1) If there must be at least one person at each table, in how many ways can five people be seated around two tables where the tables are indistinguishable?

* Needs case analysis

Recall

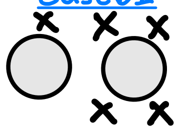
In general, if Q_r^n is the number of arrangements of 'r' elements placed around a circle chosen from a set of 'n' elements, then

$$Q_r^n = \frac{P_r^n}{r} = \frac{\frac{n!}{(n-r)!}}{r} = \frac{n!}{r\cdot(n-r)!}$$

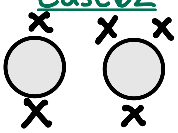
* Note that if $r=n$, then

$$Q_r^n = \frac{n!}{n(n-n)!} = \frac{n!}{n\cdot 0} = \frac{n!}{n\cdot 1} = \frac{n!}{n} = (n-1)!$$

Case 01



Case 02



Case 01 Case 02

$\binom{5}{1}Q_4^4 + \binom{5}{2}Q_3^3$

$5\cdot 3! + 10\cdot 2!$

$\rightarrow (5\cdot 6)+(10\cdot 2) \rightarrow (30)+(20) = 50$

2) (a) Let $s(r,n)$ be the number of ways to arrange r distinct objects around n indistinguishable circles so that every circle has at least one object. The numbers $s(r,n)$ are called the Stirling numbers of the first kind. Problem 1 is asking for $s(5,2)$. For part (a) of this problem, compute $s(4,n)$ for $n=1,2,3,4$ (we define $s(4,0)=0$).

(b) Expand the polynomial $x(x+1)(x+2)(x+3)$ as much as possible.

Recall: Sterling Numbers of the first Kind

Lower case 's' as notation

$s(r,n) = s((r-1),n-1) + (r-1)s(r-1,n)$

Properties of $s(r,n)$

- $s(r,0) = 0$; if $r \geq 1$
- $s(r,r) = 1$; if $r \geq 0$
- $s(r,1) = Q_r^r = (r-1)!$; for $r \geq 2$
- $s(r,r-1) = \binom{r}{2}$; for $r \geq 2$

a) $s(4,n)$

$n=1$

$s(4,1) = Q_4^4 = (4-1)! = 3! = 6$

$n=2$

$s(4,2) = s(3,1) + (3)s(3,2)$

$Q_3^3 = 2! = 2$ $\quad \binom{3}{2} = 3$

$\rightarrow 2 + (3)\binom{3}{2}$

$\rightarrow 2 + (3)(3) \rightarrow 2 + 9 = 11$

$s(4,2) = 11$

$n=3$

Note

$\boxed{s(r,r-1) = \binom{r}{2}}$ $s(4,3)$

So, $s(4,3) = \binom{4}{2} = 6$

$n=4$

Note

$\boxed{s(r,r) = 1}$ $s(4,4)$

So, $s(4,4) = 1$

b) I'll do this later!

3) Given that $s(6,1) = 120$ and $s(6,2) = 274$, use the formula $s(r,n) = s(r-1,n-1) + (r-1)s(r-1,n)$ to determine $s(7,2)$

$s(7,2) = s(6,1) + (6)s(6,2)$

$\boxed{s(r,1) = Q_r^r = (r-1)!}$ $\quad$ $\boxed{s(r,n) = s((r-1),n-1) + (r-1)s(r-1,n)}$

$Q_6^6 = (6-1)! = 5!$

120

$s(6,2) = s(5,1) + (5)s(5,2)$

$Q_5^5 = 4! = 24$

$\boxed{s(5,2) = s(4,1) + (4)s(4,2)}$

$Q_4^4 = 3! = 6$ $\quad s(4,2) = s(3,1) + (3)s(3,2)$

$Q_3^3 = 2! = 2$ $\quad \binom{3}{2} = 3$

All together

$s(7,2) = 120 + (24) + (5)(6 + 4(2 + (3)(3)))$

$s(7,2) = 199$ In total

4) Find the number of 4-combinations of $\{1,2,\ldots,12\}$ that contain no consecutive integers.

Recall: Section 1.5

General Case — The biggest it can be is $n-r+1$

Let $x = \{1,2,3,4,\ldots,n\}$ and let $1 \leq r \leq n$

Given a subset $\{S_0, S_1, \ldots, S_r\}$ of x with no consecutive elements define,

$f(\{S_0,S_1,\ldots,S_r\}) = \{S_0, S_1 - 1, S_2 - 2, \ldots, S_r - (r-1)\}$

This output is a subset of size 'r' since $\{S_0,S_1,\ldots,S_r\}$ has no consecutive elements

$$\binom{n-r+1}{r}$$

- $[1,12]$
- $n=12$; $r=4$

$\binom{12-4+1}{4}$

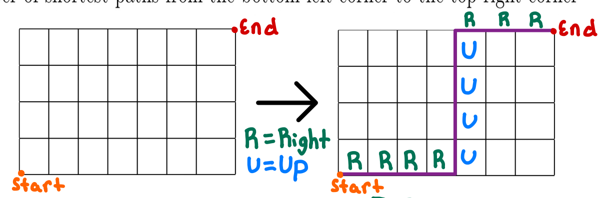
$\rightarrow \binom{9}{4} = 126$

5) Suppose that k and n are positive integers with $k \leq n$ and that $a_1,\ldots,a_n,b_1,\ldots,b_n$ are $2n$ distinct elements. Consider the n pairs $\{a_1,b_1\},\ldots,\{a_n,b_n\}$. Find the number of subsets of $\{a_1,\ldots a_n,b_1,\ldots,b_n\}$ of size k that do not contain two elements from the same pair.

$$\underbrace{\binom{n}{k}}_{\text{choose } k\text{-pairs}}\cdot\underbrace{2^k}_{\text{choose exactly one element from each pair}}$$

- 2n distinct elements
- $k \leq n$; n pairs, k size subsets

6) Find the number of shortest paths from the bottom left corner to the top right corner in the grid below.



$\underset{1}{R},\underset{2}{R},\underset{3}{R},\underset{4}{R},\underset{5}{U},\underset{6}{U},\underset{7}{U},\underset{8}{U},\underset{9}{R},\underset{10}{R},\underset{11}{R} \rightarrow$ 7 R's, 4 U's

So, $\underset{\text{Selecting for up-movements}}{\binom{11}{4}}$ or $\underset{\text{Selecting for Right-movements}}{\binom{11}{7}}$

7) (a) Let $X = \{1,b,R\}$. Find all elements of $\mathcal{P}(X)$. Recall that this is the power set of X and it is the set of all possible subets of X.

(b) Suppose X is a set with n elements. What is the cardinality of $\mathcal{P}(X)$?

a) • $X = \{1,b,R\} \rightarrow$ Produce all possible subsets of $P(X)$

So, $P(X) = \{\emptyset, \{1\}, \{b\}, \{R\}, \{1,b\}, \{1,R\}, \{b,R\}, \{1,b,R\}\}$

b) If given set n-elements, then its Power Set will contain 2^n elements. It also represents the cardinality of the power set.

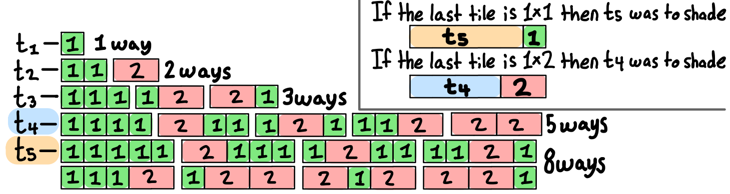
8) Find the number of 12-digit binary sequences with eight 0's and four 1's such that no two 1's are adjacent.

$\binom{n-r+1}{r}$; $n=12$, $r=4$ $\quad \binom{12-4+1}{4} \rightarrow \binom{9}{4} = 126$

9) (a) Let t_n be the number of ways to pave a $1\times n$ rectangle using 1×1 and 1×2 blocks Determine t_1, t_2, t_3, t_4, and t_5.

(b) Can you determine t_6 using what you know about t_5 and t_4?

a) 1x1 & 1x2 blocks



b) $t_6 = (t_4 + t_5) = 13$

If the last tile is 1x1 then t_5 was to shade

If the last tile is 1x2 then t_4 was to shade