For any integer $n \geq 0$,

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n}x^0 y^n$$
$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Why is it that the Binomial Coefficients appear in this formula?

Look at some terms in the middle

$$\binom{n}{r} x^{n-r} y^r$$

for some $0 \leq r \leq n$.

Well, $\underset{\text{Factor}}{(x+y)^n} = \underbrace{\underset{1}{(x+y)}\underset{2}{(x+y)}\underset{3}{(x+y)}\ldots\underset{n}{(x+y)}}$

we obtain the term $x^{n-r}y^r$ if and only if we choose r-factor to take the 'y' from and the remaining r-n-factors we took the 'x'-form

Example 01 $(x+y)^4 = \underset{1}{(x+y)}\underset{2}{(x+y)}\underset{3}{(x+y)}\underset{4}{(x+y)}$

$\binom{4}{2}$ ways to obtain xy^3

$\underset{1}{\{1,2,3\}}$ $\underset{2}{\{1,3,4\}}$ $\underset{3}{\{1,2,4\}}$ $\underset{4}{\{2,3,4\}}$

One can prove identites using the Binomial Theorem by choosing specific values of 'x' and 'y'

For instance, if $x=y=1$, $(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k$

$$\Rightarrow 2^n = \sum_{k=0}^{n} \binom{n}{k}$$

* $\binom{n}{k}$ "k" counts the layers *

	$\{1,2,3\}$	
	$\varnothing$	NULL SET
	$\{1\},\{2\},\{3\}$	$\binom{n}{1}$
	$\{1,2\},\{1,3\},\{2,3\}$	$\binom{n}{2}$
	$\{1,2,3\}$	$\binom{n}{3}$

If $x=1$ and $y=-1$

$$(1+(-1))^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} (-1)^k$$
$$\rightarrow 0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$$
$$\rightarrow \binom{4}{0} - \binom{4}{1} + \binom{4}{2} - \binom{4}{3} + \binom{4}{4}$$
$$= 0 \text{ (zero)}$$

Example 02

Show that for all positive integers 'n',

$$\sum_{r=1}^{n} r\binom{n}{r} = n2^{n-1}$$

Start with the Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Let $x=1$ in the Binomial Theorem

$$\rightarrow (1+y)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} y^k$$

Power Rule and Chain Rule $\frac{d}{dy}$ both sides to get

Power Rule

$$\frac{d}{dy} y^k = k y^{k-1}$$

$$\rightarrow (1+y)^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k y^{k-1}$$

Note: Change index

*Set $y=1$ to get...

$$\rightarrow n \cdot 2^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k$$

Recall

Power rule (x^n): $\{x^n\}' = nx^{n-1}$

THE CHAIN RULE

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Calculus 1 Lecture Notes

Theorem.

Theorem 2.64 (*The Chain Rule*). Let f and g be differentiable functions. Then

$$(f(g(x)))' = f'(g(x)) \cdot g'(x).$$

In words: *Take the derivative of the outside, keep everything inside the same, and then multiply by the derivative of the inside.*