1) For integers $1 \leq r \leq n$, give an algebraic proof that $\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$.

LHS

Definition $\boxed{\binom{n}{r} = \frac{n!}{r!(n-r)!}}$

$\binom{n}{r} = \frac{n!}{r!(n-r)!}$

RHS

$\frac{n}{r}\binom{n-1}{r-1}$

$\rightarrow \frac{n}{r} \cdot \frac{(n-1)!}{(r-1)!(n-1-(r-1))!}$

$\rightarrow \frac{n}{r} \cdot \frac{(n-1)!}{(r-1)!(n+(-1)+(-r)+1)!}$

$\rightarrow \frac{n}{r} \cdot \frac{(n-1)!}{(r-1)!(n-r)!}$

Recall: Factorial Rule $n! = n(n-1)!$

$\rightarrow \frac{n!}{r!(n-r)!}$

$\binom{n}{r} = \frac{n!}{r!(n-r)!}$ LHS = RHS $\frac{n!}{r!(n-r)!}$

2) (a) Determine the number of unordered pairs of integers from the set $\{1, 2, 3, \ldots, 6\}$.

(b) Evaluate $2^{\binom{n}{2}}$ for $n = 2, 3, 4, 5, 6$.

a) $\binom{6}{2} = \frac{6!}{2!4!} = \frac{720}{2 \cdot 24} = \frac{720}{48} = 15$

Definition $\boxed{\binom{n}{r} = \frac{n!}{r!(n-r)!}}$

b)

$n=2$: $\binom{2}{2} = \frac{2!}{2!0!} \rightarrow \frac{2}{2\cdot 1} = 2$; $2^1 = 2$

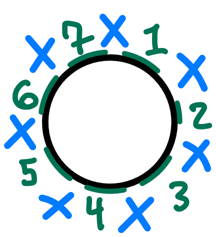
$n=3$: $\binom{3}{2} = \frac{3!}{2!1!} \rightarrow \frac{6}{2} = 3$; $2^3 = 8$

$n=4$: $\binom{4}{2} = \frac{4!}{2!2!} \rightarrow \frac{24}{4} = 6$; $2^6 = 64$

$n=5$: $\binom{5}{2} = \frac{5!}{2!3!} \rightarrow \frac{120}{2\cdot 6} \rightarrow \frac{120}{12} = 10$; $2^{10} = 1024$

$n=6$: $\binom{6}{2} = \frac{6!}{2!4!} \rightarrow \frac{720}{2\cdot 24} = \frac{720}{48}$; $2^{15} = 32,768$

3) In how many ways can 7 boys and 3 girls be seated around a table if no girls are adjacent?



Recall: In general, if Q^n_r is the number of arrangements of 'r' elements placed around a circle chosen from a set of 'n' elements, then

$$Q^n_r = \frac{P^n_r}{r} = \frac{\frac{n!}{(n-r)!}}{r} = \frac{n!}{r\cdot(n-r)!}$$

* Note that if $r=n$, then

$$Q^n_r = \frac{n!}{n(n-n)!} = \frac{n!}{n\cdot 0} = \frac{n!}{n \cdot 1} = \frac{n!}{n} = (n-1)!$$

For each girl there are n-1 ways to place a girl at an open seat. There are 7 open seats and 3 girls

So, $7\cdot 6 \cdot 5$

7boys - the number of ways to arrange them around a table

$Q^7_7 = 6! = 720$

All together

$7 \cdot 6 \cdot 5 \cdot 6! = 15,1200$

4) In a group of 12 students, 7 of them are female. If exactly 3 boys are to be selected, in how many ways can 5 students be chosen from the group to form a committee?

Group of 12 students
- 7 female, 5 boys
- exactly 3 boys needed
- 5 student committee

2female, 3 boys = 5 students

$\binom{7}{2}\binom{5}{3} = 210$

$21 \cdot 10$

5) In a group of 12 students, 7 of them are female. If at least one boy is to be selected, in how many ways can 4 students be chosen from the group to form a committee?

Case01: 3f, 1b — $\binom{7}{3}\binom{5}{1}$

Case02: 2f, 2b — $\binom{7}{2}\binom{5}{2}$

Case03: 1f, 3b — $\binom{7}{1}\binom{5}{3}$

Case04: 0f, 4b — $\binom{5}{4}$

Group of 12 students
- 7 female, 5boys
- At least 1 boy
- 4 student committee

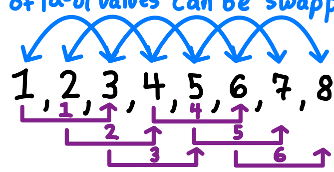
$\binom{7}{3}\binom{5}{1} + \binom{7}{2}\binom{5}{2} + \binom{7}{1}\binom{5}{3} + \binom{5}{4}$

$35 \cdot 5 \quad 21 \cdot 10 \quad 7 \cdot 10 \quad 5$

$\underbrace{175 + 210 + 70 + 5}_{460}$

6) Find the number of ordered pairs of integers (a, b) where $|a - b| = 2$ and $a, b \in \{1, 2, 3, 4, 5, 6, 7, 8\}$.

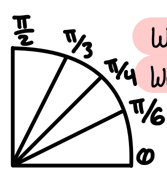
Because of |a-b| values can be swapped - 2x



6 ways where |a-b|=2 × 2 ways to reorder

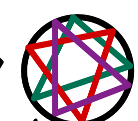
So in total $6\times 2 = 12$ ways

7) Consider a set of n equally spaced points placed on the unit circle $x^2 + y^2 = 1$ in the x, y-plane. How many triangles are there whose vertices are the points on the circle?



Wrong Idea
Wrong Approach

The proper generic approach



So $\binom{n}{3}$ ways

There are n-many ways to choose 3 points of a triangle in a circle

8) (a) How many 0-1 sequences of length 8 have exactly three 0's?
(b) How many 0-1 sequences of length 8 have at most three 0's?
(c) What is the total number of 0-1 sequences of length 8?

a) 1,0 of length 8 exactly three 0's

$\underline{\ }_0, \underline{\ }_1, \underline{\ }_2, \underline{\ }_3, \underline{\ }_4, \underline{\ }_5, \underline{\ }_6, \underline{\ }_7 \rightarrow$ so $\binom{8}{3} = 56$

b)

Case01: Three 0's — $\binom{8}{3} = 56$

Case02: Two 0's — $\binom{8}{2} = 28$

Case03: One Zero — $\binom{8}{1} = 8$

Case04: No Zeros — $\binom{8}{0} = 1$

$\rightarrow 56 + 28 + 8 + 1 = 93$

c) 0,1 → 2 states/elements; 8 positions } So $2^8 = 256$

Quiz #4 – workout

9) In a group of ten people, we must form a committee consisting of three people where one of the people is the leader of the committee and the other two people are his/her assistant. How many ways can such a committee be formed?

Group of 10
- Committee of 3
  - 1 lead
  - 2 assistants

Choose 3 from the 10 to form a committee $\binom{10}{3}$

for every committee of size 3 choose a leader $\binom{3}{1}$

$120 \cdot 3 = 360$ ways to form a committee to spec.

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10) Find the number of nonempty subsets of $\{1, 2, 3, 4, 5, 6, 7\}$ that contain only odd numbers.

$\underline{1}, 2, \underline{3}, 4, \underline{5}, 6, \underline{7}$

We choose from subset $\{1, 3, 5, 7\}$

Find all non-empty subsets

$$\underbrace{\binom{4}{1}+\binom{4}{2}+\binom{4}{3}+\binom{4}{4}}_{}$$

$$\underbrace{4 + 6 + 4 + 1}_{15}$$

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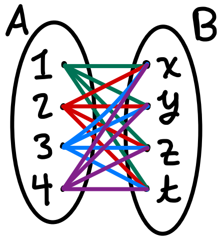
11) Let $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z, t\}$.

(a) Find the number of functions from A to B.

(b) Find the number of injective functions from A to B.

(c) Find the probability that a random function from A to B is injective.

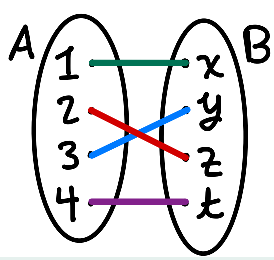
a]



For every element in set A may map to an element in set B

So, $4^4 = 256$ total functions

b]

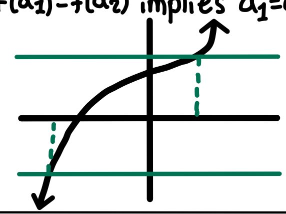


Elements of 'A' can map only to 1 element of 'B'

SO $4! = 24$

Recall

Let 'A' and 'B' be sets 'A' function $f: A \to B$ is Injective (1-to-1) if $f(a_1)=f(a_2)$ implies $a_1=a_2$



c]

1-to-1 → $\dfrac{4!}{4^4} = \dfrac{24}{256} = 0.09375$

the "random" possibilities → 4^4