California State University Sacramento - Math 101

Homework Assignment 9 - Solutions

4) The equality that we are trying to prove holds if and only if

$$\frac{1}{\binom{n}{m}} \sum_{r=m}^{n} \binom{n}{r}\binom{r}{m} = 2^{n-m}.$$

We then distribute the $\frac{1}{\binom{n}{m}}$ into the sum to get

$$\sum_{r=m}^{n} \frac{\binom{n}{r}\binom{r}{m}}{\binom{n}{m}} = 2^{n-m}. \tag{1}$$

Therefore, if we can show that (1) is true, then we know that the equality stated in the exercise is also true. Now

$$\frac{\binom{n}{r}\binom{r}{m}}{\binom{n}{m}} = \frac{\frac{n!}{r!(n-r)!}\frac{r!}{m!(r-m!)}}{\frac{n!}{m!(n-m)!}} = \frac{(n-m)!}{(n-r)!(r-m)!} = \frac{(n-m)!}{(n-r)!((n-m)-(n-r))!} = \binom{n-m}{n-r}.$$

Thus,

$$\sum_{r=m}^{n} \frac{\binom{n}{r}\binom{r}{m}}{\binom{n}{m}} = \sum_{r=m}^{n} \binom{n-m}{n-r} = 2^{n-m}$$

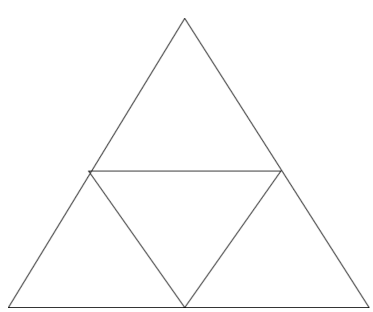
where the second equality holds because of Example 2.3.1. This shows that (1) holds and so, after multiplying both sides of (1) by $\binom{n}{m}$ we get

$$\sum_{r=m}^{n} \binom{n}{r}\binom{r}{m} = 2^{n-m}\binom{n}{m}.$$

6) By the Binomial Theorem, $(x+y)^n = \displaystyle\sum_{r=0}^{n} \binom{n}{r} x^{n-r}y^r$. Setting $x = 1$ and then taking the derivative with respect to y gives $n(1+y)^{n-1} = \displaystyle\sum_{r=0}^{n} \binom{n}{r} ry^{r-1}$. Now set $y = -1$ to get $0 = \displaystyle\sum_{r=0}^{n} \binom{n}{r} r(-1)^{r-1}$. Multiplying both sides of this equation by -1 gives

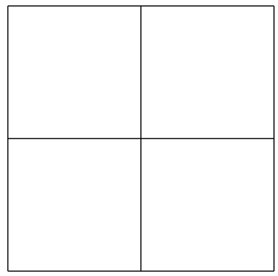
$$0 = \sum_{r=0}^{n} (-1)^r r \binom{n}{r}.$$

8) Divide the equilateral triangle into 4 smaller triangles as shown below.



By the Pigeonhole Principle, one of the smaller triangles must contain at least two of the points. Any one of the smaller triangles has side lengths $\frac{1}{2}$ and so any two points within the same smaller triangles are at most $\frac{1}{2}$ units apart.

9) Divide the square into four equal size smaller squares as shown below.



By the Pigeonhole Principle, one of the smaller squares must contain at least three of the points. The area of a small square is $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$. Any triangle inside a square will have area at most half that of the square. Thus, the area of the triangle determined by three points in the same smaller is at most $\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$.