Homework Assignment 6

1) Example 1.4.6

2) Example 1.4.7

3) Problem 18 on page 52

4) Problem 19 on page 52

5) Problem 21 on page 52

6) Problem 26 on page 53

Q1

Example 1.4.6. If there must be at least one person in each table, in how many ways can 6 people be seated

(i) around two tables?

(ii) around three tables?
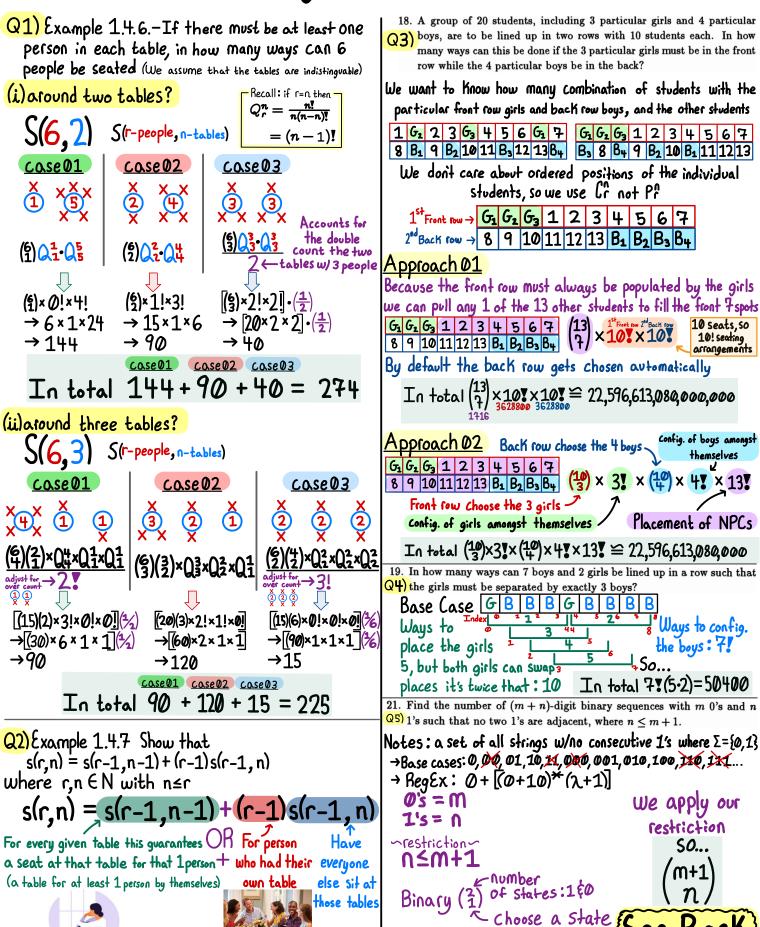
(We assume that the tables are indistinguishable.)

Q2

Example 1.4.7. Show that

$$s(r, n) = s(r - 1, n - 1) + (r - 1)s(r - 1, n)$$

where $r, n \in \mathbf{N}$ with $n \leq r$.

Q3

18. A group of 20 students, including 3 particular girls and 4 particular boys, are to be lined up in two rows with 10 students each. In how many ways can this be done if the 3 particular girls must be in the front row while the 4 particular boys be in the back?

Q4

19. In how many ways can 7 boys and 2 girls be lined up in a row such that the girls must be separated by exactly 3 boys?

Q5

21. Find the number of $(m + n)$-digit binary sequences with m 0's and n 1's such that no two 1's are adjacent, where $n \leq m + 1$.

Q6

26. Find the number of ways of forming a group of $2k$ people from n couples, where $k, n \in \mathbf{N}$ with $2k \leq n$, in each of the following cases:

(i) There are k couples in such a group;

(ii) No couples are included in such a group;

(iii) At least one couple is included in such a group;

(iv) Exactly two couples are included in such a group.

Q1) Example 1.4.6.– If there must be at least one person in each table, in how many ways can 6 people be seated (We assume that the tables are indistinguable)

(i) around two tables?

$$S(6,2)$$ $S(\text{r-people}, \text{n-tables})$

Recall: if $r = n$ then
$$Q_r^n = \frac{n!}{n(n-n)!}$$
$$= (n-1)!$$

Case01	Case02	Case03

Case01: ① ⑤ (× arrangement)
Case02: ② ④
Case03: ③ ③

$\binom{6}{1} Q_1^1 \cdot Q_5^5$ | $\binom{6}{2} Q_2^2 \cdot Q_4^4$ | $\binom{6}{3} Q_3^3 \cdot Q_3^3$

(Case03 note) Accounts for the double count the two tables w/ 3 people → divide by 2

$\binom{6}{1} \times 0! \times 4!$ | $\binom{6}{2} \times 1! \times 3!$ | $\left[\binom{6}{3} \times 2! \times 2!\right] \cdot \left(\frac{1}{2}\right)$
$\rightarrow 6 \times 1 \times 24$ | $\rightarrow 15 \times 1 \times 6$ | $\rightarrow [20 \times 2 \times 2] \cdot \left(\frac{1}{2}\right)$
$\rightarrow 144$ | $\rightarrow 90$ | $\rightarrow 40$

Case01 / Case02 / Case03

In total $144 + 90 + 40 = 274$

(ii) around three tables?

$$S(6,3)$$ $S(\text{r-people}, \text{n-tables})$

Case01	Case02	Case03

Case01: ④ ① ①
Case02: ③ ② ①
Case03: ② ② ②

$\binom{6}{4}\binom{2}{1} \times Q_4^4 \times Q_1^1 \times Q_1^1$ | $\binom{6}{3}\binom{3}{2} \times Q_3^3 \times Q_2^2 \times Q_1^1$ | $\binom{6}{2}\binom{4}{2} \times Q_2^2 \times Q_2^2 \times Q_2^2$

adjust for over count → $2!$ (① ①) | | adjust for over count → $3!$ (② ② ②)

$\left[(15)(2) \times 3! \times 0! \times 0!\right]\left(\frac{1}{2}\right)$ | $\left[(20)(3) \times 2! \times 1! \times 0!\right]$ | $\left[(15)(6) \times 0! \times 0! \times 0!\right]\left(\frac{1}{6}\right)$
$\rightarrow [(30) \times 6 \times 1 \times 1]\left(\frac{1}{2}\right)$ | $\rightarrow [(60) \times 2 \times 1 \times 1]$ | $\rightarrow [(90) \times 1 \times 1 \times 1]\left(\frac{1}{6}\right)$
$\rightarrow 90$ | $\rightarrow 120$ | $\rightarrow 15$

Case01 / Case02 / Case03

In total $90 + 120 + 15 = 225$

Q2) Example 1.4.7 Show that
$$s(r,n) = s(r-1, n-1) + (r-1)s(r-1, n)$$
where $r, n \in \mathbb{N}$ with $n \leq r$

$$s(r,n) = s(r-1, n-1) + (r-1)s(r-1, n)$$

For every given table this guarantees a seat at that table for that 1 person (a table for at least 1 person by themselves) **OR** + For person who had their own table Have everyone else sit at those tables

Q3)

We want to know how many combination of students with the particular front row girls and back row boys, and the other students

| 1 | G_2 | 2 | 3 | G_3 | 4 | 5 | 6 | G_1 | 7 | | G_1 | G_2 | G_3 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 8 | B_1 | 9 | B_2 | 10 | 11 | B_3 | 12 | 13 | B_4 | | B_3 | 8 | B_4 | 9 | B_2 | 10 | B_1 | 11 | 12 | 13 |

We don't care about ordered positions of the individual students, so we use C_r^n not P_r^n

1st Front row → | G_1 | G_2 | G_3 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |

2nd Back row → | 8 | 9 | 10 | 11 | 12 | 13 | B_1 | B_2 | B_3 | B_4 |

Approach 01

Because the front row must always be populated by the girls we can pull any 1 of the 13 other students to fill the front 7 spots

| G_1 | G_2 | G_3 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 8 | 9 | 10 | 11 | 12 | 13 | B_1 | B_2 | B_3 | B_4 |

$\binom{13}{7} \times 10! \times 10!$ (1st Front row, 2nd Back row)

10 seats, so 10! seating arrangements

By default the back row gets chosen automatically

In total $\binom{13}{7} \times 10! \times 10! \cong 22{,}596{,}613{,}080{,}000{,}000$
(1716) (3628800) (3628800)

Approach 02

| G_1 | G_2 | G_3 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 8 | 9 | 10 | 11 | 12 | 13 | B_1 | B_2 | B_3 | B_4 |

Back row choose the 4 boys   Config. of boys amongst themselves

$\binom{10}{3} \times 3! \times \binom{10}{4} \times 4! \times 13!$

Front row choose the 3 girls   Config. of girls amongst themselves   Placement of NPCs

In total $\binom{10}{3} \times 3! \times \binom{10}{4} \times 4! \times 13! \cong 22{,}596{,}613{,}080{,}000$

Q4)

Base Case
| G | B | B | B | G | B | B | B | B |

Index: 0 1 2 3 4 5 6 7 8

Ways to place the girls 5, but both girls can swap places it's twice that: 10

Ways to config. the boys: 7!

So... In total $7!(5 \cdot 2) = 50400$

Q5)

Notes: a set of all strings w/no consecutive 1's where $\Sigma = \{0,1\}$

→ Base cases: $0, 00, 01, 10, \cancel{11}, 000, 001, 010, 100, \cancel{110}, \cancel{111}...$

→ RegEx: $0 + [(0+10)^* (\lambda + 1)]$

$0's = m$
$1's = n$

~restriction~
$n \leq m+1$

Binary $\binom{2}{1}$ ← number of states: 1 & 0 ← Choose a state

We apply our restriction
so...
$$\binom{m+1}{n}$$

See Back

Q6)

26. Find the number of ways of forming a group of $2k$ people from n couples, where $k, n \in \mathbb{N}$ with $2k \leq n$, in each of the following cases:

(i) There are k couples in such a group;

(ii) No couples are included in such a group;

(iii) At least one couple is included in such a group;

(iv) Exactly two couples are included in such a group.

(i) No restriction

$$\binom{n}{k}$$

(ii)

$$\binom{n}{2k} \cdot 2^{2k}$$

(iii)

$$\binom{2n}{2k} - \binom{n}{2k} \cdot 2^{2k}$$

(iv)

$$\binom{n}{2} \binom{n-2}{2k-4} \cdot 2^{2k-4}$$