# Study Kit 01 - Homework & Exams

## CHAPTER 01 - Permutations and Combinations

### Section 1.1. Two Basic Counting Principles
### Section 1.2. Permuations
### Section 1.3. Circular Permutations
### Section 1.4. Combinations

**Homework Assignment 01 - section 1.1**

**Homework Assignment 02 - section 1.2**

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**Quiz 01 - section 1.1**

**Quiz 02 - section 1.2**

**Quiz 03 - section 1.3 - section 1.4**

# Chapter 1. Permutations and Combinations

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**Homework Assignment 1**

**1)** Let $A = \{-3, -2, -1, \ldots, 5, 6, 7\}$.
(a) Is $1 \in A$?
(b) Is $\frac{1}{2} \in A$?
(c) Find $|A|$.
(d) If $B = \{4, 6, 8, 10\}$, find $A \cup B$ and $A \cap B$.

**2)** Suppose $A_1 = \{1, 2, 3\}$, $A_2 = \{3, 4, 5\}$, and $A_3 = \{4, 5, 6\}$.
(a) Find $A_1 \cup A_2 \cup A_3$.
(b) Find $A_1 \cap A_2 \cap A_3$.
(c) True or False: $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3|$.
(d) True or False: $|A_1 \cap A_2 \cap A_3| = |A_1||A_2||A_3|$.

**3)** Suppose $A_1, A_2, \ldots, A_5$ are pairwise disjoint sets with $|A_i| = i$ for $1 \leq i \leq 5$. Determine

$$\left|\bigcup_{i=1}^{5} A_i\right|.$$

**4)** Find sets $A_1, A_2, \ldots, A_5$ such that $|A_i| = i$ for $1 \leq i \leq 5$ and

$$\left|\bigcup_{i=1}^{5} A_i\right| = 5.$$

**5)** If $A = \{x : 3 \leq x \leq 10\}$ and $\mathbb{Z}$ is the set of all integers, find

$$|A \cap \mathbb{Z}|$$

# *Chapter 1. Permutations and Combinations*

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**Homework Assignment 2**

**1)** Problem 1 on page 50

**2)** Problem 10 on page 51

**3)** Problem 11 on page 51

**4)** Problem 12 on page 51

Q1) **1. Find the number of ways to choose a pair $\{a, b\}$ of distinct numbers from the set $\{1, 2, ..., 50\}$ such that**

**(i) $|a - b| = 5$; (ii) $|a - b| \leq 5$.**

Q2) **10. Find the number of common positive divisors of $10^{40}$ and $20^{30}$.**

Q3) **11. In each of the following, find the number of positive divisors of $n$ (inclusive of $n$) which are multiples of 3:**

**(i) $n = 210$; (ii) $n = 630$; (iii) $n = 151200$.**

Q4) **12. Show that for any $n \in \mathbf{N}$, the number of positive divisors of $n^2$ is always odd.**

Q **5)** Find the number of ordered pairs $(x, y)$ of integers such that $x^2 + y^2 \leq 4$.
Remark: This problem is similar to Example 1.1.2.

Q **6)** Find the number of sequences $a_1a_2a_3$ of length 3 where $a_i \in \{0, 1, 2, 3, 4\}$.
Remark: This is a special case of Example 1.1.4.

Q **7)** Let $X = \{1, 2, \ldots, 10\}$ and let

$$S = \{(a, b, c) : a, b, c \in X, a < b \text{ and } a < c\}.$$

Find $|S|$.
Remark: This problem is similar to Example 1.1.6.

# *Chapter 1. Permutations and Combinations*

# *Section 1.2. Permutations*

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Homework Assignment 3

1) Example 1.2.1 on page 6. Q1) **Example 1.2.1.** Let $A = \{a, b, c, d\}$. All the 3-permutations of $A$ are

2) Example 1.2.2 on page 7.

3) Example 1.2.3 on page 8.

4) Example 1.2.4 on page 9.

Q2) **Example 1.2.2.** Let $E = \{a, b, c, ..., x, y, z\}$ be the set of the 26 English alphabets. Find the number of 5-letter words that can be formed from $E$ such that the first and last letters are distinct vowels and the remaining three are distinct consonants.

5) Problem 4 on page 50.

6) Problem 2(i) and 2(ii) on page 50.

7) Problem 14(i), 14(ii), 14(iii), and 14(iv) on page 51.

Q3) **Example 1.2.3.** There are 7 boys and 3 girls in a gathering. In how many ways can they be arranged in a row so that

(i) the 3 girls form a single block (i.e. there is no boy between any two of the girls)?

(ii) the two end-positions are occupied by boys and no girls are adjacent?

Q4) **Example 1.2.4.** Between 20000 and 70000, find the number of even integers in which no digit is repeated.

Q5) 4. How many 5-letter words can be formed using $A$, $B$, $C$, $D$, $E$, $F$, $G$, $H$, $I$, $J$,

(i) if the letters in each word must be distinct?

(ii) if, in addition, $A, B, C, D, E, F$ can only occur as the first, third or fifth letters while the rest as the second or fourth letters?

Q6) 2. There are 12 students in a party. Five of them are girls. In how many ways can these 12 students be arranged in a row if

(i) there are no restrictions?

(ii) the 5 girls must be together (forming a block)?

Q7) 14. Let $n, r \in \mathbf{N}$ with $r \leq n$. Prove each of the following identities:

(i) $P^n_r = nP^{n-1}_{r-1}$,

(ii) $P^n_r = (n - r + 1)P^n_{r-1}$,

(iii) $P^n_r = \frac{n}{n-r}P^{n-1}_r$, where $r < n$,

(iv) $P^{n+1}_r = P^n_r + rP^n_{r-1}$,

(v) $P^{n+1}_r = r! + r(P^n_{r-1} + P^{n-1}_{r-1} + \cdots + P^r_{r-1})$.

## *Chapter 1. Permutations and Combinations*
## 1.3. Circular Permutations
## 1.4. Combinations

California State University Sacramento - Math 101

**Homework Assignment 4**

1) Example 1.3.2

2) Example 1.3.3

3) Problem 6 on page 51

4) Example 1.4.1

5) Example 1.4.2

Q1)

**Example 1.3.2.** In how many ways can 5 boys and 3 girls be seated around a table if

(i) there is no restriction?

(ii) boy $B_1$ and girl $G_1$ are not adjacent?

(iii) no girls are adjacent?

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Q2)

**Example 1.3.3.** Find the number of ways to seat $n$ married couples around a table in each of the following cases:

(i) Men and women alternate;

(ii) Every woman is next to her husband.

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Q3)

6. Find the number of *odd* integers between 3000 and 8000 in which no digit is repeated.

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Q4)

**Example 1.4.1.** Prove that

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}, \tag{1.4.3}$$

where $n, r \in \mathbf{N}$ with $r \leq n$.

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Q5)

**Example 1.4.2.** By Example 1.1.4, there are $2^7$ binary sequences of length 7. How many such sequences are there which contain 3 0's and 4 1's?

# Math 101 - Quiz 01 (2023-02-03)

1. $A = \{2n \ : \ n \, is \, an \, integer\}$ and B = {1, 2, 3, 4, 5}. Find $A \cap B$.

2. For $1 \leq i \leq 10$, let $A_i = \{1, \ 2, \ \ldots, \ i\}$. Determine $\left|\cup_{i=10}^{10} A_i\right|$.

3. Suppose $A = \{0, \ 3, \ 6, \ 9, \ 12, \ 15, \ 18\}$ and $B = \{0, \ 2, \ 4, \ 6, \ 8\}$. Find $A \cup B$ and $A \cap B$.

4. Suppose $A$ and $B$ are disjoint sets with $|A| = 5$ and $|B| = 3$. Determine $|A \cup B|$.

5. Suppose that $A$ and $B$ are sets with $|A| = 5$ and $|B| = 3$. Can you say anything about $|A \cup B|$?

# Quiz #2 - Page 1 of 2

California State University Sacramento - Math 101

Name: ______________________

**1)** How many pairs of distinct integers $\{a, b\}$ with $a, b \in \{1, 2, \ldots, 10\}$ satisfy $|a - b| = 3$?

**2)** How many pairs of distinct integers $\{a, b\}$ with $a, b \in \{1, 2, \ldots, 10\}$ satisfy $|a - b| \leq 3$?

**3)** Find the number of positive divisor of $1800 = 2^3 \cdot 3^2 \cdot 5^2$ which are multiples of 3.

**4)** Find the number of positive divisors of $1800 = 2^3 \cdot 3^2 \cdot 5^2$ that are multiples of 6.

**5)** Let $A_1 = \{1, 2, 3\}$, $A_2 = \{2, 3\}$, and $A_3 = \{1, 2, 3, 4\}$. Find the number of 3-tuples $(a_1, a_2, a_3)$ where $a_1 \in A_1$, $a_2 \in A_2$, and $a_3 \in A_3$.

# Quiz #3

California State University Sacramento - Math 101

Name: ______________________

**1)** Determine the number of 3-permutations of a set with 7 elements.

**2)** List all 2-permutations of the set $A = \{a, b, c\}$.

**3)** Find the number of sequences $(a_1, a_2, a_3, a_4)$ that consist of distinct elements from the set $\{1, 2, 3, 4, 5\}$ where $a_1$ is even.

**4)** Find the number of sequences $(a_1, a_2, a_3, a_4)$ that consist of elements from the set $\{1, 2, 3, 4, 5\}$ where $a_1$ is even.

**5)** Find the number of odd integers between 10,000 and 20,000 where no digit is repeated.

**6)** Find the number of 3-circular permutations of a set with 7 elements.

**7)** List all 3-circular permutations of the set $A = \{x, y, z, t\}$.

**8)** Write down formulas for $P_r^n$, $Q_r^n$, and write down an equation that relates $Q_r^n$ to $P_r^n$.

**9)** Write down a formula for $C_r^n$ in terms of factorials.

**10)** Find the number of ordered pairs $(A, B)$ where $A$ is a 3-element subset of $\{1, 2, 3, 4, 5\}$, and $B$ is a 2-element subset of $\{6, 7, 8, 9, 10\}$.

**11)** Find the number of subsets of $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ that consist of exactly three even integers and three odd integers.

**12)** Find the number of $2 \times 3$ rectangles in a $4 \times 5$ grid.

# Quiz #2 - Page 1 of 2 - Solutions

California State University Sacramento - Math 101

Name: ______________________

1) How many pairs of distinct integers $\{a, b\}$ with $a, b \in \{1, 2, \ldots, 10\}$ satisfy $|a - b| = 3$?

Let us list all such pairs:

$\{1,4\}, \{2,5\}, \{3,6\}, \{4,7\}, \{5,8\}, \{6,9\}, \{7,10\}.$

There are 7 such pairs.

2) How many pairs of distinct integers $\{a, b\}$ with $a, b \in \{1, 2, \ldots, 10\}$ satisfy $|a - b| \leq 3$?

Let us assume that $a$ is smaller than $b$. We can do this because $\{a,b\}$ is a set and so we do not count $\{a,b\}$ as being different from $\{b,a\}$.

| | $a=1$ | $a=2$ | | $a=7$ | $a=8$ | $a=9$ |
|---|---|---|---|---|---|---|
| possible $a$'s $\rightarrow$ | | | | | | |
| possible $b$'s $\rightarrow$ | $b=2$ | $b=3$ | .... | $b=8$ | $b=9$ | $b=10$ |
| | $b=3$ | $b=4$ | | $b=9$ | $b=10$ | |
| | $b=4$ | $b=5$ | | $b=10$ | | |

There are $7 \cdot 3 + 2 + 1 = 24$ such pairs.

# Quiz #2 Page 2 of 2 - Solutions

**3)** Find the number of positive divisor of $1800 = 2^3 \cdot 3^2 \cdot 5^2$ which are multiples of 3.

We want to count numbers of the form

$$2^a \cdot 3^b 5^c \quad \text{where} \quad 0 \le a \le 3, \ 1 \le b \le 2, \ 0 \le c \le 2.$$

There are $4 \cdot 2 \cdot 3 = 24$ such divisors.

**4)** Find the number of positive divisors of $1800 = 2^3 \cdot 3^2 \cdot 5^2$ that are multiples of 6.

We want to count numbers of the form

$$2^a 3^b 5^c \quad \text{where} \quad 1 \le a \le 3, \ 1 \le b \le 2, \ 0 \le c \le 2.$$

There are $3 \cdot 2 \cdot 3 = 18$ such divisors.

**5)** Let $A_1 = \{1, 2, 3\}$, $A_2 = \{2, 3\}$, and $A_3 = \{1, 2, 3, 4\}$. Find the number of 3-tuples $(a_1, a_2, a_3)$ where $a_1 \in A_1$, $a_2 \in A_2$, and $a_3 \in A_3$.

There are $3 \cdot 2 \cdot 4 = 24$ such 3-tuples.

\# of choices for $a_1$

\# of choices for $a_2$

\# of choices for $a_3$

Quiz 3

1) $P^7_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = 7 \cdot 6 \cdot 5 = 210$

2) a, b    b, a    c, a
a, c    b, c    c, b

3) $(a_1, a_2, a_3, a_4)$

$a_1$: 2 or 4

$$\underset{\text{two choices for } a_1}{2} \times \underset{a_2 \text{ cannot be } a_1 \text{ so 4 choices}}{4} \times \underset{\text{3 choices for } a_3}{3} \times \underset{\text{2 choices for } a_4}{2} = 48$$

4) Taking the same approach as in #3, we first choose $a_1$, then $a_2, a_3$, and $a_4$,

$$2 \times \underbrace{5 \times 5 \times 5}_{a_2, a_3, a_4 \text{ have no restrictions}} = 250$$

5) $\underline{1}$ __ __ __ __ all must be distinct

Must be 1 to be between 10,000 and 20,000 and even

Last digit: Must be 3, 5, 7, or 9

$$\underset{\text{choices for last digit}}{4} \times \underbrace{8 \times 7 \times 6}_{\text{choices for middle 3}} = 1344$$

# Quiz 03 Solutions Page 02/03



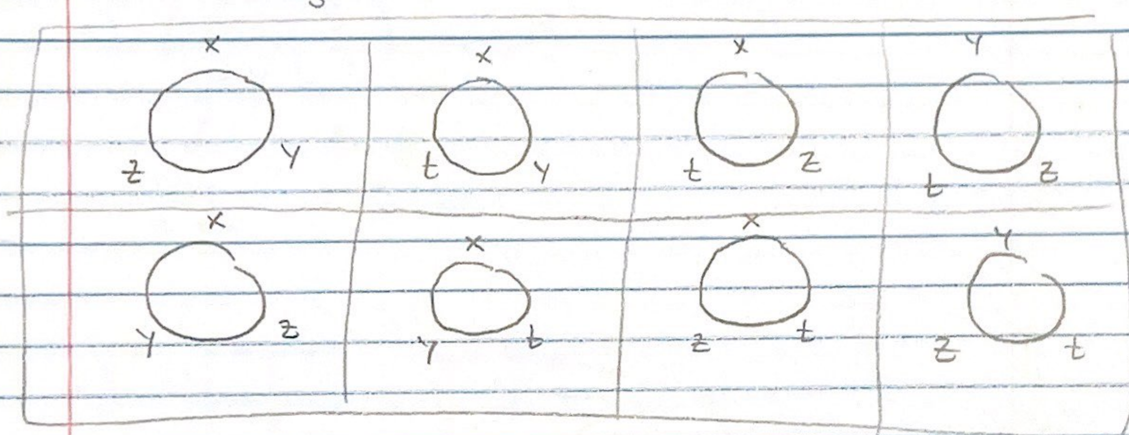
6) $Q^7_3 = \frac{7 \cdot 6 \cdot 5}{3} = 70$

or

$$Q^7_3 = \frac{P^7_3}{3} = \frac{\frac{7!}{(7-3)!}}{3} = \frac{7!/4!}{3} = \frac{7 \cdot 6 \cdot 5}{3} = 70$$

7) There will be

$$Q^4_3 = \frac{4 \cdot 3 \cdot 2}{3} = 8 \text{ possible placements}$$




8) $P^n_r = \frac{n!}{(n-r)!}$ $\quad Q^n_r = \frac{n!}{r \cdot (n-r)!}$

$$Q^n_r = \frac{1}{r} P^n_r$$

# Quiz 03 Solutions Page 03/03

9) $C^n_r = \dfrac{n!}{r!(n-r)!}$

10) $\binom{5}{3}\binom{5}{2} = 100$

choices for A

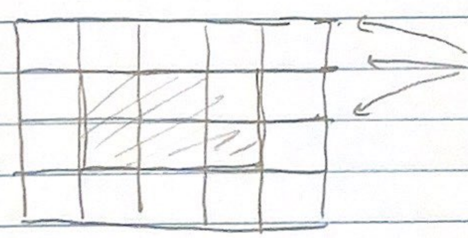
choices for B

11) $\binom{5}{3}\binom{5}{3} = 100$

choices for even

choices for odds

12)



3 choices for top of rectangle

3 choices for left side of rectangle

$3 \cdot 3 = 9$