20240416 (Tue)

# 16 – Diagonalization Theorem

## Definition

A matrix $A$ is **diagonalizable** if $A = PDP^{-1}$ (or equivalently $D = P^{-1}AP$) for some *diagonal* matrix $D$ and some invertible matrix $P$.

## Theorem: Diagonalization Theorem

Let $A$ be an $n \times n$ matrix.

1. $A$ is diagonalizable if and only if $A$ has $n$ linearly independent eigenvectors.
2. If $A$ is diagonalizable and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent eigenvectors for $A$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$, then $A = PDP^{-1}$ for

$$P = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \overset{d_1}{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \underset{d_n}{\lambda_n} \end{bmatrix}$$

**1.** Diagonalize the following, if possible.

**(a)** $B = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Eigen values: matrix is lower triangular ($\Delta$) so eigenvalues are entries on main diagonal

$\boxed{\lambda = 1, 2}$

Multiplicity 2

$\lambda = 1$ $E_1(B) = \text{Nul}(B - I) = \text{Nul}\left(\begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = 0$
$x_2$ is free
$x_3 = 0$

$\bar{x} = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Basis for $E_1(B) = \left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$

$\lambda = 2$ $E_2(B) = \text{Nul}(B - 2I) = \text{Nul}\left(\begin{bmatrix} -1 & 0 & 0 \\ 3 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$

$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = 0$
$x_2 = 0$
$x_3$ is free $\leftarrow t$

$\bar{x} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

**(b)** $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Eigenvalues: lower $\Delta$ so $\boxed{\lambda = 1, 2}$

$\lambda = 1$ $E_1(A) = \text{Nul}(A - I) = \text{Nul}\left(\begin{bmatrix} 0 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$

$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $x_1 = -1/3 x_2$, $x_2$ is free $\leftarrow t$, $x_3$ is free $\leftarrow r$ $\Big\}$ $\bar{x} = \begin{bmatrix} -1/3 t \\ t \\ r \end{bmatrix} = t\begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Basis for $E_1(A)$ is $\left\{\begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ have 2 linearly ~~dependent~~ Independent eigen vectors in $E_1(A)$

$\lambda = 2$ $E_2(A) = \text{Nul}(A - 2I) = \text{Nul}\left(\begin{bmatrix} -1 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\right)$

$\left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$ $x_1 = 0$, $x_2$ free $\leftarrow t$, $x_3 = 0$ $\Big\}$ $\bar{x} = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Basis for $E_2(A)$ is $\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$

* Combining basis we get 3 Linearly Independent eigenvectors

$\underbrace{\begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\lambda = 1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\lambda = 2}$

Note: This Combining process from diff. eigenspaces always yields a Linearly Indpendent set

* A is 3x3 and has 3 Linearly Independent eigen vectors so it $\underline{\text{is}}$ diagonalizable: $A = PDP^{-1}$ $P = \begin{bmatrix} -1/3 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$\lambda=1 \quad \lambda=1 \quad \lambda=2$

**Theorem**

Let $A$ be an $n \times n$ matrix. If $A$ has $n$ different eigenvalues, then $A$ is diagonalizable.

**2.** Explain why each of the following are diagonalizable.

**(a)** $A = \begin{bmatrix} 5 & 0 & 0 \\ 2 & 0 & 0 \\ 7 & 8 & \pi \end{bmatrix}$

'A' is a lower $\Delta$ so eigenvalues are $\lambda = 5, 0, \pi$

'A' is 3x3 with 3 diff eigenvalues, so it must be diagonizable.

'A' is 2x2

**(b)** $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Eigenvalues: find char. poly.

$P(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix}$

$= (1-\lambda)(4-\lambda) - 6 = \lambda^2 - 5\lambda - 2$

$P(\lambda) = \lambda^2 - 5\lambda - 2$

Eigenvalues: $\lambda^2 - 5\lambda - 2 = 0$

$\lambda = \frac{5 \pm \sqrt{25+8}}{2} \Rightarrow \lambda = \frac{5+\sqrt{33}}{2}, \frac{5-\sqrt{33}}{2}$

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$A = PDP^{-1}$

$P = \begin{bmatrix} \frac{-3+\sqrt{33}}{6} & \frac{-3-\sqrt{33}}{6} \\ 1 & 1 \end{bmatrix}$

$D = \begin{bmatrix} \frac{5+\sqrt{33}}{2} & 0 \\ 0 & \frac{5-\sqrt{33}}{2} \end{bmatrix}$