Spring 2023 Final Exam Study Guide

Last Week of Semester

Craig Timmons (He/Him/His)

All Sections

Here is an outline of the last week of the semester.

Monday - Homework 11 is due.

Wednesday - Quiz 12 will be based on Recurrence Relations (Homework 11) and Inclusion Exclusion (Homework 10). You may work in groups and use your notes on this quiz.

Friday - The first part of class (**1 to 1:20**) will be used like an office hour, just in the classroom. This will be the time to show me up to two completed homework assignments that can be used to replace up to two 0 with 5's for those assignments. You can study with your peers or ask me questions.

The second part of class (**1:25 to 1:50**) will be used for the individual make-up quiz. Your score on this quiz will replace your lowest quiz score provided it helps your grade. You can take this quiz for practice and do not have to turn it in. If you do turn it in and your score is below your lowest quiz score, nothing will change about your grade.

To prepare for the final exam, I recommend studying Exams 1, 2, and 3, Homework 10 and 11, and Quizzes 10, 11, and 12.

Our final exam is on Monday 5/15 from 12:45 to 2:45. You may use one 3x5 index card with writing on both the front and back for the final.

Let me know if you have any questions.

This announcement is closed for comments

California State University Sacramento - Math 101

Exam # 1

Name: ________________________

This exam is out of 12 points.

1) (a) Give a formula for P_r^n in terms of factorials. (0.25 points)

(b) Give a formula for Q_r^n in terms of factorials. (0.25 points)

(c) Give a formula for C_r^n in terms of factorials. (0.25 points)

(d) Express Q_r^n in terms of P_r^n. (0.25 points)

2) Find the number of ways that 8 boys and 3 girls can be put in a line such that the 3 girls form a single block. (1 point)

3) Find the number of **ordered** pairs (a, b), with $a, b \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, such that $|a - b| \leq 2$. (1 point)

4) Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d\}$.

(a) List all 2-permutations of A. (0.5 points)

(b) List all 3-circular permutations of B. (0.5 points)

(c) List all subsets of A of size 3. (0.25 points)

5) (a) Find the number of common positive divisors of 10^{30} and 20^{20}. (1 point)

(b) Find the number of positive divisors of $900 = 2^2 \cdot 3^3 \cdot 5^2$ that are multiples of 5. (1 point)

6) Find the number of odd integers between 1000 and 3000 that have no repeated digit. (1 point)

7) (a) Determine the number of 0-1 sequences of length 5. For instance, 10101 and 11001 are two such sequences. (0.25 points)

(b) Determine the number of 0-1 sequences of length 5 that have exactly two 1's. (0.25 points)

(c) Determine the number of 0-1 sequences of length 5 that have at most two 1's (so the sequence can have no 1's, one 1, or two 1's). (0.5 points)

(d) Let n be an arbitrary integer. Determine the number of 0-1 sequences of length n with at most one 1 (so the sequence can have no 1's, or one 1). (0.5 points)

8) Let G be a 5×6 grid.

(a) Determine the number of $1x1$ squares in G. (0.5 points)

(b) Determine the number of 2×3 rectangles in G. (0.5 points)

(c) Determine the total number of squares in G. (0.5 points)

(d) Determine the total number of rectangles in G. (0.5 points)

9) (a) Give a proof, using algebra, that $P_r^n = nP_{r-1}^{n-1}$. (0.75 points)

(b) Give a proof, using algebra, that $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$. (1 point)

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Exam # 2

Name: ______________________

Exam is out of 8 points.

1) Let $X = \{1, 2, 3, \ldots, 20\}$.

(a) What is the number of 3-permutations of X? (0.25 points)

(b) What is the number of 3-circular permutations of X? (0.25 points)

(c) What is the number of 2-combinations of X? Simplify your answer as much as possible. (0.25 points)

(d) If $\mathcal{P}(X)$ is the power set of X, how many elements are in $\mathcal{P}(X)$? (0.25 points)

(e) Find the number of 5-combinations of X with only odd numbers. (0.25 points)

(f) Find the number of 5-combinations of X that do not contain a pair of consecutive integers. (0.25 points)

2) A box contains 12 distinct marbles. Seven of the marbles are red and five of the marbles are blue. Thus, the elements of the box could be represented as the set

$$\{r_1, r_2, r_3, r_4, r_5, r_6, r_7, b_1, b_2, b_3, b_4, b_5\}.$$

(a) Find the number of ways to form a combination of two red marbles and three blue marbles. (0.25 points)

(b) Find the number of ways to form a combination of five marbles, at least three of which are blue. (0.5 points)

(c) How many ways can all 12 marbles be arranged around a circle so that no blue marbles are next to each other? (0.25 points)

(d) Find the number of permutations of all 12 marbles such that b_1 comes before b_2. (0.5 points)

3) Recall that a 0-1 sequence of length n is a sequence of the form $a_1a_2\cdots a_n$ such that each a_i is either 0 or 1.

(a) What is the total number of 0-1 sequences of length 6? Simplify your answer as much as possible. (0.25 points)

(b) Find the total number of 0-1 sequences of length 10 that have at most two 0's. Simplify your answer as much as possible. (0.5 points)

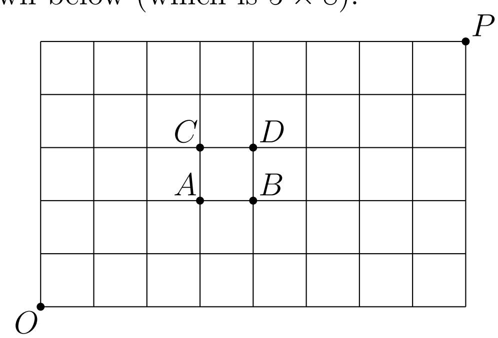
(c) 37 people are going to vote on their preference for a new menu item at Timmons' Stars and Bars SandwHich Shop. The options are A, B, C, and D. How many possible outcomes are there after the 37 people vote? In other words, how many ways can 37 elements be chosen from A, B, C, and D allowing for repetition. (0.5 points)

4) Let $1 \leq r \leq n$.

(a) Give an algebraic proof that $\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$. Write your proof carefully and use correct mathematical notation. (0.5 points)

(b) Explain why $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ is true by describing a counting process that shows the number of r-combinations of $\{1, 2, \ldots, n\}$ is equal to the right hand side of this equation. (0.5 points)

5) Consider the figure shown below (which is 5×8).



(a) Find the number of shortest routes from O to P. (0.25 points)

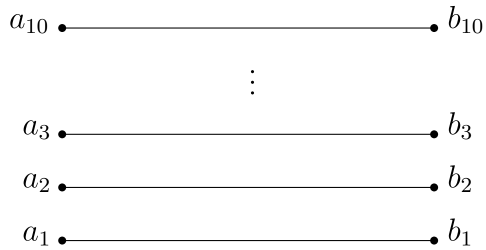
(b) Find the number of shortest routes from O to P that pass through the street AB or the street CD. (0.5 points)

6) Let $s(r, n)$ be the number of ways to arrange r distinct objects around n indistinguishable circles so that every circle has at least one object.

(a) Why is $s(r, r) = 1$? Answer in a complete sentence and do not use any math symbols. (0.25 points)

(b) Determine, with explanation (which can include pictures) the value of $s(6, 2)$. (0.75 points)

7) Consider the matching shown below.



(a) Find the number of subsets of size 5 from $\{a_1, \ldots, a_{10}, b_1, \ldots, b_{10}\}$ that do not contain any of the edges. (0.25 points)

(b) Find the number of subsets of size 5 from $\{a_1, \ldots, a_{10}, b_1, \ldots, b_{10}\}$ that contain exactly one edge. (0.5 points)

(c) Find the number of subsets of <u>size 10</u> from $\{a_1, \ldots, a_{10}, b_1, \ldots, b_{10}\}$ that contain exactly three edges. (0.25 points)

California State University Sacramento - Math 101

Exam #3

Name: ______________________

This exam is out of 7 points.

1) (a) Determine the exact value of the coefficient of x^3 in the expansion of $(x+1)^{13}$. (0.5 points)

(b) Determine the exact value of the coefficient of $x^3y^4z^2$ in the expansion of $(x+y+z)^9$. (0.5 points)

2) In the Binomial Theorem $(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$, state a value for x and value for y that produces the given formula. (0.25 points each)

(a) $0 = \sum_{r=0}^{n} \binom{n}{r} (-1)^r$

(b) $2^n = \sum_{r=0}^{n} \binom{n}{r}$

3) Show that for all integers $n \geq m \geq 1$, $\displaystyle\sum_{r=m}^{n} \binom{n-m}{n-r} = 2^{n-m}$. (0.5 points)

4) Prove $\displaystyle\sum_{r=0}^{n} r\binom{n}{r} = n2^{n-1}$ for all $n \geq 1$. (0.75 points)

5) Prove that $\binom{n}{r}\binom{n-r}{m-r} = \binom{n}{m}\binom{m}{r}$ for all integers $n \geq m \geq r \geq 1$. (0.75 points)

6) Let $X = \{1, 2, 3, 4\}$. Let $\mathcal{A}$ be the collection of all subsets of X with an even number of elements, and let $\mathcal{B}$ be the collection of all subsets of X with an odd number of elements. Remark: The empty set $\emptyset$ is one of the sets in $\mathcal{A}$ since it has 0 elements and 0 is even.

(a) List all of the elements of $\mathcal{A}$. (0.5 points)

(b) List all of the elements of $\mathcal{B}$. (0.5 points)

(c) Is the function $f(C) = X \backslash C$ a bijection from $\mathcal{A}$ to $\mathcal{B}$? Recall that $X \backslash C$ is the complement of C in X. (0.25 points)

(d) Is the function $f(C) = C \cup \{1\}$ a bijection from $\mathcal{A}$ to $\mathcal{B}$? (0.25 points)

(e) Draw a bijection between $\mathcal{A}$ and $\mathcal{B}$. Represent your bijection using an arrow diagram. (0.25 points)

7) (a) There are 50 jobs that must be assigned to 7 processors. Explain why there must be a processor that is assigned at least 8 jobs. (0.5 points)

(b) A list $\mathcal{L}$ contains 134 elements. Each element of $\mathcal{L}$ is a subset of $\{1, 2, 3, 4, 5, 6, 7\}$. Explain why the list $\mathcal{L}$ must contain two elements that are the same. (0.5 points)

(c) Suppose it takes a program 1 second to find the determinant of a 2×2 matrix. If S is the set of all 2×2 matrices of the form $\begin{pmatrix} x & y \\ z & t \end{pmatrix}$ where $x, y, z, t \in \{0, 1\}$, what is the minimum amount of time it would take the program to find the determinant of every matrix in S? (0.25 points)

8) Prove that $\displaystyle\sum_{r=0}^{n-1} \binom{2n-1}{r} = 2^{2n-2}$. (0.5 points)

1) (a) $P_r^n = \dfrac{n!}{(n-r)!}$ (b) $Q_r^n = \dfrac{n!}{r\cdot(n-r)!}$ (c) $C_r^n = \dfrac{n!}{r!\,(n-r)!}$

(d) $Q_r^n = \frac{1}{r} P_r^n$

2) $\underbrace{\overline{1}\ \overline{2}\ \overline{3}\ \overline{4}\ \overline{5}\ \overline{6}\ \overline{7}\ \overline{8}\ \overline{9}}\ \overline{10}\ \overline{11}$

9 choices for position of first girl

$$9 \cdot P_8^8 \cdot P_3^3 = 9 \cdot 8! \cdot 3!$$
$$= 2{,}177{,}280$$

3)

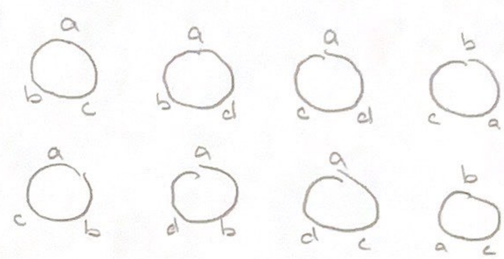
a	1	2	3	4	5	6	7	8	9	10
possible b	1,2,3	1,2,3,4	1,2,3,4,5	2,3,4,5,6	3,4,5,6,7	4,5,6,7,8	5,6,7,8,9	6,7,8,9,10	7,8,9,10	8,9,10

The number of pairs is

$$3 + 4 + 6\cdot 5 + 4 + 3 = 44.$$

4) (a) 12, 13, 14, 21, 23, 24, 31, 32, 34, 41, 42, 43

(b)



(c) {1,2,3}, {1,2,4}, {1,3,4}, {2,3,4}

5) (a) $10^{30} = 2^{30}5^{30}$ and $20^{20} = 2^{40}5^{20}$

must divide 2^{30}

must divide 5^{20}

31 choices

21 choices

$2^{\square}5^{\square}$

$21 \cdot 31 = 651$ common positive divisors

(b) $2^{\square}\,3^{\square}\,5^{\square}$ with exponents $0,1,2$ / $0,1,2,3$ / $1,2$

There are $3\times4\times2=24$ such divisors

6) $\underset{1 \text{ or } 2}{\underline{\;a_1\;}}\ \underline{\;a_2\;}\ \underline{\;a_3\;}\ \underset{1,3,5,7 \text{ or } 9}{\underline{\;a_4\;}}$

If $a_1=2$, then there are 5 choices for a_4, then $8\cdot7$ choices for a_2 and a_3.

If $a_1=1$, then there are 4 choices for a_4, and $8\cdot7$ choices for a_2 and a_3.

Total: $5\times8\times7+4\times8\times7$

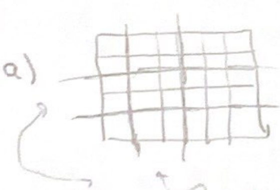
7) (a) $2^5=32$ (b) $\binom{5}{2}=10$ (c) $\binom{5}{0}+\binom{5}{1}+\binom{5}{2}=16$

↑ choose positions for 1's

_ _ _ _ _

(d) $n+1$ since only one sequence of all 0's, and $\binom{n}{1}=n$ sequences with exactly one 1.

8) (a)



$5\times6=30$ is the number of 1×1 squares

(b) $4\times4=16$

(c)

$$\underbrace{5\times6}_{\substack{1\times1\\ \text{squares}}}+\underbrace{4\times5}_{\substack{2\times2\\ \text{squares}}}+\underbrace{3\times4}_{\substack{3\times3\\ \text{squares}}}+\underbrace{2\times3}_{\substack{4\times4\\ \text{squares}}}+\underbrace{1\times2}_{\substack{5\times5\\ \text{squares}}}$$

$= 30+20+12+6+2$

$= 70$

(d) $\binom{6}{2}\binom{7}{2}$ is the number of rectangles

9) (a) $n P^{n-1}_{r-1} = n \cdot \frac{(n-1)!}{(n-1-(r-1))!} = \frac{n!}{(n-1-r+1)!} = \frac{n!}{(n-r)!} = P^n_r$

(b) $\binom{n-1}{r-1} + \binom{n-1}{r}$

$$= \frac{(n-1)!}{(n-1-(r-1))!\,(r-1)!} + \frac{(n-1)!}{(n-1-r)!\,r!}$$

multiplied first fraction by $\frac{r}{r}$

multiplied second fraction by $\frac{n-r}{n-r}$

$$= \frac{r \cdot (n-1)!}{(n-r)!\,r!} + \frac{(n-1)! \cdot (n-r)}{(n-r)!\,r!}$$

$$= \frac{r \cdot (n-1)! + (n-1)! \cdot (n-r)}{(n-r)!\,r!}$$

$$= \frac{(n-1)!\,(r+n-r)}{(n-r)!\,r!} = \frac{n!}{(n-r)!\,r!} = \binom{n}{r}$$

California State University Sacramento - Math 101
Exam # 2

Name: ______________________

Exam is out of 8 points.

1) Let $X = \{1, 2, 3, \ldots, 20\}$.
(a) What is the number of 3-permutations of X? (0.25 points)
(b) What is the number of 3-circular permutations of X? (0.25 points)
(c) What is the number of 2-combinations of X? Simplify your answer as much as possible. (0.25 points)
(d) If $\mathcal{P}(X)$ is the power set of X, how many elements are in $\mathcal{P}(X)$? (0.25 points)
(e) Find the number of 5-combinations of X with only odd numbers. (0.25 points)
(f) Find the number of 5-combinations of X that do not contain a pair of consecutive integers. (0.25 points)

(a) $P^{20}_{3} = \dfrac{20!}{(20-3)!} = \dfrac{20!}{17!} = 20 \cdot 19 \cdot 18$

(b) $Q^{20}_{3} = \dfrac{1}{3} P^{20}_{3} = \dfrac{20 \cdot 19 \cdot 18}{3} = 20 \cdot 19 \cdot 6$

(c) $\binom{20}{2} = \dfrac{20!}{2!\,18!} = \dfrac{20 \cdot 19}{2} = 10 \cdot 19 = 190$

(d) $2^{|X|} = 2^{20}$

(e) $\binom{10}{5}$ since X contains 10 odd numbers

(f) $\binom{20-5+1}{5} = \binom{16}{5}$

2) A box contains 12 distinct marbles. Seven of the marbles are red and five of the marbles are blue. Thus, the elements of the box could be represented as the set

$$\{r_1, r_2, r_3, r_4, r_5, r_6, r_7, b_1, b_2, b_3, b_4, b_5\}.$$

(a) Find the number of ways to form a combination of two red marbles and three blue marbles. (0.25 points)

(b) Find the number of ways to form a combination of five marbles, at least three of which are blue. (0.5 points)

(c) How many ways can all 12 marbles be arranged around a circle so that no blue marbles are next to each other? (0.25 points)

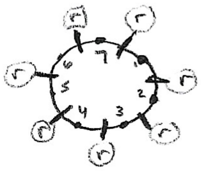
(d) Find the number of permutations of all 12 marbles such that b_1 comes before b_2. (0.5 points)

(a) $\boxed{\binom{7}{2}\binom{5}{3}}$

(b) $\boxed{\underbrace{\binom{5}{3}\binom{7}{2}}_{\text{3 blue, 2 red}} + \underbrace{\binom{5}{4}\binom{7}{1}}_{\text{4 blue, 1 red}} + \underbrace{\binom{5}{5}\binom{7}{0}}_{\text{5 blue, 0 red}}}$

(c) First place the reds around the circle

which can be done in $\frac{7!}{7} = 6!$ ways.



There are seven places between the reds

$\Rightarrow$ 7 choices for b_1, 6 choices for $b_2, \ldots,$ 3 choices for b_5

$\Rightarrow$ $\boxed{6! \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}$

(d) $\underbrace{\_\ \_\ \_\ b_1\ \_\ \_\ \_\ b_2\ \_\ \_\ \_\ \_}_{\binom{12}{2} \text{ ways to place } b_1, b_2}$ $\leftarrow$ 10! ways to line up remaining

$\boxed{\binom{12}{2} 10!}$

3) Recall that a 0-1 sequence of length n is a sequence of the form $a_1 a_2 \cdots a_n$ such that each a_i is either 0 or 1.

(a) What is the total number of 0-1 sequences of length 6? Simplify your answer as much as possible. (0.25 points)

(b) Find the total number of 0-1 sequences of length 10 that have at most two 0's. Simplify your answer as much as possible. (0.5 points)

(c) 37 people are going to vote on their preference for a new menu item at Timmons' Stars and Bars SandwHich Shop. The options are A, B, C, and D. How many possible outcomes are there after the 37 people vote? In other words, how many ways can 37 elements be chosen from A, B, C, and D allowing for repetition? (0.5 points)

(a) $2^6 = 64$

(b) $\binom{10}{0} + \binom{10}{1} + \binom{10}{2}$

$$\underbrace{\_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_}_{\text{choose zero, one, or two places to put the 0's}}$$

$= 1 + 10 + \frac{10 \cdot 9}{2}$

$= 11 + 45 = 56$

(c) A ... | B .. | C .. | D ... $\leftarrow$ An outcome corresponds to a 0-1 sequence of length $37+3$ with exactly three 1's.

$$\binom{40}{3}$$

4) Let $1 \leq r \leq n$.

(a) Give an algebraic proof that $\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$. Write your proof carefully and use correct mathematical notation. (0.5 points)

$$\frac{n}{r}\binom{n-1}{r-1} = \frac{n}{r} \cdot \frac{(n-1)!}{(r-1)!\,((n-1)-(r-1))!}$$

$$= \frac{n}{r} \cdot \frac{(n-1)!}{(r-1)!\,(n-r)!}$$

$$= \frac{n!}{r!\,(n-r)!} = \binom{n}{r}$$

(b) Explain why $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ is true by describing a counting process that shows the number of r-combinations of $\{1, 2, \ldots, n\}$ is equal to the right hand side of this equation. (0.5 points)

Consider an r-combination X of $\{1, 2, \ldots, n\}$.

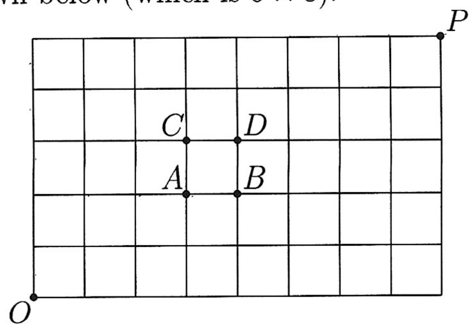
There are $\binom{n-1}{r}$ such X that do not contain 1.

$\cancel{1}$ choosing r from $2, 3, \ldots, n$

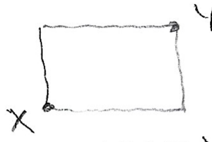
On the other hand, if X contains 1, there are $\binom{n-1}{r-1}$ ways to choose the remaining $r-1$ elements in X. Thus

$$\binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}$$

5) Consider the figure shown below (which is 5×8).



Using $n \times m$



$\binom{n+m}{n}$ shortest routes from X to Y

(a) Find the number of shortest routes from O to P. (0.25 points)

(b) Find the number of shortest routes from O to P that pass through the street AB or the street CD. (0.5 points)

(a) $\binom{13}{5}$

(b) $\underbrace{\binom{5}{2}\binom{7}{3}}_{\text{through } AB} + \underbrace{\binom{6}{3}\binom{6}{2}}_{\text{through } CD}$

6) Let $s(r, n)$ be the number of ways to arrange r distinct objects around n indistinguishable circles so that every circle has at least one object.

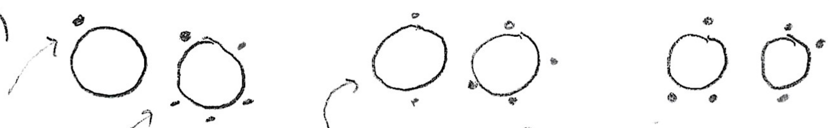
(a) Why is $s(r, r) = 1$? Answer in a complete sentence and do not use any math symbols. (0.25 points)

(b) Determine, with explanation (which can include pictures) the value of $s(6, 2)$. (0.75 points)

(a) Since the tables are indistinguishable, all placements are the same because each table must have exactly one object.

the 1/2 is due to the symmetry

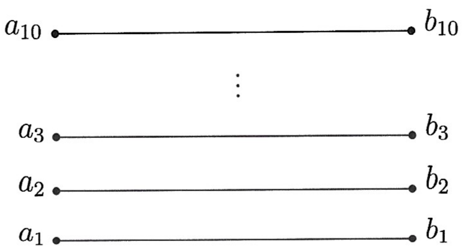
(b)



$\binom{6}{1} Q_5^5 \qquad \binom{6}{2} Q_4^4 \qquad \frac{1}{2}\binom{6}{3} \cdot Q_3^3 Q_3^3$

$$\binom{6}{1} Q_5^5 + \binom{6}{2} Q_4^4 + \frac{1}{2}\binom{6}{3} Q_3^3 Q_3^3$$

7) Consider the matching shown below.



(a) Find the number of subsets of size 5 from $\{a_1, \ldots, a_{10}, b_1, \ldots, b_{10}\}$ that do not contain any of the edges. (0.25 points)

(b) Find the number of subsets of size 5 from $\{a_1, \ldots, a_{10}, b_1, \ldots, b_{10}\}$ that contain exactly one edge. (0.5 points)

(c) Find the number of subsets of <u>size 10</u> from $\{a_1, \ldots, a_{10}, b_1, \ldots, b_{10}\}$ that contain exactly three edges. (0.25 points)

(a) $\binom{10}{5} \, 2^5$

pick 5 edges

pick a_i or b_i from each edge

(b) $10 \binom{9}{3} 2^3$

choose an edge a_j—b_j and take both a_j and b_j

pick 3 more edges

a_i—b_i

pick a_i or b_i from each edge

(c) $\binom{10}{3}\binom{7}{4}2^4$

California State University Sacramento - Math 101
Exam #3

Name: ______________________

This exam is out of 7 points.

1) (a) Determine the exact value of the coefficient of x^3 in the expansion of $(x+1)^{13}$. (0.5 points)

(b) Determine the exact value of the coefficient of $x^3y^4z^2$ in the expansion of $(x+y+z)^9$. (0.5 points)

(a) $\binom{13}{3} = \frac{13!}{3!\,10!} = \frac{13\cdot 12\cdot 11}{3\cdot 2} = 13\cdot 2\cdot 11 = 2\cdot 143 = 286$

(b) $\binom{9}{3,4,2} = \frac{9!}{3!\,4!\,2!} = \frac{9\cdot 8\cdot 7\cdot 6\cdot 5}{3\cdot 2\cdot 2} = 9\cdot 2\cdot 7\cdot 2\cdot 5$

$= 63\cdot 10\cdot 2$

$= 630\cdot 2 = 1260$

2) In the Binomial Theorem $(x+y)^n = \sum_{r=0}^{n}\binom{n}{r}x^r y^{n-r}$, state a value for x and value for y that produces the given formula. (0.25 points each)

(a) $0 = \sum_{r=0}^{n}\binom{n}{r}(-1)^r$

$x = -1$

$y = 1$

(b) $2^n = \sum_{r=0}^{n}\binom{n}{r}$

$x = 1$

$y = 1$

3) Show that for all integers $n \geq m \geq 1$, $\sum_{r=m}^{n} \binom{n-m}{n-r} = 2^{n-m}$. (0.5 points)

$$\sum_{r=m}^{n} \binom{n-m}{n-r} = \binom{n-m}{n-m} + \binom{n-m}{n-(m+1)} + \binom{n-m}{n-(m+2)} + \dots + \binom{n-m}{n-n}$$

$$= \binom{n-m}{n-m} + \binom{n-m}{n-m-1} + \binom{n-m}{n-m-2} + \dots + \binom{n-m}{0}$$

$$= 2^{n-m}$$

using $\sum_{k=0}^{N} \binom{N}{k} = 2^N$

4) Prove $\sum_{r=0}^{n} r\binom{n}{r} = n2^{n-1}$ for all $n \geq 1$. (0.75 points)

By the Binomial Theorem,

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

Taking $y=1$ and differentiating with respect to x gives

$$n(x+1)^{n-1} = \sum_{r=0}^{n} \binom{n}{r} r x^{r-1}$$

Let $x=1$ to get

$$n \cdot 2^{n-1} = \sum_{r=0}^{n} r\binom{n}{r}$$

5) Prove that $\binom{n}{r}\binom{n-r}{m-r} = \binom{n}{m}\binom{m}{r}$ for all integers $n \geq m \geq r \geq 1$. (0.75 points)

$$\binom{n}{r}\binom{n-r}{m-r} = \frac{n!}{r!\,(n-r)!} \cdot \frac{(n-r)!}{(m-r)!\,(n-r-(m-r))!}$$

$$= \frac{n!}{r!} \cdot \frac{1}{(m-r)!\,(n-m)!}$$

$$= \frac{n!}{(n-m)!} \cdot \frac{1}{r!\,(m-r)!}$$

multiply by $1 = \frac{m!}{m!}$

$$= \frac{n!}{m!\,(n-m)!} \cdot \frac{m!}{r!\,(m-r)!}$$

$$= \binom{n}{m}\binom{m}{r}$$

6) Let $X = \{1,2,3,4\}$. Let $\mathcal{A}$ be the collection of all subsets of X with an even number of elements, and let $\mathcal{B}$ be the collection of all subsets of X with an odd number of elements. Remark: The empty set $\emptyset$ is one of the sets in $\mathcal{A}$ since it has 0 elements and 0 is even.

(a) List all of the elements of $\mathcal{A}$. (0.5 points)

(b) List all of the elements of $\mathcal{B}$. (0.5 points)

(c) Is the function $f(C) = X \backslash C$ a bijection from $\mathcal{A}$ to $\mathcal{B}$? Recall that $X \backslash C$ is the complement of C in X. (0.25 points)

(d) Is the function $f(C) = C \cup \{1\}$ a bijection from $\mathcal{A}$ to $\mathcal{B}$? (0.25 points)

(e) Draw a bijection between $\mathcal{A}$ and $\mathcal{B}$. Represent your bijection using an arrow diagram. (0.25 points)

(a) $\mathcal{A} = \{ \emptyset, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3,4\} \}$

(b) $\mathcal{B} = \{ \{1\}, \{2\}, \{3\}, \{4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\} \}$

both in $\mathcal{A}$

(c) No. $f(\{1,2\}) = \{3,4\}$ so f is not even a function from $\mathcal{A}$ to $\mathcal{B}$

(d) No. The same reasoning applies.

$$f(\{1,2\}) = \{1,2\} \cup \{1\} = \{1,2\}$$

both in $\mathcal{A}$

(e)

$$
\begin{array}{ccc}
\mathcal{A} & & \mathcal{B} \\
\emptyset & \to & \{1\} \\
\{1,2\} & \to & \{2\} \\
\{1,3\} & \to & \{3\} \\
\{1,4\} & \to & \{4\} \\
\{2,3\} & \to & \{1,2,3\} \\
\{2,4\} & \to & \{1,2,4\} \\
\{3,4\} & \to & \{1,3,4\} \\
\{1,2,3,4\} & \to & \{2,3,4\}
\end{array}
$$

7) (a) There are 50 jobs that must be assigned to 7 processors. Explain why there must be a processor that is assigned at least 8 jobs. (0.5 points)

(b) A list $\mathcal{L}$ contains 134 elements. Each element of $\mathcal{L}$ is a subset of $\{1,2,3,4,5,6,7\}$. Explain why the list $\mathcal{L}$ must contain two elements that are the same. (0.5 points)

(c) Suppose it takes a program 1 second to find the determinant of a 2×2 matrix. If S is the set of all 2×2 matrices of the form $\begin{pmatrix} x & y \\ z & t \end{pmatrix}$ where $x, y, z, t \in \{0,1\}$, what is the minimum amount of time it would take the program to find the determinant of every matrix in S? (0.25 points)

(a) The average number of jobs assigned to a processer is $\frac{50}{7} > 7$ so there must be some processor that is assigned at least 8 jobs.

(b) The number of subsets of $\{1,2,3,\ldots,7\}$ is $2^7 = 128$.

Since $\mathcal{L}$ has 134 elements, the list $\mathcal{L}$ must contain two subsets that are the same.

(c)

$$\begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}$$

$\uparrow \uparrow$

0 or 1

There are $2^4 = 16$ such matrices so 16 seconds

8) Prove that $\sum_{r=0}^{n-1} \binom{2n-1}{r} = 2^{2n-2}$. (0.5 points)

We know

$$\sum_{r=0}^{2n-1} \binom{2n-1}{r} = 2^{2n-1}$$

using
$\binom{N}{M} = \binom{N}{M-N}$
for $N \geq M \geq 1$

So that

$$\sum_{r=0}^{n-1} \binom{2n-1}{r} + \sum_{r=n}^{2n-1} \binom{2n-1}{r} = 2^{2n-1}$$

$$\sum_{r=0}^{n-1} \binom{2n-1}{r} + \sum_{r=n}^{2n-1} \binom{2n-1}{2n-1-r} = 2^{2n-1}$$

$$\star$$

$$\sum_{r=0}^{n-1} \binom{2n-1}{r} + \sum_{r=0}^{n-1} \binom{2n-1}{r} = 2^{2n-1}$$

$$2 \sum_{r=0}^{n-1} \binom{2n-1}{r} = 2^{2n-1}$$

$$\sum_{r=0}^{n-1} \binom{2n-1}{r} = 2^{2n-2}$$

To see $\star$, note

$$\sum_{r=n}^{2n-1} \binom{2n-1}{2n-1-r} = \binom{2n-1}{n-1} + \binom{2n-1}{n-2} + \ldots + \binom{2n-1}{0}$$

$$= \sum_{r=0}^{n-1} \binom{2n-1}{r}$$

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Homework Assignment 10

1) Suppose that A and B are finite sets. Give a formula for $|A \cup B|$ that only involves $|A|$, $|B|$, and $|A \cap B|$.

2) Suppose that A, B, and C are finite sets. Give a formula for $|A \cup B \cup C|$ that only involves $|A|$, $|B|$, $|C|$, $|A \cap B|$, $|A \cap C|$, $|B \cap C|$, and $|A \cap B \cap C|$.

3) Find the number of integers in the set $\{1, 2, \ldots, 90\}$ that are divisible by 3 or 5.

4) Find the number of integers in the set $\{1, 2, 3, \ldots, 140\}$ that are divisible by 2, 5, or 7.

5) Problem 1 on page 173

6) Suppose 100 students play three sports; baseball, hockey, or football. Each student may play one, two, or all three sports. If 30 students played baseball, 50 students played hockey, 60 students played football, and 18 students played all three sports, how many students played at least two sports?

Exercise 4

1. A group of 100 students took examinations in Chinese, English and Mathematics. Among them, 92 passed Chinese, 75 English and 63 Mathematics; at most 65 passed Chinese and English, at most 54 Chinese and Mathematics, and at most 48 English and Mathematics. Find the largest possible number of the students that could have passed all the three subjects.

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Homework Assignment 11

1) Problem 1 on page 270

2) Problem 2 on page 270

3) Problem 3 on page 270

4) Problem 4 on page 270

5) Problem 5 on page 270

Exercise 6

1. Solve
$$a_n = 3a_{n-1} - 2a_{n-2},$$
given that $a_0 = 2$ and $a_1 = 3$.

2. Solve
$$a_n - 6a_{n-1} + 9a_{n-2} = 0,$$
given that $a_0 = 2$ and $a_1 = 3$.

3. Solve
$$a_n = \frac{1}{2}(a_{n-1} + a_{n-2}),$$
given that $a_0 = 0$ and $a_1 = 1$.

4. Solve
$$a_n - 4a_{n-1} + 4a_{n-2} = 0,$$
given that $a_0 = -\frac{1}{4}$ and $a_1 = 1$.

5. Solve
$$2a_n = a_{n-1} + 2a_{n-2} - a_{n-3},$$
given that $a_0 = 0$, $a_1 = 1$ and $a_2 = 2$.

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Homework Assignment 10 - Solutions

1) If A and B are finite sets, then $|A \cup B| = |A| + |B| - |A \cap B|$.

2) If A, B, and C are finite sets, then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

3) Let A be the integers in $\{1, 2, \ldots, 90\}$ that are divisible by 3 and B be the integers in $\{1, 2, \ldots, 90\}$ that are divisible by 5. We have

$$|A| = \frac{90}{3} = 30, |B| = \frac{90}{5} = 18, \text{ and } |A \cap B| = \frac{90}{15} = 6.$$

Therefore, the number of integers in $\{1, 2, \ldots, 90\}$ that are divisible by 3 or 5 is

$$|A \cup B| = |A| + |B| - |A \cap B| = 30 + 18 - 6 = 42.$$

4) Let A be the integers in $\{1, 2, \ldots, 140\}$ that are divisible by 2, B be the integers in $\{1, 2, \ldots, 140\}$ that are divisible by 5, and C be the integers in $\{1, 2, \ldots, 140\}$ that are divisible by 7. We have

$$|A| = \tfrac{140}{2} = 70, |B| = \tfrac{140}{5} = 28, |C| = \tfrac{140}{7} = 20,$$
$$|A \cap B| = \tfrac{140}{10} = 14, |A \cap C| = \tfrac{140}{14} = 10, |B \cap C| = \tfrac{140}{35} = 4, \text{ and } |A \cap B \cap C| = \tfrac{140}{70} = 2.$$

By the Principle of Inclusion-Exclusion, the number of integers in $\{1, 2, \ldots, 140\}$ that are divisible by 2, 5, or 7 is

$$|A \cup B \cup C| = 70 + 28 + 20 - 14 - 10 - 4 + 2 = 92.$$

5) Let C be the students who passed Chinese, E be the students who passed English, and M be the students who passed Mathematics. We are given that

$$|C| = 92, |E| = 75, |M| = 63, \tag{1}$$

and

$$|C \cap E| \leq 65, |C \cap M| \leq 54, |E \cap M| \leq 48. \tag{2}$$

By Inclusion-Exclusion,

$$|C \cup E \cup M| = |C| + |E| + |M| - |C \cap E| - |C \cap M| - |E \cap M| + |C \cap E \cap M|.$$

Using (1), this equation can be rewritten as

$$|C \cap E| + |C \cap M| + |E \cap M| + |C \cup E \cup M| = 230 + |C \cap E \cap M|.$$

Since 102 students took the exams, we know that $102 \geq |C \cup E \cup M|$. Using this inequality and (2), we have

$$102 + 65 + 54 + 48 \geq 230 + |C \cap E \cap M|$$

so $39 \geq |C \cap E \cap M|$. We conclude that at most 39 students passed all three subjects.

6) Let B be the students who played baseball, H be the students who played hockey, and F be the students who played football. By Inclusion-Exclusion,

$$\begin{aligned} 100 &= |B \cup H \cup F| = |B| + |H| + |F| - |B \cap H| - |B \cap F| - |H \cap F| + |B \cap H \cap F| \\ &= 30 + 50 + 60 - |B \cap H| - |B \cap F| - |H \cap F| + 18. \end{aligned}$$

This gives

$$|B \cap H| + |B \cap F| + |H \cap F| = 58.$$

The sum $|B \cap H| + |B \cap F| + |H \cap F|$ counts the students that played exactly two sports once, and counts the students that played exactly three sports three times. The number of students that played at least two sports is

$$|B \cap H| + |B \cap F| + |H \cap F| - 2|B \cap H \cap F| = 58 - 2(18) = 22.$$

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Homework Assignment 11 - Solutions

1) The characteristic equation is $x^2-3x+2=0$ which can be rewritten as $(x-2)(x-1)=0$. The roots of this equation are $x=2$ and $x=1$. Let

$$a_n = A \cdot 2^n + B \cdot 1^n = A \cdot 2^n + B.$$

The condition $a_0 = 2$ implies $2 = A + B$. The condition $a_1 = 3$ implies $3 = 2A + B$. The solution to this linear system is $A = B = 1$. Therefore,

$$a_n = 2^n + 1.$$

2) The characteristic equation is $x^2 - 6x + 9 = 0$ which is equivalent to $(x-3)^2 = 0$. The number 3 is a double root of this quadratic equation. Let

$$a_n = (A + Bn)3^n.$$

The condition $a_0 = 2$ implies $2 = A$ so that $a_n = (2+Bn)3^n$. The condition $a_1 = 3$ implies $3 = (2+B) \cdot 3$ which gives $B = 1$. Therefore,

$$a_n = (2-n)3^n.$$

3) The characteristic equation is $2x^2-x-1=0$ which can be rewritten as $(2x+1)(x-1)=0$. The roots of this quadratic are $-1/2$ and 1. Let

$$a_n = A \cdot 1^n + B \cdot (-1/2)^n = A + B \cdot (-1/2)^n.$$

The condition $a_0 = 0$ implies $0 = A + B$. The condition $a_1 = 1$ implies $1 = A - (1/2)B$. The solution to this system is $A = 2/3$ and $B = -2/3$. The solution to this recurrence relation is

$$a_n = \frac{2}{3} - \frac{2}{3}\left(-\frac{1}{2}\right)^n.$$

4) The characteristic equation is $x^2 - 4x + 4 = 0$ which is equivalent to $(x-2)^2 = 0$ and so 2 is a double root. Let

$$a_n = (A + Bn)2^n.$$

The initial condition $a_0 = -1/4$ implies that $-1/4 = A$. Hence, $a_n = \left(-\frac{1}{4} + Bn\right)2^n$. The condition $a_1 = 1$ implies $1 = \left(-\frac{1}{4} + B\right)2$. Solving this equation for B gives $B = 3/4$. Therefore,

$$a_n = \left(-\frac{1}{4} + \frac{3n}{4}\right)2^n.$$

5) The characteristic equation is $2x^3 - x^2 - 2x + 1 = 0$. Using factoring by grouping,

$$x^2(2x-1) - 1(2x-1) = 0 \quad \Rightarrow \quad (2x-1)(x^2-1) = 0 \quad \Rightarrow \quad (2x-1)(x-1)(x+1) = 0.$$

The roots of this equation are $x = 1/2$, $x = 1$, and $x = -1$. Let

$$a_n = A(1)^n + B(-1)^n + C(1/2)^n = A + B(-1)^n + C(1/2)^n.$$

The three initial conditions lead to the system of equations

$$\begin{aligned} A + B + C &= 0 \\ A - B + \frac{1}{2}C &= 0 \\ A + B + \frac{1}{4}C &= 2. \end{aligned}$$

The solution to this system is $A = 5/2$, $B = 1/6$, and $C = -8/3$. The solution to the congruence is

$$a_n = \frac{5}{2} + \frac{1}{6}(-1)^n - \frac{8}{3}(1/2)^n.$$

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Quiz #10

Name: ______________________

1) Compute (a) $\binom{11}{3}$ (b) $\binom{8}{0}$

2) Find the coefficient of x^3y^8 in the expansion of $(x+y)^{11}$. Simplify your answer as much as possible.

3) Prove each statement using the Binomial Theorem.

(a) $2^n = \sum_{k=0}^{n} \binom{n}{k}$ (b) $3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k$ (c) $0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$

4) Determine the exact value of $\sum_{r=1}^{10} \binom{10}{r}$.

5) Let $X = \{1,2,3,4,5\}$.

(a) If $A = \{1,2,3\}$, find $X \backslash A$.

(b) How many subsets of X have an odd number of elements?

(c) How many subsets of X have an even number of elements?

(d) Let $\mathcal{A}$ be the set of all subsets of X with an odd number of elements, and $\mathcal{B}$ be the set of all subsets of X with an even number of elements. If $f : \mathcal{A} \to \mathcal{B}$ is defined by $f(A) = X \backslash A$, find $f(\{5\})$.

6) Show that for any five points on the unit circle $x^2 + y^2 = 1$, there are at least two points that are within distance $\sqrt{2}$ of each other.

7) Suppose that there are 100 students in a class.

(a) Show that there are at least 9 students who were born in the same month.

(b) Suppose each student selects a 0-1 sequence of length 6. Must there exists two students who selected the same sequence?

(c) The class has a lottery where two numbers are chosen from $\{1, 2, \ldots, 15\}$. The winning ticket gets an A in the course. How many tickets should you purchase to ensure that you have a winning ticket?

Continued on other side

8) (a) Find the coefficient of $x^2y^3z^4$ in the expansion of $(x+y+z)^9$. You may leave your answer as a multinomial coefficent.

(b) Simplify $\binom{6}{1,2,3}$ as much as possible. You may want to use the formula

$$\binom{m}{m_1, m_2, \ldots, m_k} = \frac{(m_1 + m_2 + \cdots + m_k)!}{m_1! m_2! \cdots m_k!}$$

where $m = m_1 + m_2 + \cdots + m_k$.

9) (a) Recall that the power set of a set X is the set of all subsets of X. List all elements of the power set of $X = \{a, b, c\}$.

(b) Draw a bijection between the set of all 0-1 sequences of length 2 and all subsets of $\{x, y\}$.

(c) Draw a bijection between the set of all 0-1 sequences of length 3 and all subsets of $\{x, y, z\}$.

10) Let $X = \{1, 2, 3\}$. Suppose that five distinct subsets of X are chosen. Show that one of those subsets must contain 1.

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Quiz #11

Name: ____________________

1) Find the number of permutations of the letters in the given word.

(a) CATDOG

(b) ABBACATDOG

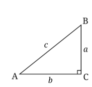
2) Find the number of pairs $\{a, b\}$ of distinct integers from the set $\{5, 6, 7, 8, \ldots, 20\}$ such that $|a - b| \leq 3$.

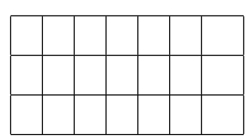
3) (a) n points are placed along the circle $x^2 + y^2 = 1$ in the x, y-plane such that distance between two consecutive points along the circle is the same. Find the number of ways to take the n points and choose 3 of them to form a triangle.

(b) The weight of a 0-1 sequence of the number of 1's in the sequence. For instance, 1010100 has weight 3. Find the number of 0-1 sequences of length 9 with weight at most two.

4) In a class of 37 students, five students are chosen to form a focus group where the focus group has one leader, and two co-leaders. How many focus groups can be formed?

5) Given the grid below, find the number of right triangles whose vertices are intersection points in the grid, and the right angle of the triangle is the lower right corner of the triangle.





California State University Sacramento - Math 101
Quiz #12

Name: ________________________

1) Solve the recurrence $a_{n+2} = a_{n+1} + 6a_n$ given that $a_0 = 0$ and $a_1 = 1$.

2) Solve the recurrence $a_{n+2} = 6a_{n+1} - 9a_n$ given that $a_0 = 1$ and $a_1 = 2$.

3) State the Inclusion-Exclusion Principle in the case of three finite sets A, B, and C.

4) Find the number of integers in the set $\{1, 2, \ldots, 180\}$ that are divisible by 4 or 9.

5) Find the number of integers in the set $\{1, 2, \ldots, 180\}$ that are divisible by 2, 3, or 5.

6) Suppose that 80 students played three sports; basketball, soccer, or volleyball. Each student may play one, two or all three sports. If 20 students played basketball, 30 played soccer, 45 played volleyball, and 5 played all three sports, how many students played at least two sports?

7) Suppose that 80 students played three sports; basketball, soccer, or volleyball. Each student may play one, two or all three sports. Assume that 45 students played basketball, 30 played soccer, and 45 played volleyball. Also, suppose that at most 15 students played both basketball and soccer, at most 10 students played both basketball and volleyball, and at most 20 students played both soccer and volleyball. Find the largest possible number of students that could have played all three sports.

8) (a) Solve the recurrence relation $a_{n+3} = a_{n+2} + a_{n+1} - a_n$ with initial conditions $a_0 = 1$, $a_1 = 1$, and $a_2 = 2$.

(b) Let a_n be the number of ways of walking up n stairs where you may take one, two, or three stairs at a time. Find a recurrence relation for a_n and initial conditions for a_1 and a_2 (we define $a_0 = 1$).

1) (a) $\binom{11}{3} = \frac{11!}{3!\,8!} = \frac{11\cdot 10\cdot 9}{3\cdot 2} = 11\cdot 5\cdot 3 = 55\cdot 3 = 165$

(b) $\binom{8}{0} = 1$

2) $\binom{11}{3} = 165$

3) (a) Letting $x=y=1$ in the Binomial Theorem gives

$$(1+1)^n = \sum_{r=0}^{n} \binom{n}{r} 1^r 1^{n-r} \quad \text{so} \quad 2^n = \sum_{r=0}^{n} \binom{n}{r}$$

(b) Let $x=2, y=1$ in the Binomial Theorem ← details left to you!

(c) Let $x=-1$ and $y=1$ in

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

to get

$$0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$$

4) $\sum_{r=1}^{10} \binom{10}{r} = -\binom{10}{0} + \sum_{r=0}^{10} \binom{10}{r} = -1 + 2^{10} = -1 + 1024$
$= 1023$

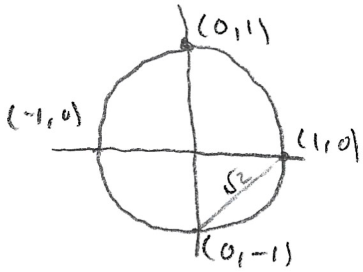
5) (a) $X \setminus A = \{4,5\}$

(b) $\binom{5}{1} + \binom{5}{3} + \binom{5}{5} = 5 + 10 + 1 = 16$

(c) $\binom{5}{0} + \binom{5}{2} + \binom{5}{4} = 1 + 10 + 5 = 16$

(d) $f(\{5\}) = \{1,2,3,4\}$

6)



By the Pigeonhole Principle, two points must be in the same quadrant and the farthest those two points can be from each other is $\sqrt{2}$.

7) (a) $\dfrac{100}{12} = 8\frac{4}{12} = 8\frac{1}{3} > 8$ ← $\frac{100}{12}$ is strictly greater than 8

By the Pigeonhole Principle, at least 9 people were born in the same month.

(b) The number of 0-1 sequences of length 6 is $2^6 = 64$ and since there are 100 students, at least two picked the same sequence.

(c) $\binom{15}{2} = \dfrac{15!}{2!\,13!} = \dfrac{15 \cdot 14}{2} = 15 \cdot 7 = 95$

If you purchase 95 tickets of all different types, you will have a winning ticket.

8) (a) $\binom{9}{2,3,4}$ (b) $\binom{6}{1,2,3} = \frac{6!}{1!2!3!} = \frac{6\cdot5\cdot4}{2} = 60$

9) (a) $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a,b\}$, $\{a,c\}$, $\{b,c\}$, $\{a,b,c\}$

(b)

$$\begin{array}{ll} 00 \longrightarrow \emptyset \\ 10 \longrightarrow \{x\} \\ 01 \longrightarrow \{y\} \\ 11 \longrightarrow \{x,y\} \end{array}$$

(c)

$$\begin{array}{l} 000 \longrightarrow \emptyset \\ 100 \longrightarrow \{x\} \\ 010 \longrightarrow \{y\} \\ 001 \longrightarrow \{z\} \\ 110 \longrightarrow \{x,y\} \\ 101 \longrightarrow \{x,z\} \\ 011 \longrightarrow \{y,z\} \\ 111 \longrightarrow \{x,y,z\} \end{array}$$

10) The subsets of X are

$\emptyset$ $\{1\}$ $\{2\}$ $\{3\}$ $\{1,2\}$ $\{1,3\}$ $\{2,3\}$ $\{1,2,3\}$

Only four subsets do not contain 1 so if a five subsets are chosen one must contain 1.

California State University Sacramento - Math 101
Quiz #11

Name: ____________________

1) Find the number of permutations of the letters in the given word.

(a) CATDOG

(b) ABBACATDOG

(a) $6! = 720$

(b) $\frac{10!}{3!2!}$

2) Find the number of pairs $\{a, b\}$ of distinct integers from the set $\{5, 6, 7, 8, \ldots, 20\}$ such that $|a - b| \leq 3$.

Assume $a < b$.

CASE 1	CASE 2	...	CASE 13		
$a = 5$	$a = 6$		$a = 17$	$a = 18$	$a = 19$
$b \in \{6, 7, 8\}$	$b \in \{7, 8, 9\}$		$b \in \{18, 19, 20\}$	$b \in \{19, 20\}$	$b \in \{20\}$
3 choices for b in each of these 13 cases				2 choices	1 choices

$3 \cdot 13 = 39$

42

3) (a) n points are placed along the circle $x^2 + y^2 = 1$ in the x, y-plane such that distance between two consecutive points along the circle is the same. Find the number of ways to take the n points and choose 3 of them to form a triangle.

(b) The weight of a 0-1 sequence of the number of 1's in the sequence. For instance, 1010100 has weight 3. Find the number of 0-1 sequences of length 9 with weight at most two.

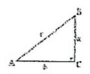
(a) $\binom{n}{3}$

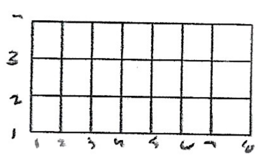
(b) $\binom{9}{0} + \binom{9}{1} + \binom{9}{2} = 1 + 9 + 36 = 46$

4) In a class of 37 students, five students are chosen to form a focus group where the focus group has one leader, and two co-leaders. How many focus groups can be formed?

$\binom{37}{5}\binom{5}{1}\binom{4}{2}$

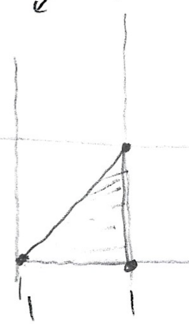
5) Given the grid below, find the number of right triangles whose vertices are intersection points in the grid, and the right angle of the triangle is the lower right corner of the triangle.





choose 2

$\underbrace{\binom{4}{2}}_{\text{Choose two horizontal lines}} \underbrace{\binom{8}{2}}_{\text{choose two vertical lines}}$



choose 2

Forms a unique right triangle

1) $x^2 = x + 6 \Rightarrow x^2 - x - 6 = 0 \Rightarrow (x-3)(x+2) = 0$

$\Rightarrow x = 3,\ x = -2$

$\Rightarrow a_n = A \cdot 3^n + B(-2)^n$

$a_0 = 0 \Rightarrow 0 = A + B \Rightarrow B = -A$

$a_1 = 1 \Rightarrow 1 = 3A - 2B$

$1 = 3A + 2A$

$A = 1/5$

$B = -1/5$

$a_n = \frac{1}{5} \cdot 3^n - \frac{1}{5} \cdot (-2)^n$

2) $x^2 = 6x - 9 \Rightarrow x^2 - 6x + 9 = 0$

$\Rightarrow (x-3)^2 = 0 \Rightarrow x = 3$

$\Rightarrow a_n = (A + Bn) \cdot 3^n$

$a_0 = 1 \Rightarrow 1 = A$

$a_n = (1 + Bn) \cdot 3^n$

$a_1 = 2 \Rightarrow 2 = (1 + B) \cdot 3$

$2 = 3 + 3B$

$-\frac{1}{3} = B$

$a_n = \left(1 - \frac{n}{3}\right) \cdot 3^n$

3) $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

4) If A_k is the number of integers in $\{1, 2, \ldots, 180\}$ divisible by k, then

$A_{36} = A_4 \cap A_9$

$A_4 = \left\lfloor \frac{180}{4} \right\rfloor = 45,\ A_9 = \left\lfloor \frac{180}{9} \right\rfloor = 20,\ A_{36} = \left\lfloor \frac{180}{36} \right\rfloor = 5$

The number of integers divisible by 4 or 9 is

$|A_4 \cup A_9| = 45 + 20 - 5 = 60$

5) Using the same method as in #4, the number of integers in $\{1,2,\dots,180\}$ that are divisible by 2, 3, or 5 is

$$\frac{180}{2}+\frac{180}{3}+\frac{180}{5}-\frac{180}{6}-\frac{180}{10}-\frac{180}{15}+\frac{180}{30}$$

$= 132$

6) Using the obvious notation, we have

$80 = |B \cup S \cup V|$, $20 = |B|$, $30 = |S|$, $45 = |V|$, $5 = |B \cap S \cap V|$.

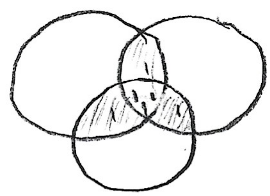
Thus,

$$80 = |B \cup S \cup V| = 20+30+45-|B\cap S|-|B\cap V|-|S\cap V|+5$$

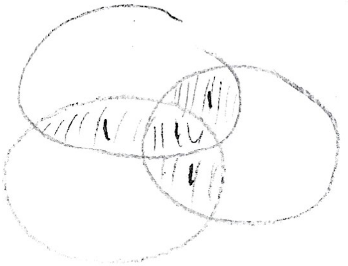
$$= 100 - |B\cap S|-|B\cap V|-|S\cap V|.$$

This implies

$$|B\cap S|+|B\cap V|+|S\cap V| = 20$$



the set $B\cap S\cap V$ is counted two additional times in this sum.



$\leftarrow 20-2|B\cap S\cap V|$

$= 20-2(5) = 10.$

10 students played at least three sports

7) As in #6,

$$80 = |B \cup S \cup V| = \overbrace{45+30+45}^{120} - |B\cap S| - |B\cap V| - |S \cap V| + |B\cap S\cap V|$$

therefore,

$$|B\cap S\cap V| = |B\cap S| + |B\cap V| + |S\cap V| - 40$$

$$\leq 15 + 10 + 20 - 40 = 5$$

At most 5 played all three sports

8) (a) $x^3 = x^2 + x - 1 \Rightarrow x^3 - x^2 - x + 1 = 0$

$\Rightarrow x^2(x-1) - 1(x-1) = 0$

$\Rightarrow (x^2-1)(x-1) = 0$

$\Rightarrow (x-1)^2(x+1) = 0$ $\quad \underbrace{x=1}_{\text{repeat root}} \quad x=-1$

$$a_n = (A+Bn)\cdot 1^n + C(-1)^n$$

$a_0 = 1 \Rightarrow$ $1 = A + C$

$a_1 = 1$ $\quad 1 = A + B - C$

$a_2 = 2$ $\quad 2 = A + 2B + C$

subtract $1 = A + C$ from $2 = A + 2B + C$

$1 = 2B \Rightarrow B = 1/2$

Solving for A and C gives $A = 3/4$, $C = 1/4$ so

$$a_n = \left(\frac{3}{4} + \frac{n}{2}\right) + \frac{1}{4}(-1)^n$$

(b)

$$a_{n+3} = a_{n+2} + a_{n+1} + a_n$$

$a_0 = 1$
$a_1 = 1$
$a_2 = 2$
$a_3 = 4$

...
stair
$n+3$

arrive at $n+3$ by
taking three steps at
once at stair n