California State University Sacramento - Math 101

Old Exam # 2

1) For integers $0 \le n \le r$, let $s(r, n)$ be the number of ways to arrange r distinct objects aroud n indistinguishable circles such that each circle has at least one object. Recall $s(r, n)$ is called the Stirling number of the first kind.

(a) Give a formula for $s(r, r)$ for $r \ge 1$.

(b) Give a formula for $s(r, r-1)$ for $r \ge 2$.

(c) Give a formula for $s(r, 1)$ for $r \ge 2$.

(d) Find $s(6, 3)$.

2) For $1 \le n \le r$, the Stirling numbers of the first kind satisfy the relation

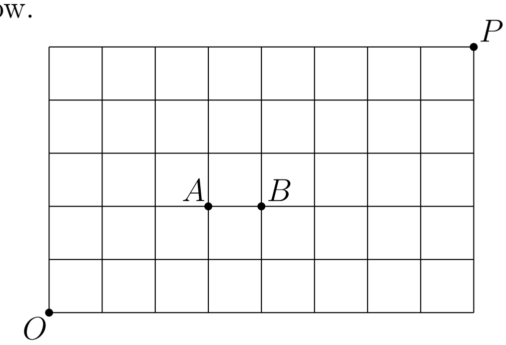
$$s(r, n) = s(r-1, n-1) + (r-1)s(r-1, n).$$

Given $s(8, 2) = 13068$ and $s(8, 3) = 13132$, find $s(9, 3)$.

3) In how many ways can the symbols A, B, C, D, E, F, G, x, and y be lined up so that x and y separated by exactly three capital letters.

4) Find the number of 15-digit binary sequences with twelve 0's and three 1's such that no two 1's are adjacent.

5) In each of the following cases, find the number of shortest routes from O to P in the street network shown below.



(a) The routes must pass through the street AB.

(b) The street AB is closed.

6) Let $X = \{1, 2, 3, \ldots, 60\}$. Find the number of 7-combinations of X that have no consecutive integers.

7) Let r and n be positive integers. Show that $\dfrac{(n+1)(n+2)\cdots(n+r)}{r!}$ is an integer.

8) Let $n > r \geq 0$. Show that $\binom{n}{r} = \frac{n}{n-r}\binom{n-1}{r}$.

9) Find the number of permutations of the letters $a, a, a, b, b, c, c, c, c, d$.

10) There are 3 types of sandwiches, namely lettuce (L), tomato (T), and onion (O) that are available in a restaurant. Assuming there is no limit on the supply of sandwiches of each type, how many ways can you order 7 sandwiches?

11) Recall H_r^n is the number of r-element multi-subsets of $M = \{\infty \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_n\}$. Give a formula for H_r^n.

12) Find the number of solutions to $6x_1 + x_2 + x_3 = 11$ where x_1, x_2, and x_3 are positive integers.

13) For integers $0 \leq n \leq r$, let $S(r, n)$ be the number of ways to put r distinct objects into n identical boxes so that no box is empty. The number $S(r, n)$ is called a *Stirling number of the second kind.*

(a) Give a formula for $S(r, r)$ for $r \geq 1$.

(b) Give a formula for $S(r, r-1)$ for $r \geq 2$.

(c) Give a formula for $S(r, 1)$ for $r \geq 2$.

(d) Find $S(5, 2)$.

14) Suppose we have r <u>distinct</u> objects and n <u>distinct</u> boxes.

(a) If $n \geq r$ and each box can hold at most one object, then how many ways are there to place the objects in the boxes?

(b) Suppose we have exactly 4 objects and exactly 3 boxes and each box can hold any number of objects. How many ways are there to place the objects in the boxes?

(c) Now suppose that we have r <u>identical</u> objects and n <u>distinct</u> boxes. If $n \geq r$ and each box can hold at most one object, then how many ways are there to place the objects in the boxes?