California State University Sacramento - Math 101

Practice for Permutations, Circular Permutations, and Combinations

1) Give formulas for P_r^n, Q_r^n, and C_r^n.

2) Explain in your own words why $r!C_r^n = P_r^n$.

3) Find the number of pairs $\{a, b\}$ of distinct integers from the set $\{1, 2, \ldots, 40\}$ such that $|a - b| \leq 4$.

4) Consider a set of n points placed on the circle $x^2 + y^2 = 1$ in the x, y-plane such that the distance between any two points along the circle is the same. How many triangles are there whose vertices are the points on the circle?

5) (a) How many 0-1 sequences of length 8 have exactly three 0's?
(b) How many 0-1 sequences of length 8 have at most three 0's?
(c) What is the total number of 0-1 sequences of length 8?

6) In a group of ten people, we must form a committee consisting of three people where one of the people is the leader of the committee and the other two people are his/her assistant. How many ways can such a committee be formed?

7) Let A be the set of all points (x, y) where x and y are integers and $1 \leq x \leq 7$, $1 \leq y \leq 4$. How many rectangles are there whose vertices are points in A? Can you find the number of squares whose vertices are points in A?

8) Show that for any r with $1 \leq r \leq n$, we have

$$\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}.$$

9) (a) Let $X = \{1, 2, 3\}$. Write the elements of $\mathcal{P}(X)$ (the power set of X).
(b) If $Y = \{1, 2, 3, 4\}$, how many elements will be in $\mathcal{P}(Y)$? List three such elements of $\mathcal{P}(Y)$.

For Fun A collection of sets $\mathcal{A}$ is called *intersecting* if $A \cap B \neq \emptyset$ for every $A, B \in \mathcal{A}$. For example, $\mathcal{A} = \{\{1, 2, 4\}, \{2, 3, 5\}, \{1, 5, 6\}, \{3, 4, 6\}\}$ is intersecting. One can check that every pair has a nonempty intersection:

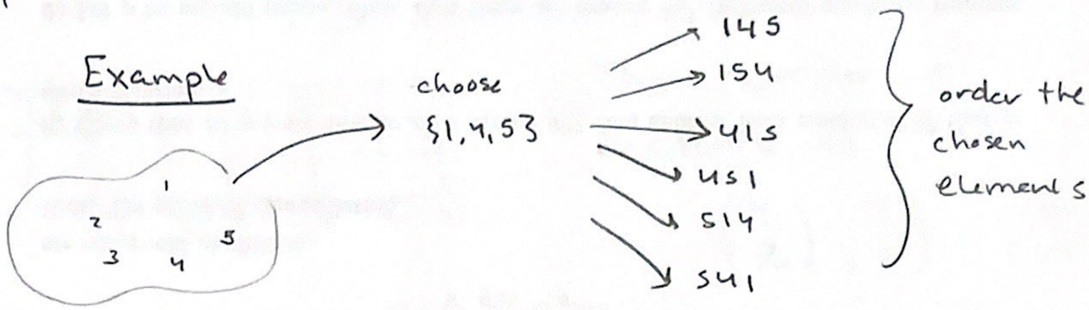
$\{1, 2, 4\} \cap \{2, 3, 5\} = \{2\}$, $\{1, 2, 4\} \cap \{1, 5, 6\} = \{1\}$, $\{1, 2, 4\} \cap \{3, 4, 6\} = \{4\}$, and so on.

If $X = \{1, 2, \ldots, n\}$ and $\mathcal{A}$ is a collection of subsets of X that is intersecting, show that $\mathcal{A}$ can contain at most 2^{n-1} subsets of X.

1) $P_r^n = \dfrac{n!}{(n-r)!}$ $\quad Q_r^n = \dfrac{n!}{r(n-r)!}$ $\quad C_r^n = \dfrac{n!}{r!(n-r)!}$

2) If we choose a subset of size r, which can be done in C_r^n ways and then order that subset, which can be done in $r!$ ways, then we get a r-permutation of $\{1,2,\dots,n\}$.

Example



3) We are counting pairs $\{a,b\}$ so we may assume $a<b$. We could not make this assumption if we were counting ordered pairs.

$\underline{a=1}$ $\quad$ $b \in \{2,3,4,5\}$

$\underline{a=2}$ $\quad$ $b \in \{3,4,5,6\}$

$\underline{a=3}$ $\quad$ $b \in \{4,5,6,7\}$

$\cdots$

$\underline{a=36}$ $\quad$ $b \in \{37,38,39,40\}$

$36 \cdot 4$ possible pairs with $1 \le a \le 36$

$\underline{a=37}$ $\quad$ $b \in \{38,39,40\}$

$\underline{a=38}$ $\quad$ $b \in \{39,40\}$

$\underline{a=39}$ $\quad$ $b \in \{40\}$

Total count is

$36 \cdot 4 + 3 + 2 + 1$

$= 150$

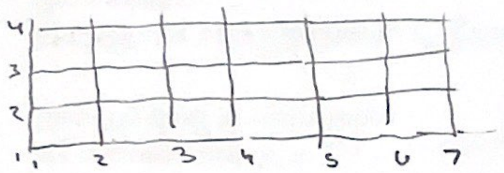
5) (a) $\binom{8}{3} = \frac{8!}{3!\,5!} = \frac{8\cdot7\cdot6}{3\cdot2} = 56$

(b) $\binom{8}{0}+\binom{8}{1}+\binom{8}{2}+\binom{8}{3} = 1+8+28+56 = 93$

(c) $2^8 = 256$

6) $\binom{10}{3}\cdot 3 = \frac{10!}{3!\,7!}\cdot 3 = \frac{10\cdot9\cdot8\cdot3}{3\cdot2} = 360$

7)

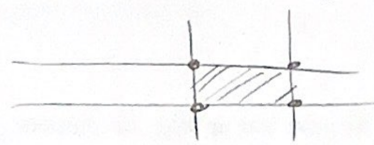


The number of rectangle is

$\binom{4}{2}\binom{7}{2}$

choose two verticals

choose two horizontals



$\frac{\#\text{ of } 1\times1 \text{ squares}}{3\times6 = 18}$

$\frac{\#\text{ of } 2\times2 \text{ squares}}{2\times5 = 10}$

$\frac{\#\text{ of } 3\times3 \text{ squares}}{1\times4 = 4}$

The number of squares is

$18+10+4 = 32$

8) $\frac{n}{r}\binom{n-1}{r-1} = \frac{n}{r} \cdot \frac{(n-1)!}{(r-1)!\,(n-1-(r-1))!}$

$= \frac{n!}{r!\,(n-r)!} = \binom{n}{r}$

9) (a) $P(X) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

(b) $P(Y)$ will contain $2^4 = 16$ elements

Three elements of $P(Y)$ are

$\{1,3,4\}$, ϕ, and $\{1,2\}$