Old Exam 3 - Solutions

1) (a) For any positive integer n and real numbers x and y,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

(b) $\binom{19}{7}$

2) $\binom{10}{3} + \binom{10}{4} = \binom{11}{4}$

3) $\binom{6}{3,2,1}$

4) If we let $y=1$ and $x=3$ in the Binomial theorem, we obtain

$$4^n = \sum_{k=0}^{n} \binom{n}{k} 3^k.$$

5) $$\sum_{r=0}^{6} \binom{12}{r}\binom{10}{6-r} = \binom{12}{0}\binom{10}{6} + \binom{12}{1}\binom{10}{5} + \binom{12}{2}\binom{10}{4}$$
$$+ \binom{12}{3}\binom{10}{3} + \binom{12}{4}\binom{10}{2} + \binom{12}{5}\binom{10}{1} + \binom{12}{6}\binom{10}{0}$$
$$= \binom{22}{6}$$ by Vandermonde's Identity.

6) $$\binom{4}{4} + \binom{5}{4} + \binom{6}{4} + \binom{7}{4}$$

$$= \binom{5}{5} + \binom{5}{4} + \binom{6}{4} + \binom{7}{4} \quad \leftarrow \binom{4}{4} = 1 = \binom{5}{5}$$

$$= \binom{6}{5} + \binom{6}{4} + \binom{7}{4} \quad \leftarrow \binom{6}{5} = \binom{5}{5} + \binom{5}{4}$$

$$= \binom{7}{5} + \binom{7}{4} \quad \leftarrow \binom{7}{5} = \binom{6}{5} + \binom{6}{4}$$

$$= \binom{8}{5} \quad \leftarrow \binom{8}{5} = \binom{7}{5} + \binom{7}{4}$$

7) <u>SKIP</u> - Has to do with Multinomial Theorem

8) Let k and n be positive integers. If kn+1 objects are placed in n boxes, then at least one box has at least k+1 objects.

9)

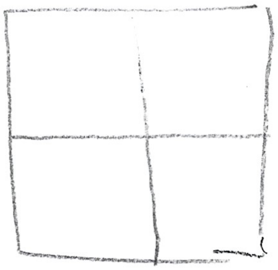
$37 = 3 \cdot 12 + 1$

↑

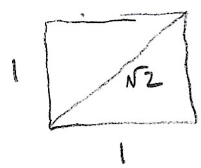
# of months

At least 4 people will have the same birthday month.

10) Divide the 2×2 square into four equal squares.

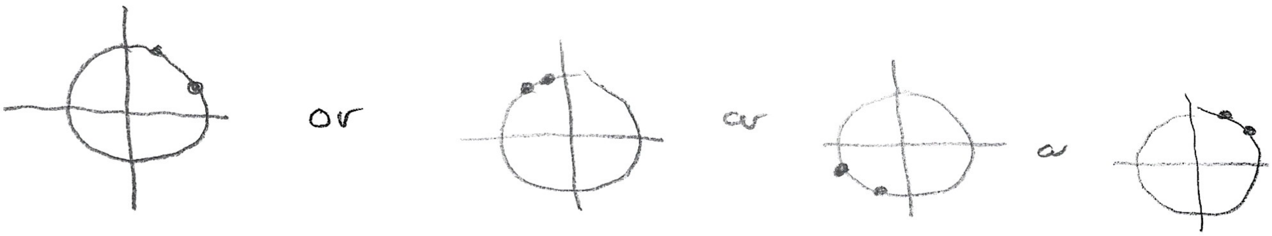


Since we have 5 points, at least one square contains two points.

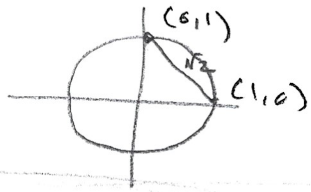


The farthest apart two points in the same square can be is $\sqrt{2}$.

11) By the Pigeonhole Principle, there must be two points in the same quadrant.



The farthest two points on the unit circle but in the same quadrant can be is $\sqrt{2}$.



12) $\displaystyle\sum_{r=0}^{10} \binom{10}{r} = 2^{10}$

using the identity $\frac{n+1}{r+1}\binom{n}{r} = \binom{n+1}{r+1}$

13) $\displaystyle\sum_{r=0}^{n} \frac{n+1}{r+1}\binom{n}{r} = \sum_{r=0}^{n} \binom{n+1}{r+1}$

$$= \binom{n+1}{1} + \binom{n+1}{2} + \dots + \binom{n+1}{n+1}$$

$$= -\binom{n+1}{0} + \binom{n+1}{0} + \binom{n+1}{1} + \dots + \binom{n+1}{n+1}$$

$$= -1 + 2^{n+1} = 2^{n+1} - 1$$