

Homework Assignment 2

- 1) Problem 1 on page 50
- 2) Problem 10 on page 51
- 3) Problem 11 on page 51
- 4) Problem 12 on page 51

Q1) 1. Find the number of ways to choose a pair $\{a, b\}$ of distinct numbers from the set $\{1, 2, \dots, 50\}$ such that
(i) $|a - b| = 5$; (ii) $|a - b| \leq 5$.

Q2) 10. Find the number of common positive divisors of 10^{40} and 20^{30} .

Q3) 11. In each of the following, find the number of positive divisors of n (inclusive of n) which are multiples of 3:
(i) $n = 210$; (ii) $n = 630$; (iii) $n = 151200$.

Q4) 12. Show that for any $n \in \mathbb{N}$, the number of positive divisors of n^2 is always odd.

Q 5) Find the number of ordered pairs (x, y) of integers such that $x^2 + y^2 \leq 4$.

Remark: This problem is similar to Example 1.1.2.

Q 6) Find the number of sequences $a_1 a_2 a_3$ of length 3 where $a_i \in \{0, 1, 2, 3, 4\}$.

Remark: This is a special case of Example 1.1.4.

Q 7) Let $X = \{1, 2, \dots, 10\}$ and let

$$S = \{(a, b, c) : a, b, c \in X, a < b \text{ and } a < c\}.$$

Find $|S|$.

Remark: This problem is similar to Example 1.1.6.

Noticed it's Unordered $\{a, b\}$ NOT ordered (a, b) So...
1.i = 45
and
1.ii = 235

(i) $|a-b|=5$

- min: 1, 6
- max: 45, 50

from min to max we have 45 possibilities of (a, b) pairs

If given a restriction of $a > b$; then, the solutions we can have follows this tuple pattern: $(6, 1), (7, 2), \dots, (50, 45)$

Because we are taking the absolute value of the difference of both 'a' and 'b' the order of either 'a' or 'b' does not matter, that is 45 more possible pairs of (a, b)

So, 45 possibilities $(a > b) + 45$ possibilities $(a < b) = 90$ possible ways of (a, b) pairings

(ii) $|a-b| \leq 5$

Let the smaller of both 'a' and 'b' be 'n' where $1 \leq n \leq 45$, then there are 5 pairs of distinct numbers from $\{1, 2, 3, \dots, 50\} : \{n, n+1\}, \{n, n+2\}, \{n, n+3\}, \{n, n+4\}, \{n, n+5\}$. Gives us: $(45)(5) = 225$

n plus offset 1 to 5 is to account for distinct numbers for either 'a' or 'b'

→ 45 (where both 'a' and 'b' are the smallest number in the set)

→ 5 (five pairs of distinct numbers from set $\{1, 2, 3, \dots, 50\}$)

$$\begin{aligned} 45 - (45+1) &= 1 \\ 45 - (45+2) &= 2 \\ &\vdots \\ 45 - (45+5) &= 5 \end{aligned}$$

When $46 \leq n < 50$

• $n = 46$ there are 4 pairs: $(46, 47), (46, 48), (46, 49), (46, 50)$

• $n = 47$ there are 3 pairs: $(47, 48), (47, 49), (47, 50)$

• $n = 48$ there are 2 pairs: $(48, 49), (48, 50)$

• $n = 49$ there is 1 pair: $(49, 50)$

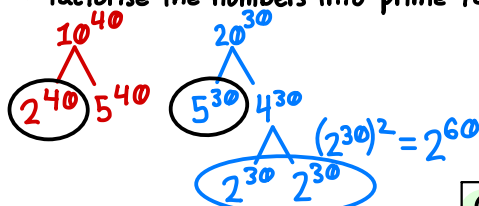
$(a < b) : [(45)(5)] + 4 + 3 + 2 + 1 = 235$ possibilities.

Because we have the absolute value account $(a > b)$:

meaning we have 235 more possibilities. So in total $235 + 235 = 470$

10. Find the number of common positive divisors of 10^{40} and 20^{30} .

* Factorise the numbers into prime factors *



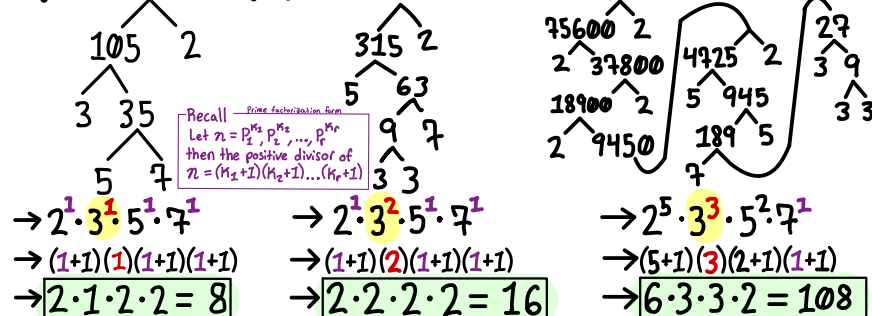
Recall Prime factorization form
Let $n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$
then the positive divisor of $n = (k_1+1)(k_2+1)\dots(k_r+1)$

So $(40+1)(30+1) = 1271$

11. In each of the following, find the number of positive divisors of n (inclusive of n) which are multiples of 3:

- (i) $n = 210$; (ii) $n = 630$; (iii) $n = 151200$

(i) $n = 210$ (ii) $n = 630$ (iii) $n = 151200$



12) Show that for any $n \in \mathbb{N}$, the number of positive divisors of n^2 is always odd. If prime factorization of $n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$ then $n^2 = (p_1^{k_1})^2 (p_2^{k_2})^2 \dots (p_r^{k_r})^2$ the resulting positive divisor of $n^2 = (2k_1+1)(2k_2+1)\dots(2k_r+1)$ follows the general form and the definition of odd $(2n+1)$.

5) Find the number of ordered pairs (x, y) of integers such that $x^2 + y^2 \leq 4$.

Case A	Case B	Case C	Case D	Case E
$x^2 + y^2 = 0$	$x^2 + y^2 = 1$	$x^2 + y^2 = 2$	$x^2 + y^2 = 3$	$x^2 + y^2 = 4$
$(0, 0)$	$(0, 1), (1, 0), (0, -1), (-1, 0)$	$(1, 1), (1, -1), (-1, 1), (-1, -1)$	NONE	$(0, 2), (2, 0), (0, -2), (-2, 0)$

So in total $1 + 4 + 4 + 0 + 4 = 13$

6) Find the number of sequences $a_1 a_2 a_3$ of length 3 where $a_i \in \{0, 1, 2, 3, 4\}$. 3 groupings $(a_1 a_2 a_3)$ of 5 elements $(0, 1, 2, 3, 4)$:

So $5 \cdot 5 \cdot 5 = 125$

* Think $p_r^n = \frac{n!}{(n-r)!} = \# \text{ of Permutations}$

7) Let $X = \{1, 2, 3, \dots, 10\}$ and let $S\{(a, b, c) : a, b, c \in X, a < b \text{ and } a < c\}$. Find $|S|$.

Key hint is $c \in X$, so we can try

$\sum_{c=1}^{10} (c-1)(c-2) \rightarrow [(1-1)(1-2) + (2-1)(2-2) + \dots + (10-1)(10-2)] = 385$