Study Kit 02 – Homework & Exams

Notifications

Announcement Details
MATH101 Combinatorics - SECTION 01

Preparing for Exam 2

CT

Craig Timmons (He/Him/His)
Mar 12, 2023 at 5:43 AM

To prepare for Exam 2 this coming Friday, I recommend studying Quizzes 4 through 7.

As of this morning, Quizzes 4 and 5 are posted in Canvas with solutions. I will return grading Quiz 6 in class tomorrow. Quiz 7 (in groups) is on Wednesday. I will post solutions to Quizzes 6 and 7 by Wednesday evening.

Quizzes

Quiz # 4

Quiz # 5

Quiz # 6

Quiz # 7

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Quiz #4

Name: ______________________

1) For integers $1 \leq r \leq n$, give an algebraic proof that

$$\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}.$$

2) (a) Determine the number of unordered pairs of integers from the set $\{1, 2, 3, \ldots, 6\}$.

(b) Evaluate $2^{\binom{n}{2}}$ for $n = 2, 3, 4, 5, 6$.

3) In how many ways can 7 boys and 3 girls be seated around a table if no girls are adjacent?

4) In a group of 12 students, 7 of them are female. If exactly 3 boys are to be selected, in how many ways can 5 students be chosen from the group to form a committee?

5) In a group of 12 students, 7 of them are female. If at least one boy is to be selected, in how many ways can 4 students be chosen from the group to form a committee?

6) Find the number of ordered pairs of integers (a, b) where $|a - b| = 2$ and $a, b \in \{1, 2, 3, 4, 5, 6, 7, 8\}$.

7) Consider a set of n equally spaced points placed on the unit circle $x^2 + y^2 = 1$ in the x, y-plane. How many triangles are there whose vertices are the points on the circle?

8) (a) How many 0-1 sequences of length 8 have exactly three 0's?
(b) How many 0-1 sequences of length 8 have at most three 0's?
(c) What is the total number of 0-1 sequences of length 8?

9) In a group of ten people, we must form a committee consisting of three people where one of the people is the leader of the committee and the other two people are his/her assistant. How many ways can such a committee be formed?

10) Find the number of nonempty subsets of $\{1, 2, 3, 4, 5, 6, 7\}$ that contain only odd numbers.

11) Let $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z, t\}$.

(a) Find the number of functions from A to B.

(b) Find the number of injective functions from A to B.

(c) Find the probability that a random function from A to B is injective.

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Quiz #5

Name: ______________________

1) If there must be at least one person at each table, in how many ways can five people be seated around two tables where the tables are indistinguishable?

2) (a) Let $s(r, n)$ be the number of ways to arrange r distinct objects around n indistinguishable circles so that every circle has at least one object. The numbers $s(r, n)$ are called the Stirling numbers of the first kind. Problem 1 is asking for $s(5, 2)$. For part (a) of this problem, compute $s(4, n)$ for $n = 1, 2, 3, 4$ (we define $s(4, 0) = 0$).

(b) Expand the polynomial $x(x + 1)(x + 2)(x + 3)$ as much as possible.

3) Given that $s(6, 1) = 120$ and $s(6, 2) = 274$, use the formula

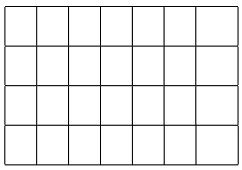
$$s(r, n) = s(r - 1, n - 1) + (r - 1)s(r - 1, n)$$

to determine $s(7, 2)$

4) Find the number of 4-combinations of $\{1, 2, \ldots, 12\}$ that contain no consecutive integers.

5) Suppose that k and n are positive integers with $k \leq n$ and that $a_1, \ldots, a_n, b_1, \ldots, b_n$ are $2n$ distinct elements. Consider the n pairs $\{a_1, b_1\}, \ldots, \{a_n, b_n\}$. Find the number of subsets of $\{a_1, \ldots a_n, b_1, \ldots, b_n\}$ of size k that do not contain two elements from the same pair.

6) Find the number of shortest paths from the bottom left corner to the top right corner in the grid below.



7) (a) Let $X = \{1, b, R\}$. Find all elements of $\mathcal{P}(X)$. Recall that this is the power set of X and it is the set of all possible subets of X.

(b) Suppose X is a set with n elements. What is the cardinality of $\mathcal{P}(X)$?

8) Find the number of 12-digit binary sequences with eight 0's and four 1's such that no two 1's are adjacent.

9) (a) Let t_n be the number of ways to pave a $1 \times n$ rectangle using 1×1 and 1×2 blocks. Determine t_1, t_2, t_3, t_4, and t_5.

(b) Can you determine t_6 using what you know about t_5 and t_4?

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Quiz #6 page 1 of 2

Name: ______________________

1) Let $X = \{1, 2, \ldots, 13, 14\}$.

(a) Find the number of 2-combinations of X. Simplify your answer as much as possible. (1 point)

(b) Find the number of 5-combinations of X that do not contain a pair of consecutive integers. Write your answer as a binomial coefficient. (1 point)

2) Find the number of 13-digit binary sequences with nine 0's and four 1's such that no two 1's are adjacent. (2 points)

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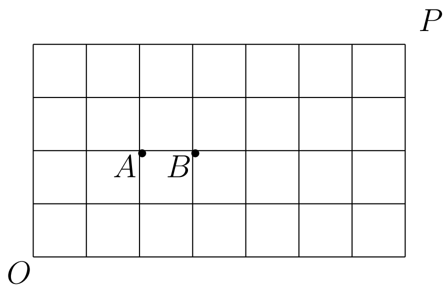
Quiz #6 page 2 of 2

3) (a) Let $X = \{\{1\}, y\}$. Find all elements of $\mathcal{P}(X)$ (the power set of X). (1 point)

(b) If Y is a set with 6 elements, how many elements are in $\mathcal{P}(Y)$? (1 point)

4) Find the number of shortest routes from O to P that pass through the street AB. (2 points)



5) Suppose that k and n are positive integers with $3 \leq k \leq n$ and that $a_1, \ldots, a_n, b_1, \ldots, b_n$ are $2n$ distinct elements. Form the n pairs $\{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_n, b_n\}$.

(a) Find the number of subsets of $\{a_1, \ldots a_n, b_1, \ldots, b_n\}$ of size k that do not contain two elements from the same pair. (1 point)

(b) Find the number of subsets of $\{a_1, \ldots a_n, b_1, \ldots, b_n\}$ of size k that contain exactly one of the pairs. (1 point)

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Quiz #7

Name: ______________________

1) Let $S = \{1, 2, 3, 4\}$ and $T = \{(x, y, z) : x, y, z \in S, x < z, y < z\}$. Determine $|T|$.

2) (a) Find the number of permutations of the 6 letters a, a, a, b, b, c.

(b) Find the number of permutations of the multi-set $\{r_1 \cdot a_1, r_2 \cdot a_2, \ldots, r_n \cdot a_n\}$.

3) Let $M = \{\infty \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_n\}$. Let H^n_r be the number of r-element multi-subsets of M. Give a formula for H^n_r.

4) Show, using algebra, that for $n \geq r \geq 1$,

$$\binom{n}{r} = \frac{n-r+1}{r}\binom{n}{r-1}.$$

5) Let $A = \{a, b, c\}$ and $B = \{u, v, x, y, z\}$.

(a) Find the number of ordered pairs of the form (α, β) where $\alpha \in A$ and $\beta \in B$.

(b) Find the number of ordered pairs of the form (A', B') where A' is a subset of A with two elements, and B' is a subset of B with three elements.

(c) Find the number of 3-combinations of $A \cup B$.

6) Consider the matching shown below.

(a) Find the number of ways to choose three edges from the matching.

(b) Find the number of ways to choose exactly one endpoint (vertex) from every edge. For example, $\{a_1, b_2, b_3, a_4, b_5\}$ contains only one vertex from each edge.

(c) Find the number of ordered pairs of the form (α, β) where α is an edge and β is a set of size 4 containing exactly one vertex from the four edges not equal to α. For example, $(\{a_5, b_5\}, \{a_4, b_3, b_2, a_1\})$.

(d) Find the number of subsets of size 5 from $\{a_1, \ldots, a_5, b_1, \ldots, b_5\}$ that contain exactly one edge.

(e) Find the number of subsets of size 5 from $\{a_1, \ldots, a_5, b_1, \ldots, b_5\}$ that do not contain any of the edges.

(f) Find the number of subsets of size 5 from $\{a_1, \ldots, a_5, b_1, \ldots, b_5\}$ that contain exactly two edges.

Quiz Workouts

Quiz # 4

Quiz # 5

Quiz # 6

Quiz # 7

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SUBJECT: **Quiz #4 - workout** DATE: / /

1) For integers $1 \le r \le n$, give an algebraic proof that
$$\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}.$$

LHS

Definition
$$\boxed{\binom{n}{r} = \frac{n!}{r!(n-r)!}}$$

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

RHS

$$\frac{n}{r}\binom{n-1}{r-1}$$
$$\to \frac{n}{r}\cdot\frac{(n-1)!}{(r-1)!(n-1-(r-1))!}$$
$$\to \frac{n}{r}\cdot\frac{(n-1)!}{(r-1)!(n+(-1)+(-r)+1)!}$$
$$\to \frac{n}{r}\cdot\frac{(n-1)!}{(r-1)!(n-r)!}$$
$$\to \frac{n!}{r!(n-r)!}$$

Recall
Factorial Rule
$$n! = n(n-1)!$$

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \quad \text{LHS} = \text{RHS} \quad \frac{n!}{r!(n-r)!}$$

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2) (a) Determine the number of unordered pairs of integers from the set $\{1, 2, 3, \ldots, 6\}$.

(b) Evaluate $2^{\binom{n}{2}}$ for $n = 2, 3, 4, 5, 6$.

a| $\binom{6}{2} = \frac{6!}{2!4!} = \frac{720}{2\cdot 24} = \frac{720}{48} = 15$

Definition
$$\boxed{\binom{n}{r} = \frac{n!}{r!(n-r)!}}$$

b|

$n=2$: $\binom{2}{2} = \frac{2!}{2!0!} \to \frac{2}{2\cdot 1} = 2$; $2^1 = 2$

$n=3$: $\binom{3}{2} = \frac{3!}{2!1!} \to \frac{6}{2} = 3$; $2^3 = 8$

$n=4$: $\binom{4}{2} = \frac{4!}{2!2!} \to \frac{24}{4} = 6$; $2^6 = 64$

$n=5$: $\binom{5}{2} = \frac{5!}{2!3!} \to \frac{120}{2\cdot 6} \to \frac{120}{12} = 10$; $2^{10} = 1024$

$n=6$: $\binom{6}{2} = \frac{6!}{2!4!} \to \frac{720}{2\cdot 24} = \frac{720}{48}$; $2^{15} = 32,768$

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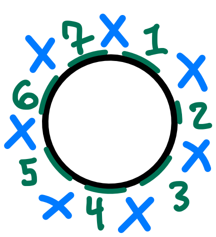
3) In how many ways can 7 boys and 3 girls be seated around a table if no girls are adjacent?



Recall

In general, if Q^n_r is the number of arrangements of 'r' elements placed around a circle chosen from a set of 'n' elements, then
$$Q^n_r = \frac{P^n_r}{r} = \frac{\frac{n!}{(n-r)!}}{r} = \frac{n!}{r\cdot(n-r)!}$$
* Note that if $r=n$, then
$$Q^n_r = \frac{n!}{n(n-n)!} = \frac{n!}{n\cdot 0} = \frac{n!}{n\cdot 1} = \frac{n!}{n} = (n-1)!$$

For each girl there are n-1 ways to place a girl at an open seat. There are 7 open seats and 3 girls

So, $7\cdot 6\cdot 5$

7boys - the number of ways to arrange them around a table

$$Q^7_7 = 6! = 720$$

All together
$$7\cdot 6\cdot 5\cdot 6! = 15,1200$$

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4) In a group of 12 students, 7 of them are female. If exactly 3 boys are to be selected, in how many ways can 5 students be chosen from the group to form a committee?

Group of 12 students
- 7 female, 5 boys
- exactly 3 boys needed
- 5 student committee

2female, 3 boys = 5 students

$$\binom{7}{2}\binom{5}{3} = 210$$
$21\cdot 10$

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5) In a group of 12 students, 7 of them are female. If at least one boy is to be selected, in how many ways can 4 students be chosen from the group to form a committee?

Case01: 3f, 1b — $\binom{7}{3}\binom{5}{1}$

Case02: 2f, 2b — $\binom{7}{2}\binom{5}{2}$

Case03: 1f, 3b — $\binom{7}{1}\binom{5}{3}$

Case04: 0f, 4b — $\binom{5}{4}$

Group of 12 students
- 7 female, 5boys
- At least 1 boy
- 4 student committee

$$\binom{7}{3}\binom{5}{1} + \binom{7}{2}\binom{5}{2} + \binom{7}{1}\binom{5}{3} + \binom{5}{4}$$
$$35\cdot 5 \qquad 21\cdot 10 \qquad 7\cdot 10 \qquad 5$$
$$\underbrace{175 + 210 + 70 + 5}_{460}$$

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6) Find the number of ordered pairs of integers (a, b) where $|a - b| = 2$ and $a, b \in \{1, 2, 3, 4, 5, 6, 7, 8\}$.

Because of $|a-b|$ values can be swapped - 2x

1, 2, 3, 4, 5, 6, 7, 8

1 2 3 4 5 6

6 ways where $|a-b|=2$ $\times$ 2 ways to reorder

So in total $6\times 2 = 12$ ways

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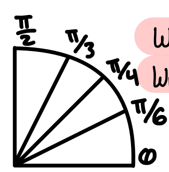
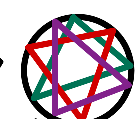
7) Consider a set of n equally spaced points placed on the unit circle $x^2 + y^2 = 1$ in the x, y-plane. How many triangles are there whose vertices are the points on the circle?



Wrong Idea
Wrong Approach

The proper generic approach



So $\binom{n}{3}$ ways

There are n-many ways to choose 3 points of a triangle in a circle

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8) (a) How many 0-1 sequences of length 8 have exactly three 0's?
(b) How many 0-1 sequences of length 8 have at most three 0's?
(c) What is the total number of 0-1 sequences of length 8?

a| 1, 0 of length 8 exactly three 0's

$\underset{0}{\_}, \underset{1}{\_}, \underset{2}{\_}, \underset{3}{\_}, \underset{4}{\_}, \underset{5}{\_}, \underset{6}{\_}, \underset{7}{\_} \to$ so $\binom{8}{3} = 56$

b|

Case01: Three 0's — $\binom{8}{3} = 56$

Case02: Two 0's — $\binom{8}{2} = 28$

Case03: One zero — $\binom{8}{1} = 8$

Case04: No zeros — $\binom{8}{0} = 1$

$\to 56 + 28 + 8 + 1 = 93$

c| 0, 1 $\to$ 2 states/elements; 8 positions } So $2^8 = 256$

Quiz #4 – workout

9) In a group of ten people, we must form a committee consisting of three people where one of the people is the leader of the committee and the other two people are his/her assistant. How many ways can such a committee be formed?

Group of 10
- Committee of 3
  - 1 lead
  - 2 assistants

Choose 3 from the 10 to form a committee $\binom{10}{3}$

for every committee of size 3 choose a leader $\binom{3}{1}$

$120 \cdot 3 = 360$ ways to form a committee to spec.

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10) Find the number of nonempty subsets of $\{1, 2, 3, 4, 5, 6, 7\}$ that contain only odd numbers.

$\underline{1}, 2, \underline{3}, 4, \underline{5}, 6, \underline{7}$

We choose from subset $\{1, 3, 5, 7\}$

Find all non-empty subsets

$\binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4}$

$\underbrace{4 + 6 + 4 + 1}$

15

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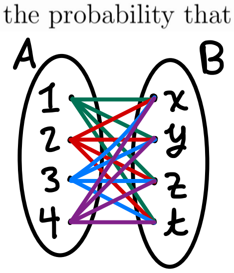
11) Let $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z, t\}$.

(a) Find the number of functions from A to B.

(b) Find the number of injective functions from A to B.

(c) Find the probability that a random function from A to B is injective.

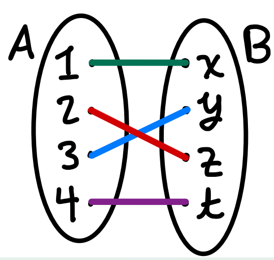
a|



For every element in set A may map to an element in set B

So, $4^4 = 256$ total functions

b|

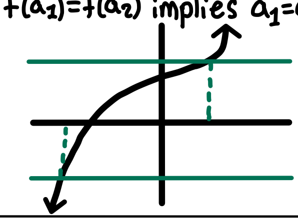


Elements of 'A' can map only to 1 element of 'B'

SO $4! = 24$

Recall

Let 'A' and 'B' be sets 'A' function $f: A \to B$ is Injective (1-to-1) if $f(a_1) = f(a_2)$ implies $a_1 = a_2$



c|

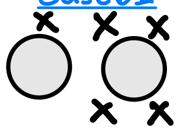
1-to-1 → $\dfrac{4!}{4^4} = \dfrac{24}{256} = 0.09375$

the "random" possibilities → 4^4

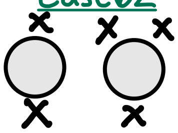
1) If there must be at least one person at each table, in how many ways can five people be seated around two tables where the tables are indistinguishable?

* Needs case analysis

Case 01



Case 02



Recall

In general, if Q^n_r is the number of arrangements of 'r' elements placed around a circle chosen from a set of 'n' elements, then

$$Q^n_r = \frac{P^n_r}{r} = \frac{\frac{n!}{(n-r)!}}{r} = \frac{n!}{r\cdot(n-r)!}$$

* Note that if $r=n$, then

$$Q^n_r = \frac{n!}{n(n-n)!} = \frac{n!}{n\cdot 0} = \frac{n!}{n\cdot 1} = \frac{n!}{n} = (n-1)!$$

Case 01 Case 02

$\binom{5}{1}Q^4_4 + \binom{5}{2}Q^3_3$

$5\cdot 3! + 10\cdot 2!$

$\rightarrow (5\cdot 6)+(10\cdot 2) \rightarrow (30)+(20) = 50$

2) (a) Let $s(r,n)$ be the number of ways to arrange r distinct objects around n indistinguishable circles so that every circle has at least one object. The numbers $s(r,n)$ are called the Stirling numbers of the first kind. Problem 1 is asking for $s(5,2)$. For part (a) of this problem, compute $s(4,n)$ for $n = 1,2,3,4$ (we define $s(4,0) = 0$).

(b) Expand the polynomial $x(x+1)(x+2)(x+3)$ as much as possible.

Recall: Sterling Numbers of the first Kind
Lower case 's' as notation

$s(r,n) = s((r-1),n-1) + (r-1)s(r-1,n)$

Properties of $s(r,n)$

- $s(r,0) = 0$; if $r \geq 1$
- $s(r,r) = 1$; if $r \geq 0$
- $s(r,1) = Q^r_r = (r-1)!$; for $r \geq 2$
- $s(r,r-1) = \binom{r}{2}$; for $r \geq 2$

a | $s(4,n)$

n = 1

$s(4,1) = Q^4_4 = (4-1)! = 3! = 6$

n = 2

$s(4,2) = s(3,1) + (3)s(3,2)$

$Q^3_3 = 2! = 2$ $\quad \binom{3}{2} = 3$

$\rightarrow 2 + (3)\binom{3}{2}$

$\rightarrow 2 + (3)(3) \rightarrow 2 + 9 = 11$

$s(4,2) = 11$

n = 3

Note $\boxed{s(r,r-1) = \binom{r}{2}}$ $s(4,3)$

So, $s(4,3) = \binom{4}{2} = 6$

n = 4

Note $\boxed{s(r,r) = 1}$ $s(4,4)$

So, $s(4,4) = 1$

b | I'll do this later!

3) Given that $s(6,1) = 120$ and $s(6,2) = 274$, use the formula $s(r,n) = s(r-1,n-1) + (r-1)s(r-1,n)$ to determine $s(7,2)$

$s(7,2) = s(6,1) + (6)s(6,2)$

$\boxed{s(r,1) = Q^r_r = (r-1)!}$ $\quad \boxed{s(r,n) = s((r-1),n-1) + (r-1)s(r-1,n)}$

$Q^6_6 = (6-1)! = 5!$

120

$s(6,2) = s(5,1) + (5)s(5,2)$

$Q^5_5 = 4! = 24$

$\boxed{s(5,2) = s(4,1) + (4)s(4,2)}$

$Q^4_4 = 3! = 6$ $\quad s(4,2) = s(3,1) + (3)s(3,2)$

$Q^3_3 = 2! = 2$ $\quad \binom{3}{2} = 3$

All together

$s(7,2) = 120 + (24) + (5)(6 + 4(2 + (3)(3)))$

$s(7,2) = 199$ In total

4) Find the number of 4-combinations of $\{1,2,\ldots,12\}$ that contain no consecutive integers.

Recall: Section 1.5

General Case — The biggest it can be is $n-r+1$

Let $x = \{1,2,3,4,\ldots,n\}$ and let $1 \leq r \leq n$

Given a subset $\{S_0, S_1, \ldots, S_r\}$ of x with no consecutive elements define,

$f(\{S_0,S_1,\ldots,S_r\}) = \{S_0, S_1-1, S_2-2, \ldots, S_r-(r-1)\}$

This output is a subset of size 'r' since $\{S_0,S_1,\ldots,S_r\}$ has no consecutive elements

$$\binom{n-r+1}{r}$$

- $[1,12]$
- $n = 12;\ r = 4$

$\binom{12-4+1}{4}$

$\rightarrow \binom{9}{4} = 126$

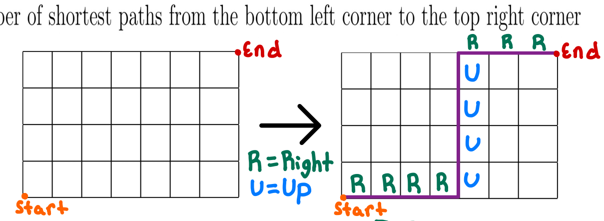
5) Suppose that k and n are positive integers with $k \leq n$ and that $a_1,\ldots,a_n,b_1,\ldots,b_n$ are $2n$ distinct elements. Consider the n pairs $\{a_1,b_1\},\ldots,\{a_n,b_n\}$. Find the number of subsets of $\{a_1,\ldots a_n,b_1,\ldots,b_n\}$ of size k that do not contain two elements from the same pair.

$$\binom{n}{k}\cdot 2^k$$

- 2n distinct elements
- $k \leq n$; n pairs, k size subsets

choose k-pairs — choose exactly one element from each pair

6) Find the number of shortest paths from the bottom left corner to the top right corner in the grid below.



$\underset{1}{R}, \underset{2}{R}, \underset{3}{R}, \underset{4}{R}, \underset{5}{U}, \underset{6}{U}, \underset{7}{U}, \underset{8}{U}, \underset{9}{R}, \underset{10}{R}, \underset{11}{R} \rightarrow$ 7 R's, 4 U's

So, $\binom{11}{4}$ or $\binom{11}{7}$

Selecting for up-movements — Selecting for Right-movements

7) (a) Let $X = \{1,b,R\}$. Find all elements of $\mathcal{P}(X)$. Recall that this is the power set of X and it is the set of all possible subets of X.

(b) Suppose X is a set with n elements. What is the cardinality of $\mathcal{P}(X)$?

a | • $X = \{1,b,R\} \rightarrow$ Produce all possible subsets of $P(X)$

So, $P(X) = \{\emptyset, \{1\}, \{b\}, \{R\}, \{1,b\}, \{1,R\}, \{b,R\}, \{1,b,R\}\}$

b | If given set n-elements, then its Power Set will contain 2^n elements. It also represents the cardinality of the power set.

8) Find the number of 12-digit binary sequences with eight 0's and four 1's such that no two 1's are adjacent.

$\binom{n-r+1}{r}$; $n = 12$, $r = 4$ $\quad \binom{12-4+1}{4} \rightarrow \binom{9}{4} = 126$

9) (a) Let t_n be the number of ways to pave a $1\times n$ rectangle using 1×1 and 1×2 blocks Determine t_1, t_2, t_3, t_4, and t_5.

(b) Can you determine t_6 using what you know about t_5 and t_4?

a | 1x1 & 1x2 blocks

t_1 — 1 — 1 way

t_2 — 1 1 | 2 — 2 ways

t_3 — 1 1 1 | 1 2 | 2 1 — 3 ways

t_4 — 1 1 1 1 | 2 1 1 | 1 2 1 | 1 1 2 | 2 2 — 5 ways

t_5 — 1 1 1 1 1 | 2 1 1 1 | 1 2 1 1 | 1 1 2 1 | 1 1 1 2 | 1 2 2 | 2 1 2 | 2 2 1 — 8 ways

b | $t_6 = (t_4 + t_5) = 13$

If the last tile is 1x1 then t_5 was to shade

t_5 | 1

If the last tile is 1x2 then t_4 was to shade

t_4 | 2

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1) Let X = {1, 2, . . ., 13, 14 }

a. Find the number of 2-combinations of X. Simplify your answer as much as possible.

b. Find the number of 5-combinations of X that do not contain a pair of consecutive integers. White your answer as a binomial coefficient. (1 point)

a| $X = \{1,2,3,\ldots,14\}$, subset of 2 combinations

- $n = 14$
- $r = 2$

So, $\binom{14}{2} = \frac{n!}{r!(n-r)!} = \frac{14!}{2!(14-2)!} = \frac{14!}{2(12)!} = 91$

b| $n = 14$

$r = 5$

$\binom{n-r+1}{r} \rightarrow \binom{14-5+1}{5}$

$\rightarrow \binom{10}{5} = 252$

Recall: Section 1.5

General Case

The biggest it can be is $n-r+1$

Let $x = \{1,2,3,4,\ldots,n\}$ and Let $1 \le r \le n$

Given a subset $\{S_0, S_1, \ldots, S_r\}$ of x with no consecutive elements define,

$f(\{S_0, S_1, \ldots, S_r\}) = \{S_0, S_1 - 1, S_2 - 2, \ldots, S_r - (r-1)\}$

This output is a subset of size 'r' since $\{S_0, S_1, \ldots, S_r\}$ has no consecutive elements

$$\binom{n-r+1}{r}$$

---

2) Find the number of 13-digit binary sequences with nine 0's and four 1's such that no two 1's are adjacent. (2 points)

- 13-digits – binary 9:0's, 4:1's, remove 2:1's

So $\binom{n-r+1}{r} \rightarrow \binom{13-4+1}{4} = \binom{10}{4} = 210$

---

3) Let X = { {1}, y }.

a. Find all elements of P(X) (the power set of X). (1 point)

b. If Y is a set with 6 elements, how many elements are in P(Y)? (1 point)

a| All elements of P(X)

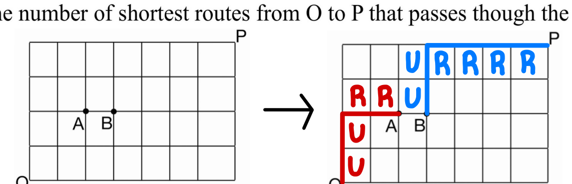
$\{\emptyset, \{\{1\}\}, \{y\}, \{\{1\}, y\}\}$

b| Y = 6 elements

So, $2^6 = 64$

---

4) Find the number of shortest routes from O to P that passes though the street AB.



From O→A – 4 steps total: 2 R's, 2 U's

From B→P – 6 steps total: 4 R's, 2U's

$\underbrace{\binom{4}{2}}_{\text{selecting for}} \cdot \underbrace{\binom{6}{2}}_{\text{selecting for}}$ OR $\underbrace{\binom{4}{2}}_{\text{selecting for}} \cdot \underbrace{\binom{6}{4}}_{\text{selecting for}}$ OR $\underbrace{\binom{4}{2}}_{\text{selecting for up}} \cdot \underbrace{\binom{6}{4}}_{\text{selecting for Right}}$ OR $\underbrace{\binom{4}{2}}_{\text{selecting for Right}} \cdot \underbrace{\binom{6}{2}}_{\text{selecting for up}}$

up-movements | Right-movements | movements | movements

90 ways

---

5) Suppose that k and n are positive integers with $3 \le k \le n$ and that $a_1, \ldots, a_n, b_1, \ldots, b_n$ are $2n$ distinct elements. Form the n pairs $\{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_n, b_n\}$.

(a) Find the number of subsets of $\{a_1, \ldots a_n, b_1, \ldots, b_n\}$ of size k that do not contain two elements from the same pair. (1 point)

(b) Find the number of subsets of $\{a_1, \ldots a_n, b_1, \ldots, b_n\}$ of size k that contain exactly one of the pairs. (1 point)

a| $\underbrace{\binom{n}{k}}_{\text{choose K-pairs}} \cdot \underbrace{2^K}_{\text{choose one from each pair}}$

b| $\underbrace{\binom{n}{1}}_{\text{Choose one Pair so have two People}} \underbrace{\binom{n-1}{K-2}}_{\text{choose K-2 pairs}} \underbrace{2^{K-2}}_{\text{Choose one from of the K-2 pairs}}$

Quiz Solutions

Quiz # 4
Quiz # 5
Quiz # 6
Quiz # 7

Quiz 4

1) (a) $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$ equal

and

$$\frac{n}{r}\binom{n-1}{r-1} = \frac{n}{r} \cdot \frac{(n-1)!}{(r-1)!(n-1-(r-1))!}$$

$$= \frac{n!}{r!(n-1-r+1)!} = \frac{n!}{r!(n-r)!}$$

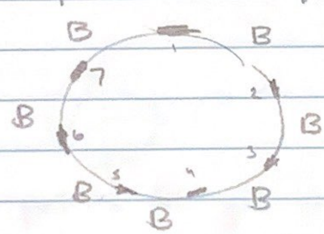
2) (a) $\binom{6}{2} = 15$

(b)

n	2	3	4	5
$2^{\binom{n}{2}}$	$2^1 = 2$	$2^{\binom{3}{2}} = 2^3 = 8$	$2^{\binom{4}{2}} = 2^6 = 64$	$2^{\binom{5}{2}} = 2^{10} = 1024$

6
$2^{\binom{6}{2}} = 2^{15} = 32,768$

3) $Q_7^7 = 6!$ ways for boys to be placed



$7 \cdot 6 \cdot 5$ ways to place the girls at the open seats

$\Rightarrow 6! \cdot 7 \cdot 6 \cdot 5$

QUIZ #4

P2

4) $\binom{5}{3}\binom{7}{2}$

choose boys choose girls

5) $\binom{5}{1}\binom{7}{3} + \binom{5}{2}\binom{7}{2} + \binom{5}{3}\binom{7}{1} + \binom{5}{4}$

\# of ways to form committee with

1 B, 3 G   2B, 2G   3B, 1G   4B

6)

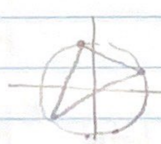
a	b
1	3
2	4
3	1 or 5
4	2 or 6
5	3 or 7
6	4 or 8
7	5
8	6

Total count is

$1+1+2+2+2+2+1+1$

$= 12$

7) $\binom{n}{3}$



choose 3 points to get a triangle

Quiz #4

8) (a) $\binom{8}{3}$ ← choose position of the 0's

(b) $\binom{8}{0}+\binom{8}{1}+\binom{8}{2}+\binom{8}{3}$

(c) 2^8 since either 0 or 1 in the 8 positions

9) $\binom{10}{3} \cdot 3$

choose 3 from 10 ↑ ↑ choose the leader

10) We are choosing from $\{1,3,5,7\}$ so

$$\binom{4}{1}+\binom{4}{2}+\binom{4}{3}+\binom{4}{4} = 2^4 - 1 = 15$$

11) (a) There are 4 choices for where to send 1 to in B
" 4 " 2
" 4 " 3
" 4 " 4

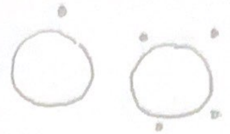
$\Rightarrow 4^4$ total functions

(b) $4!$ since 4 choices for 1, then 3 for 2, then 2 for 3.

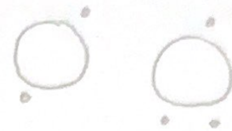
(c) $\frac{4!}{4^4} = \frac{24}{256} = \frac{3}{32} \approx 0.09375$

Quiz #5

1)



$\binom{5}{1} Q_4^4 = 5 \cdot 3! = 30$



$\binom{5}{2} Q_3^3 = 10 \cdot 2! = 20$

Total: $30 + 20 = 50$

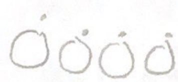
2) (a)

$s(4,1) = Q_4^4 = 3! = 6$

$s(4,2) = \binom{4}{1} Q_3^3 + \frac{1}{2}\binom{4}{2}$

$= 4 \cdot 2! + \frac{1}{2} \cdot 6 = 11$

$s(4,3) = \binom{4}{2} = 6$



$s(4,4) = 1$

$s(4,1) = 6 \quad s(4,2) = 11 \quad s(4,3) = 6 \quad s(4,4) = 1$

(b) $x(x+1)(x+2)(x+3)$

$= x(x^3 + x^2 + 2x^2 + 3x^2 + 2x + 3x + 6x + 6)$

$= \underset{\substack{\uparrow \\ s(4,4)}}{x^4} + \underset{\substack{\uparrow \\ s(4,3)}}{6x^3} + \underset{\substack{\uparrow \\ s(4,2)?}}{11x^2} + \underset{\substack{\uparrow \\ s(4,1)\,?}}{6x}$

Quiz #5

3) $s(7,2) = s(6,1) + 6 \cdot s(6,2)$

$= 120 + 6 \cdot 274 = 1764$

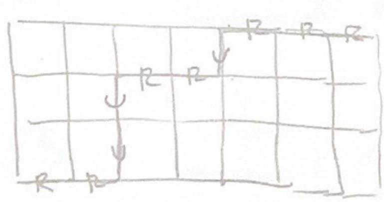
4) $\binom{12-4+1}{4} = \binom{9}{4}$

5) $\binom{n}{k} \cdot 2^k$

choose k pairs

choose exactly one element from each pair

6)



$\left.\right\}$ Will need 3 "up steps" out of the ten total steps

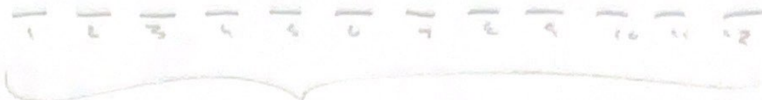
$\binom{10}{3}$

7) (a) $P(X) = \{ \phi, \{1\}, \{b\}, \{R\}, \{1,b\}, \{1,R\}, \{b,R\}, \{1,b,R\} \}$

(b) 2^n

Quiz #5

8)



choose subset of size four from the positions to place the 1's

Need no consecutive 1's so

$$\binom{12-4+1}{4} = \binom{9}{4}$$

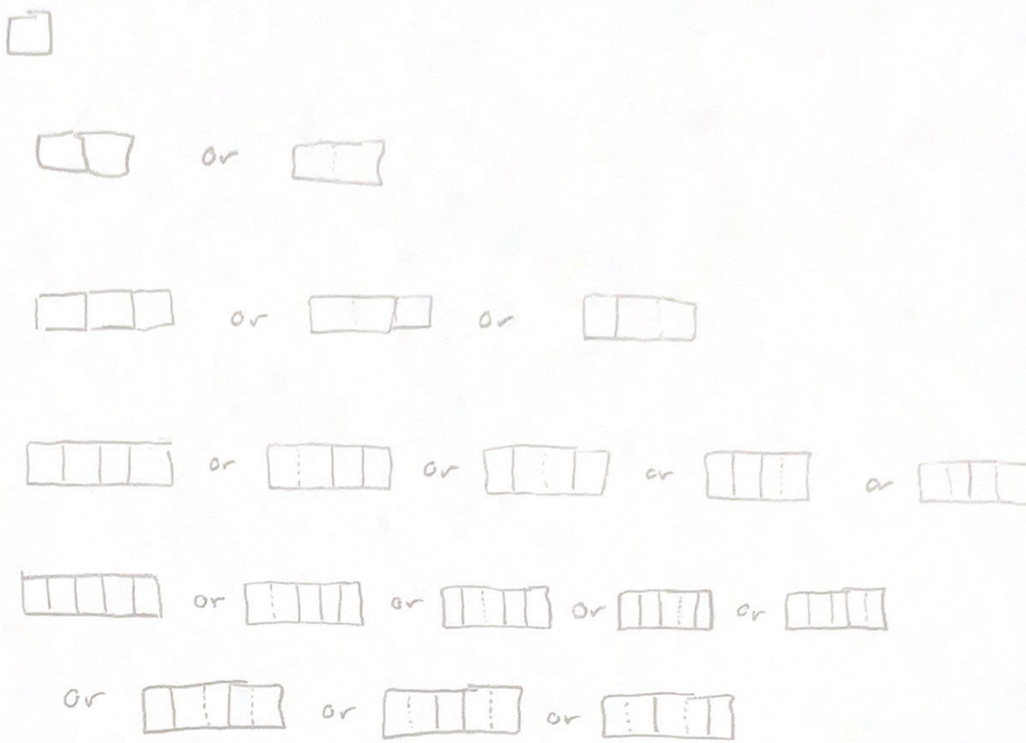
9) (a) $t_1 = 1$

$t_2 = 2$

$t_3 = 3$

$t_4 = 5$

$t_5 = 8$



(b) Yes. It turns out that $t_6 = t_5 + t_4 = 13$.

If last tile is 1x1   If last tile is 1x2

t_5 ways for shaded t_4 ways for shaded

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Quiz #6

Name: ______________________

1) Let $X = \{1, 2, \ldots, 13, 14\}$.

(a) Find the number of 2-combinations of X. Simplify your answer as much as possible. (1 point)

(b) Find the number of 5-combinations of X that do not contain a pair of consecutive integers. Write your answer as a binomial coefficient. (1 point)

(a) $\binom{14}{2} = \frac{14 \cdot 13}{2} = 7 \cdot 13 = 91$

(b) $\binom{14-5+1}{5} = \binom{10}{5}$

2) Find the number of 13-digit binary sequences with nine 0's and four 1's such that no two 1's are adjacent. (2 points)

$\underset{1}{\underline{\quad}}\ \underset{2}{\underline{\quad}}\ \underset{3}{\underline{\quad}}\ \underset{4}{\underline{\quad}}\ \underset{5}{\underline{\quad}}\ \underset{6}{\underline{\quad}}\ \underset{7}{\underline{\quad}}\ \underset{8}{\underline{\quad}}\ \underset{9}{\underline{\quad}}\ \underset{10}{\underline{\quad}}\ \underset{11}{\underline{\quad}}\ \underset{12}{\underline{\quad}}\ \underset{13}{\underline{\quad}}$

Choose four positions for the 1's such that no positions are adjacent.

$\binom{13-4+1}{4} = \binom{10}{4}$

Quiz#6

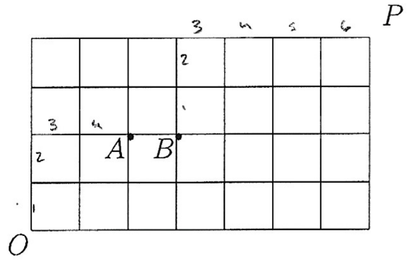
3) (a) Let $X = \{\{1\}, y\}$. Find all elements of $\mathcal{P}(X)$ (the power set of X). (1 point)

(b) If Y is a set with 6 elements, how many elements are in $\mathcal{P}(Y)$? (1 point)

(a) $\{\emptyset, \{\{1\}\}, \{y\}, \{\{1\}, y\}\}$

(b) $2^6 = 64$

4) Find the number of shortest routes from O to P that pass through the street AB. (2 points)



$$\underbrace{\binom{4}{2}}_{\text{from } O \text{ to } A} \cdot \underbrace{\binom{6}{2}}_{\text{from } B \text{ to } P} \quad \text{or} \quad \binom{4}{2} \cdot \binom{6}{4}$$

5) Suppose that k and n are positive integers with $3 \le k \le n$ and that $a_1, \dots, a_n, b_1, \dots, b_n$ are $2n$ distinct elements. Form the n pairs $\{a_1, b_1\}, \{a_2, b_2\}, \dots, \{a_n, b_n\}$.

(a) Find the number of subsets of $\{a_1, \dots a_n, b_1, \dots, b_n\}$ of size k that ~~contain~~ do not contain two elements from the same pair. (1 point)

(b) Find the number of subsets of $\{a_1, \dots a_n, b_1, \dots, b_n\}$ of size k that contain exactly one of the pairs. (1 point)

(a) $$\underbrace{\binom{n}{k}}_{\text{choose } k \text{ pairs}} \cdot \underbrace{2^k}_{\text{choose one from each pair}}$$

(b) $$\underbrace{\binom{n}{1}}_{\text{choose one pair so have two people}} \underbrace{\binom{n-1}{k-2}}_{\text{choose } k-2 \text{ pairs}} \underbrace{2^{k-2}}_{\text{choose one from of the } k-2 \text{ pairs}}$$

Quiz #7

1) If $z=1$, then there are no choices for x or y.
If $z>1$, then x and y can be any of

$$\underbrace{1,2,3,\ldots,z-1}$$

$$(z-1)\cdot(z-1)$$

↑ choices for x ↑ choices for y

$$|T| = \sum_{z=2}^{4}(z-1)^2 = 1^2+2^2+3^2 = 14$$

2) (a) $\dfrac{6!}{3!\,2!}$ (b) $\dfrac{(r_1+r_2+\ldots+r_n)}{r_1!\,r_2!\cdots r_n!}$

3) $H_r^n = \binom{n+r-1}{r}$

4) $$\frac{n-r+1}{r}\binom{n}{r-1} = \frac{n-r+1}{r}\cdot\frac{n!}{(r-1)!\,(n-(r-1))!}$$

$$= \frac{n-r+1}{r}\cdot\frac{n!}{(r-1)!\,(n-r+1)!}$$

$$= \frac{n!}{r!}\cdot\frac{n-r+1}{(n-r)!\,(n-r+1)}$$

$$= \frac{n!}{r!\,(n-r)!} = \binom{n}{r}$$

Quiz #7

5) (a) $|A||B| = 3 \cdot 5 = 15$

(b) there are $\binom{3}{2}$ choices for A' and $\binom{5}{2}$ choices for B'.

$$\Longrightarrow \binom{3}{2}\binom{5}{2} = 3 \cdot 10 = 30$$

(c) $|A \cup B| = 8$ so $\binom{8}{3}$

6) (a) $\binom{5}{3} = 10$ (b) 2^5

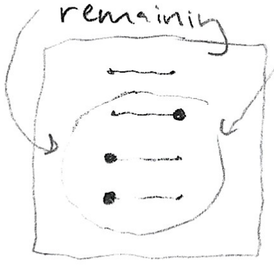
(c) There are $\binom{5}{1} = 5$ choices for α and then $2^{5-1} = 2^4$ choices for β.

(d) $5 \cdot \binom{4}{3} \cdot 2^3$

choose one edge say $a_i \longrightarrow b_i$
two elements

choose three more edges from the four remaining

then choose one endpoint from each edge



$a_i \longrightarrow b_i$

(e) 2^5

(e) $\binom{5}{2} \cdot 6$

choose edges which gives four elements
$a_i \longrightarrow b_i$
$a_j \longrightarrow b_j$

pick one more element

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Homework

Homework #4

Homework #5

Homework #6

Homework #7

1.3. Circular Permutations
1.4. Combinations

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Homework Assignment 4

1) Example 1.3.2

2) Example 1.3.3

3) Problem 6 on page 51

4) Example 1.4.1

5) Example 1.4.2

Q1) **Example 1.3.2.** **In how many ways can 5 boys and 3 girls be seated around a table if**

(i) there is no restriction?

(ii) boy B_1 and girl G_1 are not adjacent?

(iii) no girls are adjacent?

Q2) **Example 1.3.3.** **Find the number of ways to seat n married couples around a table in each of the following cases:**

(i) Men and women alternate;

(ii) Every woman is next to her husband.

Q3) **6. Find the number of *odd* integers between 3000 and 8000 in which no digit is repeated.**

Q4) **Example 1.4.1.** **Prove that**

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}, \qquad (1.4.3)$$

where $n, r \in \mathbf{N}$ with $r \leq n$.

Q5) **Example 1.4.2.** **By Example 1.1.4, there are 2^7 binary sequences of length 7. How many such sequences are there which contain 3 0's and 4 1's?**

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Homework Assignment 5

1) Problem 2(iv) on page 50

2) Problem 15 on page 51

3) Example 1.4.3 parts (i), (ii), and (iii)

4) Problem 2(iii) on page 50

5) Problem 20 on page 52

6) Problem 22 on page 52

7) Problem 23 on page 52

Q1

2. There are 12 students in a party. Five of them are girls. In how many ways can these 12 students be arranged in a row if

(iv) between two particular boys A and B, there are no boys but exactly 3 girls?

Q2

15. In a group of 15 students, 5 of them are female. If exactly 3 female students are to be selected, in how many ways can 9 students be chosen from the group

(i) to form a committee?

(ii) to take up 9 different posts in a committee?

Q3

Example 1.4.3. In how many ways can a committee of 5 be formed from a group of 11 people consisting of 4 teachers and 7 students if

(i) there is no restriction in the selection?

(ii) the committee must include exactly 2 teachers?

(iii) the committee must include at least 3 teachers?

Q4

2. There are 12 students in a party. Five of them are girls. In how many ways can these 12 students be arranged in a row if

(iii) no 2 girls are adjacent?

Q5

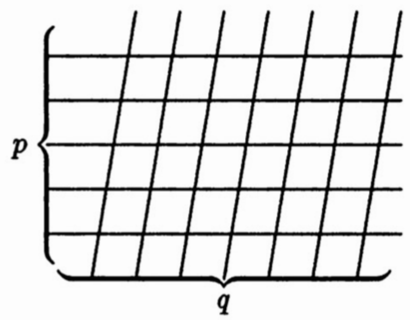
20. In a group of 15 students, 3 of them are female. If at least one female student is to be selected, in how many ways can 7 students be chosen from the group

(i) to form a committee?

(ii) to take up 7 different posts in a committee?

Q6

22. Two sets of parallel lines with p and q lines each are shown in the following diagram:



Find the number of parallelograms formed by the lines?

Q7

23. There are 10 girls and 15 boys in a junior class, and 4 girls and 10 boys in a senior class. A committee of 7 members is to be formed from these 2 classes. Find the number of ways this can be done if the committee must have exactly 4 senior students and exactly 5 boys.

Homework Assignment 6

1) Example 1.4.6

2) Example 1.4.7

3) Problem 18 on page 52

4) Problem 19 on page 52

5) Problem 21 on page 52

6) Problem 26 on page 53

Q1

Example 1.4.6. If there must be at least one person in each table, in how many ways can 6 people be seated

(i) around two tables?

(ii) around three tables?

(We assume that the tables are indistinguishable.)

Q2

Example 1.4.7. Show that

$$s(r,n) = s(r-1,n-1) + (r-1)s(r-1,n)$$

where $r, n \in \mathbf{N}$ with $n \leq r$.

Q3

18. A group of 20 students, including 3 particular girls and 4 particular boys, are to be lined up in two rows with 10 students each. In how many ways can this be done if the 3 particular girls must be in the front row while the 4 particular boys be in the back?

Q4

19. In how many ways can 7 boys and 2 girls be lined up in a row such that the girls must be separated by exactly 3 boys?

Q5

21. Find the number of $(m+n)$-digit binary sequences with m 0's and n 1's such that no two 1's are adjacent, where $n \leq m+1$.

Q6

26. Find the number of ways of forming a group of $2k$ people from n couples, where $k, n \in \mathbf{N}$ with $2k \leq n$, in each of the following cases:

(i) There are k couples in such a group;

(ii) No couples are included in such a group;

(iii) At least one couple is included in such a group;

(iv) Exactly two couples are included in such a group.

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Homework Assignment 7

1) Example 1.5.1

2) Example 1.5.2

3) Example 1.5.3 (know the result, not the proof)

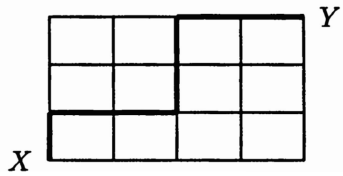
4) Problem 25 on page 53

5) Problem 27 on page 53

6) Problem 40 on page 55

7) Problem 41 on page 55

Q1 **Example 1.5.1.** A student wishes to walk from the corner X to the corner Y through streets as given in the street map shown in Figure 1.5.1. How many shortest routes are there from X to Y available to the student?



Q2 **Example 1.5.2.** Show that if $|X| = n$, then $|\mathcal{P}(X)| = 2^n$ for all $n \in \mathbf{N}$.

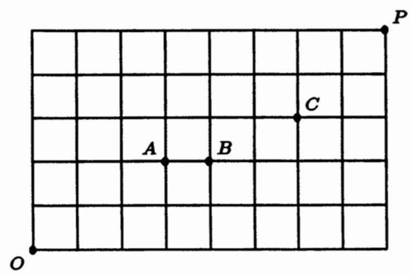
Q3 **Example 1.5.3.** Let $X = \{1, 2, ..., n\}$, where $n \in \mathbf{N}$. Show that the number of r-combinations of X which contain no consecutive integers is given by

$$\binom{n-r+1}{r},$$

where $0 \le r \le n - r + 1$.

Q4

25. In each of the following cases, find the number of shortest routes from O to P in the street network shown below:



(i) The routes must pass through the junction A;
(ii) The routes must pass through the street AB;
(iii) The routes must pass through junctions A and C;
(iv) The street AB is closed.

Q5

27. Let $S = \{1, 2, \ldots, n+1\}$ where $n \ge 2$, and let

$$T = \{(x, y, z) \in S^3 \mid x < z \quad \text{and} \quad y < z\}.$$

Show by counting $|T|$ in two different ways that

$$\sum_{k=1}^{n} k^2 = |T| = \binom{n+1}{2} + 2\binom{n+1}{3}.$$

Q6

40. Prove the identity $\binom{n}{r} = \binom{n}{n-r}$ by (BP)

Q7

41. Let $X = \{1, 2, ..., n\}$, $\mathcal{A} = \{A \subseteq X \mid n \notin A\}$, and $\mathcal{B} = \{A \subseteq X \mid n \in A\}$. Show that $|\mathcal{A}| = |\mathcal{B}|$ by (BP).

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Homework Solutions

Homework #4
Homework #5
Homework #6
Homework #7

Homework Assignment 4

Solutions page 1 of 2

Recall

In general, if Q^n_r is the number of arrangements of 'r' elements placed around a circle chosen from a set of 'n' elements, then

$$Q^n_r = \frac{P^n_r}{r} = \frac{\frac{n!}{(n-r)!}}{r} = \frac{n!}{r\cdot(n-r)!}$$

* Note that if $r=n$, then

$$Q^n_r = \frac{n!}{n(n-n)!} = \frac{n!}{n\cdot 0} = \frac{n!}{n\cdot 1} = \frac{n!}{n} = (n-1)!$$

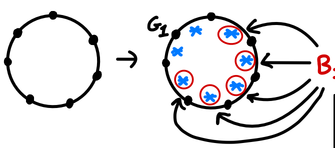
1) Example 1.3.2: In how many ways can 5 boys and 3 girls be seated around a table if

(i) there is no restriction?

$$\boxed{\frac{8!}{8} = 7!}$$

(ii) boy B_1 and girl G_1 are not adjacent?

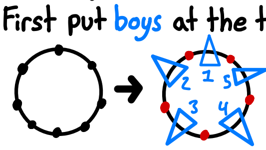
Let us place all except B_1 there $\frac{7!}{7} = 6!$ ways to place the seven people around the table (we are not placing B_1 yet)



This gives a total count of $6!\times 5$

(iii) no girls are adjacent?

First put boys at the table



$\frac{5!}{5} = 4!$ Possibilites

5·4·3 ways to place girls

$\Rightarrow$ Total is $4!\cdot 5\cdot 4\cdot 3 = \boxed{1440}$

2) Example 1.3.3 - Find the number of ways to seat 'n' married couples around a table in each of the following cases:

(i) Men and women alternate

For every partner in a pairing can be arranged $(n-1)!$ ways, for every pairing has 'n' many significant other, so in total we have:

$$\boxed{(n-1)!\,n!}$$

(ii) Every women is next to her husband

Can be thought as binary bits so,

$$\boxed{(n-1)!\,2^n}$$

3) Exercise 1.6 - Find the number of odd integers between 3000 and 8000 in which no digit is repeated

See Page 2

4) Example 1.4.1 - Prove that $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$, where 'n', $r \in \mathbb{N}$ with $r \leq n$

Recall

$$C^n_r = \frac{P^n_r}{r!} \quad \text{we can rewrite as} \quad C^n_r = \frac{n!}{r!(n-r)!}$$

By algebraic proof...

$$\binom{n-1}{r-1} + \binom{n-1}{r} = \frac{(n-1)!}{(r-1)!(n-r)!} + \frac{(n-1)!}{r!(n-1-r)!}$$

$$\rightarrow \frac{(n-r)!\,r + (n-1)!(n-r)}{r!(n-r)!}$$

$$\rightarrow \frac{(n-1)!(r+n-r)}{r!(n-r)!}$$

$$\boxed{\rightarrow \frac{n!}{r!(n-r)!} = \binom{n}{r}}$$

5) Example 1.4.2 - There are 2^7 binary sequences of length 7. How many sequences are there which contains three 0's and four 1's?

4x1's | | | | | | | | 3x0's

6 5 4 3 2 1 0

Within a range of 7 bits how many ways can we position three zeros... Once placed we are left with the obvious remaining spaces to put 1's giving us $C^4_4 = 1$, So what we care about are only the zeros $\therefore \boxed{C^7_3 = 35}$

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Homework Assignment 4 Solutions page 2 of 2

6) Write $a_1a_2a_3a_4$ for such a four digit integer. First, a_1 can be any of 3, 4, 5, 6, or 7, while a_4 can be 1, 3, 5, 7, or 9. These sets overlap so we will consider two cases.

If a_4 is 1 or 9 (2 choices), then a_1 can be 3, 4, 5, 6, or 7 (5 choices). We then have 8 choices for a_2 and 7 choices for a_3 because they must be distinct and chosen from $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}\backslash\{a_1, a_4\}$. In all, there are $2 \times 5 \times 8 \times 7 = 560$ possibilities.

Now suppose that a_4 is one of 3, 5, or 7 (3 choices). In this case there are only 4 choices for a_1. There are still 8 choices for a_2 and 7 for a_3 giving a total of $3 \times 4 \times 8 \times 7 = 672$ possibilities.

Combining the two cases gives a total count of $560 + 672 = 1232$.

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Homework Assignment 5 - Solutions page 1 of 2

1) First we place the two boys. There are 8 possible ways to do this since the first of the two boys can be 1st in the row, 2nd in the row,... all the way up to 8th in the row. There are then two ways to order these two boys (either A comes before B or B comes before A). Next we place the remaining students. There are $\binom{5}{3}$ ways to choose which three girls will go between A and B and 3! ways to order those girls. There are 7! ways to order the remaining students. This gives a total of

$$8 \cdot 2 \cdot \binom{5}{3} 3!7!$$

possibilities.

2) (i) There are $\binom{5}{3}$ ways to choose the female students and $\binom{10}{6}$ ways to choose the male students giving a total of

$$\binom{5}{3}\binom{10}{6}$$

possibilities.
(ii) As shown in part (i), there are $\binom{5}{3}\binom{10}{6}$ ways to choose the students on the committee and then 9! ways to assign these 9 students to the different posts. This gives a total of

$$\binom{5}{3}\binom{10}{6} 9!$$

possibilities.

3) Example 1.4.3 parts (i), (ii), and (iii) - See text book.

4) There are 7! ways to line up the boys. Given an arrangement of the boys, we then count how many ways we can arrange the girls. The are 8 choices for the first girl. She can go before the first boy in line, before the second boy in line, ... , before the seventh boy in line, or behind the seventh boy in line (see page 9 where a similar counting strategy is used). The next girl has 7 choices since she cannot be next to the first girl. Continuing in this fashion, we find that there are $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$ ways to line up the girls once the arrangement of the boys has been given. Therefore, there are

$$7!(8 \cdot 7 \cdot 6 \cdot 5 \cdot 4)$$

possibilities.

5) (i) We divide the problem into three cases. There are $\binom{3}{1}\binom{12}{6}$ ways to form a group with exactly one female. There are $\binom{3}{2}\binom{12}{5}$ ways to form a group with exactly two females and $\binom{3}{3}\binom{12}{4}$ ways to form a group with three females. These cases are disjoint so that we have a count of

$$\binom{3}{1}\binom{12}{6} + \binom{3}{2}\binom{12}{5} + \binom{3}{3}\binom{12}{4}.$$

(ii) Once we have chosen which 7 form a group, there are 7! ways to assign them to different posts so by part (i), we have

$$7!\left(\binom{3}{1}\binom{12}{6}+\binom{3}{2}\binom{12}{5}+\binom{3}{3}\binom{12}{4}\right)$$

possibilities.

6) Choosing two horizontal lines and two vertical lines uniquely determines a rectangle and there are

$$\binom{p}{2}\binom{q}{2}$$

ways to choose a pair of horizontal lines and a pair of vertical lines.

7) We consider three disjoint cases which depend on the number of boys chosen from the senior class.

Case 1: 4 boys are chosen from the senior class

There are $\binom{10}{4}$ ways to choose the 4 boys from the senior class, $\binom{15}{1}$ ways to choose 1 boy from the junior class, and $\binom{10}{2}$ ways to choose 2 girls from the junior class.

Case 2: 3 boys are chosen from the senior class

There are $\binom{10}{3}$ ways to choose 3 boys from the senior class, $\binom{4}{1}$ ways to choose 1 girl from the senior class, $\binom{15}{2}$ ways to choose 2 boys from the junior class, and $\binom{10}{1}$ ways to choose 1 girl from the junior class.

Case 3: 2 boys are chosen from the senior class

There are $\binom{10}{2}$ ways to choose 2 boys from the senior class, $\binom{4}{2}$ ways to choose 2 girls from the senior class, and $\binom{15}{3}$ ways to choose 3 boys from the junior class.

If we combine all three cases, we find that there are

$$\binom{10}{4}\binom{15}{1}\binom{10}{2}+\binom{10}{3}\binom{4}{1}\binom{15}{2}\binom{10}{1}+\binom{10}{2}\binom{4}{2}\binom{15}{3}$$

possibilities.

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Homework Assignment 6 - Solutions

Q1) Example 1.4.6.–If there must be at least one person in each table, in how many ways can 6 people be seated (We assume that the tables are indistinguable)

(i) around two tables?

Recall: if $r=n$ then $Q^n_r = \frac{n!}{n(n-n)!} = (n-1)!$

$S(6,2)$ S(r-people, n-tables)

case 01

$\binom{6}{1} Q^1_1 \cdot Q^5_5$

$\binom{6}{1} \times 0! \times 4!$
$\to 6 \times 1 \times 24$
$\to 144$

case 02

$\binom{6}{2} Q^2_2 \cdot Q^4_4$

$\binom{6}{2} \times 1! \times 3!$
$\to 15 \times 1 \times 6$
$\to 90$

case 03

$\frac{\binom{6}{3} Q^3_3 \cdot Q^3_3}{2}$ ← tables w/ 3 people

Accounts for the double count the two

$\left[\binom{6}{3} \times 2! \times 2!\right] \cdot \left(\frac{1}{2}\right)$
$\to [20 \times 2 \times 2] \cdot \left(\frac{1}{2}\right)$
$\to 40$

case 01 case 02 case 03

In total $144 + 90 + 40 = 274$

(ii) around three tables?

$S(6,3)$ S(r-people, n-tables)

case 01

$\frac{\binom{6}{4}\binom{2}{1} \times Q^4_4 \times Q^1_1 \times Q^1_1}{2!}$ adjust for over count

$\left[(15)(2) \times 3! \times 0! \times 0!\right](1/2)$
$\to [(30) \times 6 \times 1 \times 1](1/2)$
$\to 90$

case 02

$\binom{6}{3}\binom{3}{2} \times Q^3_3 \times Q^2_2 \times Q^1_1$

$[(20)(3) \times 2! \times 1! \times 0!]$
$\to [(60) \times 2 \times 1 \times 1]$
$\to 120$

case 03

$\frac{\binom{6}{2}\binom{4}{2} \times Q^2_2 \times Q^2_2 \times Q^2_2}{3!}$ adjust for over count

$[(15)(6) \times 0! \times 0! \times 0!](1/6)$
$\to [(90) \times 1 \times 1 \times 1](1/6)$
$\to 15$

case 01 case 02 case 03

In total $90 + 120 + 15 = 225$

Q2) Example 1.4.7 Show that
$s(r,n) = s(r-1,n-1) + (r-1)s(r-1,n)$
where $r,n \in \mathbb{N}$ with $n \le r$

$$s(r,n) = s(r-1,n-1) + (r-1)s(r-1,n)$$

For every given table this guarantees a seat at that table for that 1 person + (a table for at least 1 person by themselves)

OR

For person who had their own table

Have everyone else sit at those tables

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Homework Assignment 6 - Solutions page 2 of 2

3) There are $\binom{10}{3}$ ways to pick which seats the 3 particular girls will sit in in the first row. There are 3! ways to order these 3 particular girls in the seats that we have chosen for them. Similarly, there are $\binom{10}{4}4!$ ways to choose which seats the 4 particular boys will sit in. After these 7 students have been assigned seats, there are 13! ways to assign seats to the remaining students. Therefore, the total number of possibilities is $\binom{10}{3}3!\binom{10}{4}4!13!$.

4) There are 5 choices for which positions to put the two girls in. Once we have chosen which position that they will go in, there are 2! ways to order these two girls and 7! ways to order the 7 boys. This gives a count of $5 \cdot 2!7! = 10 \cdot 7!$.

5) We pick the positions for the n ones. Choosing the positions for the n ones is the same as choosing an n-combination from $\{1, 2, \ldots, m+n\}$ such that no two of the chosen numbers are consecutive. By Example 1.5.3, there are $\binom{n+m-n+1}{n} = \binom{m+1}{n}$ such n-combinations.

6) (i) In this case, we just need to choose k of the couples to form a group of $2k$ people consisting of k couples. Since there are n couples and we need to choose k of them, we have $\binom{n}{k}$ possibilities.

(ii) Here we will form such a group by first choosing $2k$ couples, and then choosing exactly one person from each couple. In this way, our group of $2k$ people will not include a couple. There are $\binom{n}{2k}$ ways to choose $2k$ couples from n couples, and then 2^{2k} ways to choose one person from each of these $2k$ couples giving a count of $\binom{n}{2k}2^{2k}$.

(iii) We use the Complementation Principle. The total number of ways to choose $2k$ people is $\binom{2n}{2k}$. Here we have $2n$ people to choose from since there are n couples and each couple consists of two people. By part (ii), we know that there are $\binom{n}{2k}2^{2k}$ ways to form a group where there are no couples. The opposite of a group with no couples is a group with at least one couple. Thus, by the Complementation Principle, there are

$$\binom{2n}{2k} - \binom{n}{2k}2^{2k}$$

possibilities.

(iv) First we choose which two couples will be in the group. There are $\binom{n}{2}$ ways to do this. So far we have chosen two couples and so we have four people in our group. We need to choose the remaining $2k-4$ people so that this set of people contains no couple. There are

$$\binom{n-2}{2k-4}2^{2k-4}$$

ways to do this (see part (ii)). This gives a count of

$$\binom{n}{2}\binom{n-2}{2k-4}2^{2k-4}.$$