Quiz 9 - Solutions

2) (a) $\sum_{k=0}^{n} f_{n,p}(k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$

$$= (p + (1-p))^n$$

$$= 1^n = 1$$

(b)

0	$\binom{4}{0}(0.7)^0(0.3)^4 = 0.0081$
1	$\binom{4}{1}(0.7)^1(0.3)^3 = 0.0756$
2	$\binom{4}{2}(0.7)^2(0.3)^2 = 0.2646$
3	$\binom{4}{3}(0.7)^3(0.3)^1 = 0.4116$
4	$\binom{4}{4}(0.7)^4(0.3)^0 = 0.2401$

(c)

$1 \cdot 0.0756 + 2 \cdot 0.2646 + 3 \cdot 0.4116 + 4 \cdot 0.2401$

$= 2.8$

(d) Starting with the Binomial Theorem,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

we take the derivative with respect to x to get

$$n(x+y)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k x^{k-1} y^{n-k}$$

Let $x = p$, $y = 1-p$ to obtain

$$n \cdot 1 = \sum_{k=0}^{n} \binom{n}{k} k p^{k-1} (1-p)^{n-k}.$$

Multiplying by p gives

$$np = \sum_{k=0}^{n} \binom{n}{k} k p^k (1-p)^{n-k}$$

3) Method 1

$$\binom{n}{r}\binom{n-r}{m-r} = \frac{n!}{r!(n-r)!} \cdot \frac{(n-r)!}{(m-r)!\,(n-r-(m-r))!}$$

$$= \frac{n!}{r!\cancel{(n-r)!}} \cdot \frac{\cancel{(n-r)!}}{(m-r)!\,(n-m)!}$$

$$= \frac{n!}{(n-m)!} \cdot \frac{1}{r!\,(m-r)!}$$

this step is trying to make expression look like our target $\binom{n}{m}\binom{m}{r}$.

$$= \frac{n!}{m!\,(n-m)!} \cdot \frac{m!}{r!\,(m-r)!}$$

multiplied by $1 = \frac{m!}{m!}$

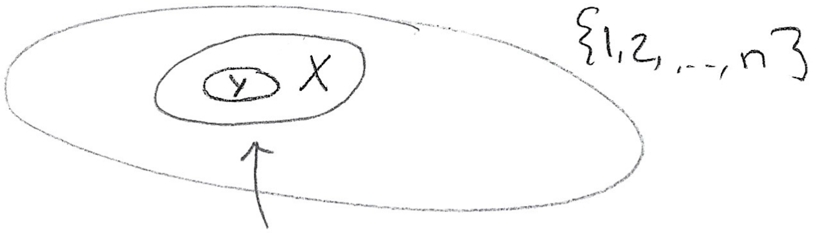
$$= \binom{n}{m}\binom{m}{r}$$

Method 2

Let N be the number of ways to choose an m-combination $X \subseteq \{1,2,\dots,n\}$, and choose an r-combination $Y \subseteq X$.

There are $\binom{n}{m}$ choices for X and then $\binom{m}{r}$ choices for $Y \subseteq X$ so $N = \binom{n}{m}\binom{m}{r}$.

However, we can also choose some r-combination for Y first, which can be done in $\binom{n}{r}$ ways, and then a m-combination X that contains Y which can be done in $\binom{n-r}{m-r}$ ways. Hence, $N = \binom{n}{r}\binom{n-r}{m-r}$.



Choosing an m-combination X and an r-combination $Y \subseteq X$

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4) $$\sum_{r=m}^{n}\binom{n-m}{n-r} = \binom{n-m}{n-m} + \binom{n-m}{n-(m+1)} + \binom{n-m}{n-(m+2)} + \dots + \binom{n-m}{n-n}$$

$$= \binom{n-m}{n-m} + \binom{n-m}{n-m-1} + \binom{n-m}{n-m-2} + \dots + \binom{n-m}{0}$$

$$= \sum_{j=0}^{n-m}\binom{n-m}{j} = 2^{n-m}$$

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5) In the Binomial Theorem, let $y=1$ to get

$$(x+1)^n = \sum_{r=0}^{n}\binom{n}{r}x^r$$

Differentiate twice with respect to x to get

$$n(x+1)^{n-1} = \sum_{r=0}^{n}\binom{n}{r}r x^{r-1}$$

$$n(n-1)(x+1)^{n-2} = \sum_{r=0}^{n}\binom{n}{r}r(r-1)x^{r-2}$$

Now let $x=1$

$$n(n-1)2^{n-2} = \sum_{r=0}^{n}\binom{n}{r}(r^2-r)$$

Now by the distributive law,

$$n(n-1)2^{n-2} = \sum_{r=0}^{n} \binom{n}{r} r^2 - \underbrace{\sum_{r=0}^{n} \binom{n}{r} r}$$

Now we add $+\sum_{r=0}^{n} \binom{n}{r} r$ to both sides

$$n(n-1)2^{n-2} + \underbrace{\sum_{r=0}^{n} \binom{n}{r} r} = \sum_{r=0}^{n} \binom{n}{r} r^2$$

We have shown that this sum is equal to $n2^{n-1}$

$$n^2 2^{n-2} - n2^{n-2} + n2^{n-1} = \sum_{r=0}^{n} \binom{n}{r} r^2$$

$$2^{n-2}\left(n^2 - n + 2n\right) = \sum_{r=0}^{n} \binom{n}{r} r^2$$

$$2^{n-2}\left(n^2 + n\right) = \sum_{r=0}^{n} \binom{n}{r} r^2$$

$$n(n+1)2^{n-2} = \sum_{r=0}^{n} \binom{n}{r} r^2$$