California State University Sacramento - Math 101

Quiz #8

Name: ____________________

1) (a) State the Binomial Theorem.

(b) Prove that $0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$ for all integers $n \geq 1$.

(a) For any positive integer $n \geq 1$, $(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$.

(b) Let $x=-1, y=1$ in the Binomial Theorem.

$$(-1+1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k 1^{n-k}$$

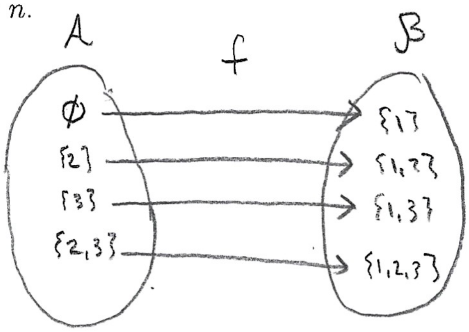
$$0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$$

2) Let $X = \{1, 2, \ldots, n\}$, $\mathcal{A} = \{A \subseteq X : 1 \notin A\}$ and $\mathcal{B} = \{B \subseteq X : 1 \in B\}$.

(a) In the case that $n = 3$, write down all of the elements of $\mathcal{A}$, all of the elements of $\mathcal{B}$, and a bijection f from $\mathcal{A}$ to $\mathcal{B}$.

(b) Give a formula for a function $f : \mathcal{A} \to \mathcal{B}$ that defines a bijection from $\mathcal{A}$ to $\mathcal{B}$ and is valid for every n.

(a)



$\mathcal{A} = \{\emptyset, \{2\}, \{3\}, \{2,3\}\}$

$\mathcal{B} = \{\{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}\}$

(b) $f(A) = A \cup \{1\}$ for $A \in \mathcal{A}$

3) What is the coefficient of x^3 in $(x+1)^{12}$? Simplify your answer as much as possible.

$$\binom{12}{3} = \frac{12!}{3!\,9!} = \frac{12\cdot 11\cdot 10}{3\cdot 2} = \frac{12\cdot 110}{6} = 220$$

4) Prove that $n2^{n-1} = \sum_{r=1}^{n} r\binom{n}{r}$ for all integers $n \geq 1$.

Let $y=1$ in $(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$ to get

$$(x+1)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$$

Differentiate with respect to x to get

$$n(x+1)^{n-1} = \sum_{r=0}^{n} \binom{n}{r} r x^{r-1}$$

Set $x=1$ to get

$$n2^{n-1} = \sum_{r=0}^{n} \binom{n}{r} r$$

which is equivalent to $n2^{n-1} = \sum_{r=1}^{n} r\binom{n}{r}$

5) Simplify $\sum_{r=1}^{n} \binom{n}{r}$ as much as possible. Your final answer should involve two terms, one of which depends on n.

$$\sum_{r=1}^{n} \binom{n}{r} = \underbrace{\binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} + \binom{n}{0}}_{2^n} - \binom{n}{0}$$

$$= 2^n - \binom{n}{0}$$

$$= 2^n - 1$$