -2 & -1 \end{array}\right]$$

So, all together - steps 2 & 3:

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 4 \\ 2 & 3 & -5 & 0 \end{array}\right] \underset{-r_1 + r_2 \rightarrow r_2}{\sim} \left[\begin{array}{ccc|c} 2 & -1 & 3 & 4 \\ 0 & 4 & -8 & -4 \end{array}\right] \underset{\frac{1}{4} r_2 \rightarrow r_2}{\sim} \left[\begin{array}{ccc|c} 2 & -1 & 3 & 4 \\ 0 & 1 & -2 & -1 \end{array}\right]$$

$$\left.\begin{array}{r} 2x - y + 3z = 4 \\ y - 2z = -1 \end{array}\right\} \Rightarrow \left.\begin{array}{r} 2x = 4 + \boxed{\text{y}} - 3z \\ \boxed{\text{y=-1+2z}} \end{array}\right\} \Rightarrow \begin{array}{c} x = \dfrac{3}{2} - \dfrac{1}{2} z \\ y = -1 + 2z \\ \text{z is free} \end{array}$$

> ✓ **Done**
>
> $$\begin{array}{c} x = \dfrac{3}{2} - \dfrac{1}{2} z \\ y = -1 + 2z \\ \text{z is free} \end{array}$$

Example solution:

$$\left(\tfrac{3}{2} - \tfrac{7}{2},\ -1 + 14,\ 7\right),\ \left(\tfrac{3}{2} - \tfrac{\pi}{2}, -1 + 2\pi, \pi\right), \ldots$$

Final Remark

A linear system can have $0,\ 1,\ \infty$-many solutions; that is to say, having exactly 2 solutions is not possible.