Example 2.3.1. Show that for all integers $n \geq 0$, HW#8

$$\sum_{r=0}^{n} \binom{n}{r} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

Example 2.3.2. Show that for all integers $n \geq 1$,

(i) $\sum_{r=0}^{n}(-1)^r\binom{n}{r} = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n} = 0,$

Example 2.3.2. Show that for all integers $n \geq 1$,

(i) $\sum_{r=0}^{n}(-1)^r\binom{n}{r} = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n} = 0,$ (2.3.2)

(ii) $\binom{n}{0}+\binom{n}{2}+\cdots+\binom{n}{2k}+\cdots = \binom{n}{1}+\binom{n}{3}+\cdots+\binom{n}{2k+1}+\cdots = 2^{n-1}$. (2.3.3)

Proof. By letting $x = 1$ and $y = -1$ in Theorem 2.2.1, we obtain

$$\sum_{r=0}^{n}\binom{n}{r}(-1)^r = (1-1)^n = 0,$$

which is (i). The identity (ii) now follows from (i) and identity (2.3.1). ∎

Remark. A subset A of a non-empty set X is called an *even-element* (resp. *odd-element*) subset of X if $|A|$ is even (resp. odd). Identity (2.3.3) says that given an n-element set X, the number of even-element subsets of X is the same as the number of odd-element subsets of X. The reader is encouraged to establish a bijection between the family of even-element subsets of X and that of odd-element subsets of X (see Problem 2.10).

Example 2.3.4. **(Vandermonde's Identity)** Show that for all $m, n, r \in \mathbf{N}$,

$$\sum_{i=0}^{r}\binom{m}{i}\binom{n}{r-i} = \binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \cdots + \binom{m}{r}\binom{n}{0}$$
$$= \binom{m+n}{r}. \qquad (2.3.5)$$

Second proof. Let $X = \{a_1, a_2, ..., a_m, b_1, b_2, ..., b_n\}$ be a set of $m+n$ objects. We shall count the number of r-combinations A of X.

Assuming that A contains exactly i a's, where $i = 0, 1, ..., r$, then the other $r - i$ elements of A are b's; and in this case, the number of ways to form A is given by $\binom{m}{i}\binom{n}{r-i}$. Thus, by (AP), we have

$$\sum_{i=0}^{r}\binom{m}{i}\binom{n}{r-i} = \binom{m+n}{r}. \quad ∎$$

Example 2.3.3. Show that for all integers $n \in \mathbf{N}$,

$$\sum_{r=1}^{n} r\binom{n}{r} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n \cdot 2^{n-1}.$$

Prove each of the following identities in Problems 24–43, where $m, n \in \mathbf{N}^*$:

24. $\sum_{r=0}^{n} 3^r\binom{n}{r} = 4^n,$

25. $\sum_{r=0}^{n}(r+1)\binom{n}{r} = (n+2)2^{n-1},$

26. $\sum_{r=0}^{n}\frac{1}{r+1}\binom{n}{r} = \frac{1}{n+1}(2^{n+1}-1),$