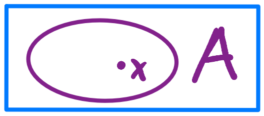
Chapter 01
Permutations and combinations

Let us start with some terms and notation

- A set is a collection of objects
- If 'A' is a set and 'x' is an element in 'A' we write

$$x \in A$$

"is an element of"



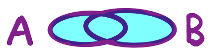
- If 'x' is not in 'A', write

$$x \notin A$$

- If 'A' and 'B' are sets,

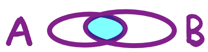
  Union of 'A' and 'B'

  $A \cup B = \{x : x \in A \text{ or } x \in B\}$



  Intersection of 'A' and 'B'

  $A \cap B = \{x : x \in A \text{ and } x \in B\}$



- The empty set, denoted

  $\varnothing$ is a set with no elements

- If $A \cap B = \varnothing$, then A and B are disjoint
- If $A_1, A_2, A_3, \ldots$ is a collection of sets, then this collection is pairwise disjoint if every pair of distinct sets is disjoint

$$A_1, A_2, A_i, \ldots, A_j$$

$$A_i \cap A_j = \varnothing$$

Cartesian Product of Sets

$A_1, A_2, \ldots, A_k$

is the set

$$A_1 \times A_2 \times A_3 \times \ldots \times A_k = \{(a_1, a_2, \ldots a_k) : a_1 \in A_1, a_2 \in A_2, \ldots a_k \in A_k\}$$

* the book's notation for cartesian product *

(choose math major, choose cs major)

$$A_1 \times A_2 \times \ldots \times A_k = \prod_{i=1}^{k} A_i$$

We write $|A|$ for the number of elements in 'A'
and this is called the "Cardinality" of 'A'.

Notation when taking unions/intersections of several sets, we often use sigms type notation
For instance,

$$\rightarrow A_1 \cap A_2 \cap A_3 = \bigcap_{i=1}^{3} A_i$$

$$\rightarrow B_0 \cup B_1 \cup \ldots = \bigcup_{i=0}^{\infty} B_i$$

Common Sets

Natural numbers

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$$

Integers

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

Whole numbers

$$\omega = \{0, 1, 2, 3, 4, \ldots\}$$

Rational

$$\mathbb{Q} = \{\tfrac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}$$

Two Basic Counting Principles

1. If $A_1, A_2, A_3, \ldots, A_3$ are pairwise (_, _, _) disjoint sets, then

$$|A_1 \cup A_2 \cup \ldots \cup A_k| = \sum_{i=1}^{k} |A_i|$$

2. If $A_1, A_2, A_3, \ldots, A_k$ (building an agenda) are finite sets, then

$$|A_1 \times A_2 \times \ldots \times A_k| = |A_1||A_2| \ldots |A_k|$$

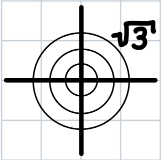
equivalently

$$\underbrace{\left|\prod_{i=1}^{k} A_i\right|}_{\text{cartesian}} = \underbrace{\prod_{i=1}^{k} |A_i|}_{\text{multiply}}$$

SUBJECT: **Section 1.1 – Two Basic Counting Principles** DATE: 2023/01/27 PAGE#:

Example 01: How many integers x,y satisfy the inequality $x^2+y^3 \leq 3$?

Here we are counting (x,y) in a circle radius $\sqrt{3}$



Count according to the value of x^2+y^2

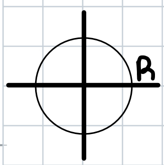
Case 01: $x^2+y^2=0$
→ only $(x,y)=(0,0)$

Case 02: $x^2+y^2=1$
→ $(x,y)=(1,0)$
→ $(x,y)=(0,1)$
→ $(x,y)=(-1,0)$
→ $(x,y)=(0,-1)$

Case 03: $x^2+y^2=2$
→ $(x,y)=(-1,1)$
→ $(x,y)=(1,-1)$ or $(\pm1,\pm1)$
→ $(x,y)=(1,1)$
→ $(x,y)=(-1,-1)$

Case 04: $x^2+y^2=3$
→ None!!!

Adding up all posibilities (case 1→4) gives a total of
→ $1+4+4=9$



$\pi R^2 +$ error term

Warm-up

A composition of a positive integer 'n' is an ordered list of positive integer that sum to 'n'

n	Composition of 'n'	Number of composition of 'n'
1	1	1
2	2, 1+1	2
3	3, 1+2, 2+1, 1+1+1	4
4	4, 2+2, 1+3, 1+1+1, 3+1, 1+1+2, 1+2+1, 2+1+1	8

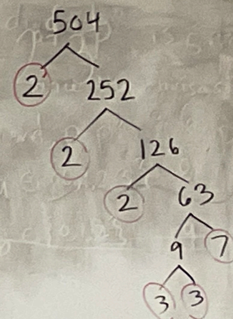
$n = 2^3 3^2$ ← divisors of 3^2 are $3^0, 3^1, 3^2$

divisors of 2^3 are $2^0, 2^1, 2^2, 2^3$

is

$$(k_1+1)(k_2+1)\cdots(k_r+1)$$

Ex Find the # of positive divisors of 504



$504 = 2^3 \cdot 3^2 \cdot 7^1$

# of pos. divisors is

$4 \cdot 3 \cdot 2 = 24$

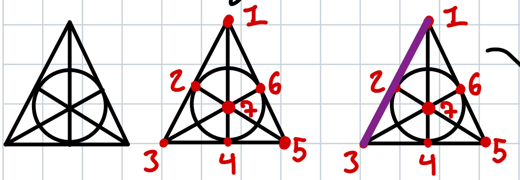
SUBJECT: 1.1 Continued

Missed previous class

DATE: 2023 / 01 / 30 PAGE#:

SUBJECT: Graphs - Error Correction - Hamming Distance DATE: 2023 / 02 / 03 PAGE#:

Before starting section 1.3, let us explore



$C_1 = \{1,2,3\}$ $C_5 = \{3,6,7\}$

$C_2 = \{1,4,7\}$ $C_6 = \{2,5,7\}$

$C_3 = \{1,5,6\}$ $C_7 = \{2,4,6\}$

$C_4 = \{3,4,5\}$

$C_1 \rightarrow$ 1 1 1 0 0 0 0

$C_2 \rightarrow$ 1 0 0 1 0 0 1

$C_3 \rightarrow$ 1 _ _ _ 1 1 _

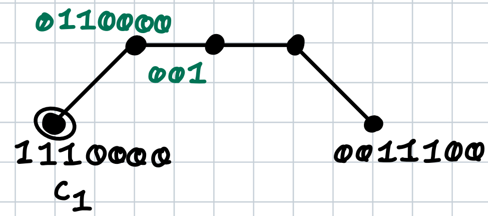
$C_4 \rightarrow$ _ _ 1 1 1 _ _

$C_5 \rightarrow$ _ _ 1 _ _ 1 1

$C_6 \rightarrow$ _ 1 _ _ 1 _ 1

$C_7 \rightarrow$ _ 1 _ 1 _ 1 _

Hamming Distance



I sent you 0110000. What is the most likely code word that was sent?

C_1 since it is the closest in Hamming distance to 0110000

What if * 0 * 0 1 1 * ? → Has to be C_3!

What if 0 0 1 0 0 0 0? → Don't know what to do???

How to build projected planes -Thano plane

13

12	11	10
9	8	7
6	5	4
3	2	1

→

$C_1 = \{13, 12, 11, 10\}$

$C_2 = \{13, 9, 8, 2\}$

$C_3 = \{13, 6, 5, 4\}$

$C_4 = \{13, 3, 2, 1\}$

$C_5 = \{12, 9, 6, 3\}$

$C_6 =$

$C_7 =$

$C_8 = \{11, 9, 5, 1\}$

$C_9 =$

$C_{10} =$

$C_{11} = \{10, 7, 5, 3\}$

$C_{12} =$

$C_{13} =$

Hamming distance to 0110000

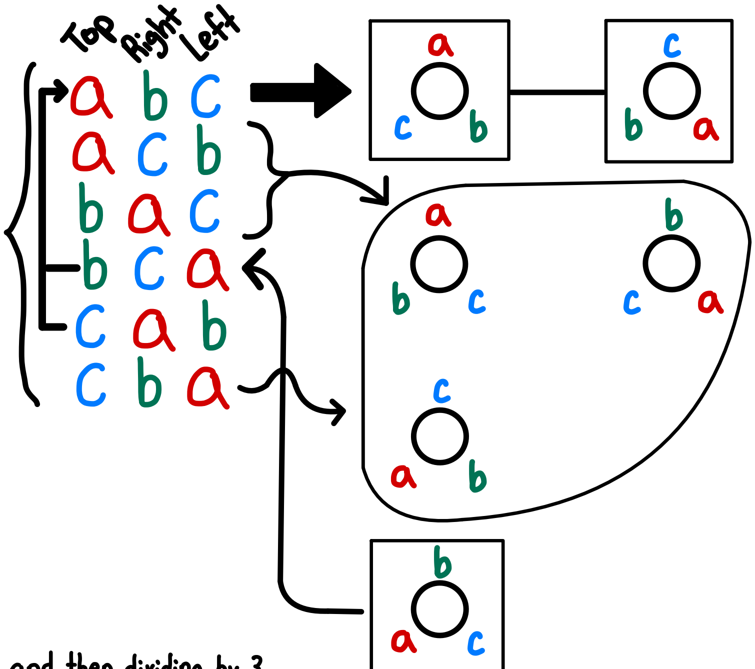
Section 1.3: Circular Permutations

SUBJECT: Section 1.3: Circular Permutations DATE: 2023/02/06 PAGE#:

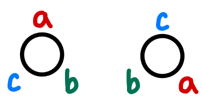
Problem: How many distinct ways can 'a', 'b', and 'c' be arranged around a circular table where two arrangements are the same if they are rotations of each other?

All 3 factorials ways of putting 'a', 'b', 'c' in a line

$$\frac{n!}{n} = (n-1)!$$



Solution: Two distinct ways



One can think of getting this count by first lining up (3! ways) and then dividing by 3.

In general, if Q_r^n is the number of arrangements of 'r' elements placed around a circle chosen from a set of 'n' elements, then

$$Q_r^n = \frac{P_r^n}{r} = \frac{\frac{n!}{(n-r)!}}{r} = \frac{n!}{r\cdot(n-r)!}$$

* Note that if $r=n$, then

$$Q_r^n = \frac{n!}{n(n-n)!} = \frac{n!}{n\cdot 0} = \frac{n!}{n\cdot 1} = \frac{n!}{n} = (n-1)! \qquad \boxed{Q_n^n = \frac{P_n^n}{n} = (n-1)!}$$

Example 01

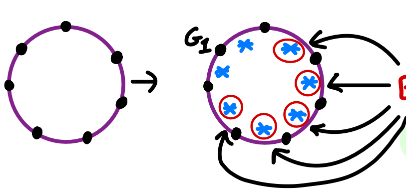
Problem: In how many ways can 5 boys and 3 girls be seated around a table if

(i) There are no restrictions

$$\frac{8!}{8} = 7!$$

(ii) Boy B_1 and girl G_1 cannot be next to each other

Let us place all except B_1 there $\frac{7!}{7} = 6!$ ways to place the seven people around the table (we are not placing B_1 yet)



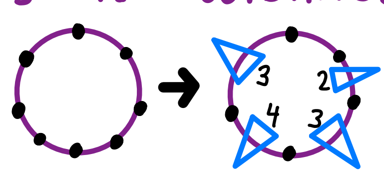
B_1 – 5 choices for B_1

This gives a total count of $6! \times 5$

(iii) No two girls sit next to each other

First put boys at the table

$\frac{5!}{5} = 4!$ Possibilites



$5\cdot 4\cdot 3$ ways to place girls

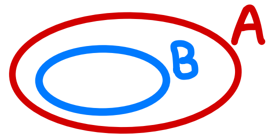
$\Rightarrow$ Total is $4!\cdot 5\cdot 4\cdot 3$

Announcements
- Friday Quiz 2
- Hw 3 Due Monday (02/13)
- Hw 4 Due Wednesday (02/15)
- Exam 1 Friday (02/17)

Section 1.4: Combinations

Let 'A' be a set.
We say that 'B' is a subset of 'A', denoted $B \subseteq A$, if every element of 'B' is also in 'A' and is denoted $P(A)$.

The set of all possible subsets of 'A' is called the <u>power set of 'A'</u>



Recap Prep For Exam 1
Section 1.1, 1.2, 1.3, 1.4
$P^n_r \; Q^n_r \; C^n_r$

Side Note: P(A)
Higher math may use alternate notation 2^A

Example 01

Sample: $P(\{a,b\}) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$
1 2 1

$(x+y)^2 = x^2+2xy+y^2$
$(x+y)^3 = x^3+3x^2y+3xy^2+y^2$
Pascal's Triangle Relation ⇒ Making the connection to pascal

$$\begin{array}{c} 1 \\ 1\;\;1 \\ 1\;\;2\;\;1 \\ 1\;\;3\;\;3\;\;1 \\ 1\;\;4\;\;6\;\;4\;\;1 \end{array}$$

If $A=\{a,b,c\}$, find $P(A)$

$P(A) = \{\underbrace{\emptyset}_{\text{zero element subset}}, \underbrace{\{a\},\{b\},\{c\}}_{\text{one element subset}}, \underbrace{\{a,b\},\{a,c\},\{b,c\}}_{\text{Two element subset}}, \underbrace{\{a,b,c\}}_{\text{Three element subset}}\}$
1 2 3 1

can be grouped in sizes

One way to view choosing subsets of $A=\{1,2,3\}$ is by expanding the polynomial $(a+1)(b+1)(c+1)$

$(a+1)(b+1)(c+1) = abc + ab + ac + bc + a + b + c + 1$
$\{a,b,c\}\; \{a,b\}\; \{a,c\}\; \{b,c\}\; \{a\}\; \{b\}\; \{c\}\; \emptyset$

* The key is to make it all be 'x' *
So when we had

$(a+1)(b+1)(c+1) = abc + ab + ac + bc + a + b + c + 1$
$\{a,b,c\}\; \{a,b\}\; \{a,c\}\; \{b,c\}\; \{a\}\; \{b\}\; \{c\}\; \emptyset$

Let $a=b=c=x$

5 in a 5-element set $(x+1)^5$ ➜ $x^3 + xx + xx + xx + x + x + x + 1$
$= x^3 + 3x^2 + 3x + 1 \longrightarrow (x+1)(x+1)(x+1)$

$$\begin{array}{cl} 1 & \\ 1\;\;1 & =2 \\ 1\;\;2\;\;1 & =4 \\ 1\;\;3\;\;3\;\;1 & =8 \\ 1\;\;4\;\;6\;\;4\;\;1 & =16 \end{array}$$
4

Pascal's Triangle Relation to Binomial Coefficients

Lets make some formulas...

A <u>combination</u> of a set 'A' is a subset of 'A'

For $0 \le r \le n$, an r-combination of 'A' is a subset with 'r' elements

$$P^n_r = \frac{n!}{(n-r)!}$$

If $A=\{a,b,c\}$, then the 2-permutations are

$$P^3_2 = \frac{3!}{(3-2)!}$$

If we ignore order

a b
a c
b a
b c
c a
c b

$$\boxed{C^3_2 = \frac{1}{2!}P^3_2}$$

In general...

$$C^n_r = \frac{P^n_r}{r!}$$

we can rewrite as

These choose numbers C^n_r are called "binomial coefficients" and C^n_r, "C-N-R", is typically written as $\binom{n}{r}$ "n-choose-r"

$$\begin{array}{c} 1 \\ 1\;\;1 \\ 1\;\;2\;\;1 \\ 1\;\;3\;\;3\;\;1 \\ 1\;\;4\;\;6\;\;4\;\;1 \end{array}$$

Note

$$C^n_r = C^n_{n-r}$$

Since,

$$C^n_r = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!\,r!} = \frac{n!}{(n-r)!(n-(n-r))!} = C^n_{n-r}$$

this means

$$\binom{n}{r} = \binom{n}{n-r}$$

example

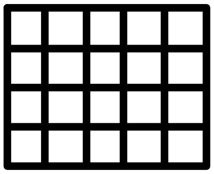
$$\binom{4}{0} + \boxed{\binom{4}{1}} + \boxed{\binom{4}{2}} + \binom{4}{3} + \binom{4}{4} = 2^4$$

<u>Identity</u>
A set with 4 elements and count by size

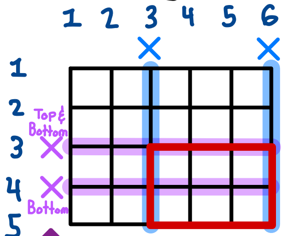
$\{\_,\_,\_,\_\}$ with a,b,c,d

Example 02

How many rectangles are in the grid below?



→



$\binom{6}{2}$ ← Ways to pick sides of rectangle

$\binom{5}{2}$ ↑ Ways to pick top & bottom of rectangle

These choose numbers C^n_r are called "binomial coefficients" and C^n_r, "C-N-R", is typically written as $\binom{n}{r}$ "n-choose-r"

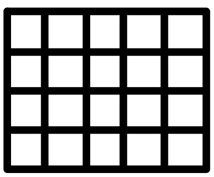
In all, there are

$$\binom{5}{2}\cdot\binom{6}{2} = \frac{5\cdot 4}{2\cdot 1}\times\frac{6\cdot 5}{2\cdot 1} \rightarrow \frac{5\cdot \cancel{4}^{2}}{{}_{1}\cancel{2}\cdot 1}\times\frac{\cancel{6}^{3}\cdot 5}{\cancel{2}_{1}\cdot 1}$$

$$\rightarrow 10\times 15 = 150$$

Example 03

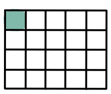
How many, non-overlapping of same sized, squares are in the grid?



→

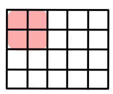
Lets do some case analysis...

Case 1: Square is 1×1



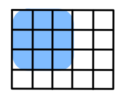
There are 4×5=20 of those

Case 2: Square is 2×2



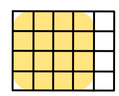
3×4=12

Case 3: Square is 3×3



2×3=6

Case 4: Square is 4×4

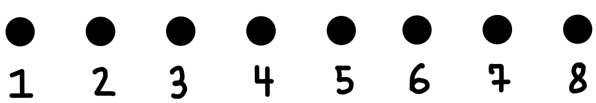


1×2=2

In all, there are 20+12+6+2=40 squares

Example 04

Let 'A' be a set with 2n elements

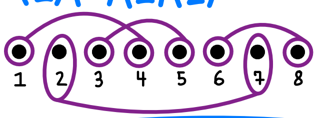


A pairing* of 'A' is a partition of 'A' into subsets of size 2.

*is unordered

How many pairings does 'A' have?

$\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{?}{2}$



{1,4}, {2,7}, {3,5}, {6,8}

$$\binom{2n}{2}\binom{2n-2}{2}\binom{2n-4}{2}\cdots\binom{2}{2}$$

↑ Choose first pair

↑ Choose second pair

Must divide by $n!$ because our initial counting orders the pairings, but pairings are not ordered so...

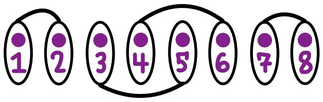
Total # of pairings is

$$\frac{\binom{2n}{2}\binom{2n-2}{2}\binom{2n-4}{2}\cdots\binom{2}{2}}{n!}$$

Last time we counted the number of pairings on pairings on a set with $2n$ elements

$\{3,5\},\{1,2\},\{4,6\},\{7,8\}$

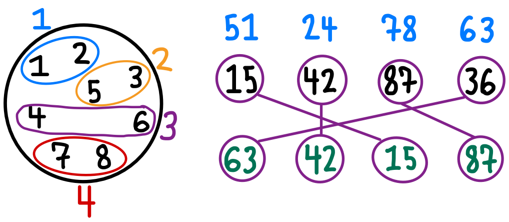
$n=4$



★ $\{1,2\},\{3,5\},\{4,6\},\{7,8\}$

what if presented in a line?

★ $\boxed{\{1,2\},\{3,4\},\{5,6\},\{7,8\}}$



$$\frac{\binom{2n}{n}\binom{2n-2}{2}\binom{2n-4}{2}\cdots\binom{2}{2}}{n!}$$

$\binom{8}{2}\quad\binom{6}{2}\quad\binom{4}{2}\quad\binom{2}{2}\quad\frac{1}{4!}$

$\boxed{\{1,2\},\{3,4\},\{5,6\},\{7,8\}}$ (a_1,a_2,a_3)

Let us count pairings in a different way...

There is $(2n)!$ ways to order all elements in the set

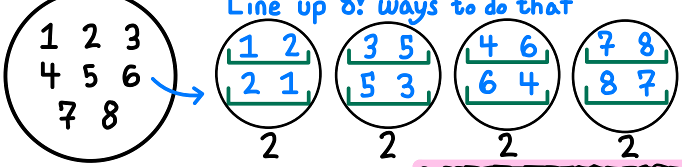
$a_1, a_2, a_3, \ldots, a_{2n} \leftarrow$ ordered list of all elements of 'A'

$\{a_1,a_2\},\{a_3,a_4\},\ldots,\{a_{2n-1},a_{2n}\}$

$a_2 \;\; a_1 \;\; a_4 \;\; a_3 \qquad a_{2n} \;\; a_{2n-1}$

Example 05

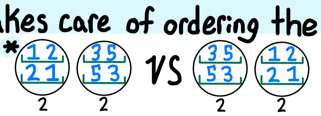
Line up $8!$ ways to do that



$$\frac{(2n)!}{2^n \cdot n!} \xleftarrow{\text{Simplifies to}} \boxed{\frac{\binom{2n}{2}\binom{2n-2}{2}\binom{2n-4}{2}\cdots\binom{2}{2}}{n!}}$$

2^n: takes care of $\{x,y\}$ vs $\{y,x\}$

$n!$: takes care of ordering the pairs



Example 06

$\{1,5\},\{4,2\},\{8,7\},\{3,6\}$

unfinished note section

Example 07

There are 10 students: 6 female, 4 male

(i) How many groups can be found with 3 female & 2 male

$$\binom{6}{3}\binom{4}{2} = 120$$

Number of ways to choose girls — Number of ways to choose boys

(ii) How many groups of 5 students can be made with at least 1 male? * Needs case analysis

case 01	case 02	case 03	case 04
-1 male-	-2 males-	-3 males-	-4 males-
$\binom{6}{4}\binom{4}{1}$	$\binom{6}{3}\binom{4}{2}$	$\binom{6}{2}\binom{4}{3}$	$\binom{6}{1}\binom{4}{4}$
60	120	60	6

Total count is

$$\underset{60}{\binom{6}{4}\binom{4}{1}} + \underset{120}{\binom{6}{3}\binom{4}{2}} + \underset{60}{\binom{6}{2}\binom{4}{3}} + \underset{6}{\binom{6}{1}\binom{4}{4}} = 246$$

(iii) Repeat Example 01 - except now the group must be ranked from 1 to 5 ordered

$$\binom{6}{3}\binom{4}{2}5!$$

Number of ways to choose girls — Number of ways to choose boys — Ways to order the five people

In total $\binom{6}{3}\binom{4}{2}5! = 14{,}400$

In Summary

→ More than 1 way to count

→

unfinished note section

Practice with counting – choose function

Exercise

In how many ways can a group of 5 people be formed from 11 people where 4 are instructors and 7 are students

(i) No restrictions/condictions

$$\binom{11}{5} = \boxed{462}$$

(ii) What if we want exactly two instructors?

Pick 2 instructors and choose the remaining 3

choose 2 instructors

$$\binom{4}{2}\binom{7}{3} = \boxed{210}$$

choose 3 students

(iii) What if we want at least three instructors?

Case	Case
3 instructors	4 instructors
2 students	1 student

$$\underbrace{\binom{4}{3}\binom{7}{2}}_{84} + \underbrace{\binom{4}{4}\binom{7}{1}}_{7} = \boxed{91}$$

**(iv) Conflict of interest where I_1 & S_1 can't be together

I_1 & S_1 cannot both be chosen, we consider cases
(3 in all)

Case 01: I_1 is chosen NOT S_1

I_1 chosen, S_1 cannot $\rightarrow \binom{9}{4} = 126$

Case 02: I_1 is NOT chosen, but S_1 is chosen

S_1 chosen, I_1 cannot $\rightarrow \binom{9}{4} = 126$

Case 03: Neither I_1 or S_1 are chosen

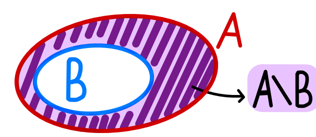
$$\binom{9}{5} = 126$$

$$\boxed{\text{In total } \underset{126}{\text{Case 01}} + \underset{126}{\text{Case 02}} + \underset{126}{\text{Case 03}} = 378}$$

Principle of Complemetation

If 'A' is a finite set and $B \subseteq A$, then $|A \setminus B| = |A| - |B|$

$$A \setminus B = \{x : x \in A, x \notin B\}$$



Attributes
* Counting the opposite
* The 1-minus rule

$$\boxed{P(A) = 1 - P(A')}$$

**Recall (iv)

In how many ways can a group of 5 people be formed from 11 people where 4 are instructors and 7 are students where I_1 & S_1 can't be together

We can use this tool, the Principle of Complementation, to count the number of 5 person groups from 4 instructors, 7 students where I_1 and S_1 are not both in the group together (case 03)

11 ← 7 students, 4 instructors
11 − 2 (1 student & 1 instructor)

Yields the same results → $\underbrace{\binom{11}{5}}_{\text{Total number of possibilities}} - \underbrace{\binom{9}{3}}_{\text{Number of groups that contain both } I_1 \text{ and } S_1} = \boxed{378}$

Note

Counting the same object in two ways can lead to combination identities

In this case...

$$\binom{11}{5} - \binom{9}{3} = \binom{9}{4} + \binom{9}{4} + \binom{9}{4}$$

Line n-up $\underline{1}, \underline{2}, \underline{3}, \ldots, \underline{r}$ $\quad P^n_r = nP^{n-1}_{r-1}$

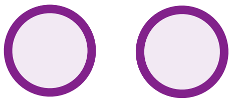
Principle of Complemetation

$$\boxed{P^n_r = nP^{n-1}_{r-1}}$$

Example: Placement type problems

If there must be atleast one person at each table, how many ways can 6 people be placed around two *indistinguishable tables?

*Can't tell the two tables apart



Divide counting into cases depending on the number per table

case 01

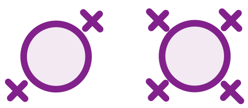
6 choices per person at own table

$Q_5^5 = 4!$

$6 \cdot 4!$

case 02

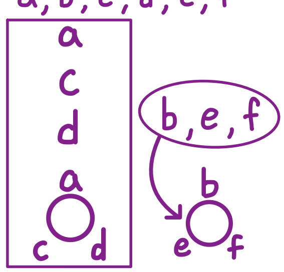
Recall: if $r=n$ then
$$Q_r^n = \frac{n!}{n(n-n)!} = (n-1)!$$



$$\binom{6}{2} \times Q_4^4 = \binom{6}{2} \cdot 3!$$

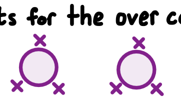
case 03

* Indistingushable tables
a, b, c, d, e, f



$$\frac{\binom{6}{3} Q_3^3 \cdot Q_3^3}{2!}$$

*accounts for the over count



The total count is

case 01 case 02 case 03
$$\binom{6}{1}Q_1^1 \cdot Q_5^5 + \binom{6}{2}Q_2^2 \cdot Q_4^4 + \frac{1}{2}\binom{6}{3}Q_3^3 \cdot Q_3^3$$

Lower case 's' as notation

~Sterling Numbers of the first Kind~

Def: Sterling Numbers of the first Kind

Given integers $0 \le n \le r$ let, $s(r,n)$ be the number of ways to place 'r' distinct objects around 'n' indistinguisable tables where no tables are empty

From our example. .

case 01 case 02 case 03
$$S(6,2) = \binom{6}{1}Q_1^1 \cdot Q_5^5 + \binom{6}{2}Q_2^2 \cdot Q_4^4 + \frac{1}{2}\binom{6}{3}Q_3^3 \cdot Q_3^3$$

Properties of s(r,n):

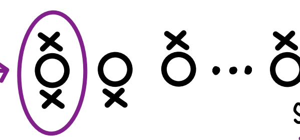
r-people r-table

- $s(r,1) = Q_r^r = (r-1)!$
- $s(r,r) = 1$
- $s(r,r-1) = \binom{r}{2}$

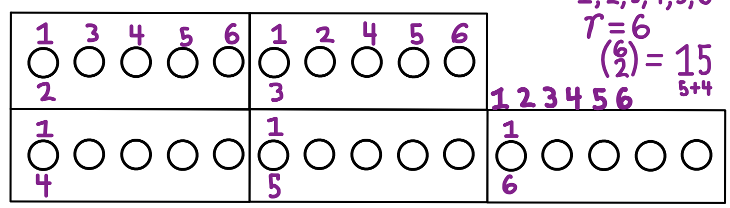
from the textbook

$s(r,0) = 0$	if $r \ge 1$
$s(r,r) = 1$	if $r \ge 0$
$s(r,1) = (r-1)!$	for $r \ge 2$
$s(r,r-1) = \binom{r}{2}$	for $r \ge 2$

Pick a pair to sit together →



$s(r,r-1) = \binom{r}{2}$
1,2,3,4,5,6
$r=6$
$\binom{6}{2} = 15$
5+4



- $s(r,r-2) = \frac{(3n-1)}{4}\binom{n}{3}$

Note: How so?
$$s(r,r-2) = 2\cdot\binom{n}{3} + \frac{1}{2}\binom{n}{2,2,n-4} = \frac{(3n-1)}{4}\binom{n}{3}$$

Claim

If $r \ge n$, then
$$S(r,n) = S((r-1),n-1) + (r-1)S(r-1,n)$$

WHY? Let 1, 2, 3, ..., r be the people

*Focus on person 1

*case 01: $S((r-1),n-1)$
Person 1 is at own table

1	r−1 people n−1 table(s)

$S(r-1,n-1)$ Ways to complete

*case 02: $(r-1)S(r-1,n)$
Person 1 is at not own table
Place all but person #1

r−1 people 1 table

Choices for who sits to left of

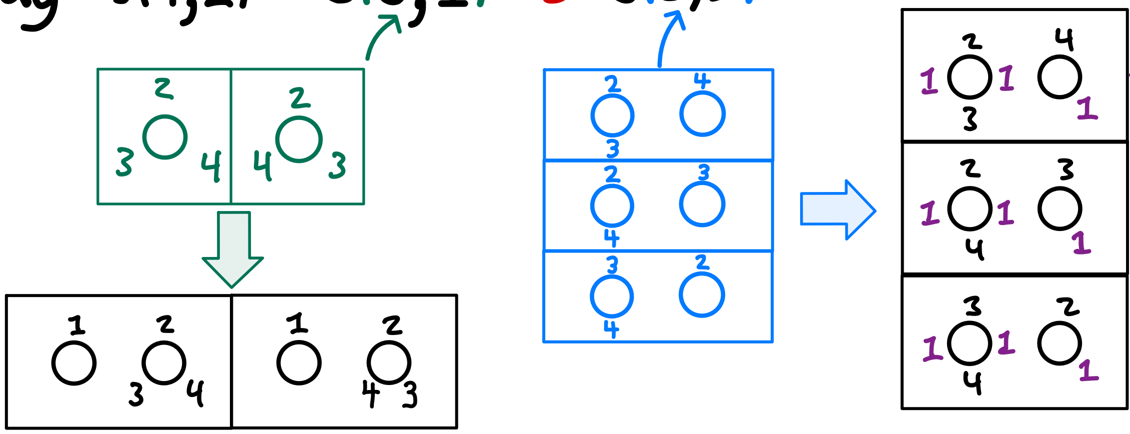
$S(r-1,n)$ Ways to complete

From last time, we talked about

$$s(r,n) = s((r-1), n-1) + (r-1)s(r-1, n)$$

Why is this true? Let us draw a specific instance of this recursion.

$$\text{Say}\quad s(4,2) = S(3,1) + 3 \cdot S(3,2)$$

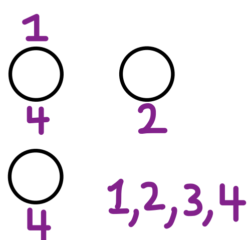


3 choices for 1

We have
$$s(4,2) = 2 + 3 \cdot 3 = 11$$

What about s(5,3) – 5 people and 3 tables

$$\overset{r,\,n}{S(5,3)} = \underbrace{S(4,2)}_{\binom{4}{3}=11} + 4 \cdot \underbrace{S(4,3)}_{\binom{4}{2}=6}$$



$$S(r, r-1) = \binom{r}{2}$$

$$\Rightarrow\ s(5,3) = 11 + 4 \cdot 6 = 35$$

Recall

Properties of s(r,n)

r – people r – table

- $s(r,1) = Q^r_r = (r-1)!$
- $s(r,r) = 1$
- $s(r,r-1) = \binom{r}{2}$
- $s(r,r-2) = \frac{(3n-1)}{4}\binom{n}{3}$

From the textbook

$s(r,0) = 0$	if $r \geq 1$
$s(r,r) = 1$	if $r \geq 0$
$s(r,1) = (r-1)!$	for $r \geq 2$
$s(r,r-1) = \binom{r}{2}$	for $r \geq 2$

$$s(r,n) = s((r-1), n-1) + (r-1)s(r-1, n)$$
$$s(5,3) = s((5-1), 3-1) + (5-1)s(5-1, 3)$$
$$= s((4), 2) + (4)s(4,3)$$

$\rightarrow s(4,2) = s(3,1) + (3)s(3,2)$

$s(r,1) = Q^r_r = (r-1)!$ (3) $s(r, r-1) = \binom{r}{2}$

$(2)! + (3) \cdot \binom{3}{2}$

$\rightarrow s(4,3) = s(3,2) + (3)s(3,3)$

$s(r,r-1) = \binom{r}{2}$ (3) $s(r,r) = 1$

$\binom{3}{2} + (3) \cdot 1$

$\rightarrow s(4,2) = \underset{2}{(2)!} + \underset{3}{(3)} \cdot \underset{3}{\binom{3}{2}}$

$\rightarrow s(4,3) = \underset{2}{\binom{3}{2}} + \underset{3}{(3)} \cdot 1$

$\rightarrow s(4,2) = 11$

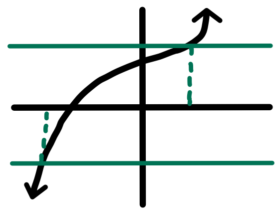
$\rightarrow s(4,3) = 6$

$$s(5,3) = 11 + (4)6$$

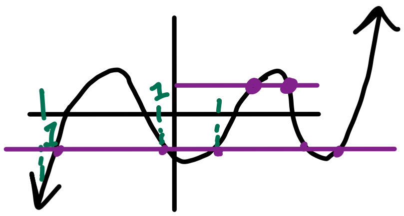
$$\therefore\ s(5,3) = 35$$

Let 'A' and 'B' be sets 'A' function $f: A \to B$ is

- **Injective** (1-to-1) if
  $f(a_1) = f(a_2)$ implies $a_1 = a_2$



- **Surjective** (onto) if
  for $a \in b$, exists an $a \in A$ with $f(a) = b$



- **Bijective** if f is both Injective (1-to-1) and Surjective (onto)

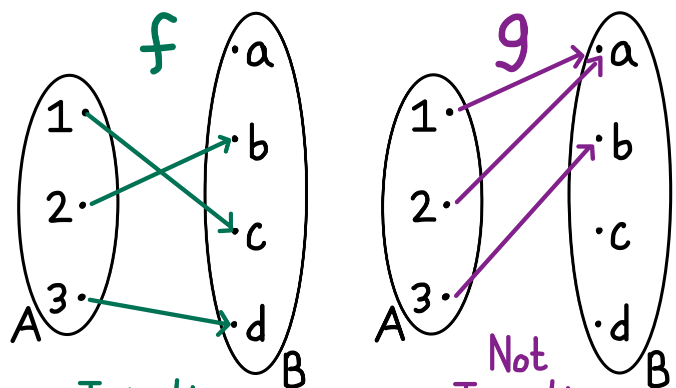
Suppose you have 2 finite sets where it has surjection

$|A| \geq |B|$ $|A| \leq |B|$

Surjection Injection

Example: Injective

Let $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$

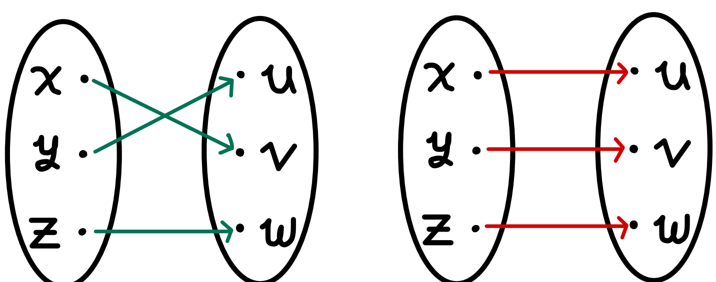


Injective Not Injective

Example: Surjective & injective

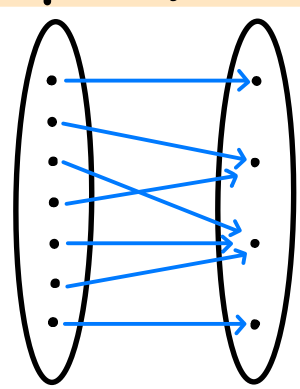
Bijective if f is both Injective (1-to-1) and Surjective (onto)

Let $A = \{x, y, z\}$, $B = \{u, v, w\}$



Surjective and injective Surjective and injective

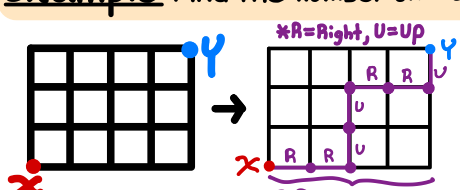
Example: Surjective not injective



Surjective (onto) if for $a \in b$, exists an $a \in A$ with $f(a) = b$

Surjective not injective

Example: Find the number shortest routes from X to Y



RRUURRU

$\binom{7}{3}$ Selecting for up

$\underbrace{\text{R R U U R R U}}$

Choose three places for when to take an U (up) - step

Selecting for Right $\binom{7}{4}$ OR $\binom{7}{3}$ Selecting for up

Example: Find the number of r-combinations (subset with r-elements) with no consecutive intetgers

Let $x = \{1, 2, 3, \ldots, n\}$,

if $x = \{1, 2, 3, 4, 5, 6\}$ and $r = 3$, the subsets are:

$\{1,3,5\}$ $\{2,4,6\}$ $\{1,3,6\}$ $\{1,4,6\}$

*** Use Binomial Cofficient ***

SUBJECT: **Section 1.5: Injection & Bijections Principles** DATE: 2023/02/27 PAGE#: 02/02

Recall from Friday, that we want to find the number of subsets of

$$\{1,2,3,\ldots,n\}$$

with no consecutive integers

For instance, if $x=\{1,2,3,4,5,6,7\}$, then

$\{1,3,5\} \to \{1,2,3\}$
$\{1,3,6\} \to \{1,2,4\}$
$\{1,3,7\} \to \{1,2,5\}$
$\{1,4,6\} \to \{1,3,4\}$
$\{1,4,7\} \to \{1,3,5\}$
$\{1,5,7\} \to \{1,4,5\}$
$\{2,4,6\} \to \{2,3,4\}$
$\{2,4,7\} \to \{2,3,5\}$
$\{2,5,7\} \to \{2,4,5\}$
$\{3,5,7\} \to \{3,4,5\}$

The subsets on the right are exactly all $\binom{5}{3}=10$ subsets $\{1,2,3,4,5\}$ of size 3

Genericly...

Leave same, Subtract 1, Subtract 2

$\{1,3,5\} \to \{1,2,3\}$
$\{1,3,6\} \to \{1,2,4\}$
$\{1,3,7\} \to \{1,2,5\}$

Subtract by the index in set

General Case

Let $x=\{1,2,3,4,\ldots,n\}$ and Let $1\le r\le n$

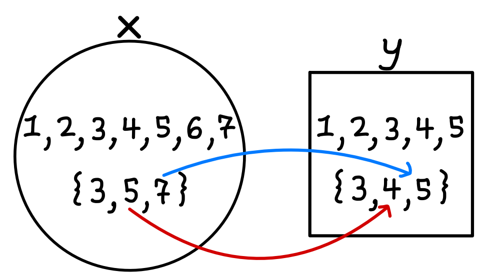
Given a subset $\{S_0,S_1,\ldots,S_r\}$ of x with no consecutive elements define,

$$f(\{S_0,S_1,\ldots,S_r\})=\{S_0,S_1-1,S_2-2,\ldots,\underline{S_r-(r-1)}\}$$

The biggest it can be is $n-r+1$

This output is a subset of size 'r' since $\{S_0,S_1,\ldots,S_r\}$ has no consecutive elements

$n=7$
$r=3$ $\quad \binom{n-r+1}{r}=\binom{5}{3}$



Note that since "S_r" can be n, but not bigger, $S_r-(r-1)$ canbe $n-r+1$, but not bigger

Therefore,

$$\underbrace{|X|}_{\text{Tricky}}=\underbrace{|Y|}_{\text{Easy}}=\binom{n-r+1}{r}$$

Tricky: all subsets of $\{1,2,3,\ldots,n\}$ with no consecutive elements

Easy: all subsets of $\{1,2,3,\ldots,n-r+1\}$ of size r

$|a-b|\ge K$

$$\binom{n-(K-1)(r-1)}{r}$$

What if we require that $|a-b|\ge 3$ for all a,b in our r-subset of $x=\{1,2,3,\ldots,n\}$?

Try $\{1,2,3,4,5,6,7,8\}$, $r=3$

same -2 -4

$\{1,4,7\} \to \{1,2,3\}$
$\{1,4,8\} \to \{1,2,4\}$
$\{1,5,8\} \to \{1,3,4\}$
$\{2,5,8\} \to \{2,3,4\}$

Guess

$$\binom{n-2(r-1)}{r}$$

Via Bijection 'x' counts the 'y'-side

Example

Let $A=\{a,b,c\}$

Arrangements & Selections with Repetitions are allowed

The 2-permutations of 'A' are : ab, ac, ba, bc, ca, cb

The 2-permutations allowed are
aa, ab, ac, ba, bb, bc, ca, cb, cc

There are $3^2=9$ such 2-permutations

In general, if $A=\{1,2,3,\ldots,n\}$ then the number r-permutations of 'A' with repitition allowed is

$$n^r$$

*no need for $0 \le r \le n$ due to being able to have repitition

SUBJECT: **Section 1.6 : Arrangements & Selections** DATE: 2023/03/01 PAGE#: 02/05

with Repetitions are allowed *order matters!

Example | Find the number of 5-permutations of a, a, a, b, c

Approach 01: "Pretend" the a's are distinguisable

Let us, for a moment, pretend the a's are different

$a_1, a_2, a_3, b, c \rightarrow a_1, a_2, a_3, b, c$

3!

There are $5! = 120$ permutations of these 5 elements

some are

$123 \leftarrow a_1, b, a_2, a_3, c$

$213 \leftarrow a_2, c, a_1, a_3, b$

$321 \leftarrow a_3, b, a_2, a_1, c$

$\vdots$

There are 3! ways to permute the a_1, a_2, a_3 among themselves

$\frac{5!}{3!}$ permutations of a,a,a,b,c

Approach 02: Count the unique items

a, a, a, b, c

$\underline{a}\ \underline{a}\ \underline{b}\ \underline{c}\ \underline{a}$, so $\binom{5}{2}2$

In general...

Suppose we have t-types of objects

r_1 of type 1

r_2 of type 2

$\vdots$

r_t of type 3

The number of permutations of all objects is

$$\frac{(r_1, r_2, r_3, \ldots, r_t)!}{r_1!\, r_2!\, r_3! \ldots r_t!}$$

Or $\binom{5}{2}2$ if we organize the b, c first and then the a's

$\underline{b}\ \underline{c}\ \underline{a}\ \underline{a}\ \underline{a}$

the reason for the 2 in $\rightarrow \binom{5}{2}2$

1.6 Continued - Multisets

A multiset is a collection of unordered objects where repitition is allowed.

$\{a,b,c\}$ ← set and multiset

$\{a,b,c,c\}$ ← multiset, but NOT a set.

Notation for multisets:

$\{a,a,b,b,b,b,c,c,c,\ldots\} = \{2\cdot a, 4\cdot b, \infty\cdot c\}$

$$\{2\cdot a, 4\cdot b, \infty\cdot c\}$$

The number 2 is called the repetition number of 'a'
4 is the repetition number of 'b'
∞ is the repetition number of 'c'

Suppose we have a multiset

$M=\{r_1\cdot a_1, r_2\cdot a_2, r_3\cdot a_3, \ldots, r_n\cdot a_n\}$

permutating all elements of M would give a total of

$$\frac{(r_1+r_2+r_3+\ldots+r_n)!}{r_1!\,r_2!\,r_3!\cdots r_n!} \text{ permutations}$$

If we permute not all of the elements M, the counting is more tricky

Example Find the sequences of length 10 with two 0's, three 1's, five 2's

$\left.\begin{array}{l}\text{Five elements total} \\ \hline \text{Two 0's} \\ \text{Three 1's}\end{array}\right\} \dfrac{5!}{2!\,3!}$ $\quad$ $\left.\begin{array}{l}\text{Two 0's} \\ \text{Three 1's}\end{array}\right\} \dfrac{5!}{2!(5-2)!}$

The total count is...

$$\frac{10!}{2!\,3!\,5!}$$

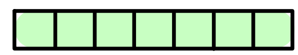
Example Find the number of ways to tile the rectangle

using blocks of size 1x1, 1x2, 1x3

We need to find the number of ways to write 7 as an ordered sum of 1, 2, and 3 where the terms can repeat

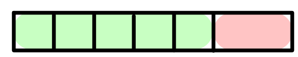
Case 01 all 1×1 blocks

$7 = 1+1+1+1+1+1+1$



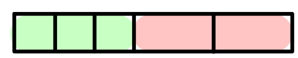
Case 02 Five 1's, One 2

$\frac{6!}{5!} = 6$



Case 03 Three 1's, Two 2's

$\frac{5!}{3!\,2!} = 10$

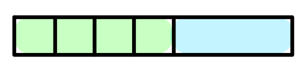


Case 04 One 1, Three 2's

$\frac{4!}{3!} = 4$

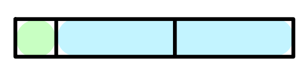
Case 05 Four 1's, One 3

$\frac{5!}{4!} = 5$



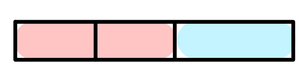
Case 06 One 1, Two 3's

$\frac{3!}{2!} = 3$



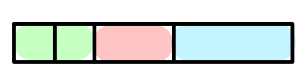
Case 07 Two 2's, One 3

$\frac{3!}{2!} = 3$



Case 08 Two 1's, One 2, One 3

$\frac{4!}{2!} = 12$



1.6 Continued - Multisets

Recall

The number of r-permutations of the multiset
$M=\{\infty\cdot a_1,\infty\cdot a_1,\infty\cdot a_1,\dots,\infty\cdot a_1$, is n^r

If $M=\{r_1\cdot a_1+r_2\cdot a_2+r_3\cdot a_3+\dots+r_n\cdot a_n\}$
Then the number of permutations of M is

$$\frac{(r_1+r_2+r_3+\dots+r_n)!}{r_1!\,r_2!\,r_3!\dots r_n!}$$

Example: "the sandwich" - taking subsets of multisets
There are three types of sandwiches
turkey (T), ham (H), and butter lental tumeric (BLT)
How many ways can a person order 6 sandwiches?

Concept: Stars & Bars
If I can see it
I can count it

Look at examples of orders:
$n=3$
$r=6$

T	H	BLT	
**	*	***	← 2T's, 1H, 3BLT's
	**	****	← 0T, 2H's, 4BLT's
***		***	← 3T's, 0H, 3BLT's

2T's, 1H, 3BLT's → 00101000
0T, 2H's, 4BLT's → 10010000
3T's, 0H, 3BLT's → 00011000
So, $\binom{8}{2}$

Each order corresponds to 0-1 sequence with exactly $6+3+1=8$ positions and has exactly 6 zeros

So, $\binom{6+3-1}{6}=\binom{8}{6}$ **or** $\binom{6+3-1}{3-1}=\binom{8}{2}$

In general, if
$M=\{\infty\cdot a_1,\infty\cdot a_2,\infty\cdot a_3,\dots,\infty\cdot a_n\}$
and H_r^n is the number of multisubsets of M with r-element is
$H_r^n=\binom{r+n-1}{r}$ OR $H_r^n=\binom{r+n-1}{n-1}$

Let $M=\{\infty\cdot a_1,\infty\cdot a_2,\dots,\infty\cdot a_n\}$. The number H_r^n of r-element multi-subsets of M is given by

$$H_r^n=\binom{r+n-1}{r}.$$

Example: Solving integers with equations

$$\underbrace{x_1+x_2+x_3}=7$$

Find the number of solutions in non-negative integers

* Same stars and bars of the previous example *

We can think of a solution as a multiset

T	H	BLT	
***	**	**	← 3+2+2=7
	***	****	← 0+3+4=7

$$\binom{\overset{r}{7}+\overset{n}{3}-1}{\underset{n-1}{3-1}}=\binom{9}{2}$$

$\binom{9}{2}=36$ is the number of non-negative integer solutions to $x_1+x_2+x_3=7$

The Binomial Coefficient

$$H_r^n=\binom{r+n-1}{r}=\binom{r+n-1}{n-1}$$

The Binomial Coefficient counts a few things

→ Number of non-negative integer solutions to $x_1+x_2+x_3+\dots+x_n=r$

→ Number of r-element multisubsets of $M=\{\infty\cdot a_1,\infty\cdot a_2,\infty\cdot a_3,\dots,\infty\cdot a_n\}$

→ Number of ways to put r-identical objects into n-distinct boxes

Box-A	Box-B	Box-C
* *	* *	* * *

The Binomial Coefficient

$$H^n_r = \binom{r+n-1}{r} = \binom{r+n-1}{n-1}$$

The Binomial Coefficient counts a few things

→ Number of non-negative integer solutions to $x_1 + x_2 + x_3 + \dots + x_n = r$

→ Number of r-element multisubsets of $M = \{\infty \cdot a_1, \infty \cdot a_2, \infty \cdot a_3, \dots, \infty \cdot a_n\}$

→ Number of ways to put r-identical objects into n-distinct boxes

Box-A	Box-B	Box-C
* *	* *	* * *

Recall: Sterling Numbers of the first Kind

Given integers $0 \le n \le r$ let, $s(r,n)$ be the number of ways to place 'r' distinct objects around 'n' indistinguisable tables where no tables are empty

Properties of $s(r,n)$: * Lower Case 's' as notation *

r - people r - table

- $s(r,1) = Q^r_r = (r-1)!$
- $s(r,r) = 1$
- $s(r, r-1) = \binom{r}{2}$

  Pick a pair to sit together →
- $s(r, r-2) = \frac{(3n-1)}{4}\binom{n}{3}$

From the textbook

$s(r,0) = 0$	if $r \ge 1$
$s(r,r) = 1$	if $r \ge 0$
$s(r,1) = (r-1)!$	for $r \ge 2$
$s(r,r-1) = \binom{r}{2}$	for $r \ge 2$

Note: How So?

$$s(r, r-2) = 2 \cdot \binom{n}{3} + \tfrac{1}{2}\binom{n}{2,2,n-4}$$
$$= \frac{(3n-1)}{4}\binom{n}{3}$$

Claim

If $r \ge n$, then

$$s(r,n) = s((r-1), n-1) + (r-1)s(r-1, n)$$

WHY? Let $1, 2, 3, \dots, r$ be the people

*Focus on person 1

case 01: $s((r-1), n-1)$

Person 1 is at own table

1	$r-1$ people $n-1$ table(s)

$s(r-1, n-1)$ Ways to complete

case 02: $(r-1)s(r-1, n)$

Person 1 is at not own table

Place all but person #1

$r-1$ people 1 table	Choices for who sits to left of

$s(r-1, n)$ Ways to complete

Stirling Numbers of the Second Kind

* Upper Case 'S' as notation *

$S(r,n)$ = Number of ways to put r-distinct objects into n-identical boxes where no boxes are empty

A B C	D E F

Example: Let $r = 4$ & try to find $S(4,n)$ for $n = 1,2,3,4$

Set will be $x = \{1,2,3,4\}$

$n=1$

$S(4,1) = 1$

1 2
3 4

$n=2$

$S(4,2) = 7$

1	2 3 4
2	1 3 4
3	1 2 4
4	1 2 3

1,2	3,4
1,3	2,4
1,4	2,3

$n=3$

$S(4,3) = 6$

1	2	3,4
1	3	2,4
1	4	2,3
2	3	1,4
2	4	1,3
3	4	1,2

$n=4$

$S(4,4) = 1$

1	2	3	4

$n \ge 5$

$S(5,3) = 0$

$1 + 7 + 6 + 1 = 15$

the number of partitions of $\{1,2,3,4\}$ = 4^{th} Bell Number

$$S(4,1) + S(4,2) + S(4,3) + S(4,4) = B_4$$

The N^{th} Bell Number is given by the formula

$$B_n = \frac{1}{e}\sum_{k=0}^{\infty}\frac{k^n}{k!}$$

Stirling Numbers of the Second Kind Properties

(i) $S(0,0) = 1$,

(ii) $S(r,0) = S(0,n) = 0$ for all $r, n \in \mathbf{N}$,

(iii) $S(r,n) > 0$ if $r \ge n \ge 1$,

(iv) $S(r,n) = 0$ if $n > r \ge 1$,

(v) $S(r,1) = 1$ for $r \ge 1$,

(vi) $S(r,r) = 1$ for $r \ge 1$.

(vii) $S(r,2) = 2^{r-1} - 1$,

(viii) $S(r,3) = \frac{1}{2}(3^{r-1} + 1) - 2^{r-1}$,

(ix) $S(r, r-1) = \binom{r}{2}$,

(x) $S(r, r-2) = \binom{r}{3} + 3\binom{r}{4}$.

Chapter 02
Binomial and Multinomial Coefficients

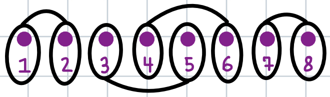
Unpolished Notes

SUBJECT: **Continued – Section 1.4: Combinations** DATE: 2023/02/13 PAGE#: 03/

Last time we counted the number of pairings on a set with $2n$ elements

$\{3,5\}, \{1,2\}, \{4,6\}, \{7,8\}$

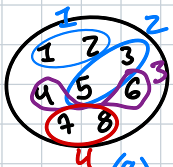
$n=4$



$*\{1,2\}, \{3,5\}, \{4,6\}, \{7,8\}$

what if presented in a line?

$\boxed{\{1,2\}\{3,4\}\{5,6\}\{7,8\}}$



$$\frac{\binom{2n}{2}\binom{2n-2}{2}\binom{2n-4}{2}\cdots\binom{2}{2}}{n!}$$

$\binom{8}{2}\ \binom{6}{2}\ \binom{4}{2}\ \binom{2}{2}\ \frac{1}{4!}$

$\boxed{\{1,2\}, \{3,5\}, \{4,6\}, \{7,8\}}$

Let us count pairings in a different way.

There are $(2n)!$ ways to order all elements in the set

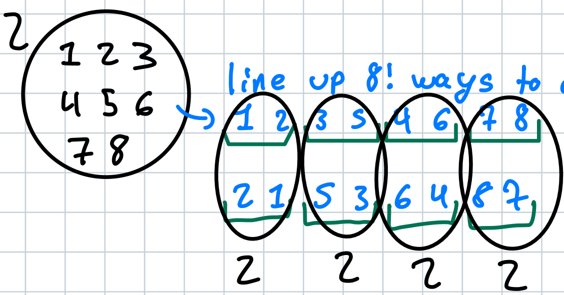
$a_1, a_2, a_3, \ldots, a_{2n}$ ← ordered list of all ements of A

$\{a_1, a_2\}\{a_3, a_4\}, \ldots, \{a_{2n-1}, a_{2n}\}$

$a_2\ a_1 \quad a_4\ a_3 \quad a_{2n}\ a_{2n-1}$

ex 3 → 1 5 4 2 8 7 3 6

Ex2



line up 8! ways to do that

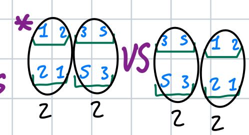
2 2 2 2

$$\frac{(2n)!}{2^n\, n!}$$

$\swarrow$ $\frac{\binom{2n}{2}\binom{2n-2}{2}\binom{2n-4}{2}\cdots\binom{2}{2}}{n!}$ Simplifies to

takes care of $\{x,y\}$ vs $\{y,x\}$

takes care of ordering the pairs



Ex 4: There are 10 students: 6 female, 4 male.

(i) How many groups can be found with 3 female & 2 male?

$\binom{6}{3}\binom{4}{2}$ # of ways to choose boys

# of ways to choose the girls

(ii) How many groups of 5 students can be made with at least 1 male?

Case 1: 1 male — $\binom{6}{4}\binom{4}{1}$

Case 2: 2 males — $\binom{6}{3}\binom{4}{2}$

Case 3: 3 males — $\binom{6}{2}\binom{4}{3}$

Case 4: 4 males — $\binom{6}{1}\binom{4}{4}$

Total count is $\binom{6}{4}\binom{4}{1} + \binom{6}{3}\binom{4}{2} + \binom{6}{2}\binom{4}{3} + \binom{6}{1}\binom{4}{4}$

(iii) Repeat Example 1 - except now the group must be ranked from 1 to 5 ordered

$\binom{6}{3}\binom{4}{2}5!$ ← ways to order the 5 people

# of ways to choose the girls

# of ways to choose boys

$$\binom{10}{5} - \binom{6}{5} = \underline{\qquad\qquad}$$

Total only females at least one male

Summary

1) more than 1 way to count

2)

Exam 1 – Friday (02/17)
Homework 1-4 main resource of study

Practice with counting – Choose function

Exercise: In how many ways can a group of 5 people be formed from 11 people where 4 are instructors and 7 are students.

(i) No restrictions/conditions

$$\binom{11}{5}$$

(ii) what if we want exactly two instructors?

Pick 2 instructors and choose the remaining 3

$$\underbrace{\binom{4}{2}}_{\text{Choose 2 instructors}}\underbrace{\binom{7}{3}}_{\text{Choose 3 students}}$$

(iii) what if we want at least three instructors?

$$\underbrace{\binom{4}{3}\binom{7}{2}}_{\text{3 instructors 2 student case}}+\underbrace{\binom{4}{4}\binom{7}{1}}_{\text{4 instructors 1 student case}}$$

(iv) Conflict of interest where S_1 & I_1 can't be together

I_1 & S_1 cannot both be chosen, we Consider cases (3 in all)

Case 01: I_1 is choosen Not S_1

$\binom{9}{4}$ ← I_1 chosen, S_1 cannot

Case 02: I_1 is not chosen, but S_1 is chosen

$\binom{9}{4}$ ← S_1 chosen, I_1 cannot

Case 03: Neither are chosen

$$\binom{9}{5}$$

$$\underbrace{\binom{11}{5}}_{\text{Total \# of possibilities}}-\underbrace{\binom{9}{3}}_{\text{\# of groups that contain both } I_1 \,\&\, S_1}$$

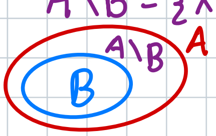
Principle of complemetation
* counting the opposite
* the 1 minus rule

$$P(A)=1-P(A')$$

Principle of complementation

If A is a finite set and $B\subseteq A$, then $|A\setminus B|=|A|-|B|$

$$A\setminus B=\{x: x\in A, x\notin B\}$$



We can use this tool to count # of 5 person groups from 4 instructors, 7 students where I_1 and S_1 are not both in the group together (case 03)

SUBJECT: Continued - Section 1.4: Combinations DATE: / / PAGE#:

Note that counting the same object in two ways can lead to combinational identities

In this case

$$\binom{11}{5} - \binom{9}{3} = \binom{9}{4} + \binom{9}{4} + \binom{9}{5}$$

$$\underbrace{\underline{1}, \underline{2}, \underline{3} \cdots \underline{r}}_{\text{Line nup}} \; P^n_r = n P^{n-1}_{r-1}$$

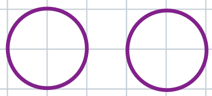
Homework 05 Due Friday - 2/24
Quiz 04 on Friday - 2/24
• will have both individual & group format

Continued - Section 1.4: Combinations

DATE: 2023/02/20

Placement type problems

Example: If there must be atleast one person at each table, how many ways can 6 people be placed around two indistinguishable tables?



Divide counting into cases depending on the # per table

Case 01

6 choices per person at own table $\rightarrow 6 \cdot 4!$

$Q^5_5 = 4!$

Case 02

$\binom{6}{2}$ choices $\times$ Q^4_4 $= \binom{6}{2} \cdot 3!$

Case 03

$$\frac{\binom{6}{3} Q^3_3 \cdot Q^3_3}{2}$$

accounts for the over count

* indistinguishable tables

a, b, c, d, e, f

The total count is

$$\binom{6}{1} Q^1_1 \cdot Q^5_5 + \binom{6}{2} Q^2_2 \cdot Q^4_4 + \frac{1}{2}\binom{6}{3} Q^3_3 \cdot Q^3_3$$

Def

Sterling Numbers of the first kind

Given integers $0 \le n \le r$ let,

$S(r,n)$ be the Number of ways to place 'r' distinct objects around 'n' indistinguishable tables where no table is empty.

From our example,

$$S(6,2) = \binom{6}{1} Q^1_1 \cdot Q^5_5 + \binom{6}{2} Q^2_2 \cdot Q^4_4 + \frac{1}{2}\binom{6}{3} Q^3_3 \cdot Q^3_3$$

Properties of $S(r,n)$:

r-people r-table

- $S(r,r) = 1$
- $S(r,1) = Q^r_r = (r-1)!$
- $S(r, r-1) = \binom{r}{2}$

Pick a pair to sit together

$S(r, r-1) = \binom{r}{2}$

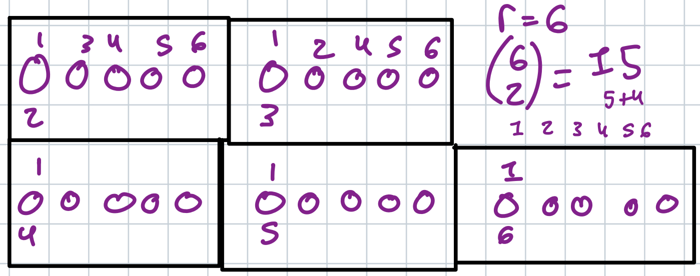
1, 2, 3, 4, 5, 6

$r = 6$

$\binom{6}{2} = 15$

5+4

1 2 3 4 5 6



SUBJECT: Continued – Section 1.4: Combinations DATE: 2023/02/20 PAGE#:

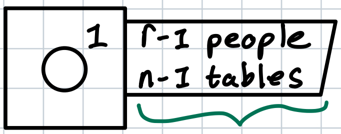
Claim: If $r \geq n$, then
$S(r,n) = S(r-1), n-1) + (r-1)\, S(r-1, n)$

WHY? let $1, 2, 3, \ldots, r$ be the people

Focus on person 1

Case 01

Person 1 is at own table

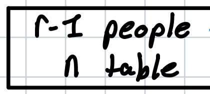


$S(r-1, n-1)$ was to complete

CASE 02

Person 1 is not at own table

- Place all but person #1

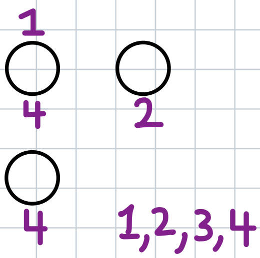


r-1 choices for who person 1 sits to left of

$S(r-1, n)$ ways to complete

What about $s(5,3)$? — 5 people & 3 tables

$S(5,3) = \underbrace{S(4,2)}_{\binom{4}{3}=11} + 4 \cdot \underbrace{S(4,3)}_{\binom{4}{2}=6}$



$S(r, r-1) = \binom{r}{2}$

$\Rightarrow s(5,3) = 11 + 4 \cdot 6 = 35$

$s(r,n) = s((r-1), n-1) + (r-1)\, s(r-1, n)$

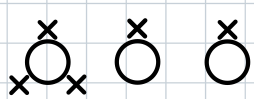
$s(5,3) = s((5-1), 3-1) + (5-1)\, s(5-1, 3)$

$\rightarrow s(4,2) + (4)\, s(4,3)$

$\rightarrow s(3,1) + (1)\, s(1,2) + 4(s(3,2) + (3)\, s(3,3))$

$\rightarrow \quad 3 + (1)\emptyset + 4(\binom{3}{2}) + 3 + 1)$

Case 01

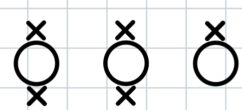


$\binom{5}{3}\binom{2}{1} Q_3^3 Q_1^1 Q_1^1$

$\left(\frac{5!}{3!(5-3)!}\right)\left(\frac{2!}{1!(2-1)!}\right) 2!\, 0!\, 0!$

$10 \cdot 2 \cdot 2$

Case 02



$\binom{5}{2}(\quad) Q_3^3 Q_1^1 Q_1^1$

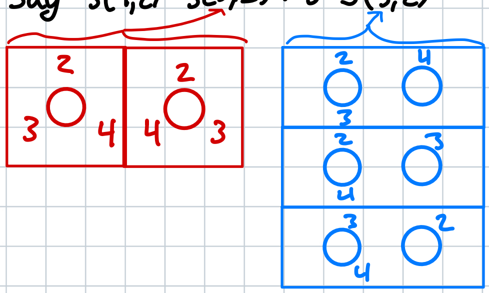
From last time, we talked about

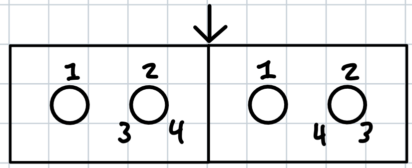
$s(r,n) = s(r-1, n-1) + (r-1)s(r-1, n)$

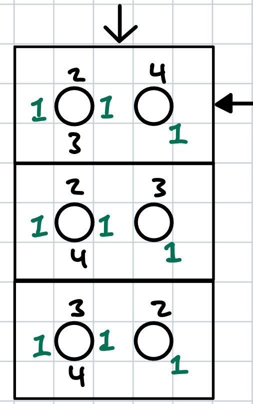
Why is this true? Let us draw a specific instance of this recursion.

Say $S(4,2) = S(3,1) + 3\cdot S(3,2)$

Note
$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$





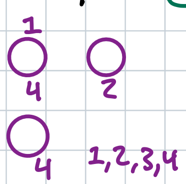


3 choices for 1

We have
$S(4,2) = 2 + 3\cdot 3$
$= 11$

What about $S(5,3)$?

$S(5,3) = \underbrace{S(4,2)}_{11} + 4\cdot \underbrace{S(4,3)}_{\binom{4}{3}=6}$

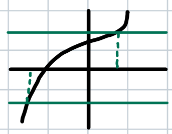


$S(r, r-1) = \binom{r}{2}$

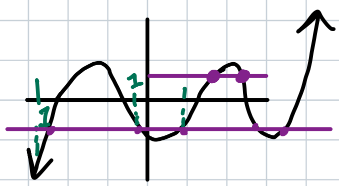
$\Rightarrow S(5,3) = 11 + 4\cdot 6 = 35$

Let A and B be sets 'A' function $f: A \to B$ is

- injective (1-to-1) if
  $f(a_1) = f(a_2)$ implies $a_1 = a_2$



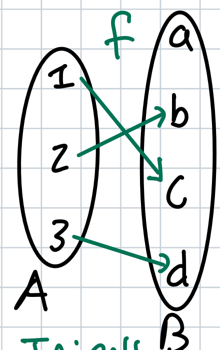
- Surjective (on-to) if
  for any $b \in B$, exists an $a \in A$ with $f(a) = b$



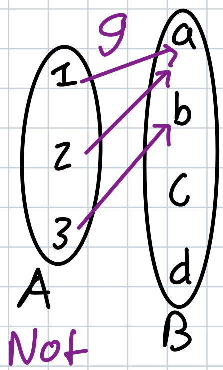
- Bijective if f is both 1-to-1
  and onto

Example

Let $A = \{1,2,3\}$, $B = \{a,b,c,d\}$



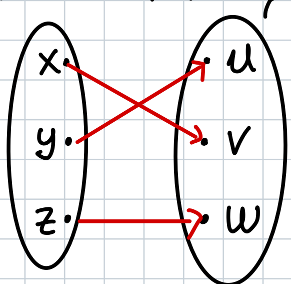
Injective



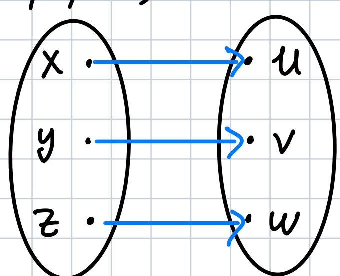
Not Injective

Example

Let $A = \{x,y,z\}$, $B = \{u,v,w\}$

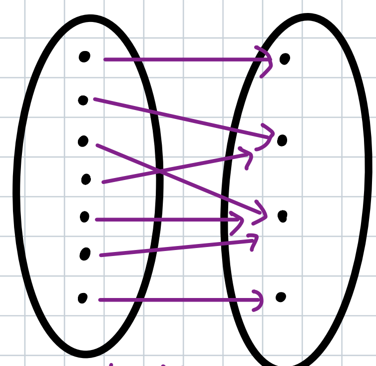


surjective & injective



surjective and injective

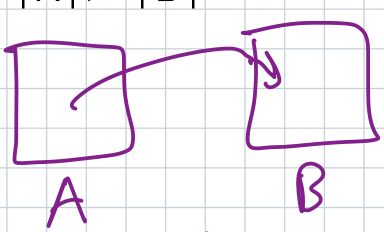
Example



surjective not injective

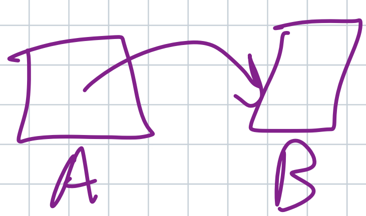
Suppose you have 2 finite sets where it has surjection

$|A| >= |B|$

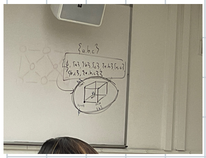


Surjection

$|A| \leq |B|$

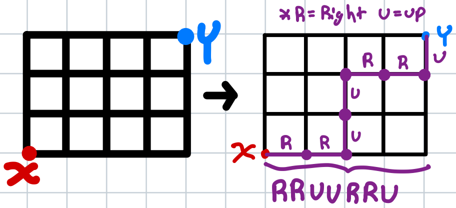


injection



Example

Find the # of Shortest routes from X to Y



$\underbrace{\_\ \_\ \_\ \_\ \_\ \_\ \_}$

Choose three places for when to take an U (up) step

Example: Let $X = \{1, 2, 3, \dots, n\}$
Find the number of r-combinations (subset with r-elements) with no consecutive integers

For example, if $X = \{1, 2, 3, 4, 5, 6\}$ and $r = 3$, the subsets

are: $\{1, 3, 5\}$ $\{2, 4, 6\}$

$\{1, 3, 6\}$

$\{1, 4, 6\}$

$\boxed{\text{* use Binomial coeff *}}$

subset w/o consecutive #s

Recall from Friday, that we went to find the # of subsets of

$\{1,2,3,\dots,n\}$

with no consecutive integers

For instance, if

$X=\{1,2,3,4,5,6,7\}$, then

$\{1,3,5\} \longrightarrow \{1,2,3\}$
$\{1,3,6\} \longrightarrow \{1,2,4\}$
$\{1,3,7\} \longrightarrow \{1,2,5\}$
$\{1,4,6\} \longrightarrow \{1,3,4\}$
$\{1,4,7\} \longrightarrow \{1,3,5\}$
$\{1,5,7\} \longrightarrow \{1,4,5\}$
$\{2,4,6\} \longrightarrow \{2,3,4\}$
$\{2,4,7\} \longrightarrow \{2,3,5\}$
$\{2,5,7\} \longrightarrow \{2,4,5\}$
$\{3,5,7\} \longrightarrow \{3,4,5\}$

The subsets on the right are exactly all $\binom{5}{3}=10$ subsets $\{1,2,3,4,5\}$ of size 3

Genericly...

Leave same, subtract 1, subtract 2

$\{1,3,5\} \longrightarrow \{1,2,3\}$
$\{1,3,6\} \longrightarrow \{1,2,4\}$
$\{1,3,7\} \longrightarrow \{1,2,5\}$

Subtract by index in set

General Case

let $X=\{1,2,3,4,\dots,n\}$ and let $1\le r\le n$

Given a subset $\{s_0,s_1,\dots,s_r\}$ of X with no consective elements, define

$f(\{s_0,s_1,s_2,\dots s_r\}) = \{s_0, s_1-1, s_2-2, \dots \underline{s_r-(r-1)}\}$

the biggest it can be is $n-r+1$

This output is a subset of size r since $\{s_0,s_1,s_2,\dots,s_r\}$ has no consecutive elements

Note that since s_r can be n, but not bigger, $s_r-(r-1)$ canbe $n-r+1$ but not bigger.

therefore,

$$\underline{|X|}=\underline{|Y|}=\binom{n-r+1}{r}$$

Tricky

all subsets of $\{1,2,\dots,n\}$ with No consecutive elements

easy

all subsets of $\{1,2,3,\dots,n-r+1\}$ of size r

$|a-b|\ge k$

$\binom{n-(k-1)(r-1)}{r}$

What if we require that $|a-b|\ge 3$ for all a,b in our r-subset of $X=\{1,2,3,\dots,n\}$?

Try $\{1,2,3,4,5,6,7,8\}$, $r=3$

Same, -2, -4

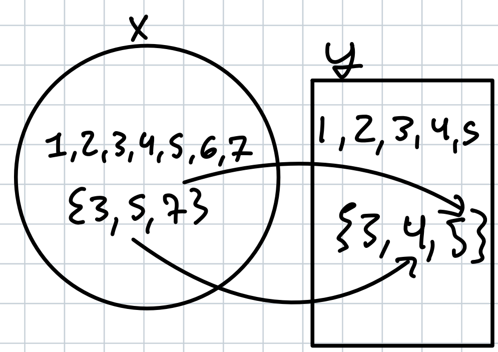
$\{1,4,7\} \rightarrow \{1,2,3\}$
$\{1,4,8\} \rightarrow \{1,2,4\}$
$\{1,5,8\} \rightarrow \{1,3,4\}$
$\{2,5,8\} \rightarrow \{2,3,4\}$

Guess:

$$\binom{n-2(r-1)}{r}$$

Via Bijection X counts the Y side

$n=7$
$r=3$
$\binom{n-r+1}{r}=\binom{5}{3}$



Repetitions are allowed

Example

Let $A = \{a, b, c\}$

The 2-permutations of A

are ab, ac, ba, bc, ca, cb

The 2-permutations allowed are

aa, ab, ac, ba, bb, bc, ca, cb, cc

There are $3^2 = 9$ such 2-permutations

In General, If $A = \{1, 2, 3, \ldots, n\}$

then the number r-permutations of 'A' with repitition allowed is

$$\boxed{n^r}$$

* No need for $0 \leq r \leq n$ due to being able to have repitition

SUBJECT: 1.6 Continued - Allowed for repititions DATE: 2023/03/01 PAGE#:

* order matters

Example: Find the # of 5-permutations of a, a, a, b, c

Approach 01: "Pretend" the a's distinguisable

Let us for a moment pretend the a's are different

$a_1, a_2, a_3, b, c \rightarrow a_1, a_2, a_3, b, c$ 3!

There are $5! = 120$ permutations of these 5 elements

Some are

123 ← a_1, b, a_2, a_3, c

a_2, c, a_1, a_3, b

321 ← a_3, b, a_2, a_1, c

⋮

There are 3! ways to permute the a_1, a_2, a_3 among themselves $\Rightarrow \frac{5!}{3!}$ Permutations of a, a, a, b, c

Approach 2:

a, a, a, b, c

$\underline{a}$ $\underline{a}$ $\frac{b}{x}$ $\frac{c}{x}$ $\underline{a}$, so $\binom{5}{2}$

In general...

Suppose we have t-types of objects

r_1 of type 1

r_2 of type 2

⋮

r_t of type t

The # of permutations of all objects is

$$\frac{(r_1, r_2, r_3, \ldots r_t)!}{r_1! r_2! r_3! \ldots r_t!}$$

or $\binom{5}{2} \cdot 2$ if we organize the b, c first and then the a's

_ _ | $\frac{c}{b}$ _ - the reason for the $\boxed{\times 2}$

SUBJECT: 1.6 Continued - Multisets DATE: 2023 03 01 PAGE#:

A multiset is a collection of unordered objects where repitition is allowed

$\{a,b,c\}$ ← set and multiset

$\{a,b,c,c\}$ ← multiset, but NOT a set

Notation for multisets:

$\{a,a,b,b,b,b,c,c,c,\ldots\} = \{2\cdot a, 4\cdot b, \infty\cdot c\}$

$\{2\cdot a, 4\cdot b, \infty\cdot c\}$

The Number 2 is called the repetition # of a
4 is repitition number of b
∞ is repitition number of c

Suppose we have a multiset

$M = \{r_1\cdot a_1, r_2\cdot a_2, r_3\cdot a_3, \ldots r_n\cdot a_n\}$

Permutating all elements of M would give a total of

$$\frac{(r_1+r_2+r_3+\ldots+r_n)!}{r_1!\,r_2!\,r_3!\ldots r_n!}$$ permutations

If we permute not all of the elements M, the counting is more tricky

Example: Find the sequences of length 10 w/ two 0's, three 1's, five 2's

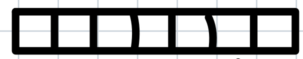
two 0s three 1's $\frac{5!}{2!\,3!}$

_ _ _ _ _ two 0's three 1's $= \frac{5!}{2!(5-2)!}$

Total count is $\frac{10!}{2!\,3!\,5!}$

order does (not) matter

Example: Find the Number of ways to tile the rectangle



using blocks of size 1x1, 1x2, and 1x3

we need to find the number of ways to write 7 as an ordered sum of 1, 2, and 3 where the terms can repeat

Case 01

$7 = \underbrace{1+1+1+1+1+1+1}_{\text{all } 1\times1 \text{ blocks}}$

Case 02 Five 1's, One 2's

$\frac{6!}{5!} = 6$

Case 03: Three 1's, Two 2's

$\frac{5!}{3!\,2!} = 10$

Case 04: One 1's, three 2's

$\frac{4!}{3!} = 4$

Case 05 Four 1's, One 3

$\frac{5!}{4!} = 5$

Case 06 one 1, two 3's

$\frac{3!}{2!} = 3$

Case 07: Two 2's, One 3

$\frac{3!}{2!} = 3$

Case 08: Two 1's, One 2, one 3

$\frac{4!}{2!} = 12$

SUBJECT: 1.6 Continued - Multisets DATE: 2023/03/03 PAGE#:

Recall that the number of r-permutations of the multiset

$$M = \{\infty \cdot a_1, \infty \cdot a_2, \infty \cdot a_3, \ldots, \infty \cdot a_n\} \text{ is } n^r$$

If $M = \{r_1 \cdot a_1, r_2 \cdot a_2, r_3 \cdot a_3, \ldots, r_n \cdot a_n\}$, Then the number of permutations of M is

$$\frac{(r_1 + r_2 + r_3 + \ldots + r_n)!}{r_1!\, r_2!\, r_3! \ldots r_n!}$$

Example: Sandwiches! - taking subsets of multisets

There are three types of sandwiches turkey (T), ham (H), and BLT (BLT). How many ways can a person order 6 sandwiches

Look at examples of orders:

"If I can see it I can count it - the Stars & Bars"

$n=3$
$r=6$

T	H	BLT	
**	*	***	← 2T's, 1H, 3BLT → 00101000
	**	****	← 0T's, 2H's, 4BLTs → 10010000
***		***	→ 00011000

So, $\binom{8}{2}$

Each order corresponds to 0-1 sequence with exactly $6+3-1=8$ positions and has exactly 6 zeros

So $\binom{6+3-1}{6} = \binom{8}{6}$ or $\binom{6+3-1}{3-1} = \binom{8}{2}$

$6+3-1=8$
6 - Denotes ?
3 - Denotes ?
1 - Denotes ?

SUBJECT: 1.6 Continued - Multisets DATE: 20230303 PAGE#:

In General, if

$M = \{\infty \cdot a_1, \infty \cdot a_2, \infty \cdot a_3, \ldots, \infty \cdot a_n\}$

and H_r^n is the number of multisubsets of M with r-element is

$$H_r^n = \binom{r+n-1}{r} \text{ or } \binom{r+n-1}{n-1}$$

Example solving integers w/ equations

Consider the equation

$$\underbrace{x_1 + x_2 + x_3 = 7}$$

Find the # of solutions in non-negative integers

* Same Stars & Bars of the previous example

we can think of a solution as a multiset

T	H	BLT	
***	**	**	$\leftarrow 3+2+2=7$
	***	****	$\leftarrow 0+3+4=7$

$$\binom{\overset{r}{7}+\overset{n}{3}-1}{\underset{n-1}{3-1}} = \binom{9}{2}$$

Is the # of non-negative integer solutions to

$$x_1 + x_2 + x_3 = 7$$

Finishes Chapter 01

SUBJECT: 1.6 Continued - Multisets DATE: 2023/03/06 PAGE#:

The binomial coefficient

$$H_r^n = \binom{r+n-1}{r} = \binom{r+n-1}{n-1}$$

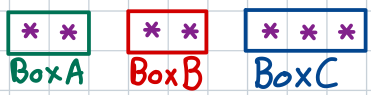
This counts a few things

- # of non-negative integer solutions to

  $x_1 + x_2 + x_3 + \dots + x_n = r$

- # of r-elements multi subsets of $M = \{\infty \cdot a_1, \infty \cdot a_2, \infty \cdot a_3, \dots, \infty \cdot a_n\}$

- # of ways to put r-identical objects into n-distinct boxes



Capital case as Notation

Stirling #s of the Second kind

$S(r,n)$ = # of ways to put r-distinct objects into n-identical boxes where ______

Example Let $r=4$ & try to find $S(4,n)$ for $n = 1, 2, 3, 4, 5, \dots$

Set will be $X = \{1, 2, 3, 4\}$

n = 1

1 2 3 4

$S(4,1) = 1$

n = 2

1	2 3 4
2	1 3 4
3	1 2 4
4	1 2 3

1, 2	3, 4
1, 3	2, 4
1, 4	2, 3

$S(4,2) = 7$

n = 3

1	2	3,4
1	3	2,4
1	4	2,3
2	3	1,4
2	4	1,3
3	4	1,2

$S(4,3) = 6$

n = 4

1	2	3	4

$S(4,4) = 1$

n ≥ 5

$S(5,3) = \emptyset$

$1 + 7 + 6 + 1 = 15$

# of partitions of $\{1, 2, 3, 4\}$ = 4th Bell Number (B_4)

$$S(4,1) + S(4,2) + S(4,3) + S(4,4) = B_4$$

N^{th} Bell # is given by the formula

$$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}$$

For $0 \le r \le n$,

$$C^n_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

is the # of r-combinations of an n-element set.

For $r > n$ or $r \le 0$,
define $\binom{n}{r} = 0$

The #s $\binom{n}{r}$ are called Binomial Coefficients

Line up items → $P^n_r = n(n-1)\dots(n-r+1)$

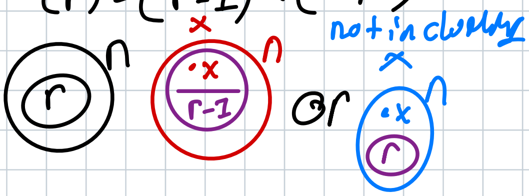
circle → $Q^n_r = \frac{1}{r} P^n_r$

line up & divide up w/ sym → $C^n_r = \frac{P^n_r}{r!}$

Relations satisfied by $\binom{n}{r}$:

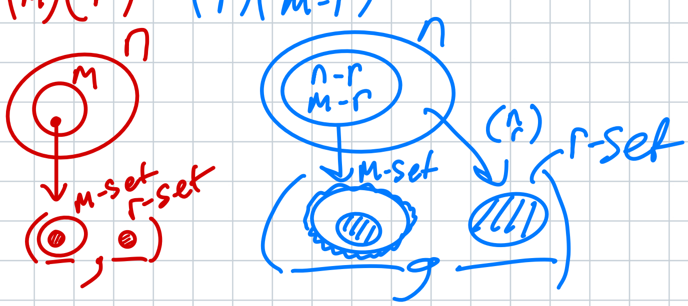
* $\cdot \binom{n}{r} = \binom{n}{n-r}$

* $\cdot \binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$



- $\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$
- $\binom{n}{r} = \frac{n-r+1}{r}\binom{n}{r-1}$
- $\binom{n}{M}\binom{M}{r} = \binom{n}{r}\binom{n-r}{M-r}$

$$\binom{n}{M}\binom{M}{r} = \binom{n}{r}\binom{n-r}{M-r}$$



For any integer $n \geq 0$,

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}x^0 y^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Why is it that the binomial coefficients appear in this formula?

Look at some term in the middle

$$\binom{n}{r} x^{n-r} y^r$$

for some $0 \leq r \leq n$.

Well, $(x+y)^n = \underbrace{\underset{1}{(x+y)}\underset{2}{(x+y)}\underset{3}{(x+y)}\cdots\underset{n}{(x+y)}}_{*}$

Factor

* we obtain the term $x^{n-r}y^r$ if and only if we choose r-factors to take the y from and the remainin $n-r$ facters we took the 'x' from

SUBJECT: DATE: / / PAGE#:

for example:

$$(x+y)^4 = (x+y)_1\,(x+y)_2\,(x+y)_3\,(x+y)_4$$

$\binom{4}{2}$ ways to obtain xy^3

$\{1,2,3\}$ $\{1,3,4\}$

$\{1,2,4\}$ $\{2,3,4\}$

one can prove identities using the Binomial therm by choosing specific values of x and y

For instance, if $x=y=1$, $(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k$

$$\Longrightarrow 2^n = \sum_{k=0}^{n} \binom{n}{k}$$ *"k" counts the layers*

$\{1,2,3\}$

$\downarrow$

$\emptyset$

$\{1\}, \{2\}, \{3\}$ $\binom{n}{1}$

$\{1,2\}\{1,3\}, \{2,3\}$

$\{1,2,3\}$

If $x=1$ and $y=-1$

$$(1+(-1))^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} (-1)^k$$

$$\Longrightarrow 0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$$

$$\rightarrow \binom{4}{0} - \binom{4}{1} + \binom{4}{2} - \binom{4}{3} + \binom{4}{4}$$

$= 0$ (zero)

SUBJECT: DATE: / / PAGE#:

Example: Show that for all positive integers n,

$$\sum_{r=1}^{n} r\binom{n}{r} = n2^{n-1}$$

* Start with the binomial theorem *

Let $x = 1$ in the Binomial theorem

$$\rightarrow (1+y)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} y^k$$

* Power rule & Chain rule *

Power Rule

$$\boxed{\frac{d}{dy} y^k = k y^{k-1}}$$

$\frac{d}{dy}$ both sides to get

$$\rightarrow n(1+y)^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k y^{k-1}$$

Note: change in index

* Set $y=1$ to get

$$\rightarrow n \cdot 2^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k$$

#3 from HW 6

We want to know how many combination of students with the particular front row girls and back row boys, and the other students

1	G_2	2	3	G_3	4	5	6	G_1	7
8	B_1	9	B_2	10	11	B_3	12	13	B_4

G_1	G_2	G_3	1	2	3	4	5	6	7
B_3	8	B_4	9	B_2	10	B_1	11	12	13

We don't care about ordered positions of the individual students, so we use C^n_r not P^n_r

1st Front row →	G_1	G_2	G_3	1	2	3	4	5	6	7
2nd Back row →	8	9	10	11	12	13	B_1	B_2	B_3	B_4

Approach 01

Because the front row must always be populated by the girls we can pull any 1 of the 13 other students to fill the front 7 spots

G_1	G_2	G_3	1	2	3	4	5	6	7
8	9	10	11	12	13	B_1	B_2	B_3	B_4

$\binom{13}{7} \times \underbrace{10!}_{\text{1st Front row}} \times \underbrace{10!}_{\text{2nd Back row}}$

10 seats, so 10! seating arrangements

By default the back row gets chosen automatically

In total $\underbrace{\binom{13}{7}}_{1716} \times \underbrace{10!}_{3628800} \times \underbrace{10!}_{3628800} \cong 22{,}596{,}613{,}080{,}000{,}000$

$\binom{10}{3}3!\binom{10}{4}4!\,13!$

$\to \frac{10!}{3!7!}\cdot 3!\cdot \frac{10!}{4!6!}\cdot 4!\cdot 13! = \binom{13}{7}(10!)^2$

Approach 02

Back row choose the 4 boys

Config. of boys amongst themselves

G_1	G_2	G_3	1	2	3	4	5	6	7
8	9	10	11	12	13	B_1	B_2	B_3	B_4

$\binom{10}{3} \times 3! \times \binom{10}{4} \times 4! \times 13!$

Front row choose the 3 girls

Config. of girls amongst themselves

Placement of NPCs

In total $\binom{10}{3}\times 3!\times\binom{10}{4}\times 4!\times 13! \cong 22{,}596{,}613{,}080{,}000$

Ex) There are m 0's and n 1's where $m \geq 2$, $n \geq 2$, and $n \geq m$.

Q1) How many ways can the 0's and 1's be put in a line so that the 0's are all consecutive?

$n+1$ ways

Scratch work

$m=3,\ n=3$

000 111

$\boxed{\begin{matrix} 000111 & 110001 \\ 100011 & 111000 \end{matrix}}$

Q2) ... put in a line such that the first position and last position must be a zero

$\binom{m+n-2}{n}$

$0\ \underbrace{\quad\quad\quad}_{m+n-2}\ 0$

Q3) ... put in a line so that no two 0's are adjacent?

m 0's, n 1's

no adj 0s

$\_\,1\,\_\,1 \ldots \_\,1 \quad \binom{n+1}{m}$