California State University Sacramento - Math 101

Exam #3

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This exam is out of 7 points.

0.5 **1)** (a) Determine the exact value of the coefficient of x^3 in the expansion of $(x+1)^{13}$. (0.5 points)

0.25 ~~(b)~~ Determine the exact value of the coefficient of $x^3y^4z^2$ in the expansion of $(x+y+z)^9$. (0.5 points)

Binomial Theorem

$$\boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k}$$

a) $(x+1)^{13} = \binom{13}{10} x^{13-10} + 1^{10}$

Coefficient

$\frac{13!}{10!(13-10)!} \Rightarrow \frac{3!}{0!3!} = \frac{13\cdot 12\cdot 11}{3!} \Rightarrow \boxed{\frac{1716}{6}}$

$= 286$

b) $\binom{9}{3,4,2}$ multinomial

$= \frac{9!}{3!4!2!}$ then simplify

0.5 **2)** In the Binomial Theorem $(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$, state a value for x and value for y that produces the given formula. (0.25 points each)

(a) $0 = \sum_{r=0}^{n} \binom{n}{r} (-1)^r$

Leap of faith Let $x = 1$ & $y = -1$

$(1+(-1))^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

LHS

$(1+(-1))^n = 0$

RHS

$\sum_{k=0}^{n} \binom{n}{k}$

(b) $2^n = \sum_{r=0}^{n} \binom{n}{r}$ Leap of faith $x = y = 1$

LHS

$(1+1)^n$

$\Rightarrow 2^n$

LHS

$\sum_{k=0}^{n} \binom{n}{0} 1^{1-0} 1^0$

$\Rightarrow \sum_{k=0}^{n} \binom{n}{k} 1^1 \cdot 1^0$

LHS = RHS $\Rightarrow \sum_{k=0}^{n} \binom{n}{k}$

1.25

~~3.05~~ 3.3

Binomial → Vandermonth
$(x+y)^n \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^n$

Vandermont theorem

~~3)~~ Show that for all integers $n \geq m \geq 1$, $\sum_{r=m}^{n} \binom{n-m}{n-r} = 2^{n-m}$. (0.5 points)

$(1+x)^{m+n} = (1+x)^m (1+x)^n$

$$\binom{n-m}{n-m} + \binom{n-m}{n-m-1} + \dots + \binom{n-m}{n-m}$$

$$= \sum_{k=0}^{n-m} \binom{n-m}{k} = 2^{n-m}$$

4) Prove $\sum_{r=0}^{n} r \binom{n}{r} = n2^{n-1}$ for all $n \geq 1$. (0.75 points)

Identity

$.2 \quad \sum_{r=0}^{n} \binom{n}{r} = 2^n$

Use this idea for #3

.2

5) Prove that $\binom{n}{r}\binom{n-r}{m-r} = \binom{n}{m}\binom{m}{r}$ for all integers $n \geq m \geq r \geq 1$. (0.75 points)

- vandermont?
- $\binom{n}{r}+\binom{n}{r-1}=\binom{n+1}{r}$? X NO
-

try algebra

simplify

into

$$\frac{n!}{r!\,(n-r)!} \cdot \frac{(n-r)!}{(n-r)!\,(n-r-m+r)!}$$

0

6) Let $X = \{1, 2, 3, 4\}$. Let $\mathcal{A}$ be the collection of all subsets of X with an even number of elements, and let $\mathcal{B}$ be the collection of all subsets of X with an odd number of elements. Remark: The empty set $\emptyset$ is one of the sets in $\mathcal{A}$ since it has 0 elements and 0 is even.

(a) List all of the elements of $\mathcal{A}$. (0.5 points)

(b) List all of the elements of $\mathcal{B}$. (0.5 points)

(c) Is the function $f(C) = X\backslash C$ a bijection from $\mathcal{A}$ to $\mathcal{B}$? Recall that $X\backslash C$ is the complement of C in X. (0.25 points)

1:1 & onto

(d) Is the function $f(C) = C \cup \{1\}$ a bijection from $\mathcal{A}$ to $\mathcal{B}$? (0.25 points)

(e) Draw a bijection between $\mathcal{A}$ and $\mathcal{B}$. Represent your bijection using an arrow diagram. (0.25 points)

a- $\emptyset$ Null set (evens)

b- $\{1\},\{2\},\{3\},\{4\}$ single set (odds) ✓

a- $\{1,2\}, \cancel{\{1,2\}}, \{1,3\}, \{1,4\}$, $\{2,3\}, \{2,4\}, \{3,4\}$ double set (evens)

b- $\{1,2,3\}, \{1,3,4\}, \{2,3,4\}$, $\{2,1,4\}$, Set of 3's (odds)

a- $\{1,2,3,4\}$ — Whole set (evens)

+1

a) $\{\emptyset\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\} \{3,4\}, \{1,2,3,4\}$

b) $\{1\}, \{2\}, \{3\}, \{4\}, \{1,2,3\}, \{1,3,4\}, \{2,3,4\}, \{2,1,4\}$

c) $f(\{\text{"the set"}\}) = X - \{\text{"The set"}\}$ ← if "The set" (C) is $\{1,2,3,4\}$

C

What?! No

$X\backslash C = \emptyset$ Null Set

d) No

A B

e) Draw arrow diagram

7) (a) There are 50 jobs that must be assigned to 7 processors. Explain why there must be a processor that is assigned at least 8 jobs. (0.5 points)

(b) A list $\mathcal{L}$ contains 134 elements. Each element of $\mathcal{L}$ is a subset of $\{1, 2, 3, 4, 5, 6, 7\}$. Explain why the list $\mathcal{L}$ must contain two elements that are the same. (0.5 points)

(c) Suppose it takes a program 1 second to find the determinant of a 2×2 matrix. If S is the set of all 2×2 matrices of the form $\begin{pmatrix} x & y \\ z & t \end{pmatrix}$ where $x, y, z, t \in \{0, 1\}$, what is the minimum amount of time it would take the program to find the determinant of every matrix in S? (0.25 points)

7a) 50 jobs
7 processors

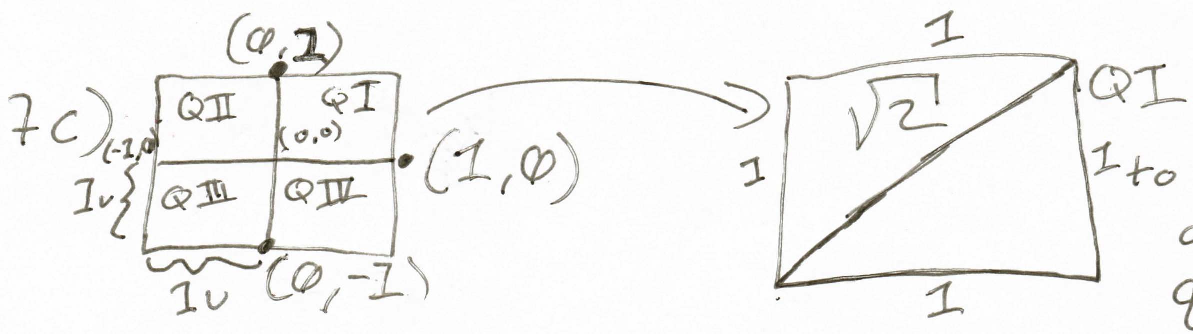
on avg
$50/7 \simeq 7.XXX$
$kn+1$

Had it been 49/7 there would have been 7 processors per 7 jobs but according to the pigeonhole principle there would be 1 more job to any 1 of the 7 processors to process.

7b)

$2^7 = 128$ possible subsets

7c)



to search in any 1 quadrant

it would take $\sqrt{2}$ time

$2^4 = 16$ possible matrices

.5

8) Prove that $\sum_{r=0}^{n-1} \binom{2n-1}{r} = 2^{2n-2}$. (0.5 points)

a1

initial thoughts

$*$ $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

will use a version of this

+1

LHS

$$? \sum_{k=0}^{n-1} \binom{2n-1}{k} \Rightarrow \binom{2(0)-1}{0} + \binom{2(1)-1}{1}$$

$\frac{-1}{0}$ $+ \dots + \binom{2n-1}{n-1}$

keep writing out sum

.1