On the day of your exam, I will choose 4 of these 6 problems. You will have to write on 3 of the 4 chosen problems.

1. Prove that there are infinitely many prime numbers, and prove that $\sqrt{2}$ is irrational.

* Source: Presentation 5 - Number Theory

2. Write a few paragraphs on the history of prime numbers. Also, state Goldbach's conjecture, the twin prime conjecture, and the prime number theorem.

3. Discuss how al-Khwarizmi would have solved $x^2 + 10x = 39$. Include pictures. (The numbers may change on exam day.)

4. Discuss Archimedes. Include a discussion on the method of exhaustion.

5. Discuss one of Zeno's paradoxes, and explain Madhava of Sangamagrama's result that proves that Zeno's logic was flawed.

6. Pick one of the mathematicians below. Write what we know about them (or at least, what we think we know) and about their place in math history.

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Infinitely many prime numbers

Proposition IX.20 from Euclid's Elements
"There are infinitely many primes"

Proof

Given any finite collection of primes, consider $N = p_1 p_2 p_3 \cdots p_k$. Now consider $N+1$

Case 1

$N+1$ is prime. Then it must be a new prime because it is larger than all p_i

Case 2

$N+1$ is composite. Then it must by Proposition VII.31, it is divisible by a prime p
This must be a new prime since if it divides $N+1$ and N,
then it would also divide $(N+1) - N = 1$, but no primes divide 1

Thus, no collection of primes is complete, implying there are infinitely many primes.

Prove that $\sqrt{2}$ is irrational

Hippasus Proof

Assume for a contradiction that $\sqrt{2}$ is rational. That is, suppose that $\sqrt{2} = \frac{b}{c}$ for some non-zero integers b and c,

Written in lowest terms (i.e. $gdc(b,c) = 1$, that b and c does not have common factors... b and c cannot be even)

Then,

$\sqrt{2}c = b$

$\rightarrow 2c^2 = b^2$

$\rightarrow \quad 2|b^2$

$\rightarrow \quad \boxed{2|b}$

$\rightarrow \quad b = 2k$ for some integer k

Thus,

$2c^2 = (2k)^2$

$\rightarrow 2c^2 = 4k^2$

$\rightarrow \quad c^2 = 2k^2$

$\rightarrow \quad 2|c^2$

$\rightarrow \quad \boxed{2|c}$

Contradiction!
b and c are even. They have common factors of 2.
This contradicts our original assumption that
b/c was in its lowest terms which meant that
b and c didn't have any common factors; as a result,
$\sqrt{2}$ is irrational.

We assumed $\sqrt{2} = \dfrac{b}{c}$ was written in lowest terms, and then proved it can't be.

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2. Write a few paragraphs on the history of prime numbers. Also, state Goldbach's conjecture, the twin prime conjecture, and the prime number theorem.

The earliest systematic proof goes as far back to the 3rd century BCE found in Euclid's Elements. In his text Euclid provided the first known proof of the infinite nature of prime numbers, setting the stage for others to explore their properties and patterns.

Euclid, in his "Elements", links prime numbers to Perfect Numbers, a positive integer that is equal to the sum of its proper divisors, and is expressed in the form $2^{n-1}(2^n-1)$.

$x^p - x$ is a multiple of p

Thanks to Fermat's Little theorem ($\text{intx}^{\text{prime}} - \text{intx}$ is a multiple of prime) → our digital world is secure with prime numbers that are hard to solve, but easy to check

State Goldbach's conjecture

Euler's reply pointed out this conjecture:

"Every positive even integer can be written as the sum of two primes."

The twin prime conjecture

Definition: Twin primes are primes that differ by 2. Examples: (3,5), (5,7), (11,13), ...

The twin prime conjecture states that: there are infinitely many twin primes.

The prime number theorem

$$\lim_{N\to\infty} \frac{\pi(N)}{\left(\frac{N}{\log(N)}\right)} = 1$$

- Euclid proved there are infinitely many primes
- Let $\pi(N)$ be the number of primes between 1 and N

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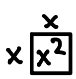
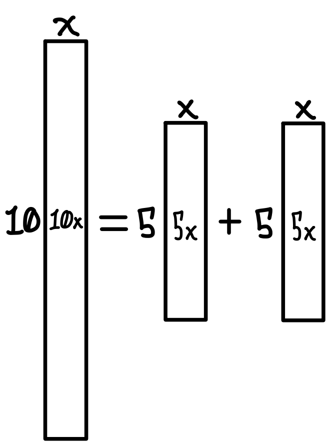
3. Discuss how al-Khwarizmi would have solved $x^2 + 10x = 39$. Include pictures. (The numbers may change on exam day.)

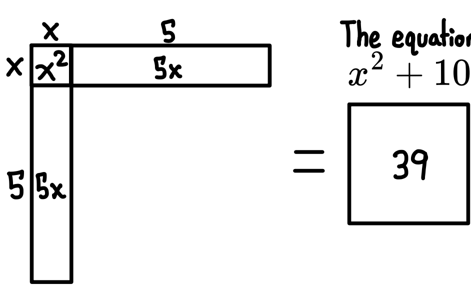


Rooted in geometry "x" represents the area of a square with side length x



The "10x" term is the area of a rectangle with side lengths 10 and x.

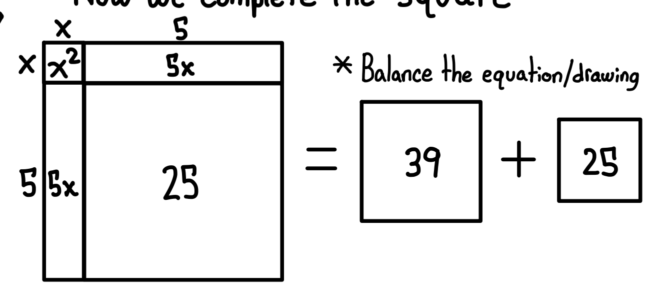
Or the combination of two rectangles, each with side lengths 5 and x



The equation in full is
$x^2 + 10x = 39$

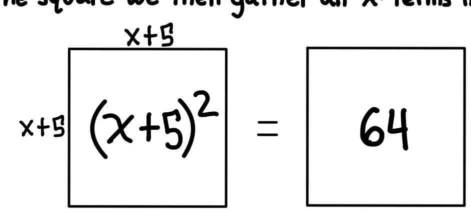
We represent "39" by a square with that area

Now we complete the square



* Balance the equation/drawing *

Completing the square allows us to rewrite as $x^2 + 10x$ as a partial square, and by completing the square we then gather all x-terms into a single square



With Al-Khwarizmi's method, by completing the square, we can demonstrate via geometry to transform the original equation $x^2+10x=39$ into an equivalent equation $(x+5)^3=64$. Now we can simply take the square root to solve...

$$(x+5)^2 = 64$$
$$x + 5 = 8$$
$$x = 3$$

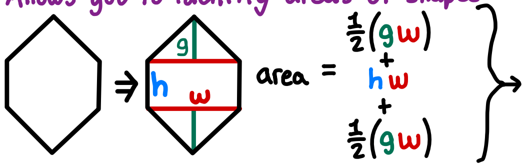
Source: Slide deck 7: early calculus

4. Discuss Archimedes. Include a discussion on the method of exhaustion.

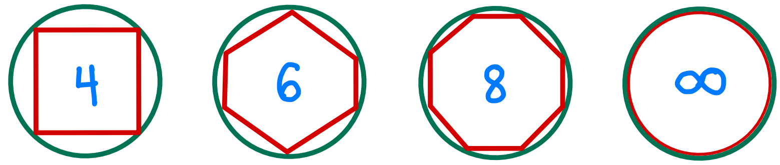
Archimedes was a Greek polymath (i.e. in addition to mathematics he is known for efforts in physics, astronomy, engineering, and as an inventor). His inventions survived time from the useful self named screw to theoretical weapons like the "Death ray" (total myth as tested on Mythbusters).

Although, developed began prior to Archimedes by earlier Greeks like Euclid and independently by Liu Hai, Archimedes mastered the "method of exhaustion" to find areas of circles, ellipse, spirals, parabolic regions, volume of spheres, and the surface area of a sphere.

Allows you to identify areas of shapes



the method of exhaustion builds on this to find the area of curved shapes.

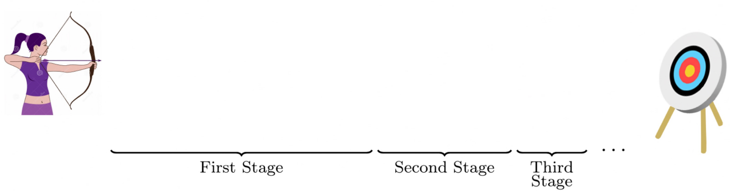


Finding by proximation ⟶

- Integral calculus is based on this method
- Calculus in general relies on the idea of approaching infinity

5. Discuss one of Zeno's paradoxes, and explain Madhava of Sangamagrama's result that proves that Zeno's logic was flawed.

The Dichotomy Paradox

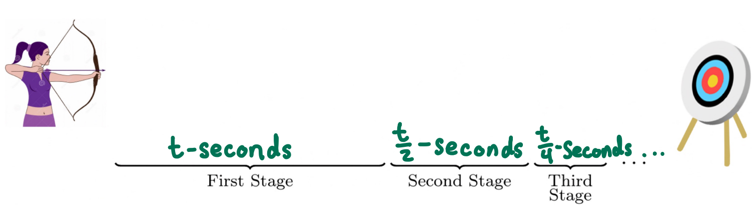


An arrow is shot it will need to travel half way to the target, then half again 1/4th, and half again 1/8th, and so on... The paradox proposes that the arrow will never reach its target...

Madhava of Sangamagrama *Infinite sums*

Theorem: $1 + r + r^2 + r^3 + \ldots + r^n = \dfrac{1 - r^{n+1}}{1 - r}$

Proof: Since $r, r^2, r^3, r^4, \ldots$ are all between 0 and 1



$D = rt$

$\downarrow$

$D/r = t$

The mistake is not recognizing that the sum of an infinite geometric series can converge to a finite value as the time interval approaches zero.

Then the total amount of time (in seconds) to complete all the stages

$$t + \frac{1}{2}t + \frac{1}{4}t + \frac{1}{8}t + \ldots = t(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots) = t\frac{1}{1 - \frac{1}{2}} = 2t$$

So the arrow completes all the stages in just 2t seconds

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Exam 2 Essay Questions

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