California State University Sacramento - Math 101

Homework Assignment 7 - Solutions

4) (i) The number of shortest paths from O to A is $\binom{5}{2}$. The number of shortest paths from A to P is $\binom{8}{3}$. By (MP), there are $\binom{5}{2}\binom{8}{3}$ routes that pass through the junction A.

(ii) By (MP), there are $\binom{5}{2}\binom{7}{3}$ such routes since there are $\binom{5}{2}$ shortest paths from O to A and $\binom{7}{3}$ shortest routes from B to P.

(iii) Again using (MP), we find that there are $\binom{5}{2}\binom{4}{1}\binom{4}{2}$ such routes. The factor $\binom{5}{2}$ counts routes from O to A. The factor $\binom{4}{1}$ counts routes from A to C. The factor $\binom{4}{2}$ counts routes from C to P.

(iv) The total number of routes from O to P is $\binom{13}{5}$. By (CP) and part (ii), the number of shortest routes which do not pass through street AB is

$$\binom{13}{5} - \binom{5}{2}\binom{7}{3}.$$

5) The first way we count the elements in T is to consider the value of z. If $z = k$ for some $k \in \{2, 3, \ldots, n+1\}$, then since $x < z$ and $y < z$, there are $(k-1)^2$ choices for the ordered pair (x, y). By (AP),

$$|T| = \sum_{k=2}^{n+1} (k-1)^2 = \sum_{k=1}^{n} k^2.$$

The second way we count the elements in T is to consider two cases which are the case when $x = y$ and the case when $x \neq y$. In the first case, the number of triples (x, x, z) with $x < z$ is $\binom{n+1}{2}$ since we pick a pair of numbers from $\{1, 2, \ldots, n+1\}$ and make z the larger one and x the smaller one. Next we count the number of triples (x, y, z) with $x < z$, $y < z$, and $x \neq y$. In this case, we choose a triple of numbers from $\{1, 2, \ldots, n+1\}$. There are $\binom{n+1}{3}$ ways to do this. In such a triple, we know that z must be the largest of the three chosen numbers. We have 2 choices for x and y since we can either make x the smallest or y the smallest. This shows that

$$|T| = \binom{n+1}{2} + 2\binom{n+1}{3}.$$

We conclude that

$$\sum_{k=1}^{n} k^2 = |T| = \binom{n+1}{2} + 2\binom{n+1}{3}.$$

6) Let $X = \{1, 2, \ldots, n\}$. We define a bijection f from the r-combinations of X to the $(n-r)$-combinations of X in the following way. Given an r-combination A, let

$$f(A) = X \backslash A.$$

For example, if $X = \{1, 2, 3, 4, 5\}$, then

$$f(\{1, 2\}) = \{3, 4, 5\},\ f(\{3\}) = \{1, 2, 4, 5\}, \text{ and } f(\{1, 2, 3, 4, 5\}) = \emptyset.$$

Since f is a bijection (a fact which we may take for granted), we have by (BP) that

$$\binom{n}{r} = \binom{n}{n-r}.$$

7) We define a bijection $f : \mathcal{A} \to \mathcal{B}$ by the following rule: given $A \in \mathcal{A}$, let

$$f(A) = A \cup \{n\}.$$

For example, if $X = \{1, 2, 3\}$, then $n = 3$,

$$\mathcal{A} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$$

and

$$\mathcal{B} = \{\{3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

In this example,

$$f(\emptyset) = \{3\},\ f(\{1\}) = \{1, 3\},\ f(\{2\}) = \{2, 3\},\ f(\{2\}) = \{2, 3\},\ \text{and}\ f(\{1, 2\}) = \{1, 2, 3\}.$$

Since f is a bijection, we have by (BP) that

$$|\mathcal{A}| = |\mathcal{B}|.$$

Important Remark: In the solutions to 6 and 7 we did not prove that f is a bijection. Since this is not a proof based course, I am not requiring you to prove that f is a bijection. However, from a mathematicians point of view, showing that f is a bijection is one of the most important parts of a complete solution.