

Homework Assignment 3

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- 2) Example 1.2.2 on page 7. **Q2)****Example 1.2.2.** Let $E = \{a, b, c, \dots, x, y, z\}$ be the set of the 26 English alphabets. Find the number of 5-letter words that can be formed from E such that the first and last letters are distinct vowels and the remaining three are distinct consonants.
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Q3)**Example 1.2.3.** There are 7 boys and 3 girls in a gathering. In how many ways can they be arranged in a row so that

- (i) the 3 girls form a single block (i.e. there is no boy between any two of the girls)?
- (ii) the two end-positions are occupied by boys and no girls are adjacent?

Q4)**Example 1.2.4.** Between 20000 and 70000, find the number of even integers in which no digit is repeated.

Q5) 4. How many 5-letter words can be formed using $A, B, C, D, E, F, G, H, I, J$,

- (i) if the letters in each word must be distinct?
- (ii) if, in addition, A, B, C, D, E, F can only occur as the first, third or fifth letters while the rest as the second or fourth letters?

Q6) 2. There are 12 students in a party. Five of them are girls. In how many ways can these 12 students be arranged in a row if

- (i) there are no restrictions?
- (ii) the 5 girls must be together (forming a block)?

Q7) 14. Let $n, r \in \mathbf{N}$ with $r \leq n$. Prove each of the following identities:

- (i) $P_r^n = n P_{r-1}^{n-1}$,
- (ii) $P_r^n = (n - r + 1) P_{r-1}^n$,
- (iii) $P_r^n = \frac{n}{n-r} P_r^{n-1}$, where $r < n$,
- (iv) $P_r^{n+1} = P_r^n + r P_{r-1}^n$,
- (v) $P_r^{n+1} = r! + r(P_{r-1}^n + P_{r-1}^{n-1} + \dots + P_{r-1}^r)$.

1) Example 1.2.1 on page 6.

Let $A = \{a, b, c, d\}$.

Find all the 3-permutations of set A.

Note

When finding permutations we use formula.

$$P_r^n = \frac{n!}{(n-r)!}$$

Where 'n' is the set or population and 'r' is the subset of 'n' or sample set

* n is the cardinality of set A, so n=4

* r is the subset of A where have sets of length 3, so r=3

$$\text{So... } P_3^4 = \frac{4!}{(4-3)!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{1} = 24$$

All together, we have 24 subsets permutations of set A's elements in sets of length 3.

2) Example 1.2.2 on page 7.

Let E = {a, b, c, ...x, y, z} be the set of 26 English alphabets.

Find the number of 5-letter words that can be formed from E such that the first and last letters are distinct vowels and the remaining three are distinct consonants.

The english alphabet consist of 5 vowels (a,e,i,o,u) and 21 consonants

- We want permutations of 2 vowels from the "vowel set" (subset of E)
- We want permutations of 3 consonants from the "consonants set" (subset of E)
- Pattern → Vowel | consonant | consonant | consonant | vowel
- Finding Permutations: $P_r^n = \frac{n!}{(n-r)!}$

$$P_2^5 = \frac{5!}{(5-2)!} \rightarrow \frac{5 \cdot 4 \cdot 3 \cdot 2}{3 \cdot 2} = \frac{120}{6} = 20 \text{ vowel permutations}$$

$$P_3^{21} = \frac{21!}{(21-3)!} \rightarrow \frac{51,090,942,171,709,440,000}{6,402,373,705,728,000} = 7980 \text{ consonant permutations}$$

So, for every distinct vowel pairings we have a distinct 3 character tuple

$$P_2^5 \times P_3^{21} = [(20)(7980)] = 159,600 \text{ unique words can be generated}$$

3) Example 1.2.3 on page 8.

There are 7 boys and 3 girls in a gathering.

In how many ways can they be arranged in a row so that:

(i) The 3 girls from a single block

- no boy between any two of the girls

$$B_1 B_2 B_3 B_4 G_1 G_2 G_3 B_5 B_6 B_7 \rightarrow B_1 B_2 B_3 B_4 \underbrace{\text{Girls Only}}_{I \text{ unit}} B_5 B_6 B_7$$

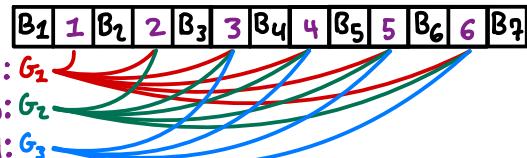
• Block of girls can be counted 1 unit

• Girls amongst themselves can be arranged 3!

So, instead of 10 independent seating the is 7 single and 1 group seat

$$(7+1)! \times 3! = 241,920$$

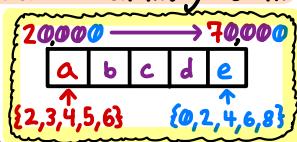
(3.ii) The two end-positions are occupied by boys and no girls are adjacent



The boys have $7!$ ways to seat themselves. G_1 has 6 possibilities, G_2 has 5 possibilities, and G_3 has 4 possibilities.
So we have $7! \cdot 6 \cdot 5 \cdot 4 = 604,800$ ways

4) Example 1.2.4 on page 9.

Between 20000 and 70000, find the number of even integers in which no digit is repeated.



Case 01: first digit is even

- 3 choices for first digit: {2,4,6}
- To account for 'a' being 1 of the five in set {0,2,4,6,8}; Leaves us with 4 choices

'bcd' can range from [0,9]. But we need to factor for leading & tail duplicates

So for case 01: $3 \times P_3^{(10-2)} \times 4 \rightarrow 3 \times P_3^8 \times 4 = 4032$

Case 02: leading digit is odd

- 'bcd' can range from [0,9]. But we need to account for leading & tail duplicates
- 2 choices for first digit: {3,5}
- Don't have to worry 'a' = 'e'

So for case 02: $2 \times P_3^{(10-2)} \times 5 \rightarrow 2 \times P_3^8 \times 5 = 5040$ All together we have $4032 + 5040 = 9072$

5) Problem 4 on page 50.

How many 5-letter words can be formed using A,B,C,D,E,F,G,H,I,J

(i) If the letters in each word must be distinct

$$P_5^{10} \rightarrow \frac{10!}{(10-5)!} = 30,240$$

(ii) If, in addition, A,B,C,D,E,F can only occur as the 1st, 3rd, or 5th letters & the rest as 2nd or 4th

$$A, B, C, D, E, F, G, H, I, J : \boxed{1 \ 2 \ 3 \ 4 \ 5}$$

$$P_6^6 \times P_4^4 \rightarrow \frac{(6!)}{(6-3)!} \times \frac{(4!)}{(4-2)!} = 1440$$

See Back

6) Problem 2(i) and 2(ii) on page 50.

There are 12 students in a party. Five of them are girls.

In how many ways can these 12 students be arranged if...

(i) there are no restrictions?

If no restrictions then,

$12!$, there are $\sim 4.79 \times 10^8$ possible ways

(ii) the 5 girls must be together (forming Voltron)?

* Voltron can be counted as 1

* The pilots of Voltron can switch lions

So... $(7+1)!5!$

$$\rightarrow 8!5! = 4,838,400$$

Possible ways



7) Problem 14(i), 14(ii), 14(iii), and 14(iv) on page 51.

14. Let $n, r \in \mathbb{N}$ with $r \leq n$. Prove each of the following identities:

$$(i) P_r^n = nP_{r-1}^{n-1},$$

$$(ii) P_r^n = (n-r+1)P_{r-1}^{n-1},$$

$$(iii) P_r^n = \frac{n}{n-r} P_r^{n-1}, \text{ where } r < n,$$

$$(iv) P_r^{n+1} = P_r^n + rP_{r-1}^n,$$

$$(i) P_r^n = nP_{r-1}^{n-1} \quad (ii) P_r^n = (n-r+1)P_{r-1}^{n-1} \quad (iii) P_r^n = \frac{n}{n-r} P_r^{n-1}, \text{ where } r < n$$

$$P_3^3 \stackrel{?}{=} 3 \cdot P_{3-1}^{3-1}$$

$$\rightarrow \frac{3!}{3!} \stackrel{?}{=} 3 \left(\frac{2!}{2!} \right)$$

$$\rightarrow 6 \stackrel{?}{=} (2 \cdot 3) \rightarrow 6$$

$$\text{LHS} = \text{RHS}$$

∴ True

$$P_3^3 \stackrel{?}{=} (3-3+1) P_{3-1}^3$$

$$\rightarrow 6 \stackrel{?}{=} P_{3-1}^3$$

$$\rightarrow 6 \stackrel{?}{=} \frac{3!}{(3-2)!}$$

$$\rightarrow 6 \stackrel{?}{=} 6$$

$$\text{LHS} = \text{RHS}$$

∴ True

$$\text{RHS} : \frac{n}{n-r} P_r^{n-1}$$

→

$$(iv) P_r^{n+1} = P_r^n + rP_{r-1}^n$$

RHS

$$\rightarrow \frac{n!}{(n-r)!} + r \frac{n!}{(n-r+1)!}$$

$$\rightarrow P_r^n + rP_{r-1}^n \rightarrow \frac{(n-r+1)}{(n-r+1)} \cdot \frac{n!}{(n-r)!} + r \frac{n!}{(n-r+1)!}$$

$$\rightarrow \frac{n!(n-r+1+r)}{(n-r+1)!}$$

$$\rightarrow \frac{(n+1)!}{(n-r+1)!} = P_r^{(n+1)}$$

Recall

$$P_r^n = \frac{n!}{(n-r)!}$$

$$\rightarrow \frac{(n-r+1)n!}{(n-r+1)!} + \frac{r \cdot n!}{(n-r+1)!}$$

$$\rightarrow \frac{(n-r+1)n! + r \cdot n!}{(n-r+1)!} = P_r^{(n+1)}$$

$$\text{LHS} = \text{RHS}$$

$$P_r^{n+1} = P_r^{n+1} \therefore \text{True}$$

14. Let $n, r \in \mathbb{N}$ with $r \leq n$. Prove each of the following identities:

- (i) $P_r^n = nP_{r-1}^{n-1}$,
- (ii) $P_r^n = (n - r + 1)P_{r-1}^n$,
- (iii) $P_r^n = \frac{n}{n-r} P_r^{n-1}$, where $r < n$,
- (iv) $P_r^{n+1} = P_r^n + rP_{r-1}^n$,

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1) Example 1.2.1 on page 6.

Example 1.2.1. Let $A = \{a, b, c, d\}$. All the 3-permutations of A are shown below:

$$\begin{array}{cccccc} abc, & acb, & bac, & bca, & cab, & cba, \\ abd, & adb, & bad, & bda, & dab, & dba, \\ acd, & adc, & cad, & cda, & dac, & dca, \\ bcd, & bdc, & cbd, & cdb, & dbc, & dcba. \end{array}$$

There are altogether 24 in number. ■

Chapter 1. Permutations and Combinations

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Let P_r^n denote the number of r -permutations of A . Thus $P_3^4 = 24$ as shown in Example 1.2.1. In what follows, we shall derive a formula for P_r^n by applying (MP).

An r -permutation of A can be formed in r steps, as described below: First, we choose an object from A and put it in the first position (see Figure 1.2.1). Next we choose an object from the remaining ones in A and put it in the second position. We proceed on until the r th-step in which we choose an object from the remaining $(n - r + 1)$ elements in A and put it in the r th-position.

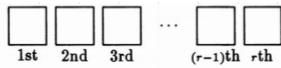


Figure 1.2.1

There are n choices in step 1, $(n - 1)$ choices in step 2, ..., $n - (r - 1)$ choices in step r . Thus by (MP),

$$P_r^n = n(n - 1)(n - 2) \cdots (n - r + 1). \quad (1.2.1)$$

If we use the factorial notation: $n! = n(n - 1) \cdots 2 \cdot 1$, then

$$P_r^n = \frac{n!}{(n - r)!}. \quad (1.2.2)$$

Remark. By convention, $0! = 1$. Note that $P_0^n = 1$ and $P_n^n = n!$.

2) Example 1.2.2 on page 7.

Example 1.2.2. Let $E = \{a, b, c, \dots, x, y, z\}$ be the set of the 26 English alphabets. Find the number of 5-letter words that can be formed from E such that the first and last letters are distinct vowels and the remaining three are distinct consonants.

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Section 1.2. Permutations

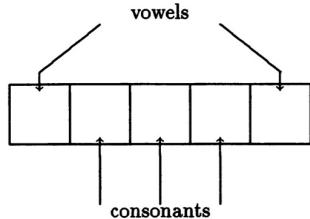


Figure 1.2.2.

Solution. There are 5 vowels and 21 consonants in E . A required 5-letter word can be formed in the following way.

Step 1. Choose a 2-permutation of $\{a, e, i, o, u\}$ and then put the first vowel in the 1st position and the second vowel in the 5th position (see Figure 1.2.2).

Step 2. Choose a 3-permutation of $E \setminus \{a, e, i, o, u\}$ and put the 1st, 2nd and 3rd consonants of the permutation in the 2nd, 3rd and 4th positions respectively (see Figure 1.2.2).

There are P_2^5 choices in Step 1 and P_3^{21} choices in Step 2. Thus by (MP), the number of such 5-letter words is given by

$$P_2^5 \times P_3^{21} = (5 \times 4) \times (21 \times 20 \times 19) = 159600. \quad \blacksquare$$

Example 1.2.3. There are 7 boys and 3 girls in a gathering. In how many ways can they be arranged in a row so that

- (i) the 3 girls form a single block (i.e. there is no boy between any two of the girls)?
- (ii) the two end-positions are occupied by boys and no girls are adjacent?

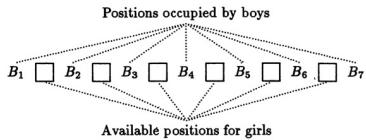
Solution. (i) Since the 3 girls must be together, we can treat them as a single entity. The number of ways to arrange 7 boys together with this entity is $(7 + 1)!$. As the girls can permute among themselves within the entity in $3!$ ways, the desired number of ways is, by (MP),

$$8! \times 3!$$

Chapter 1. Permutations and Combinations

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(ii) We first consider the arrangements of boys and then those of girls. There are $7!$ ways to arrange the boys. Fix an arbitrary one of the arrangements. Since the end-positions are occupied by boys, there are only 6 spaces available for the 3 girls G_1, G_2 and G_3 .



G_1 has 6 choices. Since no two girls are adjacent, G_2 has 5 choices and G_3 has 4. Thus by (MP), the number of such arrangements is

$$7! \times 6 \times 5 \times 4. \quad \blacksquare$$

Remark. Example 1.2.3 can also be solved by considering the arrangements for the girls first. This will be discussed in Example 1.7.2.