Chapter 01
Permutations and combinations

$\Gamma(3)=6$ HW I - Due 2/1 start of class

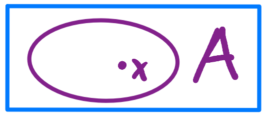
SUBJECT: Section 1.1 – Two Basic Counting Principles DATE: 2023/01/25 PAGE#: 01

Let us start with some terms and notation

- A **set** is a collection of objects
- If 'A' is a set and 'x' is an element in 'A' we write

$$x \in A$$

"is an element of"



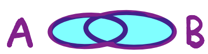
- If 'x' is not in 'A', write

$$x \notin A$$

- If 'A' and 'B' are sets,

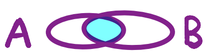
  Union of 'A' and 'B'

  $$A \cup B = \{x : x \in A \text{ or } x \in B\}$$



  Intersection of 'A' and 'B'

  $$A \cap B = \{x : x \in A \text{ and } x \in B\}$$



- The empty set, denoted

  $\varnothing$ is a set with no elements

- If $A \cap B = \varnothing$, then A and B are **disjoint**
- If $A_1, A_2, A_3, \dots$ is a collection of sets, then this collection is **pairwise disjoint** if every pair of distinct sets is disjoint

  $$A_1, A_2, A_i, \dots, A_j$$

  $$A_i \cap A_j = \varnothing$$

Cartesian Product of Sets

$A_1, A_2, \dots, A_k$

is the set

$$A_1 \times A_2 \times A_3 \times \dots \times A_k = \{(a_1, a_2, \dots a_k) : a_1 \in A_1, a_2 \in A_2, \dots a_k \in A_k\}$$

* the book's notation for cartesian product *

(choose math major, choose cs major)

$$A_1 \times A_2 \times \dots \times A_k = \prod_{i=1}^{k} A_i$$

We write $|A|$ for the number of elements in 'A' and this is called the "Cardinality" of 'A'.

Notation when taking unions/intersections of several sets, we often use sigms type notation
For instance,

$$\rightarrow A_1 \cap A_2 \cap A_3 = \bigcap_{i=1}^{3} A_i$$

$$\rightarrow B_0 \cup B_1 \cup \ldots = \bigcup_{i=0}^{\infty} B_i$$

Common Sets

Natural numbers

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$$

Integers

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

Whole numbers

$$\omega = \{0, 1, 2, 3, 4, \ldots\}$$

Rational

$$\mathbb{Q} = \{\tfrac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}$$

Two Basic Counting Principles

1. If $A_1, A_2, A_3, \ldots, A_3$ are pairwise $(\_, \_, \_)$ disjoint sets, then

$$|A_1 \cup A_2 \cup \ldots \cup A_k| = \sum_{i=1}^{k} |A_i|$$

2. If $A_1, A_2, A_3, \ldots, A_k$ (building an agenda) are finite sets, then

$$|A_1 \times A_2 \times \ldots \times A_k| = |A_1||A_2| \ldots |A_k|$$

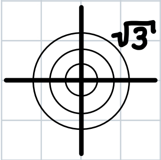
equivalently

$$\underbrace{\left|\prod_{i=1}^{k} A_i\right|}_{\text{cartesian}} = \underbrace{\prod_{i=1}^{k} |A_i|}_{\text{multiply}}$$

SUBJECT: # Section 1.1 – Two Basic Counting Principles
DATE: 2023/01/27 PAGE#:

Example 01: How many integers x,y satisfy the inequality $x^2+y^3\leq 3$?

Here we are counting (x,y) in a circle radius $\sqrt{3}$



Count according to the value of x^2+y^2

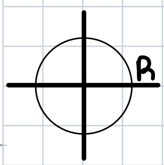
Case 01: $x^2+y^2=0$
→ only $(x,y)=(0,0)$

Case 02: $x^2+y^2=1$
→ $(x,y)=(1,0)$
→ $(x,y)=(0,1)$
→ $(x,y)=(-1,0)$
→ $(x,y)=(0,-1)$

Case 03: $x^2+y^2=2$
→ $(x,y)=(-1,1)$
→ $(x,y)=(1,-1)$ or $(\pm1,\pm1)$
→ $(x,y)=(1,1)$
→ $(x,y)=(-1,-1)$

Case 04: $x^2+y^2=3$
→ None!!!

Adding up all posibilities (case 1→4) gives a total of
→ $1+4+4=9$



$\pi R^2 +$ error term

Warm-up

A composition of a positive integer 'n' is an ordered list of positive integer that sum to 'n'

n	Composition of 'n'	Number of composition of 'n'
1	1	1
2	2, 1+1	2
3	3, 1+2, 2+1, 1+1+1	4
4	4, 2+2, 1+3, 1+1+1, 3+1, 1+1+2, 1+2+1, 2+1+1	8

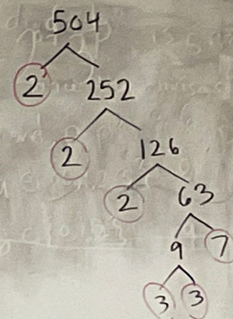
$n=2^3 3^2$ ← divisors of 3^2 are $3^0, 3^1, 3^2$

divisors of 2^3 are $2^0, 2^1, 2^2, 2^3$

is

$$(k_1+1)(k_2+1)\cdots(k_r+1)$$

<u>Ex</u> Find the # of positive divisors of 504



$504 = 2^3 \cdot 3^2 \cdot 7^1$

# of pos. divisors is

$4 \cdot 3 \cdot 2 = 24$

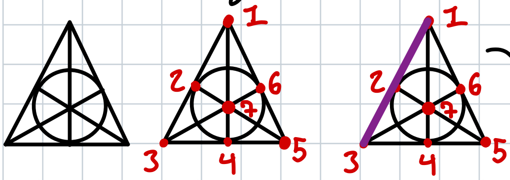
SUBJECT: 1.1 Continued

Missed previous class

DATE: 2023 / 01 / 30 PAGE#:

SUBJECT: Graphs - Error Correction - Hamming Distance DATE: 2023/02/03 PAGE#:

Before starting section 1.3, let us explore



$C_1 = \{1,2,3\}$ $C_5 = \{3,6,7\}$

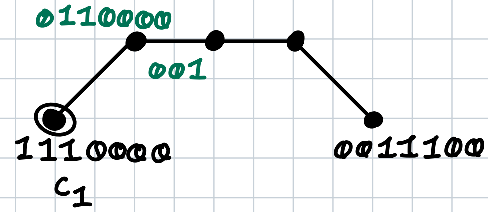
$C_2 = \{1,4,7\}$ $C_6 = \{2,5,7\}$

$C_3 = \{1,5,6\}$ $C_7 = \{2,4,6\}$

$C_4 = \{3,4,5\}$

$C_1 \to$ 1 1 1 0 0 0 0
$C_2 \to$ 1 0 0 1 0 0 1
$C_3 \to$ 1 _ _ _ 1 1 _
$C_4 \to$ _ _ 1 1 1 _ _
$C_5 \to$ _ _ 1 _ _ 1 1
$C_6 \to$ _ 1 _ _ 1 _ 1
$C_7 \to$ _ 1 _ 1 _ 1 _

Hamming Distance



I sent you 0110000. What is the most likely code word that was sent?

C_1 since it is the closest in Hamming distance to 0110000

What if * 0 * 0 1 1 * ? → Has to be C_3!

What if 0 0 1 0 0 0 0? → Don't know what to do???

How to build projected planes -Thano plane

13

12	11	10
9	8	7
6	5	4
3	2	1

$C_1 = \{13, 12, 11, 10\}$

$C_2 = \{13, 9, 8, 2\}$

$C_3 = \{13, 6, 5, 4\}$

$C_4 = \{13, 3, 2, 1\}$

$C_5 = \{12, 9, 6, 3\}$

$C_6 =$

$C_7 =$

$C_8 = \{11, 9, 5, 1\}$

$C_9 =$

$C_{10} =$

$C_{11} = \{10, 7, 5, 3\}$

$C_{12} =$

$C_{13} =$

Hamming distance to 0110000

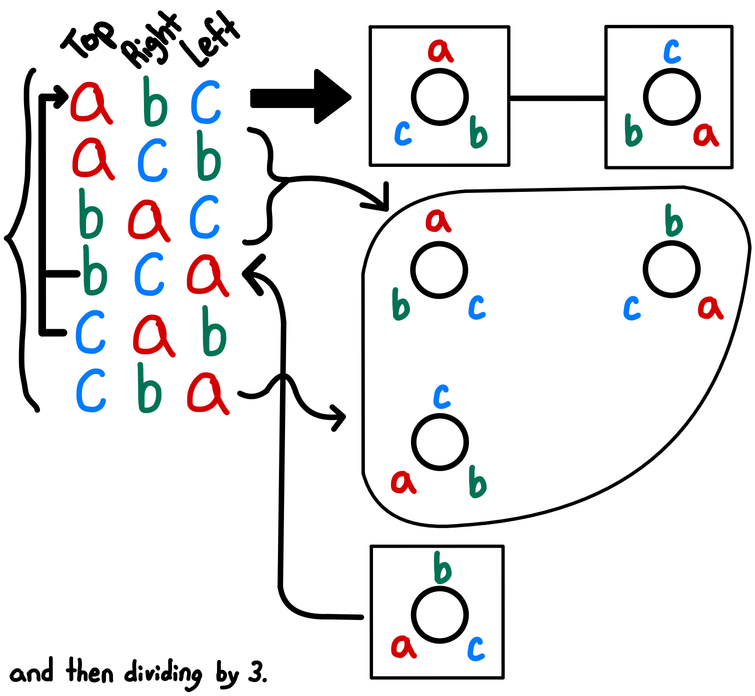
Section 1.3: Circular Permutations

DATE: 2023/02/06 PAGE#:

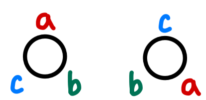
Problem: How many distinct ways can 'a', 'b', and 'c' be arranged around a circular table where two arrangements are the same if they are rotations of each other?

All 3 factorials ways of putting 'a', 'b', 'c' in a line

$$\frac{n!}{n} = (n-1)!$$



Solution: Two distinct ways



One can think of getting this count by first lining up (3! ways) and then dividing by 3.

In general, if Q^n_r is the number of arrangements of 'r' elements placed around a circle chosen from a set of 'n' elements, then

$$Q^n_r = \frac{P^n_r}{r} = \frac{\frac{n!}{(n-r)!}}{r} = \frac{n!}{r\cdot(n-r)!}$$

* Note that if $r=n$, then

$$Q^n_r = \frac{n!}{n(n-n)!} = \frac{n!}{n\cdot 0!} = \frac{n!}{n\cdot 1} = \frac{n!}{n} = (n-1)!$$

$$\boxed{Q^n_n = \frac{P^n_n}{n} = (n-1)!}$$

Example 01

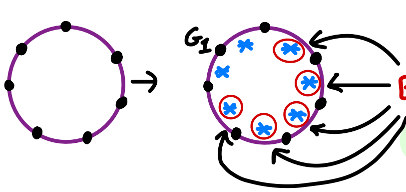
Problem: In how many ways can 5 boys and 3 girls be seated around a table if

(i) There are no restrictions

$$\frac{8!}{8} = 7!$$

(ii) Boy B_1 and girl G_1 cannot be next to each other

Let us place all except B_1 there $\frac{7!}{7} = 6!$ ways to place the seven people around the table (we are not placing B_1 yet)



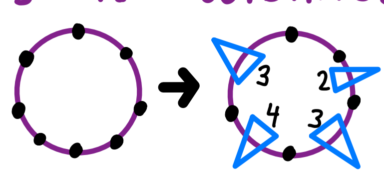
B_1 – 5 choices for B_1

This gives a total count of $6! \times 5$

(iii) No two girls sit next to each other

First put boys at the table

$\frac{5!}{5} = 4!$ Possibilites



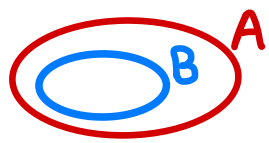
$5\cdot4\cdot3$ ways to place girls

$\Rightarrow$ Total is $4!\cdot5\cdot4\cdot3$

Section 1.4: Combinations

Let 'A' be a set.
We say that 'B' is a subset of 'A', denoted $B \subseteq A$, if every element of 'B' is also in 'A' and is denoted $P(A)$.

The set of all possible subsets of 'A' is called the power set of 'A'



Recap Prep For Exam 1
Section 1.1, 1.2, 1.3, 1.4
$P^n_r \; Q^n_r \; C^n_r$

Side Note: P(A)
Higher math may use alternate notation 2^A

Example 01

Sample: $P(\{a,b\}) = \{\underbrace{\emptyset}_{1}, \underbrace{\{a\},\{b\}}_{2}, \underbrace{\{a,b\}}_{1}\}$

$(x+y)^2 = x^2+2xy+y^2$
$(x+y)^3 = x^3+3x^2y+3xy^2+y^2$
Pascal's Triangle Relation ⇒ Making the connection to pascal

$$\begin{array}{c} 1 \\ 1\;\;1 \\ 1\;\;2\;\;1 \\ 1\;\;3\;\;3\;\;1 \\ 1\;\;4\;\;6\;\;4\;\;1 \end{array}$$

If $A=\{a,b,c\}$, find $P(A)$

$P(A) = \{\underbrace{\emptyset}_{1}, \underbrace{\{a\},\{b\},\{c\}}_{2}, \underbrace{\{a,b\},\{a,c\},\{b,c\}}_{3}, \underbrace{\{a,b,c\}}_{1}\}$

zero element subset, one element subset, Two element subset, Three element subset

can be grouped in sizes

One way to view choosing subsets of $A=\{1,2,3\}$ is by expanding the polynomial $(a+1)(b+1)(c+1)$

$(a+1)(b+1)(c+1) = abc + ab + ac + bc + a + b + c + 1$
$\{a,b,c\}\;\{a,b\}\;\{a,c\}\;\{b,c\}\;\{a\}\;\{b\}\;\{c\}\;\emptyset$

* The key is to make it all be 'x' *
So when we had

$(a+1)(b+1)(c+1) = abc + ab + ac + bc + a + b + c + 1$
$\{a,b,c\}\;\{a,b\}\;\{a,c\}\;\{b,c\}\;\{a\}\;\{b\}\;\{c\}\;\emptyset$

Let $a=b=c=x$

5 in a 5-element set $(x+1)^5$ ➡ $x^3 + xx + xx + xx + x + x + x + 1$
$= x^3 + 3x^2 + 3x + 1 \longrightarrow (x+1)(x+1)(x+1)$

$$\begin{array}{cc} 1 & \\ 1\;\;1 & =2 \\ 1\;\;2\;\;1 & =4 \\ 1\;\;3\;\;3\;\;1 & =8 \\ 1\;\;4\;\;6\;\;4\;\;1 & =16 \end{array}$$

4

Pascal's Triangle Relation to Binomial Coefficients

Lets make some formulas...

A combination of a set 'A' is a subset of 'A'

For $0 \le r \le n$, an r-combination of 'A' is a subset with 'r' elements

$$P^n_r = \frac{n!}{(n-r)!}$$

If $A=\{a,b,c\}$, then the 2-permutations are
$P^3_2 = \frac{3!}{(3-2)!}$

If we ignore order

ab
ac
ba
bc
ca
cb

$\boxed{C^3_2 = \frac{1}{2!}P^3_2}$

In general...

$$C^n_r = \frac{P^n_r}{r!}$$

we can rewrite as

These choose numbers C^n_r are called "binomial coefficients" and C^n_r, "C-N-R", is typically written as $\binom{n}{r}$
"n-choose-r"

$$\begin{array}{c} 1 \\ 1\;\;1 \\ 1\;\;2\;\;1 \\ 1\;\;3\;\;3\;\;1 \\ 1\;\;4\;\;6\;\;4\;\;1 \end{array}$$

Note

$C^n_r = C^n_{n-r}$

Since,

$$C^n_r = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!r!} = \frac{n!}{(n-r)!(n-(n-r))!} = C^n_{n-r}$$

this means

$$\binom{n}{r} = \binom{n}{n-r}$$

example

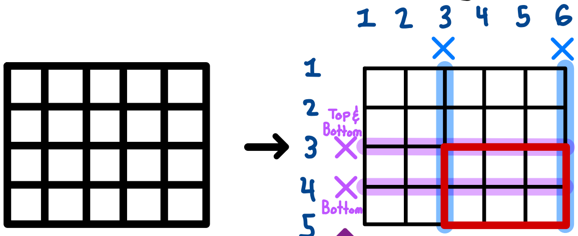
$\binom{4}{0} + \boxed{\binom{4}{1}} + \boxed{\binom{4}{2}} + \binom{4}{3} + \binom{4}{4} = 2^4$

Identity
A set with 4 elements and count by size

$\{\_,\_,\_,\_\}$ with a,b,c,d

Example 02

How many rectangles are in the grid below?



$\binom{6}{2}$ Ways to pick Sides of rectangle

$\binom{5}{2}$ Ways to pick top & bottom of rectangle

In all, there are

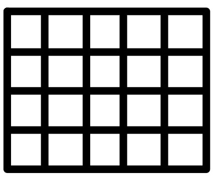
$$\binom{5}{2}\cdot\binom{6}{2} = \frac{5\cdot 4}{2\cdot 1}\times\frac{6\cdot 5}{2\cdot 1} \rightarrow \frac{5\cdot \cancel{4}^{2}}{{}_{1}\cancel{2}\cdot 1}\times\frac{\cancel{6}^{3}\cdot 5}{\cancel{2}_{1}\cdot 1}$$

$$\rightarrow 10\times 15 = 150$$

These choose numbers C^n_r are called "binomial coefficients" and C^n_r, "C-N-R", is typically written as $\binom{n}{r}$ "n-choose-r"

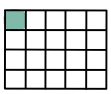
Example 03

How many, non-overlapping of same sized, squares are in the grid?



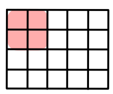
Lets do some case analysis...

Case 1: Square is 1×1



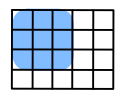
There are 4×5=20 of those

Case 2: Square is 2×2



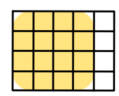
3×4=12

Case 3: Square is 3×3



2×3=6

Case 4: Square is 4×4



1×2=2

In all, there are 20+12+6+2=40 squares

Example 04

Let 'A' be a set with 2n elements

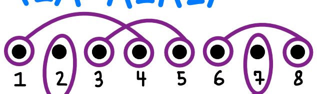
● ● ● ● ● ● ● ●
1 2 3 4 5 6 7 8

A <u>pairing</u>* of 'A' is a partition of 'A' into subsets of size 2.

*is unordered

How many pairings does 'A' have?

$\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}$



{1,4}, {2,7}, {3,5}, {6,8}

$$\binom{2n}{2}\binom{2n-2}{2}\binom{2n-4}{2}\cdots\binom{2}{2}$$

↑ Choose first pair

↑ Choose second pair

Must divide by $n!$ because our initial counting orders the pairings, but pairings are not ordered so...

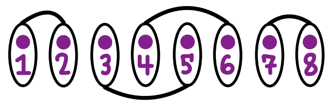
Total # of pairings is

$$\frac{\binom{2n}{2}\binom{2n-2}{2}\binom{2n-4}{2}\cdots\binom{2}{2}}{n!}$$

Last time we counted the number of pairings on pairings on a set with $2n$ elements

$\{3,5\},\{1,2\},\{4,6\},\{7,8\}$

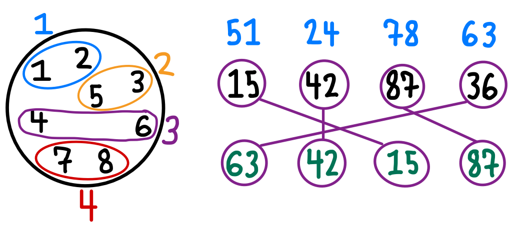
$n=4$



★ $\{1,2\},\{3,5\},\{4,6\},\{7,8\}$

what if presented in a line?

★ $\boxed{\{1,2\},\{3,4\},\{5,6\},\{7,8\}}$



$$\frac{\binom{2n}{n}\binom{2n-2}{2}\binom{2n-4}{2}\cdots\binom{2}{2}}{n!}$$

$\binom{8}{2}\quad\binom{6}{2}\quad\binom{4}{2}\quad\binom{2}{2}\quad\frac{1}{4!}$

$\boxed{\{1,2\},\{3,4\},\{5,6\},\{7,8\}}$ (a_1,a_2,a_3)

Let us count pairings in a different way...

There is $(2n)!$ ways to order all elements in the set

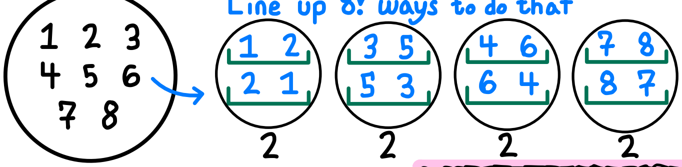
$a_1,a_2,a_3,\dots,a_{2n} \leftarrow$ ordered list of all elements of 'A'

$\{a_1,a_2\},\{a_3,a_4\},\dots,\{a_{2n-1},a_{2n}\}$

$a_2\ a_1\quad a_4\ a_3\quad a_{2n}\ a_{2n-1}$

Example 05

Line up $8!$ ways to do that

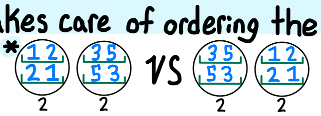


$$\frac{(2n)!}{2^n\cdot n!}$$

Simplifies to $\boxed{\frac{\binom{2n}{2}\binom{2n-2}{2}\binom{2n-4}{2}\cdots\binom{2}{2}}{n!}}$

takes care of $\{x,y\}$ vs $\{y,x\}$

takes care of ordering the pairs



Example 06

$\{1,5\},\{4,2\},\{8,7\},\{3,6\}$

unfinished note section

Example 07

There are 10 students: 6 female, 4 male

(i) How many groups can be found with 3 female & 2 male

$$\binom{6}{3}\binom{4}{2}=120$$

Number of ways to choose girls — Number of ways to choose boys

(ii) How many groups of 5 students can be made with at least 1 male? * Needs case analysis

case 01	case 02	case 03	case 04
-1 male-	-2 males-	-3 males-	-4 males-
$\binom{6}{4}\binom{4}{1}$	$\binom{6}{3}\binom{4}{2}$	$\binom{6}{2}\binom{4}{3}$	$\binom{6}{1}\binom{4}{4}$
60	120	60	6

Total count is

$$\underbrace{\binom{6}{4}\binom{4}{1}}_{60}+\underbrace{\binom{6}{3}\binom{4}{2}}_{120}+\underbrace{\binom{6}{2}\binom{4}{3}}_{60}+\underbrace{\binom{6}{1}\binom{4}{4}}_{6}=246$$

(iii) Repeat Example 01 - except now the group must be ranked from 1 to 5 ordered

$$\binom{6}{3}\binom{4}{2}5!$$

Ways to order the five people

Number of ways to choose girls — Number of ways to choose boys

In total $\binom{6}{3}\binom{4}{2}5! = 14{,}400$

In Summary

→ More than 1 way to count

→

unfinished note section

Practice with counting – choose function

Exercise

In how many ways can a group of 5 people be formed from 11 people where 4 are instructors and 7 are students

(i) No restrictions/condictions

$$\binom{11}{5} = \boxed{462}$$

(ii) What if we want exactly two instructors?

Pick 2 instructors and choose the remaining 3

choose 2 instructors

$$\binom{4}{2}\binom{7}{3} = \boxed{210}$$

choose 3 students

(iii) What if we want at least three instructions?

Case	Case
3 instructors	4 instructors
2 students	1 student

$$\underbrace{\binom{4}{3}\binom{7}{2}}_{84} + \underbrace{\binom{4}{4}\binom{7}{1}}_{7} = \boxed{91}$$

**(iv) Conflict of interest where I_1 & S_1 can't be together

I_1 & S_1 cannot both be chosen, we consider cases
(3 in all)

Case 01: I_1 is chosen NOT S_1

I_1 chosen, S_1 cannot $\rightarrow \binom{9}{4} = 126$

Case 02: I_1 is NOT chosen, but S_1 is chosen

S_1 chosen, I_1 cannot $\rightarrow \binom{9}{4} = 126$

Case 03: Neither I_1 or S_1 are chosen

$$\binom{9}{5} = 126$$

$$\boxed{\text{In total } \underset{126}{\text{Case 01}} + \underset{126}{\text{Case 02}} + \underset{126}{\text{Case 03}} = 378}$$

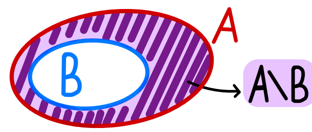
Principle of Complemetation

If 'A' is a finite set and $B \subseteq A$, then $|A \setminus B| = |A| - |B|$

$A \setminus B = \{x : x \in A, x \notin B\}$

Attributes
* Counting the opposite
* The 1-minus rule

$$P(A) = 1 - P(A')$$



**Recall (iv)

In how many ways can a group of 5 people be formed from 11 people where 4 are instructors and 7 are students where I_1 & S_1 can't be together

We can use this tool, the Principle of Complementation, to count the number of 5 person groups from 4 instructors, 7 students where I_1 and S_1 are not both in the group together (case 03)

11 ← 7 students, 4 instructors
11 − 2 (1 student & 1 instructor)

Yields the same results → $\binom{11}{5} - \binom{9}{3} = \boxed{378}$

$\binom{11}{5}$: Total number of possibilities

$\binom{9}{3}$: Number of groups that contain both I_1 and S_1

Note

Counting the same object in two ways can lead to combination identities

In this case...

$$\binom{11}{5} - \binom{9}{3} = \binom{9}{4} + \binom{9}{4} + \binom{9}{4}$$

Line n-up $\underline{1}, \underline{2}, \underline{3}, \ldots, \underline{r}$ $\quad P^n_r = nP^{n-1}_{r-1}$

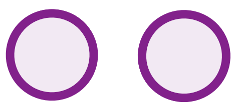
Principle of Complemetation

$$\boxed{P^n_r = nP^{n-1}_{r-1}}$$

Example: Placement type problems

If there must be atleast one person at each table, how many ways can 6 people be placed around two *indistinguishable tables?

*Can't tell the two tables apart



Divide counting into cases depending on the number per table

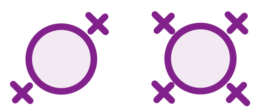
case 01

6 choices per person at own table

$Q^5_5 = 4!$

$6 \cdot 4!$

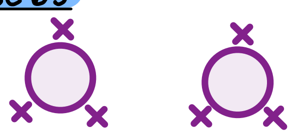
case 02



$\binom{6}{2} \times Q^4_4 = \binom{6}{2} \cdot 3!$

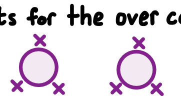
Recall: if $r=n$ then
$$Q^n_r = \frac{n!}{n(n-n)!} = (n-1)!$$

case 03

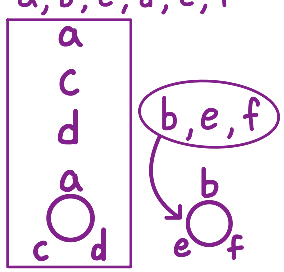


$$\frac{\binom{6}{3} Q^3_3 \cdot Q^3_3}{2!}$$

*accounts for the over count



* Indistingushable tables
a, b, c, d, e, f



The total count is

case 01 case 02 case 03
$$\binom{6}{1}Q^1_1 \cdot Q^5_5 + \binom{6}{2}Q^2_2 \cdot Q^4_4 + \frac{1}{2}\binom{6}{3}Q^3_3 \cdot Q^3_3$$

Lower case 's' as notation

~Sterling Numbers of the first Kind~

Def: Sterling Numbers of the first Kind

Given integers $0 \le n \le r$ let, $s(r,n)$ be the number of ways to place 'r' distinct objects around 'n' indistinguisable tables where no tables are empty

From our example. .

case 01 case 02 case 03
$$S(6,2) = \binom{6}{1}Q^1_1 \cdot Q^5_5 + \binom{6}{2}Q^2_2 \cdot Q^4_4 + \frac{1}{2}\binom{6}{3}Q^3_3 \cdot Q^3_3$$

Properties of s(r,n):

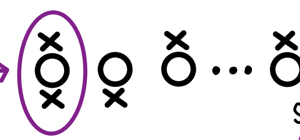
r-people r-table

- $s(r,1) = Q^r_r = (r-1)!$
- $s(r,r) = 1$
- $s(r,r-1) = \binom{r}{2}$

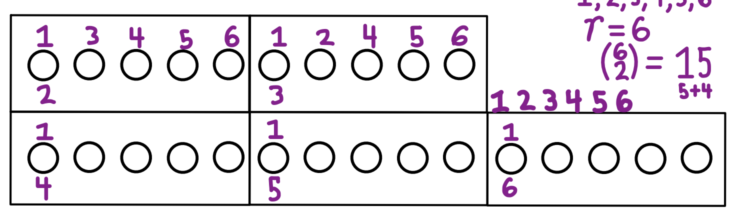
from the textbook

$s(r,0) = 0$	if $r \ge 1$
$s(r,r) = 1$	if $r \ge 0$
$s(r,1) = (r-1)!$	for $r \ge 2$
$s(r,r-1) = \binom{r}{2}$	for $r \ge 2$

Pick a pair to sit together →



$S(r,r-1) = \binom{r}{2}$
1,2,3,4,5,6
$r=6$
$\binom{6}{2} = 15$
5+4



- $s(r,r-2) = \frac{(3n-1)}{4}\binom{n}{3}$

Note: How so?
$$s(r,r-2) = 2\cdot\binom{n}{3} + \frac{1}{2}\binom{n}{2,2,n-4} = \frac{(3n-1)}{4}\binom{n}{3}$$

Claim

If $r \ge n$, then
$$S(r,n) = S((r-1),n-1) + (r-1)S(r-1,n)$$

WHY? Let $1,2,3,\dots,r$ be the people

Focus on person 1

*case 01: $S((r-1),n-1)$
Person 1 is at own table

○ 1	r−1 people n−1 table(s)

S(r-1, n-1) Ways to complete

*case 02: $(r-1)S(r-1,n)$
Person 1 is at not own table
Place all but person #1

r−1 people 1 table

Choices for who sits to left of

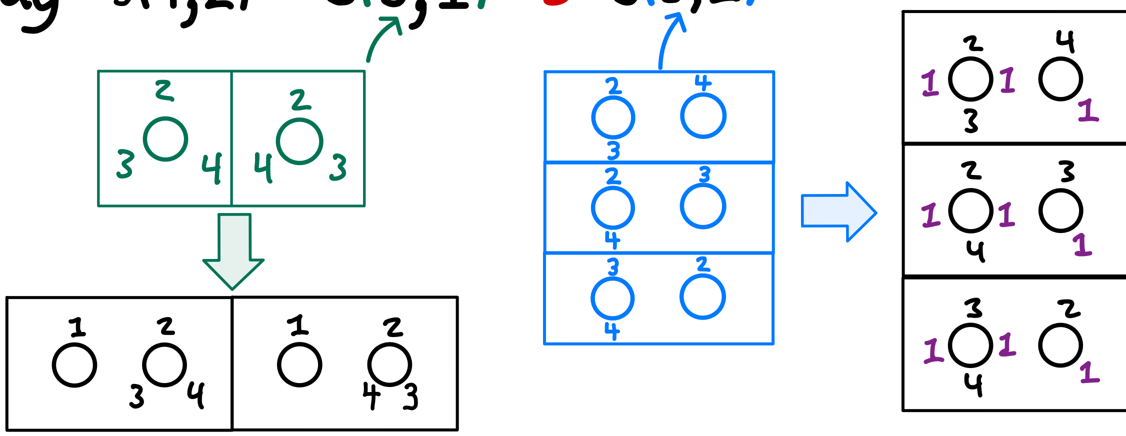
S(r-1, n) Ways to complete

From last time, we talked about

$$s(r,n) = s((r-1), n-1) + (r-1)s(r-1, n)$$

Why is this true? Let us draw a specific instance of this recursion.

$$\text{Say } s(4,2) = S(3,1) + 3 \cdot S(3,2)$$

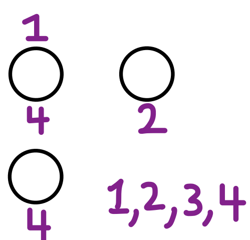


3 choices for 1

We have
$$s(4,2) = 2+3\cdot 3 = 11$$

What about $s(5,3)$ – 5 people and 3 tables

$$S(\overset{r}{5},\overset{n}{3}) = \underbrace{S(4,2)}_{\binom{4}{3}=11} + 4 \cdot \underbrace{S(4,3)}_{\binom{4}{2}=6}$$



$$S(r, r-1) = \binom{r}{2}$$

$$s(5,3) = 11 + 4\cdot 6 = 35$$

Recall

Properties of $s(r,n)$

r – people r – table

- $s(r,1) = Q_r^r = (r-1)!$
- $s(r,r) = 1$
- $s(r,r-1) = \binom{r}{2}$
- $s(r,r-2) = \frac{(3n-1)}{4}\binom{n}{3}$

From the textbook

$s(r,0)=0$	if $r \geq 1$
$s(r,r)=1$	if $r \geq 0$
$s(r,1)=(r-1)!$	for $r \geq 2$
$s(r,r-1)=\binom{r}{2}$	for $r \geq 2$

$$s(r,n) = s((r-1), n-1) + (r-1)s(r-1, n)$$

$$s(5,3) = s((5-1), 3-1) + (5-1)s(5-1, 3)$$

$$= s((4), 2) + (4)\, s(4,3)$$

$\rightarrow s(4,2) = s(3,1) + (3)s(3,2)$

$s(r,1) = Q_r^r = (r-1)!$ $\quad (3)$ $\quad s(r, r-1) = \binom{r}{2}$

$(2)! + (3)\cdot\binom{3}{2}$

$\rightarrow s(4,3) = s(3,2) + (3)s(3,3)$

$s(r, r-1) = \binom{r}{2}$ $\quad (3)$ $\quad s(r,r) = 1$

$\binom{3}{2} + (3)\cdot 1$

$\rightarrow s(4,2) = (\underset{2}{2})! + (\underset{3}{3})\cdot\binom{3}{2}_{3}$

$\rightarrow s(4,2) = 11$

$\rightarrow s(4,3) = \binom{3}{2}_{2} + (\underset{3}{3})\cdot 1$

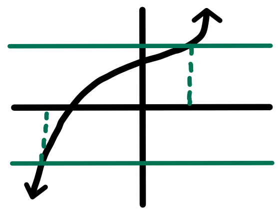
$\rightarrow s(4,3) = 6$

$$s(5,3) = 11 + (4)6$$

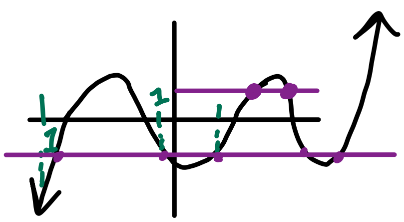
$$\therefore s(5,3) = 35$$

Let 'A' and 'B' be sets 'A' function $f: A \to B$ is

- **Injective** (1-to-1) if
  $f(a_1) = f(a_2)$ implies $a_1 = a_2$



- **Surjective** (onto) if
  for $a \in b$, exists an $a \in A$ with $f(a) = b$

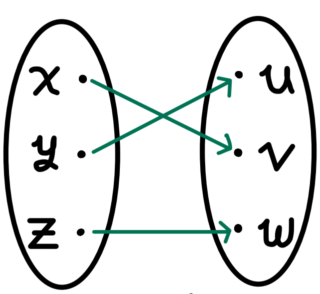


- **Bijective** if f is both
  Injective (1-to-1) and Surjective (onto)

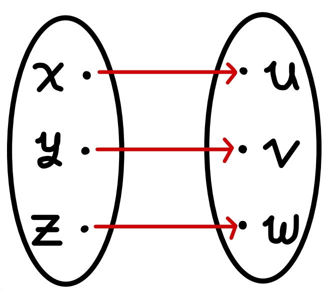
Example: Surjective & injective

Bijective if f is both Injective (1-to-1) and Surjective (onto)

Let $A = \{x, y, z\}$, $B = \{u, v, w\}$

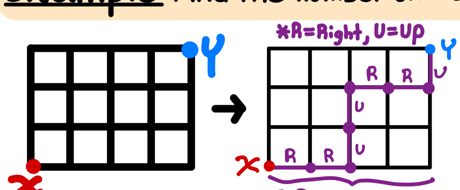


Surjective and injective



Surjective and injective

Example: Find the number shortest routes from X to Y



RRUURRU

Selecting for Right $\binom{7}{4}$ OR $\binom{7}{3}$ Selecting for up

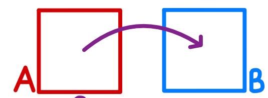
$\binom{7}{3}$ Selecting for up

$\underbrace{R\ R\ U\ U\ R\ R\ U}$

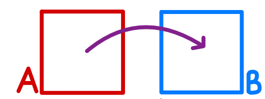
Choose three places for when to take an U (up) - step

Supposse you have 2 finite sets where it has surjection

$|A| \geq |B|$ $|A| \leq |B|$



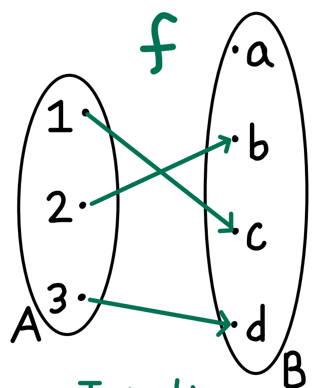
Surjection



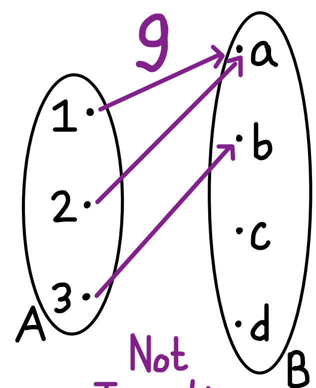
Injection

Example: Injective

Let $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$

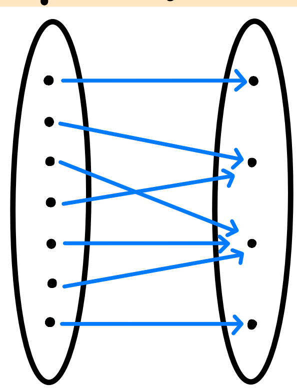


Injective



Not Injective

Example: Surjective not injective



Surjective (onto) if
for $a \in b$, exists an $a \in A$ with $f(a) = b$

Surjective not injective

Example: Find the number of r-combinations (subset with r-elements) with no consecutive intetgers

Let $x = \{1, 2, 3, \dots, n\}$,

if $x = \{1, 2, 3, 4, 5, 6\}$ and $r = 3$, the subsets are:

$\{1,3,5\}$ $\{2,4,6\}$ $\{1,3,6\}$ $\{1,4,6\}$

$\boxed{\text{* Use Binomial Cofficient *}}$

SUBJECT: Section 1.5: Injection & Bijections Principles DATE: 2023/02/27 PAGE#: 02/02

Recall from Friday, that we want to find the number of subsets of

$\{1,2,3,\dots,n\}$

with no consecutive integers

For instance, if $x=\{1,2,3,4,5,6,7\}$, then

$\{1,3,5\} \to \{1,2,3\}$
$\{1,3,6\} \to \{1,2,4\}$
$\{1,3,7\} \to \{1,2,5\}$
$\{1,4,6\} \to \{1,3,4\}$
$\{1,4,7\} \to \{1,3,5\}$
$\{1,5,7\} \to \{1,4,5\}$
$\{2,4,6\} \to \{2,3,4\}$
$\{2,4,7\} \to \{2,3,5\}$
$\{2,5,7\} \to \{2,4,5\}$
$\{3,5,7\} \to \{3,4,5\}$

The subsets on the right are exactly all $\binom{5}{3}=10$ subsets $\{1,2,3,4,5\}$ of size 3

Genericly...

Leave same, Subtract 1, Subtract 2

$$\left\{\begin{array}{l}\underbrace{\{1,3,5\}} \to \{1,2,3\}\\ \{1,3,6\} \to \{1,2,4\}\\ \underbrace{\{1,3,7\}}_{\text{Subtract by the index in set}} \to \{1,2,5\}\end{array}\right.$$

General Case

Let $x=\{1,2,3,4,\dots,n\}$ and Let $1\le r\le n$

Given a subset $\{S_0,S_1,\dots,S_r\}$ of x with no consecutive elements define,

$f(\{S_0,S_1,\dots,S_r\})=\{S_0,S_1-1,S_2-2,\dots,\underline{S_r-(r-1)}\}$ The biggest it can be is $n-r+1$

This output is a subset of size 'r' since $\{S_0,S_1,\dots,S_r\}$ has no consecutive elements

$n=7$, $r=3$ $\binom{n-r+1}{r}=\binom{5}{3}$

X: 1,2,3,4,5,6,7 — $\{3,5,7\}$ → y: 1,2,3,4,5 — $\{3,4,5\}$

Note that since "Sr" can be n, but not bigger, $S_r-(r-1)$ canbe $n-r+1$, but not bigger

Therefore,

$$\underbrace{|X|}_{\text{Tricky}}=\underbrace{|Y|}_{\text{Easy}}=\binom{n-r+1}{r}$$

Tricky: all subsets of $\{1,2,3,\dots,n\}$ with no consecutive elements

Easy: all subsets of $\{1,2,3,\dots,n-r+1\}$ of size r

$|a-b|\ge K$

$\binom{n-(K-1)(r-1)}{r}$

What if we require that $|a-b|\ge 3$ for all a,b in our r-subset of $x=\{1,2,3,\dots,n\}$?

Try $\{1,2,3,4,5,6,7,8\}$, $r=3$

same, -2, -4

$\{1,4,7\} \to \{1,2,3\}$
$\{1,4,8\} \to \{1,2,4\}$
$\{1,5,8\} \to \{1,3,4\}$
$\{2,5,8\} \to \{2,3,4\}$

Guess

$\binom{n-2(r-1)}{r}$

Via Bijection 'x' counts the 'y'-side

Example

Let $A=\{a,b,c\}$

Arrangements & Selections with Repetitions are allowed

The 2-permutations of 'A' are : ab, ac, ba, bc, ca, cb

The 2-permutations allowed are aa, ab, ac, ba, bb, bc, ca, cb, cc

There are $3^2=9$ such 2-permutations

In general, if $A=\{1,2,3,\ldots,n\}$ then the number r-permutations of 'A' with repitition allowed is

$$n^r$$

*no need for $0\leq r\leq n$ due to being able to have repitition with Repetitions are allowed *order matters!

Example Find the number of 5-permutations of a, a, a, b, c

Approach 01: "Pretend" the a's are distinguisable

Let us, for a moment, pretend the a's are different

$a_1, a_2, a_3, b, c \rightarrow \underbrace{a_1, a_2, a_3}_{3!}, b, c$

There are $5! = 120$ permutations of these 5 elements

some are

$123 \leftarrow a_1, b, a_2, a_3, c$

$213 \leftarrow a_2, c, a_1, a_3, b$

$321 \leftarrow a_3, b, a_2, a_1, c$

$\vdots$

There are $3!$ ways to permute the a_1, a_2, a_3 among themselves

$\frac{5!}{3!}$ permutations of a,a,a,b,c

Approach 02: Count the unique items

a, a, a, b, c

$\underline{a}\ \underline{a}\ \underset{\times}{\underline{b}}\ \underset{\times}{\underline{c}}\ \underline{a}$, so $\binom{5}{2}2$

In general...

Suppose we have t-types of objects

r_1 of type 1

r_2 of type 2

$\vdots$

r_t of type 3

The number of permutations of all objects is

$$\frac{(r_1, r_2, r_3, \ldots, r_t)!}{r_1!\, r_2!\, r_3! \ldots r_t!}$$

Or $\binom{5}{2}2$ if we organize the b, c first and then the a's

$\underline{b}\ \underline{c}\ \underline{a}\ \underline{a}\ \underline{a}$ the reason for the 2 in $\rightarrow \binom{5}{2}2$

SUBJECT: 1.6 Continued - Multisets DATE: 2023/03/01

A multiset is a collection of unordered objects where repitition is allowed.

$\{a, b, c\}$ ← set and multiset

$\{a, b, c, c\}$ ← multiset, but NOT a set.

Notation for multisets:

$$\{a, a, b, b, b, b, c, c, c, \ldots\} = \{2\cdot a, 4\cdot b, \infty\cdot c\}$$

$$\{2\cdot a, 4\cdot b, \infty\cdot c\}$$

The number 2 is called the repetition number of 'a'
4 is the repetition number of 'b'
∞ is the repetition number of 'c'

Suppose we have a multiset

$$M = \{r_1\cdot a_1, r_2\cdot a_2, r_3\cdot a_3, \ldots, r_n\cdot a_n\}$$

permutating all elements of M would give a total of

$$\frac{(r_1 + r_2 + r_3 + \ldots + r_n)!}{r_1!\, r_2!\, r_3! \cdots r_n!} \text{ permutations}$$

If we permute not all of the elements M, the counting is more tricky

Example Find the sequences of length 10 with two 0's, three 1's, five 2's

Five elements total / Two 0's, Three 1's $\quad \dfrac{5!}{2!\,3!}$ $\qquad$ Two 0's, Three 1's $\quad \dfrac{5!}{2!(5-2)!}$

The total count is...

$$\frac{10!}{2!\,3!\,5!}$$

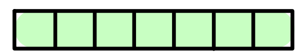
Example Find the number of ways to tile the rectangle

using blocks of size 1x1, 1x2, 1x3

We need to find the number of ways to write 7 as an ordered sum of 1, 2, and 3 where the terms can repeat

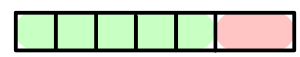
Case 01 all 1×1 blocks

$$7 = 1+1+1+1+1+1+1$$



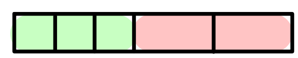
Case 02 Five 1's, One 2

$$\frac{6!}{5!} = 6$$



Case 03 Three 1's, Two 2's

$$\frac{5!}{3!\,2!} = 10$$

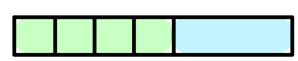


Case 04 One 1, Three 2's

$$\frac{4!}{3!} = 4$$

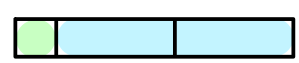
Case 05 Four 1's, One 3

$$\frac{5!}{4!} = 5$$



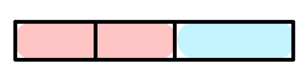
Case 06 One 1, Two 3's

$$\frac{3!}{2!} = 3$$



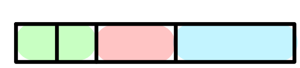
Case 07 Two 2's, One 3

$$\frac{3!}{2!} = 3$$



Case 08 Two 1's, One 2, One 3

$$\frac{4!}{2!} = 12$$



SUBJECT: 1.6 Continued - Multisets

DATE: 2023/03/03 PAGE#: 04/05

Recall

The number of r-permutations of the multiset
$M=\{\infty\cdot a_1,\infty\cdot a_1,\infty\cdot a_1,\ldots,\infty\cdot a_1$, is n^r

If $M=\{r_1\cdot a_1+r_2\cdot a_2+r_3\cdot a_3+\ldots+r_n\cdot a_n\}$
Then the number of permutations of M is

$$\frac{(r_1+r_2+r_3+\ldots+r_n)!}{r_1!\,r_2!\,r_3!\ldots r_n!}$$

Example: "the sandwich" - taking subsets of multisets
There are three types of sandwiches
turkey (T), ham (H), and butter lental tumeric (BLT)
How many ways can a person order 6 sandwiches?

Concept: Stars & Bars
If I can see it
I can count it

Look at examples of orders:

$n=3$
$r=6$

T	H	BLT	
* *	*	* * *	← 2T's, 1H, 3BLT's
	* *	* * * *	← 0T, 2H's, 4BLT's
* * *		* * *	← 3T's, 0H, 3BLT's

2T's, 1H, 3BLT's → 00101000
0T, 2H's, 4BLT's → 10010000
3T's, 0H, 3BLT's → 00011000

So, $\binom{8}{2}$

Each order corresponds to 0-1 sequence with exactly $6+3+1=8$ positions and has exactly 6 zeros

So, $\binom{6+3-1}{6}=\binom{8}{6}$ **or** $\binom{6+3-1}{3-1}=\binom{8}{2}$

In general, if

$M=\{\infty\cdot a_1,\infty\cdot a_2,\infty\cdot a_3,\ldots,\infty\cdot a_n\}$

and H_r^n is the number of multisubsets of M with r-element is

$$H_r^n=\binom{r+n-1}{r} \text{ OR } H_r^n=\binom{r+n-1}{n-1}$$

Let $M=\{\infty\cdot a_1,\infty\cdot a_2,\ldots,\infty\cdot a_n\}$. The number H_r^n of r-element multi-subsets of M is given by

$$H_r^n=\binom{r+n-1}{r}$$

Example: Solving integers with equations

$$\underbrace{x_1+x_2+x_3}=7$$

Find the number of solutions in non-negative integers

* Same stars and bars of the previous example*

We can think of a solution as a multiset

T	H	BLT	
* * *	* *	* *	← $3+2+2=7$
	* * *	* * * *	← $0+3+4=7$

$$\binom{\overset{r}{7}+\overset{n}{3}-1}{\underset{n-1}{3-1}}=\binom{9}{2}$$

$\binom{9}{2}=36$ is the number of non-negative integer solutions to $x_1+x_2+x_3=7$

The Binomial Coefficient

$$H_r^n=\binom{r+n-1}{r}=\binom{r+n-1}{n-1}$$

The Binomial Coefficient counts a few things

→ Number of non-negative integer solutions to $x_1+x_2+x_3+\ldots+x_n=r$

→ Number of r-element multisubsets of $M=\{\infty\cdot a_1,\infty\cdot a_2,\infty\cdot a_3,\ldots,\infty\cdot a_n\}$

→ Number of ways to put r-identical objects into n-distinct boxes

Box-A	Box-B	Box-C
* *	* *	* * *

The Binomial Coefficient

$$H_r^n = \binom{r+n-1}{r} = \binom{r+n-1}{n-1}$$

The Binomial Coefficient counts a few things

→ Number of non-negative integer solutions to
$x_1 + x_2 + x_3 + \dots + x_n = r$

→ Number of r-element multisubsets of
$M = \{\infty \cdot a_1, \infty \cdot a_2, \infty \cdot a_3, \dots, \infty \cdot a_n\}$

→ Number of ways to put r-identical objects into n-distinct boxes

Box-A: * * | Box-B: * * | Box-C: * * *

Recall: Sterling Numbers of the first Kind

Given integers $0 \le n \le r$ let, $s(r,n)$ be the number of ways to place 'r' distinct objects around 'n' indistinguisable tables where no tables are empty

Properties of $s(r,n)$: * Lower Case 's' as notation *

r-people, r-table

- $s(r,1) = Q_r^r = (r-1)!$
- $s(r,r) = 1$
- $s(r,r-1) = \binom{r}{2}$

Pick a pair to sit together →

- $s(r,r-2) = \frac{(3n-1)}{4}\binom{n}{3}$

from the textbook

$s(r,0) = 0$	if $r \ge 1$
$s(r,r) = 1$	if $r \ge 0$
$s(r,1) = (r-1)!$	for $r \ge 2$
$s(r,r-1) = \binom{r}{2}$	for $r \ge 2$

Note: How So?

$s(r,r-2) = 2\cdot\binom{n}{3} + \frac{1}{2}\binom{n}{2,2,n-4}$
$= \frac{(3n-1)}{4}\binom{n}{3}$

Claim

If $r \ge n$, then

$$s(r,n) = s((r-1),n-1) + (r-1)\,s(r-1,n)$$

WHY? Let $1,2,3,\dots,r$ be the people

*Focus on person 1

case 01: $s((r-1),n-1)$
Person 1 is at own table
r-1 people, n-1 table(s)
$s(r-1,n-1)$ Ways to complete

case 02: $(r-1)\,s(r-1,n)$
Person 1 is at not own table
Place all but person #1
r-1 people, 1 table
Choices for who sits to left of
$s(r-1,n)$ Ways to complete

Stirling Numbers of the Second Kind

* Upper Case 'S' as notation *

$S(r,n)$ = Number of ways to put r-distinct objects into n-identical boxes where no boxes are empty

A B C | D E F

Example: Let $r=4$ & try to find $S(4,n)$ for $n=1,2,3,4$

Set will be $x = \{1,2,3,4\}$

n=1: $S(4,1) = 1$

1	2
3	4

n=2: $S(4,2) = 7$

1	2	3	4
2	1	3	4
3	1	2	4
4	1	2	3

1,2	3,4
1,3	2,4
1,4	2,3

n=3: $S(4,3) = 6$

1	2	3,4
1	3	2,4
1	4	2,3
2	3	1,4
2	4	1,3
3	4	1,2

n=4: $S(4,4) = 1$

1	2	3	4

n≥5: $S(5,3) = 0$

$1+7+6+1 = 15$

the number of partitions of $\{1,2,3,4\}$ = 4^{th} Bell Number

$$S(4,1) + S(4,2) + S(4,3) + S(4,4) = B_4$$

The N^{th} Bell Number is given by the formula

$$B_n = \frac{1}{e}\sum_{k=0}^{\infty} \frac{k^n}{k!}$$

Stirling Numbers of the Second Kind Properties

(i) $S(0,0) = 1$,
(ii) $S(r,0) = S(0,n) = 0$ for all $r,n \in \mathbf{N}$,
(iii) $S(r,n) > 0$ if $r \ge n \ge 1$,
(iv) $S(r,n) = 0$ if $n > r \ge 1$,
(v) $S(r,1) = 1$ for $r \ge 1$,
(vi) $S(r,r) = 1$ for $r \ge 1$.
(vii) $S(r,2) = 2^{r-1} - 1$,
(viii) $S(r,3) = \frac{1}{2}(3^{r-1}+1) - 2^{r-1}$,
(ix) $S(r,r-1) = \binom{r}{2}$,
(x) $S(r,r-2) = \binom{r}{3} + 3\binom{r}{4}$.

Chapter 02
Binomial and Multinomial Coefficients

For $0 \le r \le n$,

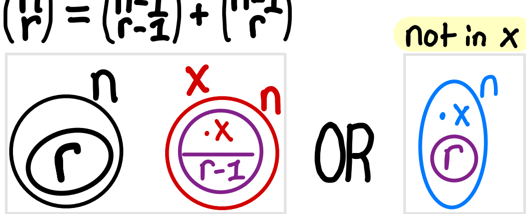
$$C_r^n = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

is the number of r-combinations of an n-element set.

For $r>0$ or $r \le 0$.
define $\binom{n}{r} = 0$
The numbers $\binom{n}{r}$ are called Binomial Coefficients

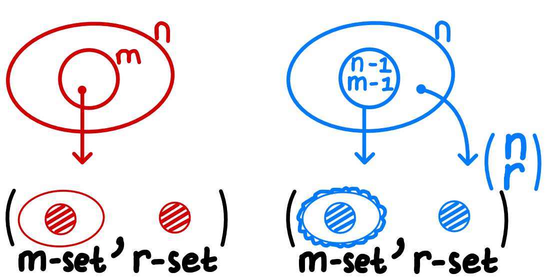
Relations satisfied by $\binom{n}{r}$:

- $\binom{n}{r} = \binom{n}{n-r}$
- $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$



- $\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$
- $\binom{n}{r} = \frac{n-r+1}{r}\binom{n}{r-1}$
- $\binom{n}{m}\binom{m}{r} = \binom{n}{r}\binom{n-1}{m-1}$

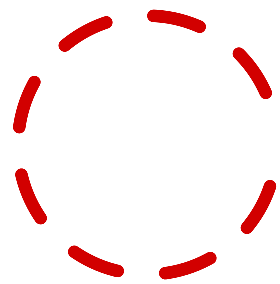
$$\binom{n}{m}\binom{m}{r} = \binom{n}{r}\binom{n-1}{m-1}$$



___ , ___ , ___ , ___

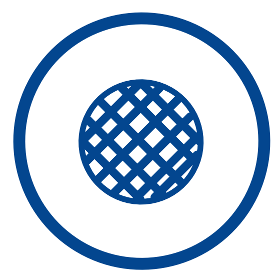
Line-up items

$$P_r^n = n(n-1)\dots(n-r+1)$$



Circle

$$Q_r^n = \frac{1}{r} P_r^n$$



Line-up & divide up with symmetry

$$C_r^n = \frac{P_r^n}{r!}$$

2.2: The Binomial Theorem

For any integer $n \geq 0$,

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n}x^0 y^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Why is it that the Binomial Coefficients appear in this formula?

Look at some terms in the middle

$$\binom{n}{r} x^{n-r} y^r$$

for some $0 \leq r \leq n$.

Well, $\underset{\text{Factor}}{(x+y)^n} = \underbrace{\underset{1}{(x+y)}\underset{2}{(x+y)}\underset{3}{(x+y)}\ldots\underset{n}{(x+y)}}$

we obtain the term $x^{n-r}y^r$ if and only if we choose r-factor to take the 'y' from and the remaining r-n-factors we took the 'x'-form

Example 01

$(x+y)^4 = \underset{1}{(x+y)}\underset{2}{(x+y)}\underset{3}{(x+y)}\underset{4}{(x+y)}$

$\binom{4}{2}$ ways to obtain xy^3

$\underset{1}{\{1,2,3\}}$ $\underset{2}{\{1,3,4\}}$ $\underset{3}{\{1,2,4\}}$ $\underset{4}{\{2,3,4\}}$

One can prove identites using the Binomial Theorem by choosing specific values of 'x' and 'y'

For instance, if $x = y = 1$, $(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k$

$\Rightarrow 2^n = \sum_{k=0}^{n} \binom{n}{k}$ * $\binom{n}{k}$ "k" counts the layers *

If $x = 1$ and $y = -1$

$(1+(-1))^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} (-1)^k$

$\rightarrow 0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$

$\rightarrow \binom{4}{0} - \binom{4}{1} + \binom{4}{2} - \binom{4}{3} + \binom{4}{4}$

$= 0$ (zero)

$\{1,2,3\}$	
$\varnothing$	NULL SET
$\{1\},\{2\},\{3\}$	$\binom{n}{1}$
$\{1,2\},\{1,3\},\{2,3\}$	$\binom{n}{2}$
$\{1,2,3\}$	$\binom{n}{3}$

Example 02

Show that for all positive integers 'n',

$$\sum_{r=1}^{n} r\binom{n}{r} = n2^{n-1}$$

Start with the Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Let $x=1$ in the Binomial Theorem

$$\rightarrow (1+y)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} y^k$$

* Power Rule and Chain Rule * $\frac{d}{dy}$ both sides to get

Power Rule: $\frac{d}{dy} y^k = k y^{k-1}$

$$\rightarrow (1+y)^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k y^{k-1}$$

Note: Change index

*Set $y=1$ to get...

$$\rightarrow n \cdot 2^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k$$

Recall

Power rule (x^n): $\{x^n\}' = nx^{n-1}$

THE CHAIN RULE

Jay Cummings
Calculus 1 Lecture Notes

Theorem.

Theorem 2.64 (*The Chain Rule*). Let f and g be differentiable functions. Then

$$(f(g(x)))' = f'(g(x)) \cdot g'(x).$$

In words: *Take the derivative of the outside, keep everything inside the same, and then multiply by the derivative of the inside.*

SUBJECT: Review and Preparation for Exam 02 DATE: 2023 / 03 / 10 PAGE#:

Example There are m 0's and n 1's where $m \geq 2$, $n \geq 2$, and $n \geq m$.

Q1) How many ways can the 0's and 1's be put in a line so that the 0's are all consecutive?

scratch work

$m=3$, $n=3$

000 111

$$m+1 \text{ ways}$$

000111	110001
100011	111000

Q2) ... put in a line such that the first position and last position must be a zero

$$\underset{\text{zero}}{\underline{0}}\ \underbrace{\binom{m+n-2}{n}}_{m+n-2}\ \underset{\text{zero}}{\underline{0}}$$

Q3) ... put in a line so that no two 0's, zeros, are adjacent?

m 0's, n 1's, no adjacent 0's

$$\_\ \underline{1}\ \_\ \underline{1} \dots \_\ \underline{1} \binom{n+1}{m}$$

#3 from HW6

We want to Know how many combination of students with the particular front row girls and back row boys, and the other students

1	G_2	2	3	G_3	4	5	6	G_1	7
8	B_1	9	B_2	10	11	B_3	12	13	B_4

G_1	G_2	G_3	1	2	3	4	5	6	7
B_3	8	B_4	9	B_2	10	B_1	11	12	13

We don't care about ordered positions of the individual students, so we use C^n_r not P^n_r

1st Front row →	G_1	G_2	G_3	1	2	3	4	5	6	7
2nd Back row →	8	9	10	11	12	13	B_1	B_2	B_3	B_4

Approach 01

Because the front row must always be populated by the girls we can pull any 1 of the 13 other students to fill the front 7 spots

G_1	G_2	G_3	1	2	3	4	5	6	7
8	9	10	11	12	13	B_1	B_2	B_3	B_4

$$\binom{13}{7} \times \underset{\text{1st Front row}}{10!} \times \underset{\text{2nd Back row}}{10!}$$

10 seats, so 10! seating arrangements

By default the back row gets chosen automatically

$$\text{In total } \underset{1716}{\binom{13}{7}} \times \underset{3628800}{10!} \times \underset{3628800}{10!} \cong 22{,}596{,}613{,}080{,}000{,}000$$

$$\binom{10}{3}3!\binom{10}{4}4!\,13!$$

$$\rightarrow \frac{10!}{3!\,7!}\cdot 3! \cdot \frac{10!}{4!\,6!}\cdot 4! \cdot 13! = \binom{13}{7}(10!)^2$$

Approach 02

G_1	G_2	G_3	1	2	3	4	5	6	7
8	9	10	11	12	13	B_1	B_2	B_3	B_4

Back row choose the 4 boys

Config. of boys amongst themselves

$$\binom{10}{3} \times 3! \times \binom{10}{4} \times 4! \times 13!$$

Front row choose the 3 girls

Config. of girls amongst themselves

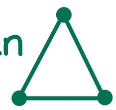
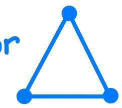
Placement of NPCs

$$\text{In total } \binom{10}{3} \times 3! \times \binom{10}{4} \times 4! \times 13! \cong 22{,}596{,}613{,}080{,}000$$

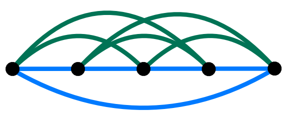
SUBJECT: # Ramsey Number R(K) DATE: 2023 / 03 / 27

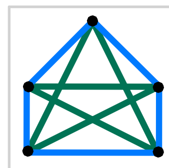
What is a Ramsey Number R(K)?

Let us start with 5 points
and try to color the connections with
green and blue such that we do not see

an  or 

This is good coloring!





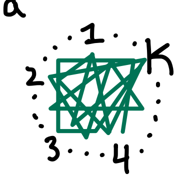
Can we do the same with six points?

No. $R(6)=6$

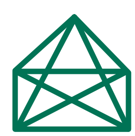
Fewest number of points such
that no matter how we color

We are forced to create a
green triangle (K_3)
or a blue triangle (K_3)

In general, $R(K)$ is the fewest number of
points such that no matter how we color the
connections between these points we are forced
to create a

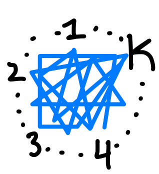
green K_K 

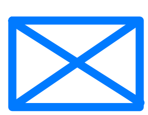
Green K_4



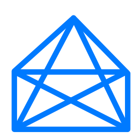
Green K_5

OR

blue K_K 

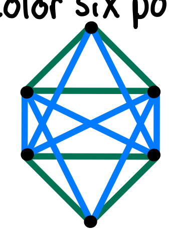


Blue K_4



Blue K_5

Can we color six points with no green K_4 or a blue K_4



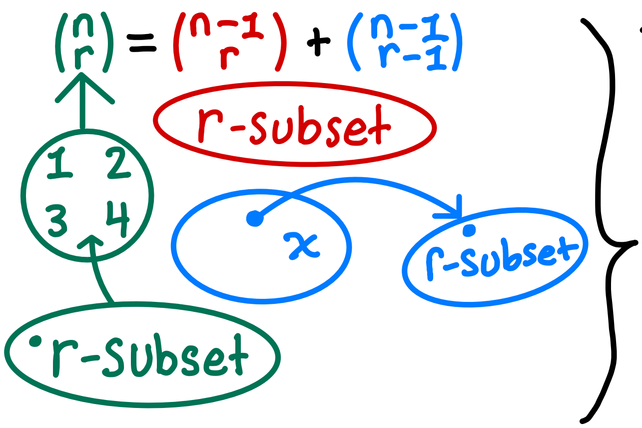
$\rightarrow R(4)>6$

it turns out that $R(4)=18$

$43 \leq R(5) \leq 48$

SUBJECT: Notes and comment from Exam 02 DATE: 2023/03/29 PAGE#:

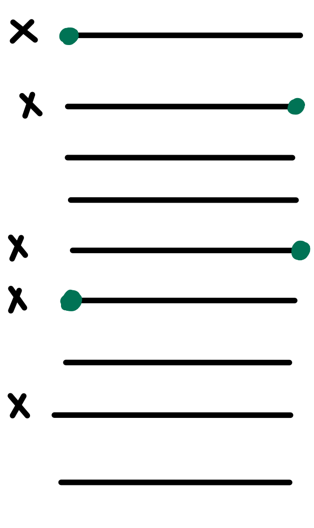
Q4b $\binom{n}{r}=\binom{n-1}{r}+\binom{n-1}{r-1}$



Is related to pascal's triangle

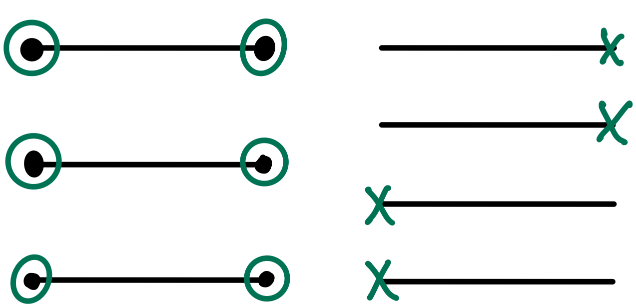
$$
\begin{array}{c}
1 \\
1\ 1 \\
1\ 2\ 1 \\
1\ 3\ \boxed{3\ 1} \\
1\ 4\ 6\ \boxed{4}\ 1
\end{array}
$$

7b



$\binom{10}{5}2^5$

7C



$\binom{10}{3}\binom{7}{4}2^4$

SUBJECT: 2.2: The Binomial Theorem Continued DATE: 2023 / 03 / 29 PAGE#:

Quick Warm-up

Does

$$\sum_{r=0}^{0}\binom{n+1}{r+1} = \sum_{r=0}^{n+1}\binom{n+1}{r} - \sum_{r=0}^{0}\binom{n+1}{r}$$

* Try writing things out *

LHS

minus-out the Empty-set

$$\sum_{r=0}^{n}\binom{n+1}{r+1} = \binom{n+1}{1} + \binom{n+1}{2} + \cdots + \binom{n+1}{n+1} = 2^{n+1} - 1$$

RHS

$$\sum_{r=0}^{n+1}\binom{n+1}{r} - 1 = \binom{n+1}{0} + \binom{n+1}{1} + \cdots + \binom{n+1}{r+1}$$

Currently we are studying the Binomial Theorem

$$^{*}(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$$

*Where 'n' is a positive integer

We have used this to prove combinatorial identites

For example, if $x=y=1$ then

$$\rightarrow 2^n = \sum_{k=0}^{n}\binom{n}{k}$$

Vandermonde's Identity

↳ Based on: $(1+x)^{m+n} = (1+x)^m(1+x)^n$

Example 03 | For all integers $m, n, r \geq 1$

$$\binom{m+n}{r} = \sum_{i=0}^{r}\binom{m}{i}\binom{n}{r-i}$$

Two ways to approach this

Approach 01: Binomial Theorem – Algebraic

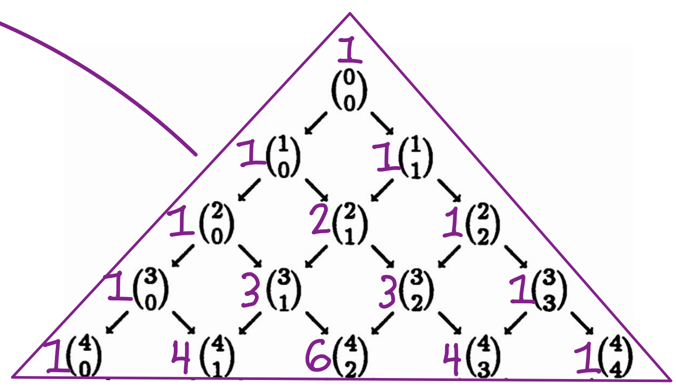
Key consequence of the Binomial Theorem is that the coefficent of x^i in $(1+x)^m$ is $\binom{m}{i}$ * Pascal's Triangle *

$(1+x)^2 = 1 + 2x + x^2$

$(1+x)^3 = 1 + 3x + 3x^2 + x^3$

$\binom{3}{1}$ $\binom{3}{2}$

```
    1
   1 1
  1 2 1
 1 3 3 1
1 4 6 4 1
```



Consider the idenity

$$\underbrace{(1+x)^{m+n}}_{\uparrow} = (1+x)^m(1+x)^n$$

Coefficent of x^r is $\binom{m+n}{r}$

$\binom{m}{0}1$	$\binom{n}{r}x^r$
$\binom{m}{1}x$	$\binom{n}{r-1}x^{r-1}$
$\binom{m}{2}x^2$	$\binom{n}{r-2}x^{r-2}$
$\vdots$	
$\binom{m}{r}x^r$	$\binom{n}{0}1$

Leverages This Rule

Key consequence of the Binomial Theorem is that the coefficent of x^i in $(1+x)^m$ is $\binom{m}{i}$

For the right hand side, the coefficent of x^r is going to be

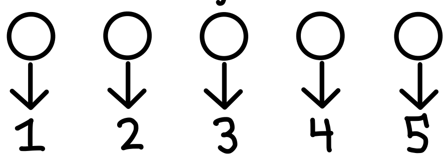
$$\binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \binom{m}{2}\binom{n}{r-2} + \cdots + \binom{m}{r}\binom{n}{0} = \sum_{i=0}^{r}\binom{m}{i}\binom{n}{r-i}$$

Coefficent of x^0 — {Coefficent of x^r}

$$\rightarrow \binom{m+n}{r} = \sum_{i=0}^{r}\binom{m}{i}\binom{n}{r-i}$$

Revisit Bijections

Recall that a bijection is a function $f: A \to B$ that is one-to-one (1:1) and "onto". If A & B are finite and there is a bijection $f: A \to B$; then $|A| = |B|$



Recall: Section 1.5: Injection & Bijections Principles
Let 'A' and 'B' be sets 'A' function $f: A \to B$ is
<u>Bijective</u> if f is both
Injective (1-to-1) and Surjective (onto)

Notation $\mathcal{A}$ and A are different
Notation $\mathcal{B}$ and B are different

<u>Example01</u> Let $x = \{1, 2, 3, \cdots, n\}$, $\mathcal{A} = \{A \subseteq x : |A| = r\}$
$\mathcal{B} = \{B \subseteq x : |B| = n - r\}$

If $n=5$ and $r=3$, write-down a bijection from $\mathcal{A}$ and $\mathcal{B}$

Here $x = \{1,2,3,4,5\}$
$\mathcal{A} = \{A \subseteq x : |A| = \overset{r}{3}\}$
$\mathcal{B} = \{B \subseteq x : |B| = \underset{n-r}{5}\}$

$\mathcal{A}$		$\mathcal{B}$
$\{1,2,3\}$	$\to$	$\{4,5\}$
$\{1,2,4\}$	$\to$	$\{3,5\}$
$\{1,2,5\}$	$\to$	$\{3,4\}$
$\{1,3,4\}$	$\to$	$\{2,5\}$
$\{1,3,5\}$	$\to$	$\{2,4\}$
$\{1,4,5\}$	$\to$	$\{2,3\}$
$\{2,3,4\}$	$\to$	$\{1,5\}$
$\{2,3,5\}$	$\to$	$\{1,4\}$
$\{2,4,5\}$	$\to$	$\{1,3\}$
$\{3,4,5\}$	$\to$	$\{1,2\}$

A formula for this bijection
$f = \mathcal{A} \to \mathcal{B}$ is
$$f(A) = X \setminus A$$
r n-r

Revisit Bijections

Example 02 Let $\mathcal{X}=\{1,2,3,\cdots,n\}$

$\mathcal{A}=\{A\subseteq\mathcal{X} : n\notin A\}$

$\mathcal{B}=\{B\subseteq\mathcal{X} : n\in B\}$

If $n=4$, find a bijection from $\mathcal{A}$ to $\mathcal{B}$ and a formula for your bijection

$\mathcal{A}: \{\emptyset\},\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}$

$\mathcal{B}: \{4\},\{1,4\},\{2,4\},\{3,4\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,3,4\}$

$f(A)=A\cup\{4\}$ $\qquad$ $f(a)=\mathcal{X}\setminus A$

also works complementry

Returning to Binomial Theorem and Vandermonde's Identity

$(1+x)^{m+n}=(1+x)^m(1+x)^n$

$x^0 \quad x^r$

$x^1 \quad x^{r-1}$

Coefficient

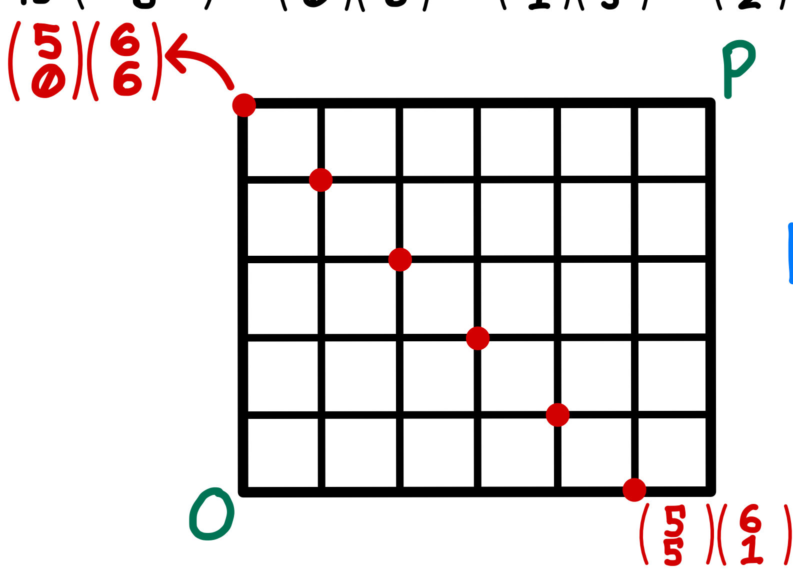
x^r is $\binom{m+n}{r}$

Vandermonde's Identity

$$\binom{m+n}{r}=\sum_{j=0}^{r}\binom{m}{j}\binom{n}{r-j}$$

For instance if $m=5$, $n=6$, $r=6$, then Vandermonde's Identity

is $\binom{5+6}{6}=\binom{5}{0}\binom{6}{6}+\binom{5}{1}\binom{6}{5}+\binom{5}{2}\binom{6}{4}+\binom{5}{3}\binom{6}{3}+\binom{5}{4}\binom{6}{2}+\binom{5}{5}\binom{6}{1}$



Number of shortest path is

$$\binom{5+6}{6}=462$$

SUBJECT: Example to help with HW08 DATE: 2023/04/05 PAGE#:

$$\sum_{r=0}^{n} \frac{(-1)^r}{r+1}\binom{n}{r} = \sum_{r=0}^{n}\frac{(-1)^r}{r+1}$$

Be able to recognize identites/prove

- $\sum_{i=0}^{n}\binom{n}{j} = 2^n \Leftarrow$ Count the number of subset $\{1,2,3,\ldots,N\}$
- Count subset by size
- $\binom{3}{0}-\binom{3}{1}+\binom{3}{2}-\binom{3}{3}=2^3$
- $\sum_{j=0}^{N}(-1)^j\binom{N}{0} = 0$

$$\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$$

$$\sum_{r=0}^{n} \frac{(-1)^r}{r+1}\binom{n}{r} = \sum_{r=0}^{n}\frac{(-1)^r}{r+1}\cdot\frac{n!}{r!(n-r)!} = \sum_{r=0}^{n}(-1)^r\frac{n!}{(r+1)!(n-r)!}$$

$$\binom{n+1}{r-1} = \frac{(n+1)!}{(r+1)!(n+1-(r+1))!}$$

$$\Rightarrow \sum_{r=0}^{n}(-1)^r\frac{n!}{(r+1)!(n-r)!}\cdot\binom{n+1}{n+1}$$

$$\Rightarrow \sum_{r=0}^{n}\frac{(-1)^r}{r+1}\binom{n+1}{n+1} \Rightarrow \frac{1}{r+1}\sum_{r=0}^{n}(-1)^r\binom{n+1}{n+1}$$

* Write it out where $r=0$ *

$$\Rightarrow \frac{1}{r+1}\left((-1)^0\binom{n+1}{0+1}+(-1)^1\binom{n+1}{1+1}+(-1)^2\binom{n+1}{2+1}+\cdots+(-1)^n\binom{n+1}{n+1}\right)$$

Missing: $(-1)^1\binom{n+1}{0}$

$$\Rightarrow \frac{1}{r+1}\left(-\binom{n+1}{0}(-1)^0\binom{n+1}{0+1}+(-1)^1\binom{n+1}{1+1}+(-1)^2\binom{n+1}{2+1}+\cdots+(-1)^n\binom{n+1}{n+1}\underbrace{+\binom{n+1}{0}}\right)$$

$$\boxed{\Rightarrow \frac{1}{r+1}(0+1) = \frac{1}{r+1} \blacksquare}$$

* The Identity *

$$\binom{N+1}{R+1}\binom{N}{R} = \binom{N+1}{R+1}$$

The world needs balance if we have $-\binom{n+1}{0}$ at the front then we need $+\binom{n+1}{0}$ in the back

SUBJECT: # Multinomial Theorem DATE: 2023 / 04 / 05 PAGE#:

Turn our attention to the Multinomial Theorem

Recall the Binomial Theorem

* Instead of terms x & y we rewrite x with subscripts *

$$(x_2+x_2)^n + \sum_{r=0}^{n}\binom{n}{r}x_1^{n-r}x_2^{r}$$

~ for $n \geq 1$ an integer ~

Real quick let us recall
how the $\binom{n}{r}$ shows up in the Binomial Theorem

If $n=5$,

$$(x_1+x_2)(x_1+x_2)(x_1+x_2)(x_1+x_2)(x_1+x_2)$$

Factor Number 1 2 3 4 5

What is the coefficent of $x_1^2x_2^3$?

5 factors choose
3 factors where
you take x_2 $\binom{5}{3}$

Coefficent of $x_1^2x_2^3$
in (x_1+x_2) is $\binom{5}{3}$

Now, lets us add an x_3

$$(x_1+x_2+x_3)^5$$
$$=(x_1+x_2+x_3)(x_1+x_2+x_3)(x_1+x_2+x_3)(x_1+x_2+x_3)(x_1+x_2+x_3)$$

Factor #: 1 x_1 2 x_3 3 x_1 4 x_1 5 x_2

Can we find the coefficent of:

- $x_1^3x_2x_3$? $\binom{5}{3}$ 1,2,3,4,5
- $x_1^2x_2^2x_3$?

1,2,3 1,2,4 1,2,5

1,3,4 1,3,5

Solution: $\frac{5!}{3!1!1!}$

SUBJECT: **Multinomial Theorem - Continued** DATE: 2023 / 04 / 07 PAGE#:

Examples involving identites related to the Binomial Theorem

Example 01 | Prove that $\sum_{i=0}^{n}\binom{n}{i}^2=\binom{2n}{n}$

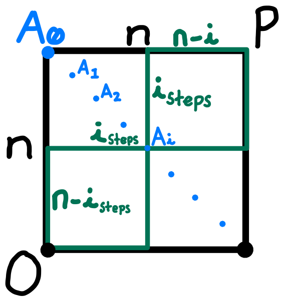
Approach 1

Use the Vandermonde's Identity

$$(1+x)^{m+n}=(1+x)^m(1+x)^n$$

OR Approach 2

Shortest Path Method (Lattice Path Argument)



Number of shortest paths through A_i is

$$\binom{n}{i}\binom{n}{i}$$

(steps chosing RIGHT)(steps chosing UP)

Example 02 | Prove that

$$\sum_{r=0}^{n}\frac{(2n)!}{(r!)^2((n-r)!)^2}=\binom{2n}{n}^2$$

Do we recognize anything?

$$\sum_{r=0}^{n}\boxed{\frac{(2n)!}{(r!)^2((n-r)!)^2}}=\binom{2n}{n}^2$$

Looks like $C_r^n=\binom{n}{r}=\frac{n!}{r!(n-r)!}$

*Especially the Binomial Coefficent in the denominator : $r!(n-r)!$

Try expanding things out!

$$\sum_{r=0}^{n}\frac{(2n)!}{(r!)^2((n-r)!)^2}=\binom{2n}{n}^2$$

LHS

$$\Rightarrow\sum_{r=0}^{n}=\frac{n!}{r!(n-r)!}\cdot\frac{n!}{r!(n-r)!}\cdot\frac{(2n)!}{n!n!}$$

$$\Rightarrow\sum_{r=0}^{n}\binom{n}{r}\binom{n}{r}\cdot\frac{(2n)!}{n!n!}\Rightarrow\frac{(2n)!}{n!n!}\sum_{r=0}^{n}\binom{n}{r}^2$$

$$\Rightarrow\frac{(2n)!}{n!n!}\cdot\binom{2n}{n}\Rightarrow\binom{2n}{n}\binom{2n}{n}\Rightarrow\binom{2n}{n}^2\therefore \text{LHS}=\text{RHS}$$

Announcements : Quiz Monday - 4/10

- Binomial Theorem & Identities
- Bijection & general formulas

Example of identities to Know

$$\sum_{r=0}^{n}\binom{n}{r}=2^n$$

Example 03 | Prove that

$$\sum_{r=0}^{n}\binom{2n}{r}=2^{2n-1}+\frac{1}{2}\binom{2n}{n}$$

Try expanding $\sum_{r=0}^{n}\binom{2n}{r}=2^{2n-1}+\frac{1}{2}\binom{2n}{n}$

Well,

$$\sum_{r=0}^{n}\binom{2n}{r}=\binom{2n}{0}+\binom{2n}{1}+\binom{2n}{2}+\cdots+\binom{2n}{n}+\binom{2n}{n+1}+\cdots+\binom{2n}{2n}$$

We could keep going

$\frac{1}{2}\binom{2n}{1}+\frac{1}{2}\binom{2n}{2n-1}$

$$\boxed{\frac{1}{2}\binom{2n}{0}+\frac{1}{2}\binom{2n}{2n}}\quad\boxed{\frac{1}{2}\binom{2n}{2}+\frac{1}{2}\binom{2n}{2n-2}}\quad\boxed{\frac{1}{2}\binom{2n}{2}+\frac{1}{2}\binom{2n}{n}}\longrightarrow$$

$$\Rightarrow\frac{1}{2}\left(\binom{2n}{0}+\binom{2n}{2n}\right)+\frac{1}{2}\left(\binom{2n}{1}+\binom{2n}{2n-1}\right)+\cdots+\frac{1}{2}\left(\binom{2n}{n}+\binom{2n}{n}\right)$$

$$\Rightarrow\frac{1}{2}\sum_{r=0}^{2n}\binom{2n}{r}+\frac{1}{2}\binom{2n}{n}$$

$$\Rightarrow\frac{1}{2}\cdot2^{2n}+\frac{1}{2}\binom{2n}{n}$$

$$\Rightarrow2^{2n-1}+\frac{1}{2}\binom{2n}{n}$$

Let us finish today by talking a bit more about the Multinomial Theorem

Look at

$(x_1+x_2+x_3)(x_1+x_2+x_3)(x_1+x_2+x_3)(x_1+x_2+x_3)(x_1+x_2+x_3)$

*If we picked all x_1

Permutations of $x_1x_2x_3$ of length 5

$=x_1x_1x_1x_1x_1+x_1x_1x_1x_1x_2+x_1x_1x_1x_1x_3$

$+x_1x_1x_1x_2x_1+x_1x_1x_1x_3x_1+\ldots$

The unique elements can be in a different position

$$\boxed{\binom{5}{4,1,0}:\ \begin{array}{l}4-x_1\\1-x_2\\0-x_3\end{array}}$$

$$\binom{5}{4,1,0}=\frac{5!}{4!1!0!}$$

Finish Chapter 02 by stating the Multinomial Theorem

$$(x_1 + x_2 + x_3 + \dots + x_m)^n = \sum_{(n_1, n_2, \dots, n_m) \in S} \binom{n}{n_1, n_2, n_3} x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m}$$

Where S is all m-tuples of non-negative integers with $n_1 + n_2 + \dots + n_m = n$

Example $(x_1 + x_2 + x_3 + x_4 + x_5)^{100}$

(20, 15, 5, 10, 50)

$$\Rightarrow \binom{100}{20, 15, 5, 10, 50} x_1^{20} x_2^{15} x_3^{5} x_4^{10} x_5^{50}$$

Let 'k' & 'n' be positive

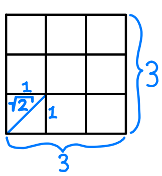
The Piegonhole Principle states that if at least $kn+1$ objects are placed in n-boxes, then at least one box has at least $k+1$ objects.

There is a chance that in this class there will be 2 students with the same birth month. Given that we have more than 12 Students in class.

Example

Show that for any ten points in the plane within a square of side length 3, there are two points within distance $\sqrt{2}$ of each other.

* Trick: form boxes/holes *



Missing additional info on this section

Example | Ten teams played in a tourment where each pair of teams play exactly once

Win : +1
Draw : 0
Loss : -1

If at least 70% of the game end in a draw, show that there are at least two teams with the same numbers of points.

There are exactly $\binom{10}{2} = \frac{10 \cdot 9}{2} = 45$ games played and at least $0.70(45) = 31.5$ had to end in a draw $\Rightarrow$ at least 32 games end in a draw

* Pigeonhole : two teams with same scores *

Now the possible scores for a team are:

$-9,-8,-7,-6,-5,-4,-3,-2,-1,0,1,2,3,4,5,6,7,8,9$

Aiming for a contridiction suppose all ten teams got a different score

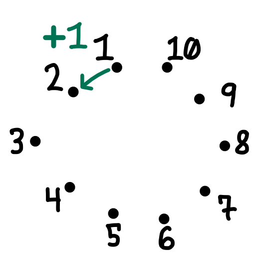
* 13 non-draws $\Rightarrow \binom{10}{2} = 45$

- ≥ 32 end in draw
- at most 13 did not end in draw

$$\underbrace{-9,-8,-7,-6,-5,-4,-3,-2,-1}_{\text{Negative scores}},0,\underbrace{1,2,3,4,5,6,7,8,9}_{\text{Positive scores}}$$

- 10 teams
- 1 team could get 0 points

Note that if five teams have five different positive scores then the number of games that did not end in a draw is at least $1+2+3+4+5=15$ which contradicts the at most 13 non-draws



Likewise, if five teams have negative scores, then the number of games that do not end in a draw is again at least 15

Therefore, at least 4 teams got a positive score and at most 4 teams got a non-negative score

$\Rightarrow$ At least two teams got Zero points, a contradiction

---

Example | Let $X \subseteq \{1,2,\ldots,99\}$ with $|X| = 10$.
Show that 'x' contains disjoint subsets Y & Z such that add the elements in Y is the same sum as adding the elements Z.

For instance : $X = \{2,7,15,19,23,50,56,60,66,99\}$

$Y = \{7,99\}$ $Z = \{50,56\}$ $7 + 99 = 50 + 56$

$\Rightarrow$ So, $\binom{99}{10}$

SUBJECT: # Chapter 4 - Inclusion & Exclusion Principle

DATE: 2023/04/24 PAGE#:

Exam 3 - Last Question Question # 8

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
$\boxed{1\ 5\ 10}\ \boxed{10\ 5\ 1}$

Absolutely Know

$$\boxed{\sum_{k=0}^{n}\binom{n}{k}=2^n}$$

$$\sum_{k=0}^{n-1}\binom{2n-1}{k}=\binom{2n-1}{0}+\binom{2n-1}{1}+\binom{2n-1}{2}+\dots+\binom{2n-1}{n-1}=\frac{1}{2}\cdot 2^{2n-1}$$

$$=\frac{1}{2}\left[\binom{2n-1}{0}+\binom{2n-1}{2n-1}\right]+\frac{1}{2}\left[\binom{2n-1}{1}+\binom{2n-1}{2n-2}\right]+\dots+\frac{1}{2}\left[\binom{2n-1}{2n-2}+\binom{2n-1}{n}\right]=\frac{1}{2}\cdot 2^{2n-1}$$

$$n=5:\ \sum_{r=0}^{4}\binom{9}{r}=\binom{9}{0}+\binom{9}{1}+\binom{9}{2}+\binom{9}{3}+\binom{9}{4}+\binom{9}{5}+\binom{9}{6}+\binom{9}{7}+\binom{9}{8}+\binom{9}{9}$$

$$=1/2$$

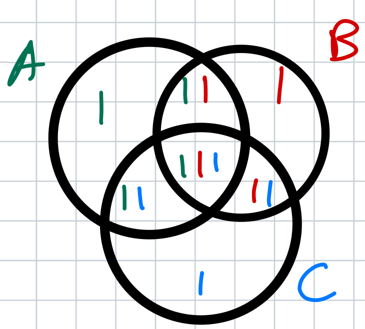
This week - lecture
05/01 - Homework 10
05/05 - Individual Quiz 11
05/08 - Homework 11
05/10 - Group Quiz 12
05/12 - Make-up Day
- Optional Quiz
- 2 homeworks for half credit

Exam 03 - 05/15 12:45-2:45

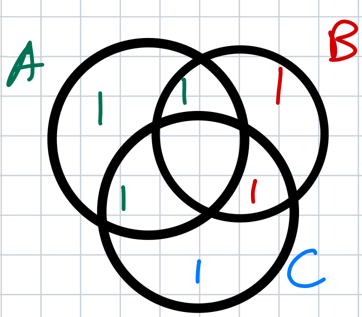
If A and B are finite sets then $|A\cup B|=|A|+|B|-|A\cap B|$

A (I (II) I) B

What if we have three sets A, B, and C?



$$|A\cup B\cup C|=|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|$$



SUBJECT: DATE: / / PAGE#:

Principle of Inclusion-Exclusion: For any finite sets $A_1, A_2, \dots, A_q$ where $q \geq 2$

Stick to this

$$* \; |A_1 \cup A_2 \cup \dots \cup A_q| = \sum_{i\leq 1}^{q} |A_i| - \sum_{1\leq i<j\leq q} |A_i \cap A_j| + \sum_{1\leq i<j<k\leq q} |A_i \cap A_j \cap A_k| + \dots + (-1)^{q+1} |A_1 \cap A_2 \cap \dots \cap A_q|$$

"Double summation"

$$\sum_{i=1}^{q-1} \sum_{j=i+1}^{q} |A_i \cap A_j| = \sum_{j=2}^{q} |A_1 \cap A_j| + \sum_{j=3}^{q} |A_2 \cap A_j| + \dots + |A_{q-1} \cap A_q|$$

$1,2;\ 1,3;\ \dots;\ 1,q$

$2,3;\ 2,4;\ \dots;\ 2,q$

$$\sum_{S \subseteq \{1,2,3,\dots,q\},\ |S|=2}$$

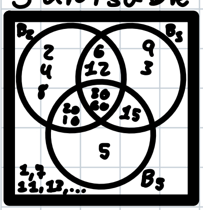
Exam 03 Q#5

Prove $\binom{n}{r}\binom{n-r}{m-r}=\binom{n}{m}\binom{m}{r}$

$$\binom{n}{r}\binom{n-r}{m-r}\overset{?}{=}\binom{n}{m}\binom{m}{r}$$

$$\frac{n!}{r!\,(n-r)!}\cdot\frac{(n-r)!}{(m-r)(n-m)!}\overset{?}{=}\frac{n!}{m!\,(n-m)!}\cdot\frac{m!}{r!\,(m-r)!}$$

Ex | Let $S=\{1,2,3,\ldots,500\}$.
Find the number of integers in S divisable by 2, 3, 5.



$B_2=\{\#$ in S divisable by 2$\}$
$B_3=\{\#$ in S divisable by 3$\}$
$B_5=\{\#$ in S divisable by 5$\}$

By I.E.,

$$|B_2\cup B_3\cup B_5|=|B_2|+|B_3|+|B_5|$$
$$-|B_2\cap B_3|-|B_2\cap B_5|-|B_3\cap B_5|$$
$$+|B_2\cap B_3\cap B_5|$$

General Fact: For $1\le k\le n$, there are $\lfloor\frac{n}{k}\rfloor$ integers in $\{1,2,\ldots,n\}$ that are divisable by k. floor$(\frac{n}{k})$

$|B_2|=\lfloor\frac{500}{2}\rfloor=250$

$|B_3|=\lfloor\frac{500}{3}\rfloor=166$

$|B_5|=\lfloor\frac{500}{5}\rfloor=100$

$|B_2\cap B_3|=\lfloor\frac{500}{6}\rfloor=83$

$|B_2\cap B_5|=\lfloor\frac{500}{10}\rfloor=50$

$|B_3\cap B_5|=\lfloor\frac{500}{15}\rfloor=33$

$|B_2\cap B_3\cap B_5|=\lfloor\frac{500}{30}\rfloor=16$

$$|B_2\cup B_3\cup B_5|=250+166+100-83-50-33-16=366$$

There are 366 integers in $\{1,2,3,\ldots,500\}$ divisable by 2, 3, or 5

SUBJECT: Inclusion-Exclusion Continue DATE: 2023 / 04 / 26 PAGE#:

Derangements

how calc sneaks into Combinatorics

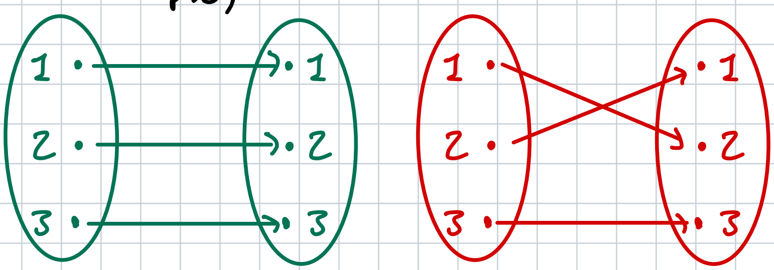
let $\mathcal{F}$ be the collection of functions that are one-to-one and onto from $\{1,2,3\}$ to $\{1,2,3\}$

How many functions in $\mathcal{F}$ have no fixed points ($f(x) \neq i$ for all i)?

fixed pt an element that does not point back to itself.

Let $A_i = \{f \in \mathcal{F} \cdot f(i) = i\}$

for example,



are the functions in A_3

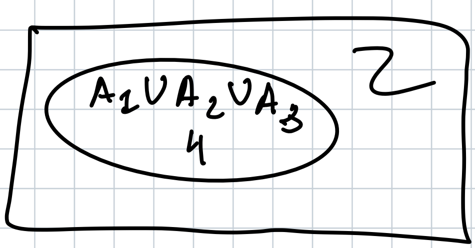
$$\Rightarrow |A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

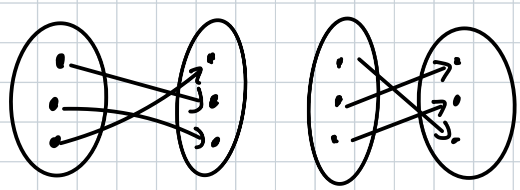
$$= (2+2+2) - 1 - 1 - 1 + 1 = 4$$

$$\Rightarrow 4 = |A_1 \cup A_2 \cup A_3|$$

But there are $3! = 6$ bijections from $\{1,2,3\}$ to $\{1,2,3\}$

$\Rightarrow$ there are 2 functions in $\mathcal{F}$ with no fixed points





2023/04/12

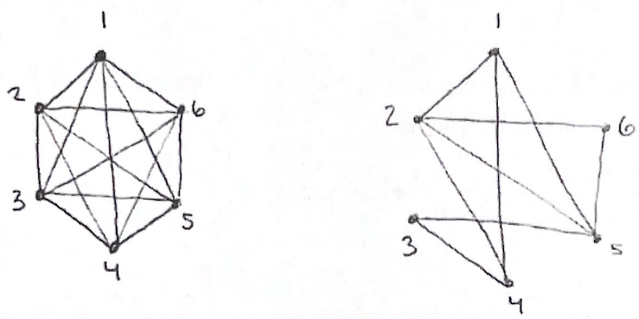
Announcements :
- No class this Friday 23/4/14
- Old exam 03 posted

California State University Sacramento - Math 101

Quiz #9

Name: ________________________

1) On the left, every two numbers from $\{1,2,3,4,5,6\}$ are connected by an edge. This is a visual representation of the *complete graph on 6 vertices* which we denote by K_6.



2-combs	H or T	Connect 2-comb. ?
{1,2}	H	Yes
{1,3}	T	No
{1,4}	H	Yes
{1,5}	H	Yes
{1,6}	T	No
{2,3}	T	No
{2,4}	H	Yes
{2,5}	H	Yes
{2,6}	H	Yes
{3,4}	H	Yes

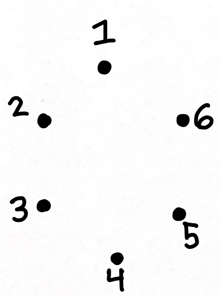
On the right is a random subgraph of K_6 that was determined by the following procedure.

Consider each 2-combination $\{a,b\}$ in the ordered list

$$\cancel{\{1,2\}},\ \cancel{\{1,3\}},\ \cancel{\{1,4\}},\ \{1,5\},\ \cancel{\{1,6\}},\ \{2,3\},\ \cancel{\{2,4\}},\ \cancel{\{2,5\}},\ \{2,6\},\ \{3,4\},\ \cancel{\{3,5\}},\ \{3,6\},\ \cancel{\{4,5\}},\ \{4,6\},\ \cancel{\{5,6\}}$$

For each $\{a,b\}$, flip of fair coin where H and T are equally likely. If the result is H, keep $\{a,b\}$ connected. If the result is T, remove the connection. The table on the right in the figure above shows this process over the first ten pairs, however, it was completed for all fifteen pairs in order to produce the random graph on the right.

Instructions: Perform this process to draw a random subgraph of K_6. You may use www.random.org/coins to perform 15 independent flips at once.



2-comb	Head or Tails	connect 2-comb.?
{1,2}	T	
{1,3}	T	
{1,4}	T	
{1,5}	H	
{1,6}	T	
{2,3}	H	
{2,4}	T	
{2,5}	T	
{2,6}	H	
{3,4}	H	
{3,5}	T	
{3,6}	H	
{4,5}	T	
{4,6}	H	
{5,6}	T	

2023/04/12

2) Let p be a real number with $0 \leq p \leq 1$ and let n be a positive integer. Define the function $f_{n,k} : \mathbb{R} \to \mathbb{R}$ by the rule

$$f_{n,k}(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & \text{if } k \in \{0, 1, 2, \ldots n\}, \\ 0 & \text{otherwise.} \end{cases}$$

The function $f_{n,k}$ is the probability density function of a Binomial random variable with parameters n and p. For $k \in \{0, 1, \ldots, n\}$, the value $f_{n,k}(k)$ is the probability of k successes in n independent trials with success probability p.

(a) Prove, using the Binomial Theorem, that $\sum_{k=0}^{n} f_{n,k}(k) = 1$.

(b) Suppose that a biased coin, which comes up heads with probability 0.7, is tossed 4 independent times. Complete the following table.

Number of heads k	Probability of k heads
0	$f_{4,0.7}(0) = \binom{4}{0}(0.7)^0(0.3)^4 = 0.0081$
1	
2	
3	
4	

(c) Use the table from part (b) to compute

$$0 \cdot f_{4,0.7}(0) + 1 \cdot f_{4,0.7}(1) + 2 \cdot f_{4,0.7}(2) + 3 \cdot f_{4,0.7}(3) + 4 \cdot f_{4,0.7}(4).$$

This value is the average number of heads we would expect to see in 4 independent tosses.

(d) Prove that for any real number p with $0 \leq p \leq 1$ and positive integer n,

$$np = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}.$$

This value is the expected number of successes if n independent trials are done where the probability of success is p.

3) Show that for integers $1 \leq r \leq m \leq n$,

$$\binom{n}{m}\binom{m}{r} = \binom{n}{r}\binom{n-r}{m-r}.$$

4) Show that for any integers $n \geq m \geq 1$,

$$\sum_{r=m}^{n} \binom{n-m}{n-r} = 2^{n-m}.$$

5) Show that for all integers $n \geq 1$,

$$\sum_{r=1}^{n} r^2 \binom{n}{r} = n(n+1)2^{n-2}.$$

Unpolished Notes

Given a Sequence $a_1, a_2, a_3 \ldots$

a <u>recurrence</u> <u>relation</u> is a formula

that expresses a_n in terms of $a_1, a_2, \ldots, a_{n-1}$

Example | $1, 2, 4, 6, 8, 16, \ldots$

$a_1 = 1,\ a_2 = 2,\ a_3 = 4, \ldots$

and in general $a_n = 2a_{n-1}$, for $n \geq 2$.

One of the most famous recurrence relation is the Fibonacci Sequence $(1, 1, 2, 3, 5, 8, 13, \ldots)$

$a_0 = 1$
$a_1 = 1$
$a_n = a_{n-1} + a_{n-2} \quad n \geq 2$

Characteristic Equations
$y'' = y' + y$

Look at the equation

$$x^2 = x + 1$$

$$\Rightarrow x^2 - x - 1 = 0$$

$$\Rightarrow x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$\Rightarrow x = \frac{1 \pm \sqrt{5}}{2}$$

$$\Rightarrow a_n = A_1\left(\frac{1+\sqrt{5}}{2}\right)^n + A_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

From $a_0 = 1$ we get

$$1 = A_1\left(\frac{1+\sqrt{5}}{2}\right)^0 + A_2\left(\frac{1-\sqrt{5}}{2}\right)^0$$

$$* \Rightarrow 1 = A_1 + A_2$$

From $a_1 = 1$, we get

$$* \ 1 = A_1\left(\frac{1+\sqrt{5}}{2}\right)^1 + A_2\left(\frac{1-\sqrt{5}}{2}\right)^1$$

The solution for A_1 and A_2 is

$$A_1 = \frac{1+\sqrt{5}}{2\sqrt{5}} \ \& \ A_2 = \frac{-1+\sqrt{5}}{2\sqrt{5}}$$

Plug into our a_n formula to get

$$a_n = \underbrace{\left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)}_{A_1}\left(\frac{1+\sqrt{5}}{2}\right)^n + \underbrace{\left(\frac{-1+\sqrt{5}}{2\sqrt{5}}\right)}_{A_2}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

roots of $x^2 = x+1$

A recurrence relation of the form

$$C_0 a_n + C_1 a_{n-1} + \dots + C_r a_{n-r} = 0$$

where the C_i's are constants and $C_0 \neq 0$ and $C_r \neq 0$ is an <u>r^{th} order linear homogeneous recurrence relation</u>

The <u>characteristic equation</u> of

$$C_0 a_n + C_1 a_{n-1} + \dots + C_r a_{n-r} = 0$$

is

$$C_0 x^r + C_1 x^{r-1} + \dots + C_r = 0$$

For instance, the characteristic equation of

$$a_n - 2a_{n-1} + 3a_{n-2} - 5a_{n-3} = 0$$

is

$$x^3 - 2x^2 + 3x - 5 = 0$$

How did we get there?

$$a_n - 2a_{n-1} + 3a_{n-2} - 5a_{n-3} = 0$$

x^3 x^2 x-linear term constant term

$$x^3 - 2x^2 + 3x - 5 = 0$$

What about

$$a_n - 2a_{n-1} - 5a_{n-3} = 0\ ?$$

x^3 x^2

$$x^3 - 2x^2 - 5 = 0$$

The roots of the characteristic equation are called the <u>characteristic roots.</u>

When the characteristic roots are all distinct, the general solution to

$$C_0 a_n + C_1 a_{n-1} + \cdots + C_r a_{n-r} = 0$$

is

$$a_n = A_1(\alpha_1)^n + A_2(\alpha_2)^n + \cdots + A_r(\alpha_r)^n$$

Characteristic roots

The initial conditions are used to determine

$A_1, A_2, \ldots, A_r$

SUBJECT: Charteristic Equation DATE: 2023/05/05 PAGE#:

Ex Solve

$$a_{n+2} - 10a_{n+1} + 25a_n = 0$$

where $a_0 = 7$, $a_1 = 1$

Recall example of characteristic equation

* why do char. eq. work?

$$a_n + 5a_{n-1} + 6a_{n-2} = 0$$

$$\Rightarrow x^2 + 5x + 6 = 0$$

Recall

Repeated roots

$$\frac{Ax+B}{x^2+1}$$

Characteristic equation is

$$x^2 - 10x + 25 = 0$$

$$(x-5)^2 = 0$$

$x = 5$ ← root of multiplicity 2

$$\Rightarrow a_n = (A + Bn) \cdot 5^n$$

$$a_0 = 7 \Rightarrow 7 = (A + B \cdot 0) \cdot 5^0$$

$$\Rightarrow 7 = A$$

Now we have $a_n = (7 + Bn) \cdot 5^n$

Now using $a_1 = 1$,

$$1 = (7 + B \cdot 1) \cdot 5^1$$

$$1 = 35 + 5B$$

$$-\frac{34}{5} = B \qquad \Rightarrow a_n = \left(7 - \frac{34}{5}n\right) \cdot 5^n$$

Start with $(x+1)^4 = \binom{4}{0} + \binom{4}{1}x^1 + \binom{4}{2}x^2 + \binom{4}{3}x^3 + \binom{4}{4}x^4$

$(x+1)(x+1)(x+1)(x+1)$

but what if $(x+1)^{\text{fractional?}}$

Let $f(x) = (x+1)^4$

draw inspiration from calculus Maclaurin series

$$\left.\begin{aligned} f(x) &= (x+4) \\ f'(x) &= 4(x+1)^3 \\ f''(x) &= 4\cdot 3(x+1)^2 \\ f'''(x) &= 4\cdot 3\cdot 2(x+1)^1 \\ f^{(4)}(x) &= 4\cdot 3\cdot 2\cdot 1 \end{aligned}\right\} \rightarrow f^{(n)}(x) = 0$$

↓ Plug in zero

$f(0) = 1$
$f'(0) = 4$
$f''(0) = 4\cdot 3$
$f'''(0) = 4\cdot 3\cdot 2$
$f^{(4)}(0) = 4\cdot 3\cdot 2\cdot 1$
$f^{(n)}(0) = 0$ for $n \geq 5$

Maclauren series for $f(x)$ is

$$\sum_{n=0}^{n} = \frac{f^{(n)}(0)x^n}{n!} = f(0) + \frac{f'(0)}{1!}x^1 + \frac{f''(0)}{2!}x^2 + \dots$$

$$= 1 + \frac{4}{1!}x^1 + \frac{4\cdot 3}{2!}x^2 + \frac{4\cdot 3\cdot 2}{3!}x^3 + \frac{4\cdot 3\cdot 2\cdot 1}{4!}x^4$$

$$= \binom{4}{0} + \binom{4}{1}x^1 + \binom{4}{2}x^2 + \binom{4}{3}x^3 + \binom{4}{4}x^4 \leftarrow \text{matches binomial theorem}$$

This can be done for any function of the form

$f(x) = (x-1)^\alpha$, where α is any real number

Newton's Binomial Theorem

$$(x+1)^\alpha = \sum_{r=0}^{\infty} \binom{\alpha}{r} x^r$$

$$\binom{\alpha}{r} = \frac{\alpha(\alpha-1)(\alpha-2)-(\alpha-r+1)}{r!}$$