chapter 6 section 5 2024-05-02 (Thu)

20 – Least Squares

when $\boxed{A\bar{x}=\bar{b}}$ is inconsistant we want a process for finding $\boxed{\text{best possible approximal solution}}$ Least square solution

$[A|\bar{b}]$

Definition: Least Squares Solution

Let A be an $m \times n$ matrix, and let $\mathbf{b}$ be in $\mathbb{R}^m$. A vector $\hat{\mathbf{x}}$ is called a **least squares solution** to $A\mathbf{x} = \mathbf{b}$ if $\text{dist}(\mathbf{b}, A\hat{\mathbf{x}}) \leq \text{dist}(\mathbf{b}, A\mathbf{x})$ for all $\mathbf{x}$ in $\mathbb{R}^n$. The number $\text{dist}(\mathbf{b}, A\hat{\mathbf{x}})$ is called the **least squares error**. this equals the smallest possible distance

1. Show that $\hat{\mathbf{x}}$ is an actual solution to $A\mathbf{x} = \mathbf{b}$ precisely when $\text{dist}(\mathbf{b}, A\hat{\mathbf{x}}) = 0$.

SKIP...

Theorem

Let A be an $m \times n$ matrix, and let $\mathbf{b}$ be in $\mathbb{R}^m$. Then $\hat{\mathbf{x}}$ is a least squares solution to $A\mathbf{x} = \mathbf{b}$ if and only if $\hat{\mathbf{x}}$ is a solution to $A^T A\mathbf{x} = A^T\mathbf{b}$.

2. Consider the system $A\mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix}$.

(a) Show that $A\mathbf{x} = \mathbf{b}$ has no solutions.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 3 \\ 1 & 0 & 1 & 8 \\ 1 & 0 & 1 & 2 \end{array}\right] \xrightarrow{-r_3+r_4 \to r_4} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 3 \\ 1 & 0 & 1 & 8 \\ 0 & 0 & 0 & -6 \end{array}\right]$$

$0 \neq -6$, so $\boxed{\text{NO SOLUTION}}$ is a contradiction and so implies no solution. Because no solutions exists the system is INCONSISTENT

(b) Find a least squares solution to $A\mathbf{x} = \mathbf{b}$.

We solve $A^T A\bar{x} = A^T\bar{b}$

$$A^T \cdot A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 0 & 2 \end{bmatrix}$$

$$A^T \cdot \bar{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix} = \begin{bmatrix} 14 \\ 4 \\ 10 \end{bmatrix}$$

Solve

$$\left[\begin{array}{ccc|c} 4 & 2 & 2 & 14 \\ 2 & 2 & 0 & 4 \\ 2 & 0 & 2 & 10 \end{array}\right] \rightsquigarrow \cdots \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

no pivot, so x_3 is free

$x_1 = 5 - x_3$
$x_2 = -3 + x_3$
x_3 is free

gives all least squares solutions

let's choose one of them

Choose $x_3 = 0$

$$\boxed{\hat{x} = \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix}}$$

$\hat{x}$ is a least squares solution to $A\bar{x} = \bar{b}$

(c) What is the least squares error?

$$A\hat{x} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 5 \\ 5 \end{bmatrix}$$

$$\text{error} = \text{distance}(\bar{b}, A\hat{x}) = \text{dist}\left(\begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 5 \\ 5 \end{bmatrix} \right) = \sqrt{(1-2)^2 + (3-2)^2 + (8-5)^2 + (2-5)^2}$$

$$= \sqrt{20} \approx 4.472 \approx \boxed{4.5}$$

3. Suppose you have the data points: $(0,2), (-3,5), (2,3), (4,12)$, and you want to model the data using a quadratic function of the form $f(t) = c_0 + c_1 t + c_2 t^2$. Use least squares to find a quadratic function that best fits the data. Then use a graphing tool to plot your answer together with the given data.

$f(0) = 2 \Rightarrow c_0 + c_1 \cdot 0 + c_2 \cdot 0 = 2$

$f(-3) = 5 \Rightarrow c_0 + c_1(-3) + c_2(-3)^2 = 5$

$f(2) = 3 \Rightarrow c_0 + c_1(2) + c_2(2)^2 = 3$

$f(4) = 12 \Rightarrow c_0 + c_1(4) + c_2(4)^2 = 12$

Need to solve for c_0, c_1, c_2

$$\left.\begin{aligned} c_0 + 0c_1 + 0c_2 &= 2 \\ c_0 - 3c_1 + 9c_2 &= 5 \\ c_0 + 2c_1 + 4c_2 &= 3 \\ c_0 + 4c_1 + 16c_2 &= 12 \end{aligned}\right\} \quad \begin{array}{ccc|c} c_0 & c_1 & c_2 & \\ \hline 1 & 0 & 0 & 2 \\ 1 & -3 & 9 & 5 \\ 1 & 2 & 4 & 3 \\ 1 & 4 & 16 & 12 \\ \hline \multicolumn{3}{c|}{A} & \bar{b} \end{array}$$

Least Squares

$$A^T \cdot A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & 2 & 4 \\ 0 & 9 & 4 & 16 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -3 & 9 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 29 \\ 3 & 29 & 45 \\ 29 & 45 & 353 \end{bmatrix}$$

Notice that there is sym along the ~~main diag~~ main diagonal

$$A^T \cdot \bar{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & 2 & 4 \\ 0 & 9 & 4 & 16 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 5 \\ 3 \\ 12 \end{bmatrix} = \begin{bmatrix} 22 \\ 39 \\ 249 \end{bmatrix}$$

Solve $A^T A x = A^T \bar{b}$

$$\left[\begin{array}{ccc|c} 4 & 3 & 29 & 22 \\ 3 & 29 & 45 & 39 \\ 29 & 45 & 353 & 249 \end{array}\right] \rightsquigarrow \cdots \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1.103 \\ 0 & 1 & 0 & 0.345 \\ 0 & 0 & 1 & 0.571 \end{array}\right] \qquad \hat{x} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1.103 \\ 0.345 \\ 0.571 \end{bmatrix}$$

So, the best fit quadratic is

$$\boxed{f(t) = 1.103 + 0.345t + 0.571t^2}$$