1) The statement of Theorem 2.2.1 on page 70. You do not need to know how to prove the Binomial Theorem but you do need to know how to use it.

For any integer $n \geq 0$,

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n}x^0 y^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Why is it that the Binomial Coefficients appear in this formula?

Look at some terms in the middle

$\binom{n}{r}x^{n-r}y^r$ for some $0 \leq r \leq n$.

Well, $\underset{\text{Factor}}{(x+y)^n} = \underbrace{(x+y)(x+y)(x+y)\ldots(x+y)}_{1 \quad 2 \quad 3 \quad n}$

we obtain the term $x^{n-r}y^r$ if and only if we choose r-factor to take the 'y' from and the remaining r-n-factors we took the 'x'-form

We use this formula, the Binomial Coefficient, to the "layers", the number of coefficents, of size base 2; for example, $(x+y)^0 = 1, \ldots, (x+y)^4 = 16$.

2) Example 2.3.1 on page 71

Example 2.3.1. Show that for all integers $n \geq 0$,

$$\sum_{r=0}^{n} \binom{n}{r} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n. \qquad (2.3.1)$$

For any integer $n \geq 0$,

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n}x^0 y^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$\rightarrow (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Recall

1st Approach: Use the definition/identity

One can prove identites using the Binomial Theorem by choosing specific values of 'x' and 'y'

For instance, if $x = y = 1$,

$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k \rightarrow (1+1)^n = (2)^n \; \square$$

3) Example 2.3.2(i) on page 72. Remark: Part (ii) of this example is a consequence of part (i)

(i) $\sum_{r=0}^{n} (-1)^r \binom{n}{r} = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0$,

(ii) $\binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{2k} + \cdots = \binom{n}{1} + \binom{n}{3} + \cdots + \binom{n}{2k+1} + \cdots = 2^{n-1}$

One can prove identites using the Binomial Theorem by choosing specific values of 'x' and 'y'

For instance, if $x = y = 1$, $(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k$

If $x = 1$ and $y = -1$

$$(1+(-1))^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} (-1)^k$$

$$\rightarrow 0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$$

$$\rightarrow \binom{4}{0} - \binom{4}{1} + \binom{4}{2} - \binom{4}{3} + \binom{4}{4}$$

$= 0$ (zero)

4) Example 2.3.3 on page 72

Example 2.3.3. Show that for all integers $n \in \mathbb{N}$,

$$\sum_{r=1}^{n} r\binom{n}{r} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n \cdot 2^{n-1}.$$

Start with the Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Let $x = 1$ in the Binomial Theorem

$$\rightarrow (1+y)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} y^k$$

* Power Rule and Chain Rule * $\frac{d}{dy}$ both sides to get

Power Rule: $\frac{d}{dy} y^k = k y^{k-1}$

$$\rightarrow n(1+y)^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k y^{k-1}$$

Note: Change index

*Set $y = 1$ to get...

$$\rightarrow n \cdot 2^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k$$

5) Example 2.3.4 on page 74, only the second proof

Example 2.3.4. **(Vandermonde's Identity)** Show that for all $m, n, r \in \mathbb{N}$,

$$\sum_{i=0}^{r} \binom{m}{i}\binom{n}{r-i} = \binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \cdots + \binom{m}{r}\binom{n}{0}$$

$$= \binom{m+n}{r}. \qquad (2.3.5)$$

Key consequence of the Binomial Theorem is that the coefficent of x^l in $(1+x)^m$ is $\binom{m}{l}$

Consider the idenity

$$(1+x)^{m+n} = (1+x)^m (1+x)^n$$

Coefficent of x^r is $\binom{m+n}{r}$

$\binom{m}{0}1$, $\binom{m}{1}x$, $\binom{m}{2}x^2$, $\vdots$, $\binom{m}{r}x^r$

$\binom{n}{r}x^r$, $\binom{n}{r-1}x^{r-1}$, $\binom{n}{r-2}x^{r-2}$, $\vdots$, $\binom{n}{0}1$

Leverages This Rule

Key consequence of the Binomial Theorem is that the coefficent of x^l in $(1+x)^m$ is $\binom{m}{l}$

For the right hand side, the coefficent of x^r is going to be

$$\binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \binom{m}{2}\binom{n}{r-2} + \cdots + \binom{m}{r}\binom{n}{0} = \sum_{i=0}^{r}\binom{m}{i}\binom{n}{r-i}$$

Coefficent of x^0 — Coefficent of x^r

$$\rightarrow \binom{m+n}{r} = \sum_{i=0}^{r}\binom{m}{i}\binom{n}{r-i} \; \square$$

Prove each of the following identities in Problems 24–43, where $m, n \in \mathbf{N}^*$:

24. $\sum_{r=0}^{n} 3^r \binom{n}{r} = 4^n$, 6) Problem 24 on page 105

$\sum_{r=0}^{n} 3^r \binom{n}{r} = 4^n$ $N^* = \{0, 1, 2, 3, \ldots\}$

LHS

$\sum_{r=0}^{n} 3^r \binom{n}{r} \rightarrow$ Binomial Theorem $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

$\rightarrow x = 1$ & $y = 3$

$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \Rightarrow (1+3)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 3^k$

Can think as

LHS

$\sum_{r=0}^{n} 3^r \binom{n}{r} = 3^0 \binom{n}{0} + 3^1 \binom{n}{1} + 3^2 \binom{n}{2} + \cdots + 3^n \binom{n}{n}$

$\rightarrow 1 + 3^1 \binom{n}{1} + 3^2 \binom{n}{2} + \cdots + 3^n \binom{n}{n}$

USE: $1^n 3^0 \binom{n}{0} + 1^{n-1} 3^1 \binom{n}{1} + 1^{n-2} 3^2 \binom{n}{2} + \cdots + 1^{n-n} 3^n \binom{n}{n}$

Using the binomial theorem above can be re-written as:

$(1+3)^n$, so $4^n = $ RHS

Binomial Theorem — Recall

$^*(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

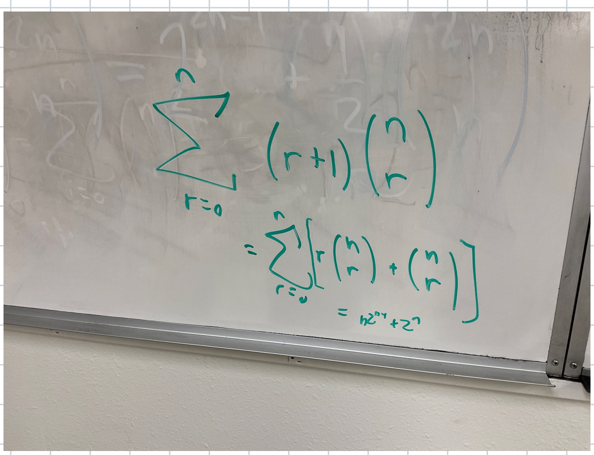
*Where 'n' is a positive integer

We have used this to prove combinatorial identites

For example, if $x = y = 1$ then

$\rightarrow 2^n = \sum_{k=0}^{n} \binom{n}{k}$

25. $\sum_{r=0}^{n} (r+1) \binom{n}{r} = (n+2) 2^{n-1}$



26. $\sum_{r=0}^{n} \frac{1}{r+1} \binom{n}{r} = \frac{1}{n+1} (2^{n+1} - 1)$