Matthew Mendoza

Prove each of the following identities in Problems 24–43, where $m, n \in \mathbf{N}^*$:

24. $\sum_{r=0}^{n} 3^r \binom{n}{r} = 4^n$,

6) Problem 24 on page 105

$\sum_{r=0}^{n} 3^r \binom{n}{r} = 4^n$

$N^* = \{0, 1, 2, 3, \ldots\}$

LHS

$\sum_{r=0}^{n} 3^r \binom{n}{r} \rightarrow$ Binomial Theorem $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

$\rightarrow x = 1 \; \& \; y = 3$

$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \Rightarrow (1+3)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 3^k$

Binomial Theorem — Recall

$^*(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

*Where 'n' is a positive integer

We have used this to prove combinatorial identites

For example, if $x=y=1$ then

$\rightarrow 2^n = \sum_{k=0}^{n} \binom{n}{k}$

Can think as

LHS

$\sum_{r=0}^{n} 3^r \binom{n}{r} = 3^0 \binom{n}{0} + 3^1 \binom{n}{1} + 3^2 \binom{n}{2} + \cdots + 3^n \binom{n}{n}$

$\rightarrow 1 + 3^1 \binom{n}{1} + 3^2 \binom{n}{2} + \cdots + 3^n \binom{n}{n}$

USE: $1^n 3^0 \binom{n}{0} + 1^{n-1} 3^1 \binom{n}{1} + 1^{n-2} 3^2 \binom{n}{2} + \cdots + 1^{n-n} 3^n \binom{n}{n}$

Using the binomial theorem above can be re-written as:

$(1+3)^n$, so $4^n = \text{RHS}$

Power rule (x^n): $\{x^n\}' = nx^{n-1}$

THE CHAIN RULE

Jay Cummings
Calculus 1 Lecture Notes

Theorem.

Theorem 2.64 (*The Chain Rule*). Let f and g be differentiable functions. Then

$$(f(g(x)))' = f'(g(x)) \cdot g'(x).$$

In words: *Take the derivative of the outside, keep everything inside the same, and then multiply by the derivative of the inside.*

25. $\sum_{r=0}^{n} (r+1)\binom{n}{r} = (n+2)2^{n-1}$

$\sum_{r=0}^{n} (r+1)\binom{n}{r}$

$\Rightarrow \sum_{r=0}^{n} \left[r\binom{n}{r} + \binom{n}{r} \right]$

$\Rightarrow n2^{n-1} + 2^n$

26. $\sum_{r=0}^{n} \frac{1}{r+1}\binom{n}{r} = \frac{1}{n+1}(2^{n+1} - 1)$