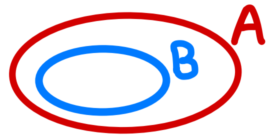
Announcements
- Friday Quiz 2
- Hw 3 Due Monday (02/13)
- Hw 4 Due Wednesday (02/15)
- Exam 1 Friday (02/17)

Section 1.4: Combinations

Let 'A' be a set.
We say that 'B' is a subset of 'A', denoted $B \subseteq A$, if every element of 'B' is also in 'A' and is denoted $P(A)$.

The set of all possible subsets of 'A' is called the <u>power set of 'A'</u>



Recap Prep For Exam 1
Section 1.1, 1.2, 1.3, 1.4
$P^n_r \; Q^n_r \; C^n_r$

Side Note: P(A)
Higher math may use alternate notation 2^A

Example 01

Sample: $P(\{a,b\}) = \{\underbrace{\emptyset}_{1}, \underbrace{\{a\},\{b\}}_{2}, \underbrace{\{a,b\}}_{1}\}$

$(x+y)^2 = x^2+2xy+y^2$
$(x+y)^3 = x^3+3x^2y+3xy^2+y^2$
Pascal's Triangle Relation ⇒ Making the connection to pascal

$$\begin{array}{c} 1 \\ 1\;\;1 \\ 1\;\;2\;\;1 \\ 1\;\;3\;\;3\;\;1 \\ 1\;\;4\;\;6\;\;4\;\;1 \end{array}$$

If $A=\{a,b,c\}$, find $P(A)$

$P(A) = \{\underbrace{\emptyset}_{1}, \underbrace{\{a\},\{b\},\{c\}}_{2}, \underbrace{\{a,b\},\{a,c\},\{b,c\}}_{3}, \underbrace{\{a,b,c\}}_{1}\}$

zero element subset, one element subset, Two element subset, Three element subset

can be grouped in sizes

One way to view choosing subsets of $A=\{1,2,3\}$ is by expanding the polynomial $(a+1)(b+1)(c+1)$

$$(a+1)(b+1)(c+1) = \underset{\{a,b,c\}}{abc} + \underset{\{a,b\}}{ab} + \underset{\{a,c\}}{ac} + \underset{\{b,c\}}{bc} + \underset{\{a\}}{a} + \underset{\{b\}}{b} + \underset{\{c\}}{c} + \underset{\emptyset}{1}$$

* The key is to make it all be 'x' *
So when we had

$$(a+1)(b+1)(c+1) = \underset{\{a,b,c\}}{abc} + \underset{\{a,b\}}{ab} + \underset{\{a,c\}}{ac} + \underset{\{b,c\}}{bc} + \underset{\{a\}}{a} + \underset{\{b\}}{b} + \underset{\{c\}}{c} + \underset{\emptyset}{1}$$

Let $a=b=c=x$

5 in a 5-element set $(x+1)^5$ ➡ $x^3 + xx + xx + xx + x + x + x + 1$
$= x^3 + 3x^2 + 3x + 1 \longrightarrow (x+1)(x+1)(x+1)$

$$\begin{array}{cl} 1 & \\ 1\;\;1 & =2 \\ 1\;\;2\;\;1 & =4 \\ 1\;\;3\;\;3\;\;1 & =8 \\ 1\;\;4\;\;6\;\;4\;\;1 & =16 \end{array}$$

4

Pascal's Triangle Relation to Binomial Coefficients

Lets make some formulas...

A <u>combination</u> of a set 'A' is a subset of 'A'

For $0 \le r \le n$, an r-combination of 'A' is a subset with 'r' elements

$$P^n_r = \frac{n!}{(n-r)!}$$

If $A=\{a,b,c\}$, then the 2-permutations are

$$P^3_2 = \frac{3!}{(3-2)!}$$

If we ignore order

a b
a c
b a
b c
c a
c b

$$\boxed{C^3_2 = \frac{1}{2!}P^3_2}$$

In general...

$$C^n_r = \frac{P^n_r}{r!}$$

we can rewrite as

These choose numbers C^n_r are called "binomial coefficients" and C^n_r, "C-N-R", is typically written as $\binom{n}{r}$ "n-choose-r"

$$\begin{array}{c} 1 \\ 1\;\;1 \\ 1\;\;2\;\;1 \\ 1\;\;3\;\;3\;\;1 \\ 1\;\;4\;\;6\;\;4\;\;1 \end{array}$$

Note

$$C^n_r = C^n_{n-r}$$

Since,

$$C^n_r = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!\,r!} = \frac{n!}{(n-r)!(n-(n-r))!} = C^n_{n-r}$$

this means

$$\binom{n}{r} = \binom{n}{n-r}$$

example

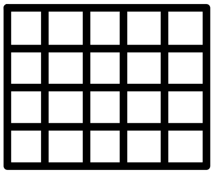
$$\binom{4}{0} + \boxed{\binom{4}{1}} + \boxed{\binom{4}{2}} + \binom{4}{3} + \binom{4}{4} = 2^4$$

<u>Identity</u>
A set with 4 elements and count by size

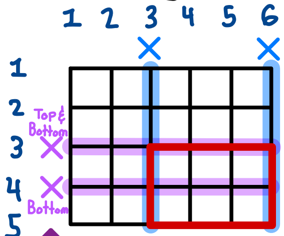
$\{\_,\_,\_,\_\}$ with a,b,c,d

Example 02

How many rectangles are in the grid below?



→



$\binom{6}{2}$ ← Ways to pick sides of rectangle

↑ $\binom{5}{2}$ Ways to pick top & bottom of rectangle

These choose numbers C^n_r are called "binomial coefficients" and C^n_r, "C-N-R", is typically written as $\binom{n}{r}$ "n-choose-r"

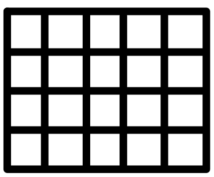
In all, there are

$$\binom{5}{2}\cdot\binom{6}{2} = \frac{5\cdot4}{2\cdot1}\times\frac{6\cdot5}{2\cdot1} \rightarrow \frac{5\cdot\cancel{4}^{2}}{{}_{1}\cancel{2}\cdot1}\times\frac{\cancel{6}^{3}\cdot5}{\cancel{2}_{1}\cdot1}$$

$$\rightarrow 10\times15 = 150$$

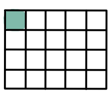
Example 03

How many, non-overlapping of same sized, squares are in the grid?



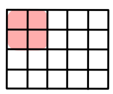
Lets do some case analysis...

Case 1: Square is 1×1



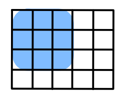
There are $4\times5=20$ of those

Case 2: Square is 2×2



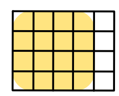
$3\times4=12$

Case 3: Square is 3×3



$2\times3=6$

Case 4: Square is 4×4

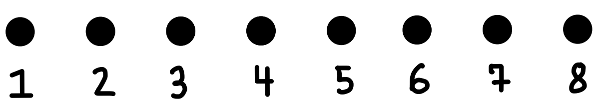


$1\times2=2$

In all, there are $20+12+6+2=40$ squares

Example 04

Let 'A' be a set with $2n$ elements

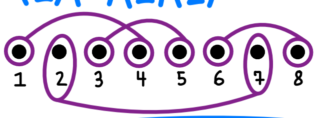


A pairing* of 'A' is a partition of 'A' into subsets of size 2.

*is unordered

How many pairings does 'A' have?

$$\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}$$



$\{1,4\},\{2,7\},\{3,5\},\{6,8\}$

$$\binom{2n}{2}\binom{2n-2}{2}\binom{2n-4}{2}\cdots\binom{2}{2}$$

↑ Choose first pair

↑ Choose second pair

Must divide by $n!$ because our initial counting orders the pairings, but pairings are not ordered so...

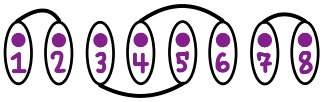
Total # of pairings is

$$\frac{\binom{2n}{2}\binom{2n-2}{2}\binom{2n-4}{2}\cdots\binom{2}{2}}{n!}$$

Last time we counted the number of pairings on pairings on a set with 2n elements

$\{3,5\}, \{1,2\}, \{4,6\}, \{7,8\}$

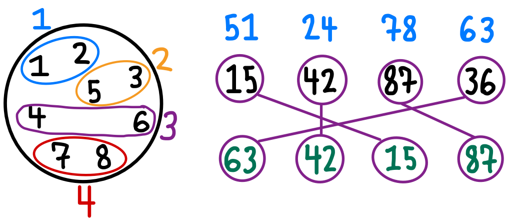
$n=4$



★ $\{1,2\}, \{3,5\}, \{4,6\}, \{7,8\}$

what if presented in a line?

★ $\boxed{\{1,2\}, \{3,4\}, \{5,6\}, \{7,8\}}$



$$\frac{\binom{2n}{n}\binom{2n-2}{2}\binom{2n-4}{2}\cdots\binom{2}{2}}{n!}$$

$\binom{8}{2} \quad \binom{6}{2} \quad \binom{4}{2} \quad \binom{2}{2} \quad \frac{1}{4!}$

$\boxed{\{1,2\}, \{3,4\}, \{5,6\}, \{7,8\}}$ (a_1, a_2, a_3)

Let us count pairings in a different way...

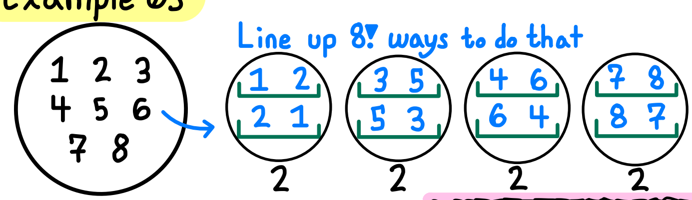
There is $(2n)!$ ways to order all elements in the set

$a_1, a_2, a_3, \ldots, a_{2n} \leftarrow$ ordered list of all elements of 'A'

$\{a_1,a_2\}, \{a_3,a_4\}, \ldots, \{a_{2n-1},a_{2n}\}$

$a_2 \; a_1 \quad a_4 \; a_3 \qquad a_{2n} \; a_{2n-1}$

Example 05



$$\frac{(2n)!}{2^n \cdot n!}$$ Simplifies to $\boxed{\frac{\binom{2n}{2}\binom{2n-2}{2}\binom{2n-4}{2}\cdots\binom{2}{2}}{n!}}$

takes care of $\{x,y\}$ vs $\{y,x\}$

takes care of ordering the pairs

* $\binom{1\,2}{2\,1}_2 \binom{3\,5}{5\,3}_2$ vs $\binom{3\,5}{5\,3}_2 \binom{1\,2}{2\,1}_2$

Example 06

$\{1,5\}, \{4,2\}, \{8,7\}, \{3,6\}$

unfinished note section

Example 07

There are 10 students: 6 female, 4 male

(i) How many groups can be found with 3 female & 2 male

$$\binom{6}{3}\binom{4}{2} = 120$$

Number of ways to choose girls ($\binom{6}{3}$); Number of ways to choose boys ($\binom{4}{2}$)

(ii) How many groups of 5 students can be made with at least 1 male? * Needs case analysis

case 01	case 02	case 03	case 04
–1 male–	–2 males–	–3 males–	–4 males–
$\binom{6}{4}\binom{4}{1}$	$\binom{6}{3}\binom{4}{2}$	$\binom{6}{2}\binom{4}{3}$	$\binom{6}{1}\binom{4}{4}$
60	120	60	6

Total count is

$$\underbrace{\binom{6}{4}\binom{4}{1}}_{60} + \underbrace{\binom{6}{3}\binom{4}{2}}_{120} + \underbrace{\binom{6}{2}\binom{4}{3}}_{60} + \underbrace{\binom{6}{1}\binom{4}{4}}_{6} = 246$$

(iii) Repeat Example 01 – except now the group must be ranked from 1 to 5 ordered

$$\binom{6}{3}\binom{4}{2}5!$$

Number of ways to choose girls; Number of ways to choose boys; Ways to order the five people

In total $\binom{6}{3}\binom{4}{2}5! = 14{,}400$

In Summary

→ More than 1 way to count

→

unfinished note section

Practice with counting – choose function

Exercise

In how many ways can a group of 5 people be formed from 11 people where 4 are instructors and 7 are students

(i) No restrictions/condictions

$$\binom{11}{5} = \boxed{462}$$

(ii) What if we want exactly two instructors?

Pick 2 instructors and choose the remaining 3

choose 2 instructors

$$\binom{4}{2}\binom{7}{3} = \boxed{210}$$

choose 3 students

(iii) What if we want at least three instructors?

Case	Case
3 instructors	4 instructors
2 students	1 student

$$\underbrace{\binom{4}{3}\binom{7}{2}}_{84} + \underbrace{\binom{4}{4}\binom{7}{1}}_{7} = \boxed{91}$$

** (iv) Conflict of interest where I_1 & S_1 can't be together

I_1 & S_1 cannot both be chosen, we consider cases
(3 in all)

Case 01 : I_1 is chosen NOT S_1

I_1 chosen
S_1 cannot

$$\binom{9}{4} = 126$$

Case 02 : I_1 is NOT chosen, but S_1 is chosen

S_1 chosen
I_1 cannot

$$\binom{9}{4} = 126$$

Case 03 : Neither I_1 or S_1 are chosen

$$\binom{9}{5} = 126$$

$$\boxed{\text{In total } \underset{126}{\text{Case 01}} + \underset{126}{\text{Case 02}} + \underset{126}{\text{Case 03}} = 378}$$

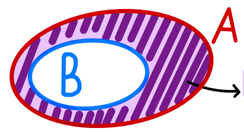
Principle of Complemetation

If 'A' is a finite set and $B \subseteq A$, then $|A \setminus B| = |A| - |B|$

$A \setminus B = \{x : x \in A, x \notin B\}$

Attributes
* Counting the opposite
* The 1-minus rule

$$\boxed{P(A) = 1 - P(A')}$$



→ $A \setminus B$

** Recall (iv)

In how many ways can a group of 5 people be formed from 11 people where 4 are instructors and 7 are students where I_1 & S_1 can't be together

⇓

We can use this tool, the Principle of Complementation, to count the number of 5 person groups from 4 instructors, 7 students where I_1 and S_1 are not both in the group together (case 03)

⇓

11 ← 7 students, 4 instructors
11 − 2 (1 student & 1 instructor)

Yields the same results → $$\underbrace{\binom{11}{5}}_{\text{Total number of possibilities}} - \underbrace{\binom{9}{3}}_{\text{Number of groups that contain both } I_1 \text{ and } S_1} = \boxed{378}$$

Note

Counting the same object in two ways can lead to combination identities

In this case...

$$\binom{11}{5} - \binom{9}{3} = \binom{9}{4} + \binom{9}{4} + \binom{9}{4}$$

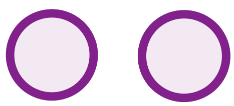
Line n-up $\underline{1}, \underline{2}, \underline{3}, \ldots, \underline{r}$ $\quad P^n_r = nP^{n-1}_{r-1}$

Principle of Complemetation

$$\boxed{P^n_r = nP^{n-1}_{r-1}}$$

Example: Placement type problems

If there must be atleast one person at each table, how many ways can 6 people be placed around two *indistinguishable tables?
*Can't tell the two tables apart



Divide counting into cases depending on the number per table

case 01

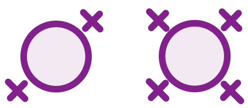
6 choices per person at own table

$Q^5_5 = 4!$

$\underline{6} \cdot \underline{4!}$

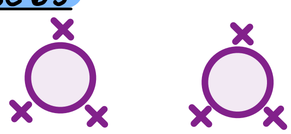
case 02

Recall: if $r = n$ then
$$Q^n_r = \frac{n!}{n(n-n)!} = (n-1)!$$



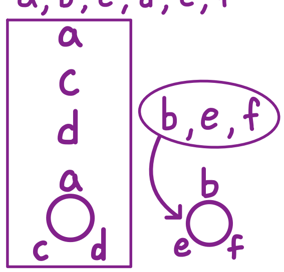
$$\binom{6}{2} \times Q^4_4 = \underline{\binom{6}{2}} \cdot \underline{3!}$$

case 03



$$\frac{\binom{6}{3} Q^3_3 \cdot Q^3_3}{2!}$$

* Indistingushable tables
a, b, c, d, e, f



*accounts for the over count

The total count is

case 01 case 02 case 03
$$\binom{6}{1}Q^1_1 \cdot Q^5_5 + \binom{6}{2}Q^2_2 \cdot Q^4_4 + \frac{1}{2}\binom{6}{3}Q^3_3 \cdot Q^3_3$$

Lower case 's' as notation

~Sterling Numbers of the first Kind~

Def: Sterling Numbers of the first Kind

Given integers $0 \le n \le r$ let, $s(r,n)$ be the number of ways to place 'r' distinct objects around 'n' indistinguisable tables where no tables are empty

From our example. .

case 01 case 02 case 03
$$S(6,2) = \binom{6}{1}Q^1_1 \cdot Q^5_5 + \binom{6}{2}Q^2_2 \cdot Q^4_4 + \frac{1}{2}\binom{6}{3}Q^3_3 \cdot Q^3_3$$

Properties of $s(r,n)$:

r-people r-table

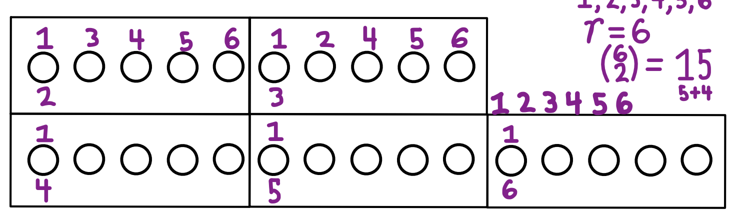
- $s(r,1) = Q^r_r = (r-1)!$
- $s(r,r) = 1$
- $s(r, r-1) = \binom{r}{2}$

from the textbook

$s(r,0) = 0$	if $r \geq 1$
$s(r,r) = 1$	if $r \geq 0$
$s(r,1) = (r-1)!$	for $r \geq 2$
$s(r,r-1) = \binom{r}{2}$	for $r \geq 2$

Pick a pair to sit together →

$s(r, r-1) = \binom{r}{2}$
1, 2, 3, 4, 5, 6
$r = 6$
$\binom{6}{2} = 15$
5+4



- $s(r, r-2) = \frac{(3n-1)}{4}\binom{n}{3}$

Note: How so?
$$s(r, r-2) = 2 \cdot \binom{n}{3} + \frac{1}{2}(2,2,^{n}n-4) = \frac{(3n-1)}{4}\binom{n}{3}$$

Claim

If $r \geq n$, then
$$S(r,n) = S((r-1), n-1) + (r-1)S(r-1, n)$$

WHY? Let $1, 2, 3, \ldots, r$ be the people

***Focus on person 1**

*case 01: $S((r-1), n-1)$
Person 1 is at own table

1	$r-1$ people $n-1$ table(s)

$S(r-1, n-1)$ Ways to complete

*case 02: $(r-1)S(r-1, n)$
Person 1 is at not own table
Place all but person #1

$r-1$ people
1 table

Choices for who sits to left of

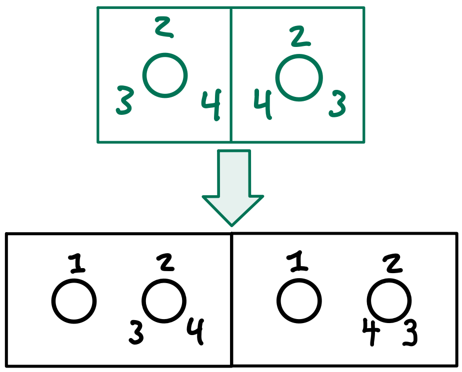
$S(r-1, n)$ Ways to complete

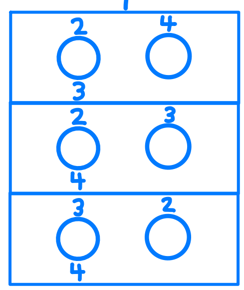
From last time, we talked about

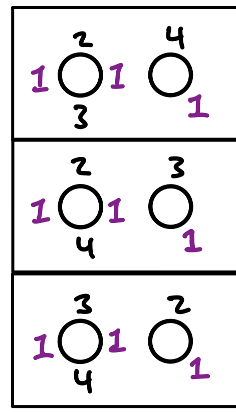
$$s(r,n) = s((r-1), n-1) + (r-1)\,s(r-1, n)$$

Why is this true? Let us draw a specific instance of this recursion.

$$\text{Say } s(4,2) = S(3,1) + 3 \cdot S(3,2)$$





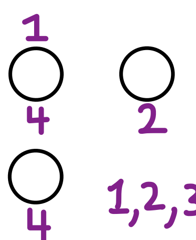


3 choices for 1

We have
$$s(4,2) = 2 + 3\cdot 3 = 11$$

What about $s(5,3)$ – 5 people and 3 tables

$$S(\overset{r}{5},\overset{n}{3}) = \underbrace{S(4,2)}_{\binom{4}{3}=11} + 4 \cdot \underbrace{S(4,3)}_{\binom{4}{2}=6}$$



$$S(r, r-1) = \binom{r}{2}$$

1,2,3,4

$$\Rightarrow s(5,3) = 11 + 4\cdot 6 = 35$$

Recall

Properties of $s(r,n)$

r – people   r – table

- $s(r,1) = Q_r^r = (r-1)!$
- $s(r,r) = 1$
- $s(r,r-1) = \binom{r}{2}$
- $s(r,r-2) = \frac{(3n-1)}{4}\binom{n}{3}$

from the textbook

$s(r,0) = 0$	if $r \geq 1$
$s(r,r) = 1$	if $r \geq 0$
$s(r,1) = (r-1)!$	for $r \geq 2$
$s(r,r-1) = \binom{r}{2}$	for $r \geq 2$

$$s(r,n) = s((r-1), n-1) + (r-1)\,s(r-1, n)$$
$$s(5,3) = s((5-1), 3-1) + (5-1)\,s(5-1, 3)$$
$$= s((4), 2) + (4)\,s(4,3)$$

$\rightarrow s(4,2) = s(3,1) + (3)s(3,2)$

$s(r,1) = Q_r^r = (r-1)!$ (3) $s(r,r-1) = \binom{r}{2}$

$(2)! + (3)\cdot\binom{3}{2}$

$\rightarrow s(4,3) = s(3,2) + (3)s(3,3)$

$s(r,r-1) = \binom{r}{2}$ (3) $s(r,r) = 1$

$\binom{3}{2} + (3)\cdot 1$

$\rightarrow s(4,2) = (2)! + (3)\cdot\binom{3}{2}$, with 2, 3, 3 written under the terms

$\rightarrow s(4,3) = \binom{3}{2} + (3)\cdot 1$, with 2, 3 written under the terms

$\rightarrow s(4,2) = 11$

$\rightarrow s(4,3) = 6$

$$s(5,3) = 11 + (4)6$$

$$\therefore s(5,3) = 35$$