California State University Sacramento - Math 101

Homework Assignment 2 - Solutions

1) (i) If a is the smaller of the two numbers, then $b = a + 5$. In order to have $b \leq 50$, we can choose a to be any number in the set $\{1, 2, \ldots, 45\}$. Once we have chosen a, the equation $b = a + 5$ determines b and so there are 45 pairs.

(ii) If a is the smaller of the two numbers, then b must be one of $a+1$, $a+2$, $a+3$, $a+4$, or $a+5$. Therefore, for any $a \in \{1, 2, \ldots, 45\}$ there are 5 choices for b. However, if $a = 46$, then we must have $b \in \{47, 48, 49, 50\}$ so that there are only four choices for b. Similarly, if $a = 47$, then $b \in \{48, 49, 50\}$ and there are three choices for b. If $a = 48$, then $b \in \{49, 50\}$ and there are two choices for b. Lastly, if $a = 49$, then $b \in \{50\}$ and there is only one choice for b. By the Addition Principle, we find that there are $45 \cdot 5 + 4 + 3 + 2 + 1 = 235$ such pairs $\{a, b\}$.

2) A divisor of $10^{40} = 2^{40}5^{40}$ is of the form 2^a5^b for some $0 \leq a \leq 40$ and $0 \leq b \leq 40$. A divisor of $20^{30} = 2^{60}5^{30}$ is of the form 2^c5^d for some $0 \leq c \leq 60$ and $0 \leq d \leq 30$. Therefore, a common divisor of 10^{40} and 20^{30} is of the form 2^x5^y where $0 \leq x \leq 40$ and $0 \leq y \leq 30$. There are $41 \cdot 31 = 1271$ common positive divisors of 10^{40} and 20^{30}.

3) (i) Since $210 = 2 \cdot 3 \cdot 5 \cdot 7$, a positive divisor of 210 that is also divisible by 3 is of the form $2^a3 \cdot 5^b7^c$ where $0 \leq a \leq 1$, $0 \leq b \leq 1$, and $0 \leq c \leq 1$. There are 2 choices for a, 2 choices for b, and 2 choices for c. This gives a total of $2^3 = 8$ positive divisors of 210 that are also multiples of 3.

(ii) A positive divisor of $630 = 2 \cdot 3^2 \cdot 5 \cdot 7$ that is also a multiple of 3 must be of the form

$$2^a3^b5^c7^d$$

where $a \in \{0, 1\}$, $b \in \{1, 2\}$, $c \in \{0, 1\}$, and $d = \{0, 1\}$. By the Multiplication Principle, there are $2 \cdot 2 \cdot 2 \cdot 2 = 16$ positive divisors of 630 that are multiples of 3.

(iii) An argument similar to that of parts (i) and (ii) gives that $151200 = 2^53^35^27$ has

$$6 \cdot 3 \cdot 3 \cdot 2$$

positive divisors that are also divisible by 3.

4) Suppose $n = p_1^{k_1}p_2^{k_2} \cdots p_r^{k_r}$ is the factorization of n into distinct primes where the k_i's are positive integers. Then

$$n^2 = p_1^{2k_1}p_2^{2k_2} \cdots p_r^{2k_r}$$

and so n^2 has $(2k_1 + 1)(2k_2 + 1) \cdots (2k_r + 1)$ positive divisors. The number

$$(2k_1 + 1)(2k_2 + 1) \cdots (2k_r + 1)$$

is odd since it is the product of r odd numbers.

5) If $x^2 + y^2 = 0$, then $(x, y) \in \{(0, 0)\}$. If $x^2 + y^2 = 1$, then

$$(x, y) \in \{(1, 0), (0, 1), (-1, 0), (0, -1)\}.$$

If $x^2 + y^2 = 2$, then

$$(x, y) \in \{(1, 1), (1, -1), (-1, 1), (-1, -1)\}.$$

It is not too hard to see that there are no integers x and y for which $x^2 + y^2 = 3$. If $x^2 + y^2 = 4$, then

$$(x, y) \in \{(2, 0), (-2, 0), (0, 2), (0, -2)\}.$$

This gives a total of $1 + 4 + 4 + 4 = 13$ pairs of integers (x, y) with $x^2 + y^2 \leq 4$.

6) There are 5 choices for a_1, 5 choices for a_2, and 5 choices for a_3. Therefore, there is a total of 5^3 sequences of the form $a_1a_2a_3$ with $a_i \in \{0, 1, 2, 3, 4\}$.

7) If $a = 1$, then $b, c \in \{2, 3, \ldots, 9, 10\}$ so there are 9^2 possible choices for b and c. If $a = 2$, then $b, c \in \{3, 4, \ldots, 9, 10\}$ so there are 8^2 possible choices for b and c in this case. Continuing in this fasion, we find that there are

$$9^2 + 8^2 + 7^2 + 6^2 + 5^2 + 4^2 + 3^2 + 2^2 + 1^2 = 285$$

such triples in S so $|S| = 285$.