California State University Sacramento - Math 101

Exam #3

Name: ____________________

This exam is out of 7 points.

1) (a) Determine the exact value of the coefficient of x^3 in the expansion of $(x+1)^{13}$. (0.5 points)

(b) Determine the exact value of the coefficient of $x^3y^4z^2$ in the expansion of $(x+y+z)^9$. (0.5 points)

(a) $\binom{13}{3} = \frac{13!}{3!\,10!} = \frac{13\cdot 12\cdot 11}{3\cdot 2} = 13\cdot 2\cdot 11 = 2\cdot 143 = 286$

(b) $\binom{9}{3,4,2} = \frac{9!}{3!\,4!\,2!} = \frac{9\cdot 8\cdot 7\cdot 6\cdot 5}{3\cdot 2\cdot 2} = 9\cdot 2\cdot 7\cdot 2\cdot 5$

$= 63\cdot 10\cdot 2$

$= 630\cdot 2 = 1260$

2) In the Binomial Theorem $(x+y)^n = \sum_{r=0}^{n}\binom{n}{r}x^r y^{n-r}$, state a value for x and value for y that produces the given formula. (0.25 points each)

(a) $0 = \sum_{r=0}^{n}\binom{n}{r}(-1)^r$

$X = -1$

$Y = 1$

(b) $2^n = \sum_{r=0}^{n}\binom{n}{r}$

$X = 1$

$Y = 1$

3) Show that for all integers $n \geq m \geq 1$, $\sum_{r=m}^{n} \binom{n-m}{n-r} = 2^{n-m}$. (0.5 points)

$$\sum_{r=m}^{n} \binom{n-m}{n-r} = \binom{n-m}{n-m} + \binom{n-m}{n-(m+1)} + \binom{n-m}{n-(m+2)} + \dots + \binom{n-m}{n-n}$$

$$= \binom{n-m}{n-m} + \binom{n-m}{n-m-1} + \binom{n-m}{n-m-2} + \dots + \binom{n-m}{0}$$

$$= 2^{n-m}$$

using $\sum_{k=0}^{N} \binom{N}{k} = 2^N$

4) Prove $\sum_{r=0}^{n} r\binom{n}{r} = n2^{n-1}$ for all $n \geq 1$. (0.75 points)

By the Binomial Theorem,

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}.$$

Taking $y=1$ and differentiating with respect to x gives

$$n(x+1)^{n-1} = \sum_{r=0}^{n} \binom{n}{r} r x^{r-1}$$

Let $x=1$ to get

$$n \cdot 2^{n-1} = \sum_{r=0}^{n} r\binom{n}{r}$$

5) Prove that $\binom{n}{r}\binom{n-r}{m-r} = \binom{n}{m}\binom{m}{r}$ for all integers $n \geq m \geq r \geq 1$. (0.75 points)

$$\binom{n}{r}\binom{n-r}{m-r} = \frac{n!}{r!\,(n-r)!} \cdot \frac{(n-r)!}{(m-r)!\,(n-r-(m-r))!}$$

$$= \frac{n!}{r!} \cdot \frac{1}{(m-r)!\,(n-m)!}$$

$$= \frac{n!}{(n-m)!} \cdot \frac{1}{r!\,(m-r)!}$$

multiply by $1 = \frac{m!}{m!}$

$$= \frac{n!}{m!\,(n-m)!} \cdot \frac{m!}{r!\,(m-r)!}$$

$$= \binom{n}{m}\binom{m}{r}$$

6) Let $X = \{1, 2, 3, 4\}$. Let $\mathcal{A}$ be the collection of all subsets of X with an even number of elements, and let $\mathcal{B}$ be the collection of all subsets of X with an odd number of elements. Remark: The empty set $\emptyset$ is one of the sets in $\mathcal{A}$ since it has 0 elements and 0 is even.

(a) List all of the elements of $\mathcal{A}$. (0.5 points)

(b) List all of the elements of $\mathcal{B}$. (0.5 points)

(c) Is the function $f(C) = X \backslash C$ a bijection from $\mathcal{A}$ to $\mathcal{B}$? Recall that $X \backslash C$ is the complement of C in X. (0.25 points)

(d) Is the function $f(C) = C \cup \{1\}$ a bijection from $\mathcal{A}$ to $\mathcal{B}$? (0.25 points)

(e) Draw a bijection between $\mathcal{A}$ and $\mathcal{B}$. Represent your bijection using an arrow diagram. (0.25 points)

(a) $\mathcal{A} = \{ \emptyset, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3,4\} \}$

(b) $\mathcal{B} = \{ \{1\}, \{2\}, \{3\}, \{4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\} \}$

both in $\mathcal{A}$

(c) No. $f(\{1,2\}) = \{3,4\}$ so f is not even a function from $\mathcal{A}$ to $\mathcal{B}$

(d) No. The same reasoning applies.

$f(\{1,2\}) = \{1,2\} \cup \{1\} = \{1,2\}$

both in $\mathcal{A}$

(e)

$\mathcal{A}$ $\quad$ $\mathcal{B}$

$\emptyset \to \{1\}$

$\{1,2\} \to \{2\}$

$\{1,3\} \to \{3\}$

$\{1,4\} \to \{4\}$

$\{2,3\} \to \{1,2,3\}$

$\{2,4\} \to \{1,2,4\}$

$\{3,4\} \to \{1,3,4\}$

$\{1,2,3,4\} \to \{2,3,4\}$

7) (a) There are 50 jobs that must be assigned to 7 processors. Explain why there must be a processor that is assigned at least 8 jobs. (0.5 points)

(b) A list $\mathcal{L}$ contains 134 elements. Each element of $\mathcal{L}$ is a subset of $\{1, 2, 3, 4, 5, 6, 7\}$. Explain why the list $\mathcal{L}$ must contain two elements that are the same. (0.5 points)

(c) Suppose it takes a program 1 second to find the determinant of a 2×2 matrix. If S is the set of all 2×2 matrices of the form $\begin{pmatrix} x & y \\ z & t \end{pmatrix}$ where $x, y, z, t \in \{0, 1\}$, what is the minimum amount of time it would take the program to find the determinant of every matrix in S? (0.25 points)

(a) The average number of jobs assigned to a processer is $\frac{50}{7} > 7$ so there must be some processar that is assigned at least 8 jobs.

(b) The number of subsets of $\{1, 2, 3, \ldots, 7\}$ is $2^7 = 128$.

Since $\mathcal{L}$ has 134 elements, the list $\mathcal{L}$ must contain two subsets that are the same.

(c) $\begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}$ There are $2^4 = 16$ such matrices so 16 seconds

0 or 1

8) Prove that $\sum_{r=0}^{n-1} \binom{2n-1}{r} = 2^{2n-2}$. (0.5 points)

We know

$$\sum_{r=0}^{2n-1} \binom{2n-1}{r} = 2^{2n-1}$$

using

$\binom{N}{M} = \binom{N}{M-N}$

for $N \geq M \geq 1$

So that

$$\sum_{r=0}^{n-1} \binom{2n-1}{r} + \sum_{r=n}^{2n-1} \binom{2n-1}{r} = 2^{2n-1}$$

$$\sum_{r=0}^{n-1} \binom{2n-1}{r} + \sum_{r=n}^{2n-1} \binom{2n-1}{2n-1-r} = 2^{2n-1}$$

$\star$

$$\sum_{r=0}^{n-1} \binom{2n-1}{r} + \sum_{r=0}^{n-1} \binom{2n-1}{r} = 2^{2n-1}$$

$$2 \sum_{r=0}^{n-1} \binom{2n-1}{r} = 2^{2n-1}$$

$$\sum_{r=0}^{n-1} \binom{2n-1}{r} = 2^{2n-2}$$

To see $\star$, note

$$\sum_{r=n}^{2n-1} \binom{2n-1}{2n-1-r} = \binom{2n-1}{n-1} + \binom{2n-1}{n-2} + \dots + \binom{2n-1}{0}$$

$$= \sum_{r=0}^{n-1} \binom{2n-1}{r}$$