Section 2.2

2024/03/07

10 – Matrix Inverses

Definition: Inverse Matrix

Let A be an $n \times n$ matrix. If there exists a matrix B such that $AB = I_n$ and $BA = I_n$, then we say A is **invertible**, and the matrix B is called the **inverse** of A, denoted by A^{-1}.

Theorem

If a matrix is invertible, then there is only one possible inverse of A.

1. Let $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$.

(a) Verify that $A^{-1} = \begin{bmatrix} -3 & 2 \\ \frac{5}{2} & \frac{-3}{2} \end{bmatrix}$ by showing $AA^{-1} = I_2 = A^{-1}A$.

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$AA^{-1} = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$-9+10=1,\ 6+(-\frac{12}{2})=0$

$$A^{-1}A = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$-9+10=1$

(b) Use the fact that $A^{-1}A = I_2$ to solve $\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

$$\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}\cdot\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix} \Rightarrow \boxed{\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}^{-1}\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}}^{I_2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}^{-1}}_{A^{-1}}\begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$A \cdot \bar{x} = \bar{b}$

A^{-1} A^{-1}

$\bar{x} = A^{-1}\cdot\bar{b}$

$$\Rightarrow I_2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}\cdot\begin{bmatrix} 3 \\ 7 \end{bmatrix} \Rightarrow \boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}}$$

2. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and assume that $ad - bc \neq 0$. Verify that $A^{-1} = \underbrace{\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}_{B}$.

$$B\cdot A = \frac{1}{ad-bc}\cdot\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} da-bc & \cancel{db-bd}^{0} \\ \cancel{-ca+ac}^{0} & -cb+ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Similarly $A\cdot B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Theorem: Formula for A^{-1} when A is 2×2

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Define $\det A = ad - bc$, which is called the **determinant** of A.

*The special number

*1. If $\det A \neq 0$, then $A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

2. If $\det A = 0$, then A is *not* invertible, i.e. A^{-1} does not exist.

Theorem

Let A be an $n \times n$ matrix. Then A is invertible if and only if its RREF is I_n. Further, when A is invertible, any sequence of row operations that transforms A to I_n will also transform I_n to A^{-1}.

A^{-1} exists

Theorem: Algorithm for finding A^{-1} when A is $n \times n$

Let A be an $n \times n$ matrix. Row reduce the augmented matrix $[A \mid I_n]$ to RREF.

- If the RREF of $[A \mid I_n]$ is $[I_n \mid B]$, then A is invertible, and $B = A^{-1}$.
- If the RREF of $[A \mid I_n]$ is $[\textbf{not } I_n \mid B]$, then A is not invertible.

3. Find the inverse of A, if it exists.

(a) $\begin{bmatrix} 0 & 3 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

By Inspection

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 3 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \not\sim I_3$$

because there can not be a pivot in 3rd col.

So, $\boxed{A^{-1} \text{ DNE}}$

(b) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$

By Algorithm

$$\left[\underbrace{\begin{matrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{matrix}}_{A}\ \middle|\ \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{array}\right] \sim \left[\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I}\ \middle|\ \underbrace{\begin{matrix} -\frac{9}{2} & 7 & -\frac{3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{matrix}}_{A^{-1}}\right]$$

We can tell $A \sim I$ so, A^{-1} exists

$$A^{-1} = \begin{bmatrix} -\frac{9}{2} & 7 & -\frac{3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix}$$

(c) $\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 4 & -3 & 7 \end{bmatrix}$