Section 1.5: Injection & Bijections Principles

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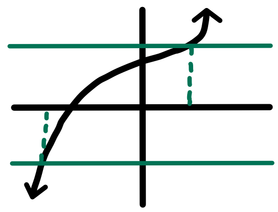
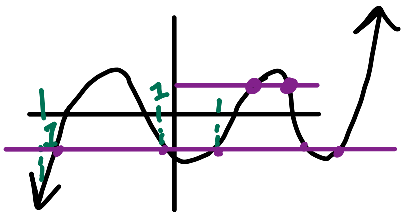
Let 'A' and 'B' be sets 'A' function $f: A \to B$ is

- **Injective** (1-to-1) if
  $f(a_1) = f(a_2)$ implies $a_1 = a_2$



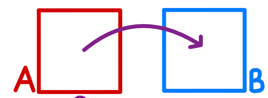
- **Surjective** (onto) if
  for $a \in b$, exists an $a \in A$ with $f(a) = b$



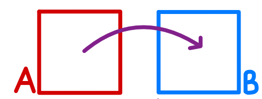
- **Bijective** if f is both
  Injective (1-to-1) and Surjective (onto)

Suppose you have 2 finite sets where it has surjection

$|A| \geq |B|$ $|A| \leq |B|$



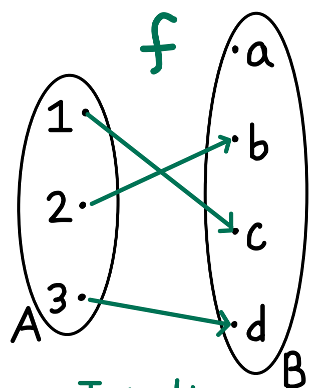
Surjection



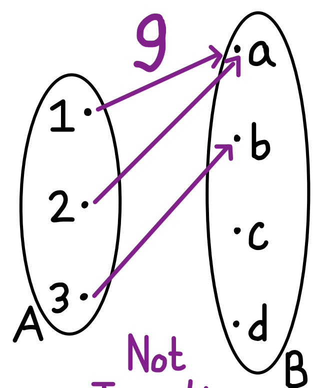
Injection

Example: Injective

Let $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$



Injective

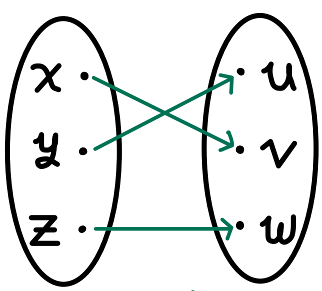


Not Injective

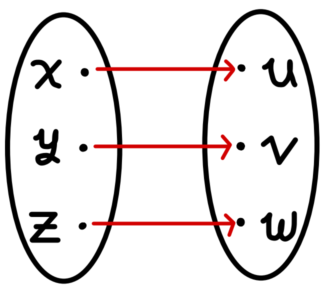
Example: Surjective & injective

Bijective if f is both Injective (1-to-1) and Surjective (onto)

Let $A = \{x, y, z\}$, $B = \{u, v, w\}$

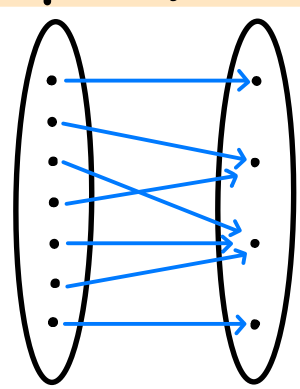


Surjective and injective



Surjective and injective

Example: Surjective not injective

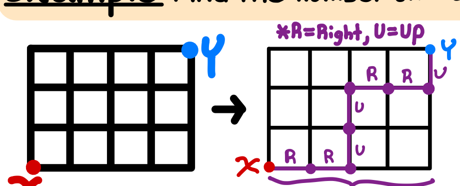


Surjective (onto) if
for $a \in b$, exists an $a \in A$ with $f(a) = b$

Surjective not injective

Example: Find the number shortest routes from X to Y

*R=Right, U=Up



RRUURRU

$\binom{7}{3}$ Selecting for up

$\underbrace{\text{R R U U R R U}}$

Choose three places for when to take an U (up)-step

Selecting for Right $\binom{7}{4}$ OR $\binom{7}{3}$ Selecting for up

Example: Find the number of r-combinations (subset with r-elements) with no consecutive intetgers

Let $x = \{1, 2, 3, \ldots, n\}$,

if $x = \{1, 2, 3, 4, 5, 6\}$ and $r = 3$, the subsets are:

$\{1,3,5\}$ $\{2,4,6\}$ $\{1,3,6\}$ $\{1,4,6\}$

*** Use Binomial Cofficient ***

Recall from Friday, that we want to find the number of subsets of

$$\{1,2,3,\dots,n\}$$

with no consecutive integers

For instance, if $x=\{1,2,3,4,5,6,7\}$, then

$\{1,3,5\} \to \{1,2,3\}$
$\{1,3,6\} \to \{1,2,4\}$
$\{1,3,7\} \to \{1,2,5\}$
$\{1,4,6\} \to \{1,3,4\}$
$\{1,4,7\} \to \{1,3,5\}$
$\{1,5,7\} \to \{1,4,5\}$
$\{2,4,6\} \to \{2,3,4\}$
$\{2,4,7\} \to \{2,3,5\}$
$\{2,5,7\} \to \{2,4,5\}$
$\{3,5,7\} \to \{3,4,5\}$

The subsets on the right are exactly all $\binom{5}{3}=10$ subsets $\{1,2,3,4,5\}$ of size 3

Genericly...

Leave same, Subtract 1, Subtract 2

$\{1,3,5\} \to \{1,2,3\}$
$\{1,3,6\} \to \{1,2,4\}$
$\{1,3,7\} \to \{1,2,5\}$

Subtract by the index in set

General Case

Let $x=\{1,2,3,4,\dots,n\}$ and Let $1\le r\le n$

Given a subset $\{S_0,S_1,\dots,S_r\}$ of x with no consecutive elements define,

$$f(\{S_0,S_1,\dots,S_r\})=\{S_0,S_1-1,S_2-2,\dots,S_r-(r-1)\}$$

The biggest it can be is $n-r+1$

This output is a subset of size 'r' since $\{S_0,S_1,\dots,S_r\}$ has no consecutive elements

$n=7$, $r=3$: $\binom{n-r+1}{r}=\binom{5}{3}$

X: 1,2,3,4,5,6,7 — $\{3,5,7\}$
Y: 1,2,3,4,5 — $\{3,4,5\}$

Note that since "S_r" can be n, but not bigger, $S_r-(r-1)$ canbe $n-r+1$, but not bigger

Therefore,

$$\underline{|X|}=\underline{|Y|}=\binom{n-r+1}{r}$$

Tricky: all subsets of $\{1,2,3,\dots,n\}$ with no consecutive elements

Easy: all subsets of $\{1,2,3,\dots,n-r+1\}$ of size r

$$|a-b|\ge K$$
$$\binom{n-(K-1)(r-1)}{r}$$

What if we require that $|a-b|\ge 3$ for all a,b in our r-subset of $x=\{1,2,3,\dots,n\}$?

Try $\{1,2,3,4,5,6,7,8\}$, $r=3$

same, -2, -4

$\{1,4,7\} \to \{1,2,3\}$
$\{1,4,8\} \to \{1,2,4\}$
$\{1,5,8\} \to \{1,3,4\}$
$\{2,5,8\} \to \{2,3,4\}$

Guess

$$\binom{n-2(r-1)}{r}$$

Via Bijection 'x' counts the 'y'-side