Study Kit 01- Homework & Exams

CHAPTER 01 - Permutations and Combinations

Section 1.1. Two Basic Counting Principles
Section 1.2. Permuations
Section 1.3. Circular Permutations
Section 1.4. Combinations

Homework Assignment 01 - section 1.1

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Quiz 01 - section 1.1

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Homework Problems

Homework Assignment 01

Homework Assignment 02

Homework Assignment 03

Homework Assignment 04

Chapter 1. Permutations and Combinations

California State University Sacramento - Math 101

Homework Assignment 1

1) Let $A = \{-3, -2, -1, \ldots, 5, 6, 7\}$.

(a) Is $1 \in A$?

(b) Is $\frac{1}{2} \in A$?

(c) Find $|A|$.

(d) If $B = \{4, 6, 8, 10\}$, find $A \cup B$ and $A \cap B$.

2) Suppose $A_1 = \{1, 2, 3\}$, $A_2 = \{3, 4, 5\}$, and $A_3 = \{4, 5, 6\}$.

(a) Find $A_1 \cup A_2 \cup A_3$.

(b) Find $A_1 \cap A_2 \cap A_3$.

(c) True or False: $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3|$.

(d) True or False: $|A_1 \cap A_2 \cap A_3| = |A_1||A_2||A_3|$.

3) Suppose $A_1, A_2, \ldots, A_5$ are pairwise disjoint sets with $|A_i| = i$ for $1 \le i \le 5$. Determine

$$\left|\bigcup_{i=1}^{5} A_i\right|.$$

4) Find sets $A_1, A_2, \ldots, A_5$ such that $|A_i| = i$ for $1 \le i \le 5$ and

$$\left|\bigcup_{i=1}^{5} A_i\right| = 5.$$

5) If $A = \{x : 3 \le x \le 10\}$ and $\mathbb{Z}$ is the set of all integers, find

$$|A \cap \mathbb{Z}|$$

Chapter 1. Permutations and Combinations

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Homework Assignment 2

1) Problem 1 on page 50

2) Problem 10 on page 51

3) Problem 11 on page 51

4) Problem 12 on page 51

Q1) **1. Find the number of ways to choose a pair $\{a, b\}$ of distinct numbers from the set $\{1, 2, ..., 50\}$ such that**

(i) $|a - b| = 5$; (ii) $|a - b| \leq 5$.

Q2) **10. Find the number of common positive divisors of 10^{40} and 20^{30}.**

Q3) **11. In each of the following, find the number of positive divisors of n (inclusive of n) which are multiples of 3:**

(i) $n = 210$; (ii) $n = 630$; (iii) $n = 151200$.

Q4) **12. Show that for any $n \in \mathbf{N}$, the number of positive divisors of n^2 is always odd.**

Q **5)** Find the number of ordered pairs (x, y) of integers such that $x^2 + y^2 \leq 4$.
Remark: This problem is similar to Example 1.1.2.

Q **6)** Find the number of sequences $a_1a_2a_3$ of length 3 where $a_i \in \{0, 1, 2, 3, 4\}$.
Remark: This is a special case of Example 1.1.4.

Q **7)** Let $X = \{1, 2, \ldots, 10\}$ and let

$$S = \{(a, b, c) : a, b, c \in X, a < b \text{ and } a < c\}.$$

Find $|S|$.
Remark: This problem is similar to Example 1.1.6.

Chapter 1. Permutations and Combinations
Section 1.2. Permutations

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Homework Assignment 3

1) Example 1.2.1 on page 6.

2) Example 1.2.2 on page 7.

3) Example 1.2.3 on page 8.

4) Example 1.2.4 on page 9.

5) Problem 4 on page 50.

6) Problem 2(i) and 2(ii) on page 50.

7) Problem 14(i), 14(ii), 14(iii), and 14(iv) on page 51.

Q1) **Example 1.2.1.** Let $A = \{a, b, c, d\}$. All the 3-permutations of A are

Q2) **Example 1.2.2.** Let $E = \{a, b, c, ..., x, y, z\}$ be the set of the 26 English alphabets. Find the number of 5-letter words that can be formed from E such that the first and last letters are distinct vowels and the remaining three are distinct consonants.

Q3) **Example 1.2.3.** There are 7 boys and 3 girls in a gathering. In how many ways can they be arranged in a row so that

(i) the 3 girls form a single block (i.e. there is no boy between any two of the girls)?

(ii) the two end-positions are occupied by boys and no girls are adjacent?

Q4) **Example 1.2.4.** Between 20000 and 70000, find the number of even integers in which no digit is repeated.

Q5) 4. How many 5-letter words can be formed using A, B, C, D, E, F, G, H, I, J,

(i) if the letters in each word must be distinct?

(ii) if, in addition, A, B, C, D, E, F can only occur as the first, third or fifth letters while the rest as the second or fourth letters?

Q6) 2. There are 12 students in a party. Five of them are girls. In how many ways can these 12 students be arranged in a row if

(i) there are no restrictions?

(ii) the 5 girls must be together (forming a block)?

Q7) 14. Let $n, r \in \mathbf{N}$ with $r \leq n$. Prove each of the following identities:

(i) $P^n_r = nP^{n-1}_{r-1}$,

(ii) $P^n_r = (n - r + 1)P^n_{r-1}$,

(iii) $P^n_r = \frac{n}{n-r}P^{n-1}_r$, where $r < n$,

(iv) $P^{n+1}_r = P^n_r + rP^n_{r-1}$,

(v) $P^{n+1}_r = r! + r(P^n_{r-1} + P^{n-1}_{r-1} + \cdots + P^r_{r-1})$.

1.3. Circular Permutations
1.4. Combinations

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Homework Assignment 4

1) Example 1.3.2

2) Example 1.3.3

3) Problem 6 on page 51

4) Example 1.4.1

5) Example 1.4.2

Q1) **Example 1.3.2.** In how many ways can 5 boys and 3 girls be seated around a table if

(i) there is no restriction?

(ii) boy B_1 and girl G_1 are not adjacent?

(iii) no girls are adjacent?

Q2) **Example 1.3.3.** Find the number of ways to seat n married couples around a table in each of the following cases:

(i) Men and women alternate;

(ii) Every woman is next to her husband.

Q3) 6. Find the number of *odd* integers between 3000 and 8000 in which no digit is repeated.

Q4) **Example 1.4.1.** Prove that

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}, \tag{1.4.3}$$

where $n, r \in \mathbf{N}$ with $r \leq n$.

Q5) **Example 1.4.2.** By Example 1.1.4, there are 2^7 binary sequences of length 7. How many such sequences are there which contain 3 0's and 4 1's?

Homework Attempts/Workouts

Homework Assignment 01
Homework Assignment 02
Homework Assignment 03
Homework Assignment 04

SUBJECT: Homework Assignment 01 DATE: 2023/01/27 PAGE#:

1) Let $A = \{-3, -2, -1, \ldots, 5, 6, 7\}$.

(a) Is $1 \in A$? – 1 is an element of A ∴ True

(b) Is $\frac{1}{2} \in A$? – 1/2 is not an element of A ∴ False

(c) Find $|A|$. – Cardinality of set A is 11

(d) If $B = \{4, 6, 8, 10\}$, find $A \cup B$ and $A \cap B$.

$A = \{-3,-2,-1,0,1,2,3,4,5,6,7\}$ $B = \{4,6,8,10\}$

(Union)

$A \cup B = \boxed{\{-3,-2,-1,0,1,2,3,4,5,6,7,8,10\}}$

(Intersection)

$A \cap B = \boxed{\{4,6\}}$

2) Suppose $A_1 = \{1, 2, 3\}$, $A_2 = \{3, 4, 5\}$, and $A_3 = \{4, 5, 6\}$.

(a) Find $A_1 \cup A_2 \cup A_3$.

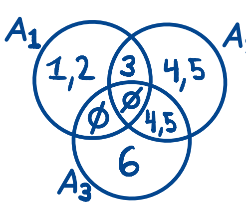
$A_1 = \{1,2,3\}$, $A_2 = \{3,4,5\}$, $A_3 = \{4,5,6\}$

$A_1 \cup A_2 \cup A_3 \Rightarrow \boxed{\{1,2,3,4,5,6\}}$

(b) Find $A_1 \cap A_2 \cap A_3$.

$A_1 = \{1,2,3\}$, $A_2 = \{3,4,5\}$, $A_3 = \{4,5,6\}$

$A_1 \cap A_2 \cap A_3 \Rightarrow$



∴ $\boxed{\varnothing \text{ Null Empty Set}}$

(c) True or False: $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3|$.

$A_1 = \{1,2,3\}$, $A_2 = \{3,4,5\}$, $A_3 = \{4,5,6\}$

cardinality

$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3|$

$6 = 3 + 3 + 3$

$6 \neq 9$

$\boxed{\therefore \text{False.}}$

(d) True or False: $|A_1 \cap A_2 \cap A_3| = |A_1||A_2||A_3|$.

$A_1 = \{1,2,3\}$, $A_2 = \{3,4,5\}$, $A_3 = \{4,5,6\}$

cardinality

$|A_1 \cap A_2 \cap A_3| = |A_1||A_2||A_3|$

$\varnothing = 3 \cdot 3 \cdot 3$

$\varnothing \neq 81$

$\boxed{\therefore \text{False.}}$

3) Suppose $A_1, A_2, \ldots, A_5$ are pairwise disjoint sets with $|A_i| = i$ for $1 \le i \le 5$.

Determine $\left| \bigcup_{i=1}^{5} A_i \right|$ Pairwise Disjoint

$\left| \bigcup_{i=1}^{5} A_i \right| \rightarrow$

$A_1 = \{1\} \rightarrow |A_i| = 1$

$A_2 = \{2,3\} \rightarrow |A_i| = 2$

$A_3 = \{4,5,6\} \rightarrow |A_i| = 3$

$A_4 = \{7,8,9,10\} \rightarrow |A_i| = 4$

$A_5 = \{11,12,13,14,15\} \rightarrow |A_i| = 5$

$\rightarrow A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5$

$\therefore \left| \bigcup_{i=1}^{5} A_i \right| \rightarrow |\{1,2,3,\ldots,13,14,15\}| = \boxed{15.}$

(Where $|A_i| = i$ for $1 \le i \le 5$)*

4) Find sets $A_1, A_2, \ldots A_5$ such that $|A_i| = i$ for $1 \le i \le 5$

and $\left| \bigcup_{i=1}^{5} A_i \right| = 5$

$\rightarrow |A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5| = 5$

(Where $|A_i| = i$ for $1 \le i \le 5$)*

$\therefore \left| \bigcup_{i=1}^{5} A_i \right| \rightarrow$

$A_1 = \{1\}$

$A_2 = \{1,2\}$

$A_3 = \{1,2,3\}$

$A_4 = \{1,2,3,4\}$

$A_5 = \{1,2,3,4,5\}$

5) If $A = \{x : 3 \le x \le 10\}$ and $\mathbb{Z}$ is the set of all integers, find $|A \cap \mathbb{Z}|$.

$A = \{3,4,5,6,7,8,9,10\}$

$\mathbb{Z} = \{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$

$\boxed{\therefore |A \cap \mathbb{Z}| \rightarrow \{3,4,5,6,7,8,9,10\} = 8}$

SUBJECT: Homework Assignment 02 DATE: 2023/02/05

Note to Professor
Noticed it's Unordered $\{a,b\}$ NOT ordered (a,b) So...
1.i = 45
and
1.ii = 235

(i) $|a-b| = 5$

1. Find the number of ways to choose a pair $\{a, b\}$ of distinct numbers from the set $\{1, 2, ..., 50\}$ such that
(i) $|a-b| = 5$; (ii) $|a-b| \le 5$.

a & b can't be the Same number from Set $\{1,2,3,\ldots 50\}$

- min: 1, 6
- max: 45, 50
- from min to max we have 45 possibilites of (a,b) pairs

If given a restriction of a>b; then, the solutions we can have follows this tuple pattern: (6,1), (7,2), ...(50,45)

Because we are taking the absolute value of the difference of both 'a' and 'b' the order of either 'a' or 'b' does not matter, that is 45 more possible pairs of (a,b)

So, 45 possibilites (a>b) + 45 possibilites (a<b) = $\boxed{\text{90 possible ways of (a,b) pairings}}$

(ii) $|a-b| \le 5$

Let the smaller of both 'a' and 'b' be 'n' where $1 \le n \le 45$, then there are 5 pairs of distinct numbers from $\{1,2,3,\ldots,50\}$: $\{n,n+1\}, \{n,n+2\}, \{n,n+3\}, \{n,n+4\}, \{n,n+5\}$. Gives us: $(45)(5) = 225$

n plus offset $1 \to 5$ is to account for distinct numbers for either 'a' or 'b'.
→ 45 (where both 'a' and 'b' are the smallest number in the set)
→ 5 (five pairs of distinct numbers from set $\{1,2,3,\ldots,50\}$

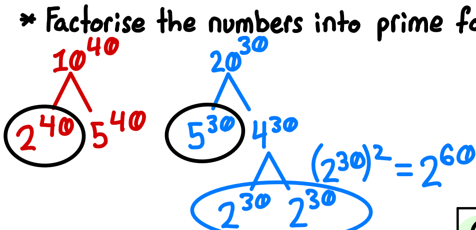
$45-(45+1)=1$
$45-(45+2)=2$
⋮
$45-(45+5)=5$

When $46 \ge n < 50$
- $n = 46$ there are 4 pairs: (46,47), (46,48), (46,49), (46,50)
- $n = 47$ there are 3 pairs: (47,48), (47,49), (47,50)
- $n = 48$ there are 2 pairs: (48,49), (48,50)
- $n = 49$ there is 1 pair: (49,50)

(a<b): $[(45)(5)]+4+3+2+1 = 235$ possibilities.
Because we have the absolute value account (a>b):
meaning we have 235 more possibilities. $\boxed{\text{So in total } 235+235=470}$

10. Find the number of common positive divisors of 10^{40} and 20^{30}.

* Factorise the numbers into prime factors *



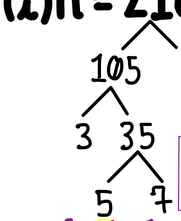
Recall — Prime factorization form
Let $n = P_1^{K_1}, P_2^{K_2}, \ldots, P_r^{K_r}$
then the positive divisor of $n = (K_1+1)(K_2+1)\ldots(K_r+1)$

$\boxed{\text{So } (40+1)(30+1) = 1271}$

11. In each of the following, find the number of positive divisors of n (inclusive of n) which are multiples of 3:

(i) $n = 210$; (ii) $n = 630$; (iii) $n = 151200$.

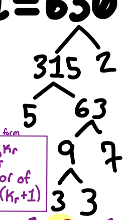
(i) $n=210$



Recall — Prime factorization form
Let $n = P_1^{K_1}, P_2^{K_2}, \ldots, P_r^{K_r}$
then the positive divisor of $n = (K_1+1)(K_2+1)\ldots(K_r+1)$

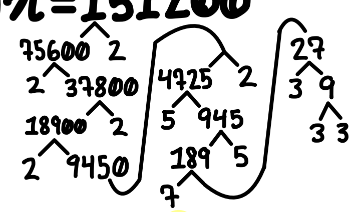
→ $2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1$
→ $(1+1)(1)(1+1)(1+1)$
→ $\boxed{2 \cdot 1 \cdot 2 \cdot 2 = 8}$

(ii) $n=630$



→ $2^1 \cdot 3^2 \cdot 5^1 \cdot 7^1$
→ $(1+1)(2)(1+1)(1+1)$
→ $\boxed{2 \cdot 2 \cdot 2 \cdot 2 = 16}$

(iii) $n=151200$



→ $2^5 \cdot 3^3 \cdot 5^2 \cdot 7^1$
→ $(5+1)(3)(2+1)(1+1)$
→ $\boxed{6 \cdot 3 \cdot 3 \cdot 2 = 108}$

12) Show that for any $n \in \mathbb{N}$, the number of positive divisors of n^2 is always odd

If prime factorization of $n = P_1^{K_1}, P_2^{K_2}, \ldots, P_r^{K_r}$ then $n^2 = (P_1^{K_1})^2 (P_2^{K_2})^2 \ldots (P_r^{K_r})^2$ the resulting positive divisor of $n^2 = (2K_1+1)(2K_2+1)\ldots(2K_r+1)$ follows the general form and the definition of odd $(2n+1)$.

5) Find the number of ordered pairs (x,y) of integers such that $x^2+y^2 \le 4$.

case A	case B	case C	case D	case E
$x^2+y^2=0$	$x^2+y^2=1$	$x^2+y^2=2$	$x^2+y^2=3$	$x^2+y^2=4$
(0,0)	(0,1)(1,0) (0,-1)(-1,0)	(-1,1)(1,1) (-1,-1)(1,-1)	NONE	(0,2)(2,0) (0,-2)(-2,0)

So in total $\boxed{1+4+4+0+4 = 13}$

6) Find the number of sequences $a_1a_2a_3$ of length 3 where $a_i \in \{0,1,2,3,4\}$.

3 groupings $(a_1a_2a_3)$ of 5 elements (0,1,2,3,4):

So $\boxed{5 \cdot 5 \cdot 5 = 125}$

*Think $P_r^n = \frac{n!}{(n-r)!}$ = # of Permutations

7) Let $X = \{1,2,3,\ldots,10\}$ and let $S\{(a,b,c): a,b,c \in X, a<b \text{ and } a<c\}$. Find $|S|$.

Key hint is $c \in X$, so we can try

$$\sum_{x=1}^{10} (x^b \cdot x^c) \to [(1\cdot1)+(2\cdot2)+(3\cdot3)+\ldots+(10\cdot10)] = 385$$

Homework Assignment 03

1) Example 1.2.1 on page 6.

Let $A=\{a,b,c,d\}$.
Find all the 3-permutations of set A.

Note
When finding permutations we use formula
$P^n_r = \frac{n!}{(n-r)!}$ Where 'n' is the set or population and 'r' is the subset of 'n' or sample set

* n is the cardinality of set A, so $n=4$
* r is the subset of A where have sets of length 3, so $r=3$

So... $P^4_3 = \frac{4!}{(4-3)!} = \frac{4\cdot3\cdot2\cdot1}{1} = \boxed{24}$

All together, we have 24 subsets permutations of set A's elements in sets of length 3.

2) Example 1.2.2 on page 7.

Let E = {a, b, c, ...x, y, z} be the set of 26 English alphabets. Find the number of 5-letter words that can be formed from E such that the first and last letters are distinct vowels and the remaining three are distinct consonants.

The english alphabet consist of 5 vowels (a,e,i,o,u) and 21 consonants

- We want permutations of 2 vowels from the "vowel set" (subset of E)
- We want permutations of 3 consonants from the "consonants set" (subset of E)
- Pattern → | Vowel | consonant | consonant | consonant | Vowel |
- Finding Permutations: $P^n_r = \frac{n!}{(n-r)!}$

$P^5_2 = \frac{5!}{(5-2)!} \rightarrow \frac{5\cdot4\cdot3\cdot2}{3\cdot2} \rightarrow \frac{120}{6} = 20$ vowel permutations

$P^{21}_3 = \frac{21!}{(21-3)!} \rightarrow \frac{51,090,942,171,709,440,000}{6,402,373,705,728,000} = 7980$ consonant permutations

So, for every distinct vowel pairings we have a distinct 3 character tuple

$P^5_2 \times P^{21}_3 = [(20)(7980)] = \boxed{159,600}$ unique words can be generated

3) Example 1.2.3 on page 8.

There are 7 boys and 3 girls in a gathering.
In how many ways can they be arranged in a row so that:
(i) The 3 girls from a single block
−no boy between any two of the girls

B_1	B_2	B_3	B_4	G_1	G_2	G_3	B_5	B_6	B_7

→

B_1	B_2	B_3	B_4	Girls Only	B_5	B_6	B_7

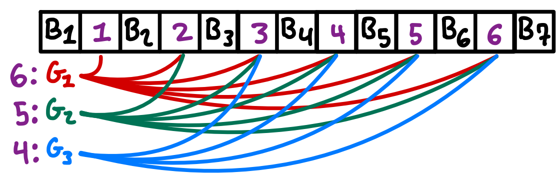
1 unit

- Block of girls can be counted 1 unit
- Girls amongst themselves can be arranged 3!

So, instead of 10 independent seating the is 7 single and 1 group seat

$(7+1)! \times 3! = \boxed{241,920}$

(3.ii) The two end-positions are occupied by boys and no girls are adjacent



The boys have 7! ways to seat themselves G_1 has 6 possibilities, G_2 has 5 possibilites, and G_3 has 4 possibilities.

So we have $7!\cdot6\cdot5\cdot4 = \boxed{604,800 \text{ ways}}$

4) Example 1.2.4 on page 9.

Between 20000 and 70000, find the number of even integers in which no digit is repeated.

20000 ⟶ 70000

a	b	c	d	e

a ↑ {2,3,4,5,6} e ↑ {0,2,4,6,8}

Case 01: first digit is even
- 3 choices for first digit {2,4,6}
- To account for 'a' being 1 of the five in set {0,2,4,6,8} Leaves us with 4 choices
- 'bcd' can range from [0,9] But we need to factor for leading & tail duplicates

So for case 01: $3\times P^{(10-2)}_3 \times 4 \rightarrow 3\times P^8_3\times 4 = 4032$

Case 02: leading digit is odd
- 'bcd' can range from [0,9]. But we need to account for leading & tail duplicates
- 2 choices for first digit: {3,5}
- Don't have to worry 'a' = 'e'

So for case 02: $3\times P^{(10-2)}_3 \times 5 \rightarrow 3\times P^8_3 \times 5 = 5040$

All together we have $\boxed{4032+5040=9072}$

5) Problem 4 on page 50.

How many 5-letter words can be formed using A,B,C,D,E,F,G,H,I,J

(i) If the letters in each word must be distinct

$\boxed{P^{10}_5 \rightarrow \frac{10!}{(10-5)!} = 30,240}$

(ii) If, in addition, A,B,C,D,E,F can only occur as the 1st, 3rd, or 5th letters & the rest as 2nd or 4th

A,B,C,D,E,F,G,H,I,J : | 1 | 2 | 3 | 4 | 5 |

$\boxed{P^6_3 \times P^4_2 \rightarrow \left(\frac{6!}{(6-3)!}\right)\left(\frac{4!}{(4-2)!}\right) = 1440}$

See Back

6) Problem 2(i) and 2(ii) on page 50.

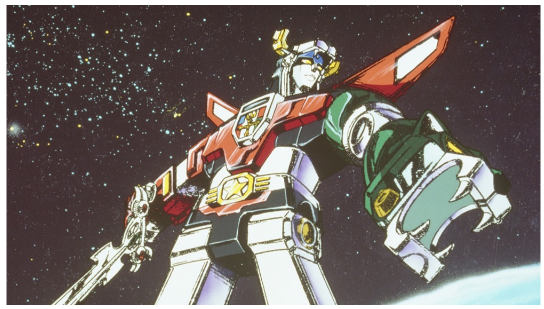
There are 12 students in a party. Five of them are girls.
In how many ways can these 12 students be arranged if...
(i) there are no restrictions?

If no restrictions then,
12!, there are ~4.79×10^8 possible ways

(ii) the 5 girls must be together (forming Voltron)?

* Voltron can be counted as 1

* The pilots of Voltron can switch lions



So... $(7+1)!\,5!$

$\rightarrow 8!\,5! = \boxed{4{,}838{,}400}$ Possible ways

7) Problem 14(i), 14(ii), 14(iii), and 14(iv) on page 51.

14. Let $n, r \in \mathbf{N}$ with $r \leq n$. Prove each of the following identities:

(i) $P^n_r = nP^{n-1}_{r-1}$,

(ii) $P^n_r = (n-r+1)P^n_{r-1}$,

(iii) $P^n_r = \frac{n}{n-r}P^{n-1}_r$, where $r < n$,

(iv) $P^{n+1}_r = P^n_r + rP^n_{r-1}$,

Note

When finding permutations we use formula

$P^n_r = \frac{n!}{(n-r)!}$ Where 'n' is the set or population and 'r' is the subset of 'n' or sample set

(i) $P^n_r = nP^{n-1}_{r-1}$

$P^3_3 \stackrel{?}{=} 3\cdot P^{(3-1)}_{(3-1)}$

$\rightarrow \frac{3!}{0!} \stackrel{?}{=} 3\left(\frac{2!}{0!}\right)$

$\rightarrow 6 \stackrel{?}{=} (2\cdot 3) \rightarrow 6$

LHS = RHS

∴ True

(ii) $P^n_r = (n-r+1)P^n_{r-1}$

$P^3_3 \stackrel{?}{=} (3-3+1)P^3_{3-1}$

$\rightarrow 6 \stackrel{?}{=} P^3_{3-1}$

$\rightarrow 6 \stackrel{?}{=} \frac{3!}{(3-2)!}$

$\rightarrow 6 \stackrel{?}{=} 6$

LHS = RHS

∴ True

(iii) $P^n_r = \frac{n}{n-r}P^{n-1}_r$, where $r < n$

RHS : $\frac{n}{n-r}P^{n-1}_r$

$\rightarrow$

(iv) $P^{n+1}_r = P^n_r + rP^n_{r-1}$

RHS

$\rightarrow P^n_r + rP^n_{r-1}$

$\rightarrow \frac{n!}{(n-r)!} + r\frac{n!}{(n-r+1)!}$

$\rightarrow \frac{(n-r+1)}{(n-r+1)}\cdot\frac{n!}{(n-r)!} + r\frac{n!}{(n-r+1)!}$

$\rightarrow \frac{(n-r+1)n!}{(n-r+1)!} + \frac{r\cdot n!}{(n-r+1)!}$

$\rightarrow \frac{(n-r+1)n! + r\cdot n!}{(n-r+1)!}$

$\rightarrow \frac{n!(n-r+1+r)}{(n-r+1)!}$

$\rightarrow \frac{(n+1)!}{(n-r+1)!} = P^{(n+1)}_r$

Recall

$P^n_r = \frac{n!}{(n-r)!}$

LHS = RHS

$P^{n+1}_r = P^{n+1}_r$ ∴ True

Homework Assignment 04

Recall

In general, if Q^n_r is the number of arrangements of 'r' elements placed around a circle chosen from a set of 'n' elements, then

$$Q^n_r = \frac{P^n_r}{r} = \frac{\frac{n!}{(n-r)!}}{r} = \frac{n!}{r\cdot(n-r)!}$$

* Note that if $r=n$, then

$$Q^n_r = \frac{n!}{n(n-n)!} = \frac{n!}{n\cdot 0} = \frac{n!}{n\cdot 1} = \frac{n!}{n} = (n-1)!$$

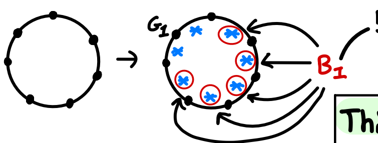
1) Example 1.3.2: In how many ways can 5 boys and 3 girls be seated around a table if

(i) there is no restriction?

$$\boxed{\frac{8!}{8} = 7!}$$

(ii) boy B_1 and girl G_1 are not adjacent?

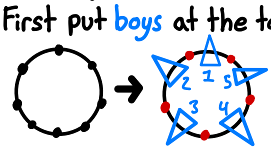
Let us place all except B_1 there $\frac{7!}{7} = 6!$ ways to place the seven people around the table (we are not placing B_1 yet)



This gives a total count of $6!\times 5$

(iii) no girls are adjacent?

First put boys at the table



$\frac{5!}{5} = 4!$ Possibilites

$5\cdot 4\cdot 3$ ways to place girls

$\Rightarrow$ Total is $4!\cdot 5\cdot 4\cdot 3 = \boxed{1440}$

2) Example 1.3.3 - Find the number of ways to seat 'n' married couples around a table in each of the following cases:

(i) Men and women alternate

For every partner in a pairing can be arranged $(n-1)!$ ways, for every pairing has 'n' many significant other, so in total we have:

$$\boxed{(n-1)!\,n!}$$

(ii) Every women is next to her husband

Can be thought as binary bits so,

$$\boxed{(n-1)!\,2^n}$$

Study #3 - Not sure why its 1232 & not 1512

*3) Exercise 1.6 - Find the number of odd integers between 3000 and 8000 in which no digit is repeated

Some case analysis would help...

$3{,}000 \longrightarrow 8{,}000$

a	b	c	d

a: {3,4,5,6,7,8} d: {1,3,5,7,9}

Case 01: First digit is even (Can't do 8, upper Limit)

- 2 choices for first digit: 4, 6
- Don't have to worry about 'a' = 'd'
- 'bc' can range from [0,9] but we need to factor that both the leading and tail digit are duplicates

So for case 01: $2\times P^{10-2}_2 \times 5 \Rightarrow 2\times P^8_2\times 5 = 840$

Case 02: Leading digit is odd

- 3 choices for first digit: 3, 5, 7
- Account for 'a' being one of the five in set {1,3,5,7,9} leaves us with 4 choices
- 'bc' can range from [0,9] but we need to factor that both the leading and tail digit is repeated

So for case 02: $3\times P^{10-2}_2\times 4 \Rightarrow 3\times P^8_2\times 4 = 672$

In total $840 + 672 = 1512$

4) Example 1.4.1 - Prove that $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$, where 'n', $r\in\mathbb{N}$ with $r\leq n$

Recall: $C^n_r = \frac{P^n_r}{r!}$ we can rewrite as $C^n_r = \frac{n!}{r!(n-r)!}$

By algebraic proof...

$$\binom{n-1}{r-1} + \binom{n-1}{r} = \frac{(n-1)!}{(r-1)!(n-r)!} + \frac{(n-1)!}{r!(n-1-r)!}$$

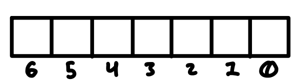
$$\rightarrow \frac{(n-r)!\,r + (n-1)!(n-r)}{r!(n-r)!}$$

$$\rightarrow \frac{(n-1)!(r+n-r)}{r!(n-r)!}$$

$$\boxed{\rightarrow \frac{n!}{r!(n-r)!} = \binom{n}{r}}$$

5) Example 1.4.2 - There are 2^7 binary sequences of length 7. How many sequences are there which contains three 0's and four 1's?

4x1's



3x0's

Within a range of 7 bits how many ways can we position three zeros... Once placed we are left with the obvious remaining spaces to put 1's giving us $C^4_4 = 1$, So what we care about are only the zeros $\therefore \boxed{C^7_3 = 35}$

Quiz Problems

Quiz 01
Quiz 02
Quiz 03

Math 101 - Quiz 01 (2023-02-03)

1. $A = \{2n\ :\ n\, is\, an\, integer\}$ and B = {1, 2, 3, 4, 5}. Find $A \cap B$.

2. For $1 \leq i \leq 10$, let $A_i = \{1,\ 2,\ \ldots,\ i\}$. Determine $\left|\cup_{i=10}^{10} A_i\right|$.

3. Suppose $A = \{0,\ 3,\ 6,\ 9,\ 12,\ 15,\ 18\}$ and $B = \{0,\ 2,\ 4,\ 6,\ 8\}$. Find $A \cup B$ and $A \cap B$.

4. Suppose A and B are disjoint sets with $|A| = 5$ and $|B| = 3$. Determine $|A \cup B|$.

5. Suppose that A and B are sets with $|A| = 5$ and $|B| = 3$. Can you say anything about $|A \cup B|$?

Quiz #2 - Page 1 of 2

California State University Sacramento - Math 101

Name: ______________________

1) How many pairs of distinct integers $\{a, b\}$ with $a, b \in \{1, 2, \ldots, 10\}$ satisfy $|a - b| = 3$?

2) How many pairs of distinct integers $\{a, b\}$ with $a, b \in \{1, 2, \ldots, 10\}$ satisfy $|a - b| \leq 3$?

3) Find the number of positive divisor of $1800 = 2^3 \cdot 3^2 \cdot 5^2$ which are multiples of 3.

4) Find the number of positive divisors of $1800 = 2^3 \cdot 3^2 \cdot 5^2$ that are multiples of 6.

5) Let $A_1 = \{1, 2, 3\}$, $A_2 = \{2, 3\}$, and $A_3 = \{1, 2, 3, 4\}$. Find the number of 3-tuples (a_1, a_2, a_3) where $a_1 \in A_1$, $a_2 \in A_2$, and $a_3 \in A_3$.

Quiz #3

California State University Sacramento - Math 101

Name: ______________________

1) Determine the number of 3-permutations of a set with 7 elements.

2) List all 2-permutations of the set $A = \{a, b, c\}$.

3) Find the number of sequences (a_1, a_2, a_3, a_4) that consist of distinct elements from the set $\{1, 2, 3, 4, 5\}$ where a_1 is even.

4) Find the number of sequences (a_1, a_2, a_3, a_4) that consist of elements from the set $\{1, 2, 3, 4, 5\}$ where a_1 is even.

5) Find the number of odd integers between 10,000 and 20,000 where no digit is repeated.

6) Find the number of 3-circular permutations of a set with 7 elements.

7) List all 3-circular permutations of the set $A = \{x, y, z, t\}$.

8) Write down formulas for P_r^n, Q_r^n, and write down an equation that relates Q_r^n to P_r^n.

9) Write down a formula for C_r^n in terms of factorials.

10) Find the number of ordered pairs (A, B) where A is a 3-element subset of $\{1, 2, 3, 4, 5\}$, and B is a 2-element subset of $\{6, 7, 8, 9, 10\}$.

11) Find the number of subsets of $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ that consist of exactly three even integers and three odd integers.

12) Find the number of 2×3 rectangles in a 4×5 grid.

Quiz Solutions

Quiz 01
Quiz 02
Quiz 03

Quiz #2 - Page 1 of 2 - Solutions

California State University Sacramento - Math 101

Name: ______________________

1) How many pairs of distinct integers $\{a, b\}$ with $a, b \in \{1, 2, \ldots, 10\}$ satisfy $|a - b| = 3$?

Let us list all such pairs:

$\{1,4\}, \{2,5\}, \{3,6\}, \{4,7\}, \{5,8\}, \{6,9\}, \{7,10\}.$

There are 7 such pairs.

2) How many pairs of distinct integers $\{a, b\}$ with $a, b \in \{1, 2, \ldots, 10\}$ satisfy $|a - b| \leq 3$?

Let us assume that a is smaller than b. We can do this because $\{a,b\}$ is a set and so we do not count $\{a,b\}$ as being different from $\{b,a\}$.

possible a's $\rightarrow$	$a = 1$	$a = 2$		$a = 7$	$a = 8$	$a = 9$
possible b's $\rightarrow$	$b = 2$	$b = 3$	....	$b = 8$	$b = 9$	$b = 10$
	$b = 3$	$b = 4$		$b = 9$	$b = 10$	
	$b = 4$	$b = 5$		$b = 10$		

There are $7 \cdot 3 + 2 + 1 = 24$ such pairs.

Quiz #2 Page 2 of 2 - Solutions

3) Find the number of positive divisor of $1800 = 2^3 \cdot 3^2 \cdot 5^2$ which are multiples of 3.

We want to count numbers of the form

$$2^a \cdot 3^b 5^c \quad \text{where} \quad 0 \le a \le 3,\ 1 \le b \le 2,\ 0 \le c \le 2.$$

There are $4 \cdot 2 \cdot 3 = 24$ such divisors.

4) Find the number of positive divisors of $1800 = 2^3 \cdot 3^2 \cdot 5^2$ that are multiples of 6.

We want to count numbers of the form

$$2^a 3^b 5^c \quad \text{where} \quad 1 \le a \le 3,\ 1 \le b \le 2,\ 0 \le c \le 2.$$

There are $3 \cdot 2 \cdot 3 = 18$ such divisors.

5) Let $A_1 = \{1, 2, 3\}$, $A_2 = \{2, 3\}$, and $A_3 = \{1, 2, 3, 4\}$. Find the number of 3-tuples (a_1, a_2, a_3) where $a_1 \in A_1$, $a_2 \in A_2$, and $a_3 \in A_3$.

There are $3 \cdot 2 \cdot 4 = 24$ such 3-tuples.

\# of choices for a_3

\# of choices for a_1

\# of choices for a_2

Quiz 3

1) $P^7_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = 7 \cdot 6 \cdot 5 = 210$

2) a, b b, a c, a
a, c b, c c, b

3) (a_1, a_2, a_3, a_4)

a_1: 2 or 4

$$\underset{\substack{\text{Two choices} \\ \text{for } a_1}}{2} \times \underset{\substack{a_2 \text{ cannot} \\ \text{be } a_1 \text{ so 4 choices}}}{4} \times \underset{\text{3 choices for } a_3}{3} \times \underset{\text{2 choices for } a_4}{2} = 48$$

4) Taking the same approach as in #3, we first choose a_1, then a_2, a_3, and a_4.

$$2 \times \underbrace{5 \times 5 \times 5}_{a_2, a_3, a_4 \text{ have no restrictions}} = 250$$

5) 1 __ __ __ __

all must be distinct

Must be 1 to be between 10,000 and 20,000 and even

Must be 3, 5, 7, or 9

$$\underset{\substack{\text{choices for} \\ \text{last digit}}}{4} \times \underbrace{8 \times 7 \times 6}_{\substack{\text{choices for} \\ \text{middle 3}}} = 1344$$

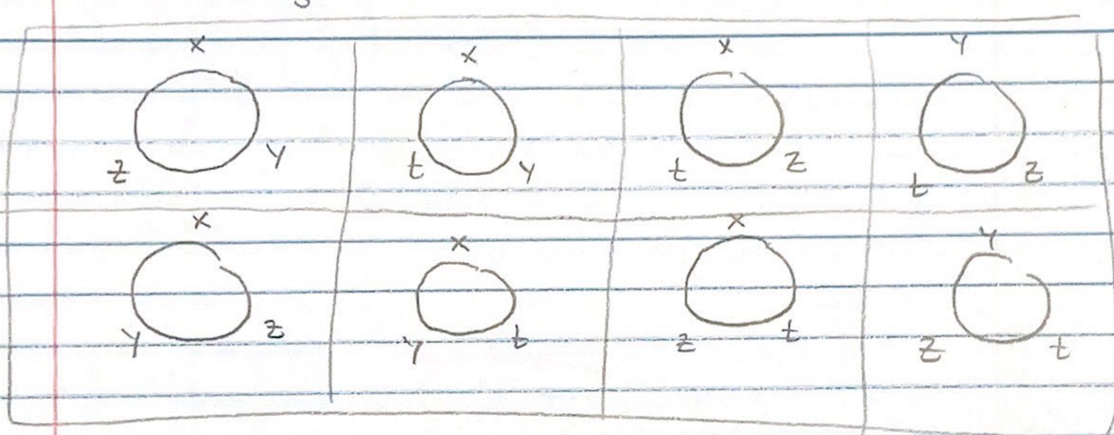
6) $Q^7_3 = \frac{7\cdot6\cdot5}{3} = 70$

or

$$Q^7_3 = \frac{P^7_3}{3} = \frac{\frac{7!}{(7-3)!}}{3} = \frac{7!/4!}{3} = \frac{7\cdot6\cdot5}{3} = 70$$

7) There will be

$$Q^4_3 = \frac{4\cdot3\cdot2}{3} \to 8 \text{ possible placements}$$



8) $P^n_r = \frac{n!}{(n-r)!}$ $\qquad Q^n_r = \frac{n!}{r\cdot(n-r)!}$

$$Q^n_r = \frac{1}{r} P^n_r$$

Quiz 03 Solutions Page 03/03

9) $C^n_r = \dfrac{n!}{r!(n-r)!}$

10) $\binom{5}{3}\binom{5}{2} = 100$

choices for A

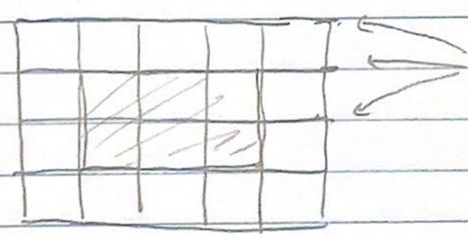
choices for B

11) $\binom{5}{3}\binom{5}{3} = 100$

choices for even

choices for odds

12)



3 choices for top of rectangle

~~3 choices for left side of rectange~~

$$3 \cdot 3 = 9$$