California State University Sacramento - Math 101

Homework Assignment 10 - Solutions

1) If A and B are finite sets, then $|A \cup B| = |A| + |B| - |A \cap B|$.

2) If A, B, and C are finite sets, then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

3) Let A be the integers in $\{1, 2, \dots, 90\}$ that are divisible by 3 and B be the integers in $\{1, 2, \dots, 90\}$ that are divisible by 5. We have

$$|A| = \frac{90}{3} = 30, |B| = \frac{90}{5} = 18, \text{ and } |A \cap B| = \frac{90}{15} = 6.$$

Therefore, the number of integers in $\{1, 2, \dots, 90\}$ that are divisible by 3 or 5 is

$$|A \cup B| = |A| + |B| - |A \cap B| = 30 + 18 - 6 = 42.$$

4) Let A be the integers in $\{1, 2, \dots, 140\}$ that are divisible by 2, B be the integers in $\{1, 2, \dots, 140\}$ that are divisible by 5, and C be the integers in $\{1, 2, \dots, 140\}$ that are divisible by 7. We have

$$|A| = \tfrac{140}{2} = 70, |B| = \tfrac{140}{5} = 28, |C| = \tfrac{140}{7} = 20,$$
$$|A \cap B| = \tfrac{140}{10} = 14, |A \cap C| = \tfrac{140}{14} = 10, |B \cap C| = \tfrac{140}{35} = 4, \text{ and } |A \cap B \cap C| = \tfrac{140}{70} = 2.$$

By the Principle of Inclusion-Exclusion, the number of integers in $\{1, 2, \dots, 140\}$ that are divisible by 2, 5, or 7 is

$$|A \cup B \cup C| = 70 + 28 + 20 - 14 - 10 - 4 + 2 = 92.$$

5) Let C be the students who passed Chinese, E be the students who passed English, and M be the students who passed Mathematics. We are given that

$$|C| = 92, |E| = 75, |M| = 63, \tag{1}$$

and

$$|C \cap E| \leq 65, |C \cap M| \leq 54, |E \cap M| \leq 48. \tag{2}$$

By Inclusion-Exclusion,

$$|C \cup E \cup M| = |C| + |E| + |M| - |C \cap E| - |C \cap M| - |E \cap M| + |C \cap E \cap M|.$$

Using (1), this equation can be rewritten as

$$|C \cap E| + |C \cap M| + |E \cap M| + |C \cup E \cup M| = 230 + |C \cap E \cap M|.$$

Since 102 students took the exams, we know that $102 \geq |C \cup E \cup M|$. Using this inequality and (2), we have

$$102 + 65 + 54 + 48 \geq 230 + |C \cap E \cap M|$$

so $39 \geq |C \cap E \cap M|$. We conclude that at most 39 students passed all three subjects.

6) Let B be the students who played baseball, H be the students who played hockey, and F be the students who played football. By Inclusion-Exclusion,

$$\begin{aligned} 100 &= |B \cup H \cup F| = |B| + |H| + |F| - |B \cap H| - |B \cap F| - |H \cap F| + |B \cap H \cap F| \\ &= 30 + 50 + 60 - |B \cap H| - |B \cap F| - |H \cap F| + 18. \end{aligned}$$

This gives

$$|B \cap H| + |B \cap F| + |H \cap F| = 58.$$

The sum $|B \cap H| + |B \cap F| + |H \cap F|$ counts the students that played exactly two sports once, and counts the students that played exactly three sports three times. The number of students that played at least two sports is

$$|B \cap H| + |B \cap F| + |H \cap F| - 2|B \cap H \cap F| = 58 - 2(18) = 22.$$