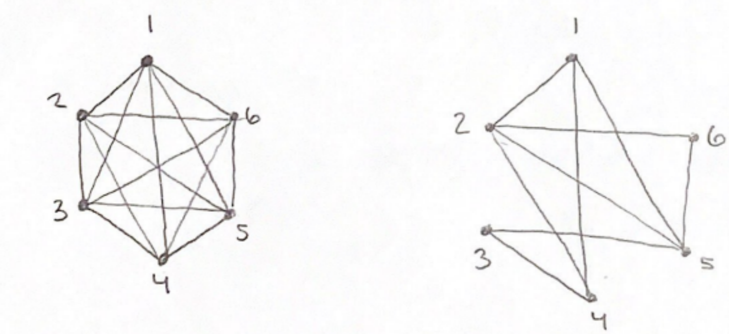
California State University Sacramento - Math 101

Quiz #9

Name: ________________________

1) On the left, every two numbers from $\{1, 2, 3, 4, 5, 6\}$ are connected by an edge. This is a visual representation of the *complete graph on 6 vertices* which we denote by K_6.



2-comb	H or T	Connect 2-comb.?
{1,2}	H	Yes
{1,3}	T	No
{1,4}	H	Yes
{1,5}	H	Yes
{1,6}	T	No
{2,3}	T	No
{2,4}	H	Yes
{2,5}	H	Yes
{2,6}	H	Yes
{3,4}	H	Yes
⋮		

On the right is a random subgraph of K_6 that was determined by the following procedure. Consider each 2-combination $\{a, b\}$ in the ordered list

$$\{1,2\},\ \{1,3\},\ \{1,4\},\ \{1,5\},\ \{1,6\},\ \{2,3\},\ \{2,4\},\ \{2,5\},\ \{2,6\},\ \{3,4\},\ \{3,5\},\ \{3,6\},\ \{4,5\},\ \{4,6\},\ \{5,6\}$$

For each $\{a, b\}$, flip of fair coin where H and T are equally likely. If the result is H, keep $\{a, b\}$ connected. If the result is T, remove the connection. The table on the right in the figure above shows this process over the first ten pairs, however, it was completed for all fifteen pairs in order to produce the random graph on the right.

Instructions: Perform this process to draw a random subgraph of K_6. You may use www.random.org/coins to perform 15 independent flips at once.

2) Let p be a real number with $0 \leq p \leq 1$ and let n be a positive integer. Define the function $f_{n,p} : \mathbb{R} \to \mathbb{R}$ by the rule

$$f_{n,p}(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & \text{if } k \in \{0, 1, 2, \ldots n\}, \\ 0 & \text{otherwise.} \end{cases}$$

The function $f_{n,p}$ is the probability density function of a Binomial random variable with parameters n and p. For $k \in \{0, 1, \ldots, n\}$, the value $f_{n,k}(k)$ is the probability of k successes in n independent trials with success probability p.

(a) Prove, using the Binomial Theorem, that $\displaystyle\sum_{k=0}^{n} f_{n,p}(k) = 1$.

(b) Suppose that a biased coin, which comes up heads with probability 0.7, is tossed 4 independent times. Complete the following table.

Number of heads k	Probability of k heads
0	$f_{4,0.7}(0) = \binom{4}{0}(0.7)^0(0.3)^4 = 0.0081$
1	
2	
3	
4	

(c) Use the table from part (b) to compute

$$0 \cdot f_{4,0.7}(0) + 1 \cdot f_{4,0.7}(1) + 2 \cdot f_{4,0.7}(2) + 3 \cdot f_{4,0.7}(3) + 4 \cdot f_{4,0.7}(4).$$

This value is the average number of heads we would expect to see in 4 independent tosses.

(d) Prove that for any real number p with $0 \leq p \leq 1$ and positive integer n,

$$np = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}.$$

This value is the expected number of successes if n independent trials are done where the probability of success is p.

3) Show that for integers $1 \leq r \leq m \leq n$,

$$\binom{n}{m}\binom{m}{r} = \binom{n}{r}\binom{n-r}{m-r}.$$

4) Show that for any integers $n \geq m \geq 1$,

$$\sum_{r=m}^{n} \binom{n-m}{n-r} = 2^{n-m}.$$

5) Show that for all integers $n \geq 1$,

$$\sum_{r=1}^{n} r^2 \binom{n}{r} = n(n+1)2^{n-2}.$$