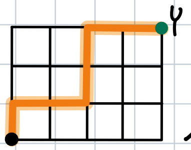
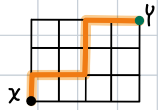
Q1) A student wishes to walk from the corner X to the corner Y through streets as given in the street map. How many shortest routes are from X to Y available to the student?





U = UP, R = Right

U R R U U R R
1 2 3 4 5 6 7

# of U's : 3
# of R's : 4

So choose based on taking a U(UP)-Step, R(Right)-Step

$\binom{7}{3}$, $\binom{7}{4}$

Q2) Example 1.5.2. Show that if $|X| = n$, then $|P(X)| = 2^n$ for all $n \in \mathbb{N}$.

Requires binary thinking where by bijection we can map all binary sequence to an integer

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Q3) **Example 1.5.3.** Let $X = \{1, 2, ..., n\}$, where $n \in \mathbf{N}$. Show that the number of r-combinations of X which contain no consecutive integers is given by

$$\binom{n-r+1}{r},$$

where $0 \le r \le n-r+1$.

General Case

Let $x = \{1,2,3,4,\dots,n\}$ and Let $1 \le r \le n$

Given a subset $\{S_0, S_1, \dots, S_r\}$ of x with no consecutive elements define,

$f(\{S_0, S_1, \dots, S_r\}) = \{S_0, S_1 - 1, S_2 - 2, \dots, \underline{S_r - (r-1)}\}$

This output is a subset of size 'r' since $\{S_0, S_1, \dots, S_r\}$ has no consecutive elements

Note that since "S_r" can be n, but not bigger, $S_r - (r-1)$ canbe $n-r+1$, but not bigger

Therefore,

$$|X| = |Y| = \binom{n-r+1}{r}$$

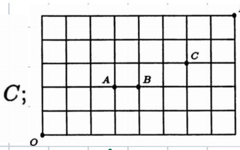
Tricky: all subsets of $\{1,2,3,\dots,n\}$ with no consecutive elements

Easy: all subsets of $\{1,2,3,\dots,n-r+1\}$ of size r

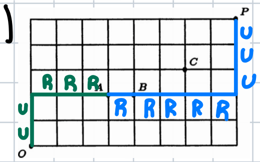
See Back side

Q4) 25. In each of the following cases, find the number of shortest routes from O to P in the street network shown below:

(i) The routes must pass through the junction A;
(ii) The routes must pass through the street AB;
(iii) The routes must pass through junctions A and C;
(iv) The street AB is closed.



(i)

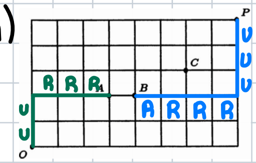


$O \to A$: $\underbrace{U\ U\ R\ R\ R}_{5}$ $\binom{5}{2}$ or $\binom{5}{3}$

$A \to P$: $\underbrace{R\ R\ R\ R\ R\ U\ U\ U}_{8}$ $\binom{8}{2}$ or $\binom{8}{5}$

$\therefore \binom{5}{2}\binom{8}{3}$

(ii)

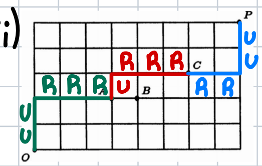


$O \to A$: $\underbrace{UURRR}_{5}$ $\binom{5}{2}$ or $\binom{5}{3}$

$B \to P$: $\underbrace{RRRRUUU}_{7}$ $\binom{7}{3}$ or $\binom{7}{4}$

$\therefore \binom{5}{2}\binom{7}{3}$

(iii)



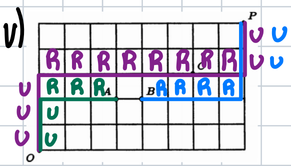
$O \to A$: $\underbrace{UURRR}_{5}$ $\binom{5}{2}$ or $\binom{5}{3}$

$A \to C$: $\underbrace{URRR}_{4}$ $\binom{4}{1}$ or $\binom{4}{3}$

$C \to P$: $\underbrace{RRUU}_{4}$ $\binom{4}{2}$

$\therefore \binom{5}{2}\binom{4}{1}\binom{4}{2}$

(iv)



$O \to P$ (avoiding points AB)

$\underbrace{UUURRRRRRRRUUU}_{13}$ $\binom{13}{5}$ or $\binom{13}{8}$

$O \to A$: $\underbrace{UURRR}_{5}$ $\binom{5}{2}$ or $\binom{5}{3}$

$B \to P$: $\underbrace{RRRRUUU}_{7}$ $\binom{7}{3}$ or $\binom{7}{4}$

$\therefore \binom{13}{5} - \left[\binom{5}{2}\binom{7}{3}\right]$

Q5) 27. Let $S = \{1, 2, \dots, n+1\}$ where $n \ge 2$, and let

$$T = \{(x, y, z) \in S^3 \mid x < z \quad \text{and} \quad y < z\}.$$

Show by counting $|T|$ in two different ways that

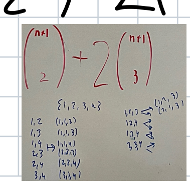
$$\sum_{k=1}^{n} k^2 = |T| = \binom{n+1}{2} + 2\binom{n+1}{3}.$$

Summation Approach

$\sum_{k=1}^{n} k^2$ or $\sum_{k=2}^{n+1} (k+1)^2$

Choosing Approach

$\binom{n+1}{2} + 2\binom{n+1}{3}$



Q6) 40. Prove the identity $\binom{n}{r} = \binom{n}{n-r}$ by (BP) ← What is BP? British Petroleum?

A combination of a set 'A' is a subset of 'A'

For $0 \le r \le n$, an r-combination of 'A' is a subset with 'r' elements

$$P^n_r = \frac{n!}{(n-r)!}$$

If $A = \{a, b, c\}$, then the 2-permutations are

$$P^3_2 = \frac{3!}{(3-2)!}$$

If we ignore order

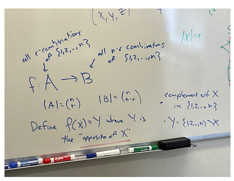
ab
ac
ba
bc
ca
cb

$$\boxed{C^3_2 = \frac{1}{2!} P^3_2}$$

In general...

$$C^n_r = \frac{P^n_r}{r!}$$

we can rewrite as



These choose numbers C^n_r are called "binomial coefficients" and C^n_r,

The Binomial Cofficient

$$\boxed{H^n_r = \binom{r+n-1}{r} = \binom{r+n-1}{n-1}}$$

"C-N-R", is typically written as $\binom{n}{r}$ "n-choose-r"

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Note

$$C^n_r = C^n_{n-r}$$

Since,

$$C^n_r = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!\,r!} = \frac{n!}{(n-r)!(n-(n-r))!} = C^n_{n-r}$$

this means

$$\binom{n}{r} = \binom{n}{n-r}$$

example

$$\binom{4}{0} + \boxed{\binom{4}{1}} + \boxed{\binom{4}{2}} + \binom{4}{3} + \binom{4}{4} = 2^4$$

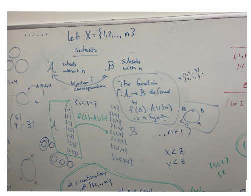
Identity

A set with 4 element and count by size

$\{a, b, c, d\}$
$\{\_, \_, \_, \_\}$

Q7) 41. Let $X = \{1, 2, ..., n\}$, $\mathcal{A} = \{A \subseteq X \mid n \notin A\}$, and $\mathcal{B} = \{A \subseteq X \mid n \in A\}$. Show that $|\mathcal{A}| = |\mathcal{B}|$ by (BP). ← British Petroleum?

Isn't this just asking what question 3 kinda asked?



General Case

Let $x = \{1, 2, 3, 4, \dots, n\}$ and Let $1 \le r \le n$

Given a subset $\{S_0, S_1, \dots, S_r\}$ of x with no consecutive elements define,

$$f(\{S_0, S_1, \dots, S_r\}) = \{S_0, S_1 - 1, S_2 - 2, \dots, S_r - (r-1)\}$$

The biggest it can be is $n-r+1$

This output is a subset of size 'r' since $\{S_0, S_1, \dots, S_r\}$ has no consecutive elements

Note that since "S_r" can be n, but not bigger, $S_r - (r-1)$ canbe $n - r + 1$, but not bigger

Therefore,

$$|X| = |Y| = \binom{n-r+1}{r}$$

Tricky: all subsets of $\{1,2,3,\dots,n\}$ with no consecutive elements

Easy: all subsets of $\{1,2,3,\dots,n-r+1\}$ of size r

$$|a-b| \ge K$$
$$\binom{n-(K-1)(r-1)}{r}$$

$$n = 7 \quad r = 3 \quad \binom{n-r+1}{r} = \binom{5}{3}$$