Study Kit Exam 03 Homework and Quizzes

Exam 3 is on 4/21. It will **focus on the Binomial Theorem and related concepts, as well as the Pigeonhole Principle.** Here are the materials that I will place in front of me when I write Exam 3. Note that this does not mean I will take the problems directly from these materials. However, the concepts and ideas will be the same.

The old exam 3 that is posted in the files section of Canvas

Homework 7 but only the bijection problems (#6 and #7)

Homework 8

Homework 9

Quizzes 8, 9, and 10. We will take Quiz 10 in groups on Wednesday.

You may bring a 3x5 card with reasonably sized writing on one side of the card to use during Exam 3.

I hope the studying is going well and that you can make it to tomorrow's class for the group quiz.

CONTAINS

- **Homework #7 - only the bijection problems (#6 and #7)**
- **Homework #8**
- **Homework #9**
- **Homework #7 Solutions**
- **Homework #8 Solutions**
- **Homework #9 Solutions**
- **Old 2016 Exam 03**
- **Old 2016 Exam 03 Soltuions**
- **Quiz #8**
- **Quiz #9**
- **Quiz #10**
- **Quiz #8 Solutions**
- **Quiz #9 Solutions**
- **Quiz #10 Soltuions**

PROBLEMS

Homework #07
only the bijection problems (#6 and #7)

Homework #08

Homework #09

Old 2016 Exam 03

Quiz #08

Quiz #09

Quiz #10

HW #7 - Bijection Problems (#6 and #7)

40. Prove the identity $\binom{n}{r} = \binom{n}{n-r}$ by (BP)

41. Let $X = \{1, 2, ..., n\}$, $\mathcal{A} = \{A \subseteq X \mid n \notin A\}$, and $\mathcal{B} = \{A \subseteq X \mid n \in A\}$. Show that $|\mathcal{A}| = |\mathcal{B}|$ by (BP).

Homework Assignment 8

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Homework Assignment 8

1) The statement of Theorem 2.2.1 on page 70. You do not need to know how to prove the Binomial Theorem but you do need to know how to use it.

2) Example 2.3.1 on page 71

Example 2.3.1. Show that for all integers $n \geq 0$,

$$\sum_{r=0}^{n} \binom{n}{r} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

3) Example 2.3.2(i) on page 72. Remark: Part (ii) of this example is a consequence of part (i).

Example 2.3.2. Show that for all integers $n \geq 1$,

(i) $\sum_{r=0}^{n}(-1)^r\binom{n}{r} = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n} = 0,$

4) Example 2.3.3 on page 72

5) Example 2.3.4 on page 74, only the second proof.

Example 2.3.3. Show that for all integers $n \in \mathbf{N}$,

$$\sum_{r=1}^{n} r\binom{n}{r} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n \cdot 2^{n-1}.$$

6) Problem 24 on page 105

7) Problem 25 on page 105. Hint: Use the results of Examples 2.3.1 and 2.3.3

8) Problem 26 on page 105. Hint: The identity that you are trying to prove is equivalent to the identity

$$\sum_{r=0}^{n} \frac{n+1}{r+1}\binom{n}{r} = 2^{n+1} - 1$$

(divide both sides of this identity by $n+1$ to get the original identity). Next use the fact that $\frac{n+1}{r+1}\binom{n}{r} = \binom{n+1}{r+1}$ and finish by using Example 2.3.1.

Example 2.3.4. **(Vandermonde's Identity)** Show that for all $m, n, r \in \mathbf{N}$,

$$\sum_{i=0}^{r}\binom{m}{i}\binom{n}{r-i} = \binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \cdots + \binom{m}{r}\binom{n}{0}$$
$$= \binom{m+n}{r}. \qquad (2.3.5)$$

Second proof. Let $X = \{a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n\}$ be a set of $m+n$ objects. We shall count the number of r-combinations A of X.

Assuming that A contains exactly i a's, where $i = 0, 1, \ldots, r$, then the other $r-i$ elements of A are b's; and in this case, the number of ways to form A is given by $\binom{m}{i}\binom{n}{r-i}$. Thus, by (AP), we have

$$\sum_{i=0}^{r}\binom{m}{i}\binom{n}{r-i} = \binom{m+n}{r}. \quad \blacksquare$$

Prove each of the following identities in Problems 24–43, where $m, n \in \mathbf{N}^*$:

24. $\sum_{r=0}^{n} 3^r\binom{n}{r} = 4^n,$

25. $\sum_{r=0}^{n}(r+1)\binom{n}{r} = (n+2)2^{n-1},$

26. $\sum_{r=0}^{n}\frac{1}{r+1}\binom{n}{r} = \frac{1}{n+1}(2^{n+1}-1),$

27. $\sum_{r=0}^{n}\frac{(-1)^r}{r+1}\binom{n}{r} = \frac{1}{n+1},$

Homework Assignment 8 - Problems

HW#8

Example 2.3.1. Show that for all integers $n \geq 0$,

$$\sum_{r=0}^{n} \binom{n}{r} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

Example 2.3.2. Show that for all integers $n \geq 1$,

(i) $\sum_{r=0}^{n}(-1)^r \binom{n}{r} = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0,$

Example 2.3.2. Show that for all integers $n \geq 1$,

(i) $\sum_{r=0}^{n}(-1)^r \binom{n}{r} = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0,$ (2.3.2)

(ii) $\binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{2k} + \cdots = \binom{n}{1} + \binom{n}{3} + \cdots + \binom{n}{2k+1} + \cdots = 2^{n-1}$.(2.3.3)

Proof. By letting $x = 1$ and $y = -1$ in Theorem 2.2.1, we obtain

$$\sum_{r=0}^{n} \binom{n}{r}(-1)^r = (1-1)^n = 0,$$

which is (i). The identity (ii) now follows from (i) and identity (2.3.1). ∎

Remark. A subset A of a non-empty set X is called an *even-element* (resp. *odd-element*) subset of X if $|A|$ is even (resp. odd). Identity (2.3.3) says that given an n-element set X, the number of even-element subsets of X is the same as the number of odd-element subsets of X. The reader is encouraged to establish a bijection between the family of even-element subsets of X and that of odd-element subsets of X (see Problem 2.10).

Example 2.3.4. **(Vandermonde's Identity)** Show that for all $m, n, r \in \mathbf{N}$,

$$\sum_{i=0}^{r} \binom{m}{i}\binom{n}{r-i} = \binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \cdots + \binom{m}{r}\binom{n}{0}$$
$$= \binom{m+n}{r}. \qquad (2.3.5)$$

Second proof. Let $X = \{a_1, a_2, ..., a_m, b_1, b_2, ..., b_n\}$ be a set of $m+n$ objects. We shall count the number of r-combinations A of X.

Assuming that A contains exactly i a's, where $i = 0, 1, ..., r$, then the other $r - i$ elements of A are b's; and in this case, the number of ways to form A is given by $\binom{m}{i}\binom{n}{r-i}$. Thus, by (AP), we have

$$\sum_{i=0}^{r} \binom{m}{i}\binom{n}{r-i} = \binom{m+n}{r}. \quad ∎$$

Example 2.3.3. Show that for all integers $n \in \mathbf{N}$,

$$\sum_{r=1}^{n} r\binom{n}{r} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n \cdot 2^{n-1}.$$

Prove each of the following identities in Problems 24–43, where $m, n \in \mathbf{N}^*$:

24. $\sum_{r=0}^{n} 3^r \binom{n}{r} = 4^n,$

25. $\sum_{r=0}^{n} (r+1)\binom{n}{r} = (n+2)2^{n-1},$

26. $\sum_{r=0}^{n} \frac{1}{r+1}\binom{n}{r} = \frac{1}{n+1}(2^{n+1} - 1),$

Homework Assignment 9

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Homework Assignment 9

1) Example 2.3.4 on page 74

2) The statement of Theorem 2.8.1

3) Example 2.8.1 on page 99

4) Problem 28 on page 106.

5) Problem 31 on page 106.

6) State the Pigeonhole Principle

7) Example 3.2.4

8) Problem 1 on page 137

9) Problem 3 on page 137

1) **Example 2.3.4. (Vandermonde's Identity)** Show that for all $m, n, r \in \mathbf{N}$,

$$\sum_{i=0}^{r}\binom{m}{i}\binom{n}{r-i}=\binom{m}{0}\binom{n}{r}+\binom{m}{1}\binom{n}{r-1}+\cdots+\binom{m}{r}\binom{n}{0}$$
$$=\binom{m+n}{r}. \tag{2.3.5}$$

2) **Theorem 2.8.1 (The Multinomial Theorem).** *For $n, m \in \mathbf{N}$,*

$$(x_1+x_2+\cdots+x_m)^n=\sum\binom{n}{n_1,n_2,\ldots,n_m}x_1^{n_1}x_2^{n_2}\cdots x_m^{n_m}$$

3) **Example 2.8.1.** For $n = 4$ and $m = 3$, we have by Theorem 2.8.1,

$(x_1 + x_2 + x_3)^4 =$

4) **106** ***Exercise 2***

28. $\displaystyle\sum_{r=m}^{n}\binom{n}{r}\binom{r}{m}=2^{n-m}\binom{n}{m}$ for $m \leq n$,

5)

31. $\displaystyle\sum_{r=0}^{n}(-1)^r r\binom{n}{r}=0,$

6)

3.2. The Pigeonhole Principle

If three pigeons are to be put into two compartments, then you will certainly agree that one of the compartments will accommodate at least two pigeons. A much more general statement of this simple observation, known as the *Pigeonhole Principle*, is given below.

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7) **Example 3.2.4.** Show that for any set of 10 points chosen within a square whose sides are of length 3 units, there are two points in the set whose distance apart is at most $\sqrt{2}$.

What are the objects? What are the boxes? These are the two questions we have to ask beforehand. It is fairly clear that we should treat the 10 given points in the set as our "objects". The conclusion we wish to arrive at is the existence of "2 points" from the set which are "close" to each other (i.e. their distance apart is at most $\sqrt{2}$ units). This indicates that "$k + 1 = 2$" (i.e., $k = 1$), and suggests also that we should partition the 3×3 square into n smaller regions, $n < 10$, so that the distance between any 2 points in a region is at most $\sqrt{2}$.

Exercise 3

8) 1. Show that among any 5 points in an equilateral triangle of unit side length, there are 2 whose distance is at most $\frac{1}{2}$ units apart.

9) 3. Given any set S of 9 points within a unit square, show that there always exist 3 distinct points in S such that the area of the triangle formed by these 3 points is less than or equal to $\frac{1}{8}$. (Beijing Math. Competition, 1963)

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Exam # 3

Name: ______________________

1) (a) State the Binomial Theorem. (4 points)

(b) Use the Binomial Theorem to determine the coefficient of x^7y^{12} in the expansion of $(x+y)^{19}$. (3 points)

2) Write the sum $\binom{10}{3} + \binom{10}{4}$ as a single binomial coefficient. (3 points)

3) Find the coefficient of $x_1^3x_2^2x_3$ in the expansion of $(x_1+x_2+x_3)^6$. (3 points)

4) Show that $\sum_{r=0}^{n} 3^r \binom{n}{r} = 4^n$ for all integers $n \geq 1$. (4 points)

5) Write the sum $\sum_{r=0}^{6} \binom{12}{r}\binom{10}{6-r}$ as a single binomial coefficient. (4 points)

6) Write the sum $\binom{4}{4} + \binom{5}{4} + \binom{6}{4} + \binom{7}{4}$ as a single binomial coefficient. (3 points)

7) Find the coefficient of x^7 in the expansion of $(1 + x + x^3)^{12}$. (5 points)

8) State the Pigeonhole Principle as stated in class. (4 points)

9) Suppose that 37 people are in a classroom. At least how many people have their birthdays in the same month? (3 points)

10) Show that for any set of five points in a 2 by 2 square, there are at least two points in the set whose distance apart is at most $\sqrt{2}$. (4 points)

11) Show that for any set of 5 points on the unit circle, there are at least two points in the set whose distance apart is at most $\sqrt{2}$. (3 points)

12) Determine the exact value of $\displaystyle\sum_{r=0}^{10}\binom{10}{r}$. (3 points)

13) Show that $\displaystyle\sum_{r=0}^{n} \frac{n+1}{r+1}\binom{n}{r} = 2^{n+1} - 1$ for all integers $n \geq 1$. (4 points)

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Quiz #8

Name: ____________________________

1) (a) State the Binomial Theorem.

(b) Prove that $0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$ for all integers $n \geq 1$.

2) Let $X = \{1, 2, \ldots, n\}$, $\mathcal{A} = \{A \subseteq X : 1 \notin A\}$ and $\mathcal{B} = \{B \subseteq X : 1 \in B\}$.

(a) In the case that $n = 3$, write down all of the elements of $\mathcal{A}$, all of the elements of $\mathcal{B}$, and a bijection f from $\mathcal{A}$ to $\mathcal{B}$.

(b) Give a formula for a function $f : \mathcal{A} \to \mathcal{B}$ that defines a bijection from $\mathcal{A}$ to $\mathcal{B}$ and is valid for every n.

3) What is the coefficient of x^3 in $(x+1)^{12}$? Simplify your answer as much as possible.

4) Prove that $n2^{n-1} = \sum_{r=1}^{n} r\binom{n}{r}$ for all integers $n \geq 1$.

5) Simplify $\sum_{r=1}^{n} \binom{n}{r}$ as much as possible. Your final answer should involve two terms, one of which depends on n.

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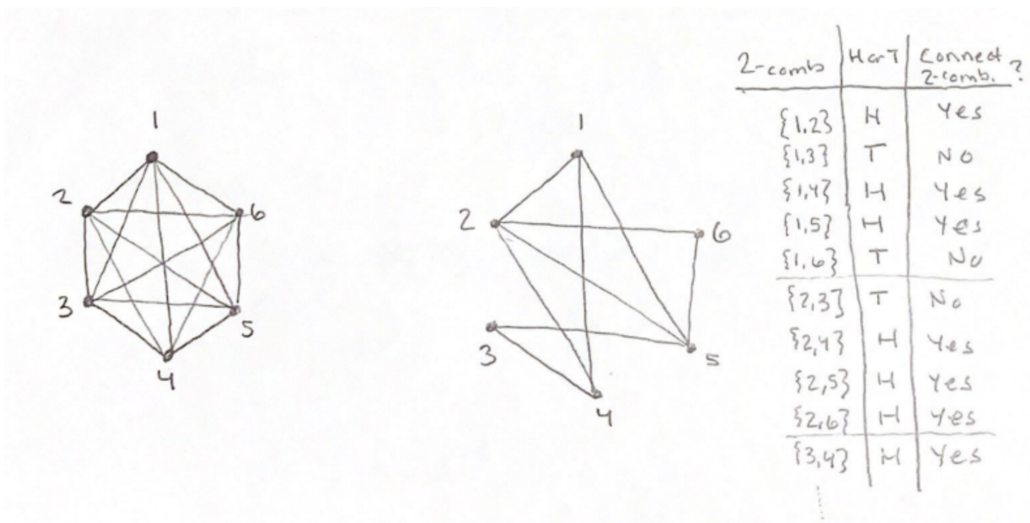
Quiz #9

Name: ______________________

1) On the left, every two numbers from $\{1, 2, 3, 4, 5, 6\}$ are connected by an edge. This is a visual representation of the *complete graph on 6 vertices* which we denote by K_6.



On the right is a random subgraph of K_6 that was determined by the following procedure.
Consider each 2-combination $\{a, b\}$ in the ordered list

$$\{1,2\},\ \{1,3\},\ \{1,4\},\ \{1,5\},\ \{1,6\},\ \{2,3\},\ \{2,4\},\ \{2,5\},\ \{2,6\},\ \{3,4\},\ \{3,5\},\ \{3,6\},\ \{4,5\},\ \{4,6\},\ \{5,6\}$$

For each $\{a, b\}$, flip of fair coin where H and T are equally likely. If the result is H, keep $\{a, b\}$ connected. If the result is T, remove the connection. The table on the right in the figure above shows this process over the first ten pairs, however, it was completed for all fifteen pairs in order to produce the random graph on the right.

Instructions: Perform this process to draw a random subgraph of K_6. You may use www.random.org/coins to perform 15 independent flips at once.

2) Let p be a real number with $0 \leq p \leq 1$ and let n be a positive integer. Define the function $f_{n,p} : \mathbb{R} \to \mathbb{R}$ by the rule

$$f_{n,p}(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & \text{if } k \in \{0, 1, 2, \ldots n\}, \\ 0 & \text{otherwise.} \end{cases}$$

The function $f_{n,p}$ is the probability density function of a Binomial random variable with parameters n and p. For $k \in \{0, 1, \ldots, n\}$, the value $f_{n,k}(k)$ is the probability of k successes in n independent trials with success probability p.

(a) Prove, using the Binomial Theorem, that $\sum_{k=0}^{n} f_{n,p}(k) = 1$.

(b) Suppose that a biased coin, which comes up heads with probability 0.7, is tossed 4 independent times. Complete the following table.

Number of heads k	Probability of k heads
0	$f_{4,0.7}(0) = \binom{4}{0}(0.7)^0(0.3)^4 = 0.0081$
1	
2	
3	
4	

(c) Use the table from part (b) to compute

$$0 \cdot f_{4,0.7}(0) + 1 \cdot f_{4,0.7}(1) + 2 \cdot f_{4,0.7}(2) + 3 \cdot f_{4,0.7}(3) + 4 \cdot f_{4,0.7}(4).$$

This value is the average number of heads we would expect to see in 4 independent tosses.

(d) Prove that for any real number p with $0 \leq p \leq 1$ and positive integer n,

$$np = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}.$$

This value is the expected number of successes if n independent trials are done where the probability of success is p.

3) Show that for integers $1 \leq r \leq m \leq n$,

$$\binom{n}{m}\binom{m}{r} = \binom{n}{r}\binom{n-r}{m-r}.$$

4) Show that for any integers $n \geq m \geq 1$,

$$\sum_{r=m}^{n} \binom{n-m}{n-r} = 2^{n-m}.$$

5) Show that for all integers $n \geq 1$,

$$\sum_{r=1}^{n} r^2 \binom{n}{r} = n(n+1)2^{n-2}.$$

Quiz #10 - Page 1 of 2

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Quiz #10

Name: ____________________

1) Compute (a) $\binom{11}{3}$ (b) $\binom{8}{0}$

2) Find the coefficient of x^3y^8 in the expansion of $(x+y)^{11}$. Simplify your answer as much as possible.

3) Prove each statement using the Binomial Theorem.

(a) $2^n = \sum_{k=0}^{n} \binom{n}{k}$ (b) $3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k$ (c) $0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$

4) Determine the exact value of $\sum_{r=1}^{10} \binom{10}{r}$.

5) Let $X = \{1, 2, 3, 4, 5\}$.

(a) If $A = \{1, 2, 3\}$, find $X \backslash A$.

(b) How many subsets of X have an odd number of elements?

(c) How many subsets of X have an even number of elements?

(d) Let $\mathcal{A}$ be the set of all subsets of X with an odd number of elements, and $\mathcal{B}$ be the set of all subsets of X with an even number of elements. If $f : \mathcal{A} \to \mathcal{B}$ is defined by $f(A) = X \backslash A$, find $f(\{5\})$.

6) Show that for any five points on the unit circle $x^2 + y^2 = 1$, there are at least two points that are within distance $\sqrt{2}$ of each other.

7) Suppose that there are 100 students in a class.

(a) Show that there are at least 9 students who were born in the same month.

(b) Suppose each student selects a 0-1 sequence of length 6. Must there exists two students who selected the same sequence?

(c) The class has a lottery where two numbers are chosen from $\{1, 2, \ldots, 15\}$. The winning ticket gets an A in the course. How many tickets should you purchase to ensure that you have a winning ticket?

Continued on other side

8) (a) Find the coefficient of $x^2y^3z^4$ in the expansion of $(x+y+z)^9$. You may leave your answer as a multinomial coefficent.

(b) Simplify $\binom{6}{1,2,3}$ as much as possible. You may want to use the formula

$$\binom{m}{m_1, m_2, \ldots, m_k} = \frac{(m_1+m_2+\cdots+m_k)!}{m_1!m_2!\cdots m_k!}$$

where $m = m_1 + m_2 + \cdots + m_k$.

9) (a) Recall that the power set of a set X is the set of all subsets of X. List all elements of the power set of $X = \{a, b, c\}$.

(b) Draw a bijection between the set of all 0-1 sequences of length 2 and all subsets of $\{x, y\}$.

(c) Draw a bijection between the set of all 0-1 sequences of length 3 and all subsets of $\{x, y, z\}$.

10) Let $X = \{1, 2, 3\}$. Suppose that five distinct subsets of X are chosen. Show that one of those subsets must contain 1.

SOLUTIONS

Homework #07
only the bijection problems (#6 and #7)

Homework #08

Homework #09

Old 2016 Exam 03

Quiz #08

Quiz #09

Quiz #10

Homework Assignment 7 - Solutions

6) Let $X = \{1, 2, \dots, n\}$. We define a bijection f from the r-combinations of X to the $(n-r)$-combinations of X in the following way. Given an r-combination A, let

$$f(A) = X \backslash A.$$

For example, if $X = \{1, 2, 3, 4, 5\}$, then

$$f(\{1,2\}) = \{3,4,5\},\ f(\{3\}) = \{1,2,4,5\},\ \text{and}\ f(\{1,2,3,4,5\}) = \emptyset.$$

Since f is a bijection (a fact which we may take for granted), we have by (BP) that

$$\binom{n}{r} = \binom{n}{n-r}.$$

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7) We define a bijection $f : \mathcal{A} \to \mathcal{B}$ by the following rule: given $A \in \mathcal{A}$, let

$$f(A) = A \cup \{n\}.$$

For example, if $X = \{1, 2, 3\}$, then $n = 3$,

$$\mathcal{A} = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$$

and

$$\mathcal{B} = \{\{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

In this example,

$$f(\emptyset) = \{3\},\ f(\{1\}) = \{1,3\},\ f(\{2\}) = \{2,3\},\ f(\{2\}) = \{2,3\},\ \text{and}\ f(\{1,2\}) = \{1,2,3\}.$$

Since f is a bijection, we have by (BP) that

$$|\mathcal{A}| = |\mathcal{B}|.$$

Important Remark: In the solutions to 6 and 7 we did not prove that f is a bijection. Since this is not a proof based course, I am not requiring you to prove that f is a bijection. However, from a mathematicians point of view, showing that f is a bijection is one of the most important parts of a complete solution.

Homework Assignment 8 - Solutions

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Homework Assignment 8 - Solutions

6) If we take $x = 1$ and $y = 3$ in the Binomial Theorem, then we have

$$\sum_{r=0}^{n} \binom{n}{r} 1^{n-r} 3^r = (1+3)^n$$

which can be rewritten as

$$\sum_{r=0}^{n} \binom{n}{r} 3^n = 4^n.$$

7) From Examples 2.3.1 and 2.3.3 we know that

$$\sum_{r=0}^{n} r\binom{n}{r} = n2^{n-1} \quad \text{and} \quad \sum_{r=0}^{n} \binom{n}{r} = 2^n.$$

Adding these two equations together gives

$$\sum_{r=0}^{n} (r+1)\binom{n}{r} = n2^{n-1} + 2^n$$

but the right hand side is equal to $(n+2)2^{n-1}$.

8) We have

$$\sum_{r=0}^{n} \frac{n+1}{r+1}\binom{n}{r} = \sum_{r=0}^{n} \binom{n+1}{r+1} = \sum_{r=1}^{n+1} \binom{n+1}{r}. \tag{1}$$

We know from Example 2.3.1 that

$$2^{n+1} = \sum_{r=0}^{n+1} \binom{n+1}{r} = \binom{n+1}{0} + \sum_{r=1}^{n+1} \binom{n+1}{r}.$$

Since $\binom{n+1}{0} = 1$, we can rewrite this last equation as

$$2^{n+1} - 1 = \sum_{r=1}^{n+1} \binom{n+1}{r}. \tag{2}$$

Combining (1) and (2) we get

$$\sum_{r=0}^{n} \frac{n+1}{r+1}\binom{n}{r} = 2^{n+1} - 1$$

which can be rewritten as $\displaystyle\sum_{r=0}^{n} \frac{1}{r+1}\binom{n}{r} = \frac{1}{n+1}(2^{n+1} - 1)$.

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

Level 1: plugging in #'s for x, y to get identity

$x = 1$
$y = 1$

$$2^n = \sum_{r=0}^{n} \binom{n}{r}$$

$x = -1$
$y = 1$

$$0 = \sum_{r=0}^{n} \binom{n}{r} (-1)^r$$

Level 2: Additional like a derivative or using a previous, or simplifying first

$$\sum_{r=m}^{n} \binom{n}{r}\binom{r}{m} \overset{?}{=} 2^{n-m} \binom{n}{m}$$

$$\sum_{r=m}^{n} \frac{n!}{\cancel{r!}\,(n-r)!} \cdot \frac{\cancel{r!}}{m!\,(r-m)!} \overset{?}{=} 2^{n-m} \frac{n!}{m!\,(n-m)!}$$

$$\frac{n!}{m!} \sum_{r=m}^{n} \frac{1}{(n-r)!\,(r-m)!} \overset{?}{=} \frac{n!}{m!} \cdot 2^{n-m} \cdot \frac{1}{(n-m)!}$$

$$\frac{n!}{m!} \sum_{r=m}^{n} \frac{(n-m)!}{(n-r)!\,(r-m)!} \overset{?}{=} \frac{n!}{m!} 2^{n-m}$$

Homework Assignment 9 - Office Hours Page 2

$$\frac{n!}{m!} \sum_{r=m}^{n} \binom{n-m}{n-r} \stackrel{?}{=} 2^{n-m} \frac{n!}{m!}$$

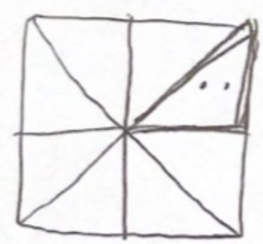
$$\binom{n-m}{n-r} = \frac{(n-m)!}{(n-r)!\,(n-m-(n-r))!}$$

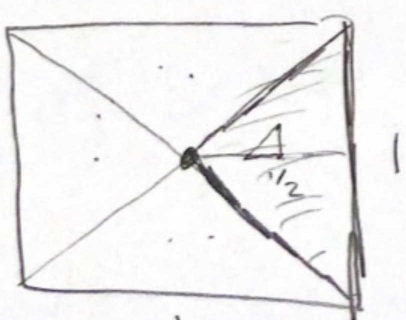
$$= \frac{(n-m)!}{(n-r)!\,(r-m)!}$$

$$\frac{n!}{m!} \Bigg(\underbrace{\binom{n-m}{n-m} + \binom{n-m}{n-m-1} + \dots + \binom{n-m}{0}}_{2^{n-m}} \Bigg) \stackrel{?}{=} 2^{n-m} \frac{n!}{m!}$$

$\frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$

Match!





Area of small triangle formed by 3 points

$\leq$ Area of 

$= \frac{1}{2}(1)\left(\frac{1}{2}\right) = \frac{1}{4}$

Old Exam 3 - Solutions

Old Exam 03 Solutions - Page 1 of 3

1) (a) For any positive integer n and real numbers x and y,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

(b) $\binom{19}{7}$

2) $\binom{10}{3} + \binom{10}{4} = \binom{11}{4}$

3) $\binom{6}{3,2,1}$

4) If we let $y=1$ and $x=3$ in the Binomial theorem, we obtain

$$4^n = \sum_{k=0}^{n} \binom{n}{k} 3^k .$$

5)
$$\sum_{r=0}^{6} \binom{12}{r}\binom{10}{6-r} = \binom{12}{0}\binom{10}{6} + \binom{12}{1}\binom{10}{5} + \binom{12}{2}\binom{10}{4}$$
$$+ \binom{12}{3}\binom{10}{3} + \binom{12}{4}\binom{10}{2} + \binom{12}{5}\binom{10}{1} + \binom{12}{6}\binom{10}{0}$$
$$= \binom{22}{6} \text{ by Vandermonde's Identity.}$$

6)
$$\binom{4}{4} + \binom{5}{4} + \binom{6}{4} + \binom{7}{4}$$
$$= \binom{5}{5} + \binom{5}{4} + \binom{6}{4} + \binom{7}{4} \quad \longleftarrow \binom{4}{4} = 1 = \binom{5}{5}$$
$$= \binom{6}{5} + \binom{6}{4} + \binom{7}{4} \quad \longleftarrow \binom{6}{5} = \binom{5}{5} + \binom{5}{4}$$
$$= \binom{7}{5} + \binom{7}{4} \quad \longleftarrow \binom{7}{5} = \binom{6}{5} + \binom{6}{4}$$
$$= \binom{8}{5} \quad \longleftarrow \binom{8}{5} = \binom{7}{5} + \binom{7}{4}$$

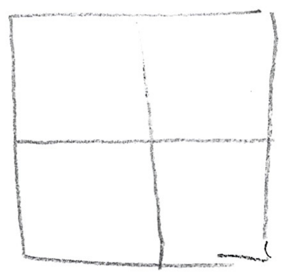
7) SKIP - Has to do with Multinomial Theorem

8) Let k and n be positive integers. If $kn+1$ objects are placed in n boxes, then at least one box has at least $k+1$ objects.

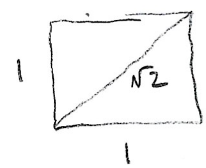
9) $37 = 3 \cdot \underbrace{12}_{\#\text{ of months}} + 1$

At least 4 people will have the same birthday month.

10) Divide the 2×2 square into four equal squares.



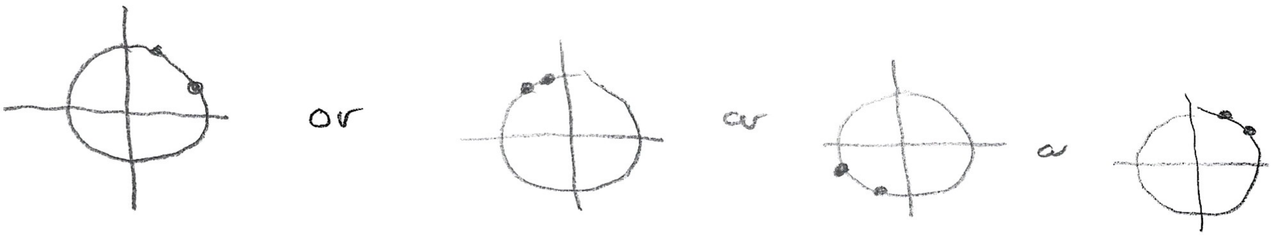
Since we have 5 points, at least one square contains two points.



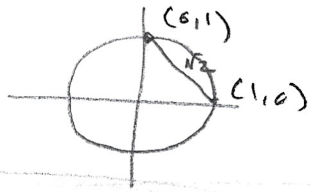
The farthest apart two points in the same square can be is $\sqrt{2}$.

Old Exam 03 Solutions - Page 3 of 3

11) By the Pigeonhole Principle, there must be two points in the same quadrant.



The farthest two points on the unit circle but in the same quadrant can be is $\sqrt{2}$.



12) $\displaystyle\sum_{r=0}^{10} \binom{10}{r} = 2^{10}$

using the identity $\frac{n+1}{r+1}\binom{n}{r} = \binom{n+1}{r+1}$

13) $$\sum_{r=0}^{n} \frac{n+1}{r+1}\binom{n}{r} = \sum_{r=0}^{n} \binom{n+1}{r+1}$$

$$= \binom{n+1}{1} + \binom{n+1}{2} + \dots + \binom{n+1}{n+1}$$

$$= -\binom{n+1}{0} + \binom{n+1}{0} + \binom{n+1}{1} + \dots + \binom{n+1}{n+1}$$

$$= -1 + 2^{n+1} = 2^{n+1} - 1$$

Quiz #08 Solutions - Page 1 of 2

California State University Sacramento - Math 101
Quiz #8

Name: ____________________

1) (a) State the Binomial Theorem.

(b) Prove that $0 = \sum_{k=0}^{n}(-1)^k\binom{n}{k}$ for all integers $n \geq 1$.

(a) For any positive integer $n \geq 1$, $(x+y)^n = \sum_{r=0}^{n}\binom{n}{r}x^r y^{n-r}$.

(b) Let $x=-1, y=1$ in the Binomial Theorem.

$$(-1+1)^n = \sum_{k=0}^{n}\binom{n}{k}(-1)^k 1^{n-k}$$

$$0 = \sum_{k=0}^{n}(-1)^k\binom{n}{k}$$

2) Let $X = \{1, 2, \ldots, n\}$, $\mathcal{A} = \{A \subseteq X : 1 \notin A\}$ and $\mathcal{B} = \{B \subseteq X : 1 \in B\}$.

(a) In the case that $n = 3$, write down all of the elements of $\mathcal{A}$, all of the elements of $\mathcal{B}$, and a bijection f from $\mathcal{A}$ to $\mathcal{B}$.

(b) Give a formula for a function $f : \mathcal{A} \to \mathcal{B}$ that defines a bijection from $\mathcal{A}$ to $\mathcal{B}$ and is valid for every n.

(a)

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f} & \mathcal{B} \\ \emptyset & \longrightarrow & \{1\} \\ \{2\} & \longrightarrow & \{1,2\} \\ \{3\} & \longrightarrow & \{1,3\} \\ \{2,3\} & \longrightarrow & \{1,2,3\} \end{array}$$

$\mathcal{A} = \{\emptyset, \{2\}, \{3\}, \{2,3\}\}$

$\mathcal{B} = \{\{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}\}$

(b) $f(A) = A \cup \{1\}$ for $A \in \mathcal{A}$

Quiz #08 Solutions - Page 2 of 2

3) What is the coefficient of x^3 in $(x+1)^{12}$? Simplify your answer as much as possible.

$$\binom{12}{3} = \frac{12!}{3!\,9!} = \frac{12\cdot 11\cdot 10}{3\cdot 2} = \frac{12\cdot 110}{6} = 220$$

4) Prove that $n2^{n-1} = \sum_{r=1}^{n} r\binom{n}{r}$ for all integers $n \geq 1$.

Let $y=1$ in $(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$ to get

$$(x+1)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$$

Differentiate with respect to x to get

$$n(x+1)^{n-1} = \sum_{r=0}^{n} \binom{n}{r} r x^{r-1}$$

Set $x=1$ to get

$$n2^{n-1} = \sum_{r=0}^{n} \binom{n}{r} r$$

which is equivalent to $n2^{n-1} = \sum_{r=1}^{n} r\binom{n}{r}$

5) Simplify $\sum_{r=1}^{n} \binom{n}{r}$ as much as possible. Your final answer should involve two terms, one of which depends on n.

$$\sum_{r=1}^{n} \binom{n}{r} = \underbrace{\binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} + \binom{n}{0}}_{} - \binom{n}{0}$$

$$= 2^n - \binom{n}{0}$$

$$= 2^n - 1$$

Quiz 9 - Solutions

2) (a) $$\sum_{k=0}^{n} f_{n,p}(k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

$$= (p + (1-p))^n$$

$$= 1^n = 1$$

(b)

0	$\binom{4}{0}(0.7)^0(0.3)^4 = 0.0081$
1	$\binom{4}{1}(0.7)^1(0.3)^3 = 0.0756$
2	$\binom{4}{2}(0.7)^2(0.3)^2 = 0.2646$
3	$\binom{4}{3}(0.7)^3(0.3)^1 = 0.4116$
4	$\binom{4}{4}(0.7)^4(0.3)^0 = 0.2401$

(c)

$$1 \cdot 0.0756 + 2 \cdot 0.2646 + 3 \cdot 0.4116 + 4 \cdot 0.2401$$

$$= 2.8$$

(d) Starting with the Binomial Theorem,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

we take the derivative with respect to x to get

$$n(x+y)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k x^{k-1} y^{n-k}$$

Let $x = p$, $y = 1-p$ to obtain

$$n \cdot 1 = \sum_{k=0}^{n} \binom{n}{k} k p^{k-1} (1-p)^{n-k}.$$

Multiplying by p gives

$$np = \sum_{k=0}^{n} \binom{n}{k} k p^k (1-p)^{n-k}$$

3) Method 1

$$\binom{n}{r}\binom{n-r}{m-r} = \frac{n!}{r!(n-r)!} \cdot \frac{(n-r)!}{(m-r)!\,(n-r-(m-r))!}$$

$$= \frac{n!}{r!\cancel{(n-r)!}} \cdot \frac{\cancel{(n-r)!}}{(m-r)!\,(n-m)!}$$

$$= \frac{n!}{(n-m)!} \cdot \frac{1}{r!\,(m-r)!}$$

this step is trying to make expression look like our target $\binom{n}{m}\binom{m}{r}$.

$$= \frac{n!}{m!\,(n-m)!} \cdot \frac{m!}{r!\,(m-r)!}$$

multiplied by $1 = \frac{m!}{m!}$

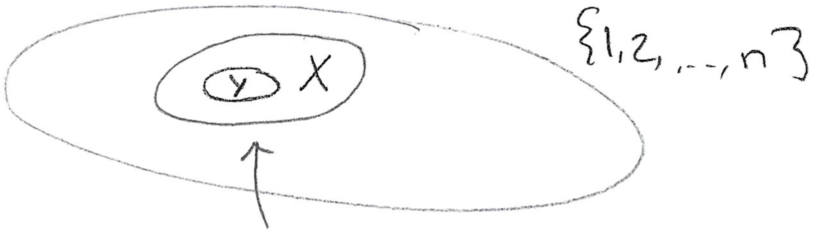
$$= \binom{n}{m}\binom{m}{r}$$

Method 2

Let N be the number of ways to choose an m-combination $X \subseteq \{1, 2, \ldots, n\}$, and choose an r-combination $Y \subseteq X$.

There are $\binom{n}{m}$ choices for X and then $\binom{m}{r}$ choices for $Y \subseteq X$ so $N = \binom{n}{m}\binom{m}{r}$.

However, we can also choose some r-combination for Y first, which can be done in $\binom{n}{r}$ ways, and then a m-combination X that contains Y which can be done in $\binom{n-r}{m-r}$ ways. Hence, $N = \binom{n}{r}\binom{n-r}{m-r}$.



Choosing an m-combination X and an r-combination $Y \subseteq X$

4) $$\sum_{r=m}^{n}\binom{n-m}{n-r} = \binom{n-m}{n-m} + \binom{n-m}{n-(m+1)} + \binom{n-m}{n-(m+2)} + \dots + \binom{n-m}{n-n}$$

$$= \binom{n-m}{n-m} + \binom{n-m}{n-m-1} + \binom{n-m}{n-m-2} + \dots + \binom{n-m}{0}$$

$$= \sum_{j=0}^{n-m}\binom{n-m}{j} = 2^{n-m}$$

5) In the Binomial Theorem, let $y=1$ to get

$$(x+1)^n = \sum_{r=0}^{n}\binom{n}{r}x^r$$

Differentiate twice with respect to x to get

$$n(x+1)^{n-1} = \sum_{r=0}^{n}\binom{n}{r} r x^{r-1}$$

$$n(n-1)(x+1)^{n-2} = \sum_{r=0}^{n}\binom{n}{r} r(r-1) x^{r-2}$$

Now let $x=1$

$$n(n-1)2^{n-2} = \sum_{r=0}^{n}\binom{n}{r}(r^2-r)$$

Now by the distributive law,

$$n(n-1)2^{n-2} = \sum_{r=0}^{n}\binom{n}{r}r^2 - \underbrace{\sum_{r=0}^{n}\binom{n}{r}r}$$

Now we add $+\sum_{r=0}^{n}\binom{n}{r}r$ to both sides

$$n(n-1)2^{n-2} + \underbrace{\sum_{r=0}^{n}\binom{n}{r}r} = \sum_{r=0}^{n}\binom{n}{r}r^2$$

We have shown that this sum is equal to $n2^{n-1}$

$$n^2 2^{n-2} - n2^{n-2} + n2^{n-1} = \sum_{r=0}^{n}\binom{n}{r}r^2$$

$$2^{n-2}\left(n^2 - n + 2n\right) = \sum_{r=0}^{n}\binom{n}{r}r^2$$

$$2^{n-2}\left(n^2+n\right) = \sum_{r=0}^{n}\binom{n}{r}r^2$$

$$n(n+1)2^{n-2} = \sum_{r=0}^{n}\binom{n}{r}r^2$$

Quiz #10 Solution - Page 1 of 3

1) (a) $\binom{11}{3} = \frac{11!}{3!8!} = \frac{11\cdot10\cdot9}{3\cdot2} = 11\cdot5\cdot3 = 55\cdot3 = 165$

(b) $\binom{8}{0} = 1$

2) $\binom{11}{3} = 165$

3) (a) Letting $x=y=1$ in the Binomial Theorem gives

$$(1+1)^n = \sum_{r=0}^{n}\binom{n}{r}1^r1^{n-r} \quad \text{so} \quad 2^n = \sum_{r=0}^{n}\binom{n}{r}$$

(b) Let $x=2, y=1$ in the Binomial Theorem ← details left to you!

(c) Let $x=-1$ and $y=1$ in

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}$$

to get

$$0 = \sum_{k=0}^{n}\binom{n}{k}(-1)^k$$

4) $\displaystyle\sum_{r=1}^{10}\binom{10}{r} = -\binom{10}{0} + \sum_{r=0}^{10}\binom{10}{r} = -1+2^{10} = -1+1024$
$= 1023$

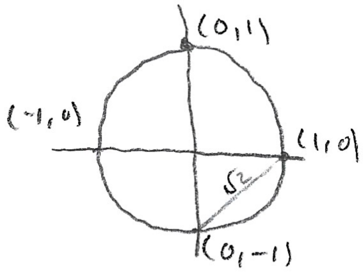
5) (a) $X\setminus A = \{4,5\}$

(b) $\binom{5}{1}+\binom{5}{3}+\binom{5}{5} = 5+10+1 = 16$

(c) $\binom{5}{0}+\binom{5}{2}+\binom{5}{4} = 1+10+5 = 16$

(d) $f(\{5\}) = \{1,2,3,4\}$

6)



By the Pigeonhole Principle, two points must be in the same quadrant and the farthest those two points can be from each other is $\sqrt{2}$.

7) (a) $\frac{100}{12} = 8\frac{4}{12} = 8\frac{1}{3} > 8 \leftarrow \frac{100}{12}$ is strictly greater than 8

By the Pigeonhole Principle, at least 9 people were born in the same month.

(b) The number of 0-1 sequences of length 6 is $2^6 = 64$ and since there are 100 students, at least two picked the same sequence.

(c) $\binom{15}{2} = \frac{15!}{2!\,13!} = \frac{15 \cdot 14}{2} = 15 \cdot 7 = 95$

If you purchase 95 tickets of all different types, you will have a winning ticket.

Quiz #10 Solution - Page 3 of 3

8) (a) $\binom{9}{2,3,4}$ (b) $\binom{6}{1,2,3} = \frac{6!}{1!2!3!} = \frac{6 \cdot 5 \cdot 4}{2} = 60$

9) (a) $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a,b\}$, $\{a,c\}$, $\{b,c\}$, $\{a,b,c\}$

(b)

$00 \longrightarrow \emptyset$
$10 \longrightarrow \{x\}$
$01 \longrightarrow \{y\}$
$11 \longrightarrow \{x,y\}$

(c)

$000 \longrightarrow \emptyset$
$100 \longrightarrow \{x\}$
$010 \longrightarrow \{y\}$
$001 \longrightarrow \{z\}$
$110 \longrightarrow \{x,y\}$
$101 \longrightarrow \{x,z\}$
$011 \longrightarrow \{y,z\}$
$111 \longrightarrow \{x,y,z\}$

10) The subsets of X are

$\emptyset$ $\{1\}$ $\{2\}$ $\{3\}$ $\{1,2\}$ $\{1,3\}$ $\{2,3\}$ $\{1,2,3\}$

Only four subsets do not contain 1 so if a five subsets are chosen one must contain 1.

Study Kit Exam 03 Notes

Announcements
- Friday Quiz 2
- Hw 3 Due Monday (02/13)
- Hw 4 Due Wednesday (02/15)
- Exam 1 Friday (02/17)

Let 'A' be a set.
We say that 'B' is a subset of 'A', denoted $B \subseteq A$, if every element of 'B' is also in 'A' and is denoted $P(A)$.

The set of all possible subsets of 'A' is called the power set of 'A'

Recap Prep For Exam 1
Section 1.1, 1.2, 1.3, 1.4
P^n_r Q^n_r C^n_r

Side Note: P(A)
Higher math may use alternate notation 2^A

Example 01

Sample: $P(\{a,b\}) = \{\underbrace{\emptyset}_{1}, \underbrace{\{a\},\{b\}}_{2}, \underbrace{\{a,b\}}_{1}\}$

$(x+y)^2 = x^2+2xy+y^2$
$(x+y)^3 = x^3+3x^2y+3xy^2+y^2$
Pascal's Triangle Relation ⇒ Making the connection to pascal

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

If $A=\{a,b,c\}$, find $P(A)$

$P(A) = \{\underbrace{\emptyset}_{\text{zero element subset}}, \underbrace{\{a\},\{b\},\{c\}}_{\text{one element subset}}, \underbrace{\{a,b\},\{a,c\},\{b,c\}}_{\text{Two element subset}}, \underbrace{\{a,b,c\}}_{\text{Three element subset}}\}$
(sizes: 1, 2, 3, 1)

can be grouped in sizes

One way to view choosing subsets of $A=\{1,2,3\}$ is by expanding the polynomial $(a+1)(b+1)(c+1)$

$(a+1)(b+1)(c+1) = abc + ab + ac + bc + a + b + c + 1$
$\{a,b,c\}\ \{a,b\}\ \{a,c\}\ \{b,c\}\ \{a\}\ \{b\}\ \{c\}\ \emptyset$

* The key is to make it all be 'x' *
So when we had

$(a+1)(b+1)(c+1) = abc + ab + ac + bc + a + b + c + 1$
$\{a,b,c\}\ \{a,b\}\ \{a,c\}\ \{b,c\}\ \{a\}\ \{b\}\ \{c\}\ \emptyset$

Let $a=b=c=x$

5 in a 5-element set $(x+1)^5$ ➜ $x^3 + xx + xx + xx + x + x + x + 1$
$= x^3 + 3x^2 + 3x + 1$ ⟶ $(x+1)(x+1)(x+1)$

1
1 1 = 2
1 2 1 = 4
1 3 3 1 = 8
1 4 6 4 1 = 16
4

Pascal's Triangle Relation to Binomial Coefficients

Lets make some formulas...

A combination of a set 'A' is a subset of 'A'

For $0 \le r \le n$, an r-combination of 'A' is a subset with 'r' elements

$$P^n_r = \frac{n!}{(n-r)!}$$

If $A=\{a,b,c\}$, then the 2-permutations are
$$P^3_2 = \frac{3!}{(3-2)!}$$

If we ignore order

ab
ac
ba
bc
ca
cb

$$\boxed{C^3_2 = \frac{1}{2!}P^3_2}$$

In general...

$$C^n_r = \frac{P^n_r}{r!}$$

we can rewrite as

These choose numbers C^n_r are called "binomial coefficients" and C^n_r, "C-N-R", is typically written as $\binom{n}{r}$ "n-choose-r"

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Note

$$C^n_r = C^n_{n-r}$$

Since,
$$C^n_r = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!r!} = \frac{n!}{(n-r)!(n-(n-r))!} = C^n_{n-r}$$

this means
$$\binom{n}{r} = \binom{n}{n-r}$$

example

$$\binom{4}{0} + \boxed{\binom{4}{1}} + \boxed{\binom{4}{2}} + \binom{4}{3} + \binom{4}{4} = 2^4$$

Identity
A set with 4 elements and count by size
$\{\_,\_,\_,\_\}$ with a,b,c,d

For $0 \le r \le n$,

$$C_r^n = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

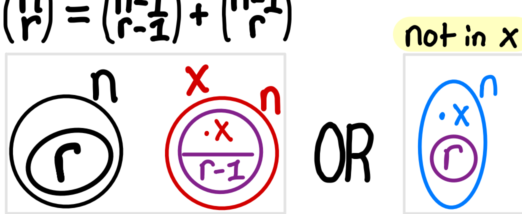
is the number of r-combinations of an n-element set.

For $r>0$ or $r \le 0$. define $\binom{n}{r} = 0$

The numbers $\binom{n}{r}$ are called Binomial Coefficients

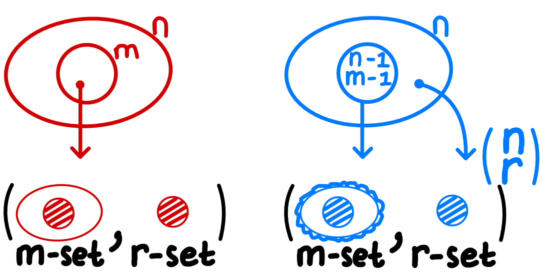
Relations satisfied by $\binom{n}{r}$:

- $\binom{n}{r} = \binom{n}{n-r}$
- $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$



- $\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$
- $\binom{n}{r} = \frac{n-r+1}{r}\binom{n}{r-1}$
- $\binom{n}{m}\binom{m}{r} = \binom{n}{r}\binom{n-m}{m-r}$

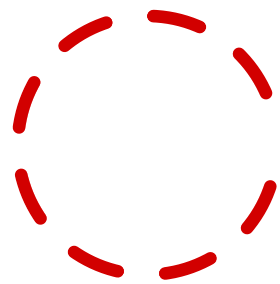
$$\binom{n}{m}\binom{m}{r} = \binom{n}{r}\binom{n-r}{m-r}$$



___ , ___ , ___ , ___

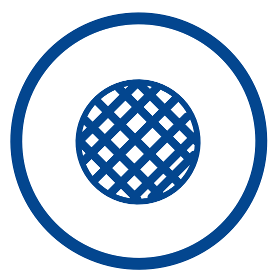
Line-up items

$$P_r^n = n(n-1)\ldots(n-r+1)$$



Circle

$$Q_r^n = \frac{1}{r} P_r^n$$

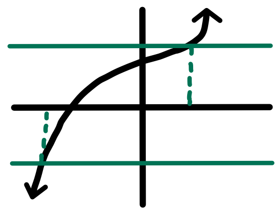


Line-up & divide up with symmetry

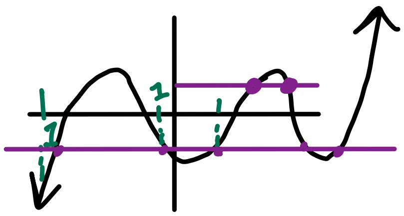
$$C_r^n = \frac{P_r^n}{r!}$$

Let 'A' and 'B' be sets 'A' function $f: A \to B$ is

- **Injective** (1-to-1) if
  $f(a_1) = f(a_2)$ implies $a_1 = a_2$



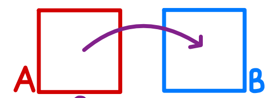
- **Surjective** (onto) if
  for $a \in b$, exists an $a \in A$ with $f(a) = b$



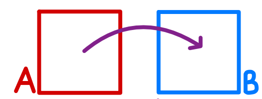
- **Bijective** if f is both
  Injective (1-to-1) and Surjective (onto)

Suppose you have 2 finite sets where it has surjection

$|A| \geq |B|$ $|A| \leq |B|$



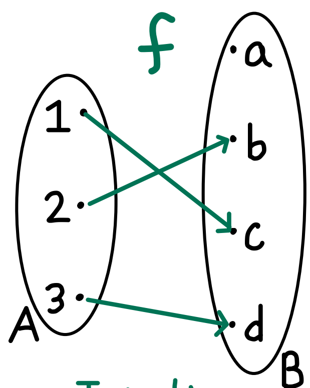
Surjection



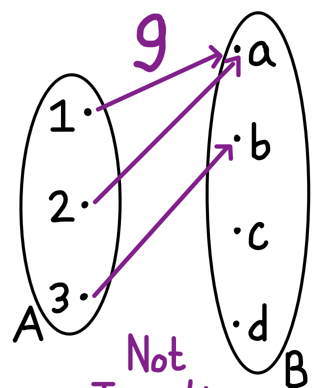
Injection

Example: Injective

Let $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$



Injective

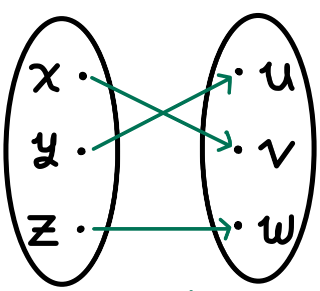


Not Injective

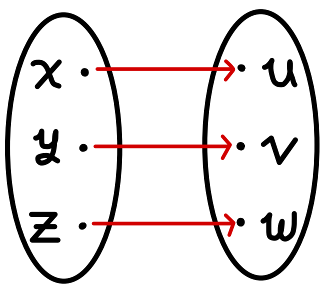
Example: Surjective & injective

Bijective if f is both Injective (1-to-1) and Surjective (onto)

Let $A = \{x, y, z\}$, $B = \{u, v, w\}$

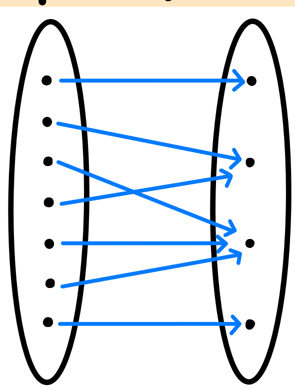


Surjective and injective



Surjective and injective

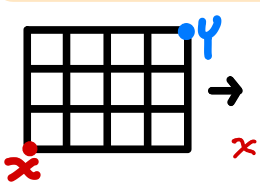
Example: Surjective not injective



Surjective (onto) if
for $a \in b$, exists an $a \in A$ with $f(a) = b$

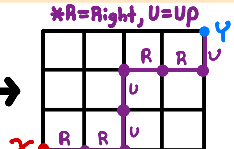
Surjective not injective

Example: Find the number shortest routes from X to Y



→

*R=Right, U=UP



RRUURRU

$\binom{7}{3}$ Selecting for up

$\underbrace{\text{R R U U R R U}}$

Choose three places for when to take an U (up) - step

Selecting for Right $\binom{7}{4}$ OR $\binom{7}{3}$ Selecting for up

Example: Find the number of r-combinations (subset with r-elements) with no consecutive intetgers

Let $x = \{1, 2, 3, \ldots, n\}$,

if $x = \{1, 2, 3, 4, 5, 6\}$ and $r = 3$, the subsets are:

$\{1,3,5\}$ $\{2,4,6\}$ $\{1,3,6\}$ $\{1,4,6\}$

$\boxed{\text{* Use Binomial Cofficient *}}$

SUBJECT: Section 1.5: Injection & Bijections Principles

DATE: 2023/02/27

Recall from Friday, that we want to find the number of subsets of

$$\{1,2,3,\dots,n\}$$

with no consecutive integers

For instance, if $x=\{1,2,3,4,5,6,7\}$, then

$$\{1,3,5\} \to \{1,2,3\}$$
$$\{1,3,6\} \to \{1,2,4\}$$
$$\{1,3,7\} \to \{1,2,5\}$$
$$\{1,4,6\} \to \{1,3,4\}$$
$$\{1,4,7\} \to \{1,3,5\}$$
$$\{1,5,7\} \to \{1,4,5\}$$
$$\{2,4,6\} \to \{2,3,4\}$$
$$\{2,4,7\} \to \{2,3,5\}$$
$$\{2,5,7\} \to \{2,4,5\}$$
$$\{3,5,7\} \to \{3,4,5\}$$

The subsets on the right are exactly all $\binom{5}{3}=10$ subsets $\{1,2,3,4,5\}$ of size 3

Genericly...

Leave same, Subtract 1, Subtract 2

$$\{1,3,5\} \to \{1,2,3\}$$
$$\{1,3,6\} \to \{1,2,4\}$$
$$\{1,3,7\} \to \{1,2,5\}$$

Subtract by the index in set

General Case

Let $x=\{1,2,3,4,\dots,n\}$ and Let $1\le r\le n$

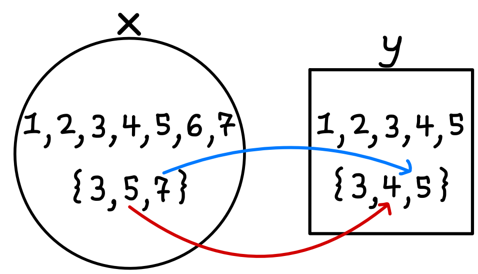
Given a subset $\{S_0,S_1,\dots,S_r\}$ of x with no consecutive elements define,

$$f(\{S_0,S_1,\dots,S_r\})=\{S_0,S_1-1,S_2-2,\dots,\underline{S_r-(r-1)}\}$$

The biggest it can be is $n-r+1$

This output is a subset of size 'r' since $\{S_0,S_1,\dots,S_r\}$ has no consecutive elements

$$n=7 \quad r=3 \quad \binom{n-r+1}{r}=\binom{5}{3}$$



Note that since "S_r" can be n, but not bigger, $S_r-(r-1)$ canbe $n-r+1$, but not bigger

Therefore,

$$|X|=|Y|=\binom{n-r+1}{r}$$

Tricky: all subsets of $\{1,2,3,\dots,n\}$ with no consecutive elements

Easy: all subsets of $\{1,2,3,\dots,n-r+1\}$ of size r

$$|a-b|\ge K$$
$$\binom{n-(K-1)(r-1)}{r}$$

What if we require that $|a-b|\ge 3$ for all a,b in our r-subset of $x=\{1,2,3,\dots,n\}$?

Try $\{1,2,3,4,5,6,7,8\}$, $r=3$

same, -2, -4

$$\{1,4,7\} \to \{1,2,3\}$$
$$\{1,4,8\} \to \{1,2,4\}$$
$$\{1,5,8\} \to \{1,3,4\}$$
$$\{2,5,8\} \to \{2,3,4\}$$

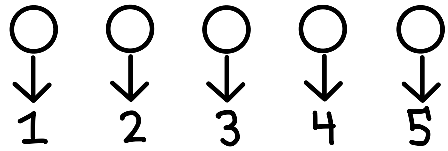
Guess

$$\binom{n-2(r-1)}{r}$$

Via Bijection 'x' counts the 'y'-side

Revisit Bijections

Recall that a bijection is a function $f : A \to B$ that is one-to-one (1:1) and "onto". If A & B are finite and there is a bijection $f : A \to B$; then $|A| = |B|$



Recall: Section 1.5: Injection & Bijections Principles
Let 'A' and 'B' be sets 'A' function $f : A \to B$ is
<u>Bijective</u> if f is both
Injective (1-to-1) and Surjective (onto)

Notation $\mathcal{A}$ and A are different
Notation $\mathcal{B}$ and B are different

<u>Example01</u> Let $X = \{1, 2, 3, \cdots, n\}$, $\mathcal{A} = \{A \subseteq X : |A| = r\}$
$\mathcal{B} = \{B \subseteq X : |B| = n - r\}$

If $n = 5$ and $r = 3$, write-down a bijection from $\mathcal{A}$ and $\mathcal{B}$

Here $X = \{1, 2, 3, 4, 5\}$
$\mathcal{A} = \{A \subseteq X : |A| = \underbrace{3}_{r}\}$
$\mathcal{B} = \{B \subseteq X : |B| = \underbrace{5}_{n-r}\}$

$\mathcal{A}$		$\mathcal{B}$
$\{1,2,3\}$	$\to$	$\{4,5\}$
$\{1,2,4\}$	$\to$	$\{3,5\}$
$\{1,2,5\}$	$\to$	$\{3,4\}$
$\{1,3,4\}$	$\to$	$\{2,5\}$
$\{1,3,5\}$	$\to$	$\{2,4\}$
$\{1,4,5\}$	$\to$	$\{2,3\}$
$\{2,3,4\}$	$\to$	$\{1,5\}$
$\{2,3,5\}$	$\to$	$\{1,4\}$
$\{2,4,5\}$	$\to$	$\{1,3\}$
$\{3,4,5\}$	$\to$	$\{1,2\}$

A formula for this bijection
$f = \mathcal{A} \to \mathcal{B}$ is

$$f(A) = X \setminus A$$

r n-r

SUBJECT: # Revisit Bijections
DATE: 2023/04/03 PAGE#: 02/02

<u>Example02</u> Let $\mathcal{X}=\{1,2,3,\cdots,n\}$

$\mathcal{A}=\{A\subseteq\mathcal{X} : n\notin A\}$
$\mathcal{B}=\{B\subseteq\mathcal{X} : n\notin B\}$

If $n=4$, find a bijection from $\mathcal{A}$ to $\mathcal{B}$ and a formula for your bijection

$\mathcal{A}: \{\emptyset\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}$

$\mathcal{B}: \{4\}, \{1,4\}, \{2,4\}, \{3,4\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}$

$f(A)=A\cup\{4\}$ $\qquad$ $f(a)=\mathcal{X}\setminus A$

also works complementry

Returning to Binomial Theorem and Vandermonde's Identity

$$(1+x)^{m+n}=(1+x)^m(1+x)^n$$

Coefficient

$x^{\emptyset}$ $\quad$ x^r

x^1 $\quad$ x^{r-1}

x^r is $\binom{m+n}{r}$

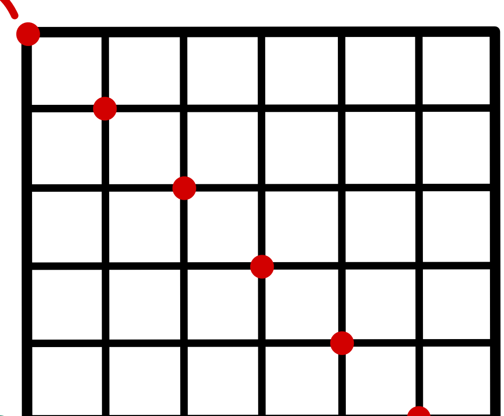
Vandermonde's Identity

$$\binom{m+n}{r}=\sum_{j=\emptyset}^{r}\binom{m}{j}\binom{n}{r-j}$$

For instance if $m=5$, $n=6$, $r=6$, then Vandermonde's Identity

is $\binom{5+6}{6}=\binom{5}{\emptyset}\binom{6}{6}+\binom{5}{1}\binom{6}{5}+\binom{5}{2}\binom{6}{4}+\binom{5}{3}\binom{6}{3}+\binom{5}{4}\binom{6}{2}+\binom{5}{5}\binom{6}{1}$

$\binom{5}{\emptyset}\binom{6}{6}$ $\qquad$ P



O $\qquad$ $\binom{5}{5}\binom{6}{1}$

Number of shortest path is

$$\binom{5+6}{6}=462$$

For any integer $n \geq 0$,

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \dots + \binom{n}{n}x^0 y^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Why is it that the Binomial Coefficients appear in this formula?

Look at some terms in the middle

$$\binom{n}{r} x^{n-r} y^r$$

for some $0 \leq r \leq n$.

Well, $\underset{\text{Factor}}{(x+y)^n} = \underbrace{\underset{1}{(x+y)}\underset{2}{(x+y)}\underset{3}{(x+y)}\dots\underset{n}{(x+y)}}$

we obtain the term $x^{n-r}y^r$ if and only if we choose r-factor to take the 'y' from and the remaining r-n-factors we took the 'x'-form

Example 01 | $(x+y)^4 = \underset{1}{(x+y)}\underset{2}{(x+y)}\underset{3}{(x+y)}\underset{4}{(x+y)}$

$\binom{4}{2}$ ways to obtain xy^3

$\underset{1}{\{1,2,3\}}$ $\underset{2}{\{1,3,4\}}$ $\underset{3}{\{1,2,4\}}$ $\underset{4}{\{2,3,4\}}$

One can prove identites using the Binomial Theorem by choosing specific values of 'x' and 'y'

For instance, if $x=y=1$, $(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k$

$\Rightarrow 2^n = \sum_{k=0}^{n} \binom{n}{k}$ * $\binom{n}{k}$ "k" counts the layers *

{1,2,3}

If $x=1$ and $y=-1$

$(1+(-1))^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} (-1)^k$

$\to 0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$

$\to \binom{4}{0} - \binom{4}{1} + \binom{4}{2} - \binom{4}{3} + \binom{4}{4}$

$= 0$ (zero)

$\varnothing$	NULL SET
{1}, {2}, {3}	$\binom{n}{1}$
{1,2}, {1,3}, {2,3}	$\binom{n}{2}$
{1,2,3}	$\binom{n}{3}$

Example 02

Show that for all positive integers 'n',

$$\sum_{r=1}^{n} r\binom{n}{r} = n2^{n-1}$$

Start with the Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Let $x=1$ in the Binomial Theorem

$$\to (1+y)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} y^k$$

* Power Rule and Chain Rule * $\frac{d}{dy}$ both sides to get

Power Rule: $\frac{d}{dy} y^k = k y^{k-1}$

$$\to (1+y)^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k y^{k-1}$$

Note: Change index

*Set $y=1$ to get...

$$\to n \cdot 2^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k$$

Recall

Power rule (x^n): $\{x^n\}' = nx^{n-1}$

THE CHAIN RULE

Jay Cummings
Calculus 1 Lecture Notes

Theorem.

Theorem 2.64 (*The Chain Rule*). Let f and g be differentiable functions. Then

$$(f(g(x)))' = f'(g(x)) \cdot g'(x).$$

In words: *Take the derivative of the outside, keep everything inside the same, and then multiply by the derivative of the inside.*

SUBJECT: 2.2: The Binomial Theorem Continued DATE: 2023 / 03 / 29 PAGE#:

Quick Warm-up

Does

$$\sum_{r=0}^{\varnothing}\binom{n+1}{r+1} = \sum_{r=0}^{n+1}\binom{n+1}{r} - \sum_{r=0}^{\varnothing}\binom{n+1}{r}$$

* Try writing things out *

LHS

minus-out the Empty-set

$$\sum_{r=0}^{n}\binom{n+1}{r+1} = \binom{n+1}{1} + \binom{n+1}{2} + \cdots + \binom{n+1}{n+1} = 2^{n+1} - 1$$

RHS

$$\sum_{r=0}^{n+1}\binom{n+1}{r} - 1 = \binom{n+1}{0} + \binom{n+1}{1} + \cdots + \binom{n+1}{r+1}$$

Currently we are studying the Binomial Theorem

$$^*(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$$

*Where 'n' is a positive integer

We have used this to prove combinatorial identites

For example, if $x=y=1$ then

$$\rightarrow 2^n = \sum_{k=0}^{n}\binom{n}{k}$$

Vandermonde's Identity

↳ Based on: $(1+x)^{m+n} = (1+x)^m(1+x)^n$

Example 03 | For all integers $m, n, r \geq 1$

$$\binom{m+n}{r} = \sum_{i=0}^{r}\binom{m}{i}\binom{n}{r-i}$$

Two ways to approach this

Approach 01: Binomial Theorem – Algebraic

Key consequence of the Binomial Theorem is that the coefficent of x^i in $(1+x)^m$ is $\binom{m}{i}$ * Pascal's Triangle *

$$(1+x)^2 = 1+2x+x^2$$

$$(1+x)^3 = 1+\underbrace{3x}_{\binom{3}{1}}+\underbrace{3x^2}_{\binom{3}{2}}+x^3$$

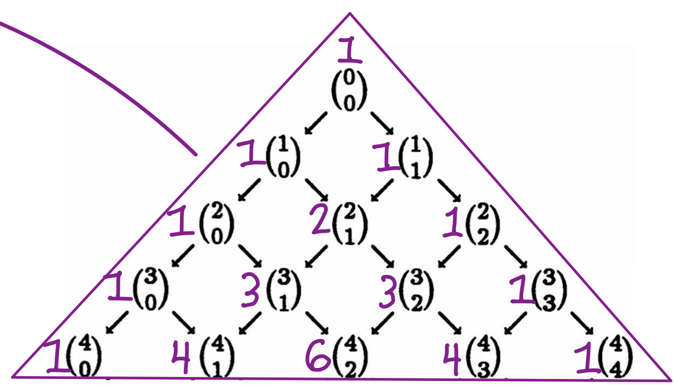
1

1 1

1 2 1

1 $3\binom{3}{1}$ 3 1

1 4 6 4 1



Consider the idenity

$$\underbrace{(1+x)^{m+n}}_{\text{Coefficent of } x^r \text{ is } \binom{m+n}{r}} = (1+x)^m(1+x)^n$$

$\binom{m}{0}1$	$\binom{n}{r}x^r$
$\binom{m}{1}x$	$\binom{n}{r-1}x^{r-1}$
$\binom{m}{2}x^2$	$\binom{n}{r-2}x^{r-2}$
$\vdots$	
$\binom{m}{r}x^r$	$\binom{n}{0}1$

Leverages This Rule

Key consequence of the Binomial Theorem is that the coefficent of x^i in $(1+x)^m$ is $\binom{m}{i}$

For the right hand side, the coefficent of x^r is going to be

$$\binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \binom{m}{2}\binom{n}{r-2} + \cdots + \binom{m}{r}\binom{n}{0} = \sum_{i=0}^{r}\binom{m}{i}\binom{n}{r-i}$$

Coefficent of x^0 {Coefficent of x^r}

$$\rightarrow \binom{m+n}{r} = \sum_{i=0}^{r}\binom{m}{i}\binom{n}{r-i}$$

SUBJECT: Example to help with HW08 DATE: 2023/04/05 PAGE#:

$$\sum_{r=0}^{n} \frac{(-1)^r}{r+1}\binom{n}{r} = \sum_{r=0}^{n}\frac{(-1)^r}{r+1}$$

Be able to recognize identites/prove

- $\sum_{i=0}^{n}\binom{n}{j} = 2^n \Leftarrow$ Count the number of subset $\{1,2,3,\dots,N\}$
- Count subset by size
- $\binom{3}{0}-\binom{3}{1}+\binom{3}{2}-\binom{3}{3}=2^3$
- $\sum_{j=0}^{N}(-1)^j\binom{N}{0}=0$

$$\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$$

$$\sum_{r=0}^{n}\frac{(-1)^r}{r+1}\binom{n}{r}=\sum_{r=0}^{n}\frac{(-1)^r}{r+1}\cdot\frac{n!}{r!(n-r)!}=\sum_{r=0}^{n}(-1)^r\frac{n!}{(r+1)!(n-r)!}$$

$$\binom{n+1}{r-1}=\frac{(n+1)!}{(r+1)!(n+1-(r+1))!}$$

$$\Rightarrow\sum_{r=0}^{n}(-1)^r\frac{n!}{(r+1)!(n-r)!}\cdot\binom{n+1}{n+1}$$

$$\Rightarrow\sum_{r=0}^{n}\frac{(-1)^r}{r+1}\binom{n+1}{n+1}\Rightarrow\frac{1}{r+1}\sum_{r=0}^{n}(-1)^r\binom{n+1}{n+1}$$

* Write it out where $r=0$ *

$$\Rightarrow\frac{1}{r+1}\left((-1)^0\binom{n+1}{0+1}+(-1)^1\binom{n+1}{1+1}+(-1)^2\binom{n+1}{2+1}+\cdots+(-1)^n\binom{n+1}{n+1}\right)$$

Missing: $(-1)^1\binom{n+1}{0}$

$$\Rightarrow\frac{1}{r+1}\left(-\binom{n+1}{0}(-1)^0\binom{n+1}{0+1}+(-1)^1\binom{n+1}{1+1}+(-1)^2\binom{n+1}{2+1}+\cdots+(-1)^n\binom{n+1}{n+1}\underbrace{+\binom{n+1}{0}}\right)$$

$$\boxed{\Rightarrow\frac{1}{r+1}(0+1)=\frac{1}{r+1}\ \blacksquare}$$

* The Identity *

$$\binom{N+1}{R+1}\binom{N}{R}=\binom{N+1}{R+1}$$

The world needs balance if we have $-\binom{n+1}{0}$ at the front then we need $+\binom{n+1}{0}$ in the back

SUBJECT: # Multinomial Theorem

DATE: 2023 / 04 / 05 PAGE#:

Turn our attention to the Multinomial Theorem

Recall the Binomial Theorem

* Instead of terms x & y we rewrite x with subscripts *

$$(x_2+x_2)^n + \sum_{r=0}^{n}\binom{n}{r}x_1^{n-r}x_2^{r}$$

~ for $n \geq 1$ an integer ~

Real quick let us recall how the $\binom{n}{r}$ shows up in the Binomial Theorem

If $n=5$,

$$(x_1+x_2)(x_1+x_2)(x_1+x_2)(x_1+x_2)(x_1+x_2)$$

Factor Number 1 2 3 4 5

What is the coefficent of $x_1^2x_2^3$?

5 factors choose 3 factors where you take x_2 $\binom{5}{3}$

Coefficent of $x_1^2x_2^3$ in (x_1+x_2) is $\binom{5}{3}$

Now, lets us add an x_3

$$(x_1+x_2+x_3)^5$$
$$=(x_1+x_2+x_3)(x_1+x_2+x_3)(x_1+x_2+x_3)(x_1+x_2+x_3)(x_1+x_2+x_3)$$

Factor #: 1 x_1 2 x_3 3 x_1 4 x_1 5 x_2

Can we find the coefficent of:

- $x_1^3x_2x_3$?
- $x_1^2x_2^2x_3$?

$\binom{5}{3}$ 1, 2, 3, 4, 5

1, 2, 3   1, 2, 4   1, 2, 5

1, 3, 4   1, 3, 5

Solution: $\dfrac{5!}{3!1!1!}$

Multinomial Theorem - Continued

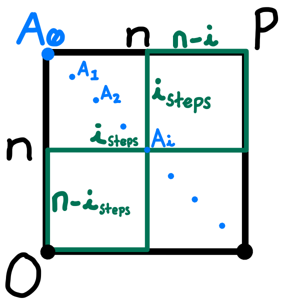
Examples involving identites related to the Binomial Theorem

Example 01 Prove that $\sum_{i=0}^{n}\binom{n}{i}^2=\binom{2n}{n}$

Approach 1
Use the Vandermonde's Identity

$$(1+x)^{m+n}=(1+x)^m(1+x)^n$$

OR **Approach 2**
Shortest Path Method (Lattice Path Argument)



Number of shortest paths through A_i is

$$\binom{n}{i}\binom{n}{i}$$

(steps chosing RIGHT)(steps chosing UP)

Example 02 Prove that

$$\sum_{r=0}^{n}\frac{(2n)!}{(r!)^2((n-r)!)^2}=\binom{2n}{n}^2$$

Do we recognize anything?

$$\sum_{r=0}^{n}\boxed{\frac{(2n)!}{(r!)^2((n-r)!)^2}}=\binom{2n}{n}^2$$

Looks like $C_r^n=\binom{n}{r}=\frac{n!}{r!(n-r)!}$
*Especially the Binomial Coeffcent in the denominator : $r!(n-r)!$

Try expanding things out!

$$\sum_{r=0}^{n}\frac{(2n)!}{(r!)^2((n-r)!)^2}=\binom{2n}{n}^2$$

LHS

$$\Rightarrow\sum_{r=0}^{n}=\frac{n!}{r!(n-r)!}\cdot\frac{n!}{r!(n-r)!}\cdot\frac{(2n)!}{n!n!}$$

$$\Rightarrow\sum_{r=0}^{n}\binom{n}{r}\binom{n}{r}\cdot\frac{(2n)!}{n!n!}\Rightarrow\frac{(2n)!}{n!n!}\sum_{r=0}^{n}\binom{n}{r}^2$$

$$\Rightarrow\frac{(2n)!}{n!n!}\cdot\binom{2n}{n}\Rightarrow\binom{2n}{n}\binom{2n}{n}\Rightarrow\binom{2n}{n}^2\therefore LHS=RHS$$

Announcements : Quiz Monday - 4/10
- Binomial Theorem & Identities
- Bijection & general formulas

Example of identities to Know

$$\sum_{r=0}^{n}\binom{n}{r}=2^n$$

Example 03 Prove that

$$\sum_{r=0}^{n}\binom{2n}{r}=2^{2n-1}+\frac{1}{2}\binom{2n}{n}$$

Try expanding

$$\sum_{r=0}^{n}\binom{2n}{r}=2^{2n-1}+\frac{1}{2}\binom{2n}{n}$$

Well,

$$\sum_{r=0}^{n}\binom{2n}{r}=\binom{2n}{0}+\binom{2n}{1}+\binom{2n}{2}+\cdots+\binom{2n}{n}+\binom{2n}{n+1}+\cdots+\binom{2n}{2n}$$

We could Keep going

$$\frac{1}{2}\binom{2n}{1}+\frac{1}{2}\binom{2n}{2n-1}$$

$$\boxed{\frac{1}{2}\binom{2n}{0}+\frac{1}{2}\binom{2n}{2n}}\quad\boxed{\frac{1}{2}\binom{2n}{2}+\frac{1}{2}\binom{2n}{2n-2}}\boxed{\frac{1}{2}\binom{2n}{2}+\frac{1}{2}\binom{2n}{n}}\longrightarrow$$

$$\Rightarrow\frac{1}{2}\left(\binom{2n}{0}+\binom{2n}{2n}\right)+\frac{1}{2}\left(\binom{2n}{1}+\binom{2n}{2n-1}\right)+\cdots+\frac{1}{2}\left(\binom{2n}{n}+\binom{2n}{n}\right)$$

$$\Rightarrow\frac{1}{2}\sum_{r=0}^{2n}\binom{2n}{r}+\frac{1}{2}\binom{2n}{n}$$

$$\Rightarrow\frac{1}{2}\cdot2^{2n}+\frac{1}{2}\binom{2n}{n}$$

$$\Rightarrow2^{2n-1}+\frac{1}{2}\binom{2n}{n}$$

Let us finish today by talking a bit more about the Multinomial Theorem

Look at

$$(x_1+x_2+x_3)(x_1+x_2+x_3)(x_1+x_2+x_3)(x_1+x_2+x_3)(x_1+x_2+x_3)$$

*If we picked all x_1
Permutations of $x_1x_2x_3$ of length 5

$$=x_1x_1x_1x_1x_1+x_1x_1x_1x_1x_2+x_1x_1x_1x_1x_3+x_1x_1x_1x_2x_1+x_1x_1x_1x_3x_1+\ldots$$

The unique elements can be in a different position

$$\binom{5}{4,1,0}: \begin{array}{l}4-x_1\\1-x_2\\0-x_3\end{array}$$

$$\binom{5}{4,1,0}=\frac{5!}{4!1!0!}$$

Announcements:
- Wednesday Group Quiz 04/12
- Homework 08 due Friday

SUBJECT: Chapter 03: The Pigeonhole Principle & Ramsey Number DATE: 2023 / 04 / 10 PAGE#:

Finish Chapter 02 by stating the Multinomial Theorem

$$(x_1 + x_2 + x_3 + \dots + x_m) = \sum_{(n_1, n_2, \dots, n_m) \in S} \binom{n}{n_1, n_2, n_3} x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m}$$

Where S is all m-tuples of non-negative integers with $n_1 + n_2 + \dots + n_m = n$

Example $(x_1 + x_2 + x_3 + x_4 + x_5)^{100}$

$(20, 15, 5, 10, 50)$

$$\Rightarrow \boxed{\binom{100}{20, 15, 5, 10, 50} x_1^{20} x_2^{15} x_3^{5} x_4^{10} x_5^{50}}$$

Let 'K' & 'n' be positive

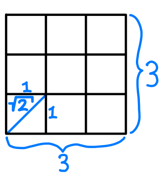
The Piegonhole Principle states that if at least $Kn+1$ objects are placed in n-boxes, then at least one box has at least $K+1$ objects.

There is a chance that in this class there will be 2 students with the same birth month. Given that we have more than 12 students in class.

Example

Show that for any ten points in the plane within a square of side length 3, there are two points within distance $\sqrt{2}$ of each other.

* Trick: form boxes/holes *



SUBJECT: Chapter 03: The Pigeonhole Principle – Continued DATE: 2023 / 04 / 17 PAGE#:

Example Ten teams played in a tourment where each pair of teams play exactly once

Win : +1
Draw : 0
Loss : -1

If at least 7% of the game end in a draw, show that there are at least two teams with the same numbers of points.

There are exactly $\binom{10}{2} = \frac{10 \cdot 9}{2} = 45$ games played and at least $0.70(45) = 31.5$ had to end in a draw $\Rightarrow$ at least 32 games end in a draw

* Pigeonhole : two teams with same scores *

Now the possible scores for a team are:

$-9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$

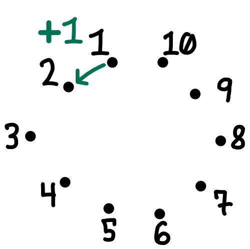
Aiming for a contridiction suppose all ten teams got a different score

* 13 non-draws $\Rightarrow \binom{10}{2} = 45$ — ≥ 32 end in draw; at most 13 did not end in draw

$\underbrace{-9, -8, -7, -6, -5, -4, -3, -2, -1}_{\text{Negative scores}}, 0, \underbrace{1, 2, 3, 4, 5, 6, 7, 8, 9}_{\text{Positive scores}}$

- 10 teams
- 1 team could get 0 point

Note that if five teams have five different positive scores then the number of games that did not end in a draw is <u>at least</u> $1+2+3+4+5 = 15$ which contradicts the at most 13 non-draws



Likewise, if five teams have negative scores, then the number of games that do not end in a draw is again at least 15

Therefore, at least 4 teams got a positive score and at most 4 teams got a non-negative score

$\Rightarrow$ At least two teams got Zero points, a contradiction

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Example Let $X \subseteq \{1, 2, \ldots, 99\}$ with $|X| = 10$.
Show that 'x' contains disjoint subsets Y & Z such that add the elements in Y is the same sum as adding the elements Z.

For instance : $X = \{2, 7, 15, 19, 23, 50, 56, 60, 66, 99\}$

$Y = \{7, 99\}$ $Z = \{50, 56\}$ $7 + 99 = 50 + 56$

$\Rightarrow$ So, $\binom{99}{10}$

Study Kit Exam 03 Homework and Quizzes WORKOUTS