California State University Sacramento - Math 101

Exam # 2

Name: ____________________

Exam is out of 8 points.

1) Let $X = \{1, 2, 3, \ldots, 20\}$.

(a) What is the number of 3-permutations of X? (0.25 points)

(b) What is the number of 3-circular permutations of X? (0.25 points)

(c) What is the number of 2-combinations of X? Simplify your answer as much as possible. (0.25 points)

(d) If $\mathcal{P}(X)$ is the power set of X, how many elements are in $\mathcal{P}(X)$? (0.25 points)

(e) Find the number of 5-combinations of X with only odd numbers. (0.25 points)

(f) Find the number of 5-combinations of X that do not contain a pair of consecutive integers. (0.25 points)

(a) $P^{20}_{3} = \frac{20!}{(20-3)!} = \frac{20!}{17!} = 20 \cdot 19 \cdot 18$

(b) $Q^{20}_{3} = \frac{1}{3} P^{20}_{3} = \frac{20 \cdot 19 \cdot 18}{3} = 20 \cdot 19 \cdot 6$

(c) $\binom{20}{2} = \frac{20!}{2!\,18!} = \frac{20 \cdot 19}{2} = 10 \cdot 19 = 190$

(d) $2^{|X|} = 2^{20}$

(e) $\binom{10}{5}$ since X contains 10 odd numbers

(f) $\binom{20-5+1}{5} = \binom{16}{5}$

2) A box contains 12 distinct marbles. Seven of the marbles are red and five of the marbles are blue. Thus, the elements of the box could be represented as the set

$$\{r_1, r_2, r_3, r_4, r_5, r_6, r_7, b_1, b_2, b_3, b_4, b_5\}.$$

(a) Find the number of ways to form a combination of two red marbles and three blue marbles. (0.25 points)

(b) Find the number of ways to form a combination of five marbles, at least three of which are blue. (0.5 points)

(c) How many ways can all 12 marbles be arranged around a circle so that no blue marbles are next to each other? (0.25 points)

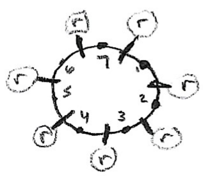
(d) Find the number of permutations of all 12 marbles such that b_1 comes before b_2. (0.5 points)

(a) $\boxed{\binom{7}{2}\binom{5}{3}}$

(b) $\boxed{\underbrace{\binom{5}{3}\binom{7}{2}}_{\text{3 blue, 2 red}} + \underbrace{\binom{5}{4}\binom{7}{1}}_{\text{4 blue, 1 red}} + \underbrace{\binom{5}{5}\binom{7}{0}}_{\text{5 blue, 0 red}}}$

(c) First place the reds around the circle

which can be done in $\frac{7!}{7} = 6!$ ways.



There are seven places between the reds

$\Rightarrow$ 7 choices for b_1, 6 choices for b_2, ..., 3 choices for b_5

$\Rightarrow \boxed{6! \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}$

(d) $\underbrace{\text{— — — } \underset{b_1}{\text{—}} \text{ — — — } \underset{b_2}{\text{—}} \text{ — — — —}}_{\binom{12}{2} \text{ ways to place } b_1, b_2}$ $\leftarrow$ 10! ways to line up remaining

$\boxed{\binom{12}{2} 10!}$

3) Recall that a 0-1 sequence of length n is a sequence of the form $a_1 a_2 \cdots a_n$ such that each a_i is either 0 or 1.

(a) What is the total number of 0-1 sequences of length 6? Simplify your answer as much as possible. (0.25 points)

(b) Find the total number of 0-1 sequences of length 10 that have at most two 0's. Simplify your answer as much as possible. (0.5 points)

(c) 37 people are going to vote on their preference for a new menu item at Timmons' Stars and Bars SandwHich Shop. The options are A, B, C, and D. How many possible outcomes are there after the 37 people vote? In other words, how many ways can 37 elements be chosen from A, B, C, and D allowing for repetition. (0.5 points)

(a) $2^6 = 64$

(b) $\binom{10}{0} + \binom{10}{1} + \binom{10}{2}$

$\underbrace{-\ -\ -\ -\ -\ -\ -\ -\ -\ -}$

choose zero, one, or two places to put the 0's

$= 1 + 10 + \frac{10 \cdot 9}{2}$

$= 11 + 45 = 56$

(c) $A \ldots \mid B \ldots \mid C \ldots \mid D \ldots$ ← An outcome corresponds to a 0-1 sequence of length $37+3$ with exactly three 1's.

$\binom{40}{3}$

4) Let $1 \leq r \leq n$.

(a) Give an algebraic proof that $\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$. Write your proof carefully and use correct mathematical notation. (0.5 points)

$$\frac{n}{r}\binom{n-1}{r-1} = \frac{n}{r} \cdot \frac{(n-1)!}{(r-1)!\,((n-1)-(r-1))!}$$

$$= \frac{n}{r} \cdot \frac{(n-1)!}{(r-1)!\,(n-r)!}$$

$$= \frac{n!}{r!\,(n-r)!} = \binom{n}{r}$$

(b) Explain why $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ is true by describing a counting process that shows the number of r-combinations of $\{1, 2, \ldots, n\}$ is equal to the right hand side of this equation. (0.5 points)

Consider an r-combination X of $\{1, 2, \ldots, n\}$.

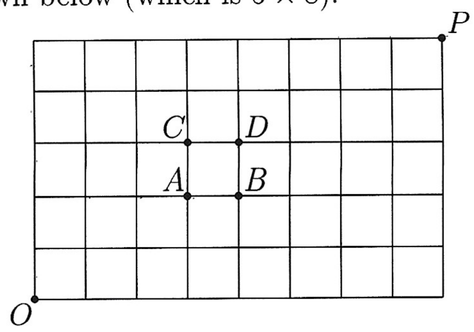
There are $\binom{n-1}{r}$ such X that do not contain 1.

$\cdot 1$ choosing r from $2, 3, \ldots, n$

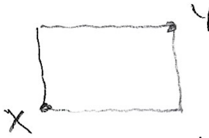
On the other hand, if X contains 1, there are $\binom{n-1}{r-1}$ ways to choose the remaining $r-1$ elements in X. Thus

$$\binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}$$

5) Consider the figure shown below (which is 5×8).



Using $n \times m$



$\binom{n+m}{n}$ shortest routes from X to Y

(a) Find the number of shortest routes from O to P. (0.25 points)

(b) Find the number of shortest routes from O to P that pass through the street AB or the street CD. (0.5 points)

(a) $\binom{13}{5}$

(b) $\underbrace{\binom{5}{2}\binom{7}{3}}_{\text{through } AB} + \underbrace{\binom{6}{3}\binom{6}{2}}_{\text{through } CD}$

6) Let $s(r, n)$ be the number of ways to arrange r distinct objects around n indistinguishable circles so that every circle has at least one object.

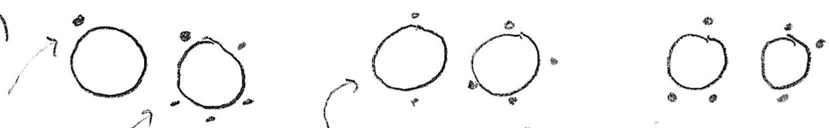
(a) Why is $s(r, r) = 1$? Answer in a complete sentence and do not use any math symbols. (0.25 points)

(b) Determine, with explanation (which can include pictures) the value of $s(6, 2)$. (0.75 points)

(a) Since the tables are indistinguishable, all placements are the same because each table must have exactly one object.

the 1/2 is due to the symmetry

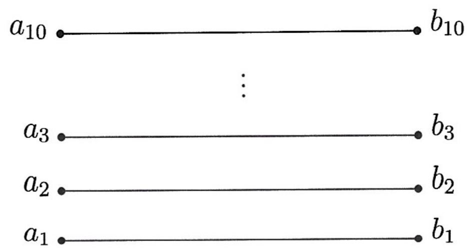
(b)



$\binom{6}{1} Q^5_5$ $\qquad$ $\binom{6}{2} Q^4_4$ $\qquad$ $\frac{1}{2}\binom{6}{3} \cdot Q^3_3 Q^3_3$

$$\binom{6}{1} Q^5_5 + \binom{6}{2} Q^4_4 + \frac{1}{2}\binom{6}{3} Q^3_3 Q^3_3$$

7) Consider the matching shown below.



(a) Find the number of subsets of size 5 from $\{a_1, \ldots, a_{10}, b_1, \ldots, b_{10}\}$ that do not contain any of the edges. (0.25 points)

(b) Find the number of subsets of size 5 from $\{a_1, \ldots, a_{10}, b_1, \ldots, b_{10}\}$ that contain exactly one edge. (0.5 points)

(c) Find the number of subsets of <u>size 10</u> from $\{a_1, \ldots, a_{10}, b_1, \ldots, b_{10}\}$ that contain exactly three edges. (0.25 points)

(a) $\binom{10}{5} 2^5$

pick 5 edges

pick a_i or b_i from each edge

(b) $10 \binom{9}{3} 2^3$

choose an edge a_j—b_j and take both a_j and b_j

pick 3 more edges

pick a_i or b_i from each edge

a_i b_i

(c) $\binom{10}{3} \binom{7}{4} 2^4$