20240502 (Thu)

6.5 Least Squares

When $\boxed{A\bar{x} = \bar{b}}$ is inconsistant we want a process for finding $\boxed{\text{best possible approximat solution}}$

$\boxed{[A \mid \bar{b}]}$

least squares solution

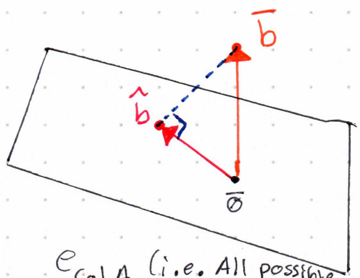
- Handout 20 was provided

Studying $A\bar{x} = \bar{b}$

How do we find a least squares solution?

Idea:

- We can solve $A\bar{x} = \bar{b}$ precisely when $\bar{b}$ is in column A (column space A).
- If $\bar{b}$ is not in Column A (column space Of A) then the closest vector to $\bar{b}$ that is in ColA is $\text{projection}_{\text{Col}A} \bar{b}$.



$e_{\text{Col}A}$ (i.e. All possible $A\bar{x}$'s)

$\bar{w}$ is in colA $\Leftrightarrow A\bar{x} = \bar{w}$ has a solution

Let $\hat{b} = \text{Proj}_{\text{Col}(A)} \bar{b}$.
Instead of solving $A\bar{x} = \bar{b}$
we will solve $A\bar{x} = \hat{b}$.

Let $\hat{x}$ be a solution to $A\bar{x} = \hat{b}$
So, $A\hat{x} = \hat{b}$.

How do we find $\hat{x}$?

Notice $\bar{b} - \hat{b}$ is orthogonal to ColA, so $\bar{b} - \hat{b}$ is orthogonal to every column of A.
write $A = [\bar{a}_1 \ \bar{a}_2 \dots \bar{a}_n]$
then

$$\bar{a}_1 \cdot (\bar{b} - \hat{b}) = 0 \Leftrightarrow a_1^T \cdot (\bar{b} - \hat{b}) = 0$$

$$\vdots \qquad\qquad \vdots$$

$$\bar{a}_n \cdot (\bar{b} - \hat{b}) = 0 \Leftrightarrow \bar{a}_n^T \cdot (\bar{b} - \hat{b}) = 0$$

Thus,

$$A^T(\bar{b} - \hat{b}) = \bar{0}$$
$$A^T\bar{b} - A^T\hat{b} = \bar{0}$$
$$A^T\bar{b} = A^T\hat{b}$$
$$A\hat{x} = \hat{b}$$
$$A^T\bar{b} = A^TA\hat{x}$$

Theorem

$\hat{x}$ is a least squares solution to $A\bar{x} = \bar{b} \Longleftrightarrow \hat{x}$ is an actual solution to

$$\underbrace{A^TA}_{\text{new } A}\bar{x} = \underbrace{A^T\bar{b}}_{\text{new } \bar{b}}$$

Last Time:

Theorem

$\hat{x}$ is a best possible approx solution to $A\hat{x} \approx \bar{b}$ $\left\{ \begin{array}{l} \hat{x} \text{ is a least squares solution} \\ \text{to } A\bar{x} = \bar{b} \\ [A \mid \bar{b}] \end{array} \right.$ $\Longleftrightarrow$ $\hat{x}$ is a (actual) Solution to $\underbrace{A^T \cdot A}_{\text{new}A}\hat{x} = \underbrace{A^T \bar{b}}_{\text{new}\,\bar{b}}$

Review
#11
$\text{proj}_W(\bar{v})$

$$\bar{v} = \begin{bmatrix} -19 \\ -14 \\ -14 \end{bmatrix}$$

$$W = \text{Span}\left(\underbrace{\begin{bmatrix} -2 \\ 6 \\ -2 \end{bmatrix}}_{W_1}, \underbrace{\begin{bmatrix} 6 \\ 6 \\ 12 \end{bmatrix}}_{W_2} \right)$$

$$\text{proj}_W(\bar{v}) = \text{proj}_{w_1}(\bar{v}) + \text{proj}_{w_2}(\bar{v})$$

$$= \frac{\bar{v} \cdot \bar{w}_1}{\bar{w}_1 \cdot \bar{w}_1}\bar{w}_1 + \frac{\bar{v} \cdot \bar{w}_2}{\bar{w}_2 \cdot \bar{w}_2}\bar{w}_2$$

$$= \frac{-18}{44}\begin{bmatrix} -2 \\ 6 \\ -2 \end{bmatrix} + \frac{-366}{216}\begin{bmatrix} 6 \\ 6 \\ 12 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{36}{44} \\ \frac{-108}{44} \\ \frac{36}{44} \end{bmatrix} + \begin{bmatrix} -2196/216 \\ -2196/216 \\ -4392/216 \end{bmatrix} = \Big[$$

$$= \begin{bmatrix} 36/44 - 2196/216 \\ -108/44 - 2196/216 \\ 36/44 - 4392/216 \end{bmatrix}$$

<u>Sidework</u>

$\bar{v} \cdot \bar{w}_1 = 19 \cdot 2 - 14 \cdot 6 + 14 \cdot 2 = -18$

$\bar{w}_1 \cdot \bar{w}_1 = (-2)^2 + (6)^2 + (2)^2 = 44$

$\bar{v} \cdot \bar{w}_2 = -19 \cdot 6 - 14 \cdot 6 - 14 \cdot 12 = -366$

$\bar{w}_2 \cdot \bar{w}_2 = 6^2 + 6^2 + (12)^2 = 216$

HW 12, Q4

$$A = \begin{bmatrix} 49 & 24 \\ -112 & -55 \end{bmatrix}$$

$$P = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$$

corresponding

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \rightarrow \begin{bmatrix} 7 & 0 \\ 0 & -1 \end{bmatrix}$$

eigenvalues

$$A = PDP^{-1}$$

<u>eigenvalues</u>

$$P(\lambda) = \det(A - \lambda I)$$

$$= \det\left(\begin{bmatrix} 49-\lambda & 24 \\ -122 & -55-\lambda \end{bmatrix}\right)$$

$$= (49-\lambda)(-55-\lambda) - 24(-112)$$

$$= \lambda^2 - 6\lambda - 2695 + 2688$$

$$= \lambda^2 - 6\lambda - 7$$

$P(\lambda) = 0$

$$\lambda^2 - 6\lambda - 7 = 0$$

$$(\lambda + 7)(\lambda - 1) = 0$$

$$\boxed{\lambda = -7, 1}$$