20240402 P1

Last Time
- Intro to determinants via cofactor expansion
- we saw

$$\det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & & a_{33} & \vdots \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix} = a$$

product of entries

Section 3.2 Properties of Determinants

We'll first study how rows operations affect determinants.
Let's investigate 2x2 case

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. $\det A = ad - bc$

Interchange Rows

Consider $B = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

$$\begin{aligned} \det B &= cb - da \\ &= bc - ad \\ &= -ad + bc \\ &= -(ad - bc) \\ &= -\det A \end{aligned}$$

Thus $A \underset{r_1 \leftrightarrow r_2}{\sim} B$ then $-\det A = \det B$

Scaling a Row

Consider $B = \begin{bmatrix} ka & kb \\ c & d \end{bmatrix}$

$$\begin{aligned} \det B &= ka \cdot d - kb \cdot c \\ &= k(ad - bc) \\ &= k \cdot \det A \end{aligned}$$

Thus,

$A \underset{kr_1 \to r_1}{\sim} B \Rightarrow k \cdot \det A = \det B$

Replacement

Consider $B = \begin{bmatrix} a & b \\ ka+c & kb+d \end{bmatrix}$

$$\begin{aligned} \det B &= a(kb+d) - b(ka+c) \\ &= \boxed{akb} + ad \boxed{-bka} - bc \\ &= ad - bc \\ &= \det A \end{aligned}$$

Thus,

$A \underset{kr_1 + r_2 \to r_2}{\sim} B \Rightarrow \det A = \det B$

Theorem: Let A be $n \times n$

① **Replacement:** If B is the result of performing a replacement row operation on A, then $A \underset{kr_i + r_j \to r_i}{\sim} B \Rightarrow \det A = \det B$

② **Interchange:** If B is the result of interchanging 2 rows of A then,
$A \underset{r_i \leftrightarrow r_j}{\sim} B \Rightarrow -\det A = \det B$

③ **Scaling:** If B is the result of scaling one row of A by k, then

$A \underset{kr_i \to r_i}{\sim} B \Rightarrow k \cdot \det A = \det B$

$\left(\det A = \frac{1}{k} \det B\right)$ if $k \neq 0$

purple

Reduction to triangular form

$* \begin{vmatrix} xxx \\ xxx \\ xxx \end{vmatrix}$ ← straight lines mean determinants

Example A Compute

$$\begin{vmatrix}1&-4&2\\-2&8&-9\\-1&7&0\end{vmatrix}$$

So,

$$\underbrace{\begin{vmatrix}1&-4&2\\-2&8&-9\\-1&7&0\end{vmatrix}}_{\det\left(\begin{bmatrix}1&-4&2\\-2&8&9\\-1&7&0\end{bmatrix}\right)} \overset{\substack{2r_1+r_2\to r_2\\ r_1+r_3\to r_3\\ \text{Replacement}\\ \text{So determinants are equal}}}{=} \begin{vmatrix}1&-4&2\\0&0&-5\\0&3&2\end{vmatrix} = \overset{\substack{r_2\leftrightarrow r_3\\ \text{Interchange}}}{-} \begin{vmatrix}1&-4&2\\0&3&2\\0&0&-5\end{vmatrix}$$

$$= -(1\cdot 3\cdot(-5))$$

$$= \boxed{15}$$

Example B

$$\begin{vmatrix}2&-8&6&8\\3&-9&5&10\\-1&4&-3&-6\\-3&0&1&-2\end{vmatrix}$$

So, $\overset{\substack{\frac{1}{2}r_1\to r_1\\ \text{Scale}}}{2}\begin{vmatrix}2&-4&3&4\\3&-9&5&10\\-1&4&-3&-6\\-3&0&1&-2\end{vmatrix} \overset{\substack{-3r_1+r_2\to r_2\\ r_1+r_3\to r_3\\ 3r_1+r_4\to r_4\\ \text{Replacement}}}{=} 2\begin{vmatrix}1&-4&3&4\\0&3&-4&-2\\0&0&0&-2\\0&-12&10&10\end{vmatrix} \overset{\substack{4r_2+r_4\to r_4\\ \text{replace}}}{=}$

$$= 2\begin{vmatrix}1&-4&3&4\\0&3&-4&2\\0&0&0&-2\\0&0&-6&2\end{vmatrix} \overset{\substack{r_3\leftrightarrow r_4\\ \text{Interchange}}}{=} -2\begin{vmatrix}1&-4&3&4\\0&3&-4&2\\0&0&-6&2\\0&0&0&-2\end{vmatrix} = -2(1\cdot3\cdot(-6)\cdot(-2))$$

$$= \boxed{-72}$$

*Note

$A \xrightarrow{kr_i\to r_i} B$

$k\det A = \det B$

$\det A = \frac{1}{k}\det B$

Section 3.2: Properties of Determinants – Invertability & Determinants

Recall: A^{-1} exists $\Longleftrightarrow A \sim I$

Assume A^{-1} exists. What can we say about $\det A$.

$$A^{-1} \text{ exists} \Rightarrow A \sim A_1 \sim A_2 \sim \cdots \sim A_k = I$$

$$\Rightarrow \det A = c_1 \det A_1 = c_1c_2\det A_2 = \cdots = c_1c_2\cdots c_k \det I$$

for some $c_1, c_2, \ldots, c_k$

Note: $\det I = 1$
and none of $c_1, c_2, \ldots c_k$ are equal to zero

Recap: A^{-1} exists $\Rightarrow \det A = c_1c_2\ldots c_k$ where $c_1,\ldots,c_k$ are not zero

$$\Rightarrow \det A \neq 0$$

Similar idea shows $\det A \neq 0 \Rightarrow A^{-1}$ exists

* Theorem $\underset{A^{-1}\text{ exists}}{\underline{A \text{ is invertable}}} \Longleftrightarrow \det A \neq 0$.

Other Properties of the determinant

Theorem

Transpose: for 2x2

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \det\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right)$$

- $\det(A^T) = \det A$
- Important $*$ $\det(A \cdot B) = (\det A)(\det B)$
- for $k = 1, 2, 3, \ldots$; $\det(A^k) = (\det A)^k$
- if $\det A \neq 0$, then $\det(A^{-1}) = \frac{1}{\det A} = (\det A)^{-1}$

Note: Typically $\det(A+B) \neq \det A + \det B$

Example Suppose you know $\det A = 7$, $\det B = 1/2$

Compute

(a) $\det(BA)$

(b) $\det(B^3)$

(c) $\det(A^{-1})$

(a) $\det(BA) = \det B \cdot \det A$

$= 1/2 \cdot 7$

$= 7/2$

(b) $\det(B^3) =$

(c) Since $\det A \neq 0$, A^{-1} exists

$$A \cdot A^{-1} = I$$

$$\det(A \cdot A^{-1}) = \det I$$

$$(\det A) \cdot (\det A^{-1}) = 1$$

$$\det A^{-1} = \frac{1}{\det A} = \boxed{1/7}$$