Example

Let $A=\{a,b,c\}$

Arrangements & Selections with Repetitions are allowed

The 2-permutations of 'A' are: ab, ac, ba, bc, ca, cb

The 2-permutations allowed are aa, ab, ac, ba, bb, bc, ca, cb, cc

There are $3^2=9$ such 2-permutations

In general, if $A=\{1,2,3,\ldots,n\}$ then the number r-permutations of 'A' with repitition allowed is

$$n^r$$

*no need for $0 \leq r \leq n$ due to being able to have repitition

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with Repetitions are allowed *order matters!

Example | Find the number of 5-permutations of a, a, a, b, c

Approach 01: "Pretend" the a's are distinguisable

Let us, for a moment, pretend the a's are different

$a_1, a_2, a_3, b, c \rightarrow a_1, a_2, a_3, b, c$

$3!$

There are $5! = 120$ permutations of these 5 elements

some are

$123 \leftarrow a_1, b, a_2, a_3, c$

$213 \leftarrow a_2, c, a_1, a_3, b$

$321 \leftarrow a_3, b, a_2, a_1, c$

$\vdots$

There are $3!$ ways to permute the a_1, a_2, a_3 among themselves

$\frac{5!}{3!}$ permutations of a,a,a,b,c

Approach 02: Count the unique items

a, a, a, b, c

a a b c a, so $\binom{5}{2}2$

In general...

Suppose we have t-types of objects

r_1 of type 1

r_2 of type 2

$\vdots$

r_t of type 3

The number of permutations of all objects is

$$\frac{(r_1, r_2, r_3, \dots, r_t)!}{r_1!\, r_2!\, r_3! \dots r_t!}$$

Or $\binom{5}{2}2$ if we organize the b, c first and then the a's

b c a a a

the reason for the 2 in $\rightarrow \binom{5}{2}2$

1.6 Continued - Multisets

A multiset is a collection of unordered objects where repitition is allowed.

$\{a,b,c\}$ ← set and multiset

$\{a,b,c,c\}$ ← multiset, but NOT a set.

Notation for multisets:

$$\{a,a,b,b,b,b,c,c,c,\ldots\} = \{2\cdot a, 4\cdot b, \infty\cdot c\}$$

$$\{2\cdot a, 4\cdot b, \infty\cdot c\}$$

The number 2 is called the repetition number of 'a'
4 is the repetition number of 'b'
∞ is the repetition number of 'c'

Suppose we have a multiset

$$M=\{r_1\cdot a_1, r_2\cdot a_2, r_3\cdot a_3, \ldots, r_n\cdot a_n\}$$

permutating all elements of M would give a total of

$$\frac{(r_1+r_2+r_3+\ldots+r_n)!}{r_1!\,r_2!\,r_3!\cdots r_n!} \text{ permutations}$$

If we permute not all of the elements M, the counting is more tricky

Example Find the sequences of length 10 with two 0's, three 1's, five 2's

Two 0's
Three 1's
Five elements total

$$\frac{5!}{2!\,3!}$$

Two 0's
Three 1's

$$\frac{5!}{2!(5-2)!}$$

The total count is...

$$\frac{10!}{2!\,3!\,5!}$$

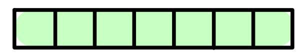
Example Find the number of ways to tile the rectangle

using blocks of size 1x1, 1x2, 1x3

We need to find the number of ways to write 7 as an ordered sum of 1, 2, and 3 where the terms can repeat

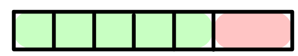
Case 01 all 1×1 blocks

$$7 = 1+1+1+1+1+1+1$$



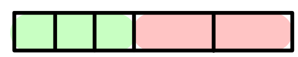
Case 02 Five 1's, One 2

$$\frac{6!}{5!} = 6$$



Case 03 Three 1's, Two 2's

$$\frac{5!}{3!\,2!} = 10$$

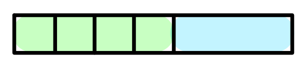


Case 04 One 1, Three 2's

$$\frac{4!}{3!} = 4$$

Case 05 Four 1's, One 3

$$\frac{5!}{4!} = 5$$

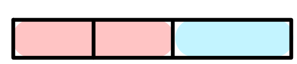


Case 06 One 1, Two 3's

$$\frac{3!}{2!} = 3$$

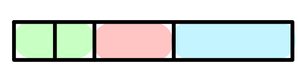
Case 07 Two 2's, One 3

$$\frac{3!}{2!} = 3$$



Case 08 Two 1's, One 2, One 3

$$\frac{4!}{2!} = 12$$



1.6 Continued - Multisets

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Recall

The number of r-permutations of the multiset $M=\{\infty\cdot a_1, \infty\cdot a_1, \infty\cdot a_1, \ldots, \infty\cdot a_1$, is n^r

If $M=\{r_1\cdot a_1 + r_2\cdot a_2 + r_3\cdot a_3 + \ldots + r_n\cdot a_n\}$

Then the number of permutations of M is

$$\frac{(r_1+r_2+r_3+\ldots+r_n)!}{r_1!\,r_2!\,r_3!\ldots r_n!}$$

Example: "the sandwich" - taking subsets of multisets
There are three types of sandwiches
turkey (T), ham (H), and butter lental tumeric (BLT)
How many ways can a person order 6 sandwiches?

Concept: Stars & Bars
If I can see it
I can count it

Look at examples of orders:

$n=3$
$r=6$

T	H	BLT	
**	*	***	← 2T's, 1H, 3BLT's
	**	****	← 0T, 2H's, 4BLT's
***		***	← 3T's, 0H, 3BLT's

2T's, 1H, 3BLT's → 00101000
0T, 2H's, 4BLT's → 10010000
3T's, 0H, 3BLT's → 00011000

So, $\binom{8}{2}$

Each order corresponds to 0-1 sequence with exactly $6+3+1=8$ positions and has exactly 6 zeros

So, $\binom{6+3-1}{6}=\binom{8}{6}$ or $\binom{6+3-1}{3-1}=\binom{8}{2}$

In general, if

$M=\{\infty\cdot a_1, \infty\cdot a_2, \infty\cdot a_3, \ldots, \infty\cdot a_n\}$

and H_r^n is the number of multisubsets of M with r-element is

$H_r^n=\binom{r+n-1}{r}$ OR $H_r^n=\binom{r+n-1}{n-1}$

Let $M=\{\infty\cdot a_1, \infty\cdot a_2, \ldots, \infty\cdot a_n\}$. The number H_r^n of r-element multi-subsets of M is given by

$$H_r^n=\binom{r+n-1}{r}.$$

Example: Solving integers with equations

$$\underbrace{x_1+x_2+x_3}=7$$

Find the number of solutions in non-negative integers

* Same stars and bars of the previous example*

We can think of a solution as a multiset

T	H	BLT	
***	**	**	← $3+2+2=7$
	***	****	← $0+3+4=7$

$$\binom{\overset{r}{7}+\overset{n}{3}-1}{\underset{n-1}{3-1}}=\binom{9}{2}$$

$\binom{9}{2}=36$ is the number of non-negative integer solutions to $x_1+x_2+x_3=7$

The Binomial Coefficient

$$H_r^n=\binom{r+n-1}{r}=\binom{r+n-1}{n-1}$$

The Binomial Coefficient counts a few things

→ Number of non-negative integer solutions to $x_1+x_2+x_3+\ldots+x_n=r$

→ Number of r-element multisubsets of $M=\{\infty\cdot a_1, \infty\cdot a_2, \infty\cdot a_3, \ldots, \infty\cdot a_n\}$

→ Number of ways to put r-identical objects into n-distinct boxes

Box-A	Box-B	Box-C
**	**	***

The Binomial Coefficient

$$H_r^n = \binom{r+n-1}{r} = \binom{r+n-1}{n-1}$$

The Binomial Coefficient counts a few things

→ Number of non-negative integer solutions to
$x_1 + x_2 + x_3 + \dots + x_n = r$

→ Number of r-element multisubsets of
$M = \{\infty \cdot a_1, \infty \cdot a_2, \infty \cdot a_3, \dots, \infty \cdot a_n\}$

→ Number of ways to put r-identical objects into n-distinct boxes

Box-A	Box-B	Box-C
* *	* *	* * *

Recall: Sterling Numbers of the first Kind
Given integers $0 \le n \le r$ let,
$s(r,n)$ be the number of ways to place 'r' distinct objects around 'n' indistinguisable tables where no tables are empty

Properties of $s(r,n)$: * Lower Case 's' as notation *

r - people r - table

- $s(r,1) = Q_r^r = (r-1)!$
- $s(r,r) = 1$
- $s(r,r-1) = \binom{r}{2}$

Pick a pair to sit together →

- $s(r,r-2) = \frac{(3n-1)}{4}\binom{n}{3}$

from the textbook

$s(r,0) = 0$	if $r \ge 1$
$s(r,r) = 1$	if $r \ge 0$
$s(r,1) = (r-1)!$	for $r \ge 2$
$s(r,r-1) = \binom{r}{2}$	for $r \ge 2$

Note: How So?
$s(r,r-2) = 2 \cdot \binom{n}{3} + \frac{1}{2}\binom{n}{2,2,n-4}$
$= \frac{(3n-1)}{4}\binom{n}{3}$

Claim

If $r \ge n$, then
$s(r,n) = s((r-1),n-1) + (r-1)s(r-1,n)$

WHY? Let $1,2,3,\dots,r$ be the people

*Focus on person 1

case 01: $s((r-1),n-1)$
Person 1 is at own table
$r-1$ people
$n-1$ table(s)
$s(r-1,n-1)$ Ways to complete

case 02: $(r-1)s(r-1,n)$
Person 1 is at not own table
Place all but person #1
$r-1$ people
1 table
Choices for who sits to left of
$s(r-1,n)$ Ways to complete

Stirling Numbers of the Second Kind

* Upper Case 'S' as notation *

$S(r,n)$ = Number of ways to put r-distinct objects into n-identical boxes where no boxes are empty

A B C	D E F

Example: Let $r=4$ & try to find $S(4,n)$ for $n=1,2,3,4$

Set will be $x = \{1,2,3,4\}$

$n=1$
$S(4,1) = 1$

1 2 3 4	

$n=2$
$S(4,2) = 7$

1	2 3 4
2	1 3 4
3	1 2 4
4	1 2 3
1,2	3,4
1,3	2,4
1,4	2,3

$n=3$
$S(4,3) = 6$

1	2	3,4
1	3	2,4
1	4	2,3
2	3	1,4
2	4	1,3
3	4	1,2

$n=4$
$S(4,4) = 1$

1	2	3	4

$n \ge 5$
$S(5,3) = 0$

$1+7+6+1 = 15$
the number of partitions of $\{1,2,3,4\}$ = 4^{th} Bell Number

$$S(4,1) + S(4,2) + S(4,3) + S(4,4) = B_4$$

The N^{th} Bell Number is given by the formula

$$B_n = \frac{1}{e}\sum_{k=0}^{\infty}\frac{k^n}{k!}$$

Stirling Numbers of the Second Kind Properties

(i) $S(0,0) = 1$,
(ii) $S(r,0) = S(0,n) = 0$ for all $r,n \in \mathbf{N}$,
(iii) $S(r,n) > 0$ if $r \ge n \ge 1$,
(iv) $S(r,n) = 0$ if $n > r \ge 1$,
(v) $S(r,1) = 1$ for $r \ge 1$,
(vi) $S(r,r) = 1$ for $r \ge 1$.
(vii) $S(r,2) = 2^{r-1} - 1$,
(viii) $S(r,3) = \frac{1}{2}(3^{r-1}+1) - 2^{r-1}$,
(ix) $S(r,r-1) = \binom{r}{2}$,
(x) $S(r,r-2) = \binom{r}{3} + 3\binom{r}{4}$.