Exam 1—Math 100

Spring 2024

Name Matthew Mendoza

Instructor: J. Wiscons

29.5

Points. The exam is out of **35 points**.

Time. This is an in-class 50-minute exam.

Rules for the exam. *Please read these carefully!* Violation of the rules will be reported to the Sacramento State Office of Student Conduct.

1. You are allowed:

- to use a calculator for *basic arithmetical* computations;
- to use one 3×5 notecard of notes.

2. You are not allowed:

- to use any resources on this exam except those listed above;
- to look at another person's exam or their work;
- to let another person see your exam or your work.

3. Please justify all of your work and show all steps unless indicated otherwise.

4. Let me know if you have any questions at all!

1. [4pt] Row reduce the following matrix to **RREF**. *Show all work.*

$$\begin{bmatrix} 1 & -3 & 1 & 0 \\ 1 & -1 & 5 & 6 \\ -2 & 8 & 2 & 3 \end{bmatrix}$$

REF

$$\begin{bmatrix} 1 & -3 & 1 & 0 \\ 1 & -1 & 5 & 6 \\ -2 & 8 & 2 & 3 \end{bmatrix} \begin{matrix} -1r_1 + r_2 \to r_2 \\ 2r_1 + r_3 \to r_3 \end{matrix} \begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & 2 & 4 & 6 \\ 0 & 2 & 4 & 3 \end{bmatrix} \; 2^{-1}r_2 \to r_2 \begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 3 \end{bmatrix} \; -2r_2 + r_3 \begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

RREF

$$\begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -3 \end{bmatrix} \; r_3 + r_2 \to r_2 \begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} \; 3r_2 + r_1 \to r_1 \begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

3.5

x_3 Free

scale

2. [2pt] Use your answer to Problem **1** to determine if the system below is consistent or not. *Explain.*

2

$$\left.\begin{aligned} x_1 - 3x_2 + x_3 &= 0 \\ x_1 - x_2 + 5x_3 &= 6 \\ -2x_1 + 8x_2 + 2x_3 &= 3 \end{aligned}\right\} \text{augmented matrix} \left\{ \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 1 & -1 & 5 & 6 \\ -2 & 8 & 2 & 3 \end{array}\right] \right. \begin{matrix} -1r_1 + r_2 \to r_2 \\ 2r_1 + r_3 \to r_3 \end{matrix} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 2 & 4 & 6 \\ 0 & 2 & 4 & 3 \end{array}\right]$$

$$2^{-1}r_2 \to r_2 \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 3 \end{array}\right] -2r_2 + r_3 \to r_3 \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & \boxed{-3} \end{array}\right]$$

Contradiction $0 \neq -3$ ✓
So No solutions
∴ Inconsistent !

Inconsistent.

3. [6pt] Solve the system, and **write your answer in parametric vector form**. *Show all work.*

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -7 & 0 \\ 0 & -1 & 1 & 2 & 0 \\ -2 & 6 & -1 & -3 & 0 \end{array}\right] 2r_1 + r_3 \to r_3 \left[\begin{array}{cccc|c} 1 & 0 & 0 & -7 & 0 \\ 0 & -1 & 1 & 2 & 0 \\ 0 & 6 & -1 & -17 & 0 \end{array}\right] -1r_2 \to r_2 \left[\begin{array}{cccc|c} 1 & 0 & 0 & -7 & 0 \\ 0 & 1 & -1 & -2 & 0 \\ 0 & 6 & -1 & -17 & 0 \end{array}\right] -6r_2 + r_3 \to r_3$$

REF

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -7 & 0 \\ 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & -15 & 0 \end{array}\right] -15^{-1}r_3 \to r_3 \left[\begin{array}{cccc|c} 1 & 0 & 0 & -7 & 0 \\ 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{matrix} 2r_3 + r_2 \to r_2 \\ 7r_3 + r_1 \to r_1 \end{matrix} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \end{array}\right]$$

careful

~~No Solution~~

x_3 is free

3.5 Parametric vector form

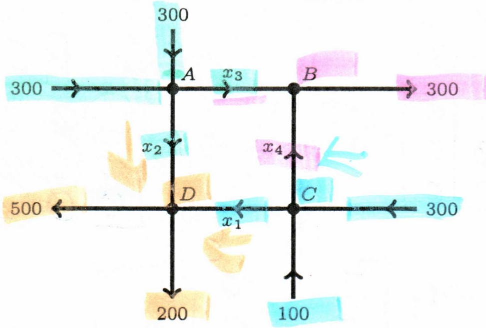
$x_1 = 0$
$x_2 = 1 = 0$
~~$x_3 = 0$~~
x_3 is free
$x_4 = 0$

correct idea

Solution? No Solution
$x = 0$
$x_2 = 1$
~~$x_3 = 0$~~
x_3 free
$x_4 = 0$

9

4. [8pt] The network below shows the traffic flow in vehicles per hour over various one-way streets. There are four intersections labeled A, B, C, D.



(a) Write an equation for the flow at each of the four intersections; the first is done for you.

IN = OUT A: $x_2 + x_3 = 600$

B: $x_3 + x_4 = 300$ B: $x_3 + x_4 = 300$

C: $300 + 100 = x_1 + x_4$ C: $x_1 + x_4 = 400$

D $x_1 + x_2 = 500 + 200$ D: $x_1 + x_2 = 700$

(b) Write the system you found in (a) as an augmented matrix.

As Augmented Matrix

$$\left.\begin{aligned} A&: x_2 + x_3 = 600 \\ B&: x_3 + x_4 = 300 \\ C&: x_1 + x_4 = 400 \\ D&: x_1 + x_2 = 700 \end{aligned}\right\} \left[\begin{array}{cccc|c} 0 & 1 & 1 & 0 & 600 \\ 0 & 0 & 1 & 1 & 300 \\ 1 & 0 & 0 & 1 & 400 \\ 1 & 1 & 0 & 0 & 700 \end{array}\right]$$

(c) The augmented matrix you found in (b) can be reduced to the matrix below. Use this to write out the general flow pattern. *You do not need to reduce the matrix yourself.*

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 400 \\ 0 & 1 & 0 & -1 & 300 \\ 0 & 0 & 1 & 1 & 300 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Implies Infinite solutions

x_4 is free

Parametric vector form

$x_1 = 400 - x_4$

$x_2 = 300 + x_4$ ✓

$x_3 = 300 - x_4$

x_4 is free

x_4 can be infinate because $x_2 = 300 + x_4$

So $300 \geq x_4 >$

come back

(d) What is the largest possible value for x_4? Why?

Maximum

when x_4 is 400

because $x_1 = 400 - x_4$ when $x_4 = 0$?

x_4 is free

good but $x_3 = 300 - x_4$ shows $x_4 \leq 300$

Minimum

when $x_4 = 0$

is 300 because

$x_2 = 300$

7.5

5. [10pt] Let $\mathbf{a}_1 = \begin{bmatrix}1\\0\\-2\end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix}0\\3\\3\end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix}2\\0\\-2\end{bmatrix}$. } augmented matrix $\left\{ \left[\begin{array}{ccc|c}1&0&2&0\\0&3&0&0\\-2&3&-2&0\end{array}\right] \right.$

(a) Determine if $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are linearly independent. *Show all work, and explain your answer.*

Trivial Solution all zeros

$$\left[\begin{array}{ccc|c}1&0&2&0\\0&3&0&0\\-2&3&-2&0\end{array}\right] \xrightarrow{2r_1+r_3\to r_3} \left[\begin{array}{ccc|c}1&0&2&0\\0&3&0&0\\0&3&2&0\end{array}\right] \xrightarrow{3^{-1}r_2\to r_2} \left[\begin{array}{ccc|c}1&0&2&0\\0&1&0&0\\0&3&2&0\end{array}\right] \xrightarrow{-3r_2+r_3\to r_3} \left[\begin{array}{ccc|c}1&0&2&0\\0&1&0&0\\0&0&2&0\end{array}\right]$$

REF

$$\left[\begin{array}{ccc|c}1&0&2&0\\0&1&0&0\\0&0&2&0\end{array}\right] \xrightarrow{2^{-1}r_1\to r_1} \left[\begin{array}{ccc|c}1&0&2&0\\0&1&0&0\\0&0&1&0\end{array}\right] \xrightarrow{-2r_3+r_1\to r_1} \left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right]$$

RREF ✓

Solution
$x_1=0$, $x_2=0$, $x_3=0$ } Trivial Solution ∴ a_1, a_2, a_3 are all $\boxed{\text{linearly Independent}}$

(b) Determine if $\mathbf{b} = \begin{bmatrix}1\\3\\6\end{bmatrix}$ is a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$. If it is, find values for x_1, x_2, x_3 such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$. *Show all work.*

$A\bar{x} = b$

$$\left[\begin{array}{ccc|c}1&0&2&1\\0&3&0&3\\-2&3&-2&6\end{array}\right] \xrightarrow{2r_1+r_3\to r_3} \left[\begin{array}{ccc|c}1&0&2&1\\0&3&0&3\\0&3&2&8\end{array}\right] \xrightarrow{3^{-1}r_2\to r_2} \left[\begin{array}{ccc|c}1&0&2&1\\0&1&0&1\\0&3&2&8\end{array}\right] \xrightarrow{-3r_2+r_3\to r_3} \left[\begin{array}{ccc|c}1&0&2&1\\0&1&0&1\\0&0&2&5\end{array}\right]$$

REF

$$\xrightarrow{2^{-1}r_3\to r_3} \left[\begin{array}{ccc|c}1&0&2&1\\0&1&0&1\\0&0&1&\frac{5}{2}\end{array}\right] \xrightarrow{-2r_3+r_1\to r_1} \left[\begin{array}{ccc|c}1&0&0&-\cancel{5}\,4\\0&1&0&1\\0&0&1&\frac{5}{2}\end{array}\right]$$

RREF

$$\underset{a_1}{-5}\begin{bmatrix}1\\0\\-2\end{bmatrix} + \underset{a_2}{1}\begin{bmatrix}0\\3\\3\end{bmatrix} + \underset{a_3}{\tfrac{5}{2}}\begin{bmatrix}2\\0\\-2\end{bmatrix} = b$$

$-8.\frac{5}{2} \Rightarrow -5$

$$\boxed{-\cancel{5}\,4\begin{bmatrix}1\\0\\-2\end{bmatrix} + 1\begin{bmatrix}0\\3\\3\end{bmatrix} + \frac{5}{2}\begin{bmatrix}2\\0\\-2\end{bmatrix} = \begin{bmatrix}1\\3\\6\end{bmatrix}}$$

6. [1pt each] True or False: **Circle one.** You do not need to justify your answer.

(a) **True or False:** $\begin{bmatrix}2&2&1\\0&1&0\\1&1&5\end{bmatrix} \cdot \begin{bmatrix}1\\2\\1\end{bmatrix} = \begin{bmatrix}7\\2\\8\end{bmatrix}$.

$3\times\underbrace{3 \quad 3}_{\text{Match}}\times 1$ ✓ ~~(2·1)+(2·0)+(1·3)~~

True

(b) **True or False:** It is possible to find vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ in $\mathbb{R}^3$ that are linearly independent.

(c) **True or False:** Geometrically, Span$\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\3\\0\end{bmatrix}\right\}$ represents a line through the origin in $\mathbb{R}^3$.

(d) **True or False:** $\begin{bmatrix}1\\2\\3\end{bmatrix}$ is in Span$\left\{\begin{bmatrix}1\\3\\1\end{bmatrix}, \begin{bmatrix}0\\2\\0\end{bmatrix}, \begin{bmatrix}1\\5\\1\end{bmatrix}\right\}$.

(e) **True or False:** Every vector in $\mathbb{R}^3$ can be expressed as a linear combination of $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}$?

13