California State University Sacramento - Math 101

Homework Assignment 3

1) Example 1.2.1 on page 6. Q1) **Example 1.2.1.** Let $A = \{a, b, c, d\}$. All the 3-permutations of A are

2) Example 1.2.2 on page 7.

3) Example 1.2.3 on page 8.

4) Example 1.2.4 on page 9.

Q2) **Example 1.2.2.** Let $E = \{a, b, c, ..., x, y, z\}$ be the set of the 26 English alphabets. Find the number of 5-letter words that can be formed from E such that the first and last letters are distinct vowels and the remaining three are distinct consonants.

5) Problem 4 on page 50.

6) Problem 2(i) and 2(ii) on page 50.

7) Problem 14(i), 14(ii), 14(iii), and 14(iv) on page 51.

Q3) **Example 1.2.3.** There are 7 boys and 3 girls in a gathering. In how many ways can they be arranged in a row so that

(i) the 3 girls form a single block (i.e. there is no boy between any two of the girls)?

(ii) the two end-positions are occupied by boys and no girls are adjacent?

Q4) **Example 1.2.4.** Between 20000 and 70000, find the number of even integers in which no digit is repeated.

Q5) 4. How many 5-letter words can be formed using A, B, C, D, E, F, G, H, I, J,

(i) if the letters in each word must be distinct?

(ii) if, in addition, A, B, C, D, E, F can only occur as the first, third or fifth letters while the rest as the second or fourth letters?

Q6) 2. There are 12 students in a party. Five of them are girls. In how many ways can these 12 students be arranged in a row if

(i) there are no restrictions?

(ii) the 5 girls must be together (forming a block)?

Q7) 14. Let $n, r \in \mathbf{N}$ with $r \leq n$. Prove each of the following identities:

(i) $P_r^n = nP_{r-1}^{n-1}$,

(ii) $P_r^n = (n - r + 1)P_{r-1}^n$,

(iii) $P_r^n = \frac{n}{n-r}P_r^{n-1}$, where $r < n$,

(iv) $P_r^{n+1} = P_r^n + rP_{r-1}^n$,

(v) $P_r^{n+1} = r! + r(P_{r-1}^n + P_{r-1}^{n-1} + \cdots + P_{r-1}^r)$.

1) Example 1.2.1 on page 6.

Let $A=\{a,b,c,d\}$.
Find all the 3-permutations of set A.

Note
When finding permutations we use formula
$P^n_r = \frac{n!}{(n-r)!}$ Where 'n' is the set or population and 'r' is the subset of 'n' or sample set

* n is the cardinality of set A, so $n=4$
* r is the subset of A where have sets of length 3, so $r=3$

So... $P^4_3 = \frac{4!}{(4-3)!} = \frac{4\cdot3\cdot2\cdot1}{1} = \boxed{24}$

All together, we have 24 subsets permutations of set A's elements in sets of length 3.

2) Example 1.2.2 on page 7.

Let E = {a, b, c, ...x, y, z} be the set of 26 English alphabets. Find the number of 5-letter words that can be formed from E such that the first and last letters are distinct vowels and the remaining three are distinct consonants.

The english alphabet consist of 5 vowels (a,e,i,o,u) and 21 consonants

- We want permutations of 2 vowels from the "vowel set" (subset of E)
- We want permutations of 3 consonants from the "consonants set" (subset of E)
- Pattern → | Vowel | consonant | consonant | consonant | Vowel |
- Finding Permutations: $P^n_r = \frac{n!}{(n-r)!}$

$P^5_2 = \frac{5!}{(5-2)!} \rightarrow \frac{5\cdot4\cdot3\cdot2}{3\cdot2} \rightarrow \frac{120}{6} = 20$ vowel permutations

$P^{21}_3 = \frac{21!}{(21-3)!} \rightarrow \frac{51{,}090{,}942{,}171{,}709{,}440{,}000}{6{,}402{,}373{,}705{,}728{,}000} = 7980$ consonant permutations

So, for every distinct vowel pairings we have a distinct 3 character tuple

$P^5_2 \times P^{21}_3 = \boxed{(20)(7980)} = \boxed{159{,}600}$ unique words can be generated

3) Example 1.2.3 on page 8.

There are 7 boys and 3 girls in a gathering.
In how many ways can they be arranged in a row so that:
(i) The 3 girls from a single block
– no boy between any two of the girls

| B_1 | B_2 | B_3 | B_4 | G_1 | G_2 | G_3 | B_5 | B_6 | B_7 | → | B_1 | B_2 | B_3 | B_4 | Girls Only | B_5 | B_6 | B_7 |

1 unit

- Block of girls can be counted 1 unit
- Girls amongst themselves can be arranged 3!

So, instead of 10 independent seating the is 7 single and 1 group seat

$(7+1)! \times 3! = \boxed{241{,}920}$

(3.ii) The two end-positions are occupied by boys and no girls are adjacent

| B_1 | 1 | B_2 | 2 | B_3 | 3 | B_4 | 4 | B_5 | 5 | B_6 | 6 | B_7 |

6: G_1
5: G_2
4: G_3

The boys have 7! ways to seat themselves G_1 has 6 possibilities, G_2 has 5 possibilites, and G_3 has 4 possibilities.

So we have $7!\cdot6\cdot5\cdot4 = \boxed{604{,}800 \text{ ways}}$

4) Example 1.2.4 on page 9.

Between 20000 and 70000, find the number of even integers in which no digit is repeated.

20000 ⟶ 70000
| a | b | c | d | e |
a: {2,3,4,5,6} e: {0,2,4,6,8}

Case 01: first digit is even

- 3 choices for first digit {2,4,6}
- To account for 'a' being 1 of the five in set {0,2,4,6,8} Leaves us with 4 choices
- 'bcd' can range from [0,9] But we need to factor for leading & tail duplicates

So for case 01: $3\times P^{(10-2)}_3 \times 4 \rightarrow 3\times P^8_3\times4 = 4032$

Case 02: leading digit is odd

- 'bcd' can range from [0,9]. But we need to account for leading & tail duplicates
- 2 choices for first digit: {3,5}
- Don't have to worry 'a' = 'e'

So for case 02: $3\times P^{(10-2)}_3\times5 \rightarrow 3\times P^8_3\times5 = 5040$

All together we have $\boxed{4032+5040=9072}$

5) Problem 4 on page 50.

How many 5-letter words can be formed using A,B,C,D,E,F,G,H,I,J

(i) If the letters in each word must be distinct

$\boxed{P^{10}_5 \rightarrow \frac{10!}{(10-5)!} = 30{,}240}$

(ii) If, in addition, A,B,C,D,E,F can only occur as the 1st, 3rd, or 5th letters & the rest as 2nd or 4th

A,B,C,D,E,F,G,H,I,J : | 1 | 2 | 3 | 4 | 5 |

$\boxed{P^6_3 \times P^4_2 \rightarrow \left(\frac{6!}{(6-3)!}\right)\left(\frac{4!}{(4-2)!}\right) = 1440}$

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6) Problem 2(i) and 2(ii) on page 50.

There are 12 students in a party. Five of them are girls.
In how many ways can these 12 students be arranged if...
(i) there are no restrictions?

If no restrictions then,
$12!$, there are $\sim 4.79 \times 10^8$ possible ways

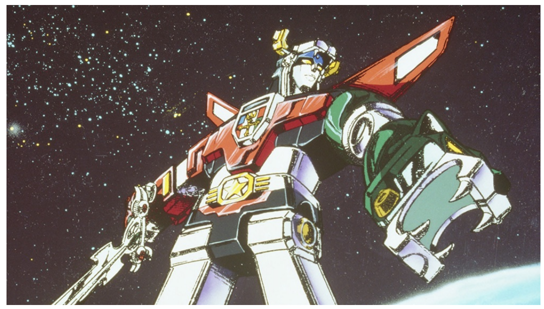
(ii) the 5 girls must be together (forming Voltron)?

* Voltron can be counted as 1

* The pilots of Voltron can switch lions

So... $(7+1)!\,5!$

$\rightarrow 8!\,5! = \boxed{4{,}838{,}400 \text{ Possible ways}}$



7) Problem 14(i), 14(ii), 14(iii), and 14(iv) on page 51.

14. Let $n, r \in \mathbf{N}$ with $r \leq n$. Prove each of the following identities:

(i) $P_r^n = nP_{r-1}^{n-1}$,

(ii) $P_r^n = (n-r+1)P_{r-1}^n$,

(iii) $P_r^n = \frac{n}{n-r}P_r^{n-1}$, where $r < n$,

(iv) $P_r^{n+1} = P_r^n + rP_{r-1}^n$,

Note

When finding permutations we use formula

$P_r^n = \frac{n!}{(n-r)!}$ Where 'n' is the set or population and 'r' is the subset of 'n' or sample set

(i) $P_r^n = nP_{r-1}^{n-1}$

$P_3^3 \overset{?}{=} 3 \cdot P_{(3-1)}^{(3-1)}$

$\rightarrow \frac{3!}{0!} \overset{?}{=} 3\left(\frac{2!}{0!}\right)$

$\rightarrow 6 \overset{?}{=} (2 \cdot 3) \rightarrow 6$

LHS = RHS

$\therefore$ True

(ii) $P_r^n = (n-r+1)P_{r-1}^n$

$P_3^3 \overset{?}{=} (3-3+1)P_{3-1}^3$

$\rightarrow 6 \overset{?}{=} P_{3-1}^3$

$\rightarrow 6 \overset{?}{=} \frac{3!}{(3-2)!}$

$\rightarrow 6 \overset{?}{=} 6$

LHS = RHS

$\therefore$ True

(iii) $P_r^n = \frac{n}{n-r}P_r^{n-1}$, where $r < n$

RHS : $\frac{n}{n-r}P_r^{n-1}$

$\rightarrow$

(iv) $P_r^{n+1} = P_r^n + rP_{r-1}^n$

RHS

$\rightarrow P_r^n + rP_{r-1}^n$

$\rightarrow \frac{n!}{(n-r)!} + r\frac{n!}{(n-r+1)!}$

$\rightarrow \frac{(n-r+1)}{(n-r+1)} \cdot \frac{n!}{(n-r)!} + r\frac{n!}{(n-r+1)!}$

$\rightarrow \frac{(n-r+1)n!}{(n-r+1)!} + \frac{r \cdot n!}{(n-r+1)!}$

$\rightarrow \frac{(n-r+1)\,n! + r \cdot n!}{(n-r+1)!}$

$\rightarrow \frac{n!(n-r+1+r)}{(n-r+1)!}$

$\rightarrow \frac{(n+1)!}{(n-r+1)!} = P_r^{(n+1)}$

Recall

$P_r^n = \frac{n!}{(n-r)!}$

LHS = RHS

$P_r^{n+1} = P_r^{n+1}$ $\therefore$ True

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14. Let $n, r \in \mathbf{N}$ with $r \leq n$. Prove each of the following identities:

(i) $P_r^n = nP_{r-1}^{n-1}$,

(ii) $P_r^n = (n-r+1)P_{r-1}^n$,

(iii) $P_r^n = \frac{n}{n-r}P_r^{n-1}$, where $r < n$,

(iv) $P_r^{n+1} = P_r^n + rP_{r-1}^n$,

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1) Example 1.2.1 on page 6.

Example 1.2.1. Let $A = \{a, b, c, d\}$. All the 3-permutations of A are shown below:

$$\begin{array}{cccccc} abc, & acb, & bac, & bca, & cab, & cba, \\ abd, & adb, & bad, & bda, & dab, & dba, \\ acd, & adc, & cad, & cda, & dac, & dca, \\ bcd, & bdc, & cbd, & cdb, & dbc, & dcb. \end{array}$$

There are altogether 24 in number. ∎

Let P_r^n denote the number of r-permutations of A. Thus $P_3^4 = 24$ as shown in Example 1.2.1. In what follows, we shall derive a formula for P_r^n by applying (MP).

An r-permutation of A can be formed in r steps, as described below: First, we choose an object from A and put it in the first position (see Figure 1.2.1). Next we choose an object from the remaining ones in A and put it in the second position. We proceed on until the rth-step in which we choose an object from the remaining $(n - r + 1)$ elements in A and put it in the rth-position.

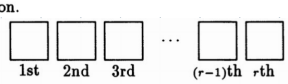


Figure 1.2.1

There are n choices in step 1, $(n - 1)$ choices in step 2, ..., $n - (r - 1)$ choices in step r. Thus by (MP),

$$P_r^n = n(n-1)(n-2)\cdots(n-r+1). \qquad (1.2.1)$$

If we use the factorial notation: $n! = n(n-1)\cdots 2 \cdot 1$, then

$$P_r^n = \frac{n!}{(n-r)!}. \qquad (1.2.2)$$

Remark. By convention, $0! = 1$. Note that $P_0^n = 1$ and $P_n^n = n!$.

2) Example 1.2.2 on page 7.

Example 1.2.2. Let $E = \{a, b, c, ..., x, y, z\}$ be the set of the 26 English alphabets. Find the number of 5-letter words that can be formed from E such that the first and last letters are distinct vowels and the remaining three are distinct consonants.

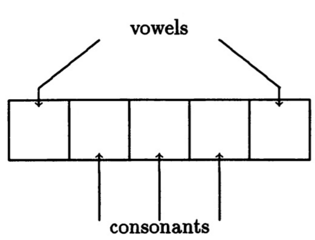


Figure 1.2.2.

Solution. There are 5 vowels and 21 consonants in E. A required 5-letter word can be formed in the following way.

Step 1. Choose a 2-permutation of $\{a, e, i, o, u\}$ and then put the first vowel in the 1^{st} position and the second vowel in the 5^{th} position (see Figure 1.2.2).

Step 2. Choose a 3-permutation of $E \backslash \{a, e, i, o, u\}$ and put the 1st, 2nd and 3rd consonants of the permutation in the 2nd, 3rd and 4th positions respectively (see Figure 1.2.2).

There are P_2^5 choices in Step 1 and P_3^{21} choices in Step 2. Thus by (MP), the number of such 5-letter words is given by

$$P_2^5 \times P_3^{21} = (5 \times 4) \times (21 \times 20 \times 19) = 159600. \ \blacksquare$$

Example 1.2.3. There are 7 boys and 3 girls in a gathering. In how many ways can they be arranged in a row so that

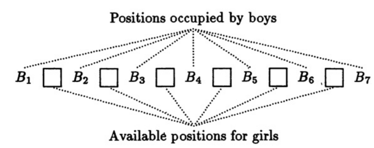
(i) the 3 girls form a single block (i.e. there is no boy between any two of the girls)?

(ii) the two end-positions are occupied by boys and no girls are adjacent?

Solution. (i) Since the 3 girls must be together, we can treat them as a single entity. The number of ways to arrange 7 boys together with this entity is $(7+1)!$. As the girls can permute among themselves within the entity in 3! ways, the desired number of ways is, by (MP),

$$8! \times 3!.$$

(ii) We first consider the arrangements of boys and then those of girls. There are 7! ways to arrange the boys. Fix an arbitrary one of the arrangements. Since the end-positions are occupied by boys, there are only 6 spaces available for the 3 girls G_1, G_2 and G_3.



G_1 has 6 choices. Since no two girls are adjacent, G_2 has 5 choices and G_3 has 4. Thus by (MP), the number of such arrangements is

$$7! \times 6 \times 5 \times 4. \ \blacksquare$$

Remark. Example 1.2.3 can also be solved by considering the arrangements for the girls first. This will be discussed in Example 1.7.2.