Q1 Vandermonde's Identity

For all integers $m, n, r \geq 1$

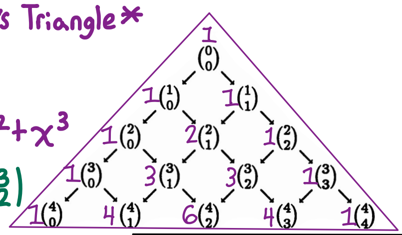
$$\binom{m+n}{r}=\sum_{i=0}^{r}\binom{m}{i}\binom{n}{r-i}$$

By: Binomial Theorem – Algebraic

Key consequence of the Binomial Theorem is that the coefficent of x^l in $(1+x)^m$ is $\binom{m}{l}$ *Pascal's Triangle*

$(1+x)^2=1+2x+x^2$

$(1+x)^3=1+\underbrace{3x}_{\binom{3}{1}}+\underbrace{3x^2}_{\binom{3}{2}}+x^3$



Consider the idenity

$$\underbrace{(1+x)^{m+n}}_{\text{Coefficent of } x^r \text{ is } \binom{m+n}{r}}=(1+x)^m(1+x)^n$$

Binomial Theorem

$$\boxed{(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k}$$

$\binom{m}{0}1$	$\binom{n}{r}x^r$
$\binom{m}{1}x$	$\binom{n}{r-1}x^{r-1}$
$\binom{m}{2}x^2$	$\binom{n}{r-2}x^{r-2}$
$\vdots$	
$\binom{m}{r}x^r$	$\binom{n}{0}1$

Leverages This Rule

$\boxed{\text{Key consequence of the Binomial Theorem is that the coefficent of } x^l \text{ in } (1+x)^m \text{ is } \binom{m}{l}}$

For the right hand side, the coefficent of x^r is going to be

$$\binom{m}{0}\binom{n}{r}+\binom{m}{1}\binom{n}{r-1}+\binom{m}{2}\binom{n}{r-2}+\cdots+\binom{m}{r}\binom{n}{0}=\sum_{i=0}^{r}\binom{m}{i}\binom{n}{r-i}$$

Coefficent of x^0 Coefficent of x^r

$$\rightarrow \binom{m+n}{r}=\sum_{i=0}^{r}\binom{m}{i}\binom{n}{r-i}$$

Q2 Multinomial Theorem: For $n, m \in \mathbb{N}$

$$(x_1+x_2+\cdots+x_m)^n=\sum\binom{n}{n_1,n_2,\ldots,n_m}x_1^{n_1}x_2^{n_2}x_m^{n_m}$$

$$(x_1+x_2+\cdots+x_m)^n=\sum\underbrace{\binom{n}{n_1,n_2,\ldots,n_m}}_{\left\{\frac{n!}{n_1!\,n_2!\ldots n_m!}\right\}}x_1^{n_1}x_2^{n_2}x_m^{n_m}$$

$(n_1,n_2,\cdots,n_m)\in S$

Where S is all m-tuples of non-negative integers with $n_1+n_2+\ldots+n_m=n$

Q3 For $n=4$ and $m=3$, we have by Theorem 2.8.1

$$(x_1+x_2+x_3)^4$$

Multinomial Theorem

$$\boxed{(x_1+x_2+\cdots+x_m)^n=\sum\binom{n}{n_1,n_2,\ldots,n_m}x_1^{n_1}x_2^{n_2}x_m^{n_m}}$$

$n=4$ and $m=3$ ← number of positions

adds up to 4

$$\left\{\frac{n!}{n_1!\,n_2!\ldots n_m!}\right\}$$

$$(x_1+x_2+x_3)^4=\underbrace{\binom{4}{4,0,0}}_{\frac{4!}{4!0!0!}=1}x_1^4+\underbrace{\binom{4}{3,1,0}}_{\frac{4!}{3!1!0!}=4}x_1^3x_2^1+\underbrace{\binom{4}{3,1,0}}_{\frac{4!}{3!1!0!}=4}x_1^3x_2^1$$

$$+\underbrace{\binom{4}{2,2,0}}_{\frac{4!}{2!2!0!}=6}x_1^2x_2^2x_3^0+\underbrace{\binom{4}{2,1,1}}_{\frac{4!}{2!1!1!}=12}x_1^2x_2^1x_3^1+\underbrace{\binom{4}{2,0,2}}_{\frac{4!}{2!0!2!}=6}x_1^2x_2^0x_3^2$$

$$+\underbrace{\binom{4}{1,3,0}}_{\frac{4!}{1!3!0!}=4}x_1^1x_2^3x_3^0+\underbrace{\binom{4}{1,2,1}}_{\frac{4!}{1!2!1!}=12}x_1^1x_2^2x_3^1+\underbrace{\binom{4}{1,1,2}}_{\frac{4!}{1!1!2!}=12}x_1^1x_2^1x_3^2$$

$$+\underbrace{\binom{4}{1,0,3}}_{\frac{4!}{1!0!3!}=4}x_1^1x_2^0x_3^3+\underbrace{\binom{4}{0,4,0}}_{\frac{4!}{0!4!0!}=1}x_1^0x_2^4x_3^0+\underbrace{\binom{4}{0,3,1}}_{\frac{4!}{0!3!1!}=4}x_1^0x_2^3x_3^1$$

$$+\underbrace{\binom{4}{0,2,2}}_{\frac{4!}{0!2!2!}=6}x_1^0x_2^2x_3^2+\underbrace{\binom{4}{0,1,3}}_{\frac{4!}{0!1!3!}=4}x_1^0x_2^1x_3^3+\underbrace{\binom{4}{0,0,4}}_{\frac{4!}{0!0!4!}=1}x_1^0x_2^0x_3^4$$

$$=x_1^4+4x_1^3x_2^1+4x_1^3x_2^1$$

$$+6x_1^2x_2^2x_3^0+12x_1^2x_2^1x_3^1+6x_1^2x_2^0x_3^2$$

$$+4x_1^1x_2^3x_3^0+12x_1^1x_2^2x_3^1+12x_1^1x_2^1x_3^2$$

$$+4x_1^1x_2^0x_3^3+1x_1^0x_2^4x_3^0+4x_1^0x_2^3x_3^1$$

$$+6x_1^0x_2^2x_3^2+4x_1^0x_2^1x_3^3+1x_1^0x_2^0x_3^4$$

Q4 28. $\displaystyle\sum_{r=m}^{n}\binom{n}{r}\binom{r}{m}=2^{n-m}\binom{n}{m}$ for $m \leq n$,

$$\sum_{r=m}^{n}\binom{n}{r}\binom{r}{m}=2^{n-m}\binom{n}{m}$$

LHS: $\displaystyle\sum_{r=m}^{n}\binom{n}{r}\binom{r}{m}\Rightarrow\sum_{r=m}^{n}\binom{n}{m}\binom{n-m}{r-m}\Rightarrow\binom{n}{m}\sum_{r=m}^{n}\binom{n-m}{r-m}$

$\displaystyle\Rightarrow\binom{n}{m}\sum_{r=m}^{n}\binom{n-m}{r}$

Identitiy

$$\boxed{\sum_{r=0}^{n}\binom{n}{r}=2^n}$$

$\Rightarrow\binom{n}{m}2^{n-m}$ $\therefore$ LHS = RHS

Q5 31. $\displaystyle\sum_{r=0}^{n}(-1)^r r\binom{n}{r}=0,$

$$\sum_{r=0}^{n}(-1)^r r\binom{n}{r}=0$$

LHS: $\displaystyle\sum_{r=0}^{n}(-1)^r r\binom{n}{r}\Rightarrow\sum_{r=0}^{n}(-1)^r r\binom{n-1}{r-1}$

$\displaystyle\rightarrow -n\sum_{r-1}^{n}(-1)^r\binom{n-1}{r}$

* Identity: $\displaystyle\sum_{r=0}^{n}(-1)^r\binom{n}{r}=\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\cdots+(-1)^n\binom{n}{n}=0$

$\displaystyle -n\sum_{r-1}^{n}(-1)^r\binom{n-1}{r}=0$ So LHS = RHS

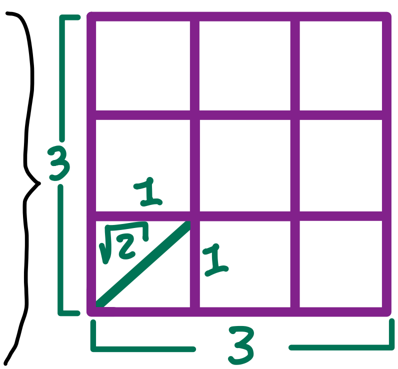
6) State the Pigeonhole Principle

Let K & n be positive the Pigeonhole Principle states that if atleast Kn+1 objects are placed in n-boxes, then at least one box has at least K+1 objects

Pigion Hole Principle Example
There is a chance that in this class there will be 2 students with the same month.
Given that we have more than 12 students in class.

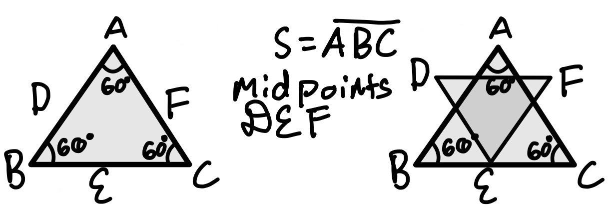
Q7 **Example 3.2.4.** Show that for any set of 10 points chosen within a square whose sides are of length 3 units, there are two points in the set whose distance apart is at most $\sqrt{2}$.

What are the objects? What are the boxes? These are the two questions we have to ask beforehand. It is fairly clear that we should treat the 10 given points in the set as our "objects". The conclusion we wish to arrive at is the existence of "2 points" from the set which are "close" to each other (i.e. their distance apart is at most $\sqrt{2}$ units). This indicates that "$k+1=2$" (i.e., $k=1$), and suggests also that we should partition the 3×3 square into n smaller regions, $n < 10$, so that the distance between any 2 points in a region is at most $\sqrt{2}$.



Exercise 3

1. Show that among any 5 points in an equilateral triangle of unit side length, there are 2 whose distance is at most $\frac{1}{2}$ units apart.



If BA = I
then BD = I/2

3. Given any set S of 9 points within a unit square, show that there always exist 3 distinct points in S such that the area of the triangle formed by these 3 points is less than or equal to $\frac{1}{8}$. (Beijing Math. Competition, 1963)