2024 03 05 PI

2.1 Matrix Operations

Addition and Scalar multiplication

$$A=\begin{bmatrix}1 & 3 & 6\\ -2 & 0 & 8\end{bmatrix} \qquad \overset{2\times 1}{C=\begin{bmatrix}1\\3\end{bmatrix}}$$

$$\overset{2\times 3}{B=\begin{bmatrix}7 & 2 & 0\\ 7 & 1 & -1\end{bmatrix}} \qquad \overset{1\times 2}{D=\begin{bmatrix}1 & 3\end{bmatrix}}$$

By Example

Addition

$$A+B=\begin{bmatrix}8 & 5 & 6\\ 5 & 1 & 7\end{bmatrix}$$ add corresponding entries

scalar multiplication

$$-3B=\begin{bmatrix}-21 & -6 & 0\\ -21 & -3 & 3\end{bmatrix}$$ multiply all enteries by the scalar

$$A-3B=\begin{bmatrix}-20 & -3 & 6\\ -23 & -3 & 11\end{bmatrix}$$
$A+(-3B)$

$A+C$ is undefined (DNE)

To add matrices they have to have the same dimensions

$C=D$? NO $C\neq D$

They have to have the same dimensions

Matrix Multiplication

Recall: we already know how to multiply a matrix by a vector.

e.g. $$\begin{bmatrix}3 & -5\\ 2 & 1\end{bmatrix}\begin{bmatrix}1\\4\end{bmatrix}=1\begin{bmatrix}3\\2\end{bmatrix}+4\begin{bmatrix}-5\\1\end{bmatrix}$$

Definition: ~~IR IAR~~ Let 'A' be $m\times n$ and 'B' be $n\times p$

Must match otherwise it's undefined (UND)

write

$$B=[\bar{b}_1, \bar{b}_2, \dots, \bar{b}_p]$$

Then

$$AB=[A\bar{b}_1, A\bar{b}_2, \dots, A\bar{b}_p] \quad m\times p$$

* I.e. $col_i(AB)=A\cdot col_i(B)$

Example: Let $A=\begin{bmatrix}3 & -5\\ 2 & 1\end{bmatrix}$, $B=\begin{bmatrix}1 & 3 & 0\\ 4 & 2 & -3\end{bmatrix}$

① Compute AB

Match? $2\times\underbrace{2 \quad 2}\times 3$ Yes, so is NOT UND, can compute

- $A\,col_1(B)=\begin{bmatrix}3 & -5\\ 2 & 1\end{bmatrix}\begin{bmatrix}1\\4\end{bmatrix}=1\begin{bmatrix}3\\2\end{bmatrix}+4\begin{bmatrix}-5\\1\end{bmatrix}=\begin{bmatrix}-17\\6\end{bmatrix}$
- $A\,col_2(B)=\begin{bmatrix}3 & -5\\ 2 & 1\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix}=3\begin{bmatrix}3\\2\end{bmatrix}+2\begin{bmatrix}-5\\1\end{bmatrix}=\begin{bmatrix}-1\\8\end{bmatrix}$
- $A\,col_3(B)=\begin{bmatrix}3 & -5\\ 2 & 1\end{bmatrix}\begin{bmatrix}0\\-3\end{bmatrix}=0\begin{bmatrix}3\\2\end{bmatrix}+(-3)\begin{bmatrix}-5\\1\end{bmatrix}=\begin{bmatrix}15\\-3\end{bmatrix}$

$$\therefore AB=\begin{bmatrix}-17 & -1 & 15\\ 6 & 8 & -3\end{bmatrix}$$

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② B·A Match?

2x3 2x3

NO, can't compute, therefore UND or DNE

Another view of matrix mult.

Example: Let A, B be as before. Compute AB

$$A\cdot B = \begin{bmatrix} 3 & -5 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 4 & 2 & -3 \end{bmatrix} = \begin{bmatrix} -17 & -1 & 15 \\ 6 & 8 & -3 \end{bmatrix}$$

AB,

so

$$AB = \begin{bmatrix} -17 & -1 & 15 \\ 6 & 8 & -3 \end{bmatrix}$$

(Row, col)

(1,1): $3\cdot 1 + (-5)4 = -17$

(2,1): $2\cdot 1 + 1\cdot 4 = 6$

(1,2): $3\cdot 3 + (-5)2 = -1$

(2,2): $2\cdot 3 + 1\cdot 2 = 8$

(1,3): $3\cdot 0 + (-5)(-3) = 15$

(2,3): $2\cdot 0 + 1(-3) = -13$

Remark: (i,j)-entry of AB is "dot product" of $Row_i(A)$ with $col_j(B)$

Example Let

$$A = \begin{bmatrix} -2 & 1 \\ 3 & 0 \\ 5 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 0 & 3 & -1 \\ 4 & -1 & 2 & 0 \end{bmatrix}$$

Compute AB

Match?

3x2 2x4

yes

$$\begin{array}{cc} & B\begin{bmatrix} 2 & 0 & 3 & -1 \\ 4 & -1 & 2 & 0 \end{bmatrix} \\ AB = A\begin{bmatrix} -2 & 1 \\ 3 & 0 \\ 5 & -2 \end{bmatrix} & \begin{bmatrix} 0 & -1 & -4 & 2 \\ 6 & 0 & 9 & -3 \\ 2 & 2 & 11 & -5 \end{bmatrix} \end{array}$$

Answer AB

BA is undefined

2x4 3x2

Does Not match, can't compute, therefore UND or DNE

Properties of Matrix Arithmetic

Many familiar properties are true: read theorems 1, 2 (pg 95, 99).

For example

$$A(B+C) = AB + AC \text{ (when defined)}$$

But, some familiar properties fail...

Example: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 \\ 1/2 & 1/2 \end{bmatrix}$

① $AB = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$

$BA = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 0 & 0 \end{bmatrix}$

! So, in general $AB \neq BA$

② $AB =$ from before $= \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$

$A \cdot C = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$

! In general, if $AB = AC$ then we can <u>NOT</u> conclude $B = C$

Transpose of a matrix

Def: If 'A' is $m \times n$, then the <u>transpose</u> of 'A', denoted A^T, is the $n \times m$ matrix where $row_i(A^T) = Col_i(A)$ * TL;DR: Interchange rows and columns

Example:

$A = \begin{bmatrix} \boxed{\begin{matrix} 1 \\ 0 \\ 4 \end{matrix}} & \boxed{\begin{matrix} 2 \\ -3 \\ 5 \end{matrix}} \end{bmatrix}$ $\quad A^T \begin{bmatrix} \boxed{1 \; 0 \; 4} \\ \boxed{2 \; -3 \; 5} \end{bmatrix}$

$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ $\quad B^T \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$

Theorem: Prop of transpose

- $(A^T)^T = A$
- $(A+B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$

DATE / NO. 2024/03/07

TITLE

2.1 Cont'd, 2.2 Inverse of a matrix

<u>Def</u>: I_n is the $n \times n$ matrix with 1's on the main diagonal and 0's everywhere else

$$I = I_n \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

* For example,

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

* I_n is called the <u>identity matrix</u>

! Notice: If A is $m \times n$

$\underline{A\,col_1(I_n)}$ $A\,col_2(I_n)$ ...

$I\,col_1(A$ [Missing]

<u>Powers of a matrix</u>

<u>Ex</u> Let $A = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

Compute A^2 and A^3

$$A^2 = AA = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A^3 = A\underbrace{AA}_{AA^2} = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Zero matrix often simply denoted as 0

So, $A^3 = 0$ Zero matrix

2.2 Inverse of a matrix

Question: How would you solve

$5x = 7$

$5x = 7 \Rightarrow \frac{1}{5} 5x = \frac{1}{5} 7$

$\Rightarrow 1x = \frac{7}{5}$

$\Rightarrow x = \frac{7}{5}$

Q: Can we apply a similar method to solve $A\bar{x} = \bar{b}$?
For example

$$\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \bar{x} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

Can we multiply both sides by A^{-1}?