2024-05-02 (Thu)

19 – Orthogonal Projections

Definition: Projection Onto a Line

Let $L = \text{Span}\{\mathbf{u}\}$ for some nonzero $\mathbf{u}$ in $\mathbb{R}^n$. For any $\mathbf{y}$ in $\mathbb{R}^n$, define the **orthogonal projection of y onto** L to be

$$\text{proj}_L(\mathbf{y}) = \left(\frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}.$$

Note: we sometimes write $\text{proj}_{\mathbf{u}}(\mathbf{y})$ *in place of* $\text{proj}_L(\mathbf{y})$

1. Let $\mathbf{y} = \begin{bmatrix}3\\1\end{bmatrix}$ and $L = \text{Span}\left\{\begin{bmatrix}4\\3\end{bmatrix}\right\}$.

$\uparrow \bar{u}$

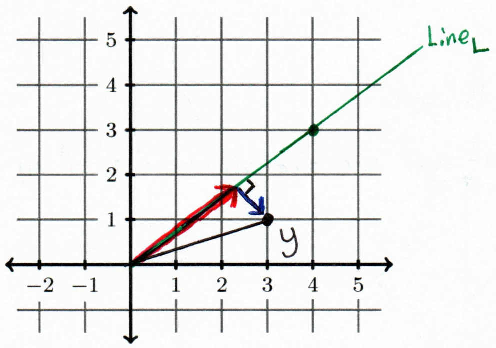
(a) Compute $\text{proj}_L(\mathbf{y})$.

$$\text{Proj}_L(\bar y) = \frac{\bar y \cdot \bar u}{\bar u \cdot \bar u}\bar u = \frac{15}{25}\begin{bmatrix}4\\3\end{bmatrix} \Rightarrow \frac{3}{5}\begin{bmatrix}4\\3\end{bmatrix} \Rightarrow \begin{bmatrix}12/5\\9/5\end{bmatrix} = \boxed{\begin{bmatrix}2.4\\1.8\end{bmatrix}}$$

$\bar y \cdot \bar u \Rightarrow \begin{bmatrix}3\\1\end{bmatrix}\cdot\begin{bmatrix}4\\3\end{bmatrix} = 12+3 = 15$

$\bar u \cdot \bar u \Rightarrow \begin{bmatrix}4\\3\end{bmatrix}\cdot\begin{bmatrix}4\\3\end{bmatrix} = 4^2+3^2 = 25$

(b) Let $\hat{\mathbf{y}} = \text{proj}_L(\mathbf{y})$ and $\mathbf{b} = \mathbf{y} - \cancel{\text{proj}_L(\mathbf{y})}\ \hat{\mathbf{y}}$. Graph L, $\mathbf{y}$, $\hat{\mathbf{y}}$, and $\mathbf{b}$.



Definition

A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is said to be **orthogonal** if each pair of vectors in the set is orthogonal. If the set is orthogonal *and* every vector is a unit vector, then the set is said to be **orthonormal**.

2. Let $\mathbf{v}_1 = \begin{bmatrix}1\\1/3\\1/3\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}-2\\4\\2\end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix}-1\\-4\\7\end{bmatrix}$. Verify that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is orthogonal but *not* orthonormal.

$\bar V_1 \cdot \bar V_2 \Rightarrow -2 + \frac{4}{3} + \frac{2}{3} = 0$ ✓ $\rightarrow$ $\bar V_1, \bar V_2$ are orthogonal

$\bar V_1 \cdot \bar V_3 \Rightarrow -1 - \frac{4}{3} + \frac{7}{3} = 0$ ✓ $\rightarrow$ $\bar V_1, \bar V_3$ are orthogonal

$\bar V_2 \cdot \bar V_3 \Rightarrow 2 - 16 + 14 = 0$ ✓ $\rightarrow$ $\bar V_2, \bar V_3$ are orthogonal

$\{\bar V_1, \bar V_2, \bar V_3\}$ is orthogonal

Note: $\|\bar V_1\| = \sqrt{1^2 + (\frac{1}{3})^2 + (\frac{1}{3})^2} = \sqrt{\frac{11}{9}} \neq 1$ so, $\bar V_1$ is not a unit vector so the set is <u>NOT</u> orthogonal

3. Give an example of an orthonormal set of three vectors in $\mathbb{R}^3$.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Theorem

If a set of nonzero vectors forms an orthogonal set, then the vectors are linearly independent.

Definition: Projection Onto a Subspace

Let W be a subspace of $\mathbb{R}^n$, and let $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be any *orthogonal* basis for W. For any $\mathbf{y}$ in $\mathbb{R}^n$, define the **orthogonal projection of y onto** W to be

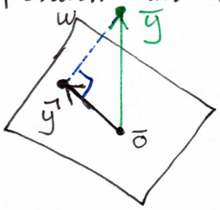
$$\text{proj}_W(\mathbf{y}) = \text{proj}_{\mathbf{u}_1}(\mathbf{y}) + \cdots + \text{proj}_{\mathbf{u}_k}(\mathbf{y}) = \left(\frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right) \mathbf{u}_1 + \cdots + \left(\frac{\mathbf{y} \cdot \mathbf{u}_k}{\mathbf{u}_k \cdot \mathbf{u}_k}\right) \mathbf{u}_k.$$

Note: $\text{proj}_W(\mathbf{y})$ *gives the same answer no matter which orthogonal basis you use.*

4. Let $W = \text{Span}\{\mathbf{w}_1, \mathbf{w}_2\}$ for $\mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$.

(a) Verify that $\mathbf{w}_1, \mathbf{w}_2$ is an orthogonal basis for W.

- $\overline{w}_1 \cdot \overline{w}_2 \Rightarrow -1+3-2=0 \rightarrow \overline{w}_1, \overline{w}_2$ are orthogonal
- By theorem they are Linearly Independent and we are given that they span W, so they form a basis for W.



(b) Compute $\text{proj}_W(\mathbf{y})$ for $\mathbf{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$.

$\text{proj}_W(\bar{y}) = \text{proj}_{\bar{w}_1}(\bar{y}) + \text{proj}_{\bar{w}_2}(\bar{y})$

$$= \frac{\bar{y} \cdot \bar{w}_1}{\bar{w}_1 \cdot \bar{w}_1} \overline{w}_1 + \frac{y \cdot \bar{w}_2}{\bar{w}_2 \cdot \bar{w}_2} \overline{W}_2 \Rightarrow \frac{-1+4+3}{1^2+1^2+1^2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1+12+(-6)}{(-1)^2+3^2+(-2)^2} \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$$

$$\Rightarrow \frac{6}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{7}{14} \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} = \boxed{\begin{bmatrix} 3/2 \\ 7/2 \\ 1 \end{bmatrix}}$$

(c) Let $\hat{\mathbf{y}} = \text{proj}_W(\mathbf{y})$ and $\mathbf{b} = \mathbf{y} - \text{proj}_W(\mathbf{y})$. Use GeoGebra to graph W, $\mathbf{y}$, $\hat{\mathbf{y}}$, and $\mathbf{b}$.

Theorem: Best Approximation Theorem

Distance

Let W be a subspace of $\mathbb{R}^n$, and let $\hat{\mathbf{y}} = \text{proj}_W(\mathbf{y})$. Then $\hat{\mathbf{y}}$ is the vector of W that is *closest* to $\mathbf{y}$ in the sense that $\text{dist}(\mathbf{y}, \hat{\mathbf{y}}) \leq \text{dist}(\mathbf{y}, \mathbf{w})$ for all $\mathbf{w}$ in W.