California State University Sacramento - Math 101

Homework Assignment 9

1) Example 2.3.4 on page 74

2) The statement of Theorem 2.8.1

3) Example 2.8.1 on page 99

4) Problem 28 on page 106.

5) Problem 31 on page 106.

6) State the Pigeonhole Principle

7) Example 3.2.4

8) Problem 1 on page 137

9) Problem 3 on page 137

1) **Example 2.3.4. (Vandermonde's Identity)** Show that for all $m, n, r \in \mathbf{N}$,

$$\sum_{i=0}^{r} \binom{m}{i}\binom{n}{r-i} = \binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \cdots + \binom{m}{r}\binom{n}{0} = \binom{m+n}{r}. \tag{2.3.5}$$

2) **Theorem 2.8.1 (The Multinomial Theorem).** *For $n, m \in \mathbf{N}$,*

$$(x_1 + x_2 + \cdots + x_m)^n = \sum \binom{n}{n_1, n_2, \ldots, n_m} x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m}$$

3) **Example 2.8.1.** For $n = 4$ and $m = 3$, we have by Theorem 2.8.1,

$(x_1 + x_2 + x_3)^4 =$

4) **106** ***Exercise 2***

28. $\sum_{r=m}^{n} \binom{n}{r}\binom{r}{m} = 2^{n-m}\binom{n}{m}$ for $m \le n$,

5) 31. $\sum_{r=0}^{n} (-1)^r r \binom{n}{r} = 0,$

6)

3.2. The Pigeonhole Principle

If three pigeons are to be put into two compartments, then you will certainly agree that one of the compartments will accommodate at least two pigeons. A much more general statement of this simple observation, known as the *Pigeonhole Principle*, is given below.

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7) **Example 3.2.4.** Show that for any set of 10 points chosen within a square whose sides are of length 3 units, there are two points in the set whose distance apart is at most $\sqrt{2}$.

What are the objects? What are the boxes? These are the two questions we have to ask beforehand. It is fairly clear that we should treat the 10 given points in the set as our "objects". The conclusion we wish to arrive at is the existence of "2 points" from the set which are "close" to each other (i.e. their distance apart is at most $\sqrt{2}$ units). This indicates that "$k + 1 = 2$" (i.e., $k = 1$), and suggests also that we should partition the 3×3 square into n smaller regions, $n < 10$, so that the distance between any 2 points in a region is at most $\sqrt{2}$.

Exercise 3

8) 1. Show that among any 5 points in an equilateral triangle of unit side length, there are 2 whose distance is at most $\frac{1}{2}$ units apart.

9) 3. Given any set S of 9 points within a unit square, show that there always exist 3 distinct points in S such that the area of the triangle formed by these 3 points is less than or equal to $\frac{1}{8}$. (Beijing Math. Competition, 1963)

Q1 Vandermonde's Identity

For all integers $m, n, r \geq 1$

$$\binom{m+n}{r}=\sum_{i=0}^{r}\binom{m}{i}\binom{n}{r-i}$$

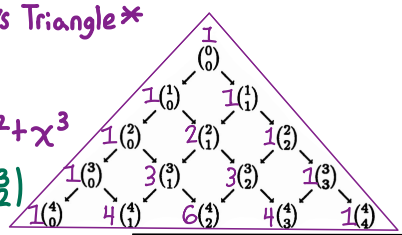
By: Binomial Theorem – Algebraic

Key consequence of the Binomial Theorem is that the coefficent of x^{i} in $(1+x)^m$ is $\binom{m}{i}$ *Pascal's Triangle*

$(1+x)^2 = 1+2x+x^2$

$(1+x)^3 = 1+3x+3x^2+x^3$

$\binom{3}{1}$ $\binom{3}{2}$



Consider the idenity

$$\underbrace{(1+x)^{m+n}}=(1+x)^m(1+x)^n$$

Binomial Theorem

$$\boxed{(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^{k}}$$

Coefficent of x^r is $\binom{m+n}{r}$

$\binom{m}{0}1$ $\binom{n}{r}x^r$

$\binom{m}{1}x$ $\binom{n}{r-1}x^{r-1}$

$\binom{m}{2}x^2$ $\binom{n}{r-2}x^{r-2}$

$\vdots$

$\binom{m}{r}x^r$ $\binom{n}{0}1$

Leverages This Rule

$\boxed{\text{Key consequence of the Binomial Theorem is that the coefficent of } x^{i} \text{ in } (1+x)^m \text{ is } \binom{m}{i}}$

For the right hand side, the coefficent of x^r is going to be

$$\binom{m}{0}\binom{n}{r}+\binom{m}{1}\binom{n}{r-1}+\binom{m}{2}\binom{n}{r-2}+\cdots+\binom{m}{r}\binom{n}{0}=\sum_{i=0}^{r}\binom{m}{i}\binom{n}{r-i}$$

Coefficent of x^0 $\left\{\text{Coefficent of } x^r\right\} \to \binom{m+n}{r}=\sum_{i=0}^{r}\binom{m}{i}\binom{n}{r-i}$

Q2 Multinomial Theorem: For $n, m \in N$

$$(x_1+x_2+\cdots+x_m)^n=\sum\binom{n}{n_1,n_2,\ldots,n_m}x_1^{n_1}x_2^{n_2}\cdots x_m^{n_m}$$

$$(x_1+x_2+\cdots+x_m)^n=\sum\underbrace{\binom{n}{n_1,n_2,\ldots,n_m}}x_1^{n_1}x_2^{n_2}\cdots x_m^{n_m}$$

$(n_1,n_2,\cdots,n_m)\in S$ $\left\{\frac{n!}{n_1!\,n_2!\ldots n_m!}\right\}$

Where S is all m-tuples of non-negative integers with $n_1+n_2+\ldots+n_m=n$

Q3 For $n=4$ and $m=3$, we have by Theorem 2.8.1

$$(x_1+x_2+x_3)^4$$

Multinomial Theorem

$$\boxed{(x_1+x_2+\cdots+x_m)^n=\sum\binom{n}{n_1,n_2,\ldots,n_m}x_1^{n_1}x_2^{n_2}\cdots x_m^{n_m}}$$

$n=4$ and $m=3$ ← number of positions; adds up to 4 $\left\{\frac{n!}{n_1!\,n_2!\ldots n_m!}\right\}$

$$(x_1+x_2+x_3)^4=\binom{4}{4,0,0}x_1^4+\binom{4}{3,1,0}x_1^3x_2^1+\binom{4}{3,1,0}x_1^3x_2^1$$

$\frac{4!}{4!0!0!}=1$, $\frac{4!}{3!1!0!}=4$, $\frac{4!}{3!1!0!}=4$

$$+\binom{4}{2,2,0}x_1^2x_2^2x_3^0+\binom{4}{2,1,1}x_1^2x_2^1x_3^1+\binom{4}{2,0,2}x_1^2x_2^0x_3^2$$

$\frac{4!}{2!2!0!}=6$, $\frac{4!}{2!1!1!}=12$, $\frac{4!}{2!0!2!}=6$

$$+\binom{4}{1,3,0}x_1^1x_2^3x_3^0+\binom{4}{1,2,1}x_1^1x_2^2x_3^1+\binom{4}{1,1,2}x_1^1x_2^1x_3^2$$

$\frac{4!}{1!3!0!}=4$, $\frac{4!}{1!2!1!}=12$, $\frac{4!}{1!1!2!}=12$

$$+\binom{4}{1,0,3}x_1^1x_2^0x_3^3+\binom{4}{0,4,0}x_1^0x_2^4x_3^0+\binom{4}{0,3,1}x_1^0x_2^3x_3^1$$

$\frac{4!}{1!0!3!}=4$, $\frac{4!}{0!4!0!}=1$, $\frac{4!}{0!3!1!}=4$

$$+\binom{4}{0,2,2}x_1^0x_2^2x_3^2+\binom{4}{0,1,3}x_1^0x_2^1x_3^3+\binom{4}{0,0,4}x_1^0x_2^0x_3^4$$

$\frac{4!}{0!2!2!}=6$, $\frac{4!}{0!1!3!}=4$, $\frac{4!}{0!0!4!}=1$

$$\begin{aligned}
&= x_1^4+4x_1^3x_2^1+4x_1^3x_2^1\\
&+6x_1^2x_2^2x_3^0+12x_1^2x_2^1x_3^1+6x_1^2x_2^0x_3^2\\
&+4x_1^1x_2^3x_3^0+12x_1^1x_2^2x_3^1+12x_1^1x_2^1x_3^2\\
&+4x_1^1x_2^0x_3^3+1x_1^0x_2^4x_3^0+4x_1^0x_2^3x_3^1\\
&+6x_1^0x_2^2x_3^2+4x_1^0x_2^1x_3^3+1x_1^0x_2^0x_3^4
\end{aligned}$$

Q4

28. $\sum_{r=m}^{n}\binom{n}{r}\binom{r}{m}=2^{n-m}\binom{n}{m}$ for $m \leq n$,

$$\sum_{r=m}^{n}\binom{n}{r}\binom{r}{m}=2^{n-m}\binom{n}{m}$$

LHS: $\sum_{r=m}^{n}\binom{n}{r}\binom{r}{m} \Rightarrow \sum_{r=m}^{n}\binom{n}{m}\binom{n-m}{r-m} \Rightarrow \binom{n}{m}\sum_{r=m}^{n}\binom{n-m}{r-m}$

$\Rightarrow \binom{n}{m}\sum_{r=m}^{n}\binom{n-m}{r-m}$

Identity $\boxed{\sum_{r=0}^{n}\binom{n}{r}=2^n}$

$\Rightarrow \binom{n}{m}2^{n-m}$ ∴ LHS = RHS

Q5

31. $\sum_{r=0}^{n}(-1)^r r\binom{n}{r}=0,$

$$\sum_{r=0}^{n}(-1)^r r\binom{n}{r}=0$$

LHS: $\sum_{r=0}^{n}(-1)^r r\binom{n}{r} \Rightarrow \sum_{r=0}^{n}(-1)^r r\binom{n-1}{r-1}$

$\to -n\sum_{r-1}^{n}(-1)^r\binom{n-1}{r-1}$

* Identity: $\sum_{r=0}^{n}(-1)^r\binom{n}{r}=\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\cdots+(-1)^n\binom{n}{n}=0$

$-n\sum_{r-1}^{n}(-1)^r\binom{n-1}{r-1}=0$ So LHS = RHS

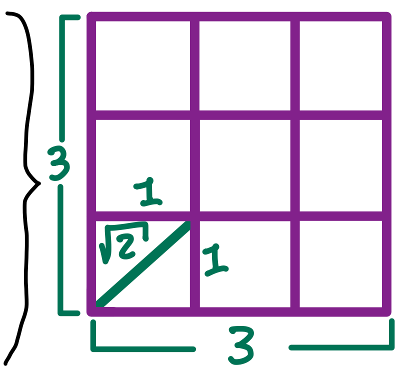
6) State the Pigeonhole Principle

Let K&n be positive the Pigeonhole Principle states that if atleast Kn+1 objects are placed in n-boxes, then at least one box has at least K+1 objects

Pigion Hole Principle Example
There is a chance that in this class there will be 2 students with the same month.
Given that we have more than 12 students in class.

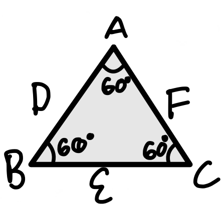
Q7 **Example 3.2.4.** Show that for any set of 10 points chosen within a square whose sides are of length 3 units, there are two points in the set whose distance apart is at most $\sqrt{2}$.

What are the objects? What are the boxes? These are the two questions we have to ask beforehand. It is fairly clear that we should treat the 10 given points in the set as our "objects". The conclusion we wish to arrive at is the existence of "2 points" from the set which are "close" to each other (i.e. their distance apart is at most $\sqrt{2}$ units). This indicates that "$k + 1 = 2$" (i.e., $k = 1$), and suggests also that we should partition the 3×3 square into n smaller regions, $n < 10$, so that the distance between any 2 points in a region is at most $\sqrt{2}$.

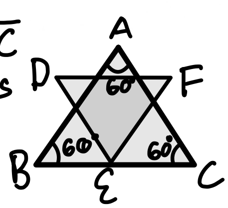


Exercise 3

1. Show that among any 5 points in an equilateral triangle of unit side length, there are 2 whose distance is at most $\frac{1}{2}$ units apart.



$S = \overline{ABC}$
Midpoints DEF



If BA = 1
then BD = 1/2

3. Given any set S of 9 points within a unit square, show that there always exist 3 distinct points in S such that the area of the triangle formed by these 3 points is less than or equal to $\frac{1}{8}$. (Beijing Math. Competition, 1963)

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

<u>Level 1</u>: plugging in #'s for x, y to get identity

$x = 1$
$y = 1$

$x = -1$
$y = 1$

$$2^n = \sum_{r=0}^{n} \binom{n}{r} \qquad 0 = \sum_{r=0}^{n} \binom{n}{r} (-1)^r$$

<u>Level 2</u>: Additional like a derivative or using a previous, or simplifying first

$$\sum_{r=m}^{n} \binom{n}{r}\binom{r}{m} \overset{?}{=} 2^{n-m}\binom{n}{m}$$

$$\sum_{r=m}^{n} \frac{n!}{\cancel{r!}\,(n-r)!} \cdot \frac{\cancel{r!}}{m!\,(r-m)!} \overset{?}{=} 2^{n-m} \frac{n!}{m!\,(n-m)!}$$

$$\frac{n!}{m!} \sum_{r=m}^{n} \frac{1}{(n-r)!\,(r-m)!} \overset{?}{=} \frac{n!}{m!} \cdot 2^{n-m} \cdot \frac{1}{(n-m)!}$$

$$\frac{n!}{m!} \sum_{r=m}^{n} \frac{(n-m)!}{(n-r)!\,(r-m)!} \overset{?}{=} \frac{n!}{m!}\, 2^{n-m}$$

$$\frac{n!}{m!} \sum_{r=m}^{n} \binom{n-m}{n-r} \stackrel{?}{=} 2^{n-m} \frac{\cancel{n!}}{m!}$$

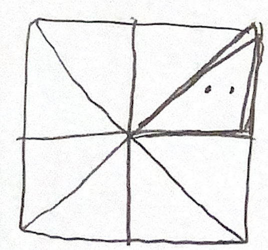
$$\binom{n-m}{n-r} = \frac{(n-m)!}{(n-r)!\,(n-m-(n-r))!}$$

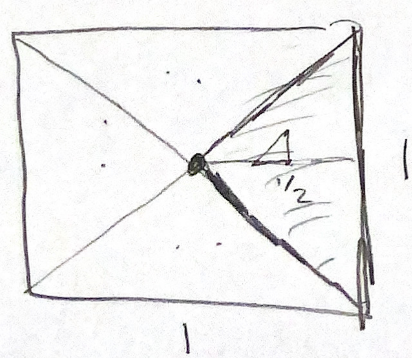
$$= \frac{(n-m)!}{(n-r)!\,(r-m)!}$$

$$\frac{n!}{m!} \left(\underbrace{\binom{n-m}{n-m} + \binom{n-m}{n-m-1} + \dots + \binom{n-m}{0}}_{2^{n-m}} \right) \stackrel{?}{=} 2^{n-m} \frac{n!}{m!}$$

$\frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$

Match!

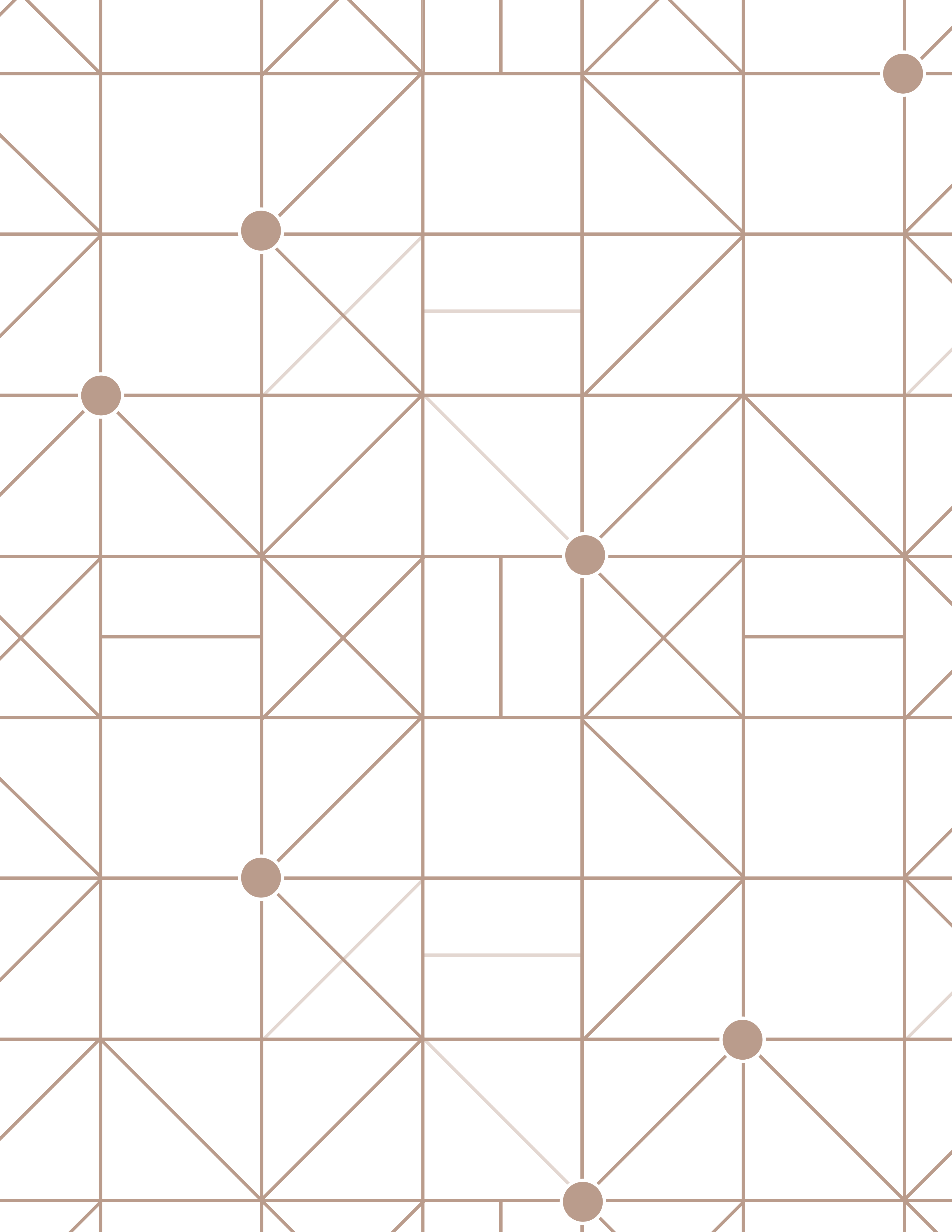




Area of small triangle formed by 3 points

$\leq$ Area of

$= \frac{1}{2}(1)\left(\frac{1}{2}\right) = \frac{1}{4}$



Q3

Example 2.8.1. For $n = 4$ and $m = 3$, we have by Theorem 2.8.1,

$$\begin{aligned}(x_1 + x_2 + x_3)^4 = & \binom{4}{4,0,0}x_1^4 + \binom{4}{3,1,0}x_1^3x_2 + \binom{4}{3,0,1}x_1^3x_3 \\ & + \binom{4}{2,2,0}x_1^2x_2^2 + \binom{4}{2,1,1}x_1^2x_2x_3 + \binom{4}{2,0,2}x_1^2x_3^2 \\ & + \binom{4}{1,3,0}x_1x_2^3 + \binom{4}{1,2,1}x_1x_2^2x_3 + \binom{4}{1,1,2}x_1x_2x_3^2 \\ & + \binom{4}{1,0,3}x_1x_3^3 + \binom{4}{0,4,0}x_2^4 + \binom{4}{0,3,1}x_2^3x_3 \\ & + \binom{4}{0,2,2}x_2^2x_3^2 + \binom{4}{0,1,3}x_2x_3^3 + \binom{4}{0,0,4}x_3^4 \\ = & \, x_1^4 + 4x_1^3x_2 + 4x_1^3x_3 + 6x_1^2x_2^2 + 12x_1^2x_2x_3 + 6x_1^2x_3^2 \\ & + 4x_1x_2^3 + 12x_1x_2^2x_3 + 12x_1x_2x_3^2 + 4x_1x_3^3 + x_2^4 \\ & + 4x_2^3x_3 + 6x_2^2x_3^2 + 4x_2x_3^3 + x_3^4.\end{aligned}$$

Because of Theorem 2.8.1, the numbers of the form (2.8.2) are usually called the *multinomial coefficients*. Since multinomial coefficients are generalizations of binomial coefficients, it is natural to ask whether some results about binomial coefficients can be generalized to multinomial coefficients. We end this chapter with a short discussion on this.

1° The identity $\binom{n}{n_1} = \binom{n}{n-n_1}$ for binomial coefficients may be written as $\binom{n}{n_1,n_2} = \binom{n}{n_2,n_1}$ (here of course $n_1 + n_2 = n$). By identity (2.8.5), it is easy to see in general that

$$\binom{n}{n_1, n_2, \ldots, n_m} = \binom{n}{n_{\alpha(1)}, n_{\alpha(2)}, \ldots n_{\alpha(m)}} \qquad (2.8.7)$$

where $\{\alpha(1), \alpha(2), \ldots, \alpha(m)\} = \{1, 2, \ldots, m\}$.

2° The identity $\binom{n}{n_1} = \binom{n-1}{n_1-1} + \binom{n-1}{n_1}$ for binomial coefficients may be written:

$$\binom{n}{n_1, n_2} = \binom{n-1}{n_1 - 1, n_2} + \binom{n-1}{n_1, n_2 - 1}.$$

In general, we have:

$$\begin{aligned}\binom{n}{n_1, n_2, \ldots, n_m} = & \binom{n-1}{n_1 - 1, n_2, \ldots, n_m} + \binom{n-1}{n_1, n_2 - 1, \ldots, n_m} + \cdots \\ & + \binom{n-1}{n_1, n_2, \ldots, n_m - 1}. \end{aligned} \qquad (2.8.8)$$

Q3

3° For binomial coefficients, we have the identity $\sum_{r=0}^{n} \binom{n}{r} = 2^n$. By letting $x_1 = x_2 = \cdots = x_m = 1$ in the multinomial theorem, we have

$$\sum \binom{n}{n_1, n_2, \ldots, n_m} = m^n \tag{2.8.9}$$

where the sum is taken over all m-ary sequences $(n_1, n_2, \ldots, n_m)$ of nonnegative integers with $\sum_{i=1}^{m} n_i = n$.

Identity (2.8.9) simply says that the sum of the coefficients in the expansion of $(x_1 + x_2 + \cdots + x_m)^n$ is given by m^n. Thus, in Example 2.8.1, the sum of the coefficients in the expansion of $(x_1 + x_2 + x_3)^4$ is 81, which is 3^4.

4° In the binomial expansion $(x_1 + x_2)^n = \sum_{r=0}^{n} \binom{n}{r} x_1^r x_2^{n-r}$, the number of distinct terms is $n + 1$. How many distinct terms are there in the expansion of $(x_1 + x_2 + \cdots + x_m)^n$? To answer this question, let us first look at Example 2.8.1. The distinct terms obtained in the expansion of $(x_1 + x_2 + x_3)^4$ are shown on the left column below:

$$\begin{array}{rcl}
x_1^4 & \rightarrow & \{4 \cdot x_1\} \\
x_1^3 x_2 & \rightarrow & \{3 \cdot x_1, x_2\} \\
x_1^3 x_3 & \rightarrow & \{3 \cdot x_1, x_3\} \\
x_1^2 x_2^2 & \rightarrow & \{2 \cdot x_1, 2 \cdot x_2\} \\
x_1^2 x_2 x_3 & \rightarrow & \{2 \cdot x_1, x_2, x_3\} \\
x_1^2 x_3^2 & \rightarrow & \{2 \cdot x_1, 2 \cdot x_3\} \\
x_1 x_2^3 & \rightarrow & \{x_1, 3 \cdot x_2\} \\
x_1 x_2^2 x_3 & \rightarrow & \{x_1, 2 \cdot x_2, x_3\} \\
x_1 x_2 x_3^2 & \rightarrow & \{x_1, x_2, 2 \cdot x_3\} \\
x_1 x_3^3 & \rightarrow & \{x_1, 3 \cdot x_3\} \\
x_2^4 & \rightarrow & \{4 \cdot x_2\} \\
x_2^3 x_3 & \rightarrow & \{3 \cdot x_2, x_3\} \\
x_2^2 x_3^2 & \rightarrow & \{2 \cdot x_2, 2 \cdot x_3\} \\
x_2 x_3^3 & \rightarrow & \{x_2, 3 \cdot x_3\} \\
x_3^4 & \rightarrow & \{4 \cdot x_3\}.
\end{array}$$

Observe that each of them corresponds to a unique 4-element multi-subset of $M = \{\infty \cdot x_1, \infty \cdot x_2, \infty \cdot x_3\}$, and vice versa, as shown on the right column

Q3

above. Thus by (BP), the number of distinct terms in the expansion of $(x_1 + x_2 + x_3)^4$ is equal to the number of 4-element multi-subsets of M, which is $H_4^3 = \binom{4+3-1}{4} = \binom{6}{4} = 15$. In general, one can prove that (see Problem 2.62)

> the number of distinct terms in the expansion of $(x_1 + x_2 + \cdots + x_m)^n$ is given by $H_n^m = \binom{n+m-1}{n}$.

In particular, for binomial expansion, we have $H_n^2 = \binom{2+n-1}{n} = n+1$, which agrees with what we mentioned before.

5° It follows from (2.7.1) and (2.7.2) that for a given positive integer n, the maximum value of the binomial coefficients $\binom{n}{r}$, $r = 0, 1, ..., n$, is equal to

$$\begin{cases} \binom{n}{\frac{n}{2}} & \text{if } n \text{ is even,} \\ \binom{n}{\frac{n-1}{2}} = \binom{n}{\frac{n+1}{2}} & \text{if } n \text{ is odd.} \end{cases}$$

What can we say about the maximum value of multinomial coefficients $\binom{n}{n_1, n_2, \ldots, n_m}$? This problem has recently been solved by Wu [W]. For $n, m \geq 2$, let

$$M(n,m) = \max\left\{ \binom{n}{n_1, n_2, \ldots, n_m} \middle| n_i \in \mathbf{N}^* \text{ and } \sum_{i=1}^{m} n_i = n \right\}.$$

Case 1. $m|n$.

Let $n = mr$ for some $r \in \mathbf{N}$. Then

$$M(n,m) = \binom{n}{\underbrace{r, r, \ldots, r}_{m}} = \frac{n!}{(r!)^m},$$

and $\binom{n}{\underbrace{r, r, \ldots, r}_{m}}$ is the only term attaining this maximum value.

Case 2. $m \nmid n$.

Suppose that $n = mr + k$ for some $r, k \in \mathbf{N}$ with $1 \leq k \leq m - 1$. Then

$$M(n,m) = \binom{n}{\underbrace{r, r, \ldots, r}_{m-k}, \underbrace{(r+1), (r+1), \ldots, (r+1)}_{k}}$$
$$= \frac{n!}{(r!)^{m-k}((r+1)!)^k} = \frac{n!}{(r+1)^k (r!)^m},$$

Q3

and $\binom{n}{n_1, n_2, \ldots, n_m}$, where $\{n_1, n_2, \ldots, n_m\} = \{(m-k) \cdot r, k \cdot (r+1)\}$ as multi-sets, are the $\binom{m}{k}$ terms attaining this maximum value.

For instance, in Example 2.8.1, we have

$$n = 4, \quad m = 3, \quad r = 1 \text{ and } k = 1.$$

Thus the maximum coefficient is

$$M(4,3) = \frac{4!}{2!(1!)^3} = 12,$$

which is attained by the following $\binom{m}{k} = 3$ terms:

$$\binom{4}{1,1,2}, \binom{4}{1,2,1} \text{ and } \binom{4}{2,1,1}.$$

Exercise 2

1. The number 4 can be expressed as a sum of one or more positive integers, taking order into account, in 8 ways:

$$\begin{aligned} 4 &= 1+3 = 3+1 = 2+2 = 1+1+2 \\ &= 1+2+1 = 2+1+1 = 1+1+1+1. \end{aligned}$$

   In general, given $n \in \mathbf{N}$, in how many ways can n be so expressed?

2. Find the number of $2n$-digit binary sequences in which the number of 0's in the first n digits is equal to the number of 1's in the last n digits.

3. Let $m, n, r \in \mathbf{N}$. Find the number of r-element multi-subsets of the multi-set

$$M = \{a_1, a_2, \ldots, a_n, m \cdot b\}$$

   in each of the following cases:
   (i) $r \leq m$, $r \leq n$;
   (ii) $n \leq r \leq m$;
   (ii) $m \leq r \leq n$.

4. Ten points are marked on a circle. How many distinct convex polygons of three or more sides can be drawn using some (or all) of the ten points as vertices? (Polygons are distinct unless they have exactly the same vertices.) (AIME, 1989/2)