SUBJECT: # Homework Assignment 03
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1) Example 1.2.1 on page 6.

Let $A=\{a,b,c,d\}$.

Find all the 3-permutations of set A.

Note

When finding permutations we use formula

$P^n_r = \frac{n!}{(n-r)!}$ Where 'n' is the set or population and 'r' is the subset of 'n' or sample set

* n is the cardinality of set A, so $n=4$
* r is the subset of A where have sets of length 3, so $r=3$

$$\text{So...}\ P^4_3 = \frac{4!}{(4-3)!} = \frac{4\cdot3\cdot2\cdot1}{1} = \boxed{24}$$

All together, we have 24 subsets permutations of set A's elements in sets of length 3.

2) Example 1.2.2 on page 7.

Let E = {a, b, c, ...x, y, z} be the set of 26 English alphabets. Find the number of 5-letter words that can be formed from E such that the first and last letters are distinct vowels and the remaining three are distinct consonants.

The english alphabet consist of 5 vowels (a,e,i,o,u) and 21 consonants

- We want permutations of 2 vowels from the "vowel set" (subset of E)
- We want permutations of 3 consonants from the "consonants set" (subset of E)
- Pattern →

Vowel	consonant	consonant	consonant	Vowel

- Finding Permutations: $P^n_r = \frac{n!}{(n-r)!}$

$$P^5_2 = \frac{5!}{(5-2)!} \rightarrow \frac{5\cdot4\cdot3\cdot2}{3\cdot2} \rightarrow \frac{120}{6} = 20 \text{ vowel permutations}$$

$$P^{21}_3 = \frac{21!}{(21-3)!} \rightarrow \frac{51{,}090{,}942{,}171{,}709{,}440{,}000}{6{,}402{,}373{,}705{,}728{,}000} = 7980 \text{ consonant permutations}$$

So, for every distinct vowel pairings we have a distinct 3 character tuple

$$P^5_2 \times P^{21}_3 = [(20)(7980)] = \boxed{159{,}600 \text{ unique words can be generated}}$$

3) Example 1.2.3 on page 8.

There are 7 boys and 3 girls in a gathering.

In how many ways can they be arranged in a row so that:

(i) The 3 girls from a single block

– no boy between any two of the girls

B1	B2	B3	B4	G1	G2	G3	B5	B6	B7

→

B1	B2	B3	B4	Girls Only	B5	B6	B7

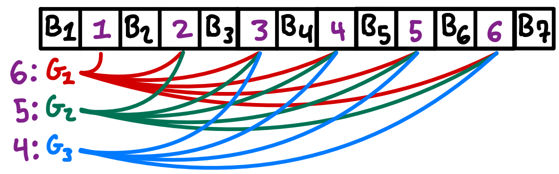
1 unit

- Block of girls can be counted 1 unit
- Girls amongst themselves can be arranged 3!

So, instead of 10 independent seating the is 7 single and 1 group seat

$$(7+1)! \times 3! = \boxed{241{,}920}$$

(3.ii) The two end-positions are occupied by boys and no girls are adjacent



The boys have 7! ways to seat themselves G_1 has 6 possibilities, G_2 has 5 possibilites, and G_3 has 4 possibilities.

So we have $7!\cdot6\cdot5\cdot4 = \boxed{604{,}800 \text{ ways}}$

4) Example 1.2.4 on page 9.

Between 20000 and 70000, find the number of even integers in which no digit is repeated.

20000 ⟶ 70000

a	b	c	d	e

a: {2,3,4,5,6} e: {0,2,4,6,8}

Case 01: first digit is even

- 3 choices for first digit {2,4,6}
- To account for 'a' being 1 of the five in set {0,2,4,6,8} Leaves us with 4 choices
- 'bcd' can range from [0,9] But we need to factor for leading & tail duplicates

So for case 01: $3\times P^{(10-2)}_3 \times 4 \rightarrow 3\times P^8_3 \times 4 = 4032$

Case 02: leading digit is odd

- 'bcd' can range from [0,9]. But we need to account for leading & tail duplicates
- 2 choices for first digit: {3,5}
- Don't have to worry 'a' = 'e'

So for case 02: $3\times P^{(10-2)}_3 \times 5 \rightarrow 3\times P^8_3 \times 5 = 5040$

All together we have $\boxed{4032+5040=9072}$

5) Problem 4 on page 50.

How many 5-letter words can be formed using A,B,C,D,E,F,G,H,I,J

(i) If the letters in each word must be distinct

$$\boxed{P^{10}_5 \rightarrow \frac{10!}{(10-5)!} = 30{,}240}$$

(ii) If, in addition, A,B,C,D,E,F can only occur as the 1^{st}, 3^{rd}, or 5^{th} letters & the rest as 2^{nd} or 4^{th}

A,B,C,D,E,F,G,H,I,J :

1	2	3	4	5

$$\boxed{P^6_3 \times P^4_2 \rightarrow \left(\frac{6!}{(6-3)!}\right)\left(\frac{4!}{(4-2)!}\right) = 1440}$$

See Back

6) Problem 2(i) and 2(ii) on page 50.

There are 12 students in a party. Five of them are girls. In how many ways can these 12 students be arranged if...

(i) there are no restrictions?

If no restrictions then, $12!$, there are $\sim 4.79\times10^8$ possible ways

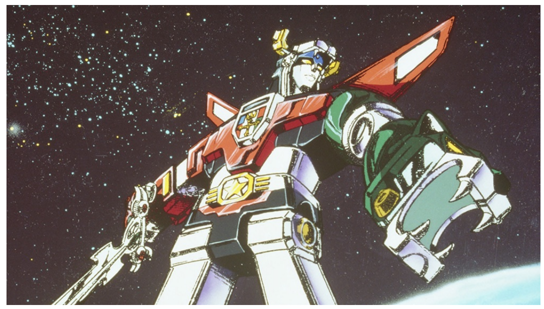
(ii) the 5 girls must be together (forming Voltron)?

* Voltron can be counted as 1

* The pilots of Voltron can switch lions

So... $(7+1)!\,5!$

$\rightarrow 8!\,5! = \boxed{4{,}838{,}400 \text{ Possible ways}}$



7) Problem 14(i), 14(ii), 14(iii), and 14(iv) on page 51.

14. Let $n, r \in \mathbf{N}$ with $r \le n$. Prove each of the following identities:

(i) $P_r^n = nP_{r-1}^{n-1}$,

(ii) $P_r^n = (n-r+1)P_{r-1}^n$,

(iii) $P_r^n = \frac{n}{n-r}P_r^{n-1}$, where $r < n$,

(iv) $P_r^{n+1} = P_r^n + rP_{r-1}^n$,

Note

When finding permutations we use formula

$P_r^n = \frac{n!}{(n-r)!}$ Where 'n' is the set or population and 'r' is the subset of 'n' or sample set

(i) $P_r^n = nP_{r-1}^{n-1}$

$P_3^3 \overset{?}{=} 3\cdot P_{(3-1)}^{(3-1)}$

$\rightarrow \frac{3!}{0!} \overset{?}{=} 3\left(\frac{2!}{0!}\right)$

$\rightarrow 6 \overset{?}{=} (2\cdot3) \rightarrow 6$

LHS = RHS

$\therefore$ True

(ii) $P_r^n = (n-r+1)P_{r-1}^n$

$P_3^3 \overset{?}{=} (3-3+1)P_{3-1}^3$

$\rightarrow 6 \overset{?}{=} P_{3-1}^3$

$\rightarrow 6 \overset{?}{=} \frac{3!}{(3-2)!}$

$\rightarrow 6 \overset{?}{=} 6$

LHS = RHS

$\therefore$ True

(iii) $P_r^n = \frac{n}{n-r}P_r^{n-1}$, where $r<n$

RHS : $\frac{n}{n-r}P_r^{n-1}$

$\rightarrow$

(iv) $P_r^{n+1} = P_r^n + rP_{r-1}^n$

RHS

$\rightarrow P_r^n + rP_{r-1}^n$

$\rightarrow \frac{n!}{(n-r)!} + r\frac{n!}{(n-r+1)!}$

$\rightarrow \frac{(n-r+1)}{(n-r+1)}\cdot\frac{n!}{(n-r)!} + r\frac{n!}{(n-r+1)!}$

$\rightarrow \frac{(n-r+1)n!}{(n-r+1)!} + \frac{r\cdot n!}{(n-r+1)!}$

$\rightarrow \frac{(n-r+1)\,n! + r\cdot n!}{(n-r+1)!}$

$\rightarrow \frac{n!(n-r+1+r)}{(n-r+1)!}$

$\rightarrow \frac{(n+1)!}{(n-r+1)!} = P_r^{(n+1)}$

Recall

$P_r^n = \frac{n!}{(n-r)!}$

LHS = RHS

$P_r^{n+1} = P_r^{n+1}$ $\therefore$ True