OLD EXAM #2

1) (a) $s(r,r)$

r distinct objects

1 2 ... r

r identical tables

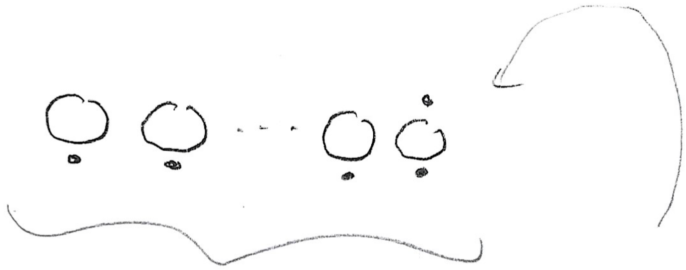
1 object per table

$$s(r,r) = 1$$

(b) $s(r, r-1)$

r distinct objects

1 2 .... r



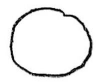
r-1 tables so exactly one table has two objects

$$s(r,r) = \binom{r}{2}$$

(c) $s(r,1)$

r distinct objects

1 2 .... r



1 table $\rightarrow$ $Q^r_r = (r-1)!$

$$\Longrightarrow s(r,1) = (r-1)!$$

(d) $s(6,3)$

CASE 1

$\binom{6}{4} Q^4_4 = \binom{6}{4} 3!$

CASE 2

$\binom{6}{2} Q^2_2 \binom{4}{3} Q^3_3$

$= \binom{6}{2} \cdot 1! \binom{4}{3} \cdot 2!$

CASE 3

3! ways to permute the tables

$\underbrace{\binom{6}{2} Q^2_2 \binom{4}{2} Q^2_2 \cdot Q^2_2}_{3!}$

Using the fact that $Q_2^2 = 1$ and $Q_3^3 = 2$,

the total count is

$$\binom{6}{4} 3! + \binom{6}{2}\binom{4}{3} \cdot 2 + \frac{1}{3!}\binom{6}{2}\binom{4}{2}.$$

2) Using the recursion,

$$\begin{aligned} s(9,3) &= s(8,2) + 8 \cdot s(8,3) \\ &= 13{,}068 + 8(13{,}132) \\ &= 118{,}124 \end{aligned}$$

← given

3) 

5 choices for position of x, y and then 2 choices for x first or y first

$\Rightarrow \; 5 \cdot 2 \cdot \underbrace{7!}_{\text{line up } A, B, \ldots, G}$

4) $\binom{15-3+1}{3} = \binom{13}{3}$

5) (a) $\binom{5}{2} \cdot \binom{7}{3}$

$\uparrow$ or $\binom{5}{3}$ $\quad$ $\uparrow$ or $\binom{7}{4}$

(b) $\binom{13}{5} - \binom{5}{3}\binom{7}{4}$

6) $\binom{60-7+1}{7} = \binom{54}{7}$

7) Trick: Recognize as $\binom{n+r}{r}$ simplified

$$\binom{n+r}{r} = \frac{(n+r)!}{r!\,(n+r-r)!} = \frac{(n+r)!}{r!\,n!}$$

$$= \frac{(n+r)(n+r-1)\cdots(n+2)(n+1)}{r!}$$

is an integer since it is the # of subsets of size r from a set with n+r elements

8) $\frac{n}{n-r}\binom{n-1}{r} = \frac{n}{n-r} \cdot \frac{(n-1)!}{r!\,(n-1-r)!}$

$$= \frac{n!}{(n-r)!\,r!} = \binom{n}{r}$$

9) $\dfrac{10!}{3!\,2!\,4!}$

10)

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$\longleftrightarrow \underbrace{001000010}$

0-1 sequence
with length 9
and exactly two 1s

$\binom{9}{2}$

There is a bijection between orders and 0-1 sequences of length 9 with exactly two 1s.

11) $H_r^n = \binom{n+r-1}{r}$

12) Note x_1 must be 1 so that

$$6 \cdot 1 + x_2 + x_3 = 11$$
$$x_2 + x_3 = 5$$

$$\left.\begin{array}{l} x_2 = 1, x_3 = 4 \\ x_2 = 2, x_3 = 3 \\ x_2 = 3, x_3 = 2 \\ x_2 = 4, x_3 = 1 \end{array}\right\}$$

4 positive integer solutions

$$\left.\begin{array}{l} (1,1,4) \\ (1,2,3) \\ (1,3,2) \\ (1,4,1) \end{array}\right\} (x_1, x_2, x_3)$$

13) (a) $S(r,r)$

r distinct objects $\quad$ r identical boxes

$1 \ 2 \ \cdots \ r$

one item per box

$$S(r,r)=1$$

(b) Exactly one box must have two objects so

$$S(r,r-1)=\binom{r}{2}.$$

(c) There is only one box so $S(r,1)=1$.

(d) $S(5,2)$

objects

1, 2, 3, 4, 5

CASE 1

$\binom{5}{4}$

CASE 2

$\binom{5}{3}$

$$\Longrightarrow S(5,2)=\binom{5}{4}+\binom{5}{3}$$
$$=5+10=15$$

14) (a)

Box 1 $\quad$ Box 2 $\quad$ Box n

$\left.\begin{array}{c}1\\2\\\vdots\\r\end{array}\right\}$ r distinct objects

n choices for object 1,

$n-1$ " " 2

$n-2$ " " 3

$\vdots$

$n-r+1$ " " r

Boxes hold at most one object

$$\Longrightarrow n(n-1)(n-2)\cdots(n-r+1)$$

14) (b)

3 choices for object 1,
3 " " 2,
3 " " 3
3 " " 4

each box can hold any # of items

$$3^4$$

(c) $\binom{n}{r}$ $\longleftarrow$ choose the r boxes to have an object