Section 2.8 (continued) 2024/03/26

- Subspaces of $\mathbb{R}^n$
  - What is a subspace
  - How to check if something is in subspace
  - Example of subspaces: $\text{Span}\{\bar{v}_1, \dots, \bar{v}_k\}$, $NUL(A)$, $\mathbb{R}^n$, $Col(A)$

- Basis for a subspace
  - About efficiently describing subspaces
  - Example: determining if something a basis for $\mathbb{R}^n$
  - Finding a basis for $Nul(A)$
  - Finding a basis for $col(A)$
  - Finding a basis for $\text{Span}\{\bar{v}_1, \dots \bar{v}_k\}$

Section 2.9 Dimension and Rank 2024/03/26

Section 3.1 Determinants 2024/03/28

Recall: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then we define $\det A = ad - bc$

and saw that A^{-1} exists $\Leftrightarrow$ $\det A \neq 0$ (zero)

We want to generalize this to larger matries.

Notation: we'll write $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ to mean $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$

Definition: Let A b $n \times n$

① A_{ij} is $(n-1)\times(n-1)$ the matrix obtained from A by deleting the i^{th} row and j^{th} column

Matrix

② The (i,j)-cofactor of A, denoted C_{ij}, is

Number

$$C_{ij} = (-1)^{i+j} \cdot \det(A_{ij})$$

Example

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 3 & 1 & 7 \\ 0 & 4 & 0 \end{bmatrix}$$

$$A_{32} = \begin{bmatrix} 1 & 0 \\ 3 & 7 \end{bmatrix}$$

$$C_{32} = (-1)^{3+2} \cdot \det A_{32}$$
$$= (-1) \cdot [1 \cdot 7 - 0 \cdot 3]$$
$$= \boxed{-7}$$

3 x 3 Determinants

Def If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{23} & a_{33} \end{bmatrix}$

Then we define

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\det\left(\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}\right) \quad = a_{11}\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{31} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{bmatrix} a_{21} & a_{22} \\ a_{31} + & a_{32} \end{bmatrix}$$

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$\det\left(\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & 2 & 3 \end{bmatrix}\right) \to \begin{vmatrix} \boxed{1 \quad 5 \quad 0} \\ 2 \quad 4 \quad -1 \\ 0 \quad 2 \quad 3 \end{vmatrix} = a_{11}C_{11} + a_{12}C_{12} + \cancel{a_1}$

$= 1 \cdot (-1)^2 \underbrace{\begin{vmatrix} 4 & -1 \\ 2 & 3 \end{vmatrix}}_{\det A_1} + 5(-1)^3 \cdot \underbrace{\begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix}}_{\det A_{12}} + \cancel{0 \cdot (-1)^4 \begin{vmatrix} 2 & 4 \\ 0 & 2 \end{vmatrix}}$

$= (12 - (-2)) - 5(6 - 0) = \boxed{-16}$

n×n Determinates

<u>Def</u> If $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{n1} & a_{n2} & \cdots & a \end{bmatrix}$

then we define

$\det A = \boxed{a_{11}C_{11} + a_{12}C_{12} + \cdots a_{1n}C_{1n}}$ cofactor exp along 1st row

In fact, you can use cofactor exp. along any row or any column.

Theorem $\det A = \boxed{a_i C_i + a_{i2}C_{i2} + \cdots + a_{in}C_{in}}$ along ith row

or

$\det A = \boxed{a_{1j}C_{1j} + a_{2j}C_{2c} + \cdots + a_{nj}C_{nj}}$ along jth column

! want to choose a row or column with lots of zerow

Example: Find $\det A$ for $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 6 & 7 & 8 \end{bmatrix}$

along 2nd row

$\det A = -4(16-21) + 5(8-18) = 20 - 50 = -30$

Example: Find $\det A$ for $A \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 6 & 7 & 8 \end{bmatrix}$ along the 3rd col

$\det A = 3C_{13} + 0C_{23} + 8C_{33}$

$= 3(-1)^4 \begin{vmatrix} 4 & 5 \\ 6 & 7 \end{vmatrix} + 0 + 8(-1)^6 \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$

$= 3(28-30) + 8(5-8)$

$= -6 - 24 = -30$

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Example Let

$$A = \begin{bmatrix} 0 & 4 & 0 & 5 \\ 2 & 2 & 3 & 0 \\ 0 & 0 & -3 & 0 \\ 2 & -1 & -2 & 0 \end{bmatrix}$$

Compute $\det A$ along 4 column

along 4 column

$$\det A = 5C_{14} + 0C_{24} + 0_{34} + 0C_{44}$$

$$= 5(-1)^5 \cdot \begin{vmatrix} 2 & 2 & 3 \\ 0 & 0 & -3 \\ 2 & -1 & -2 \end{vmatrix}$$

$$= -5\left[0C_{21} + 0C_{22} - 3C_{23}\right]$$

$$= -5\left[-3(-1)^5 \cdot \begin{vmatrix} 2 & 2 \\ 2 & -1 \end{vmatrix}\right]$$

$$= (-5)(-3)(-1)(-2-4)$$

$$= (-15)(-6) = \boxed{90}$$

Def A square matrix 'A' is called "upper triangular" if all enteries below the main diagonal one 0 (zero).

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ 0 & 0\,0 & 0 & 0 & a_{nn} \end{bmatrix}$$

upper

lower

main diagonal

We similarly define lower triangular matrices to have all zeros above the main diagonal

For Example, $\begin{bmatrix} 7 & \pi & 0 \\ 0 & -2 & 7 \\ 0 & 0 & 3 \end{bmatrix}$ is upper triangular.

Theorem: If 'A' is upper or lower triangular, then $\det A$ = product of numb. on main diag

$$= a_{11} \cdot a_{22} \cdot a_{33} \cdot a_{nn}$$

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Last Time

- Intro to determinants via cofactor expansion
- we saw

$$\det\left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & & a_{33} & \vdots \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}\right) = a$$

product of entries

Section 3.2 Properties of Determinants

We'll first study how rows operations affect determinants.
Let's investigate 2x2 case

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. $\det A = ad - bc$

Interchange Rows

Consider $B = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

$$\det B = cb - da$$
$$= bc - ad$$
$$= -ad + bc$$
$$= -(ad - bc)$$
$$= -\det A$$

Thus $A \underset{r_1 \leftrightarrow r_2}{\sim} B$ then $-\det A = \det B$

Scaling a Row

Consider $B = \begin{bmatrix} ka & kb \\ c & d \end{bmatrix}$

$$\det B = ka \cdot d - kb \cdot c$$
$$= k(ad - bc)$$
$$= k \cdot \det A$$

Thus,

$A \underset{kr_1 \to r_1}{\sim} B \Rightarrow k \cdot \det A = \det B$

Replacement

Consider $B = \begin{bmatrix} a & b \\ ka+c & kb+d \end{bmatrix}$

$$\det B = a(kb+d) - b(ka+c)$$
$$= \boxed{akb} + ad \boxed{-bka} - bc$$
$$= ad - bc$$
$$= \det A$$

Thus,

$A \underset{kr_1 + r_2 \to r_2}{\sim} B \Rightarrow \det A = \det B$

Theorem: Let A be $n \times n$

① Replacement: If B is the result of performing a replacement row operation on A, then $A \underset{kr_i + r_j \to r_i}{\sim} B \Rightarrow \det A = \det B$

② Interchange: If B is the result of interchanging 2 rows of A then,
$A \underset{r_i \leftrightarrow r_j}{\sim} B \Rightarrow -\det A = \det B$

③ Scaling: If B is the result of scaling one row of A by k, then

$A \underset{kr_i \to r_i}{\sim} B \Rightarrow k \cdot \det A = \det B$

$\left(\det A = \frac{1}{k} \det B\right)$ if $k \neq 0$

Purple