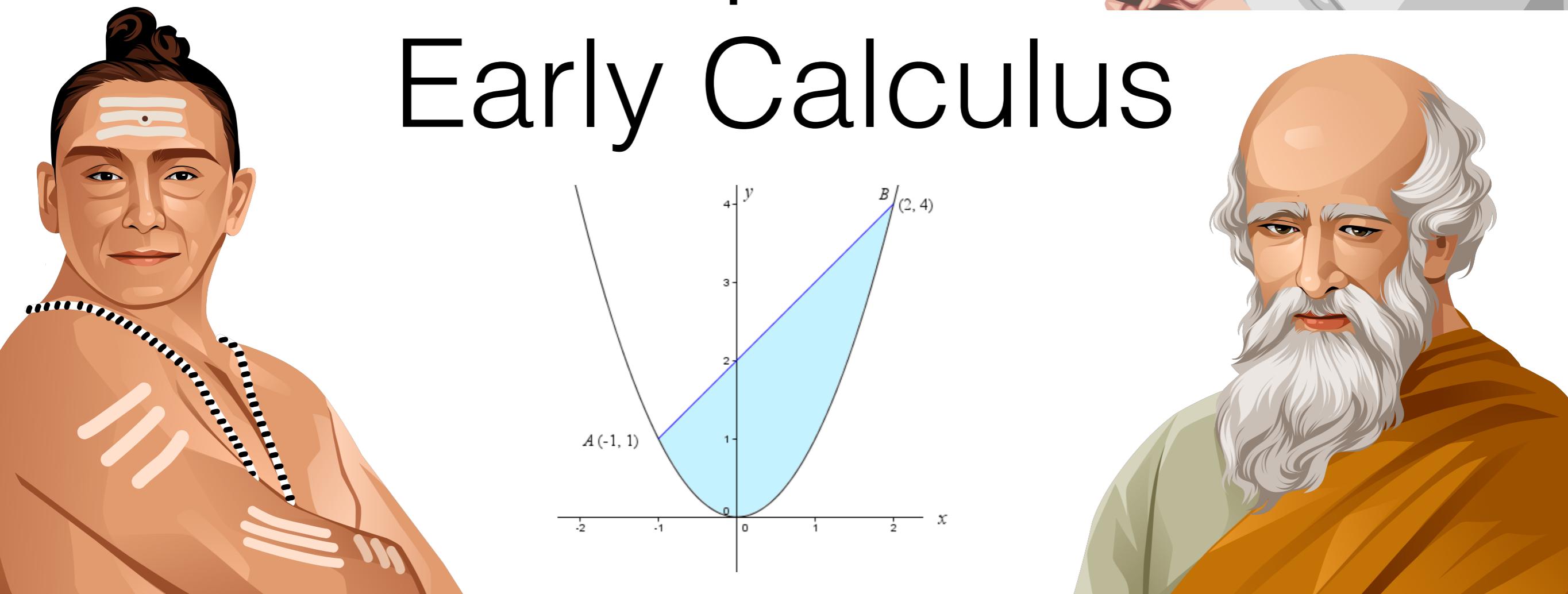
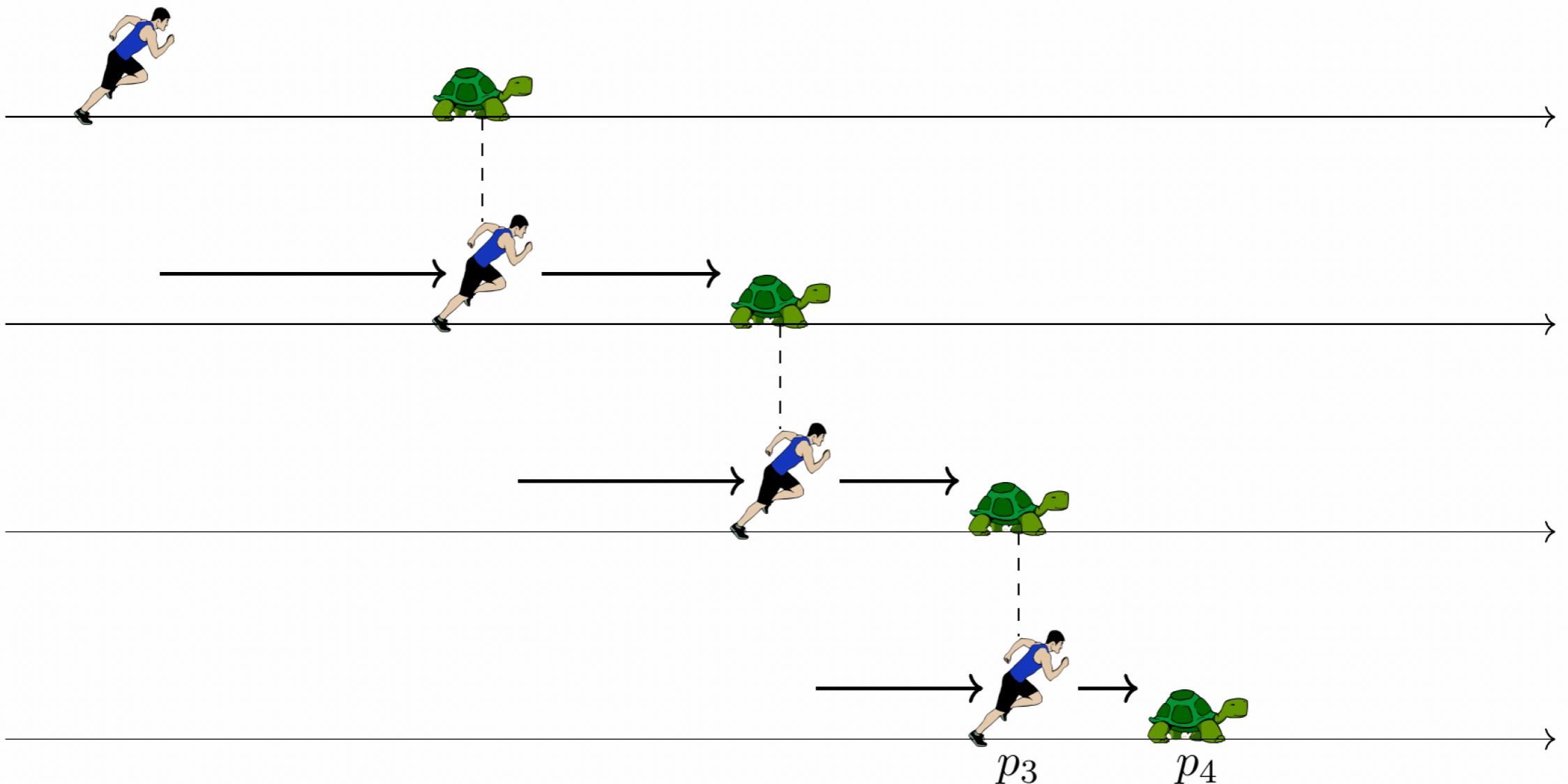


Chapter 7:

Early Calculus

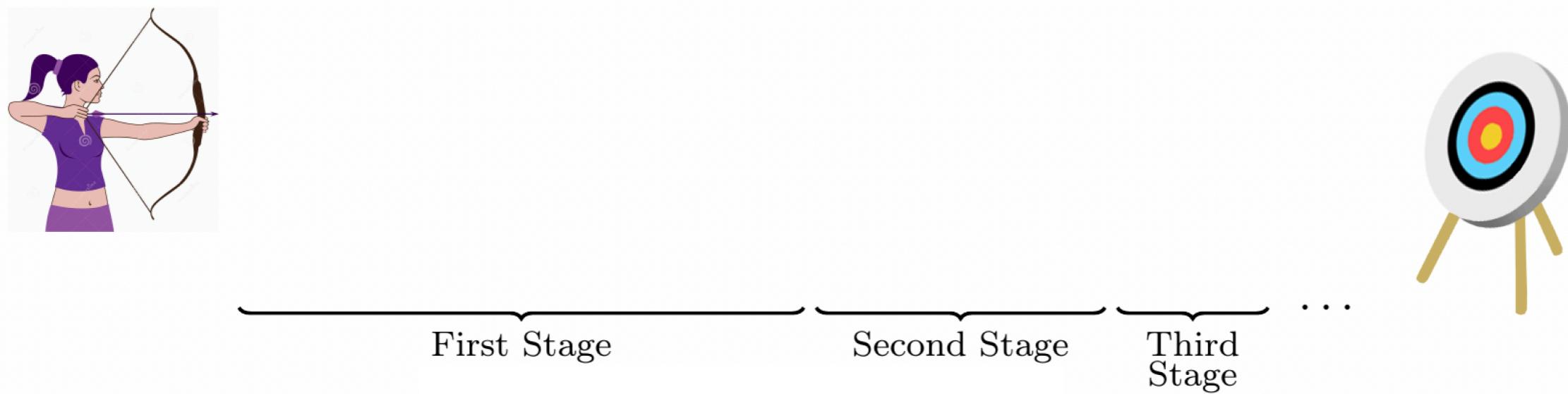


Zeno's Paradoxes



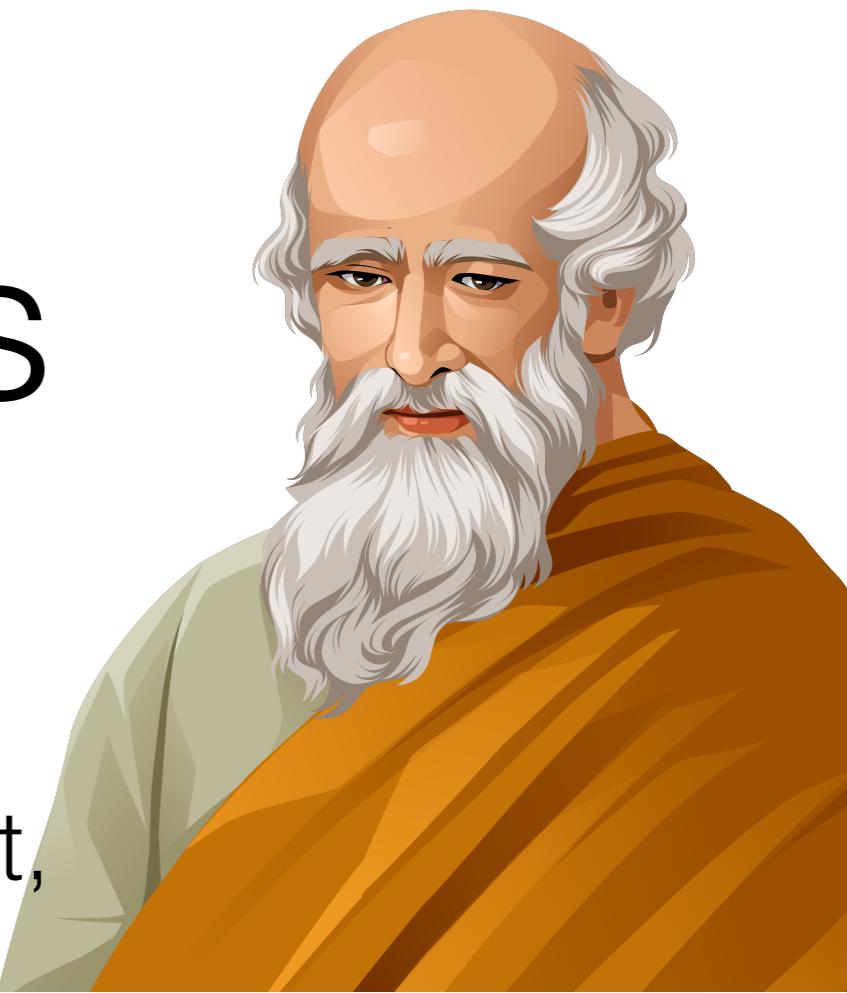
Zeno: If Achilles is always behind, he can never surpass the tortoise, so he is guaranteed to lose.

Zeno's Paradoxes



Archimedes

- Archimedes (~287 BC – ~212 BC, Syracuse) was the greatest Greek mathematician. Also a great physicist, astronomer, engineer and inventor.
- He was the GMOAT for nearly 2000 years. And maybe still is the GMOAT.
- Nearly all we know about Archimedes is due to three documents. The third of these was lost until 1906.



BIG LITERARY FIND IN CONSTANTINOPLE

Savant Discovers Books by Archimedes, Copied About 900 A. D.

IT OPENS A BIG FIELD

Whether the Turks Destroyed the Libraries When They Took the City Always a Disputed Question.

COPENHAGEN, July 15.—Y. L. Heiberg, Professor of Philology in the University of Copenhagen, made a most interesting discovery in the Convent of the Holy Grave at Constantinople a few weeks ago.

While studying old manuscripts in the convent he discovered a number of palimpsests which, in addition to prayers and psalms of the twelfth century, included works by Archimedes.

The Archimedes manuscript was a copy made about the year 900 by a monk and later conveyed to Constantinople.

The Turkish authorities did not permit Prof. Heiberg to remove the manuscript. He was permitted, however, to make a copy of it, and this will shortly be published.

The fact that Prof. Heiberg copied the Archimedes manuscript apparently indicates that it consisted, entirely or in part, of works by Archimedes that have hitherto been lost, for he would hardly have taken the trouble to transcribe the books on plane geometry, solid geometry, arithmetic, and mechanics which have come down to us from among the writings by the great Greek. Perhaps, even, the

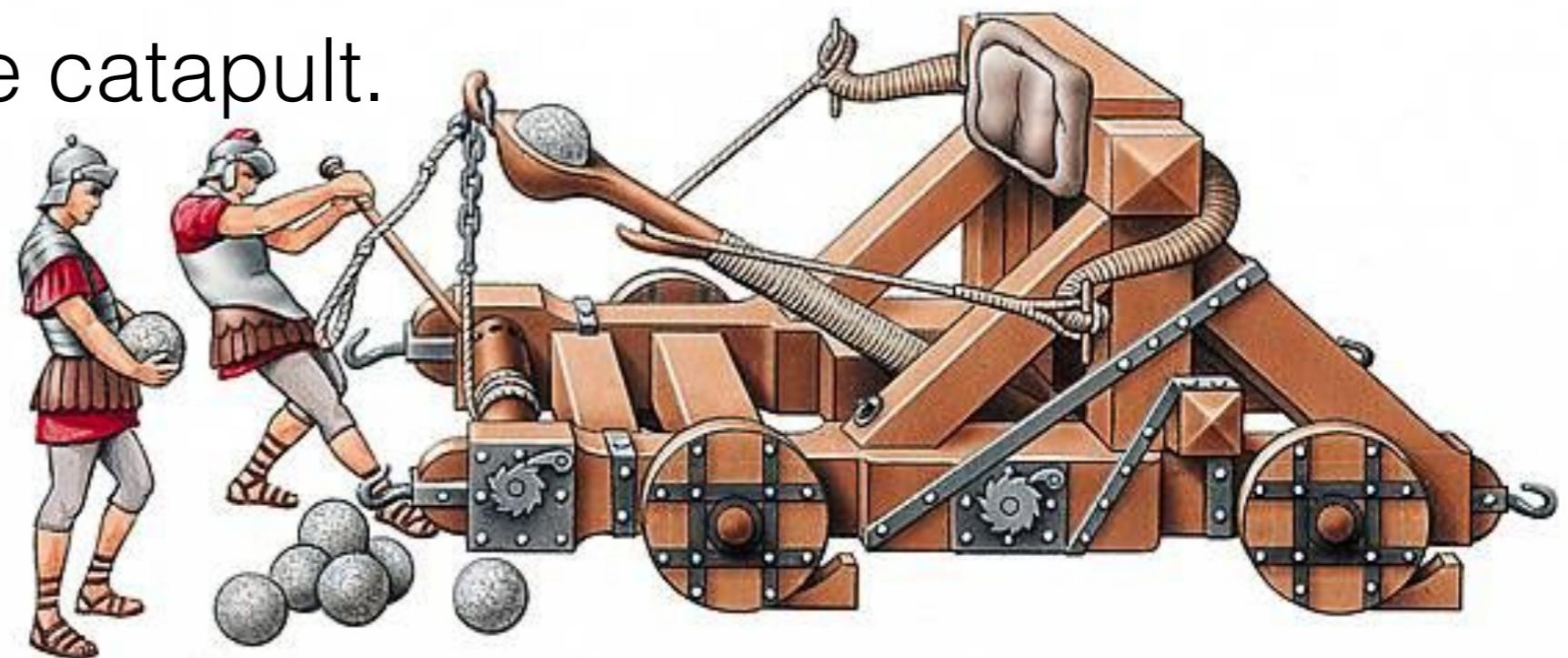
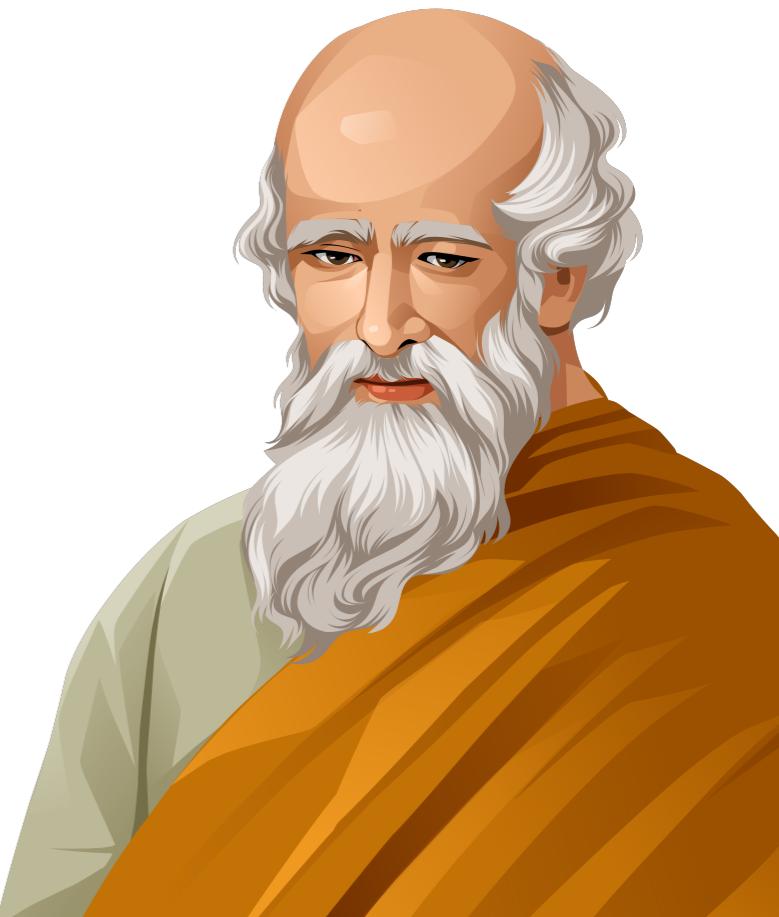
Archimedes

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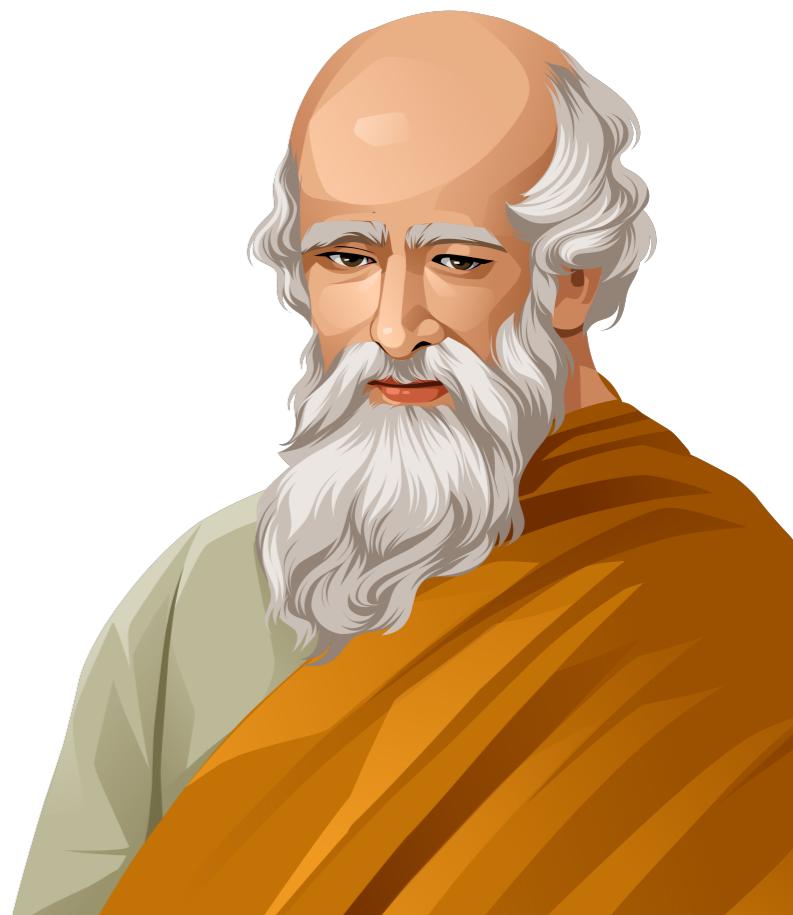


Archimedes

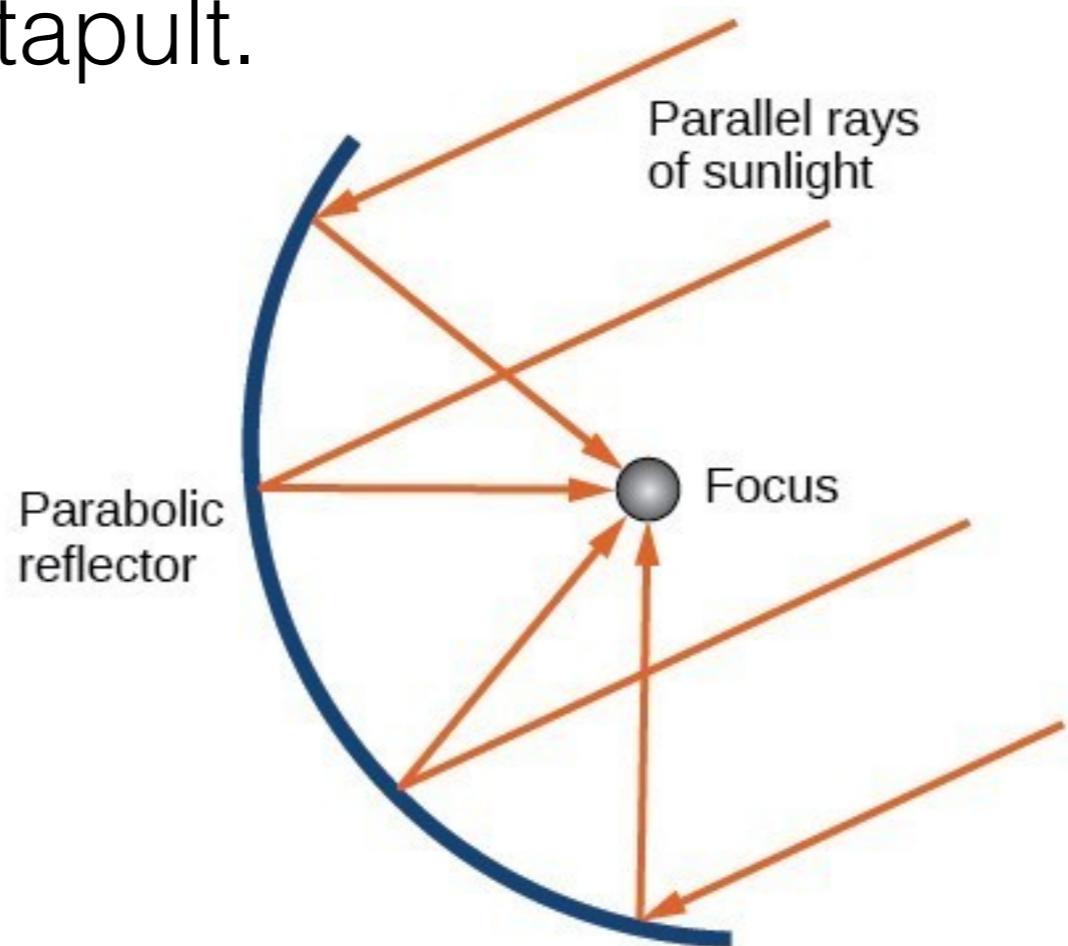
- Defended his hometown of Syracuse against the Romans using his “war machines.”
- Variable-range catapult.



Archimedes

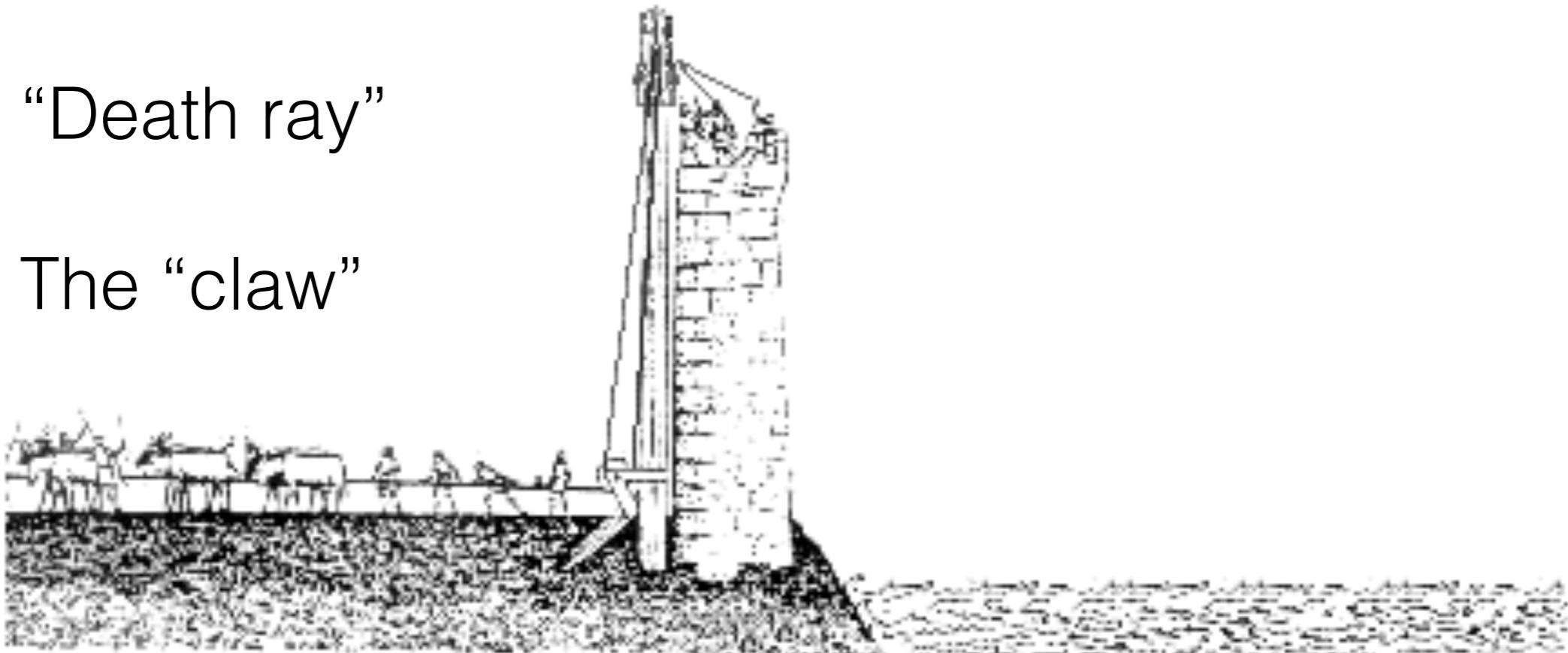
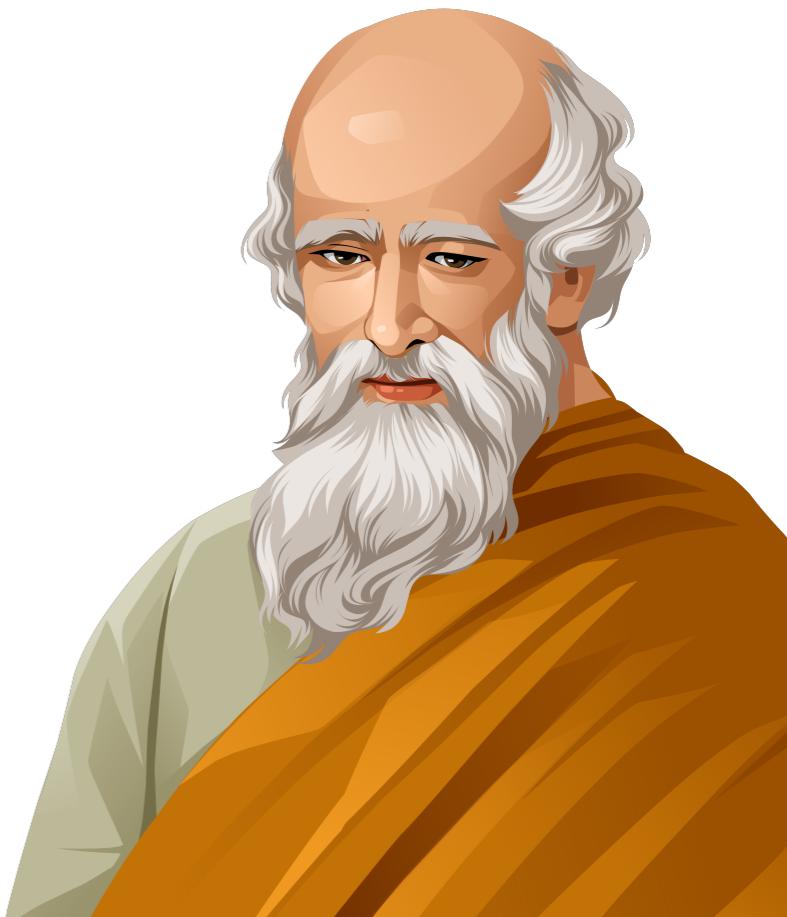


- Defended his hometown of Syracuse against the Romans using his “war machines.”
- Variable-range catapult.
- “Death ray”

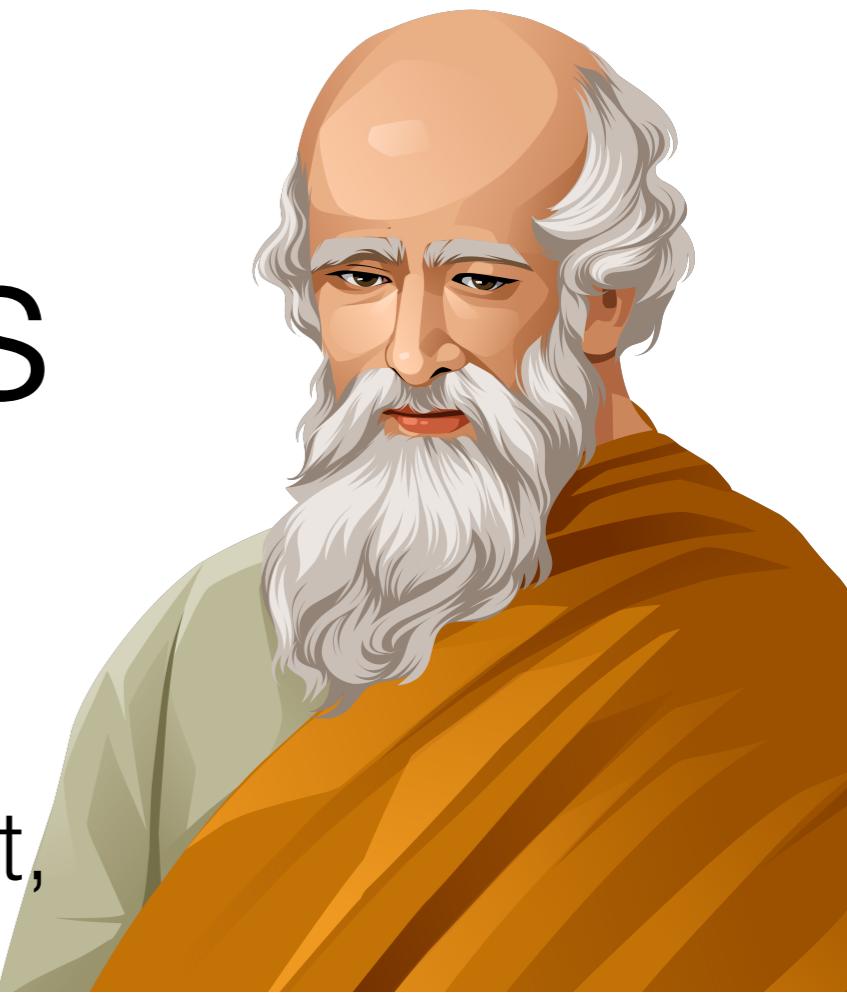


Archimedes

- Defended his hometown of Syracuse against the Romans using his “war machines.”
- Variable-range catapult.
- “Death ray”
- The “claw”



Archimedes



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Archimedes



Archimedes

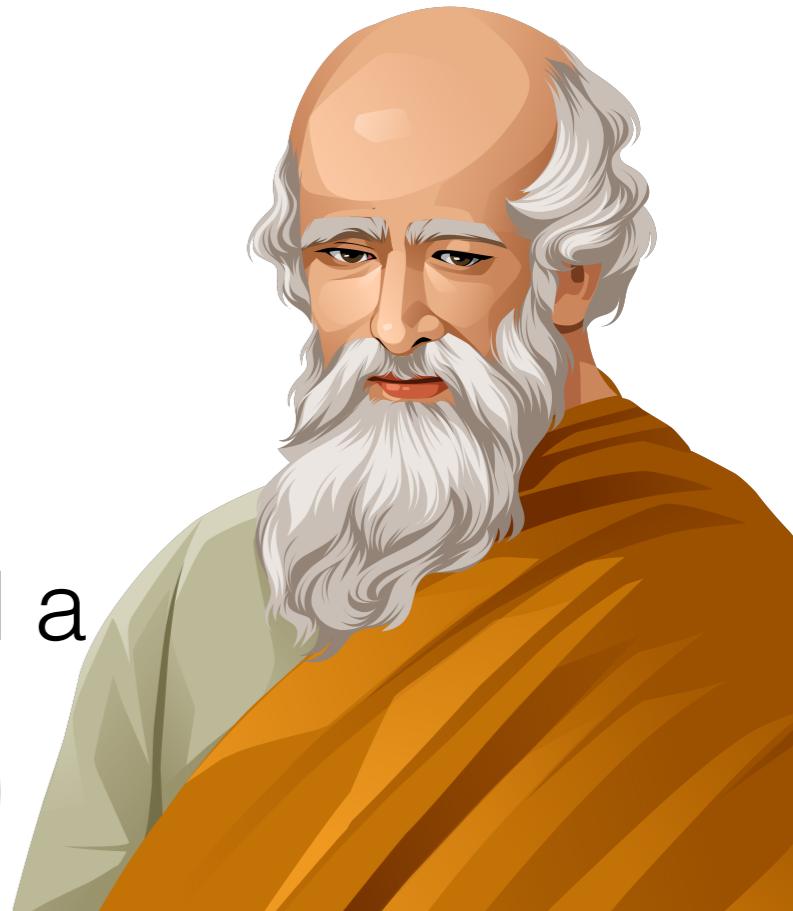
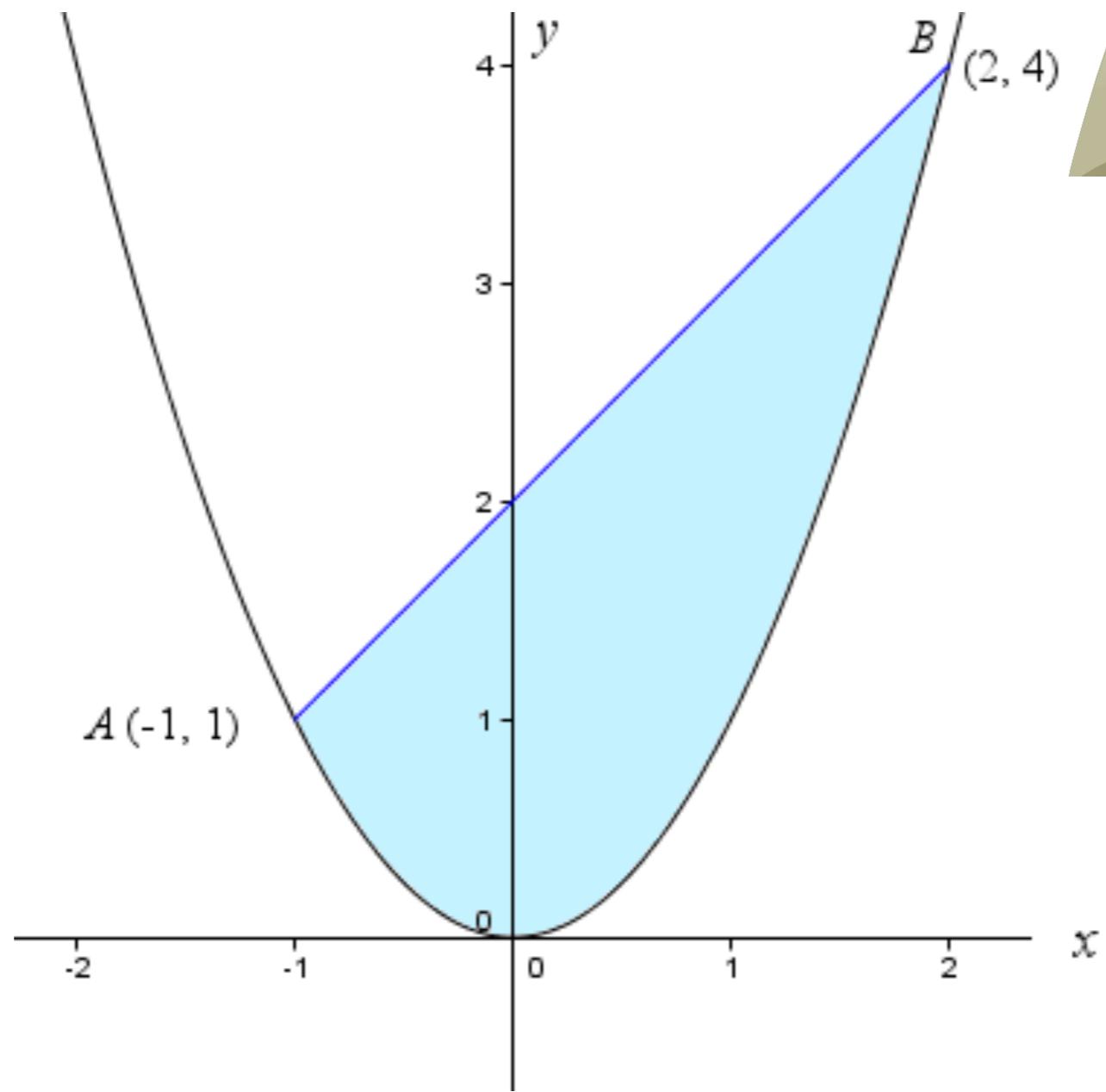


Archimedes

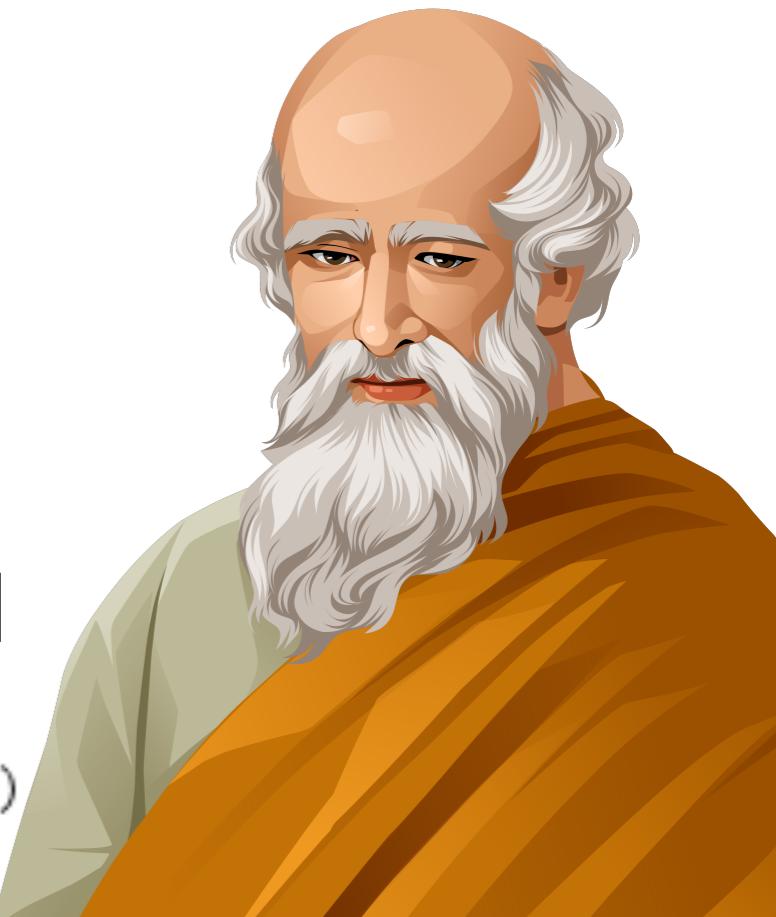


Archimedes

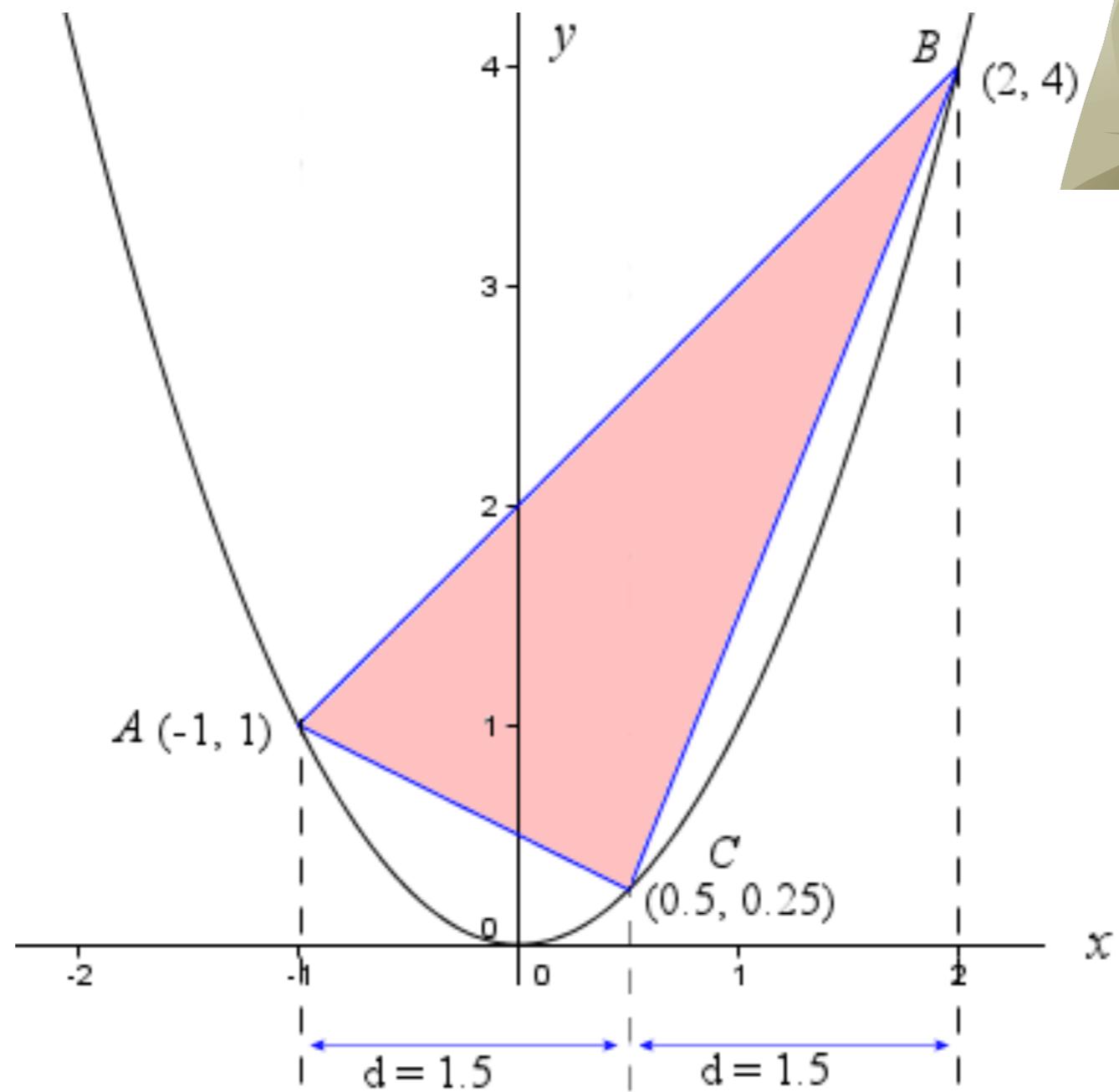
- Find the area between a parabola and a line:



Archimedes

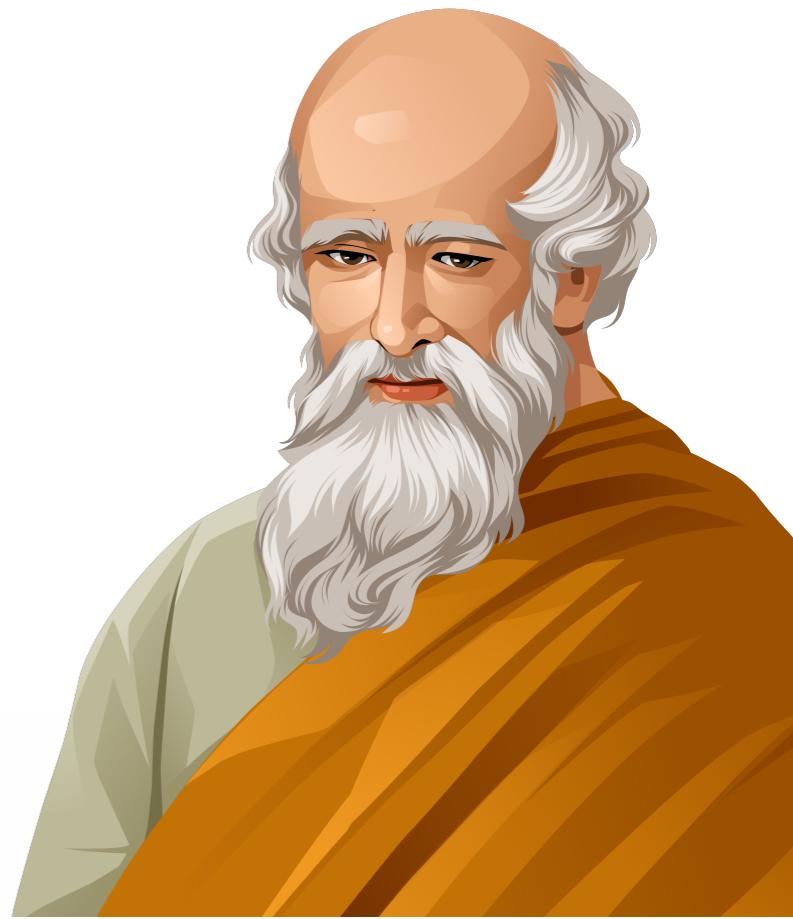
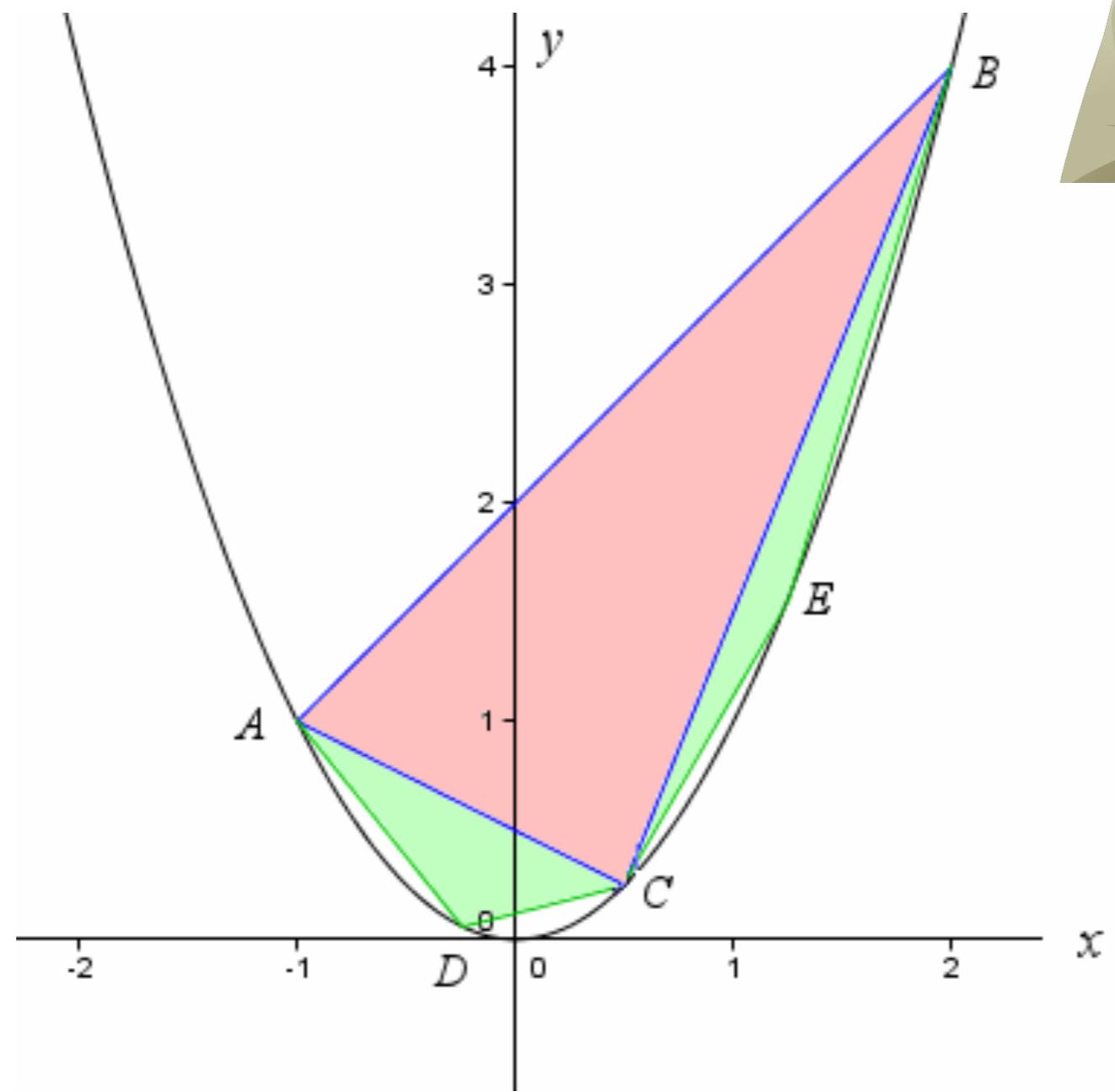


- Add a triangle with the same base and height.



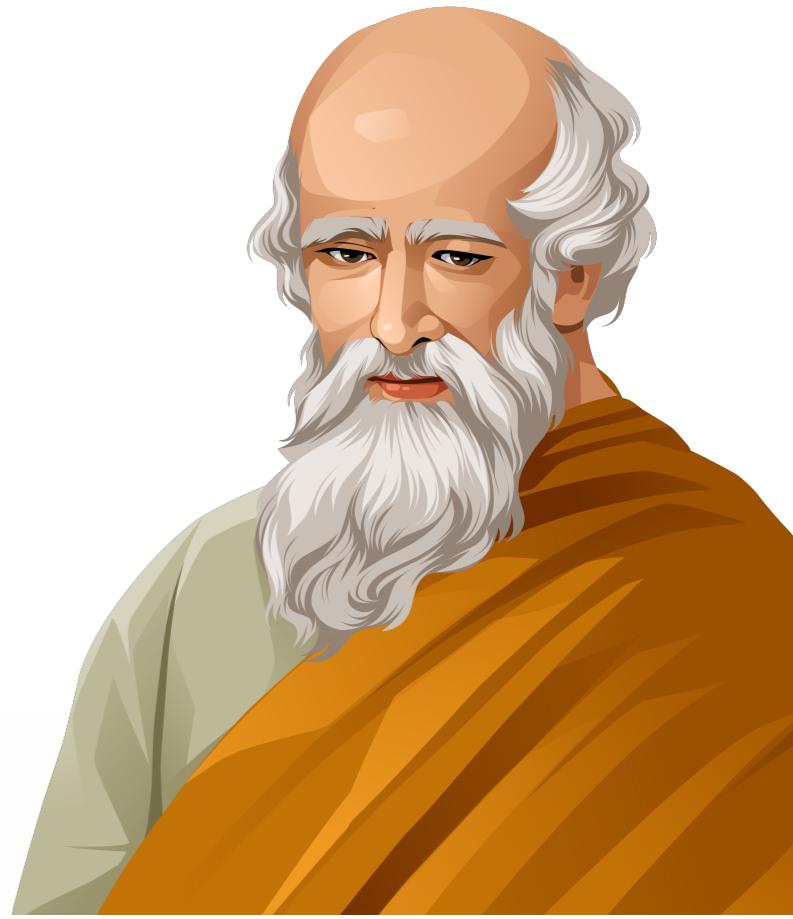
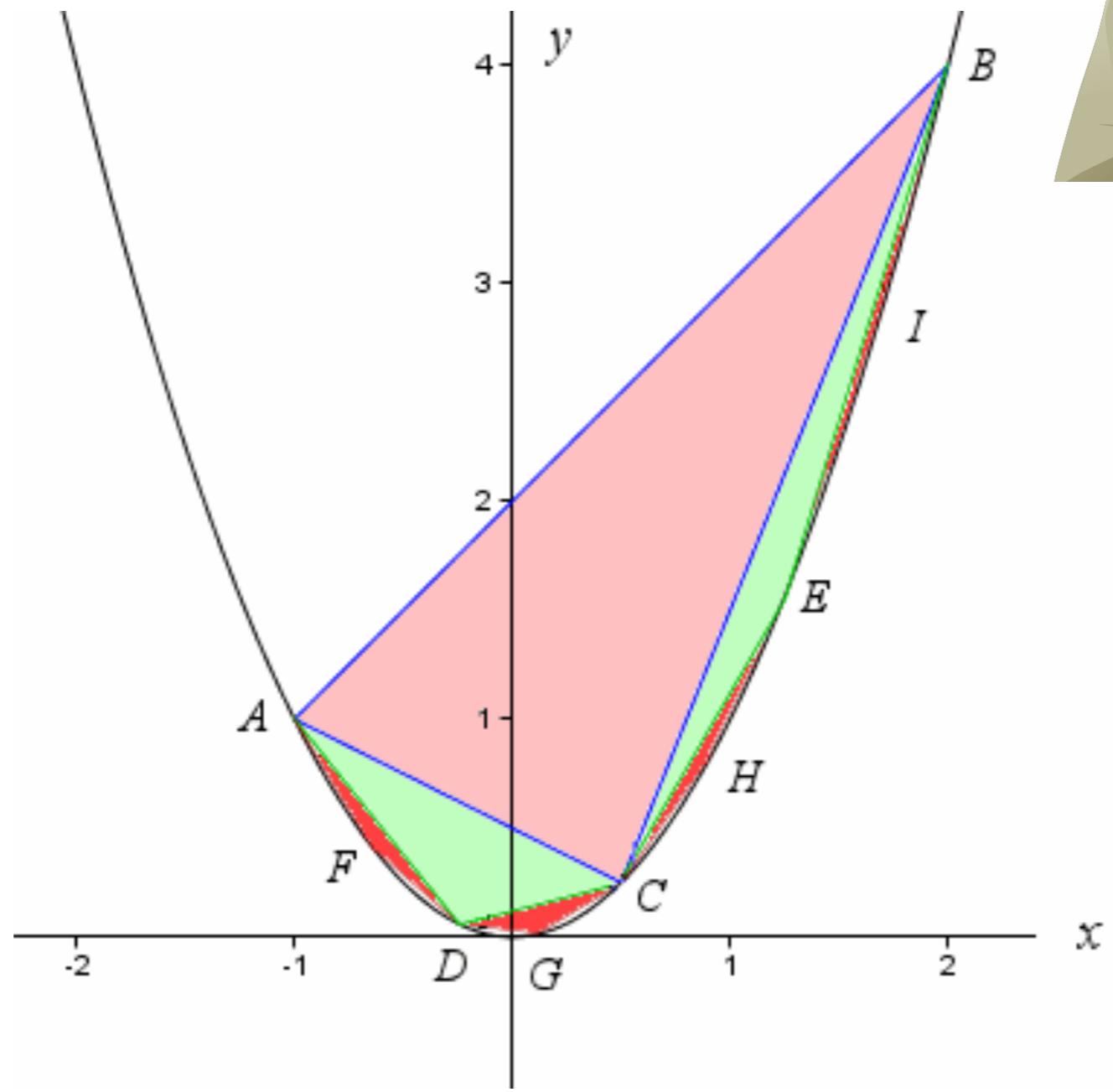
Archimedes

- To the empty places, place similar triangles.



Archimedes

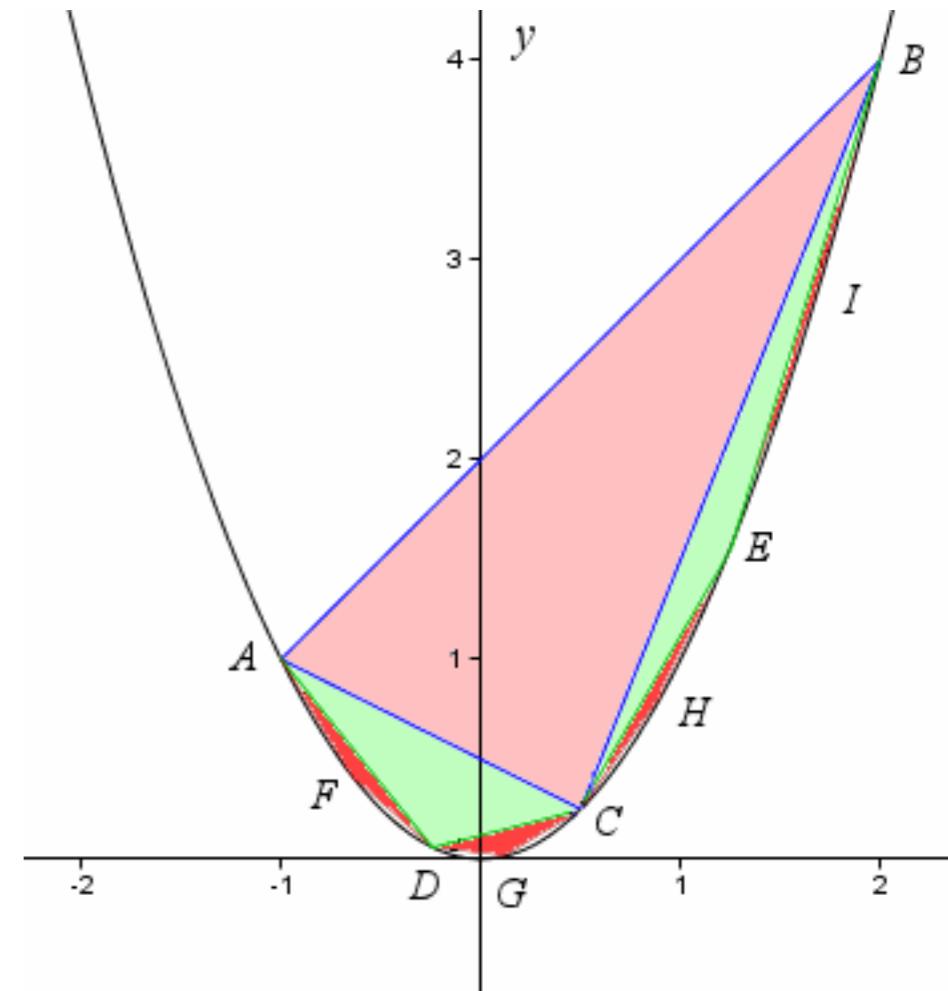
- Continue this forever.



Archimedes

- Archimedes showed that each triangle has $1/8$ the area as in the previous step.
- Since there are twice as many triangles each step, the total area each step is $1/4$ as much as in the previous step.
- Thus, if A is the area of the triangle, the area between the line and parabola is

$$A + \frac{1}{4}A + \frac{1}{4^2}A + \frac{1}{4^3}A + \frac{1}{4^4}A + \dots$$



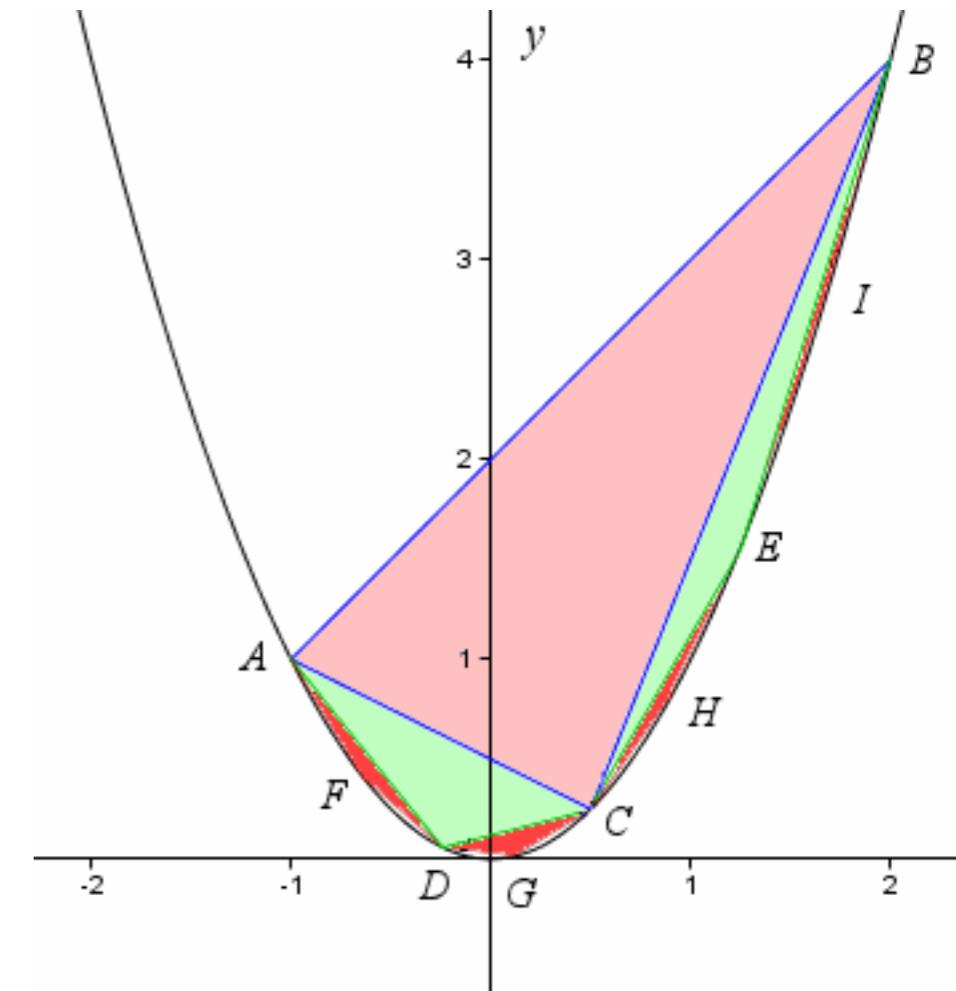
Archimedes

- First, he rewrote this as

$$A + A \left(\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots \right)$$

- Finding the area of a triangle is easy, so all Archimedes needed was to find

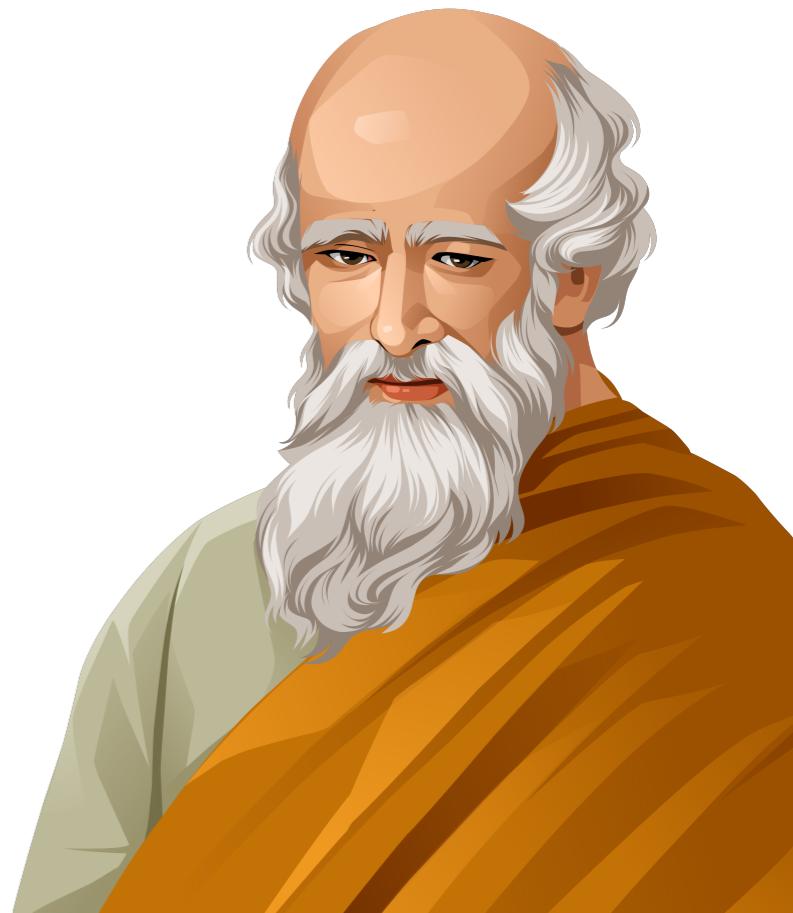
$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \dots$$



- How do you determine this before knowing a formula?

Archimedes

Finding $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \dots$



Partial sums: $\frac{1}{4} = 0.25$

$$\frac{1}{4} + \frac{1}{16} = 0.3125$$

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} = 0.328125$$

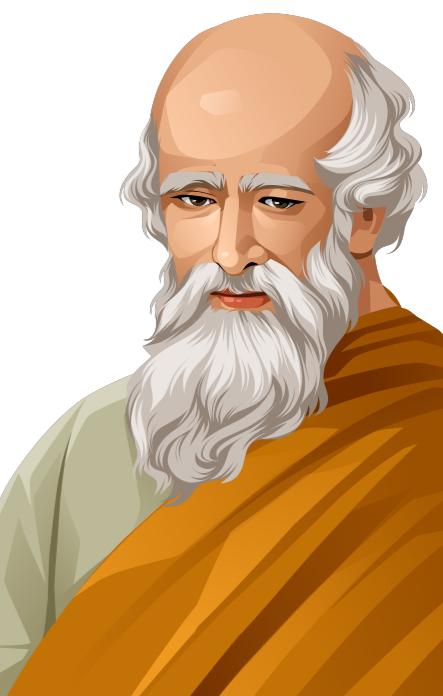
$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} = 0.33203125$$

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} = 0.3330078125$$

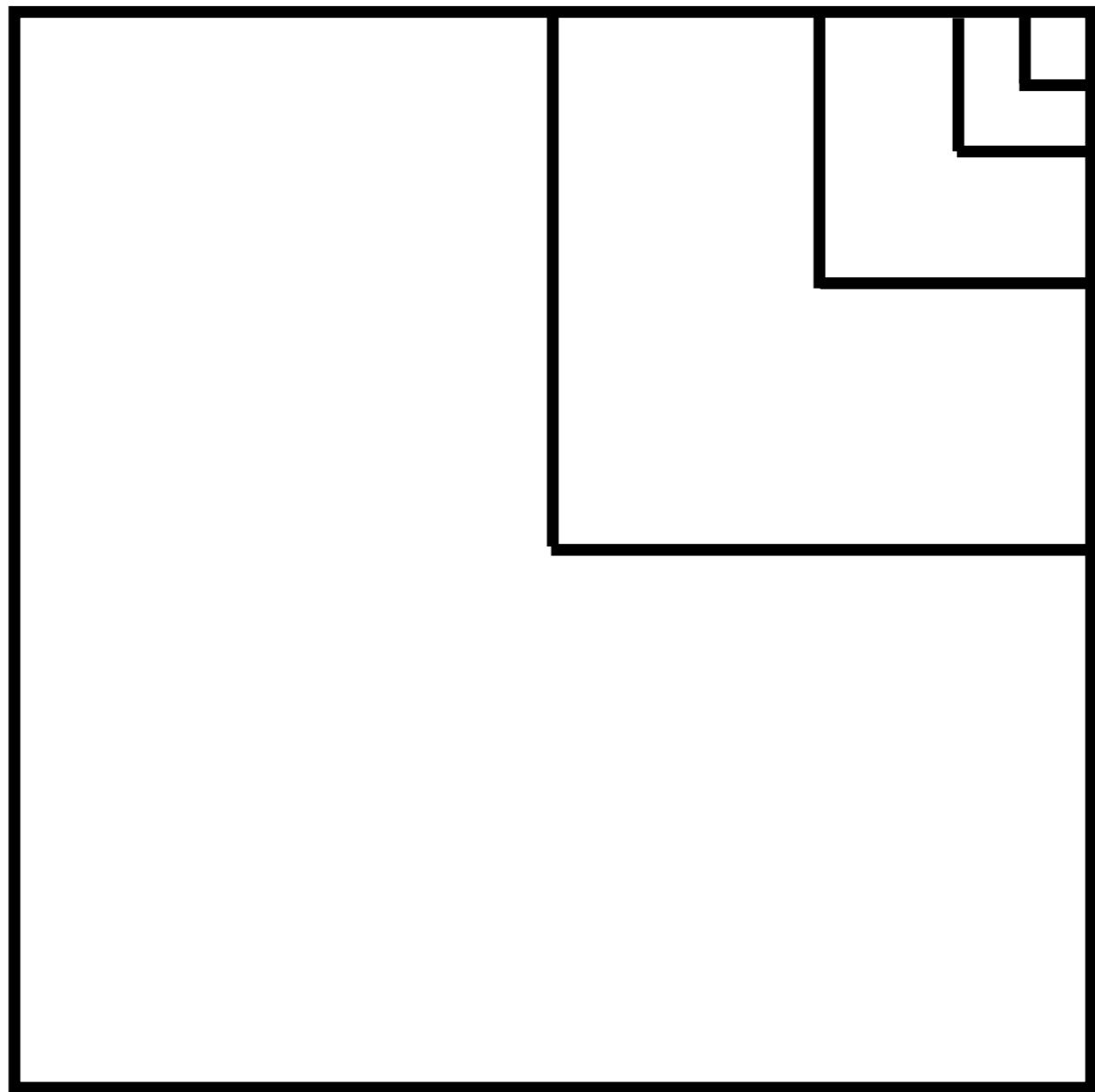
Archimedes

Showing that

$$\frac{1}{3} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \dots$$



Archimedes' logic relied on this drawing.

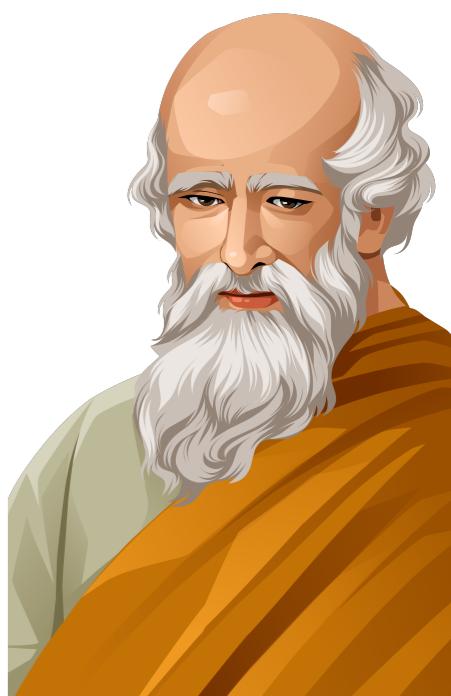


Think Like A
Math Historian

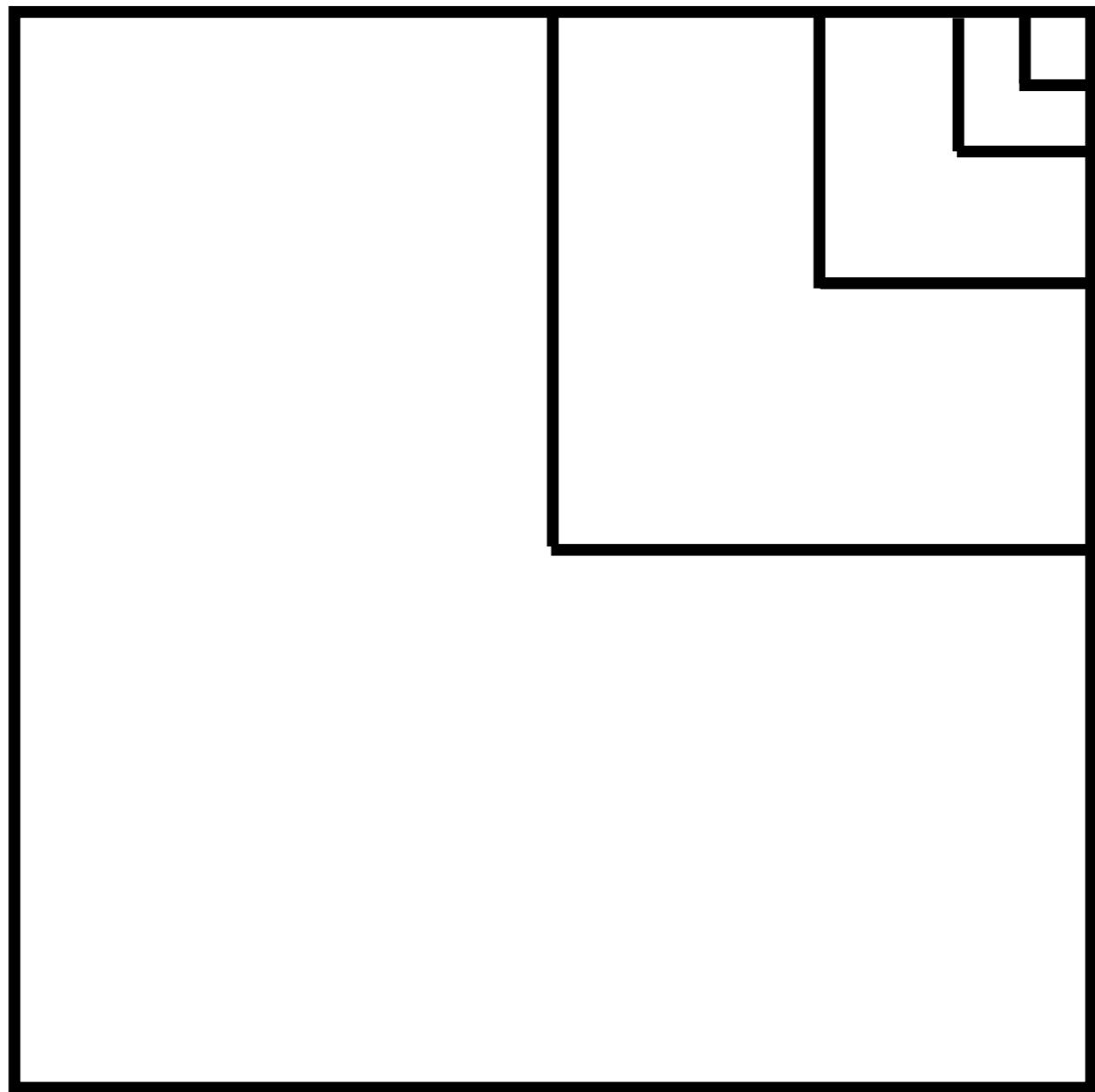
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Archimedes

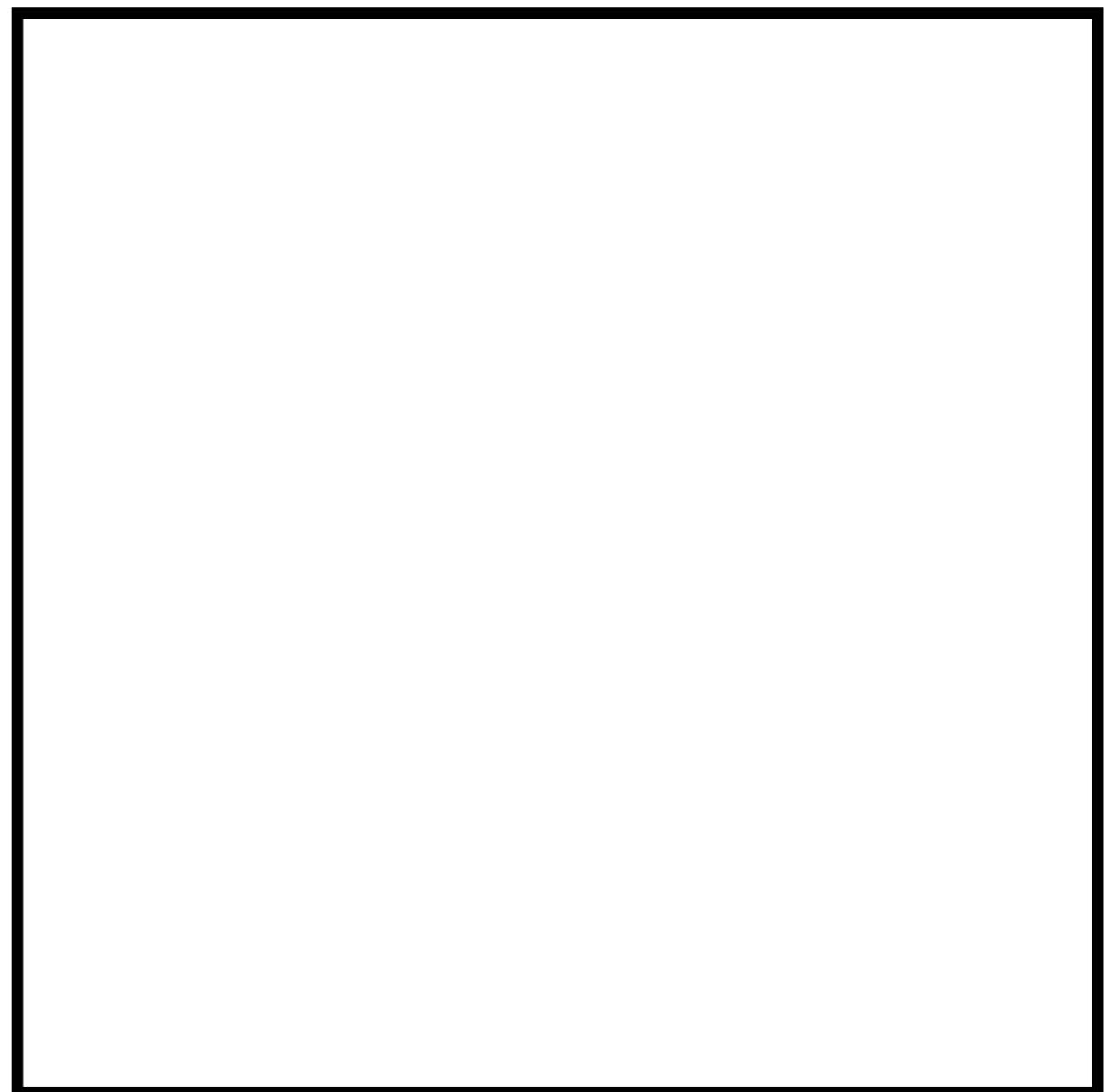
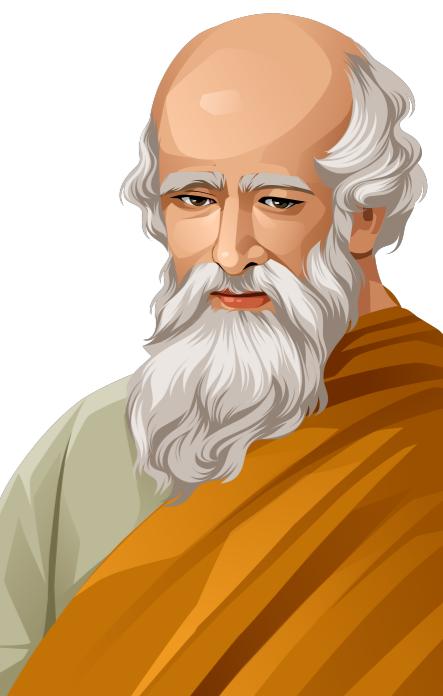
Showing that $\frac{1}{3} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \dots$



Archimedes

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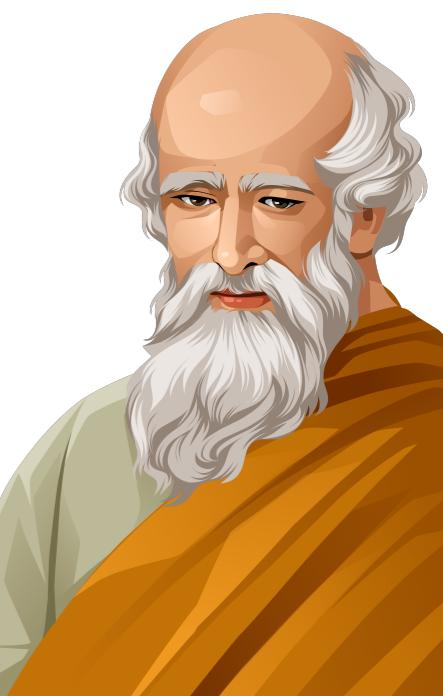
$$\frac{1}{3} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \dots$$



Archimedes

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Archimedes

Showing that

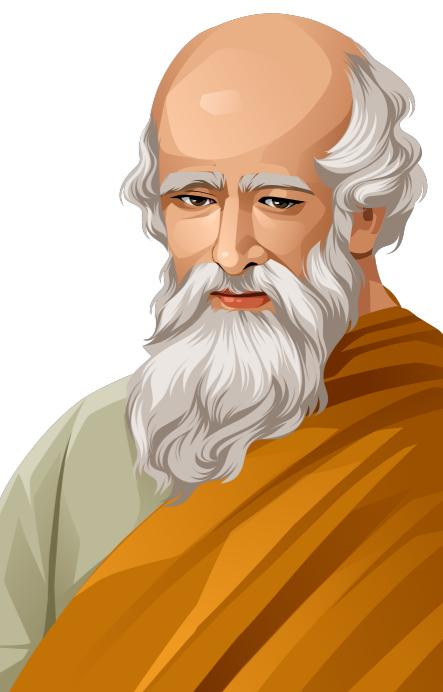
$$\frac{1}{3} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \dots$$



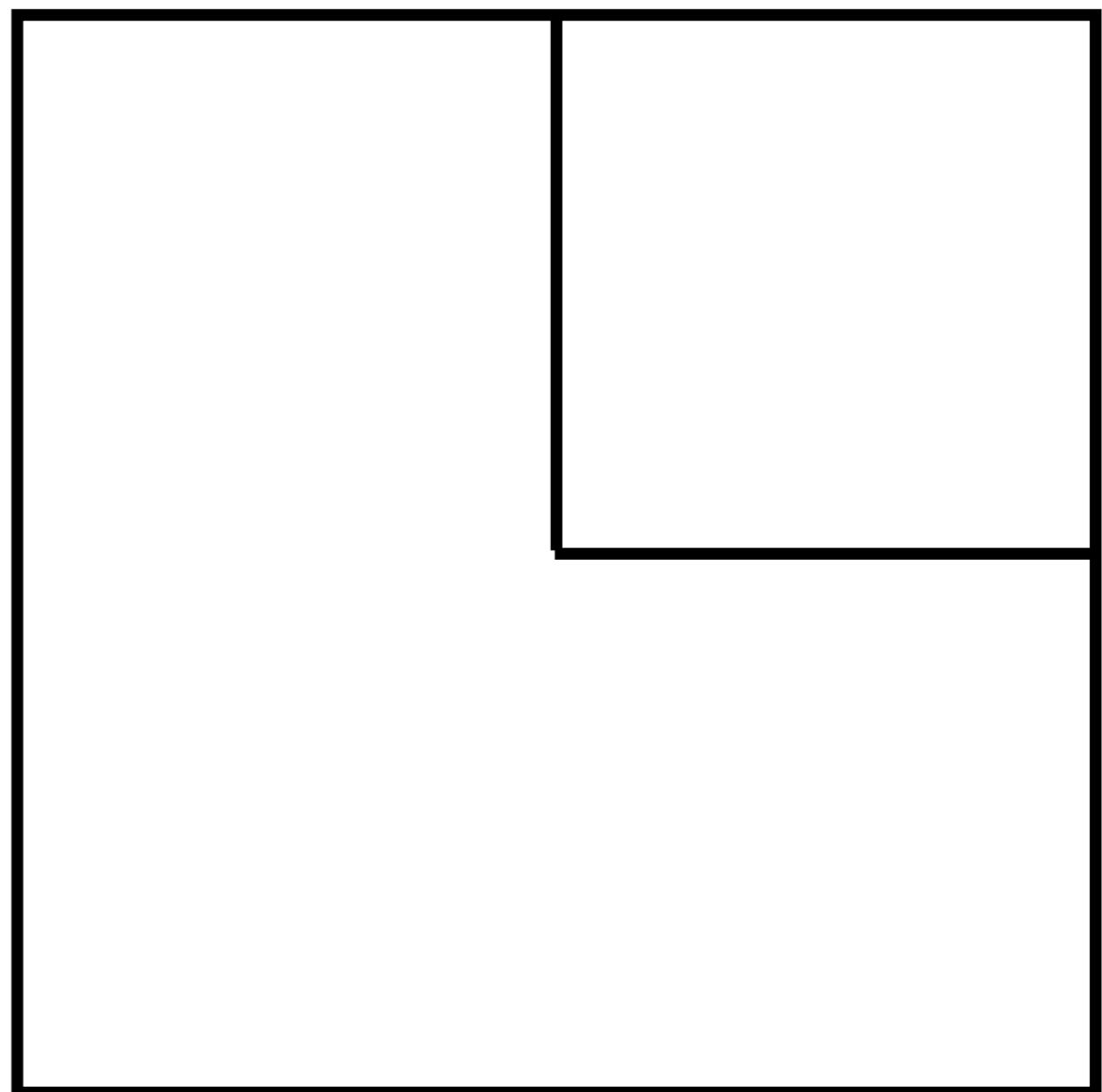
1 =

Archimedes

Showing that $\frac{1}{3} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \dots$

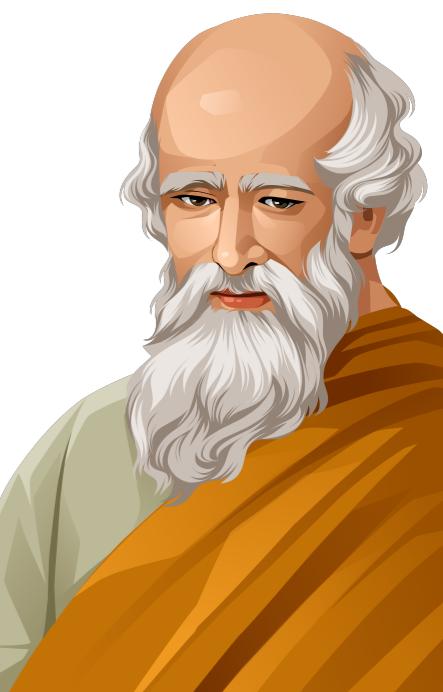


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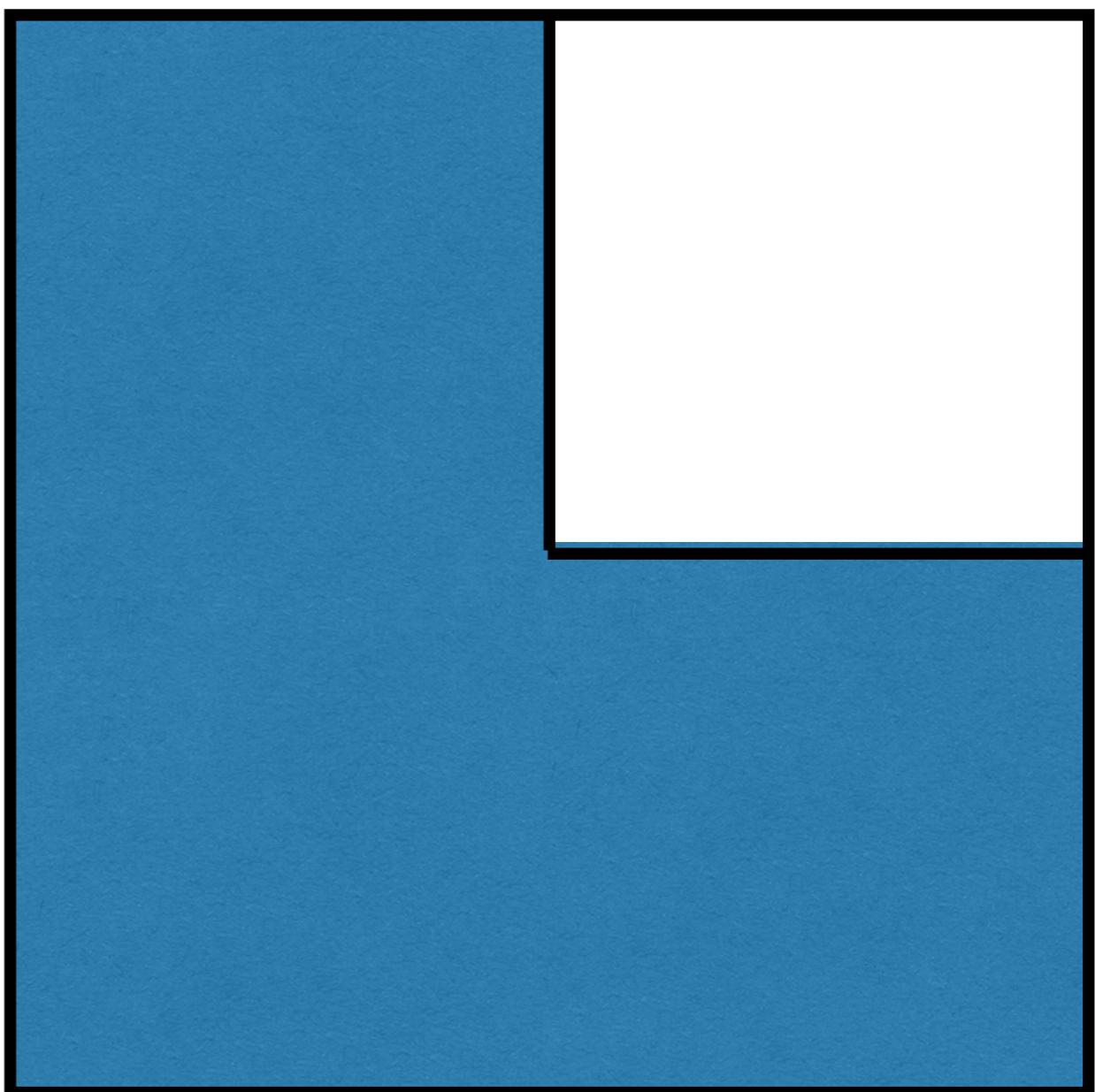


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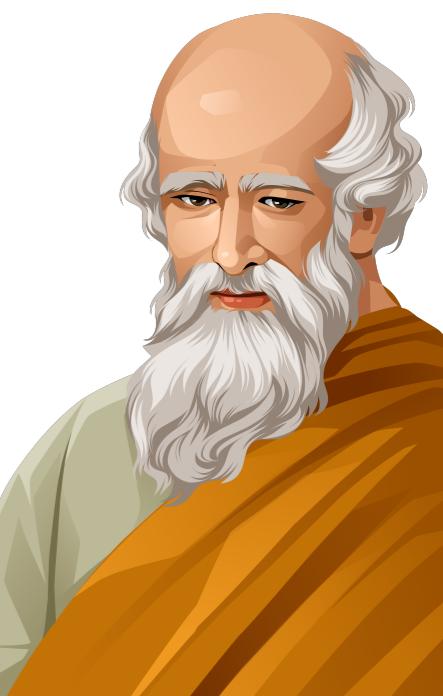
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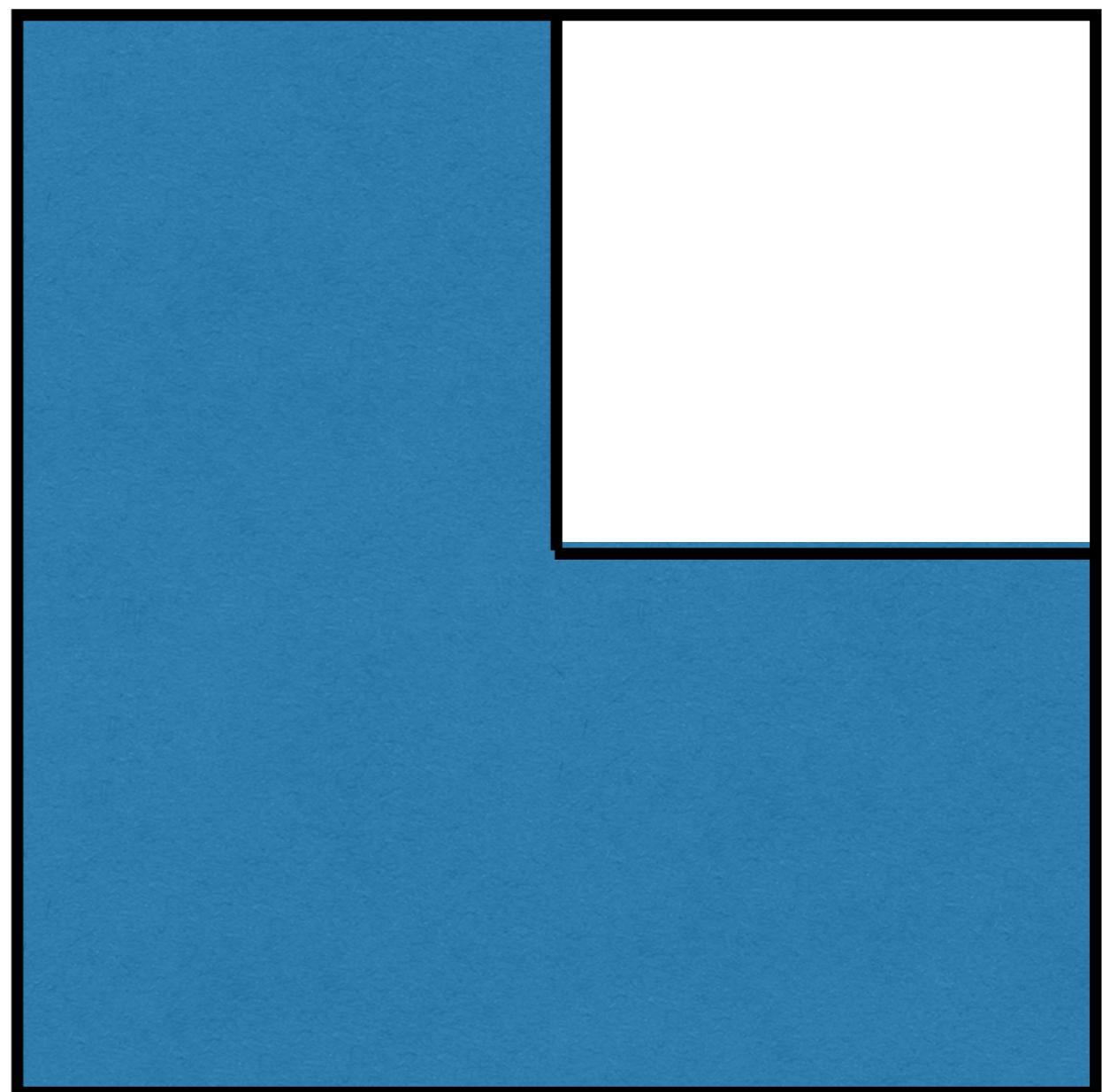
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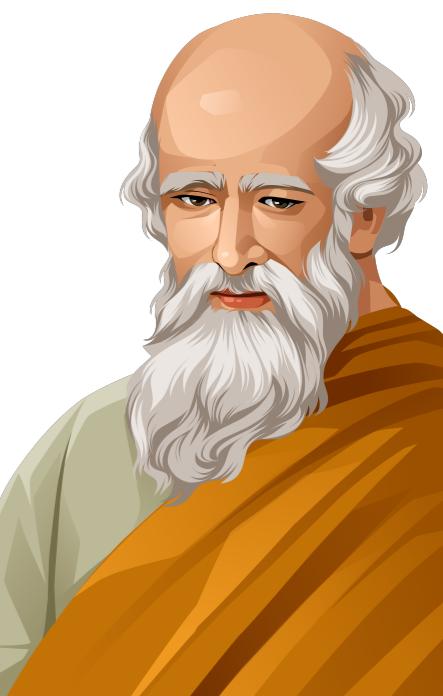
$$1 = \frac{3}{4}$$



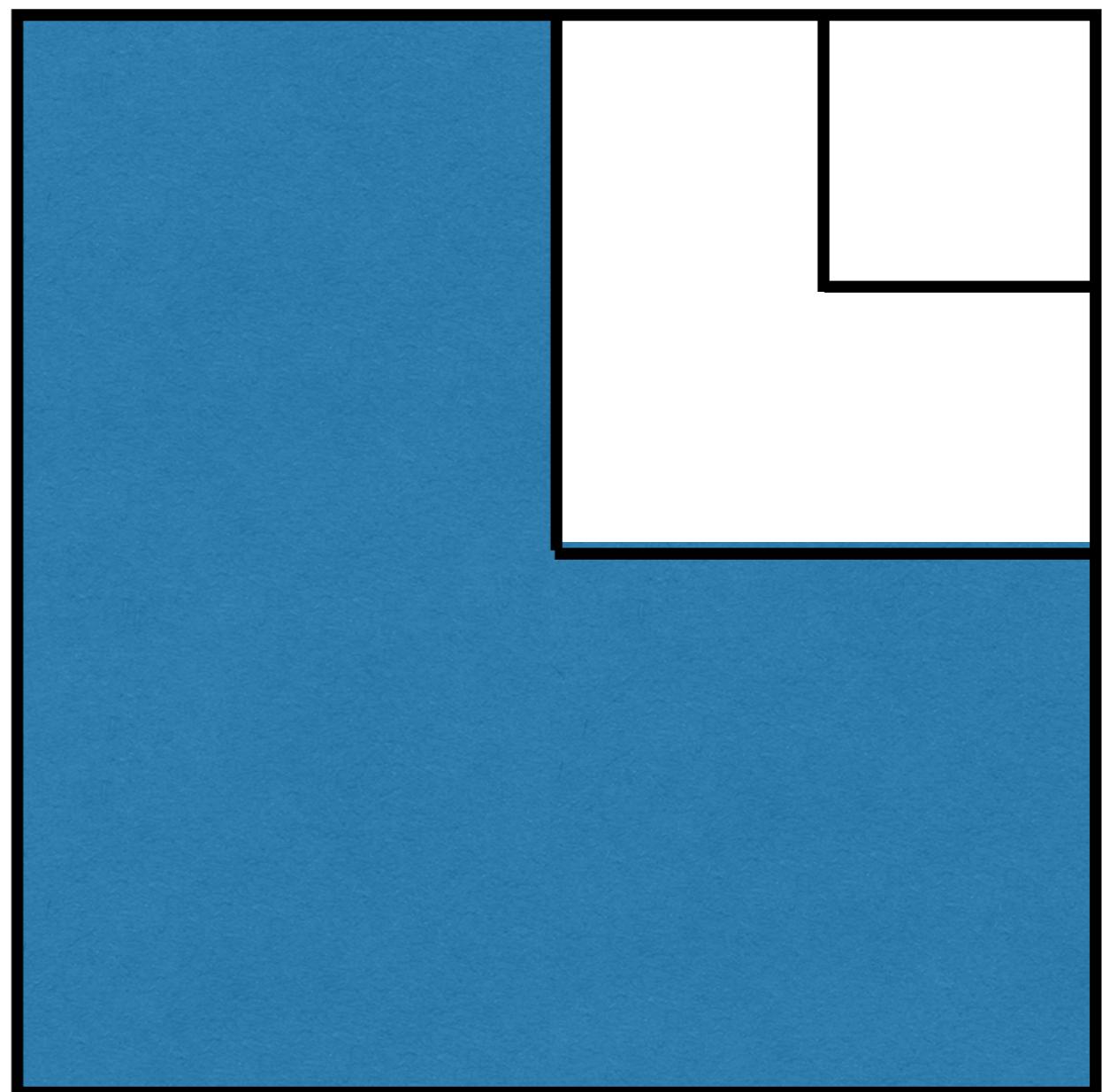
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$$\frac{1}{3} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \dots$$



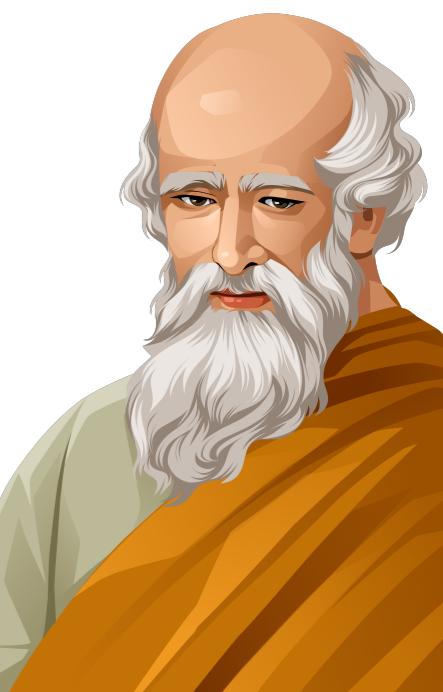
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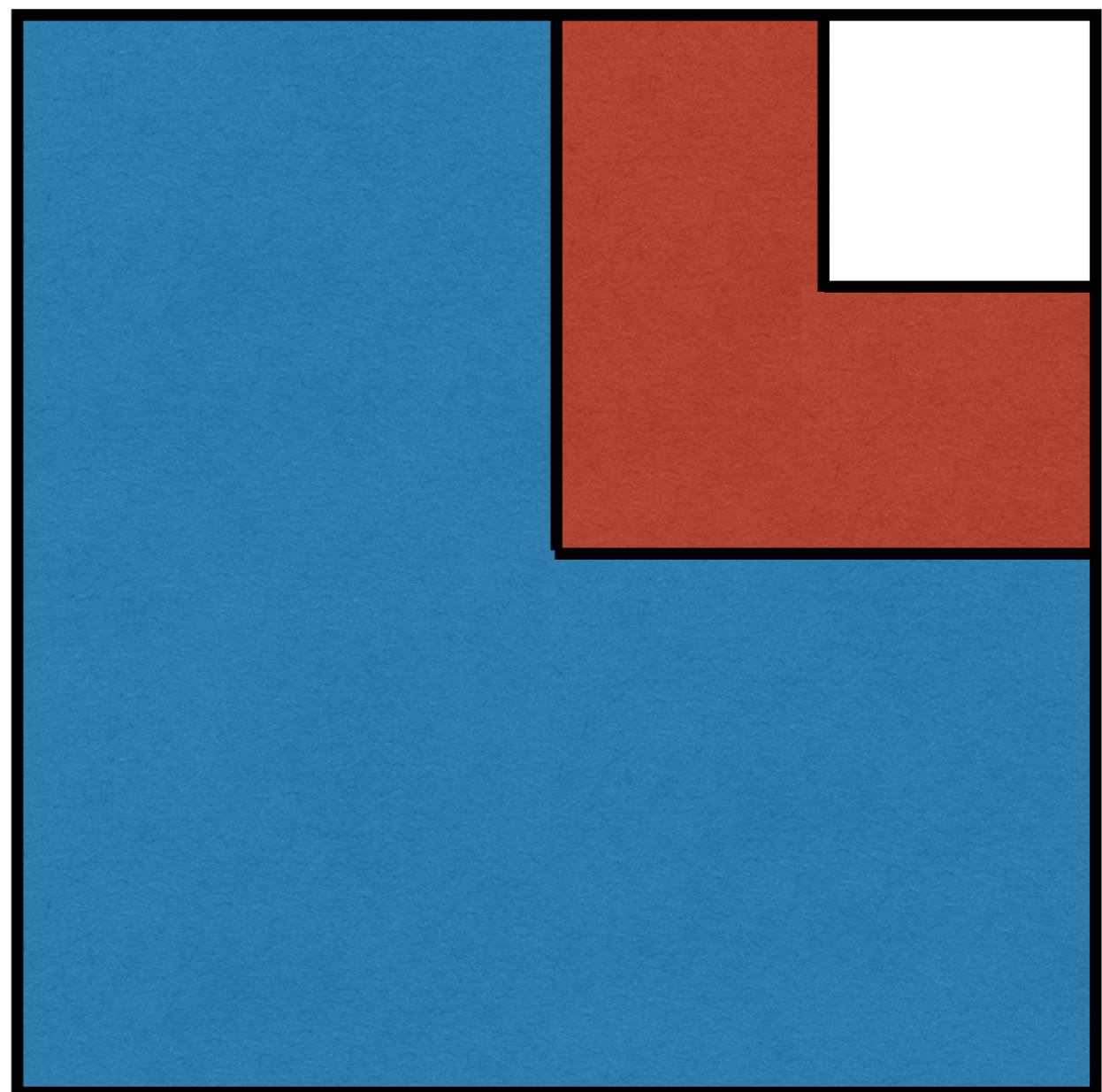
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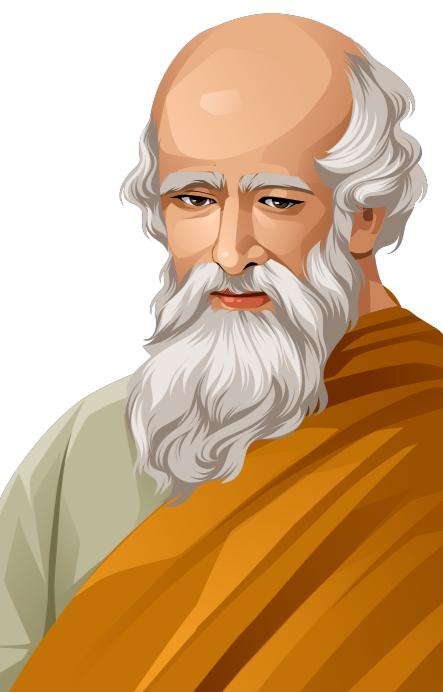
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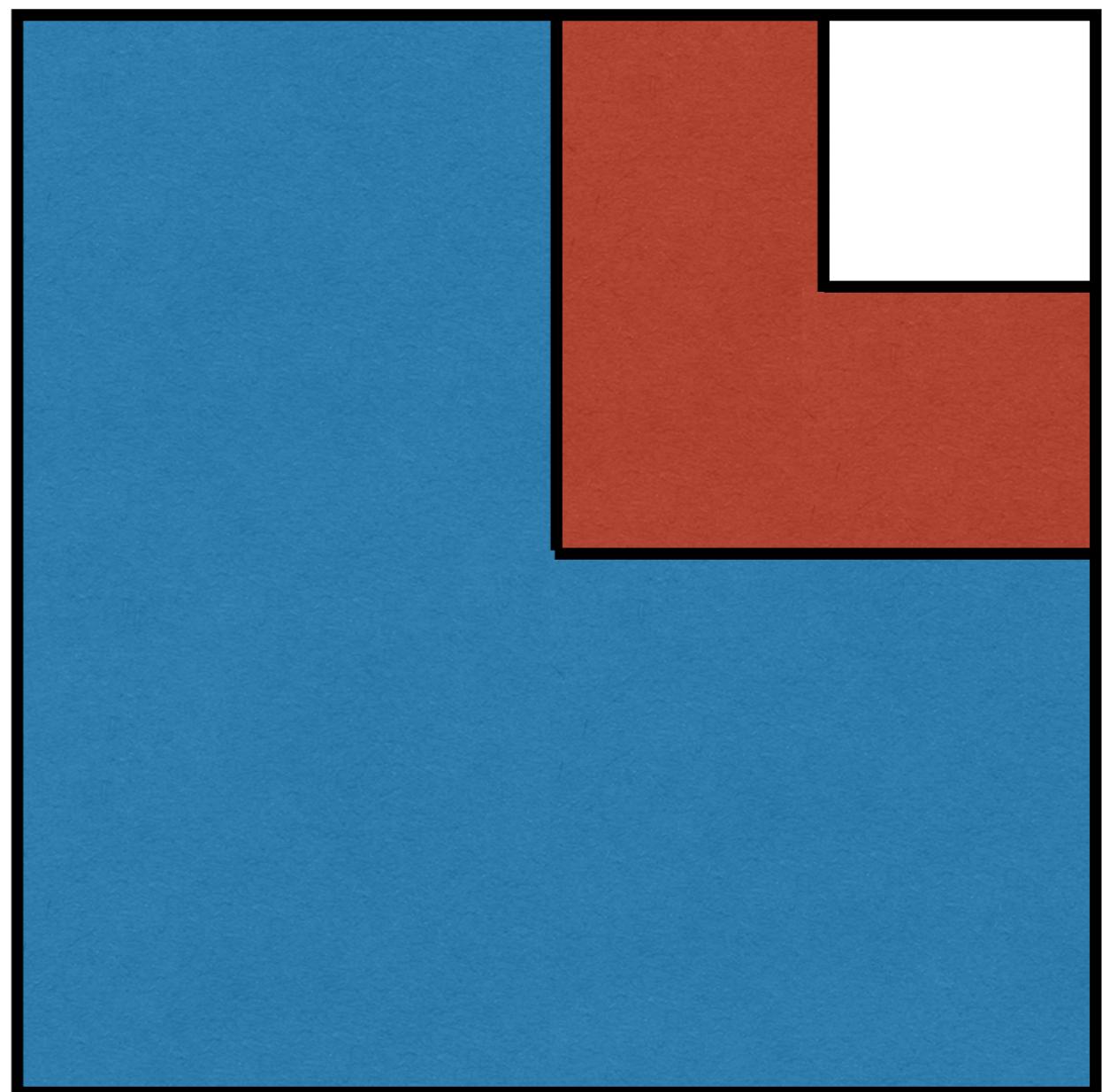
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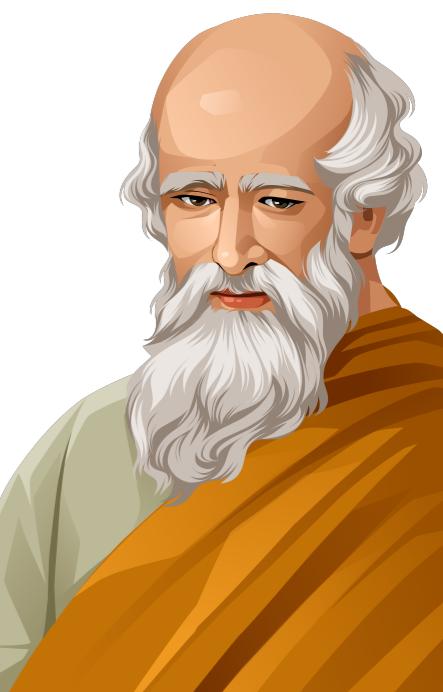
$$1 = \frac{3}{4} + \frac{3}{16}$$



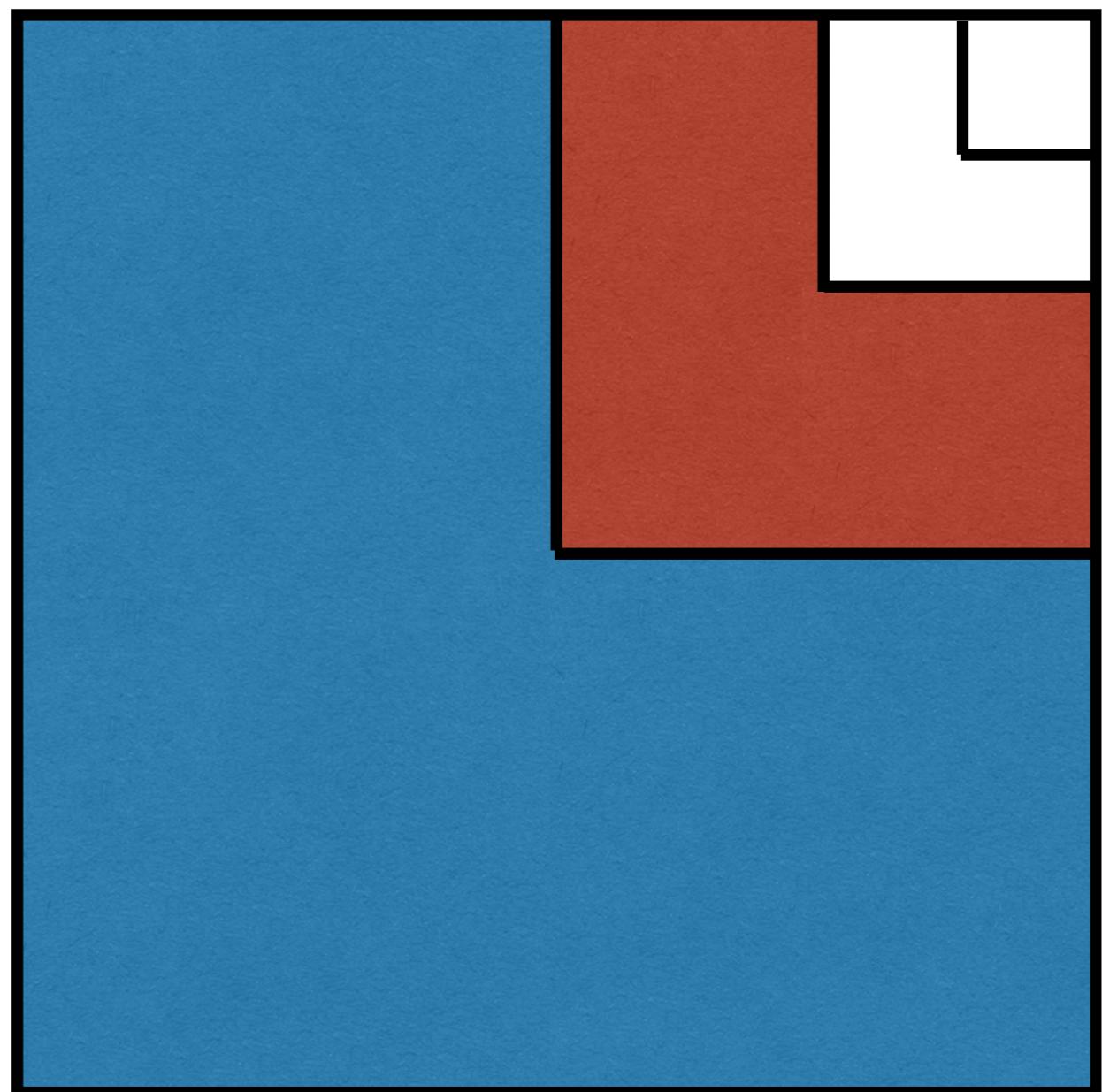
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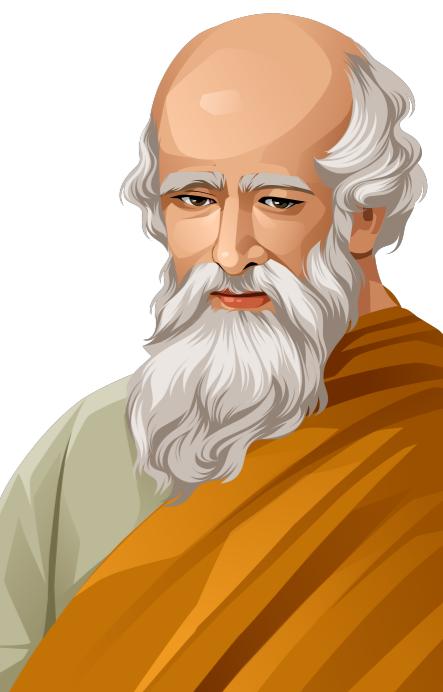
$$1 = \frac{3}{4} + \frac{3}{16}$$



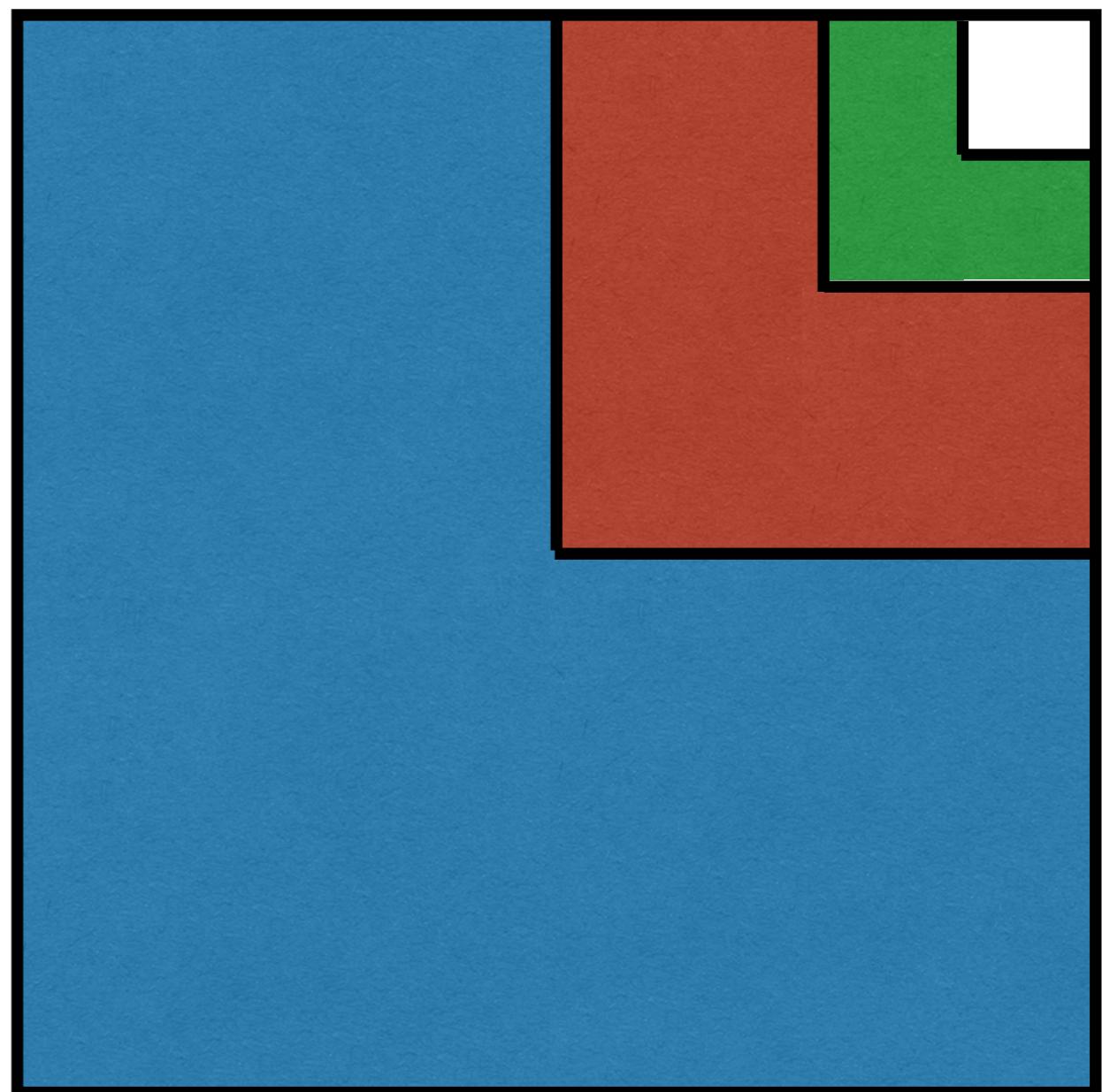
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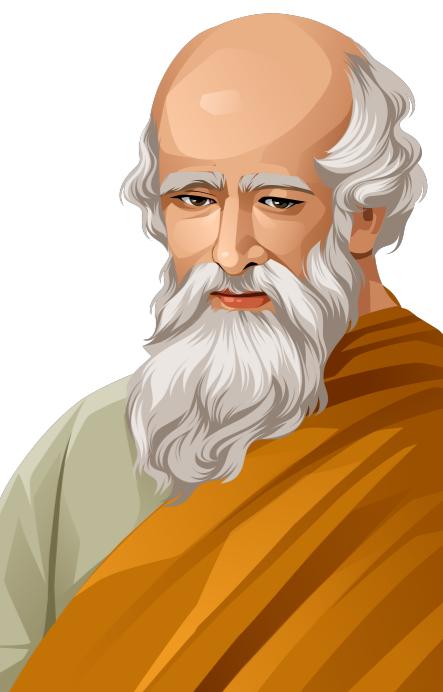


$$1 = \frac{3}{4} + \frac{3}{16}$$

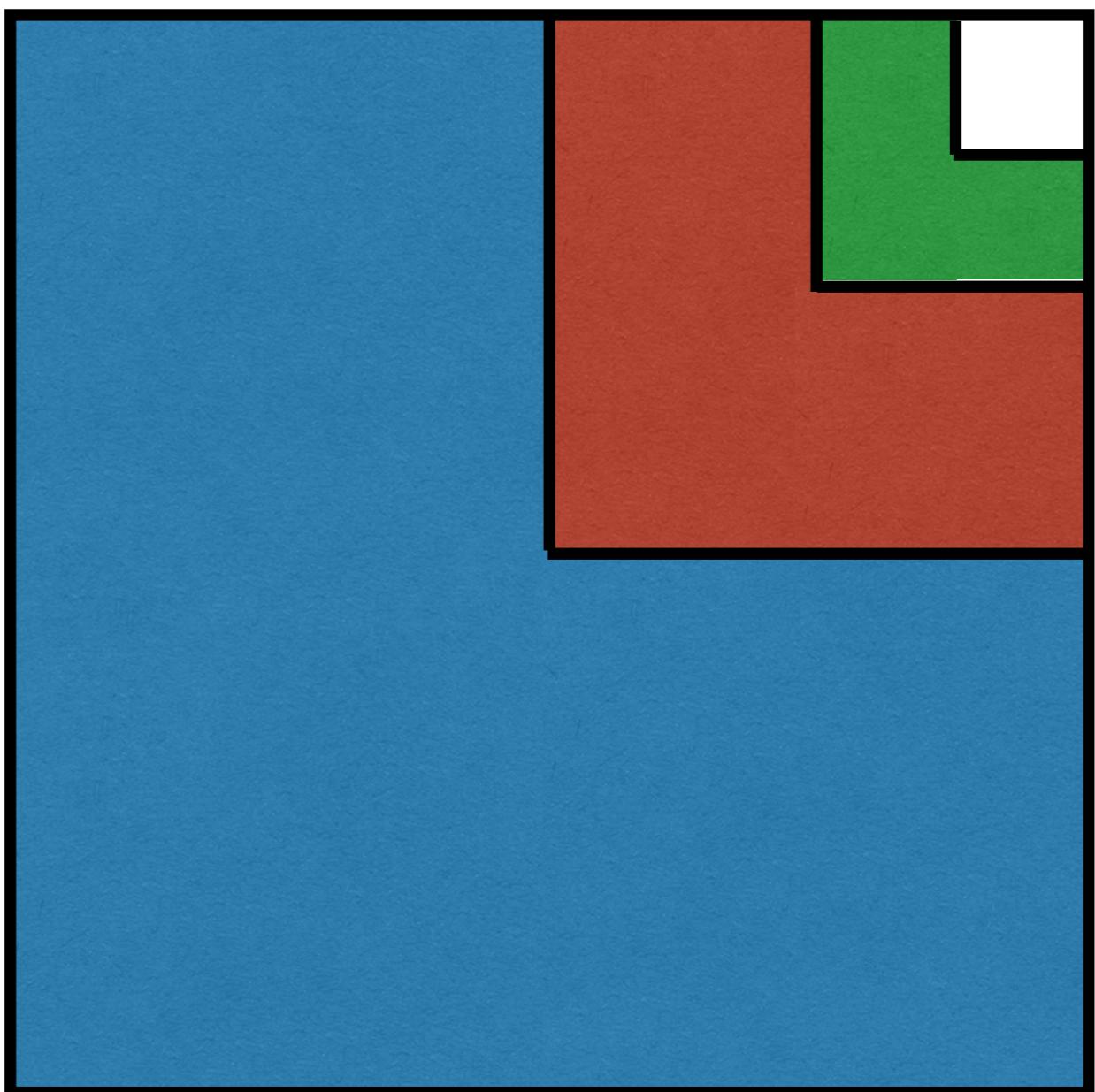


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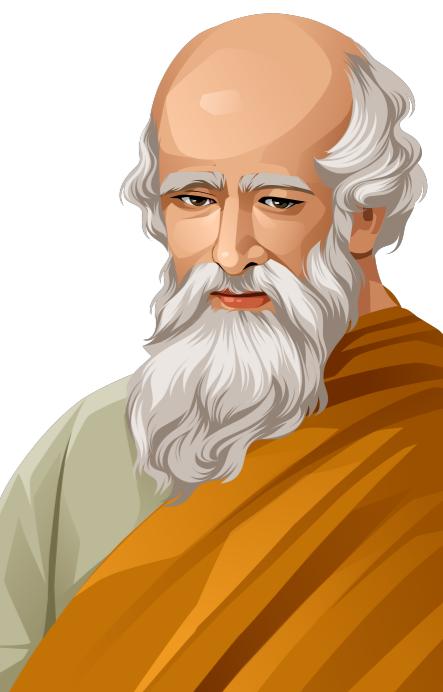


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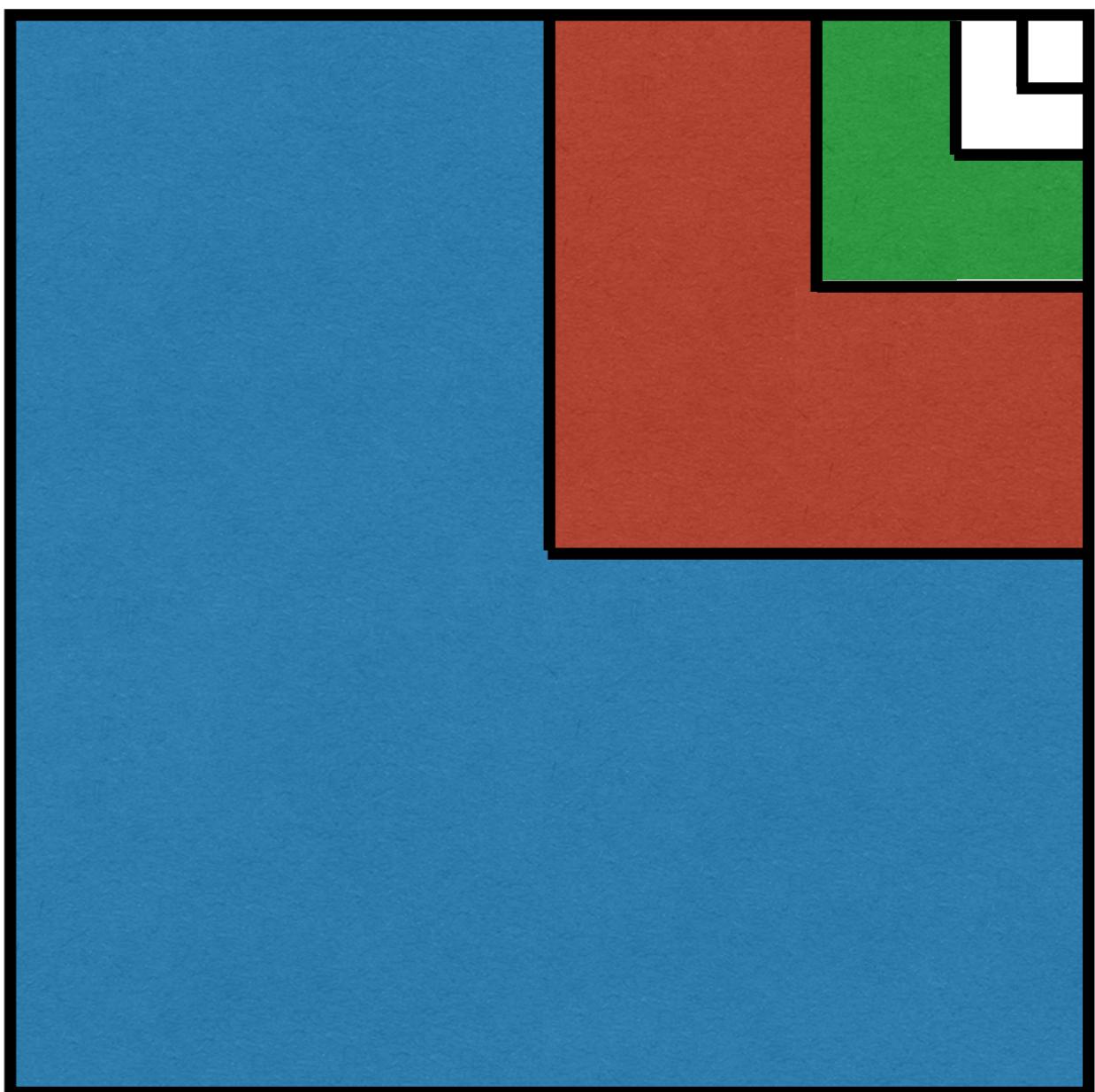


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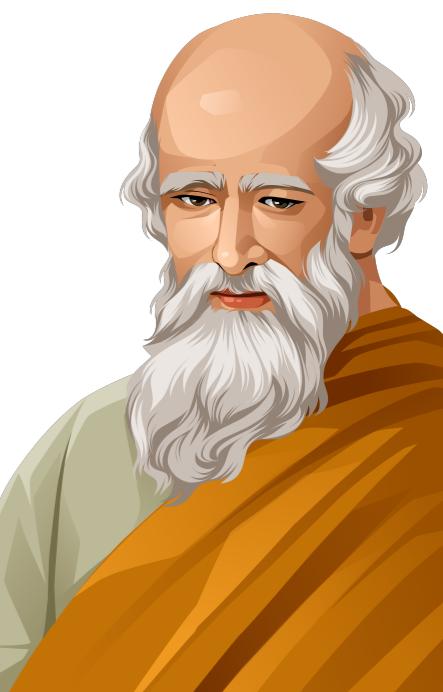


$$1 = \frac{3}{4} + \frac{3}{16} + \frac{3}{64}$$

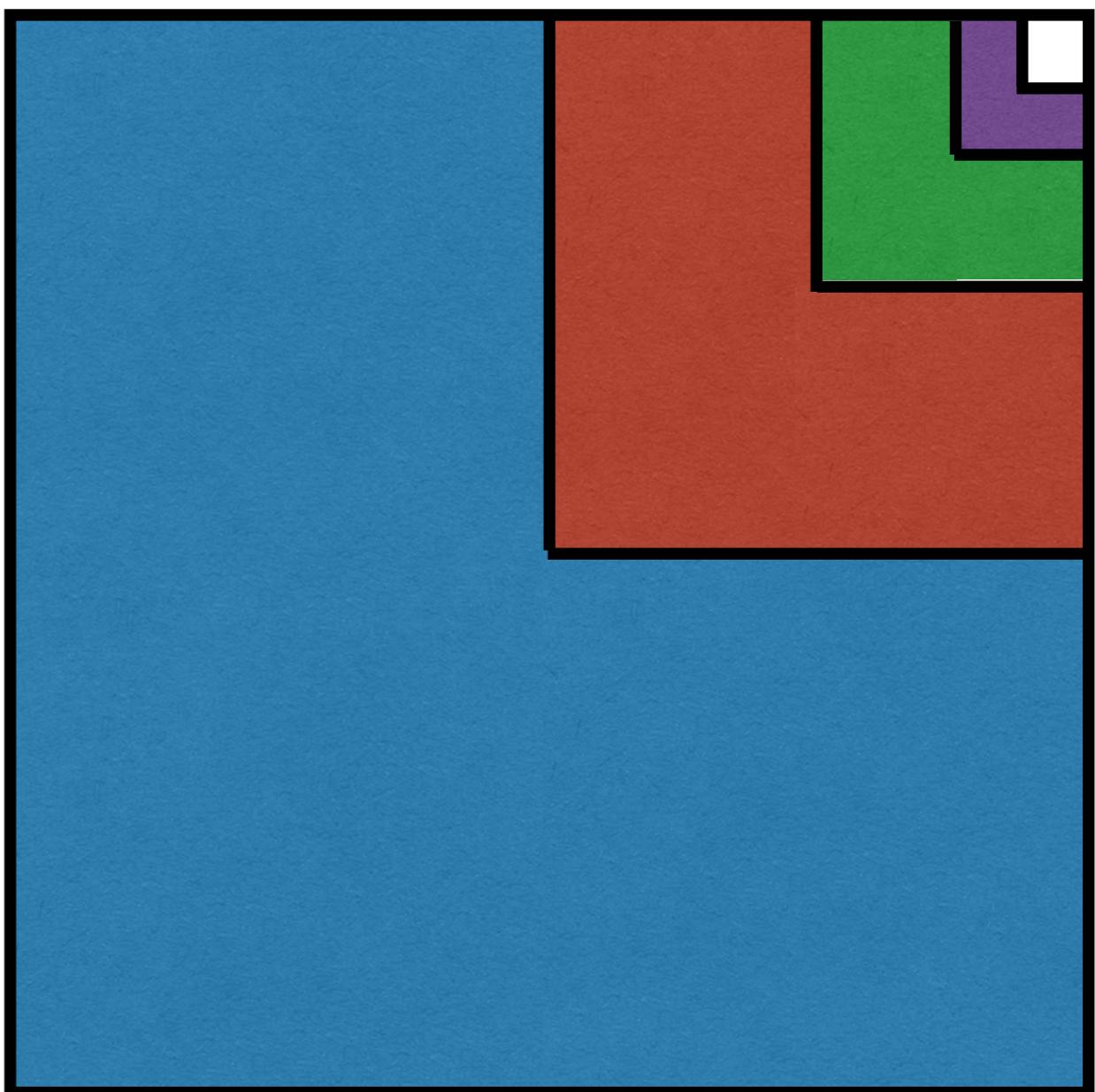


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$$1 = \frac{3}{4} + \frac{3}{16} + \frac{3}{64}$$

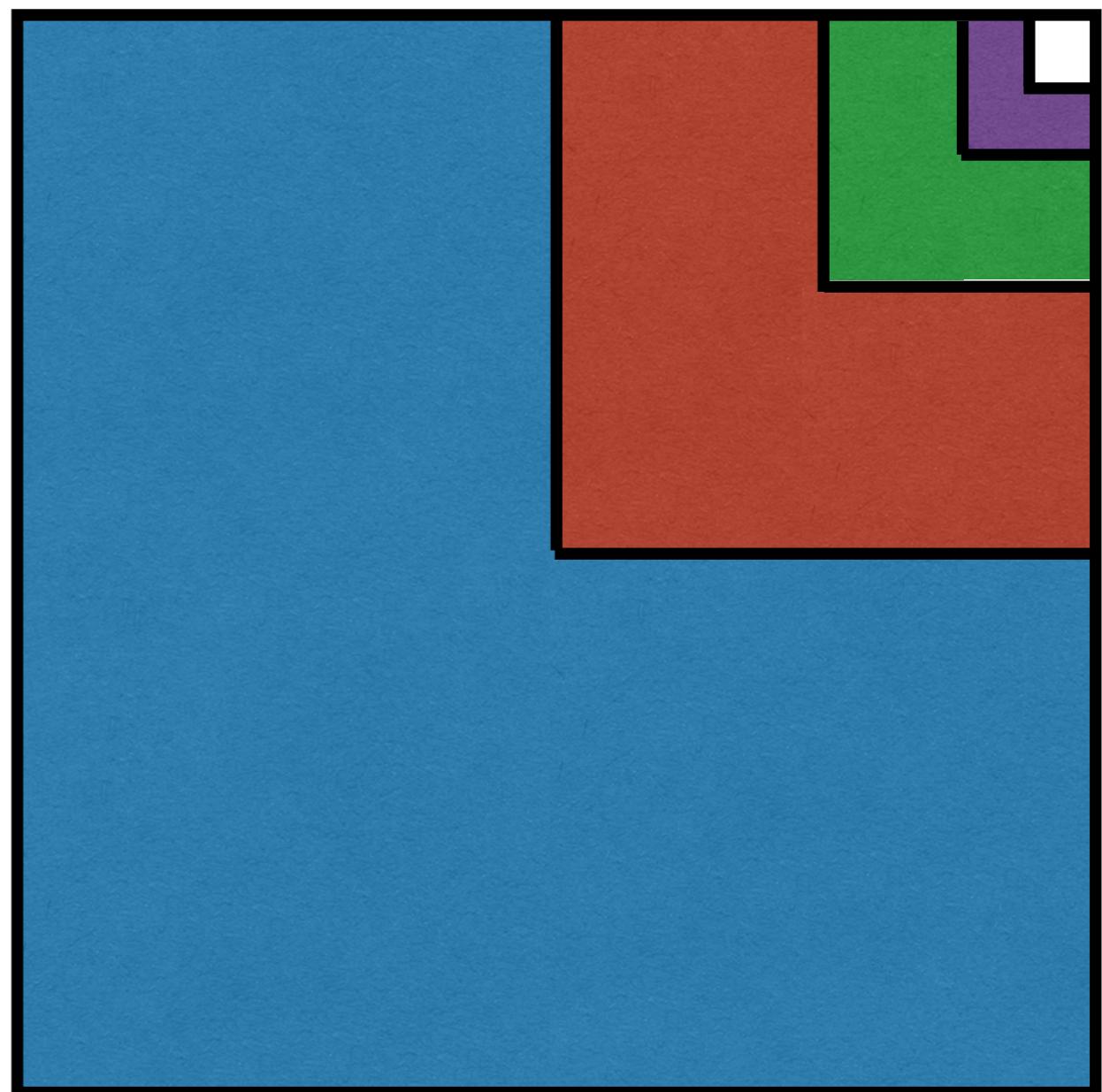


Archimedes

Showing that

$$\frac{1}{3} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \dots$$

$$1 = \frac{3}{4} + \frac{3}{16} + \frac{3}{64} + \frac{3}{256}$$

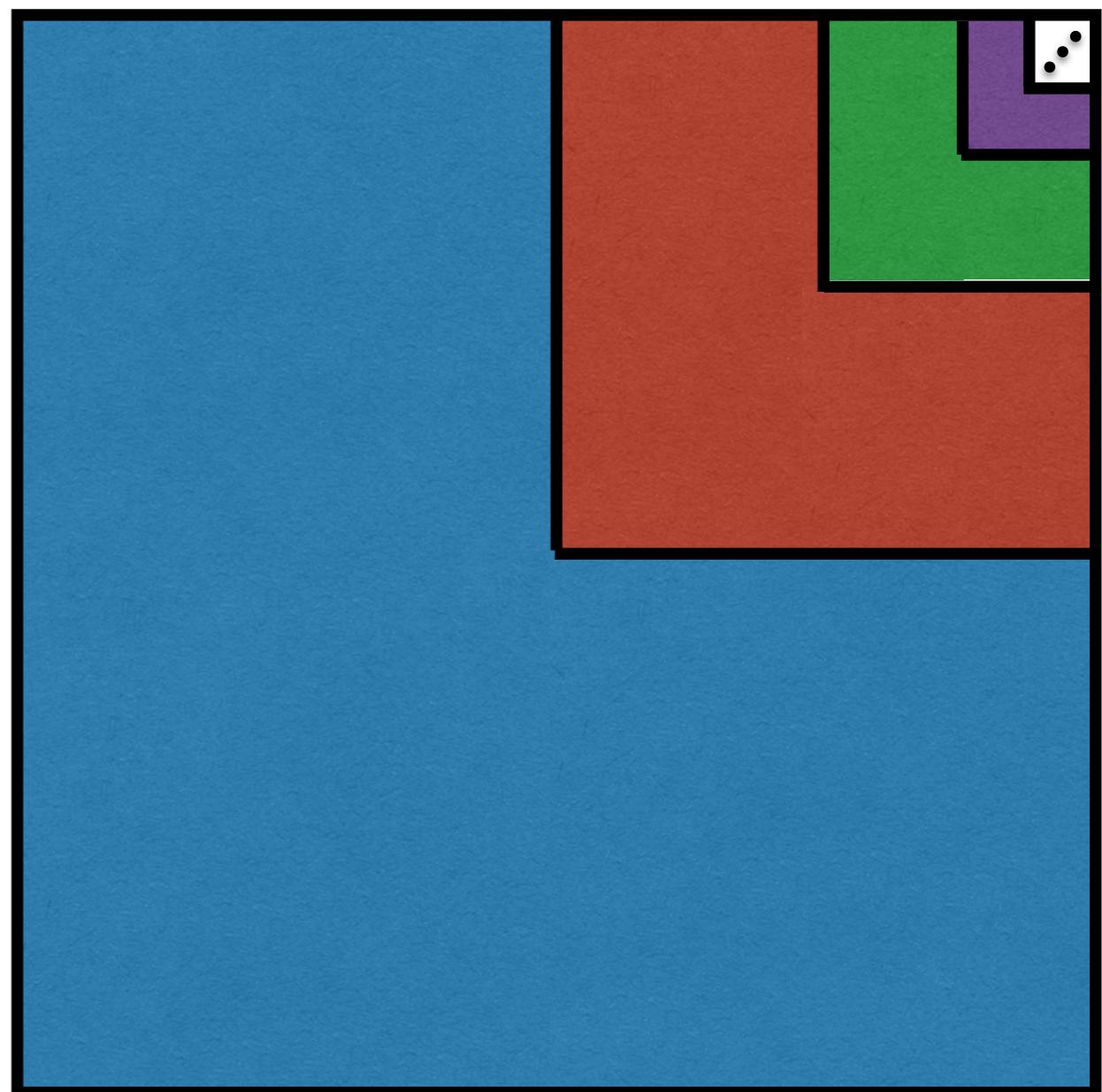


Archimedes

Showing that

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$$1 = \frac{3}{4} + \frac{3}{16} + \frac{3}{64} + \frac{3}{256}$$

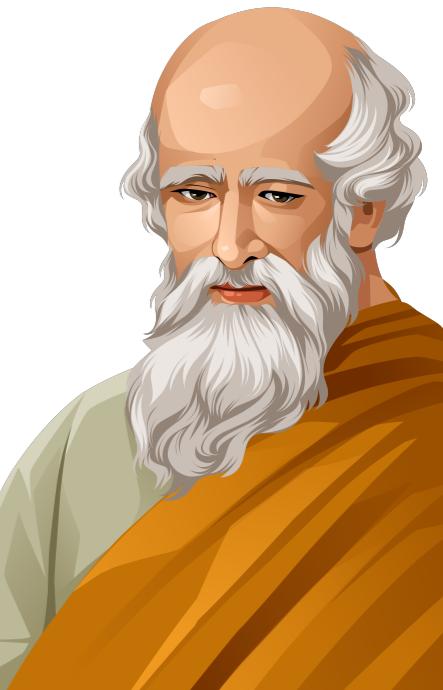
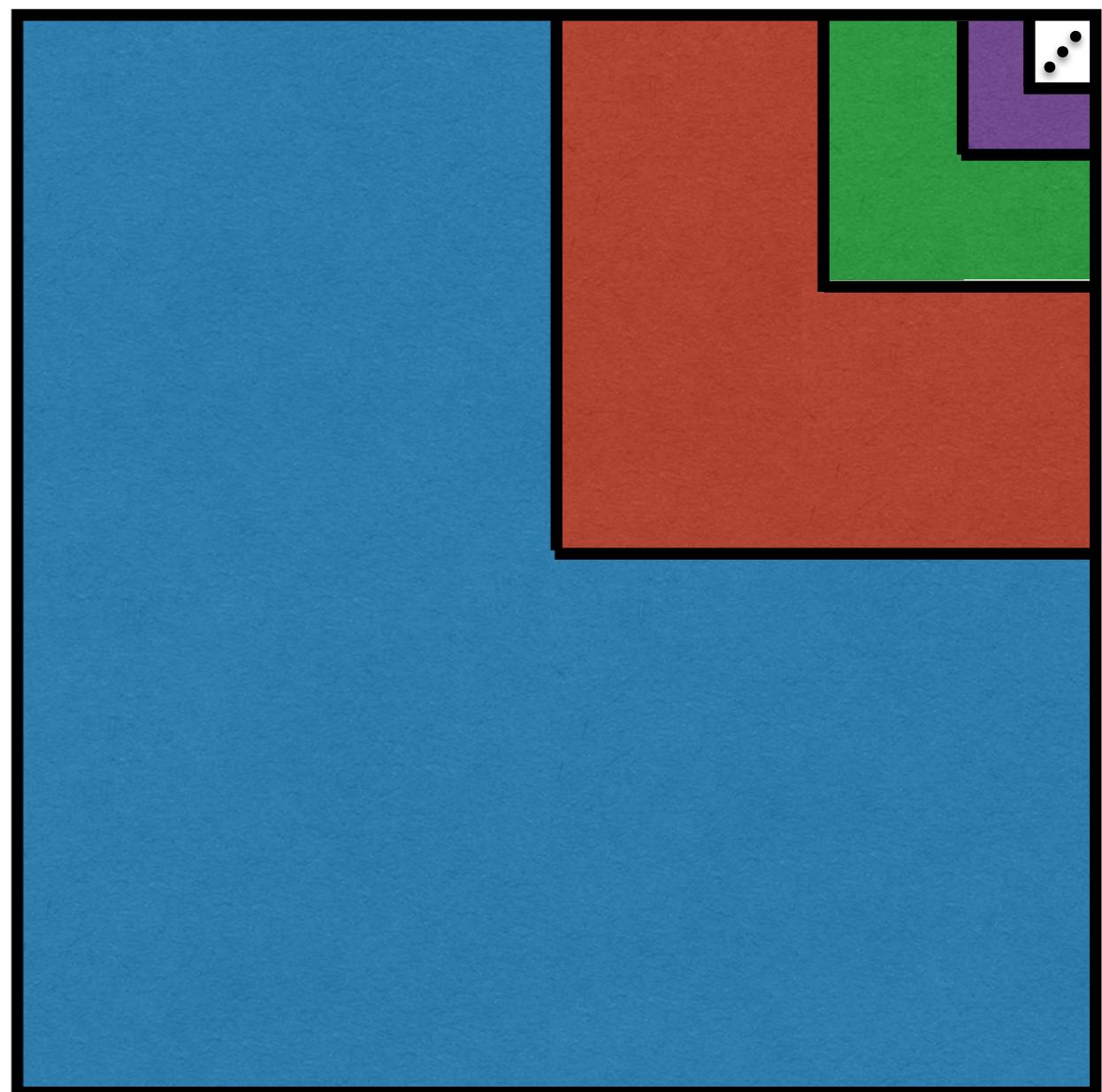


Archimedes

Showing that

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$$1 = \frac{3}{4} + \frac{3}{16} + \frac{3}{64} + \frac{3}{256} + \dots$$

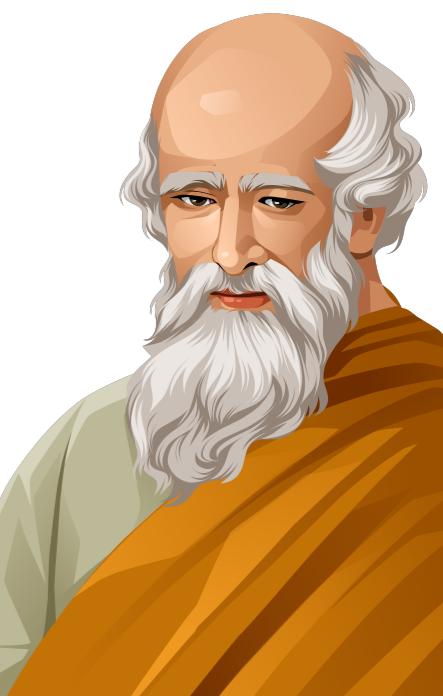
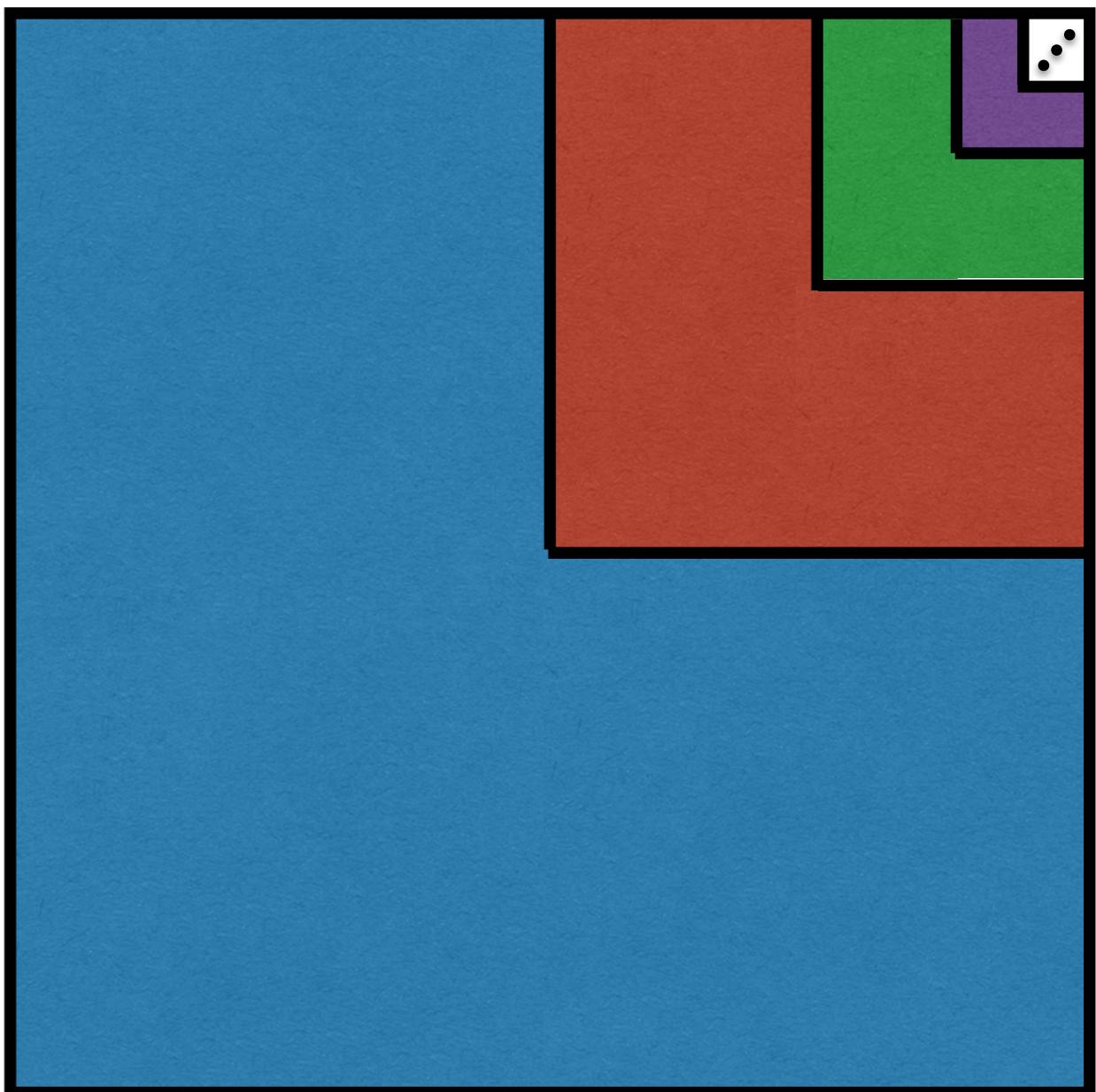


Archimedes

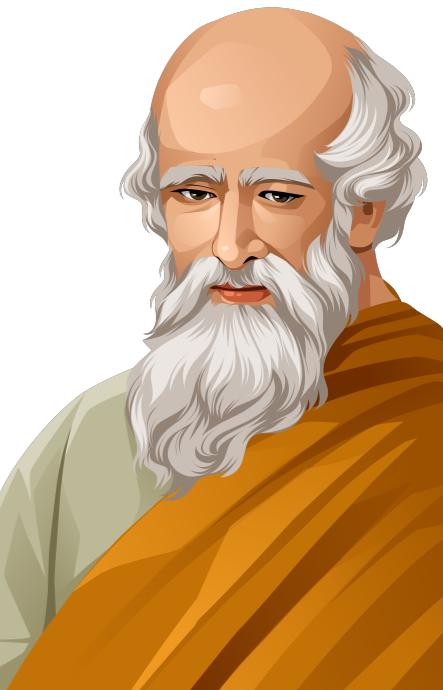
Showing that $\frac{1}{3} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \dots$

$$1 = \frac{3}{4} + \frac{3}{16} + \frac{3}{64} + \frac{3}{256} + \dots$$

Divide both sides by 3
to get the result.



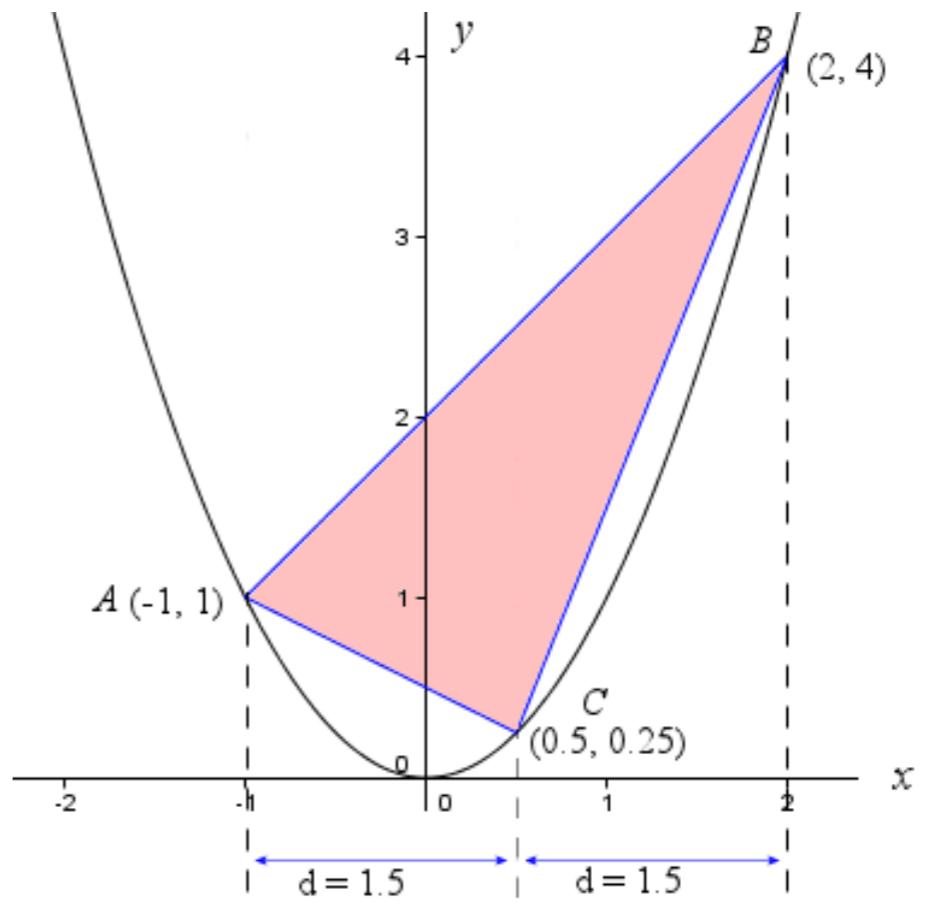
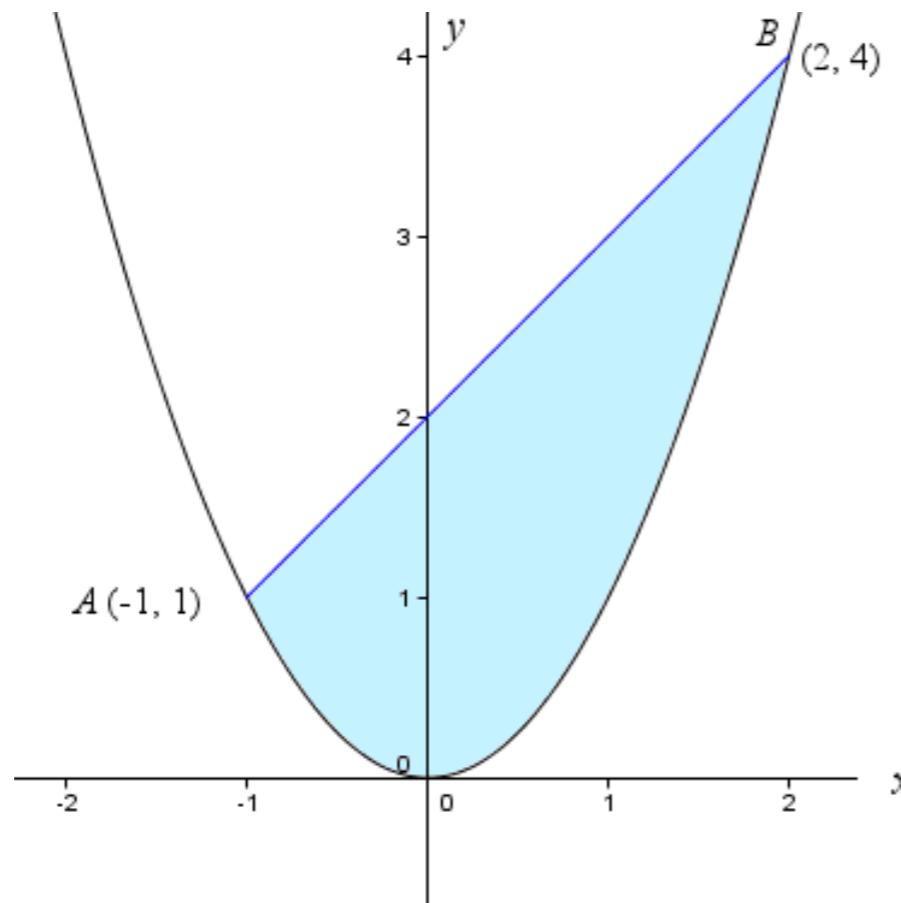
Archimedes



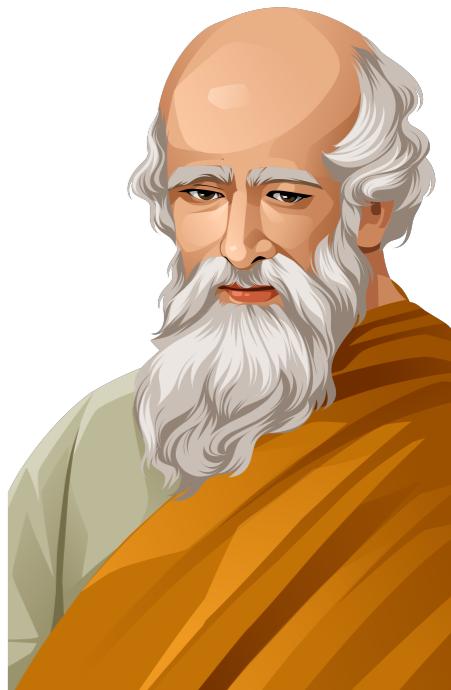
- Thus, the area between the line a parabola is

$$A + A \left(\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots \right) = \frac{4}{3}A,$$

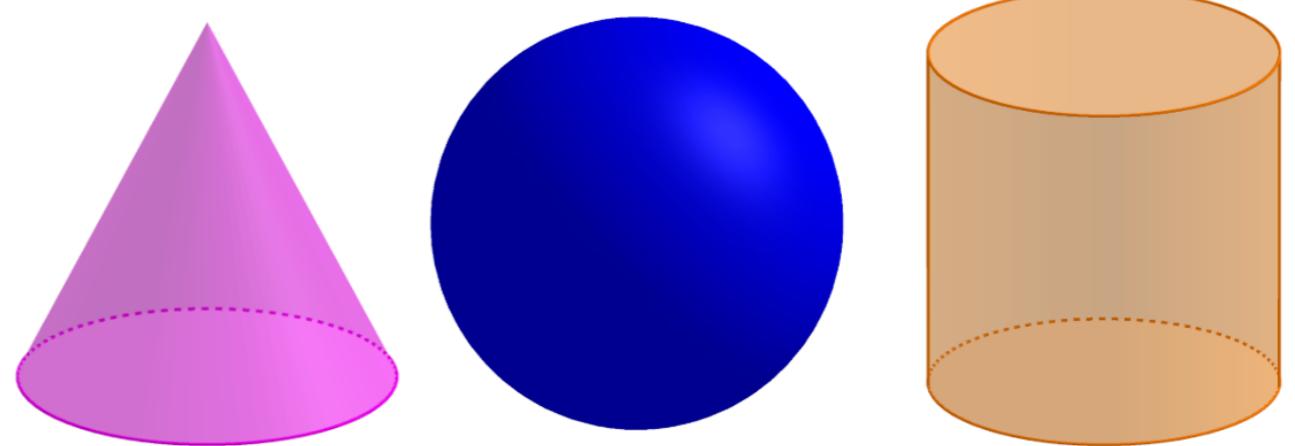
where A is the area of that inscribed triangle.



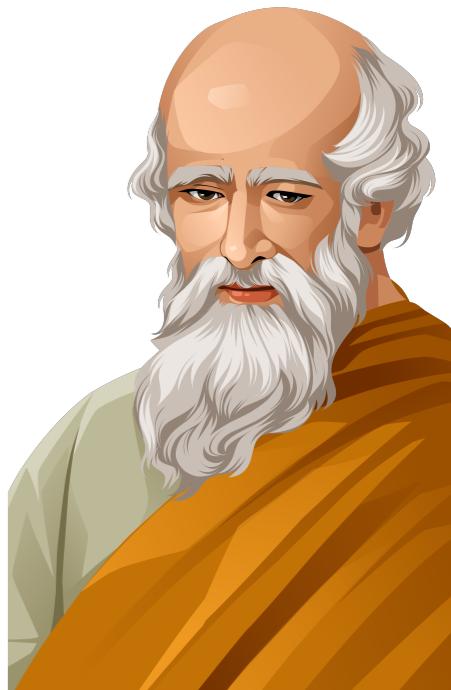
Archimedes



- According to Archimedes, his greatest theorem is that the volumes of a cone, sphere and cylinder are in a 1-to-2-to-3 proportion.
- He proved it using his “mechanical method.”
- He asked that it be carved onto his tombstone, and it was.



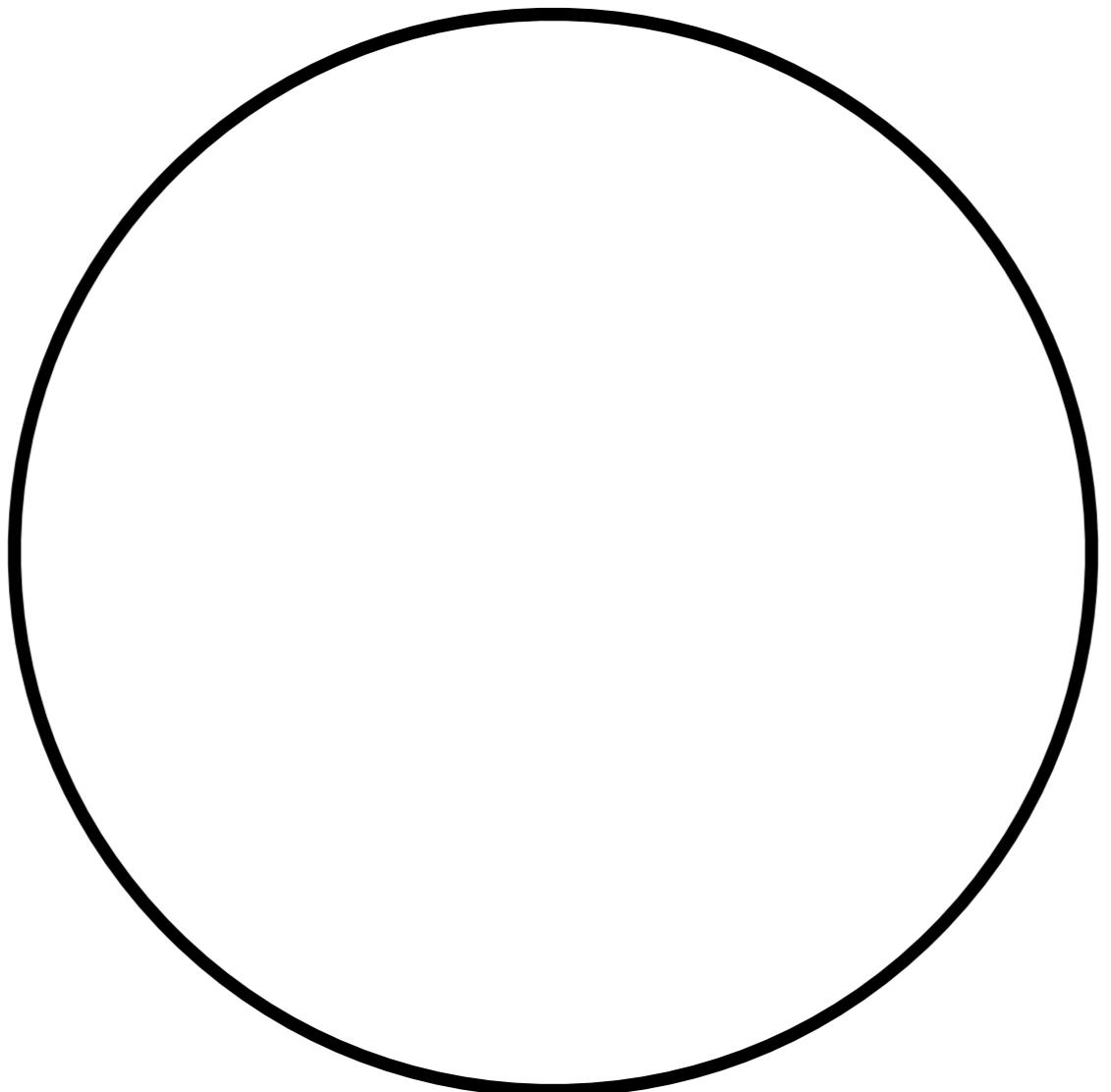
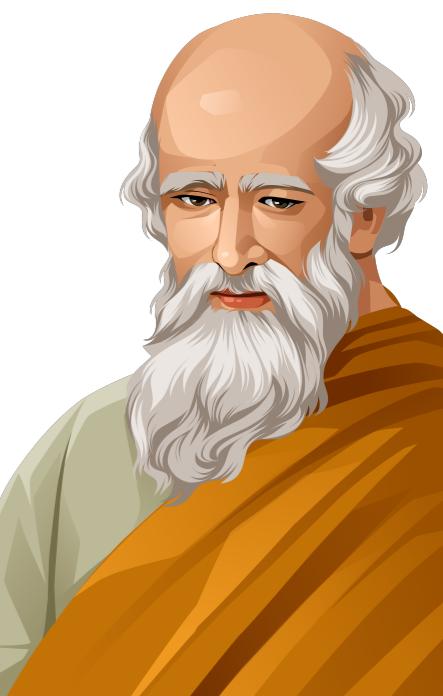
Archimedes



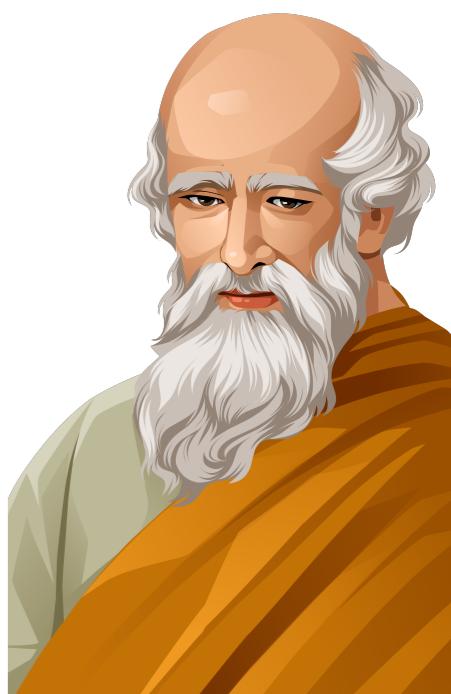
- The “method of exhaustion” is a technique to solve geometry problems involving curvature. It began before Archimedes by Greeks Antiphon, Eudoxus and Euclid.
- After Archimedes it was developed independently in China by Liu Hui.
- But Archimedes truly mastered it and used it to find:
 - Area of a circle, ellipse, spiral and parabolic region;
 - Volume of a sphere, and some solids of revolution;
 - The surface area of a sphere.

Archimedes

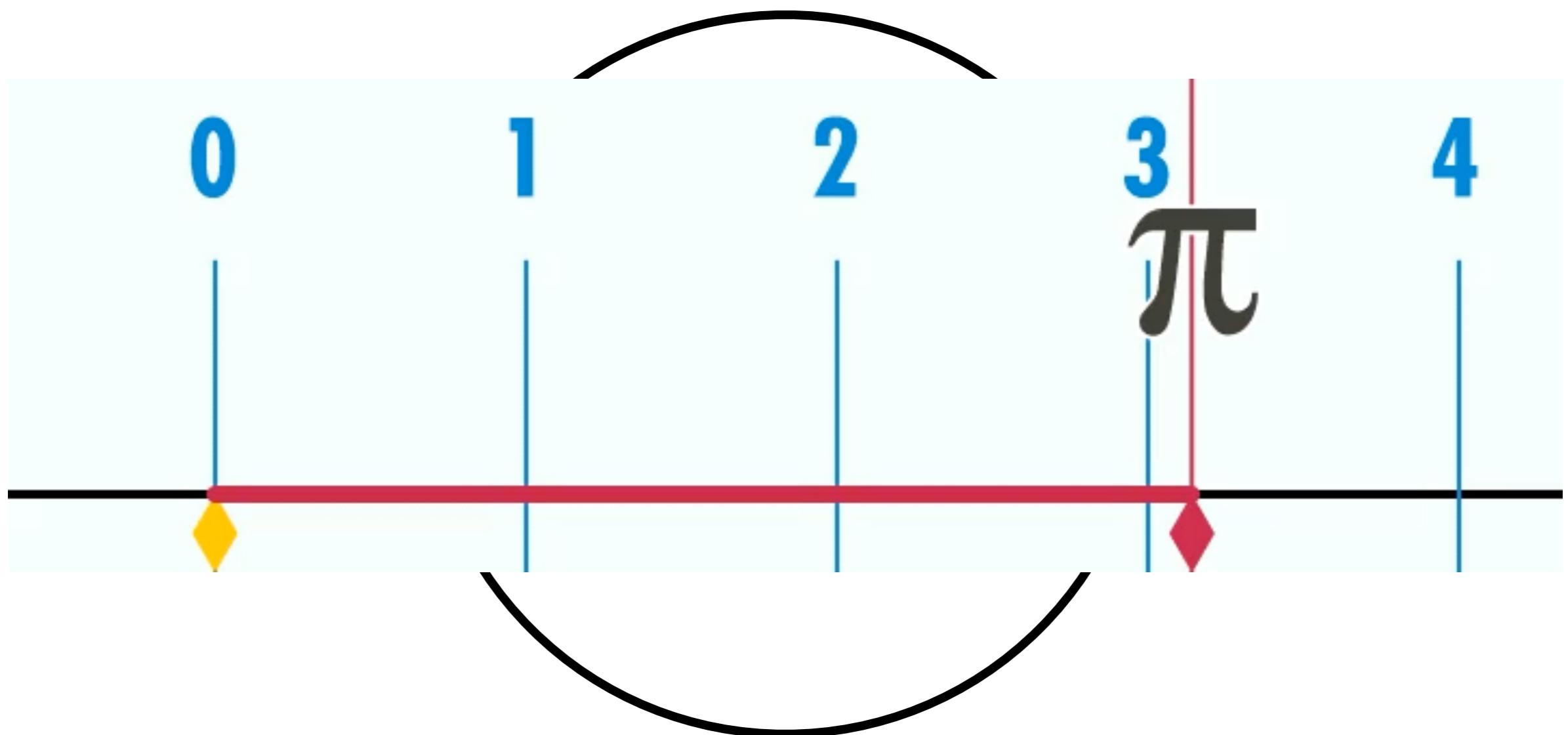
Finding π : The circumference of a circle of diameter 1



Archimedes

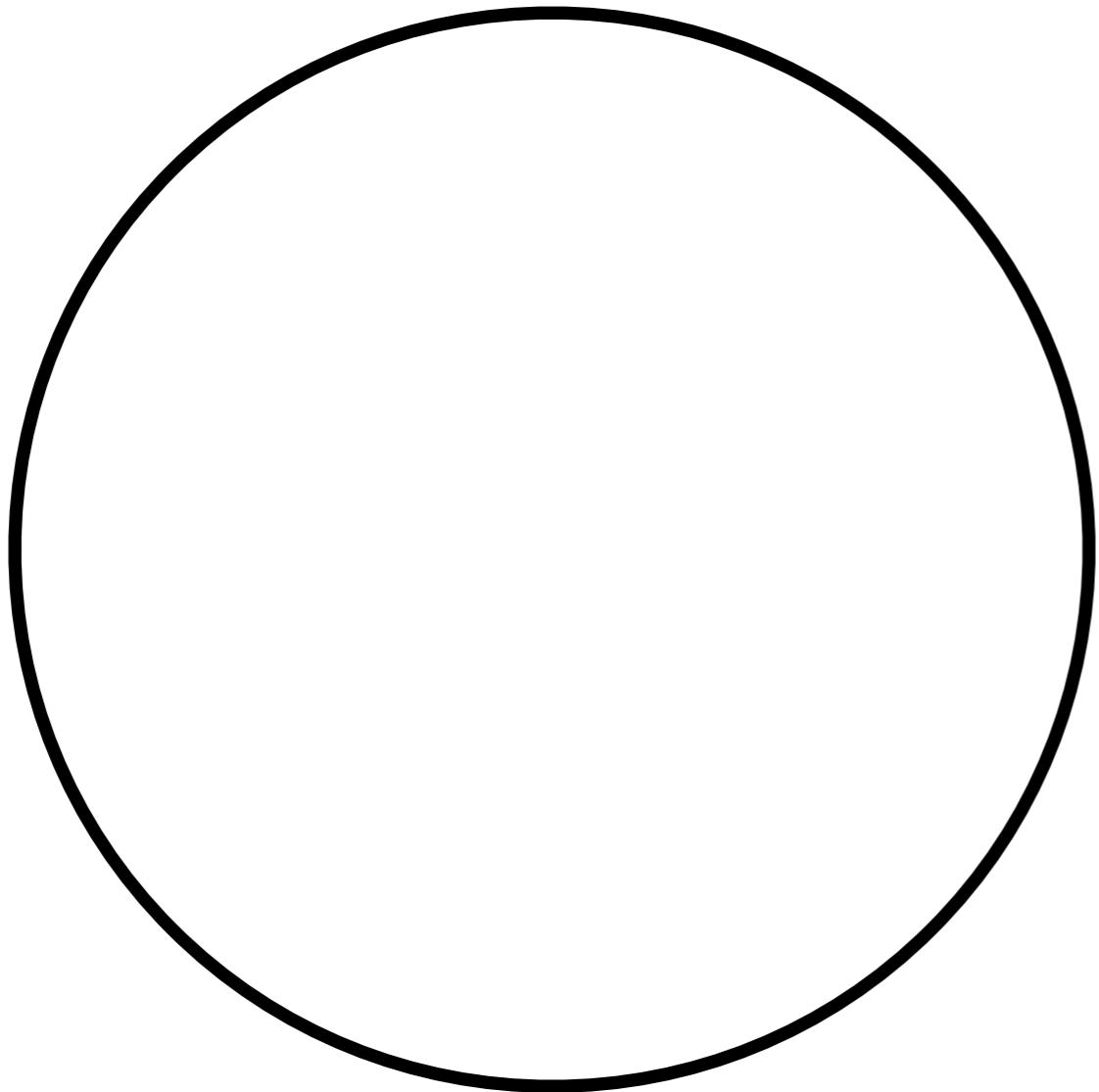
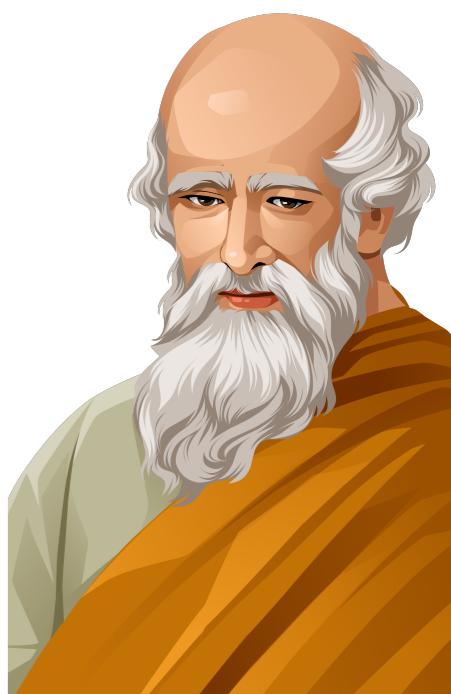


Finding π



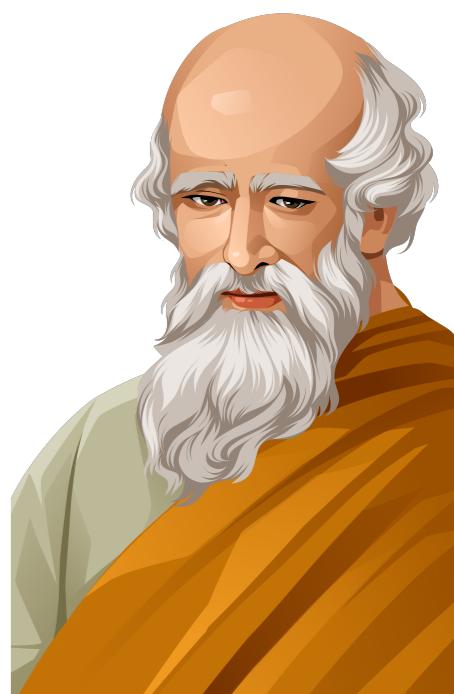
Archimedes

Finding π =3.14159...



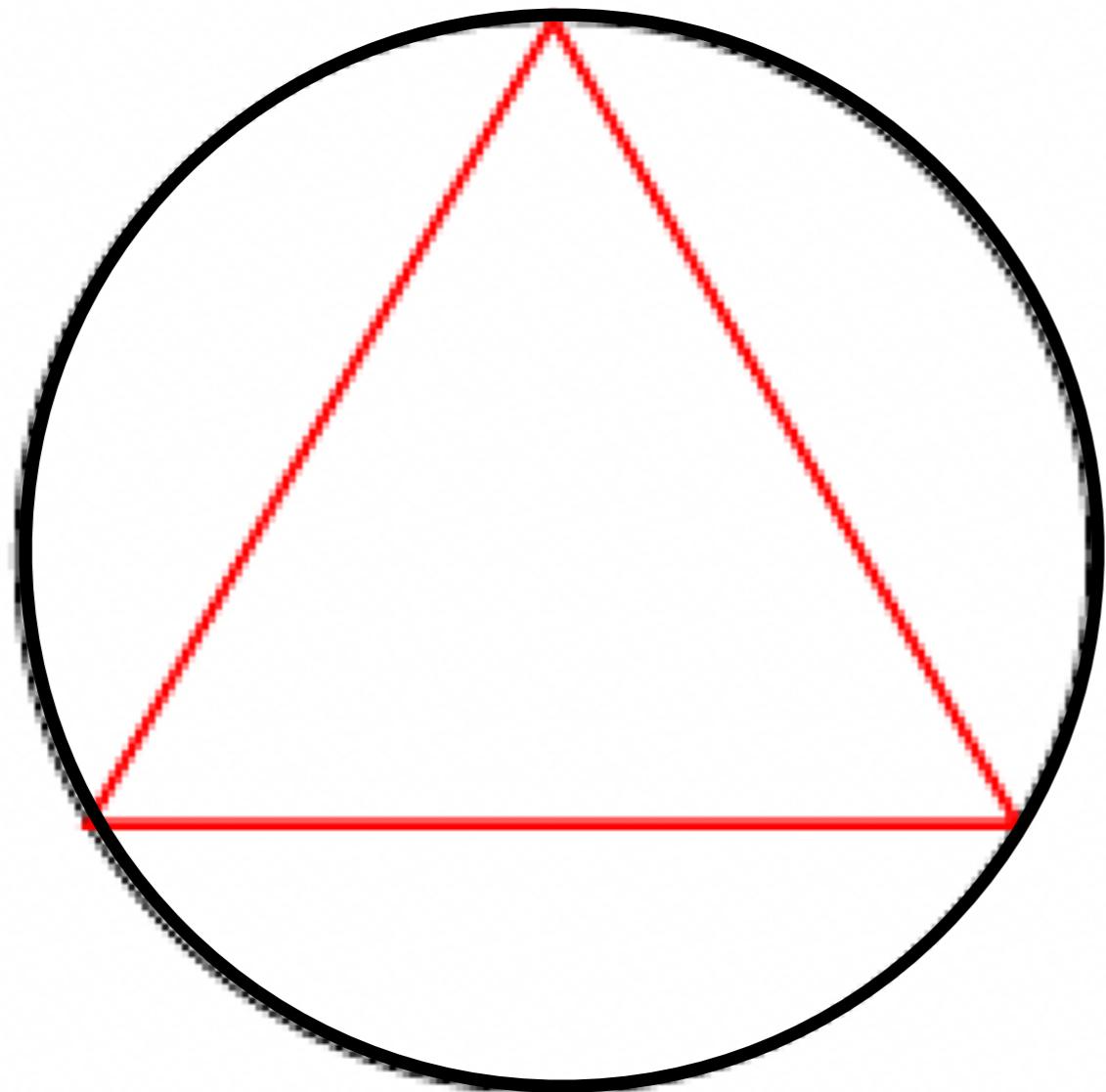
Archimedes

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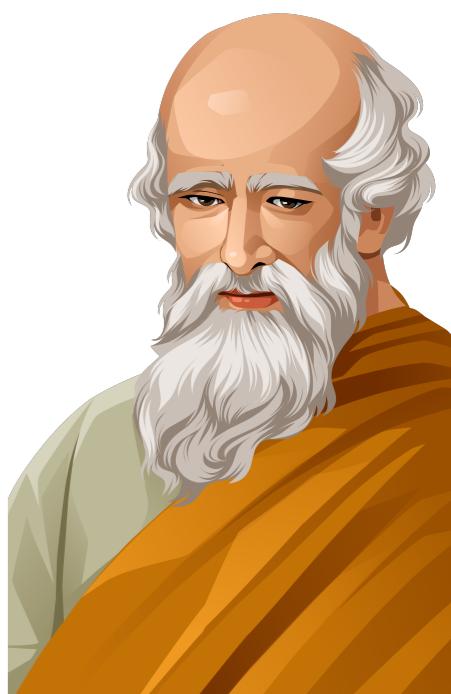
Sides = 3

Perimeter = 2.598



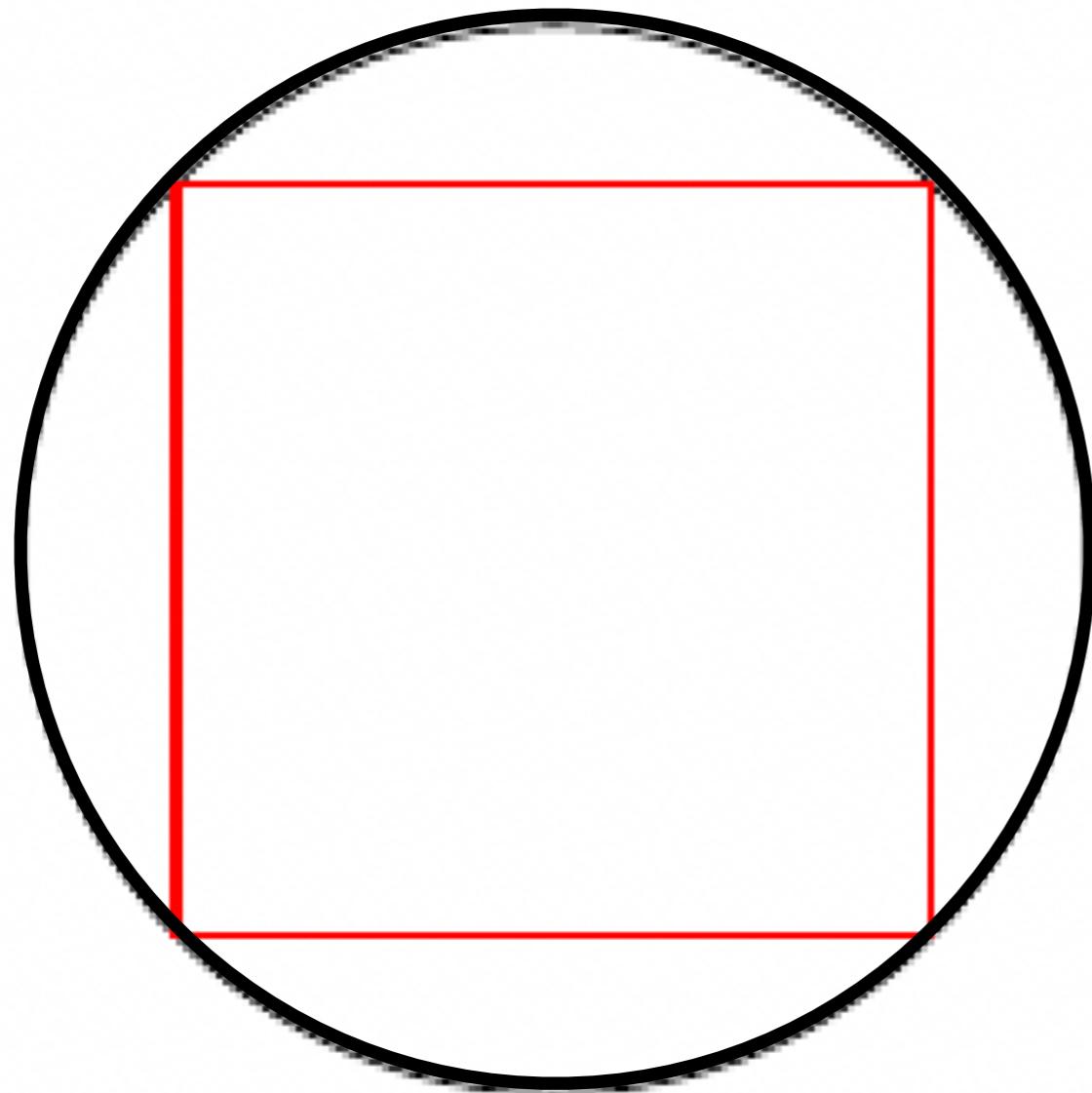
Archimedes

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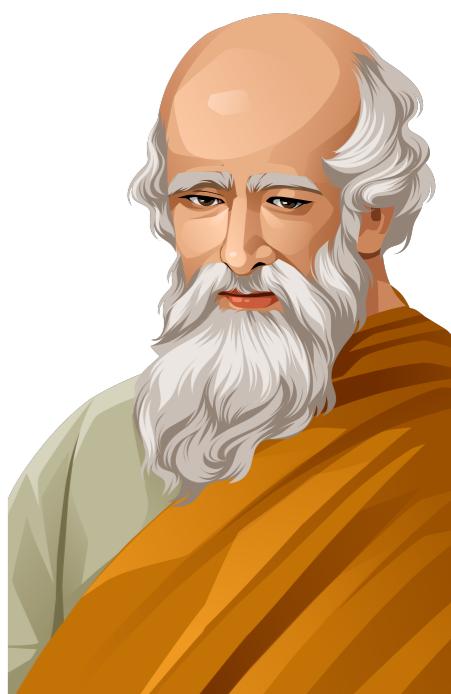
Sides = 4

Perimeter = 2.828



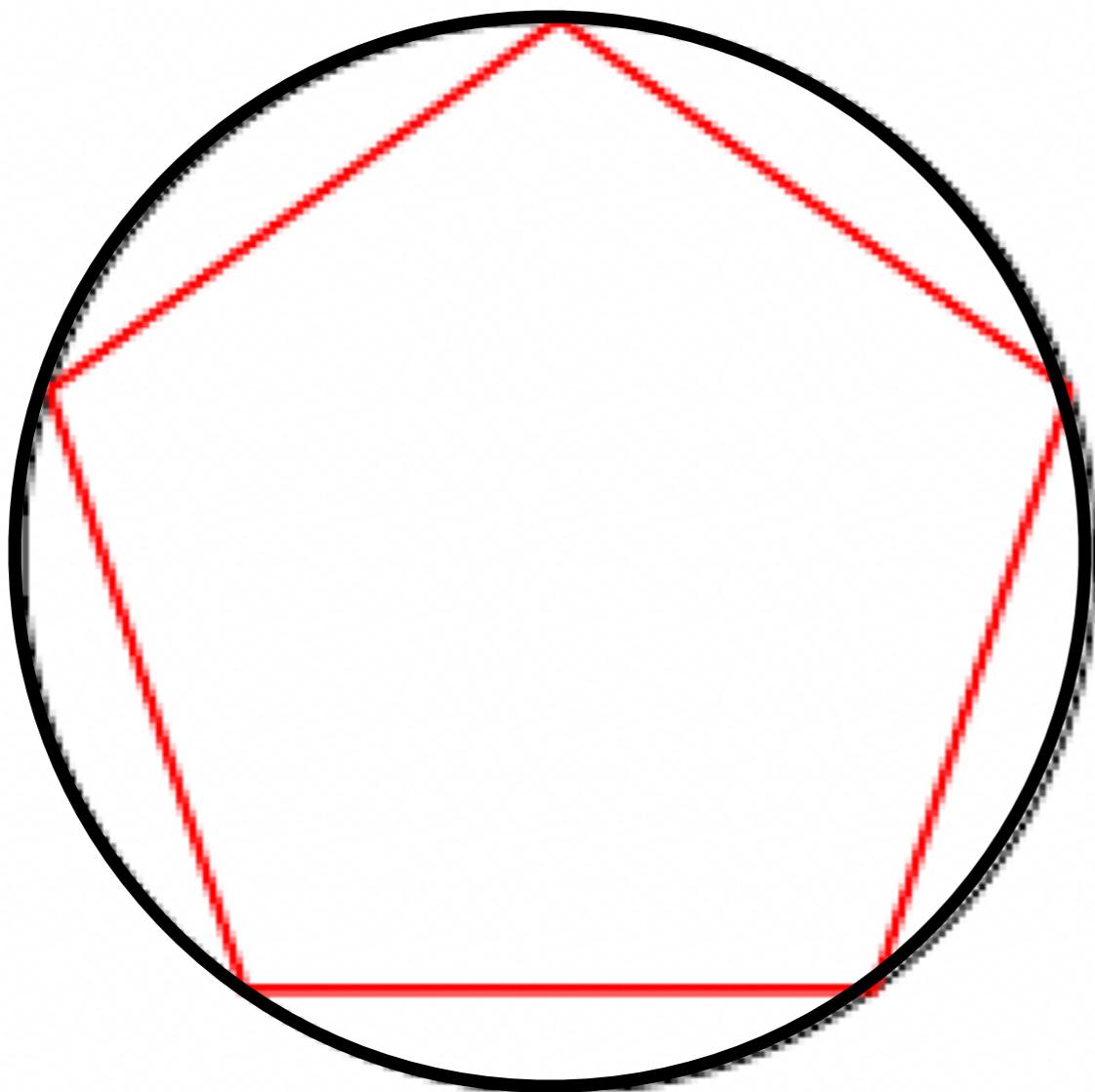
Archimedes

Finding π = 3.14159...



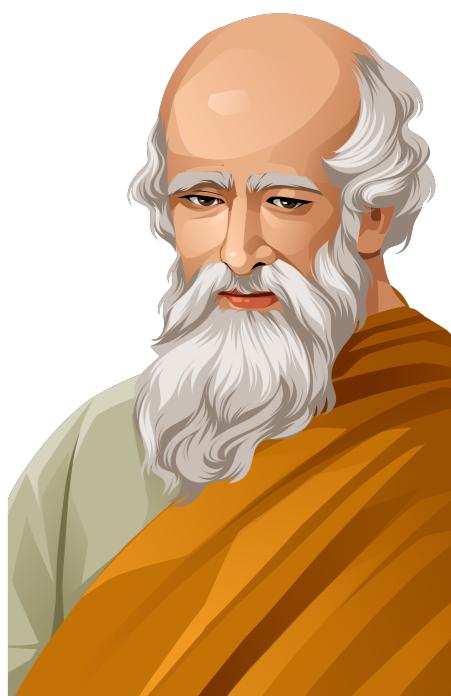
Sides = 5

Perimeter = 2.939



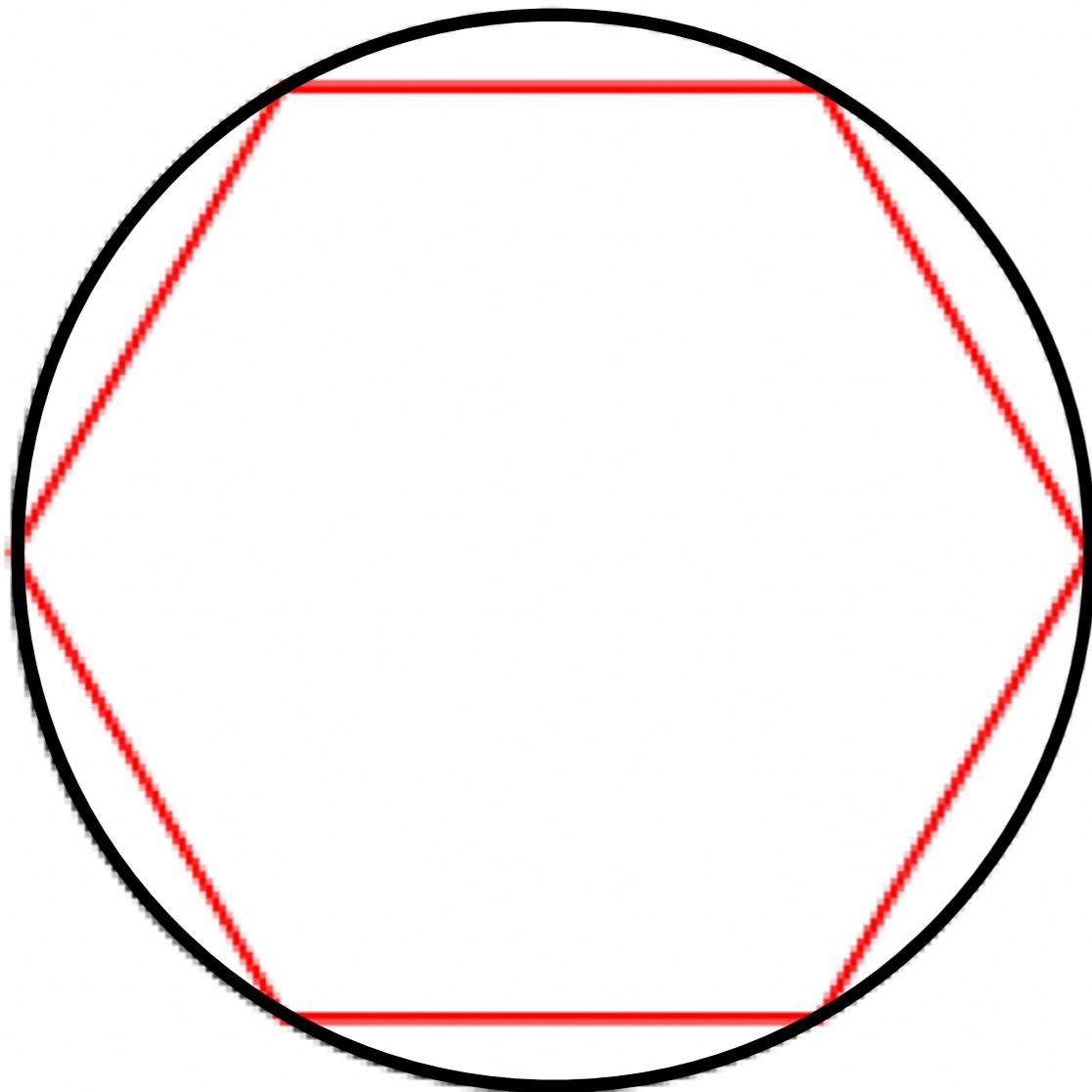
Archimedes

Finding π = 3.14159...



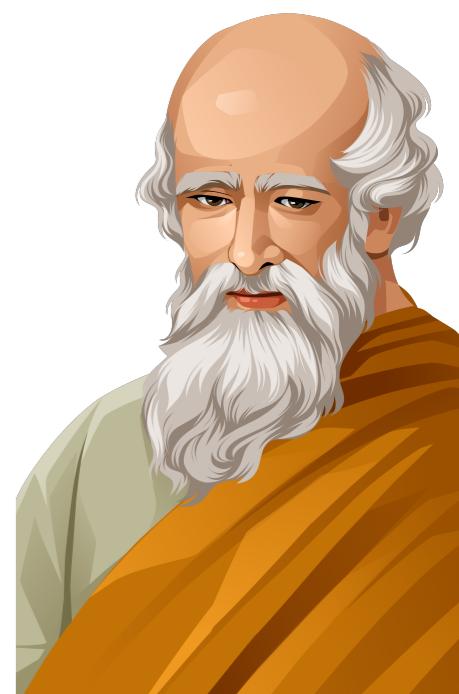
Sides = 6

Perimeter = 3.000



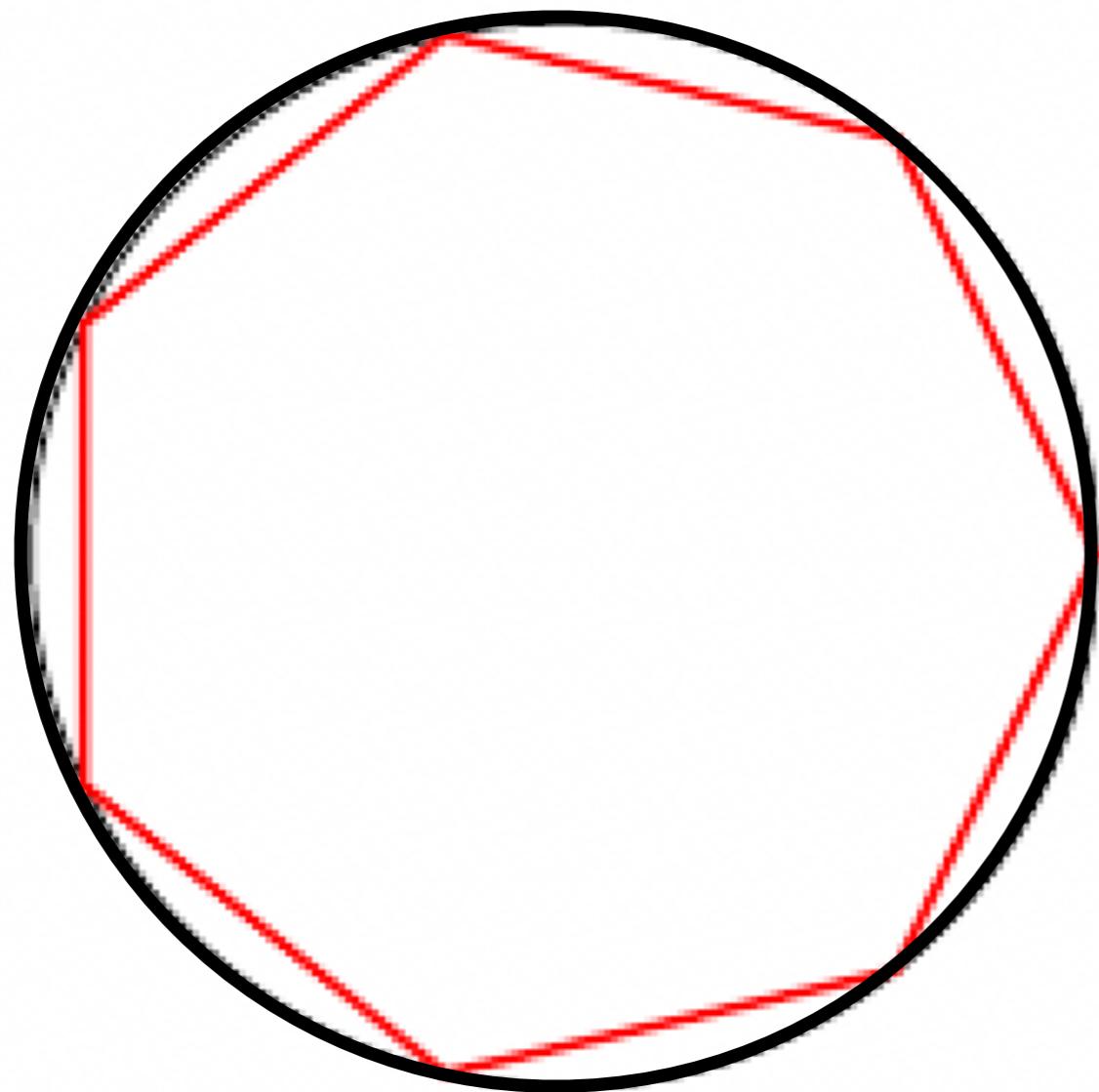
Archimedes

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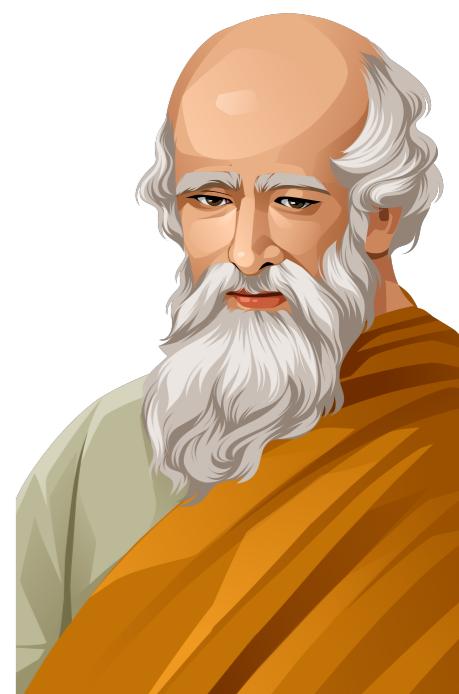
Sides = 7

Perimeter = 3.037



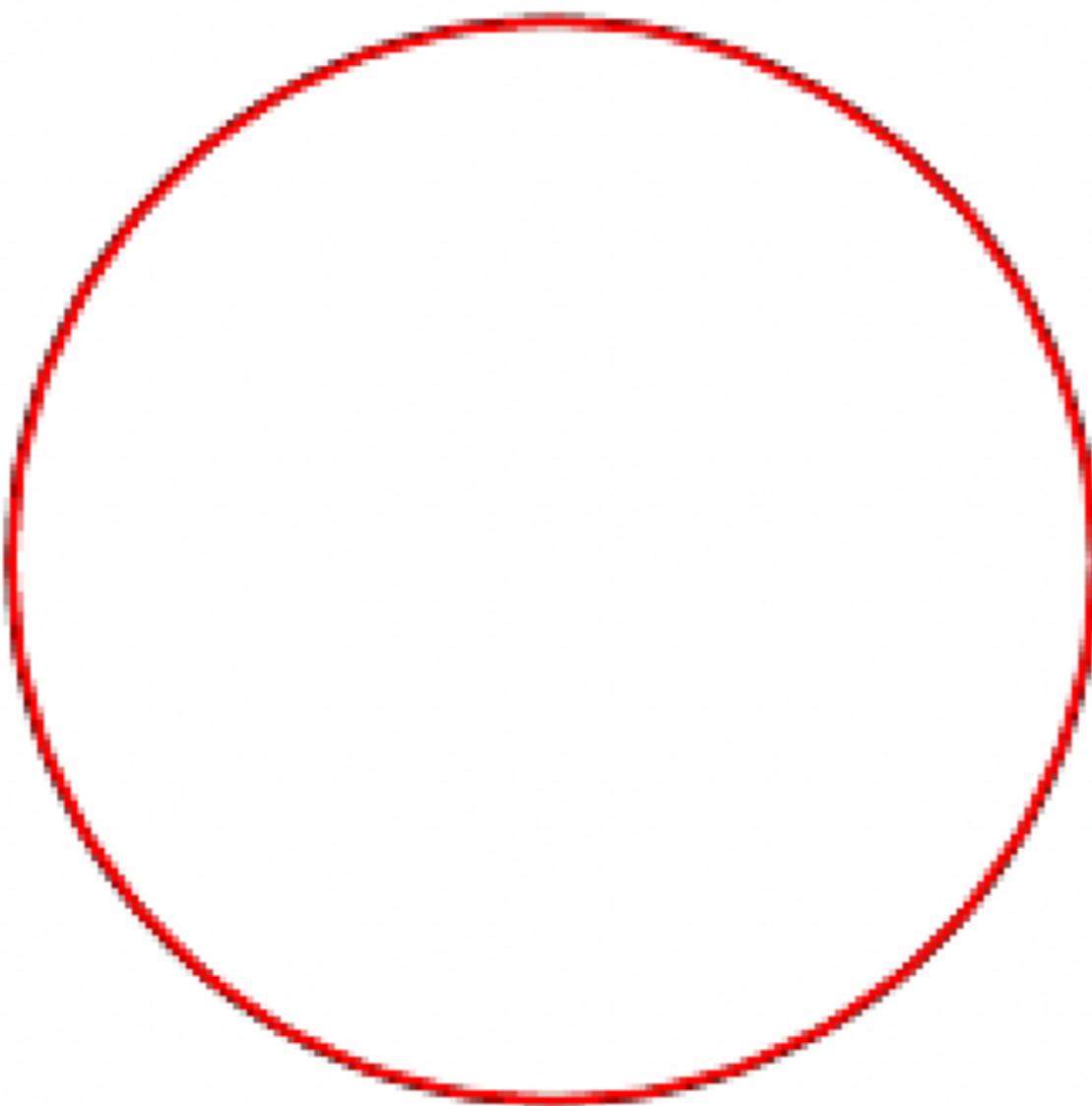
Archimedes

Finding π = 3.14159...



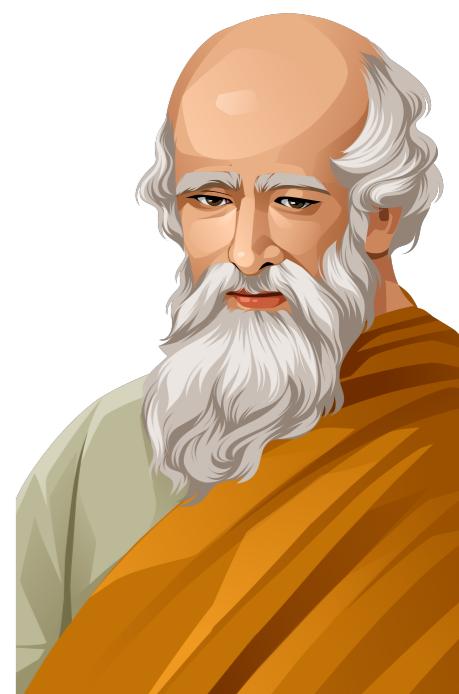
Sides = 10

Perimeter = 3.090



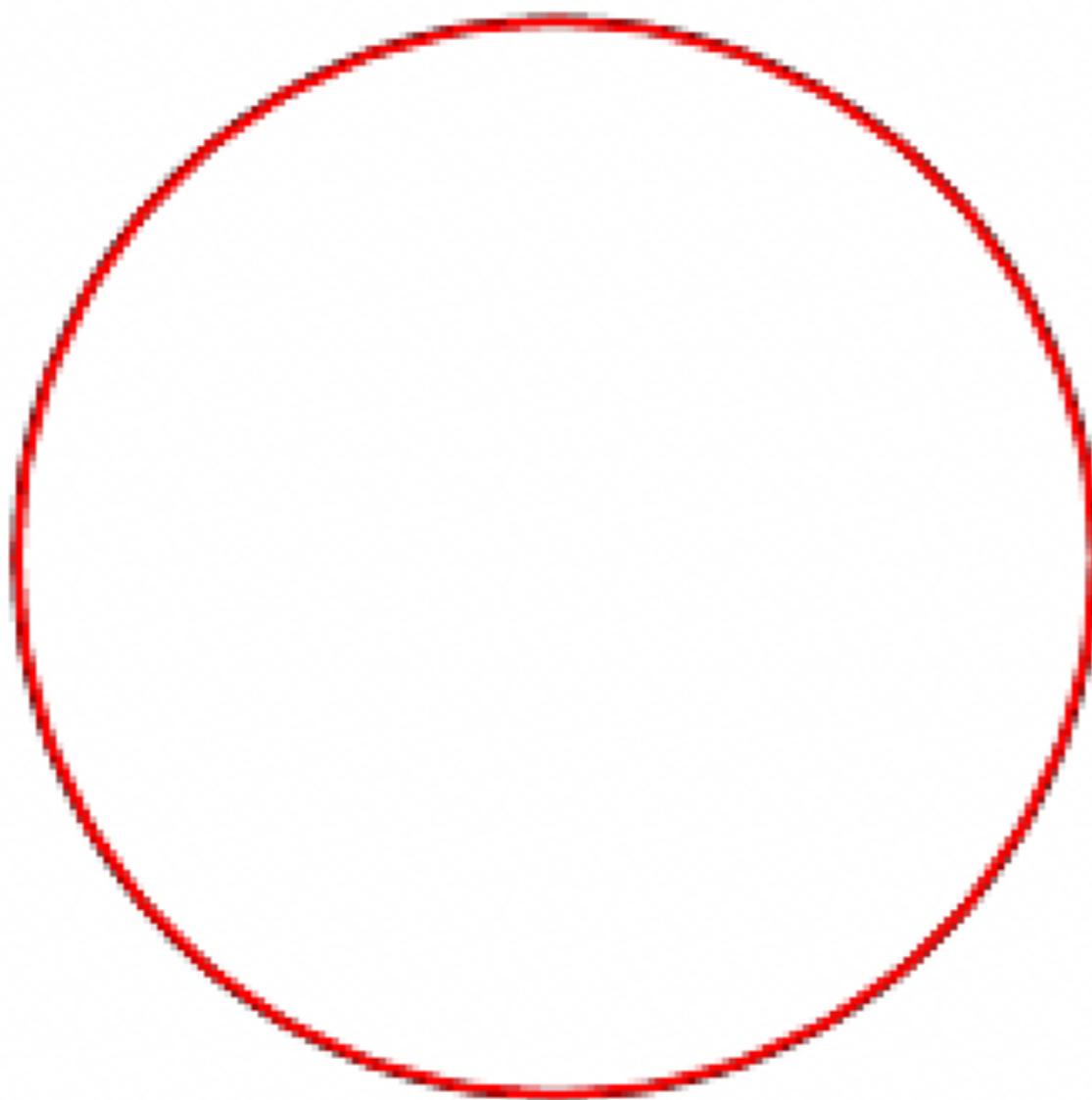
Archimedes

Finding π = 3.14159...



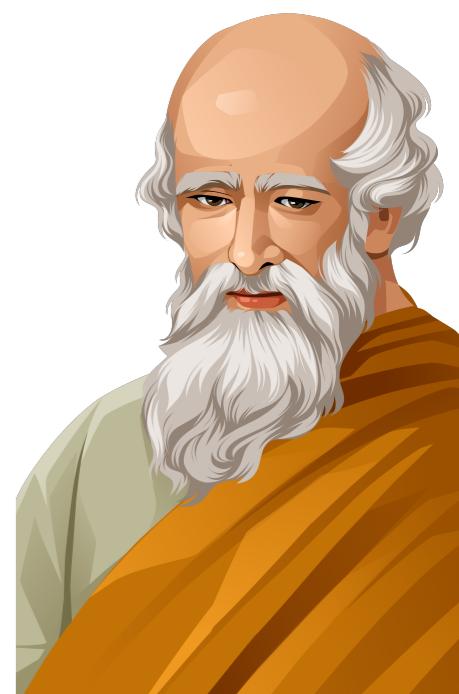
Sides = 50

Perimeter = 3.140



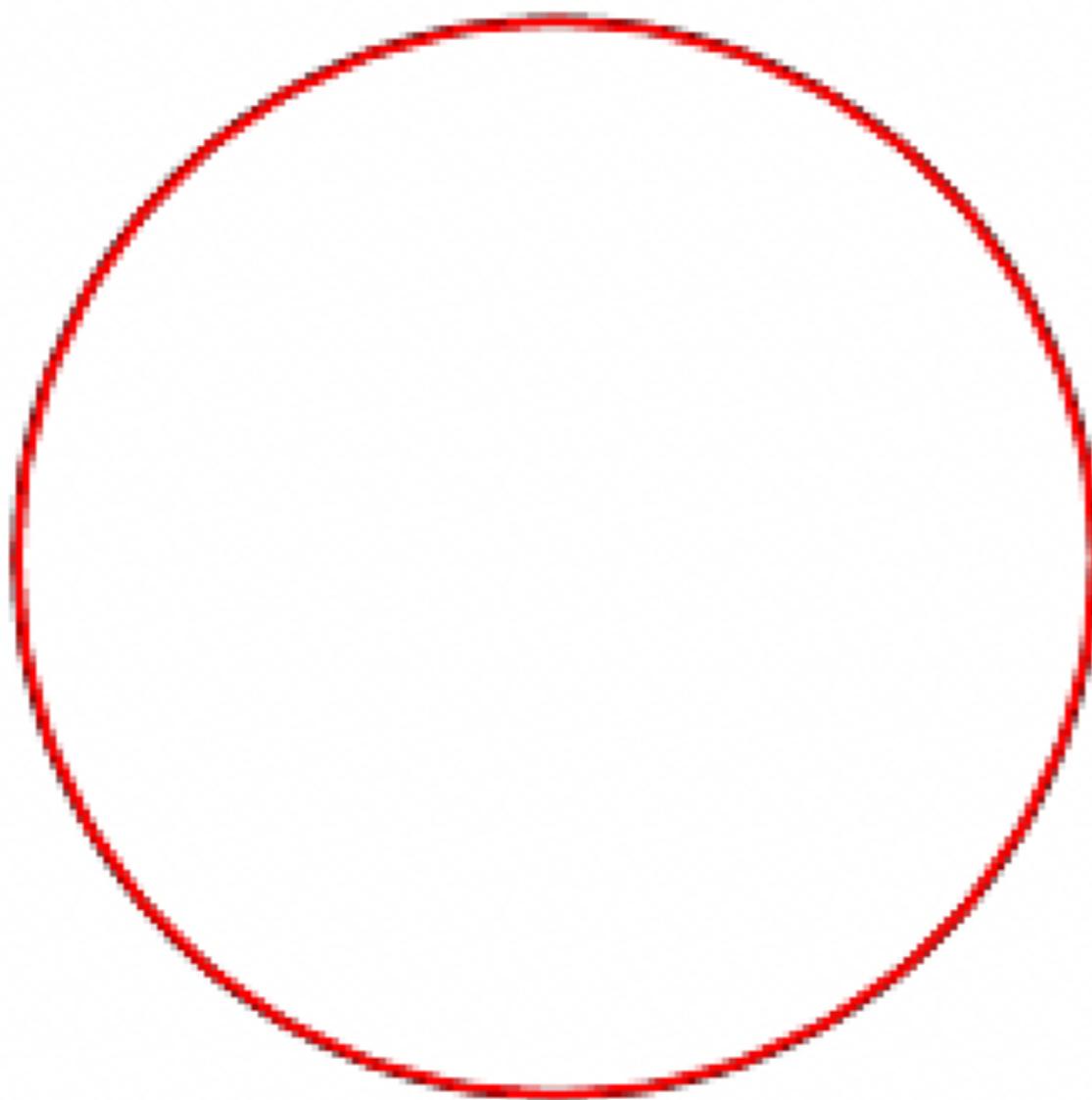
Archimedes

Finding π = 3.14159...



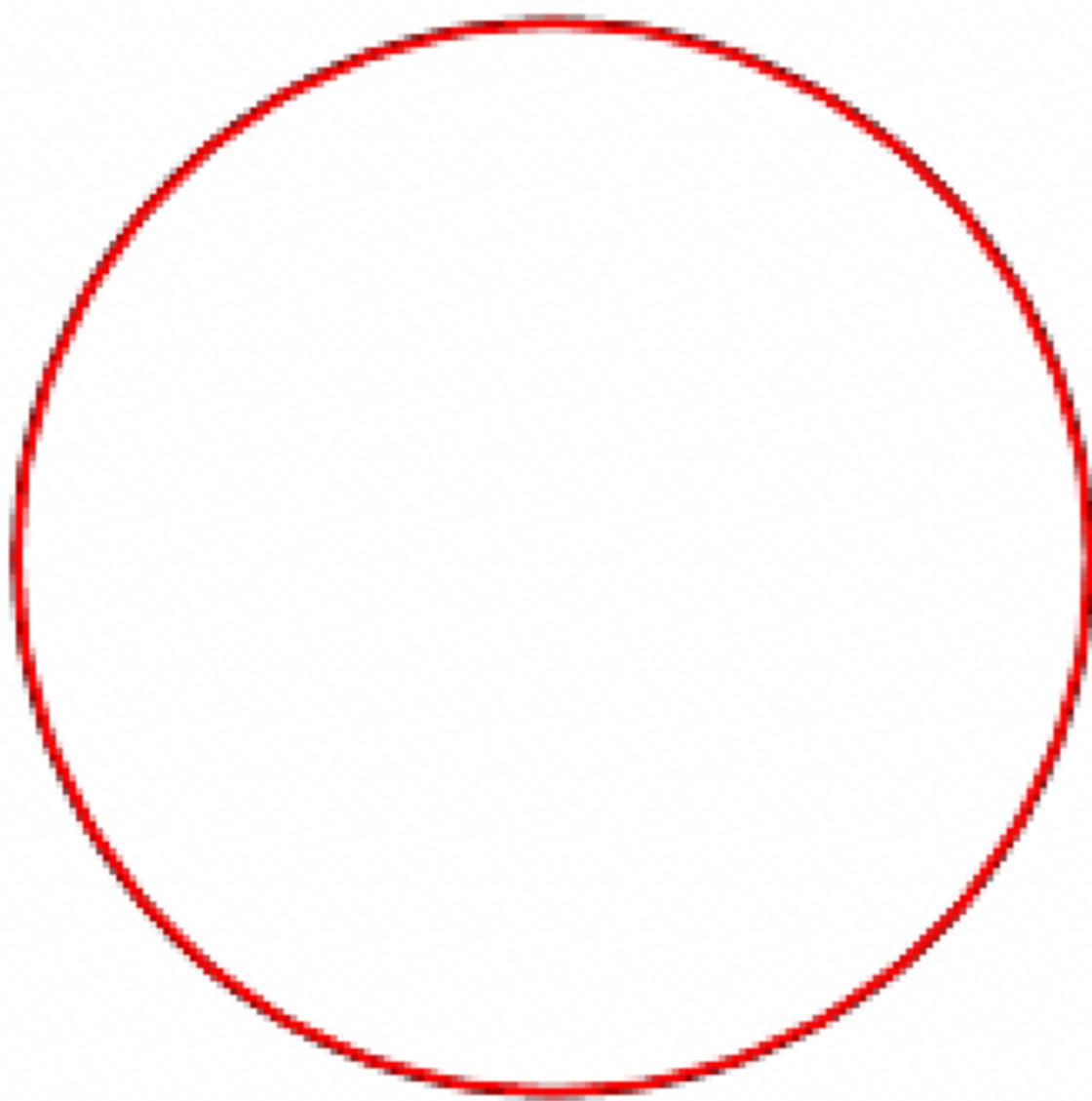
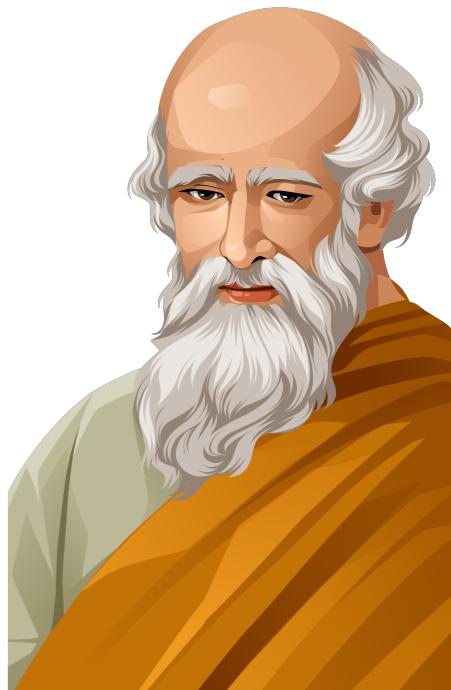
Sides = 100

Perimeter = 3.141



Archimedes

Finding π = 3.14159...



Sides = 100

Perimeter = 3.141

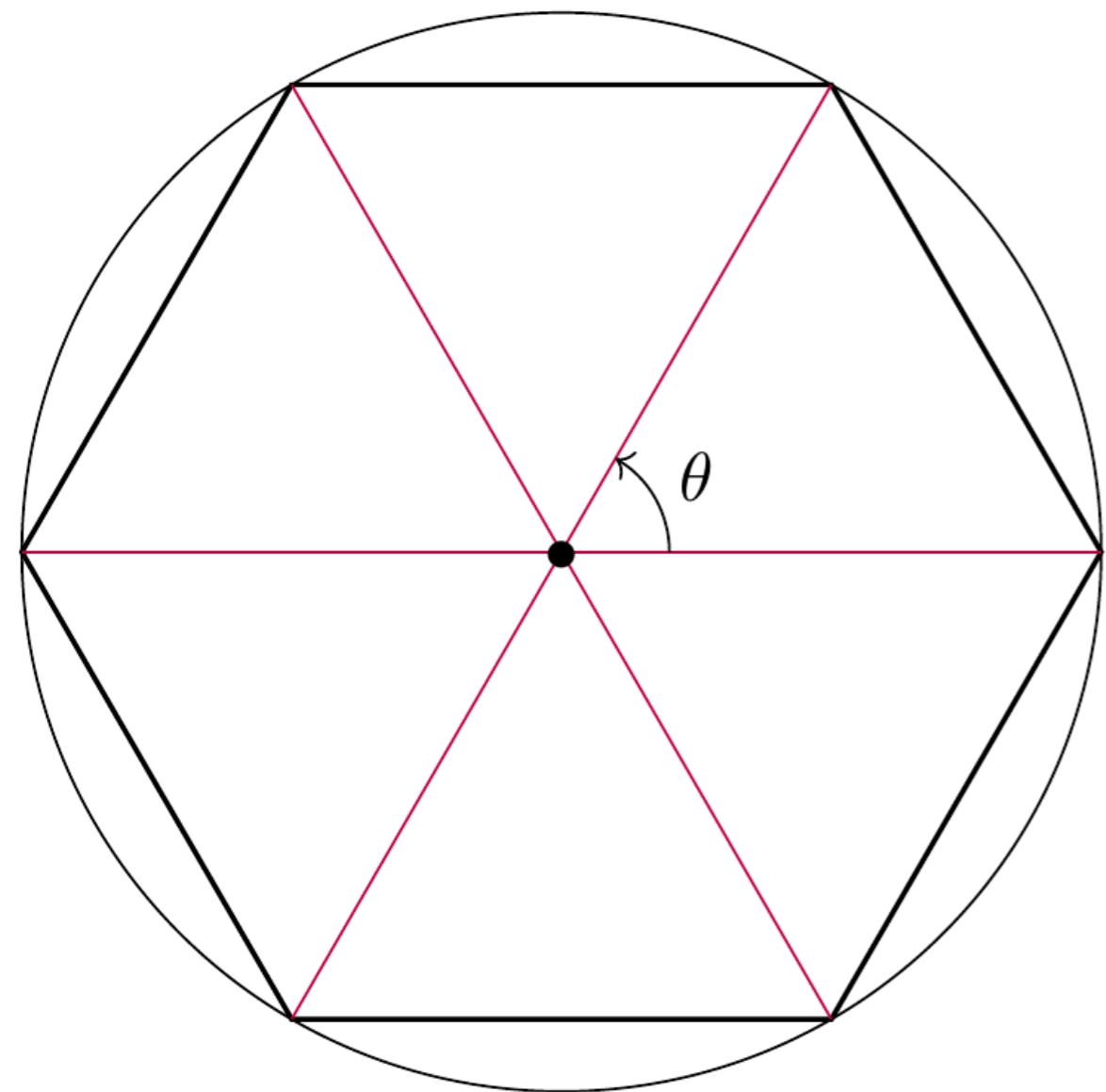
Using a 96-gon,
Archimedes proved

$$3\frac{10}{71} < \pi < 3\frac{10}{70}$$

Archimedes

Why is π also in the circle's area formula? $A = \pi r^2$.

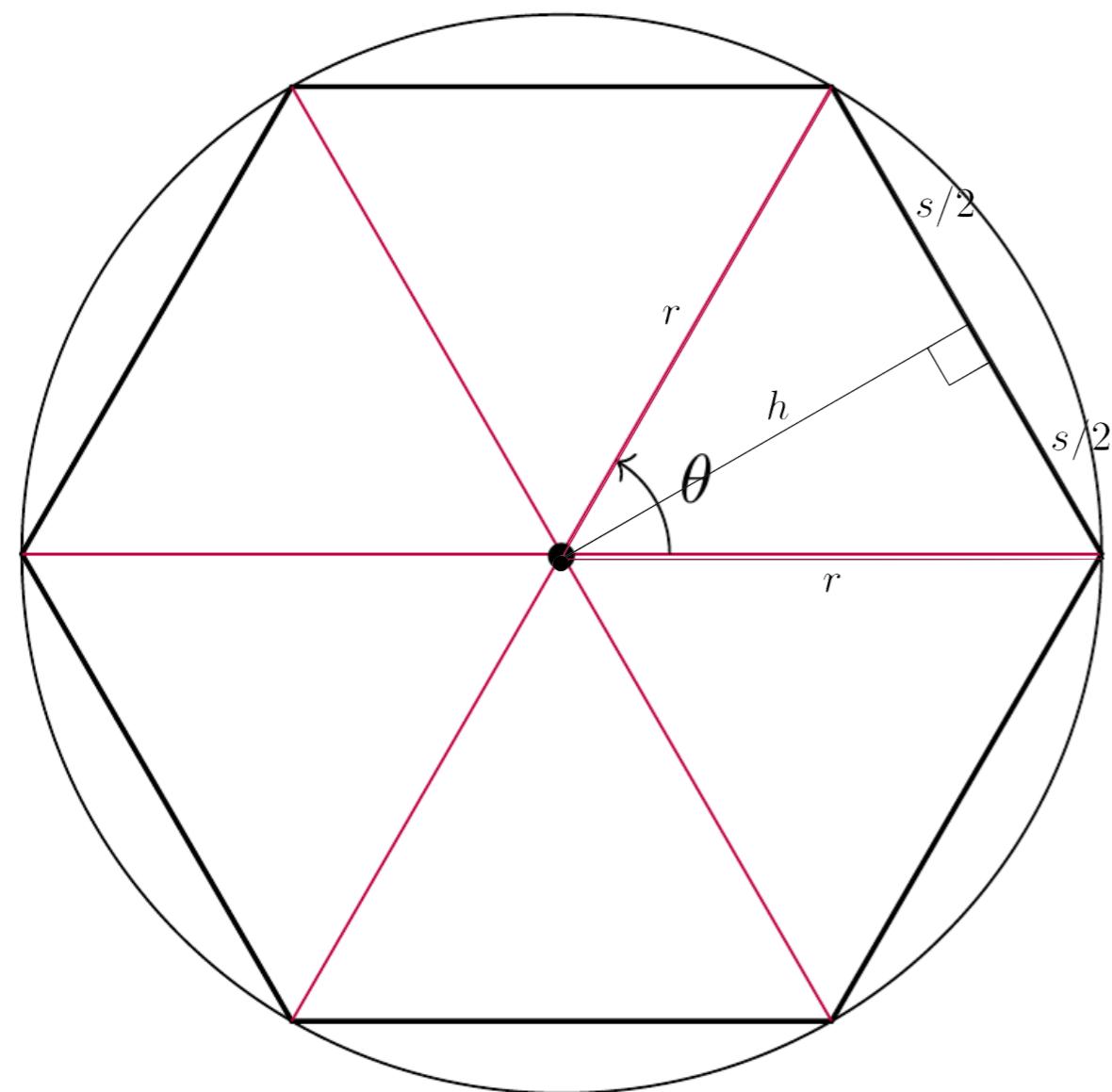
Archimedes: Consider any n -gon inscribed in the circle



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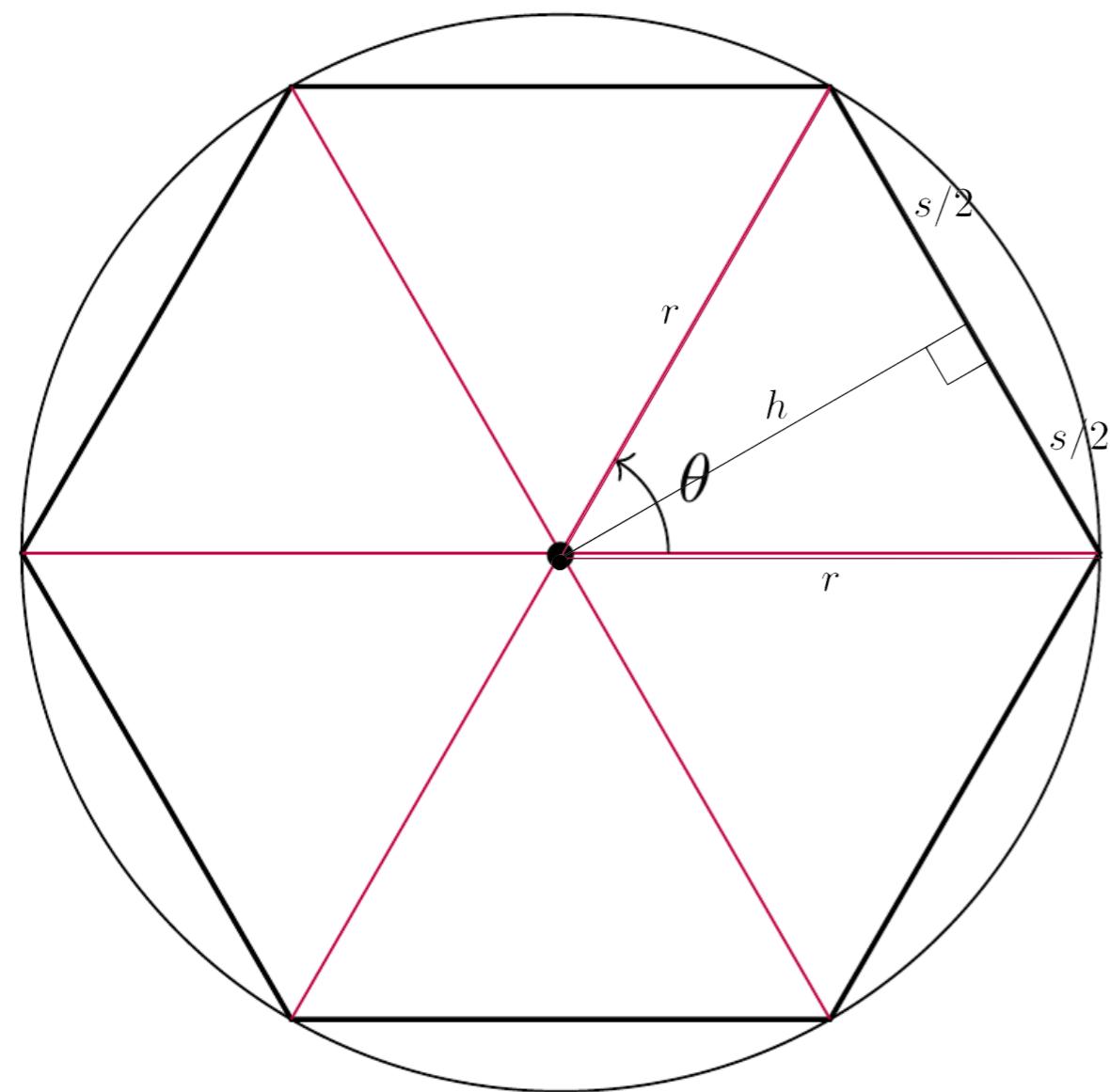
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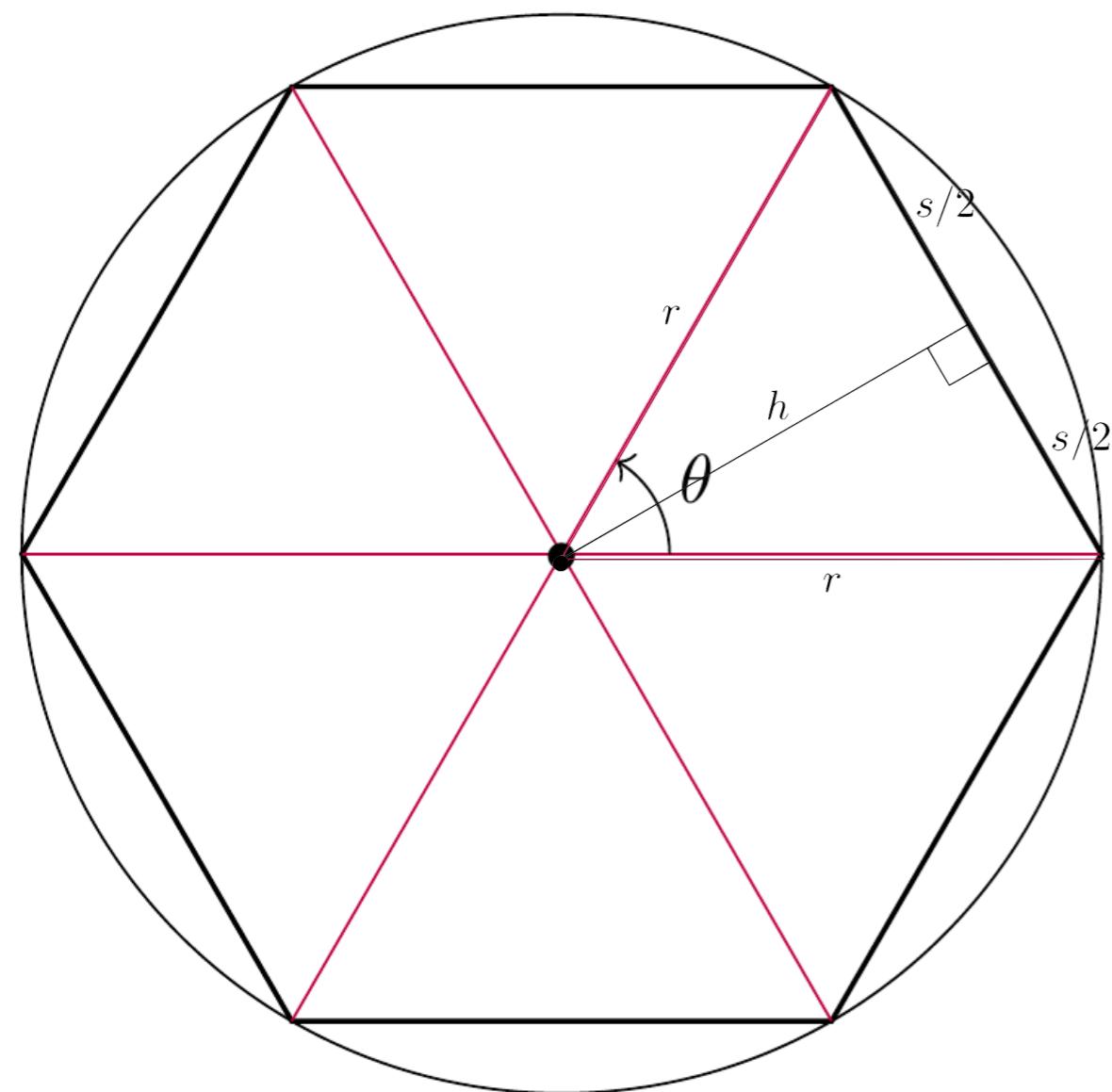


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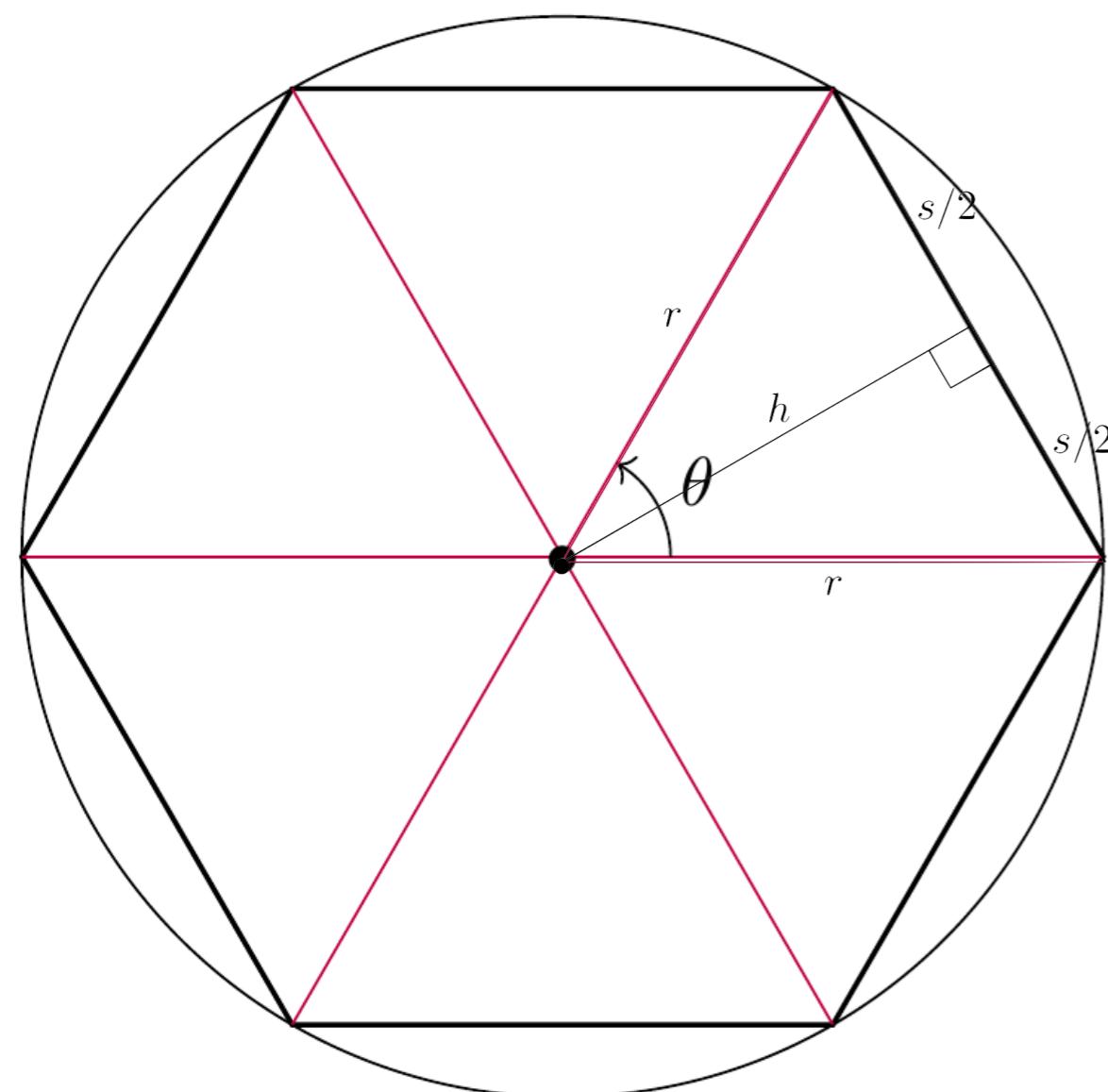


Note: sn is the perimeter of the n -gon

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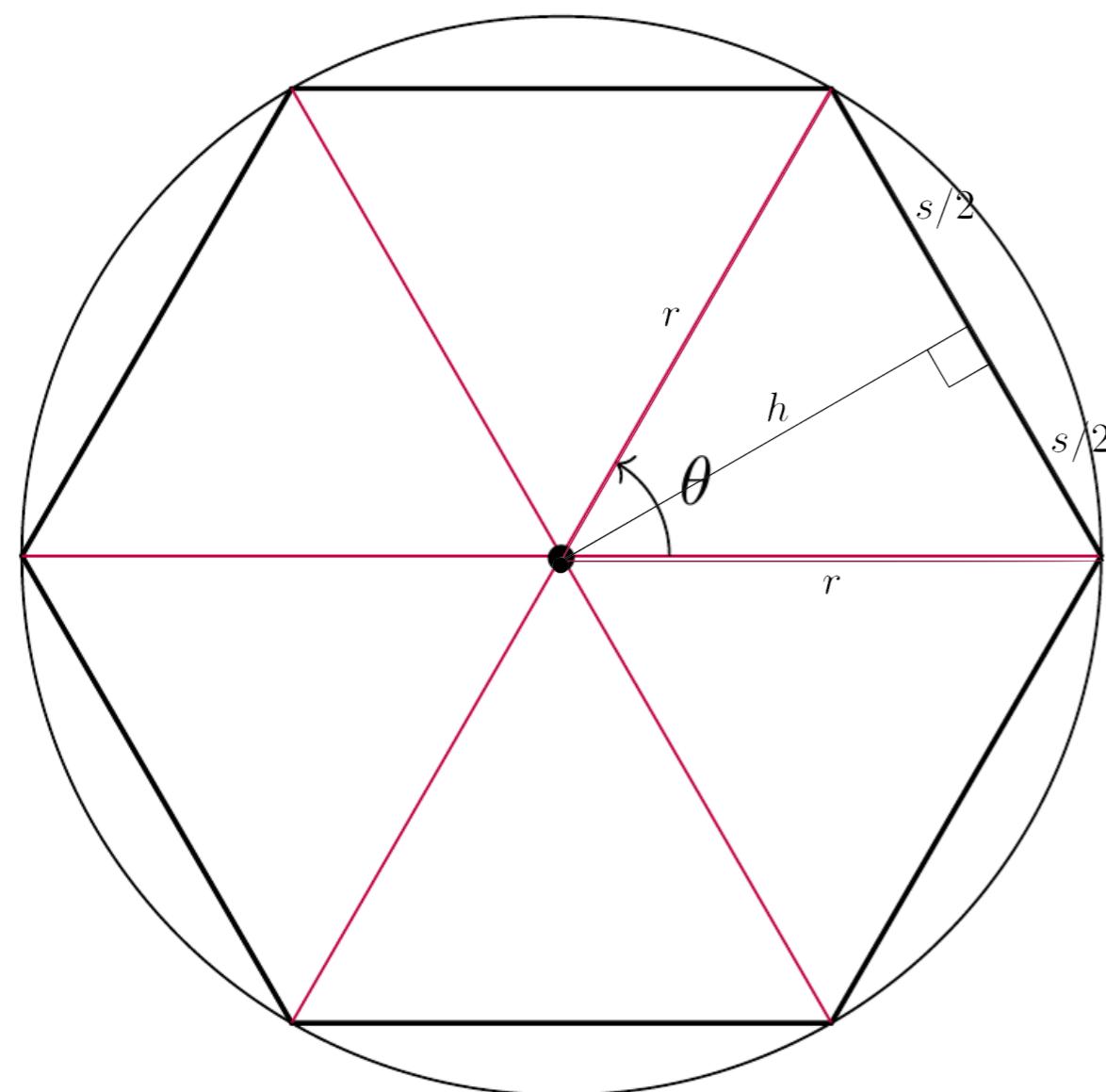


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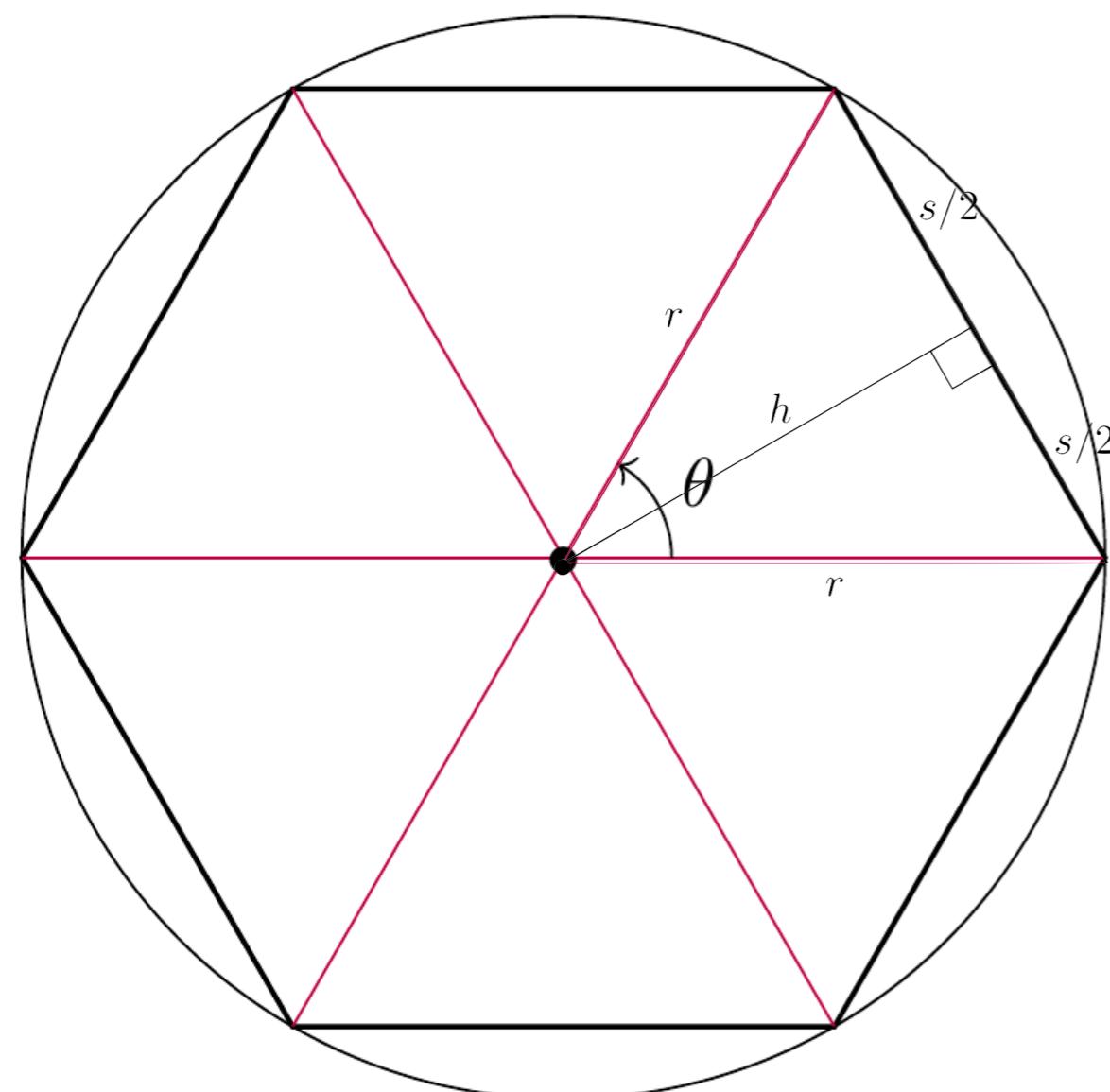
$$\begin{aligned}\text{Area of inscribed } n\text{-gon} \\ = \frac{1}{2} (\text{perimeter of } n\text{-gon}) \cdot h\end{aligned}$$

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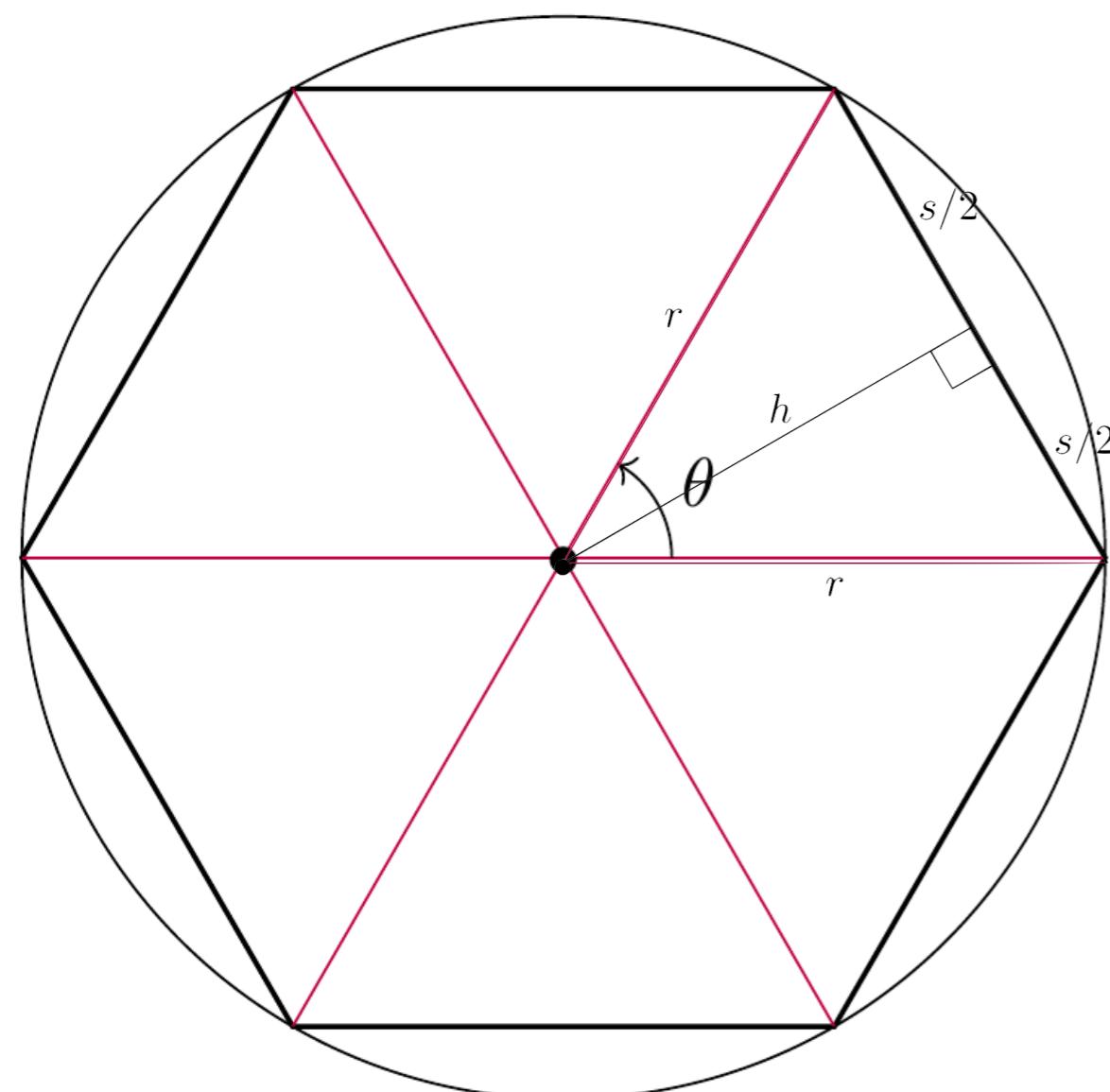
Archimedes avoided using limits

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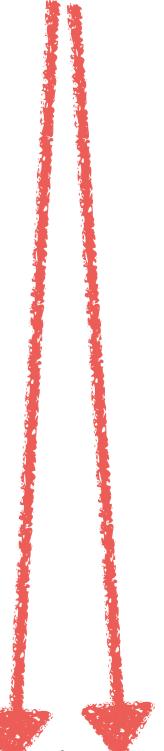
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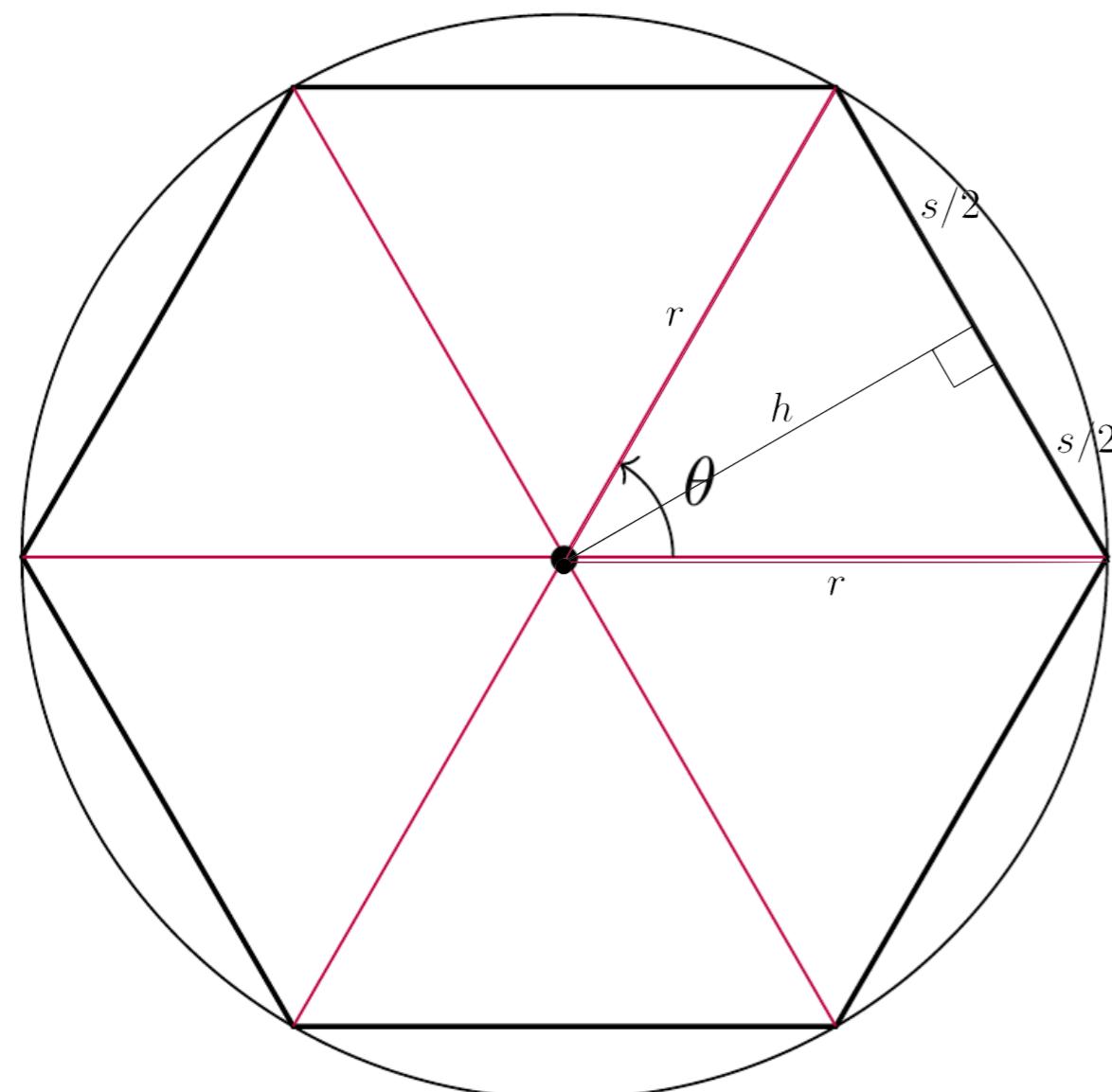
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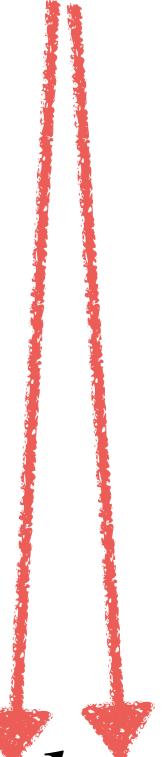


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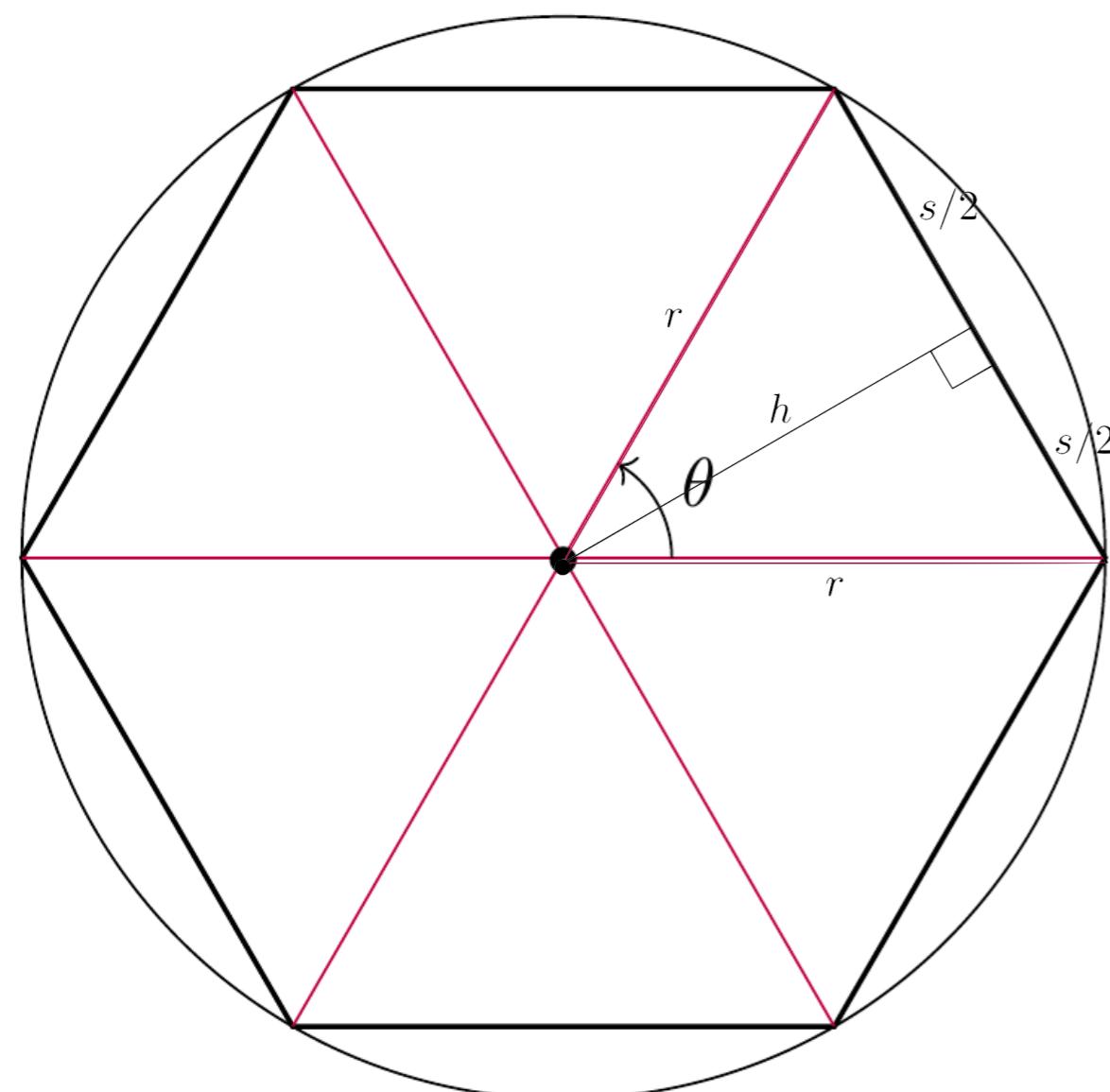
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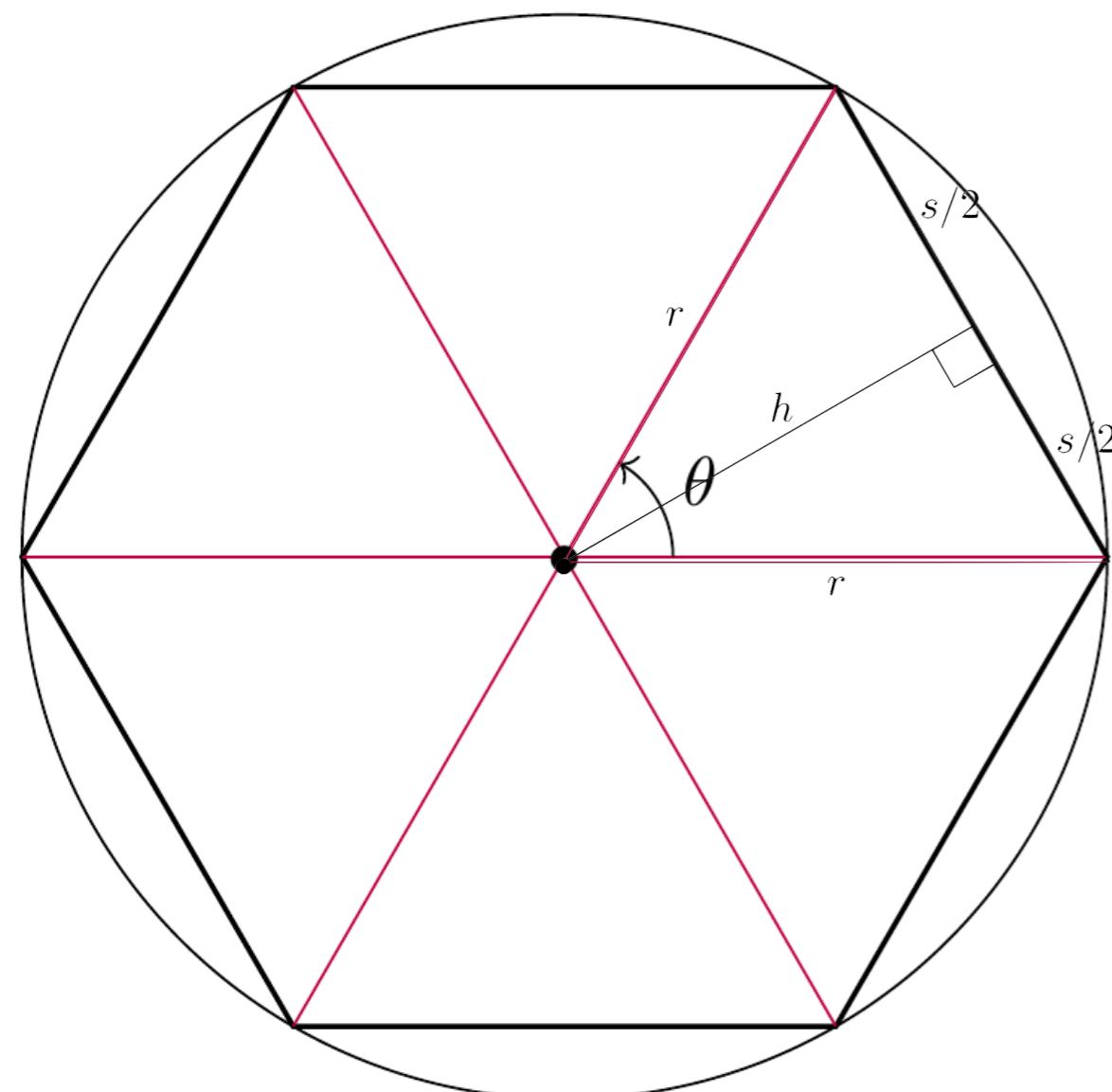
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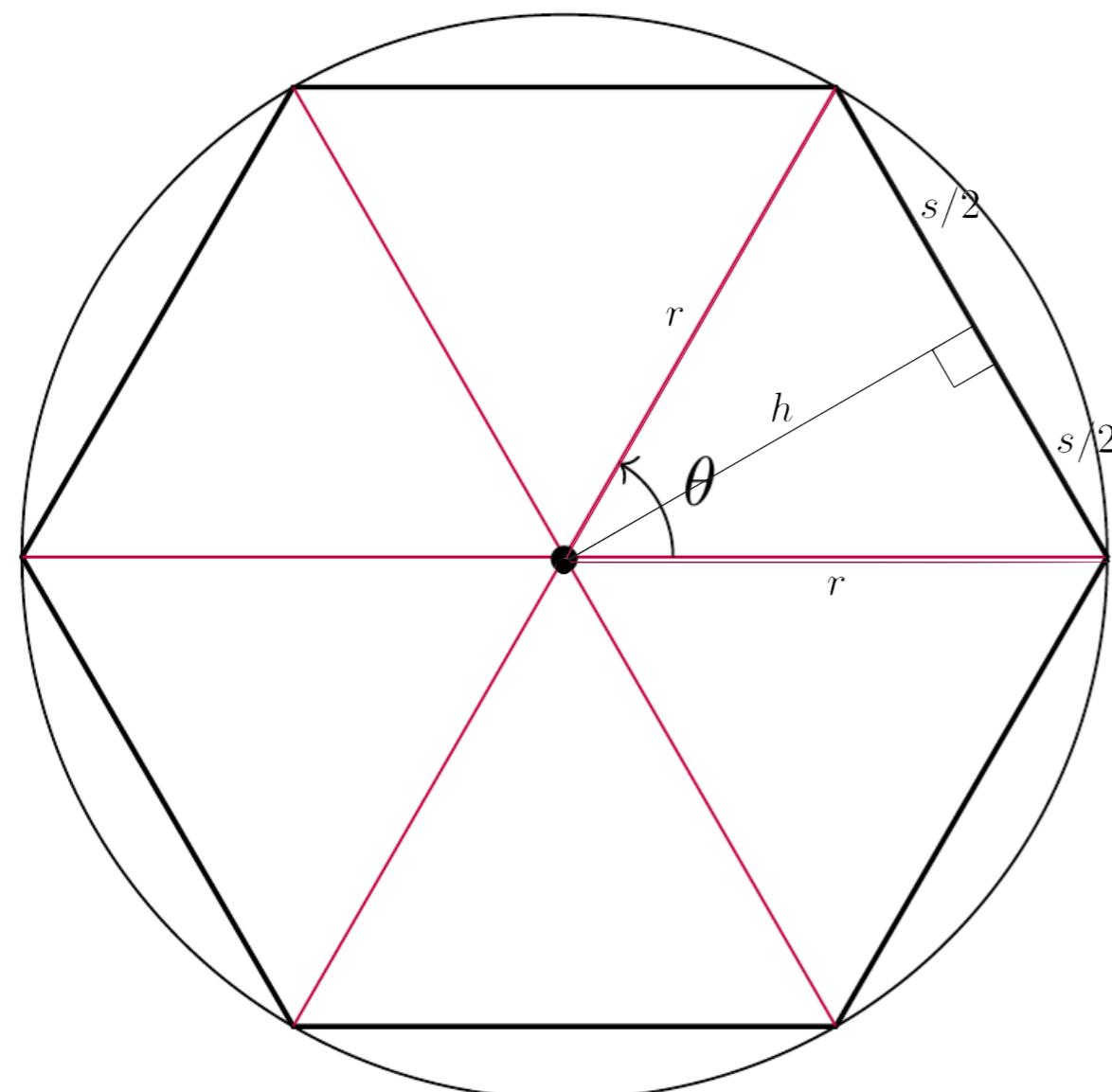
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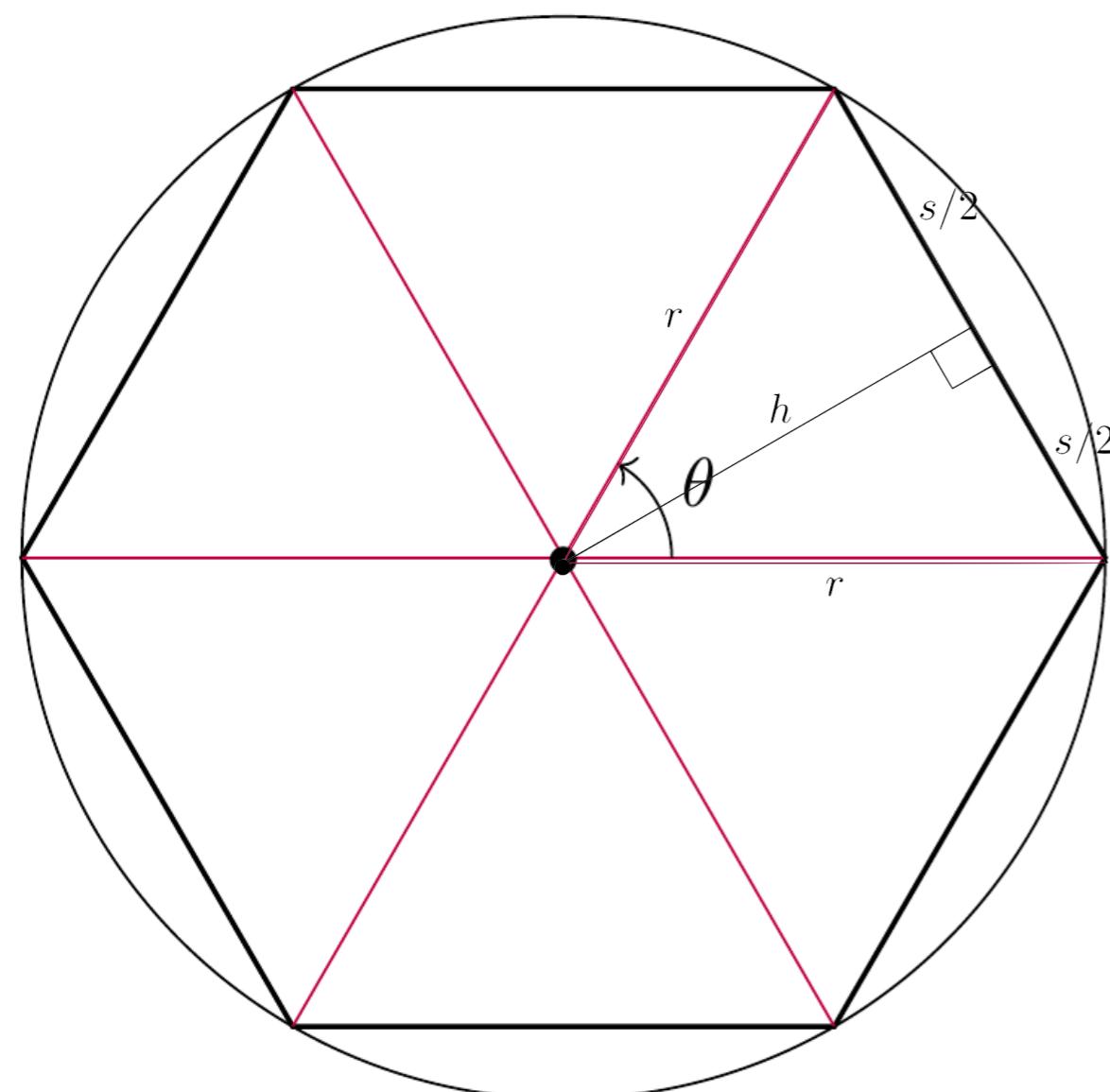
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Letting $n \rightarrow \infty$, $h \rightarrow r$

$$\text{Area of the circle} = \frac{1}{2} \cdot (2\pi r) \cdot r$$

Note:

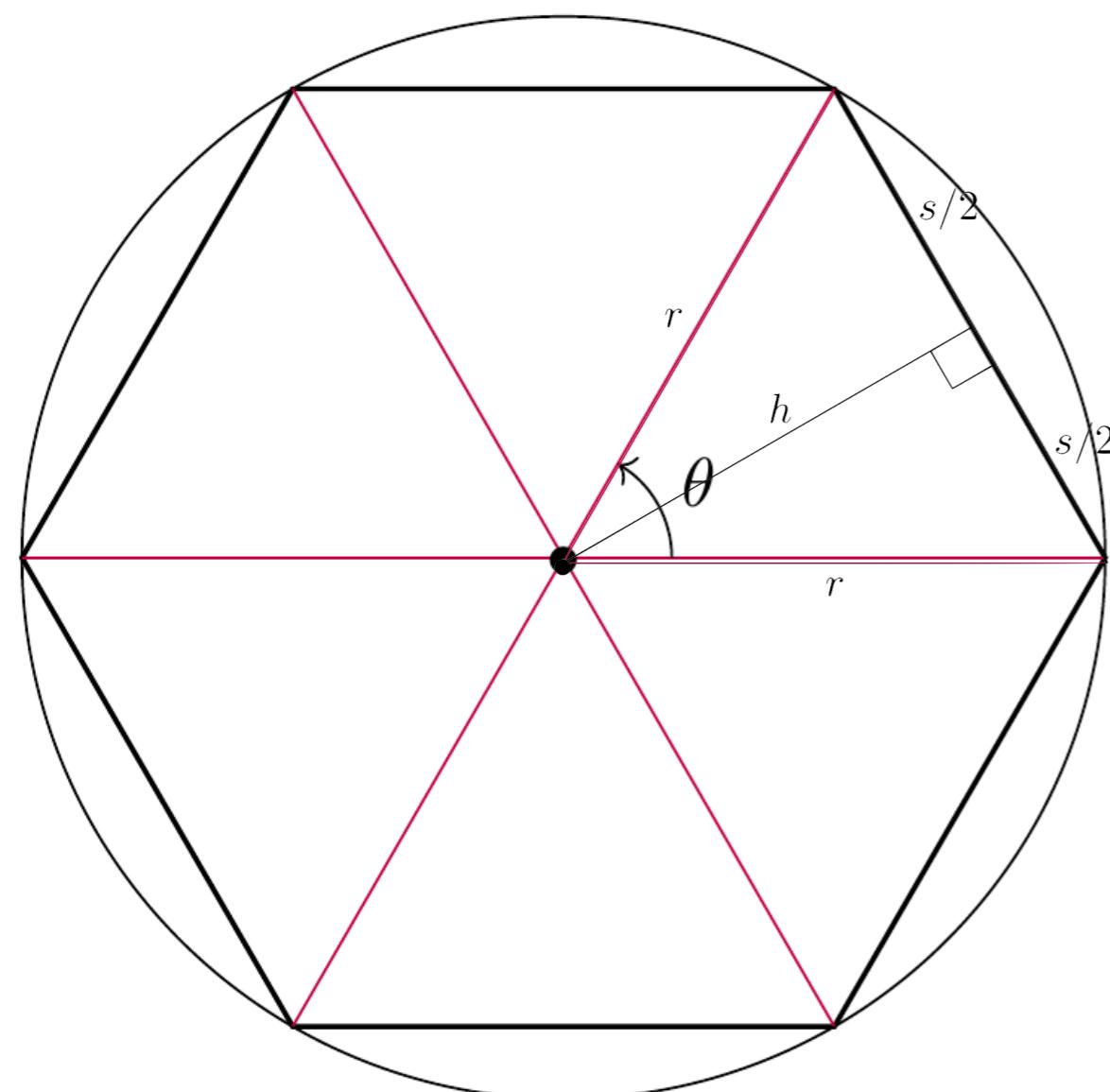
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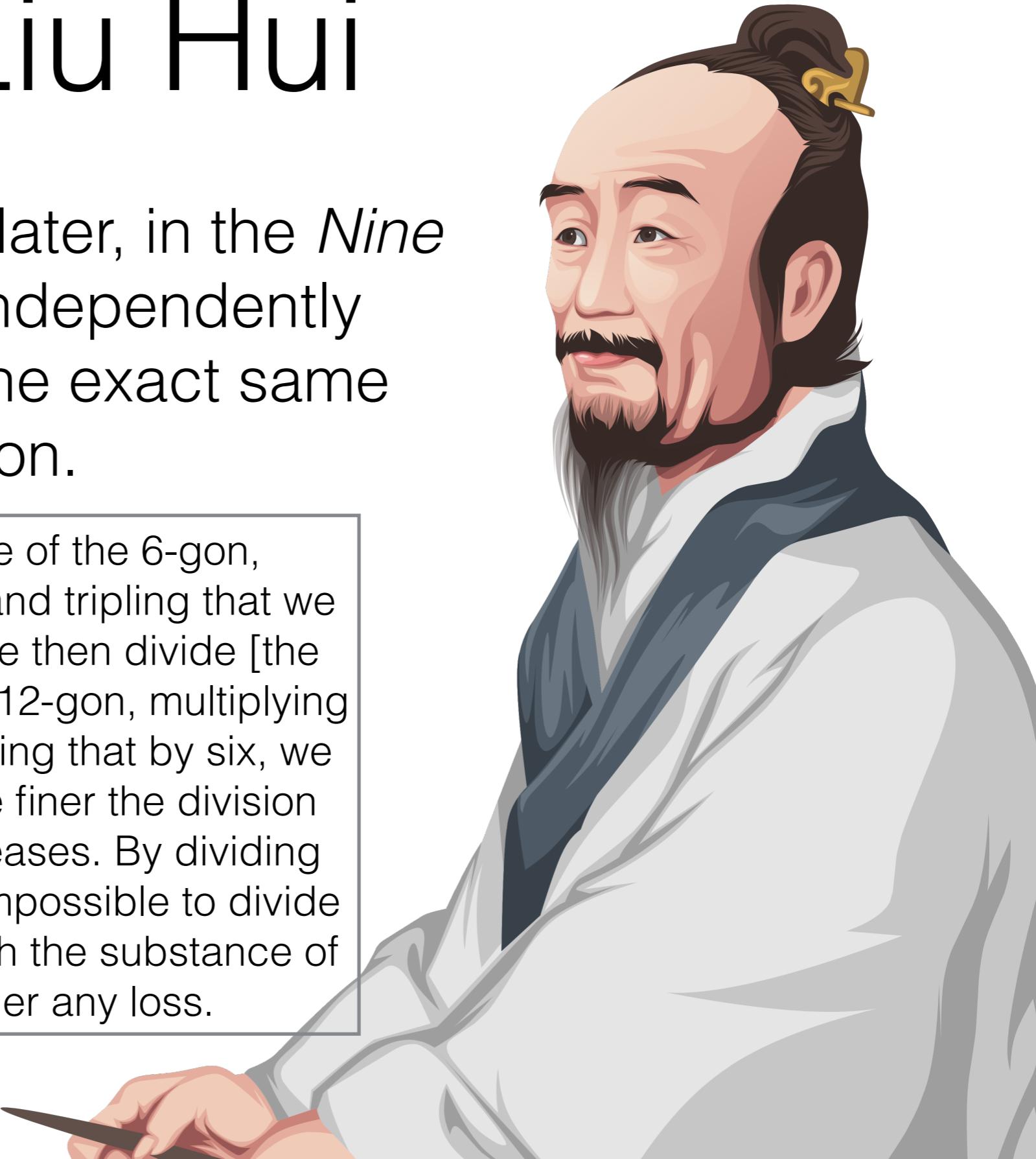
Letting $n \rightarrow \infty$, $h \rightarrow r$

$$\begin{aligned}\text{Area of the circle} &= \frac{1}{2} \cdot (2\pi r) \cdot r \\ &= \pi r^2\end{aligned}$$

Liu Hui

- Hundreds of years later, in the *Nine Chapters*, Liu Hui independently developed nearly the exact same method of exhaustion.

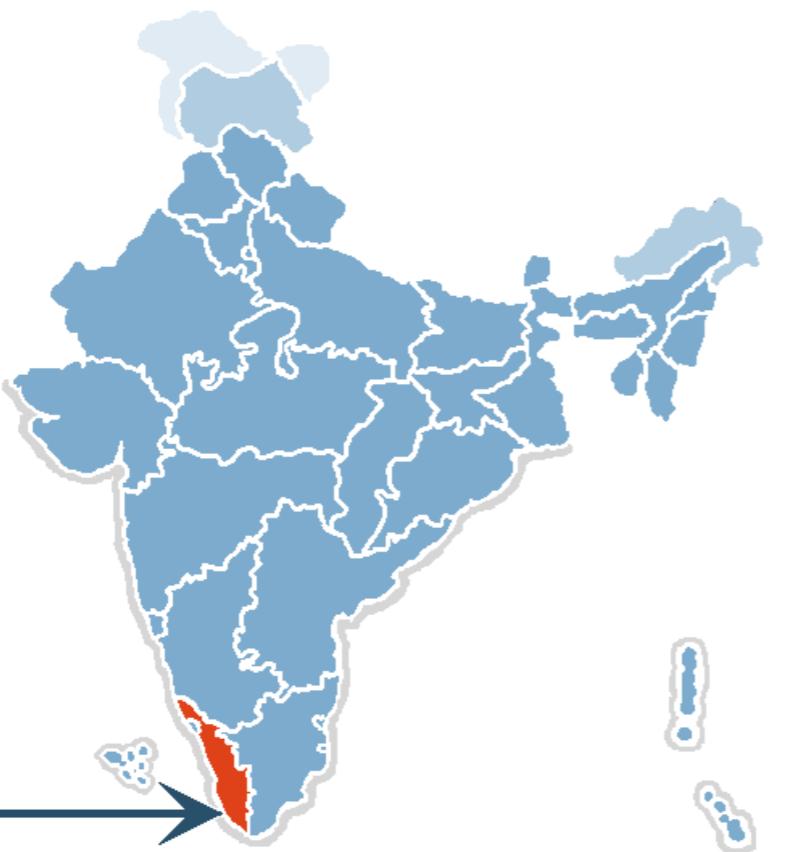
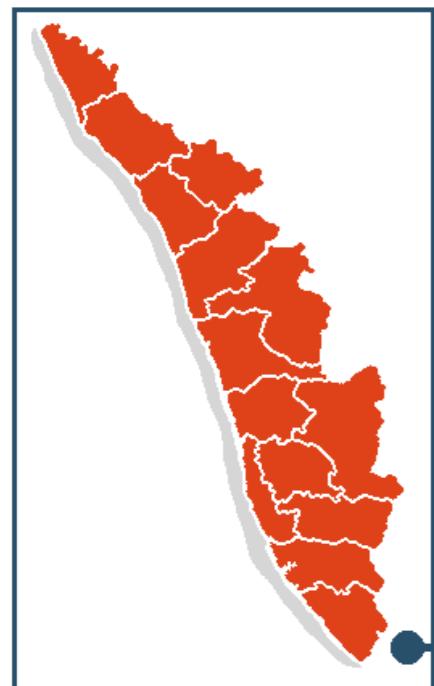
Making a figure, taking the side of the 6-gon, multiplying it by half the diameter and tripling that we obtain the area of the 12-gon. If we then divide [the circle] again, taking the side of the 12-gon, multiplying it by half the diameter and multiplying that by six, we obtain the area of the 24-gon. The finer the division becomes, the more the loss decreases. By dividing again and again until it becomes impossible to divide (further) we obtain coincidence with the substance of the disk and there is no longer any loss.



Madhava of Sangamagrama



Madhava of Sangamagrama



Madhava of Sangamagrama



- Mathematician-astronomer who founded the influential Kerala School in southern India.
- Probably the greatest medieval mathematician from India.
- First, though, we discuss one result from another mathematician at his Kerala School.

The Kerala School

The Kerala School

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Theorem. Let k be a fixed positive integer. For large positive integers n ,

$$1^k + 2^k + 3^k + \dots + n^k \approx \frac{n^{k+1}}{k+1}.$$

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Theorem. Let k be a fixed positive integer. For large positive integers n ,

$$1^k + 2^k + 3^k + \dots + n^k \approx \frac{n^{k+1}}{k+1}.$$

Note the resemblance to $\int_0^n x^k dx = \frac{n^{k+1}}{k+1}$.

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First they proved the $k = 1$ case. Note that

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$$n^2 - s_1 = s_1 - n$$

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$$\begin{aligned} s_1 \cdot n &= 1 \cdot n + 2 \cdot n + 3 \cdot n + \dots + (n-1)n + nn \\ -s_2 &= -1 \cdot 1 - 2 \cdot 2 - 3 \cdot 3 - \dots - (n-1)(n-1) - nn \end{aligned}$$

The Kerala School

Theorem. $1^k + 2^k + 3^k + \dots + n^k \approx \frac{n^{k+1}}{k+1}$.

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$$n-1 \text{ rows} \left\{ = \begin{array}{l} 1 + 2 + 3 + (n-3) + (n-2) + (n-1) \\ 1 + 2 + 3 + (n-3) + (n-2) \\ 1 + 2 + 3 + (n-3) \\ \vdots \\ 1 + 2 + 3 \\ 1 + 2 \\ 1 \end{array} \right.$$

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$$\text{Solving, } \boxed{s_2 \approx \frac{n^3}{3}} \qquad \qquad \qquad = \frac{s_2}{2}$$

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The Kerala School

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(Not rigorous by today's standards, but they had the main ideas and clearly understood induction.)



Madhava of Sangamagrama



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- Next two results: If $|r| < 1$, then

$$1 + r + r^2 + r^3 + r^4 + \dots = \frac{1}{1 - r}$$

and

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Madhava of Sangamagrama

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Proof. Madhava proved these in the case $0 < r < 1$.

In this case, note that r, r^2, r^3, r^4, \dots are all between 0 and 1.



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Theorem. $1 + r + r^2 + r^3 + r^4 + \dots = \frac{1}{1 - r}$

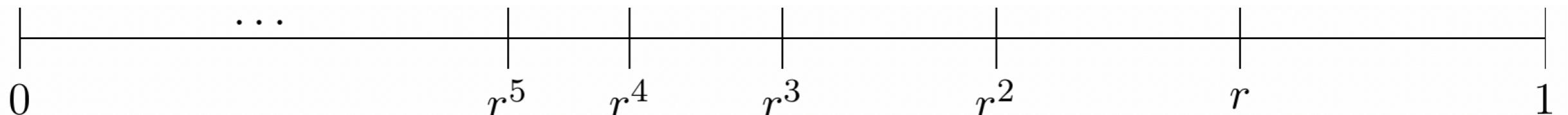
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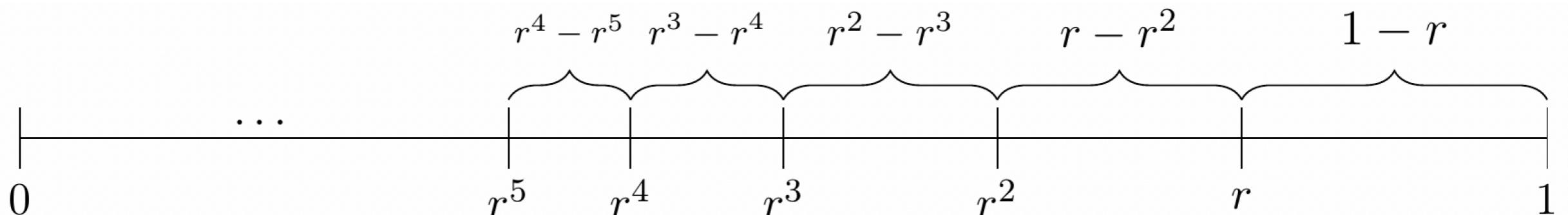




Madhava of Sangamagrama

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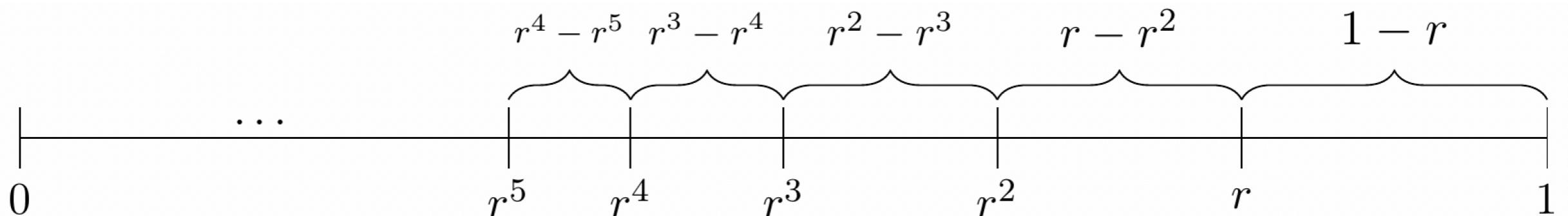




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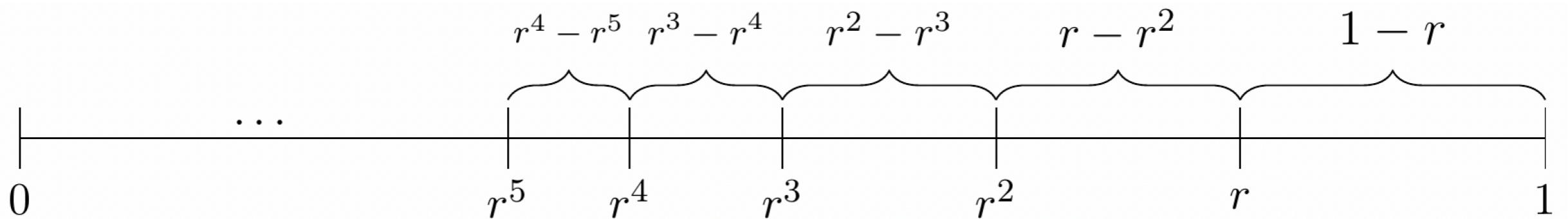
$$(1 - r) + (r - r^2) + (r^2 - r^3) + (r^3 - r^4) + (r^4 - r^5) + \dots = 1$$



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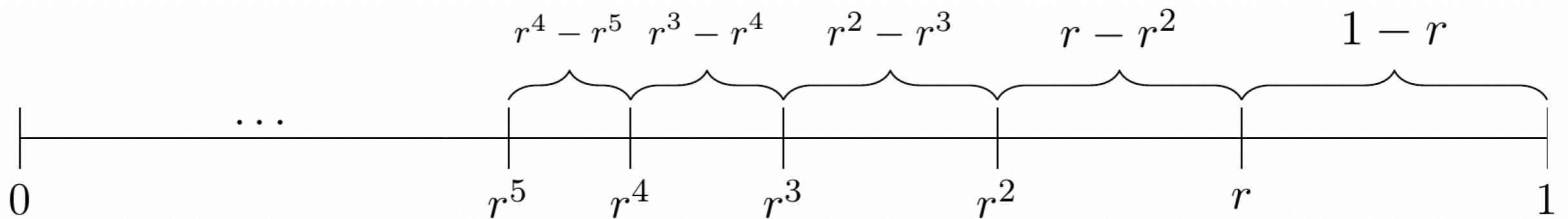
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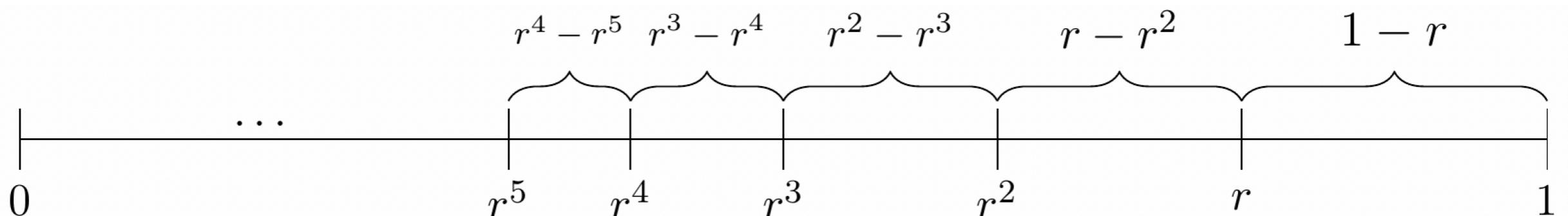
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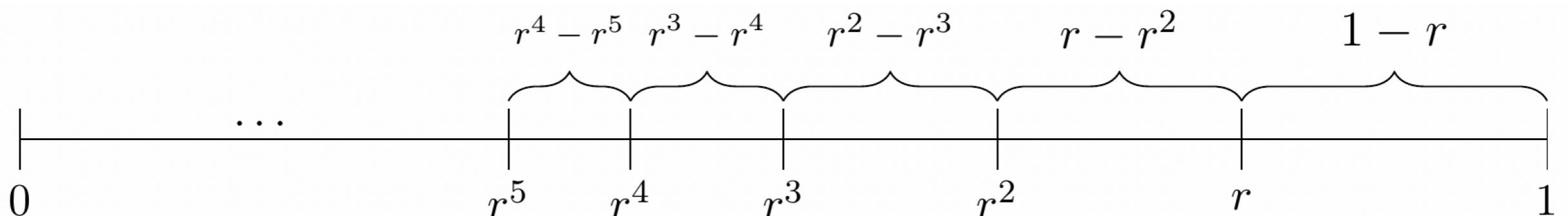
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$$1 + r + r^2 + r^3 + r^4 + \dots = \frac{1}{1 - r} \text{ Q.E.D.}$$



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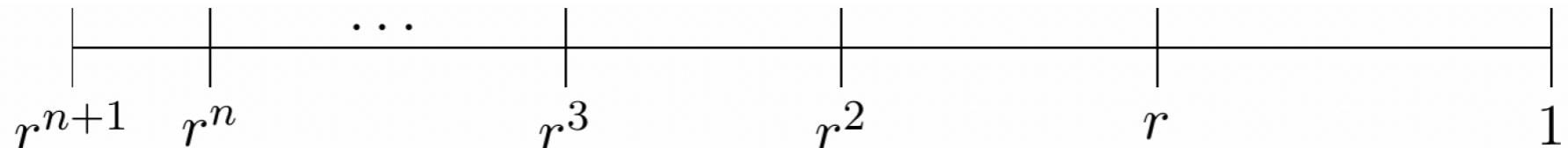
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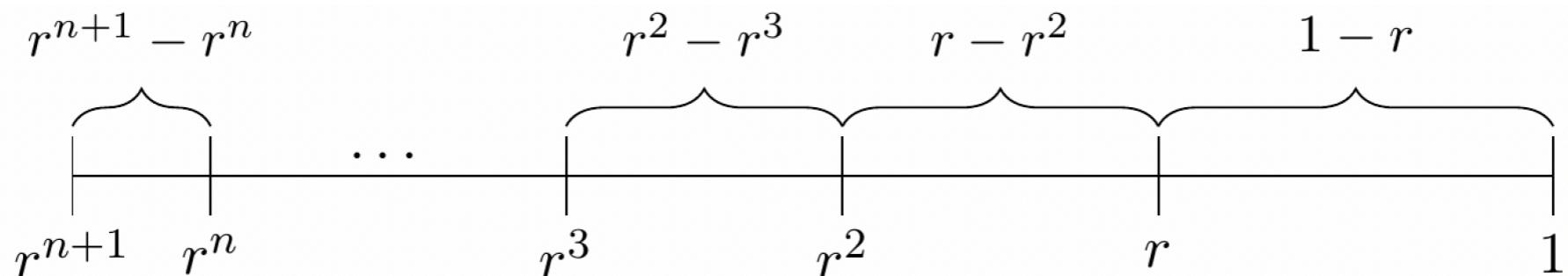




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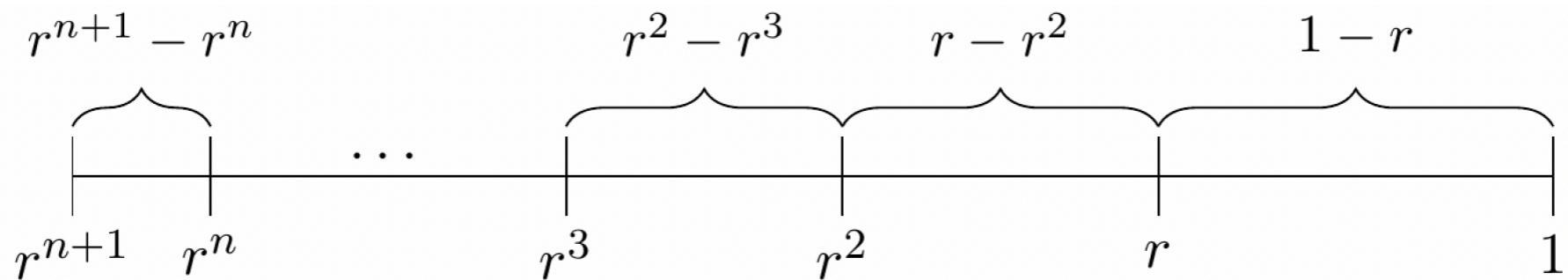




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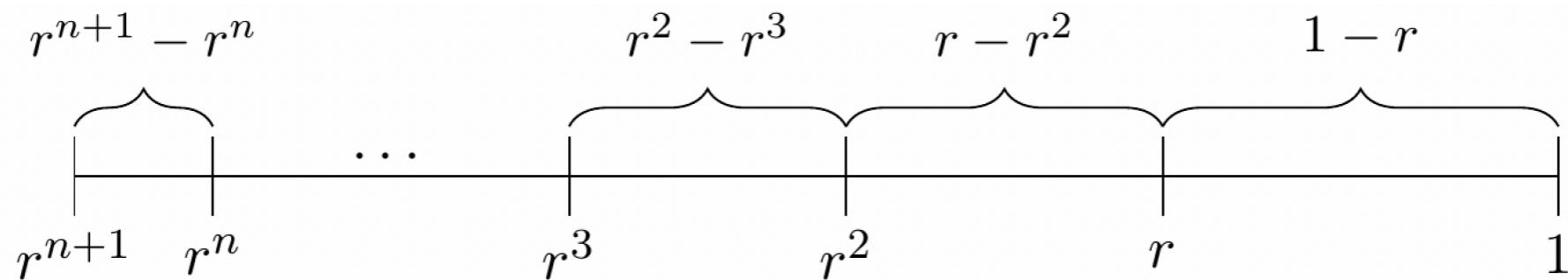
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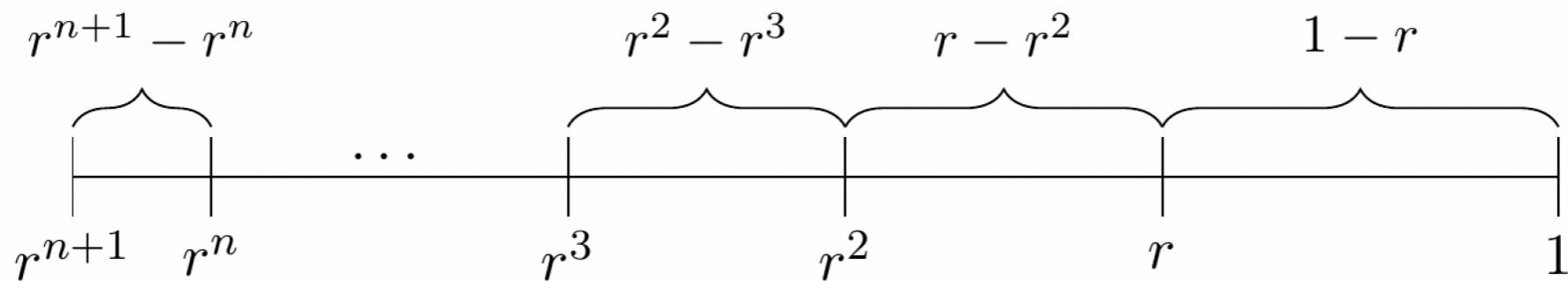
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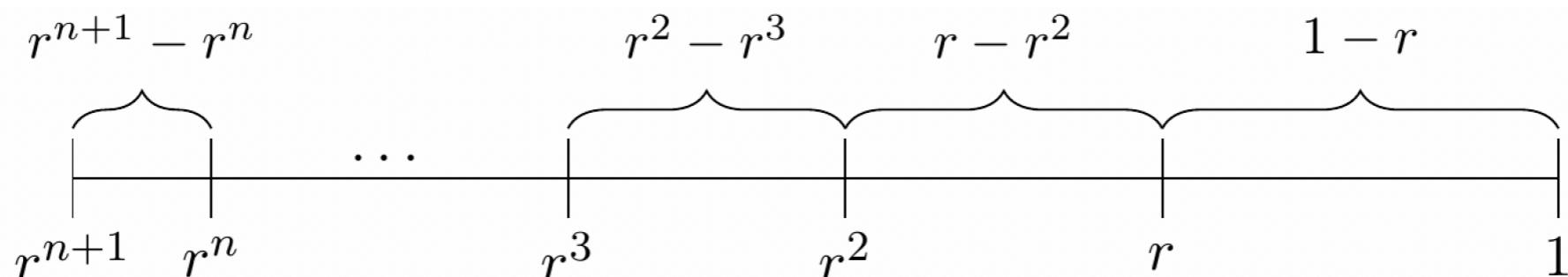
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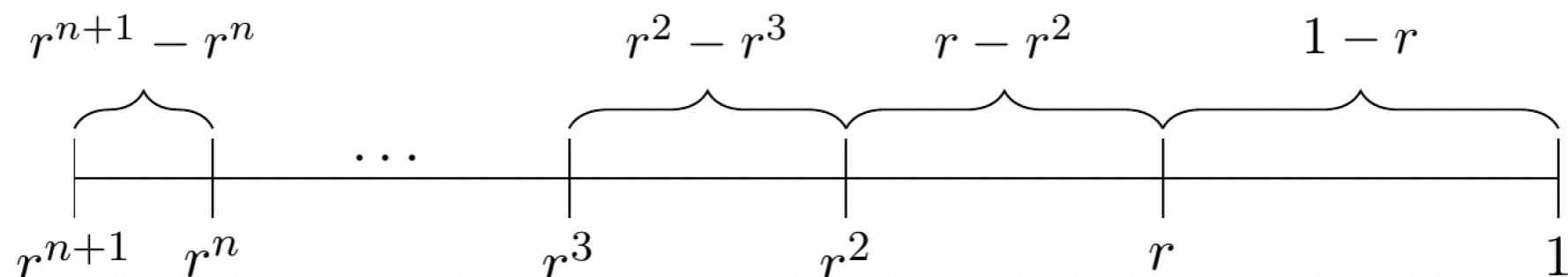
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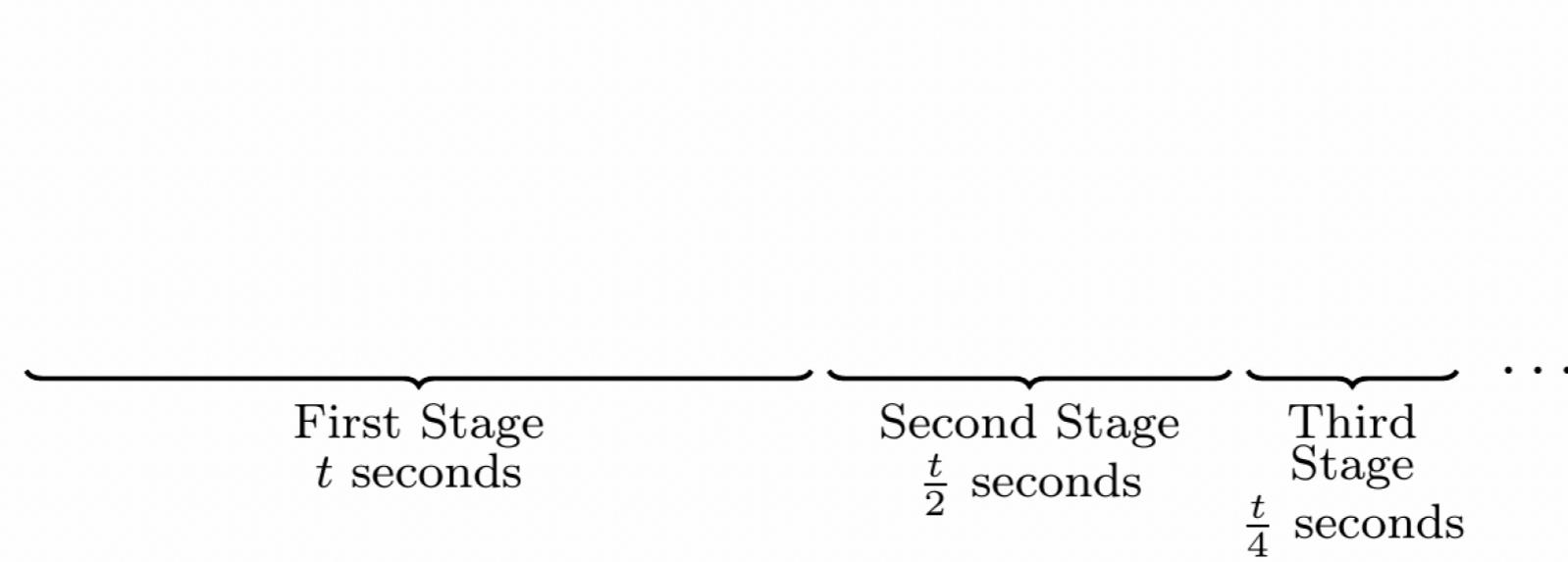
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Note: This shows why Zeno was wrong.



Madhava of Sangamagrama

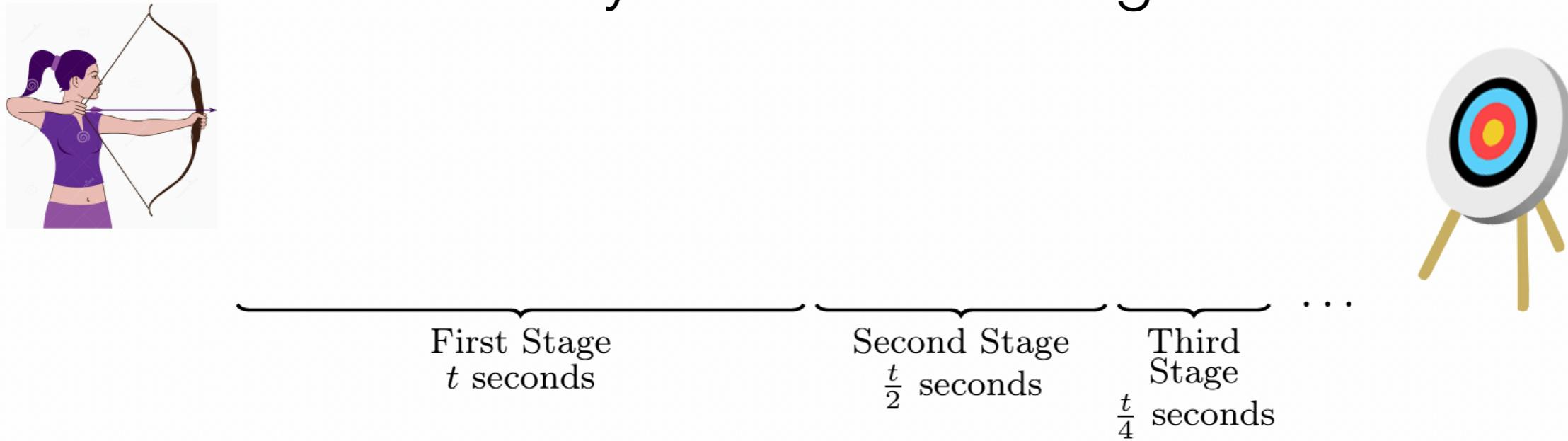
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Madhava of Sangamagrama

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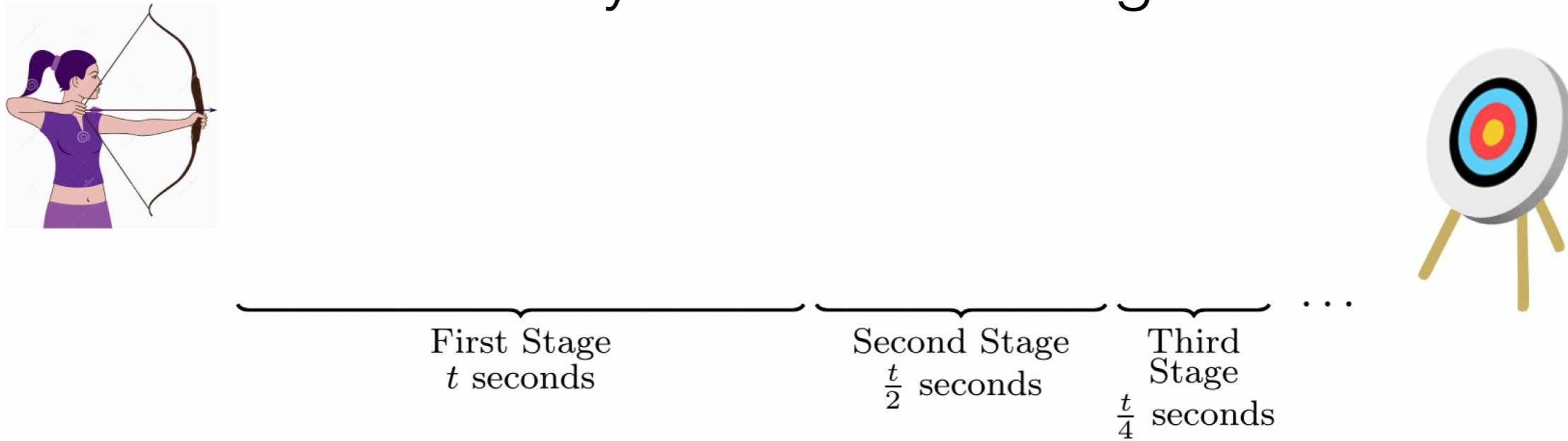
Then the total amount of time (in seconds) to complete all the stages is

$$t + \frac{1}{2}t + \frac{1}{4}t + \frac{1}{8}t + \dots = t\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\right) = t \frac{1}{1 - \frac{1}{2}} = 2t.$$



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So the arrow completes all the stages in just $2t$ seconds.



Madhava of Sangamagrama



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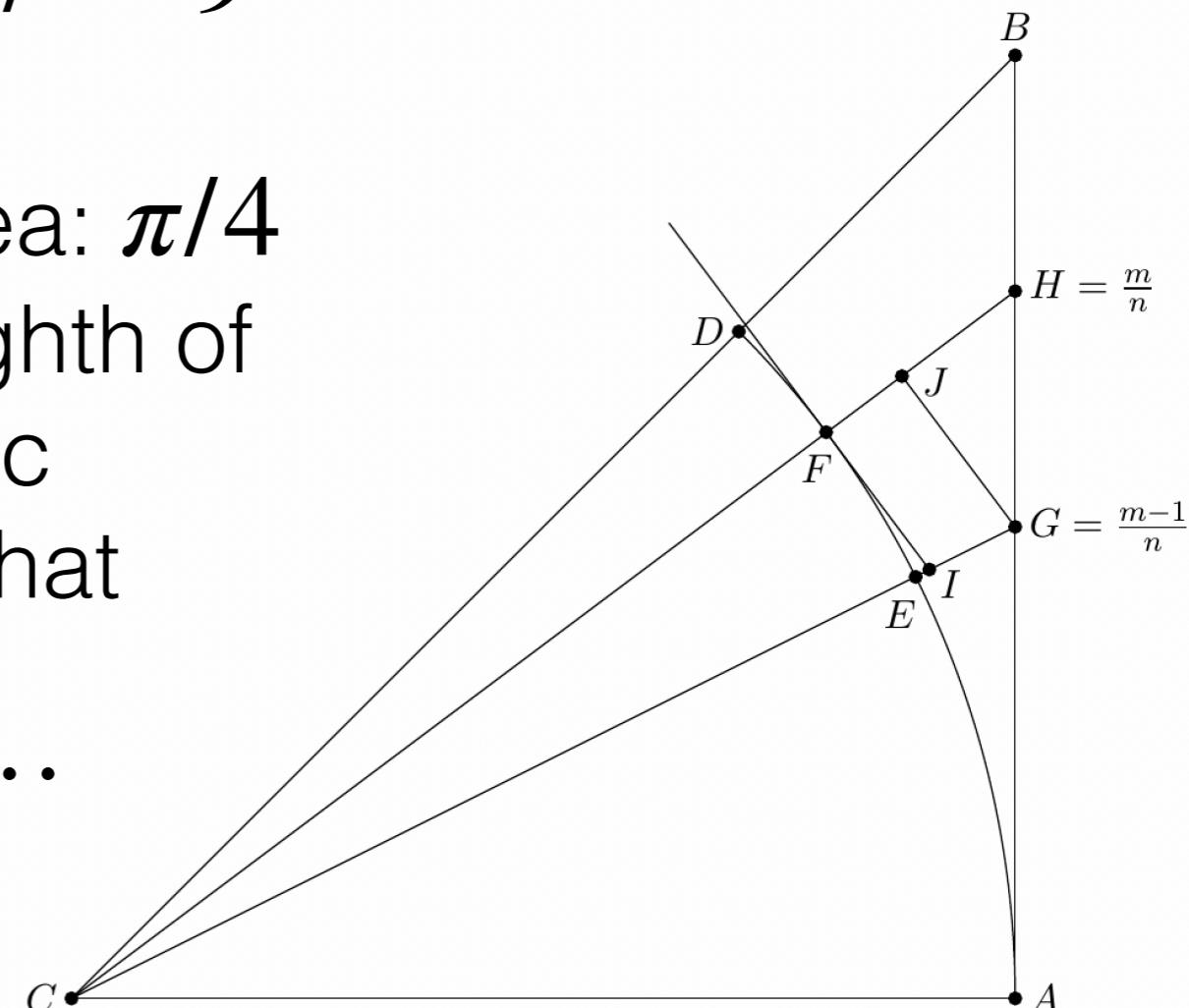
Madhava used geometric series to prove a big theorem:

Theorem. $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$

Proof. It's complicated. Basic idea: $\pi/4$ equals the length of an eighth of a unit circle. Use geometric series in reverse to show that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

also equals this.



Bhāskara II



Bhāskara II

Bhāskara II was a prominent mathematician-astronomer.

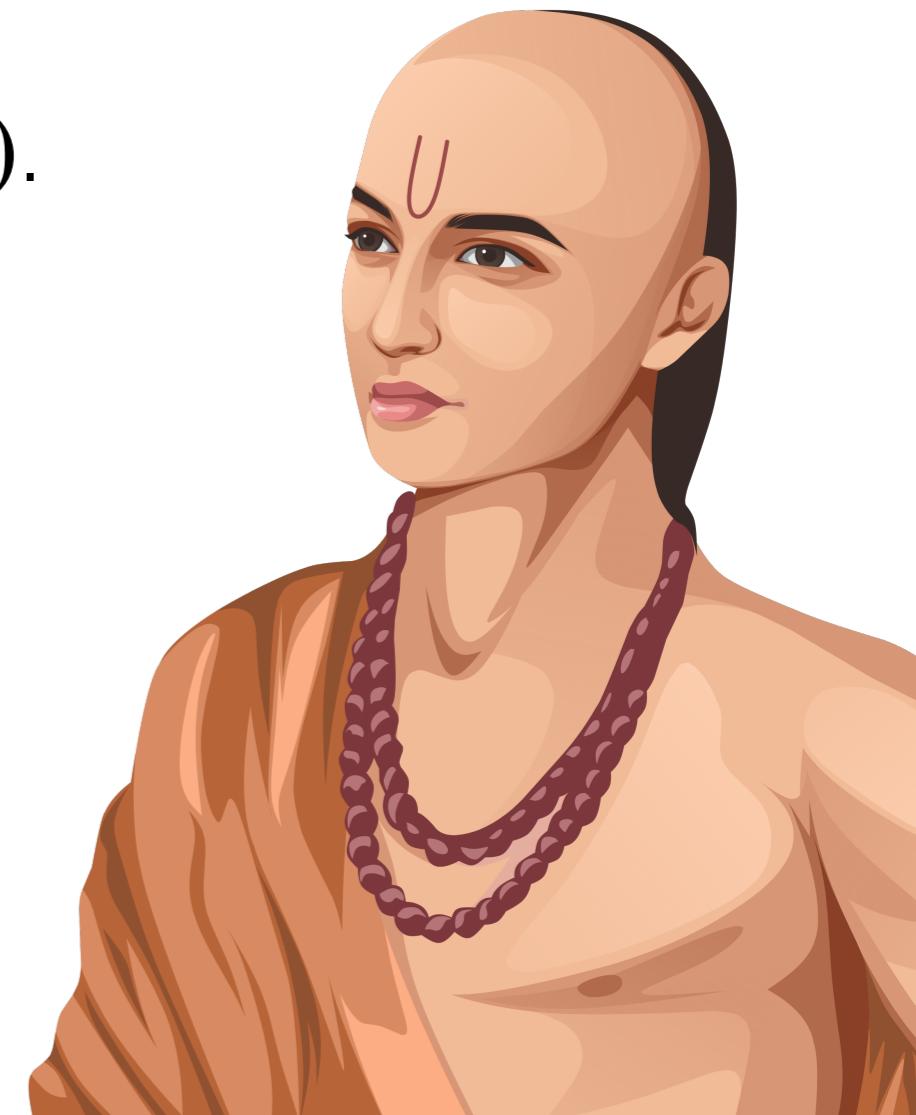


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- He found that if $x \approx y$, then $\frac{\sin(y) - \sin(x)}{y - x} \approx \cos(y)$.

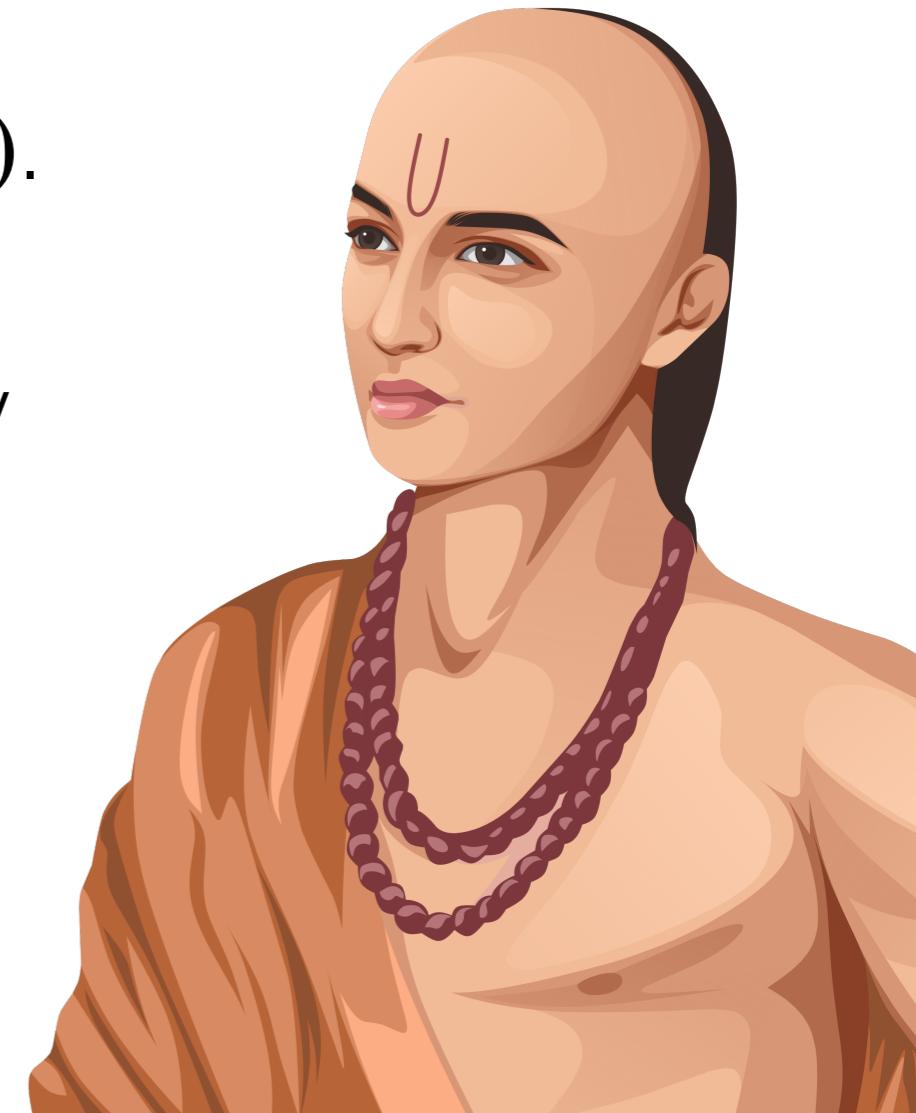
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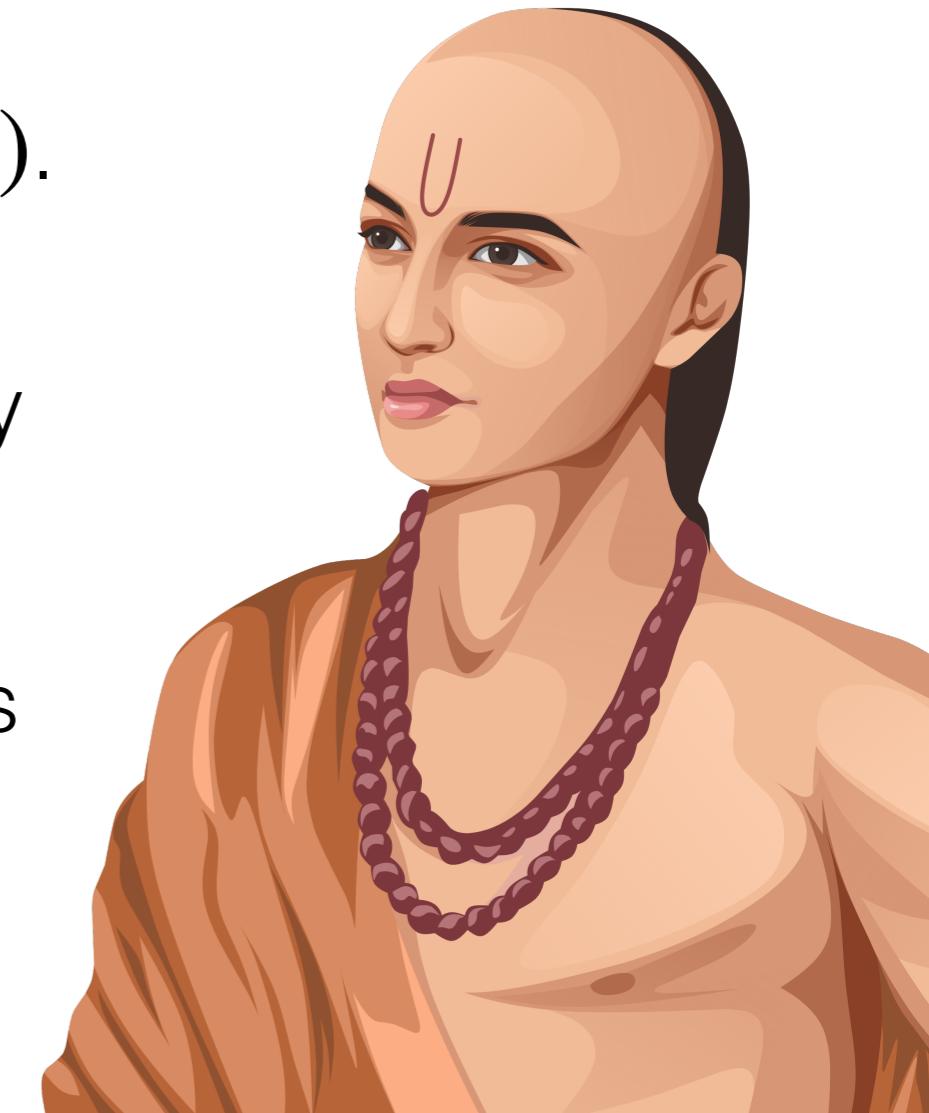
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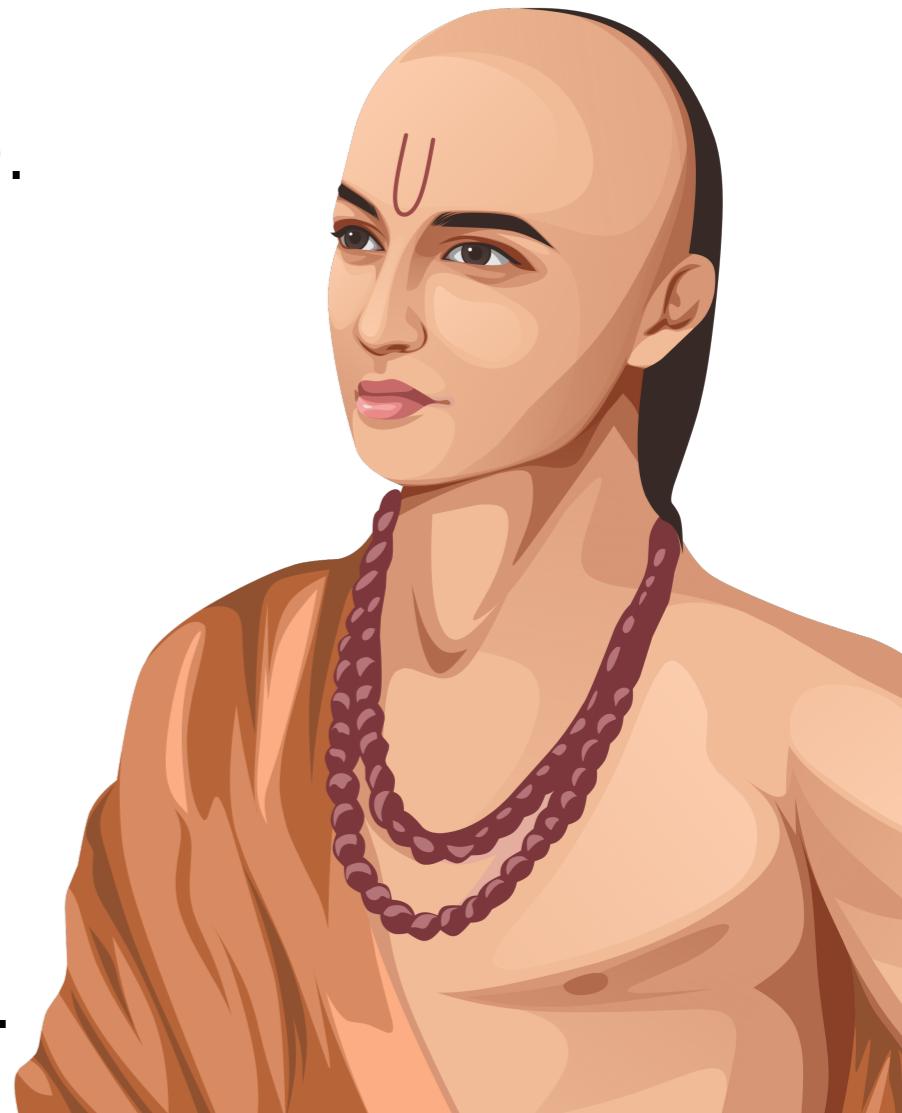
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- He had some ideas resembling the study of an infinitesimal unit of time.
- He knew that when a function reaches its maximum value, its differential vanishes.
- He used the main idea of Rolle's theorem.



Cavalieri's Principle

Cavalieri's Principle

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- Basic idea: If you move around parts of a figure, that doesn't change the figure's area or volume.

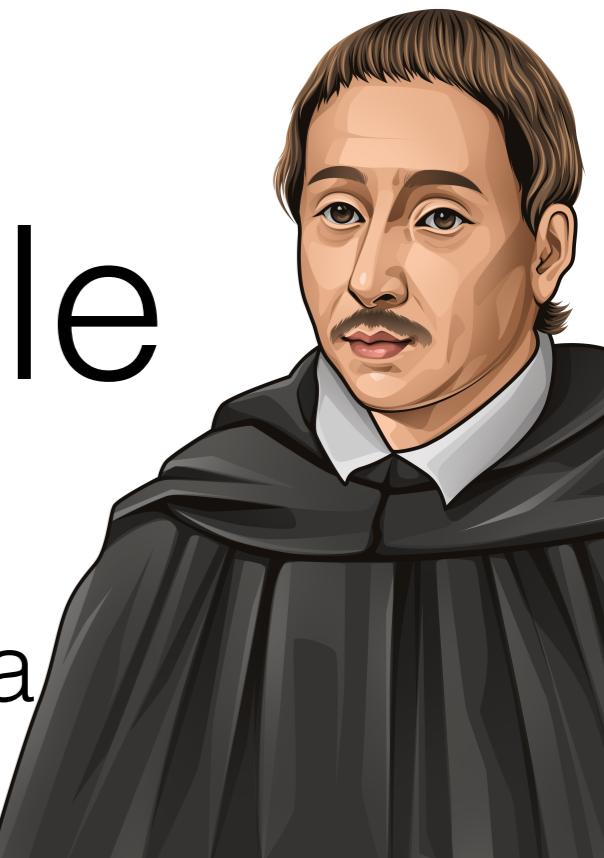
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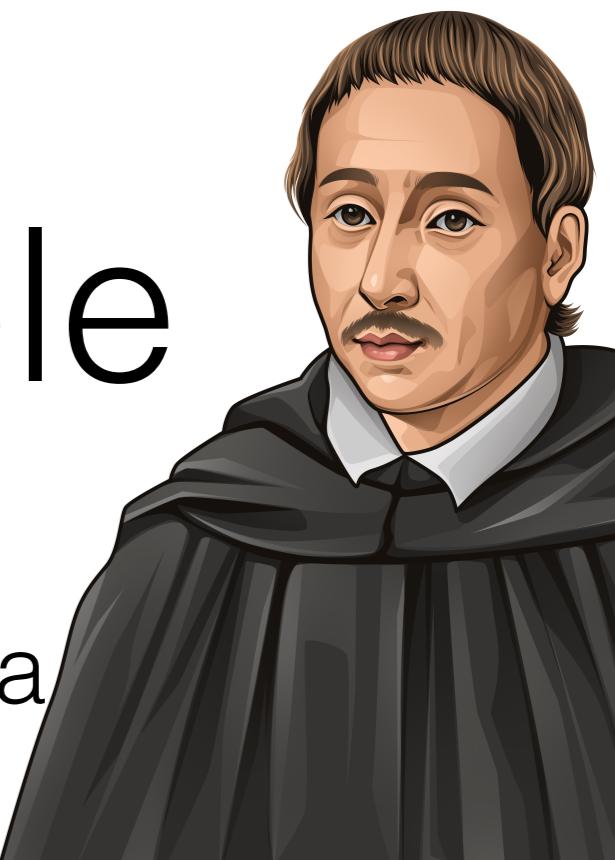


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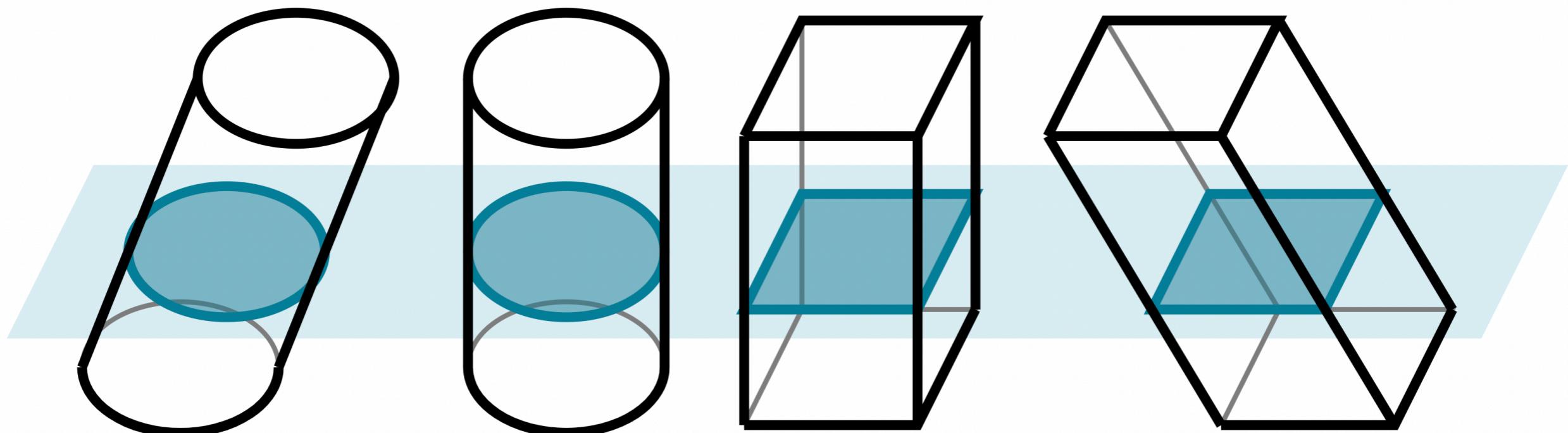
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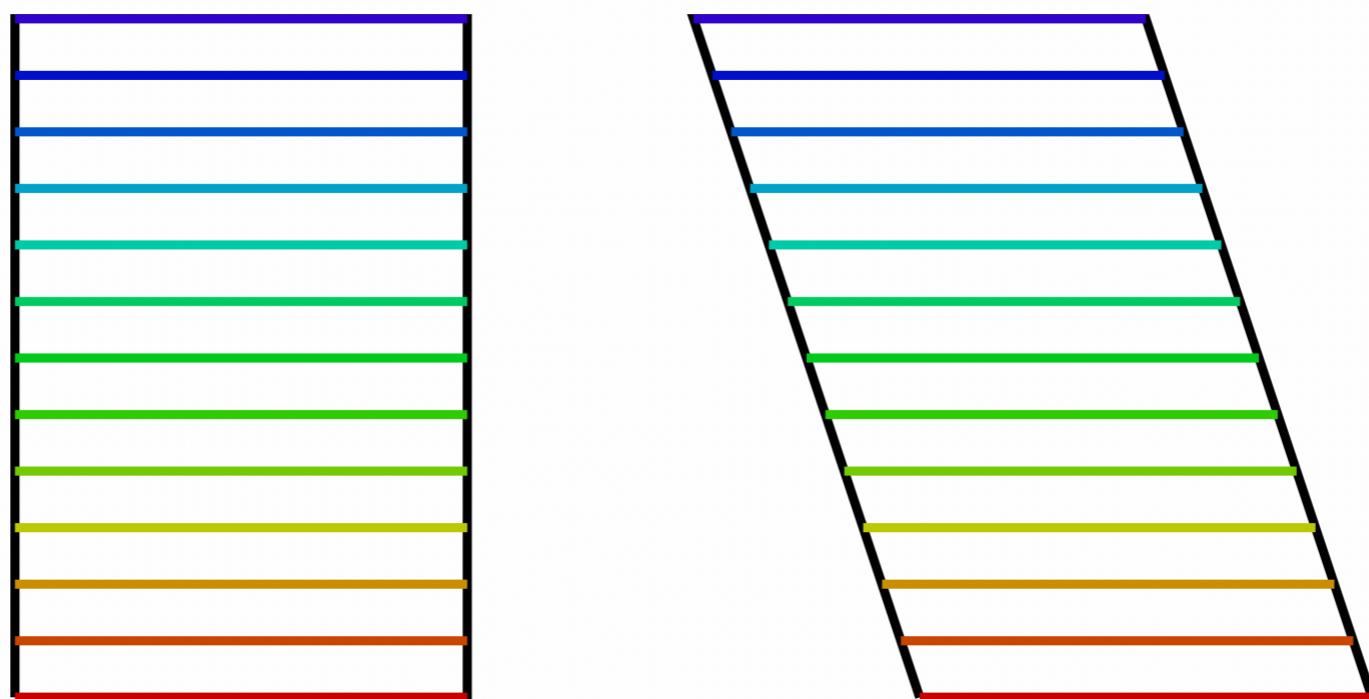
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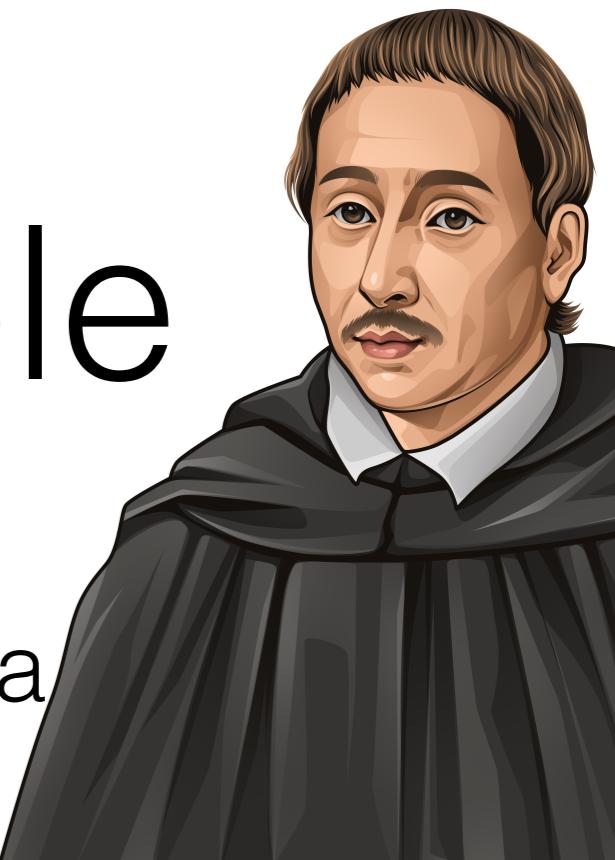


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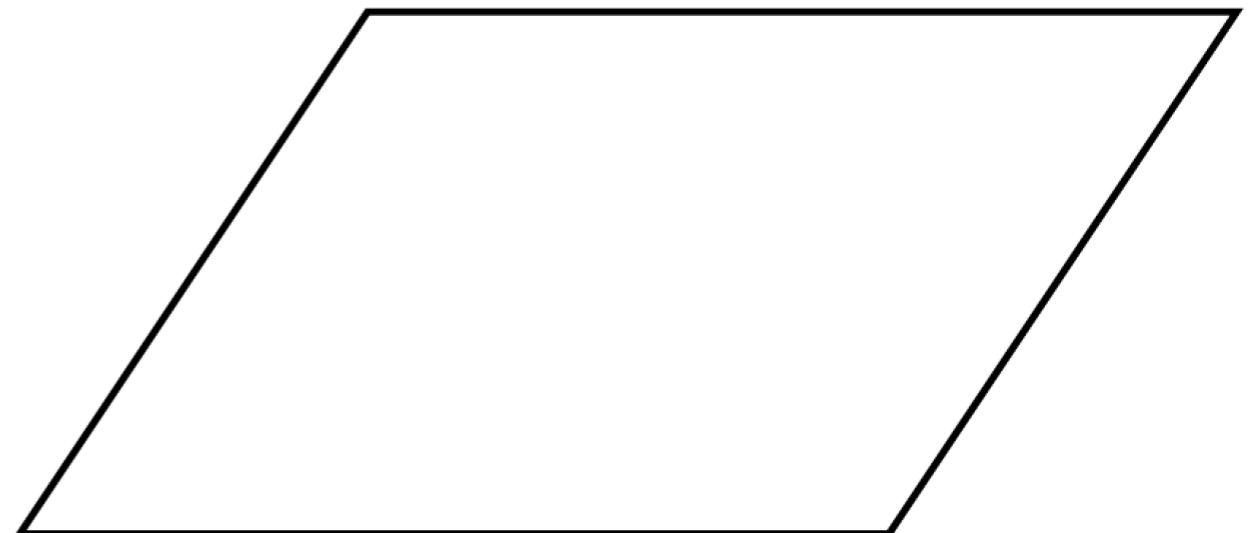
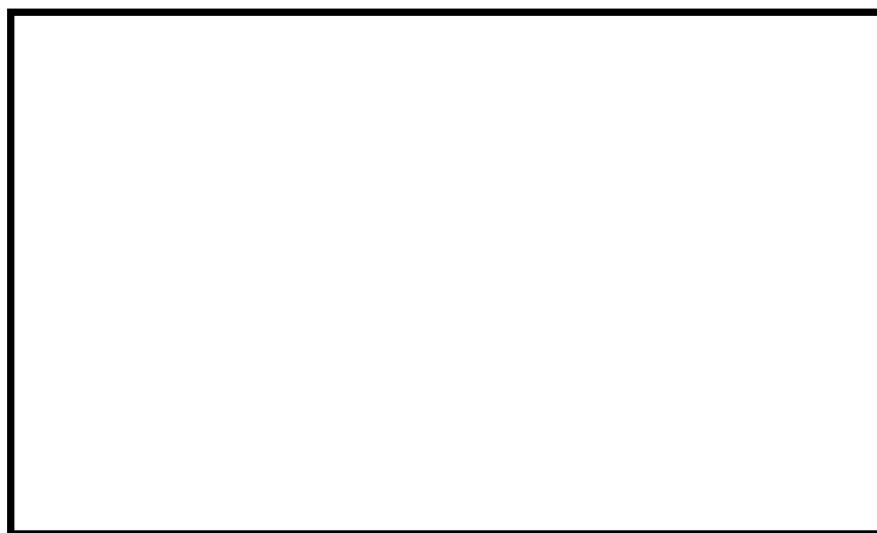


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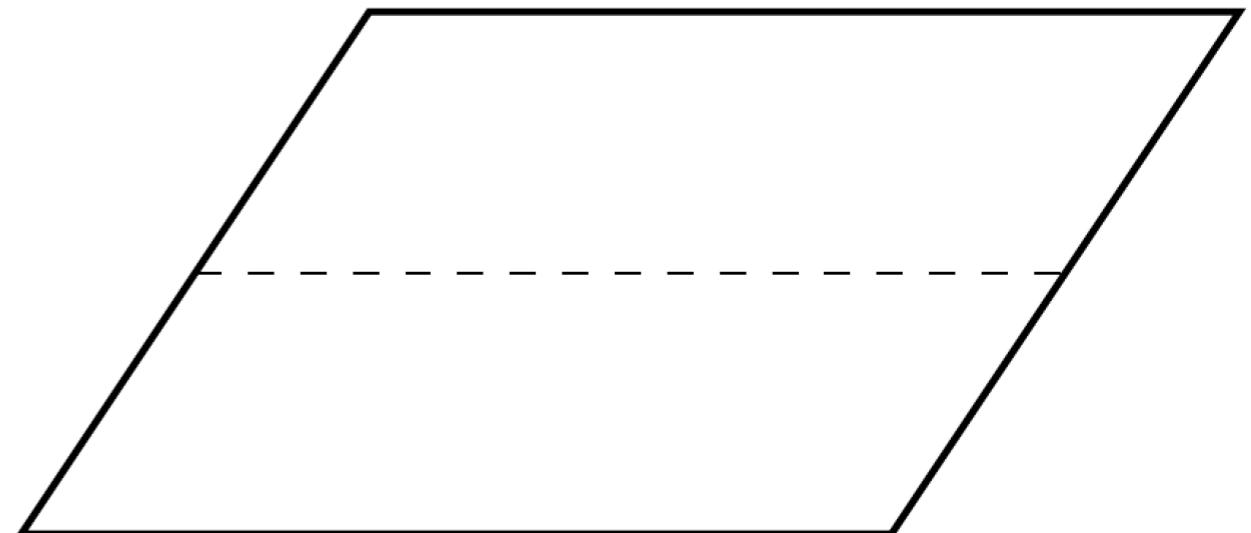
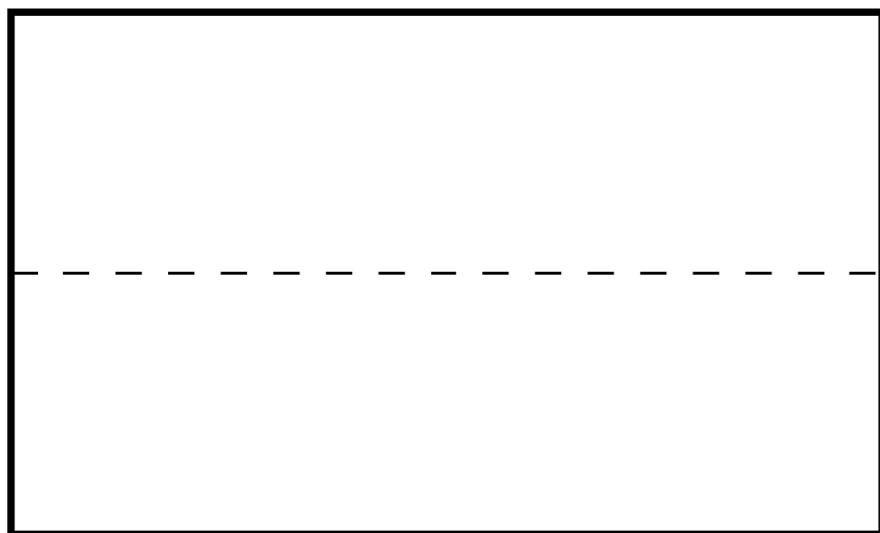
Cavalieri's Principle

- Classic example: This rectangle and parallelogram have the same area.



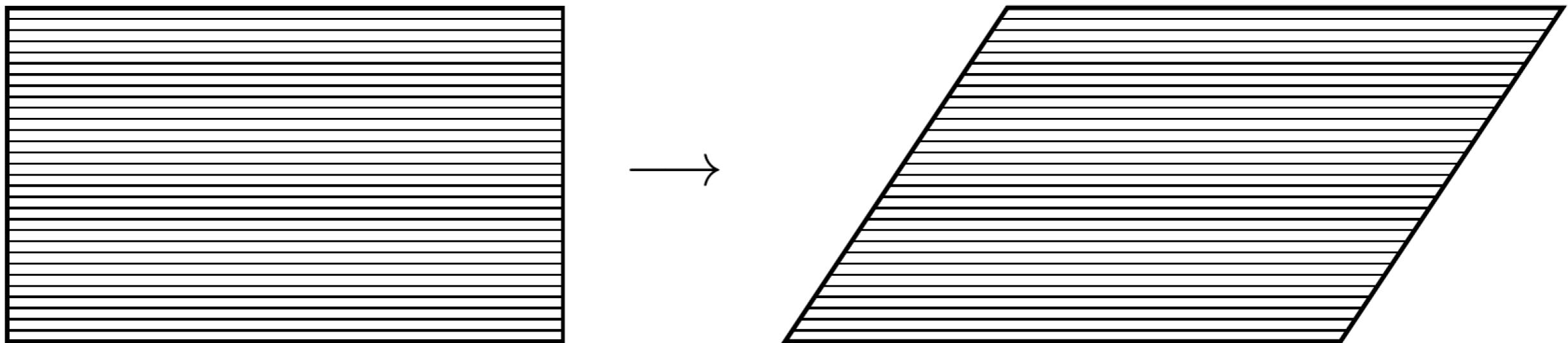
Cavalieri's Principle

- They have the same base and height, and every horizontal slice produces cross-sections of the same width.



Cavalieri's Principle

- Therefore, Cavalieri's principle says that they must have the same area, because their areas are made up of the same infinite set of lines.



Cavalieri's Principle



Cavalieri's Principle

- Cavalieri big theorem, in modern notation,

is that $\int_0^a x^n dx = \frac{a^{n+1}}{n+1}$

for $n = 1, 2, 3, \dots, 9$. This is a special case of the power rule for integrals.



Cavalieri's Principle



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Cavalieri's Principle



- Cavalieri big theorem, in modern notation, is that $\int_0^a x^n \, dx = \frac{a^{n+1}}{n+1}$ for $n = 1, 2, 3, \dots, 9$. This is a special case of the power rule for integrals.
- I will show you how he did this for the case $n = 2$. That is, we will prove that $\int_0^a x^2 \, dx = \frac{a^3}{3}$.
- To do this, we will instead think about the area from $x = 0$ to $x = 2a$.

Cavalieri's Principle



Cavalieri's Principle

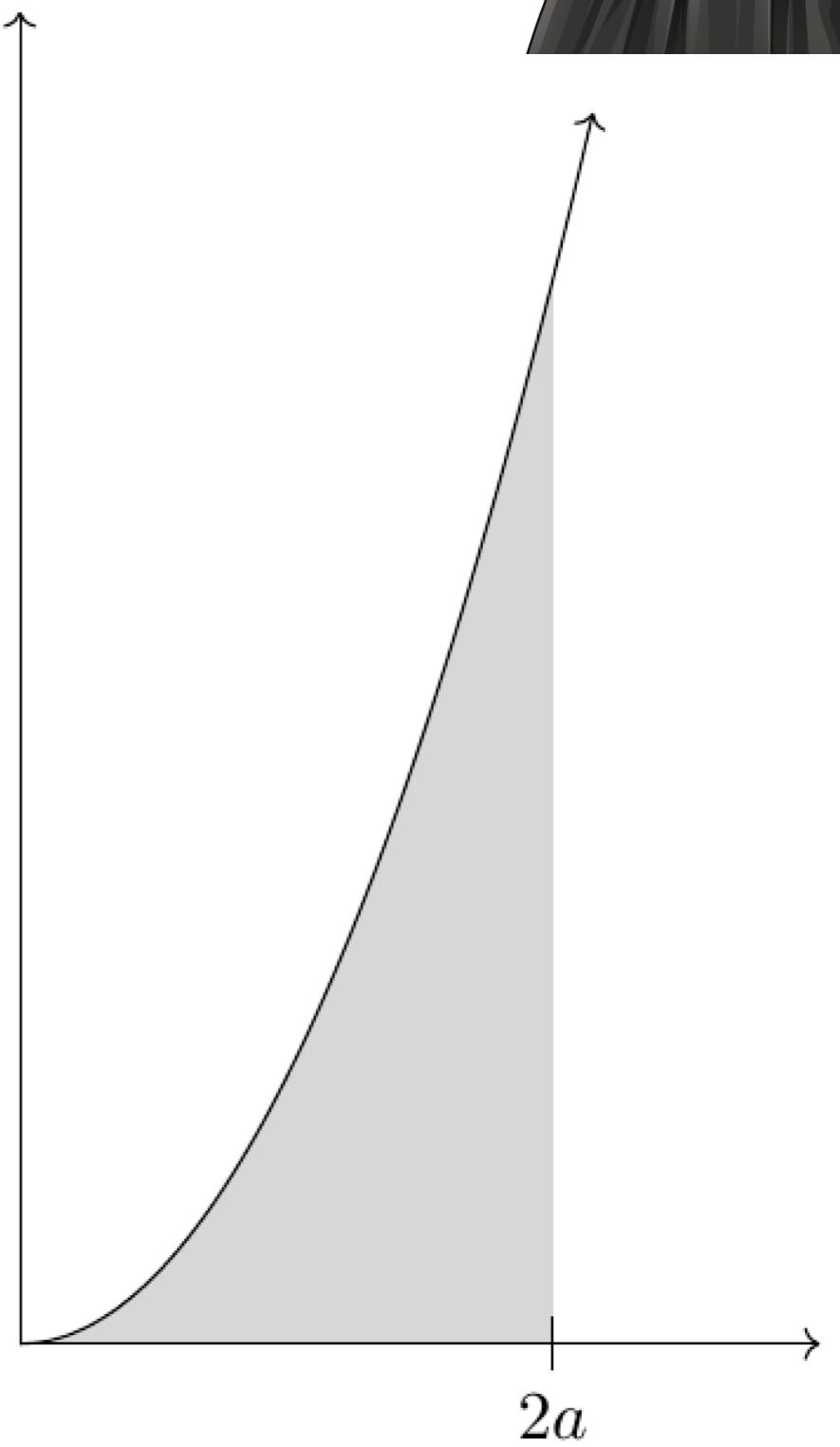


- **Theorem.** $\int_0^a x^2 \, dx = \frac{a^3}{3}.$

Cavalieri's Principle



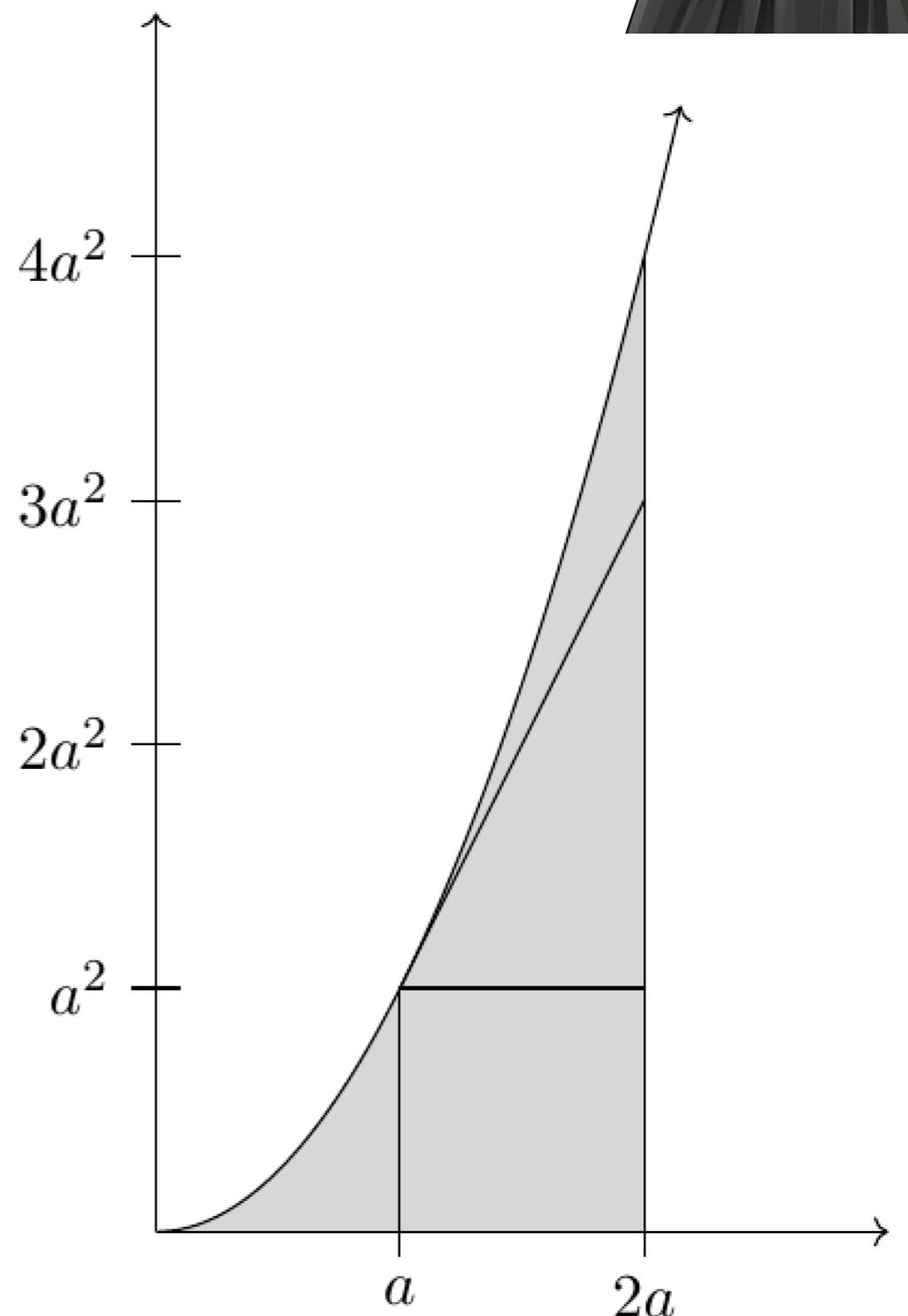
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Cavalieri's Principle



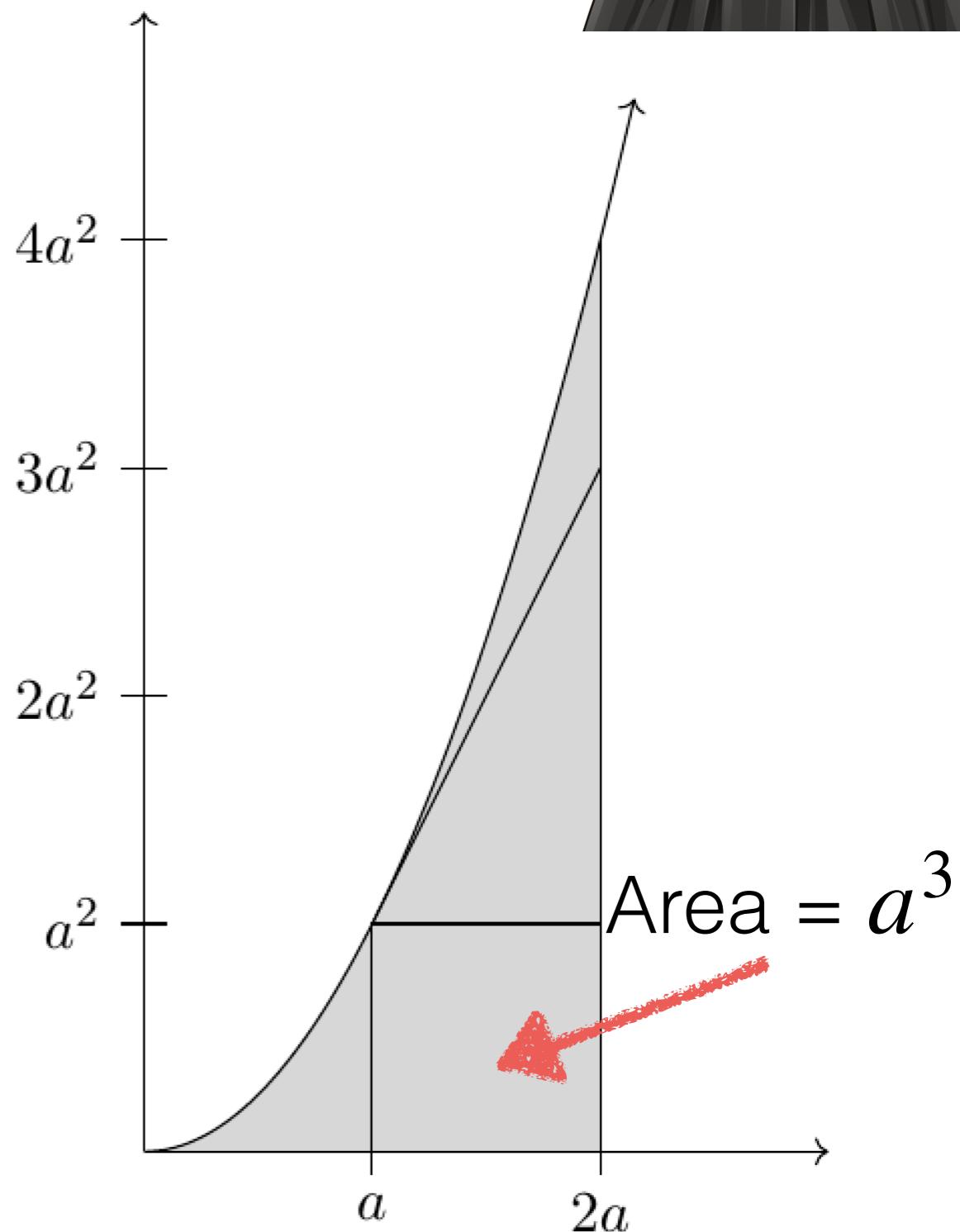
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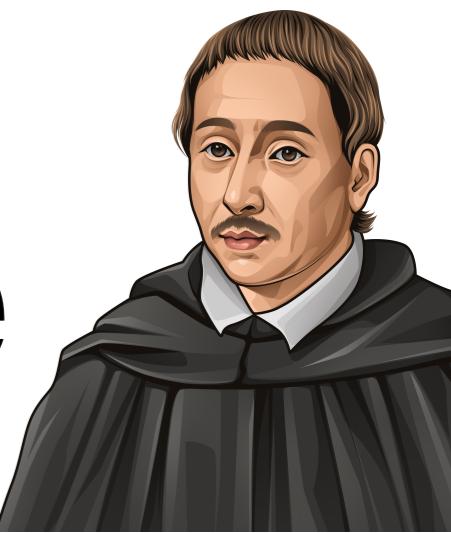
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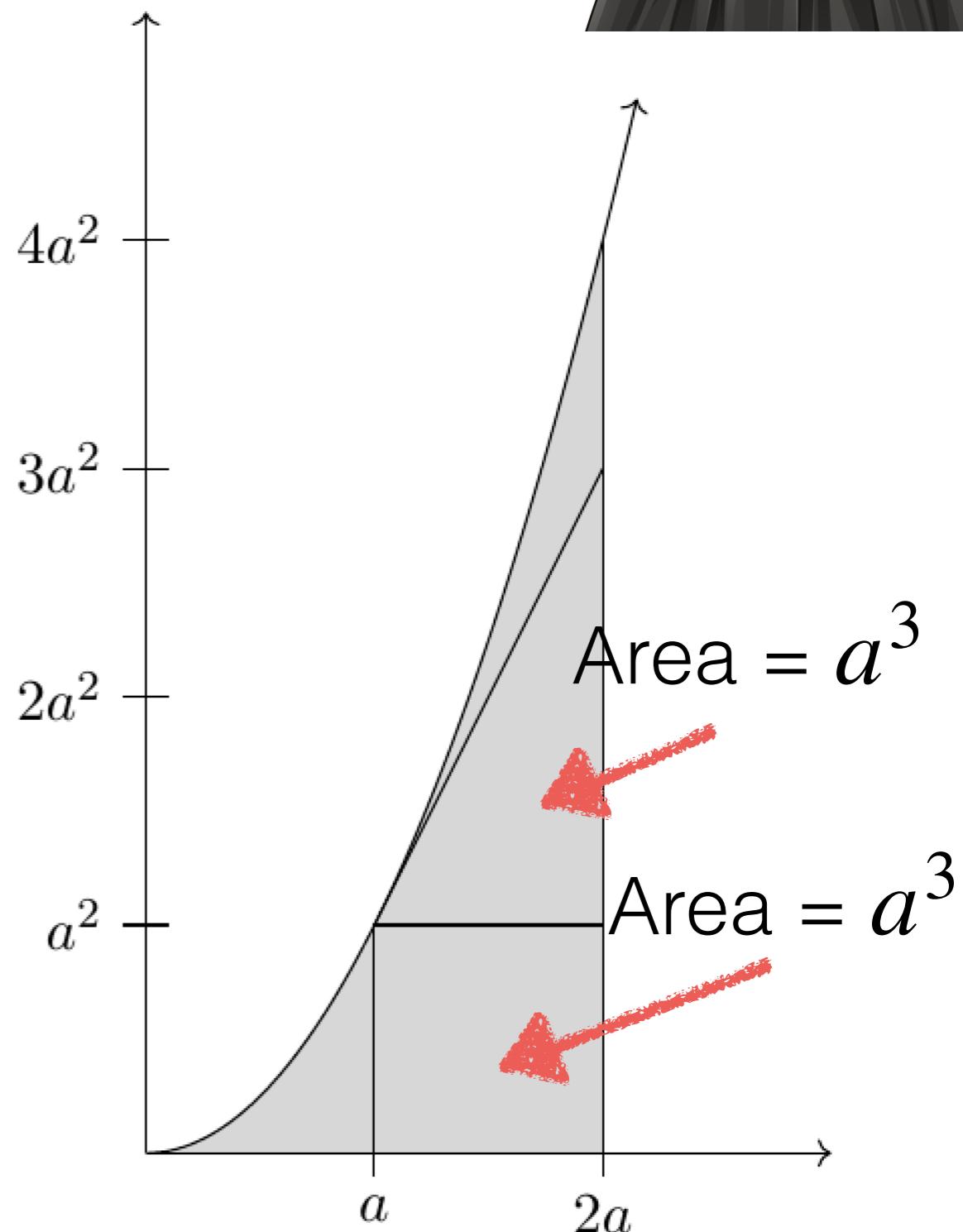
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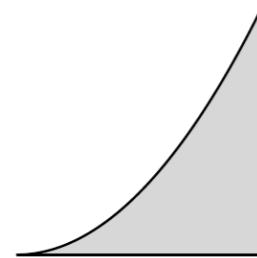


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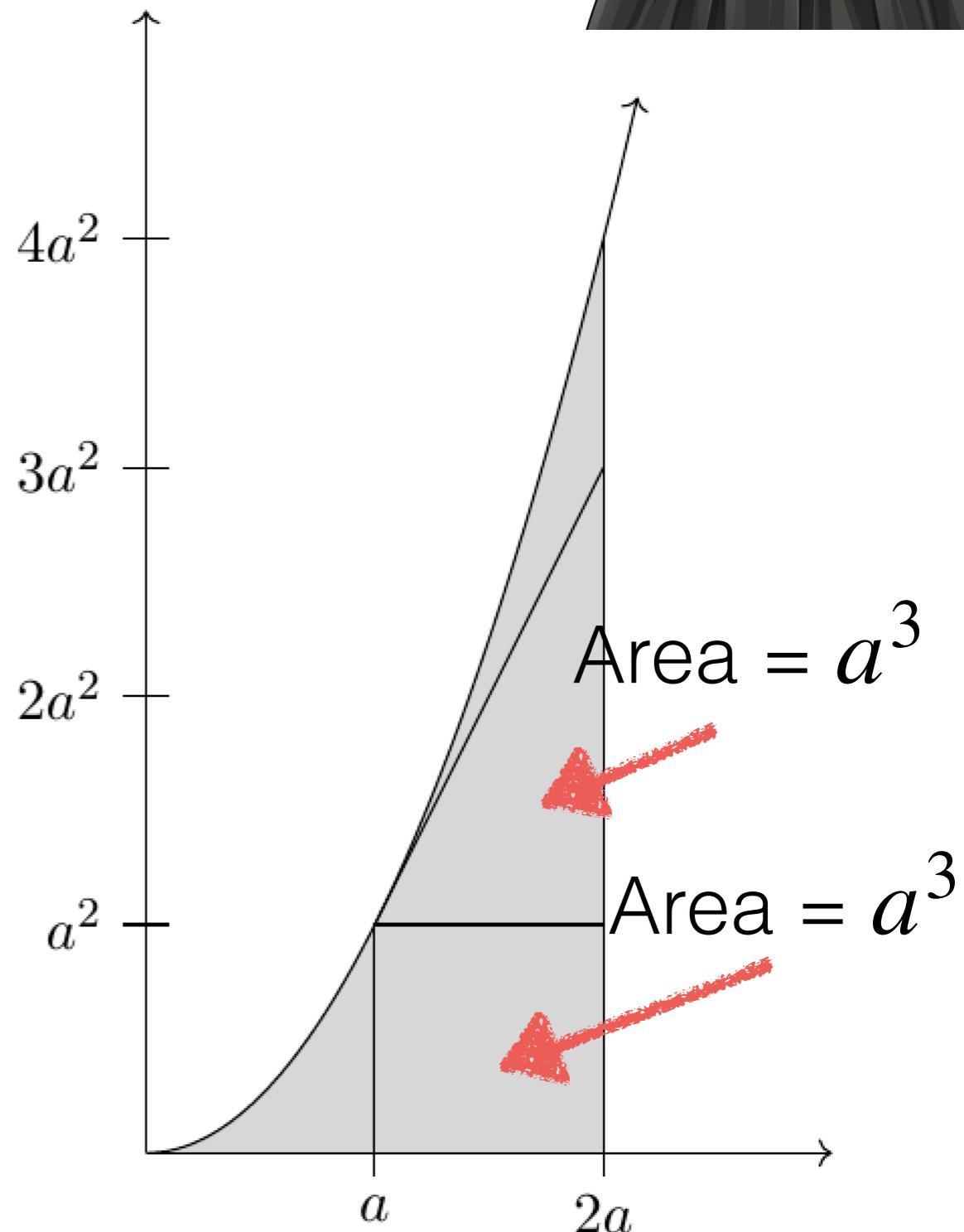


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- Need to find:



and



Cavalieri's Principle

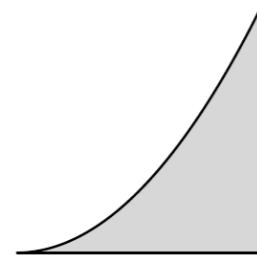


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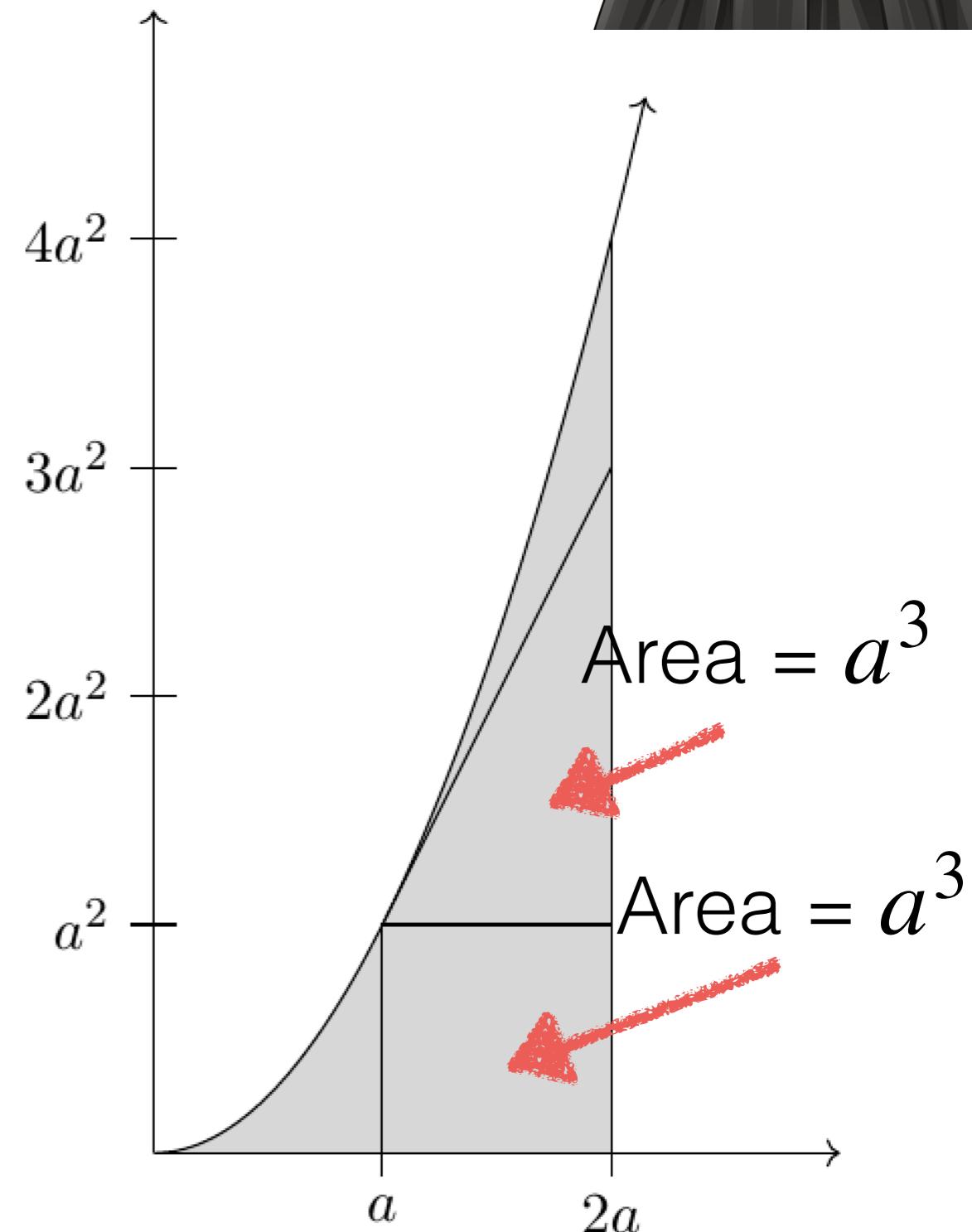
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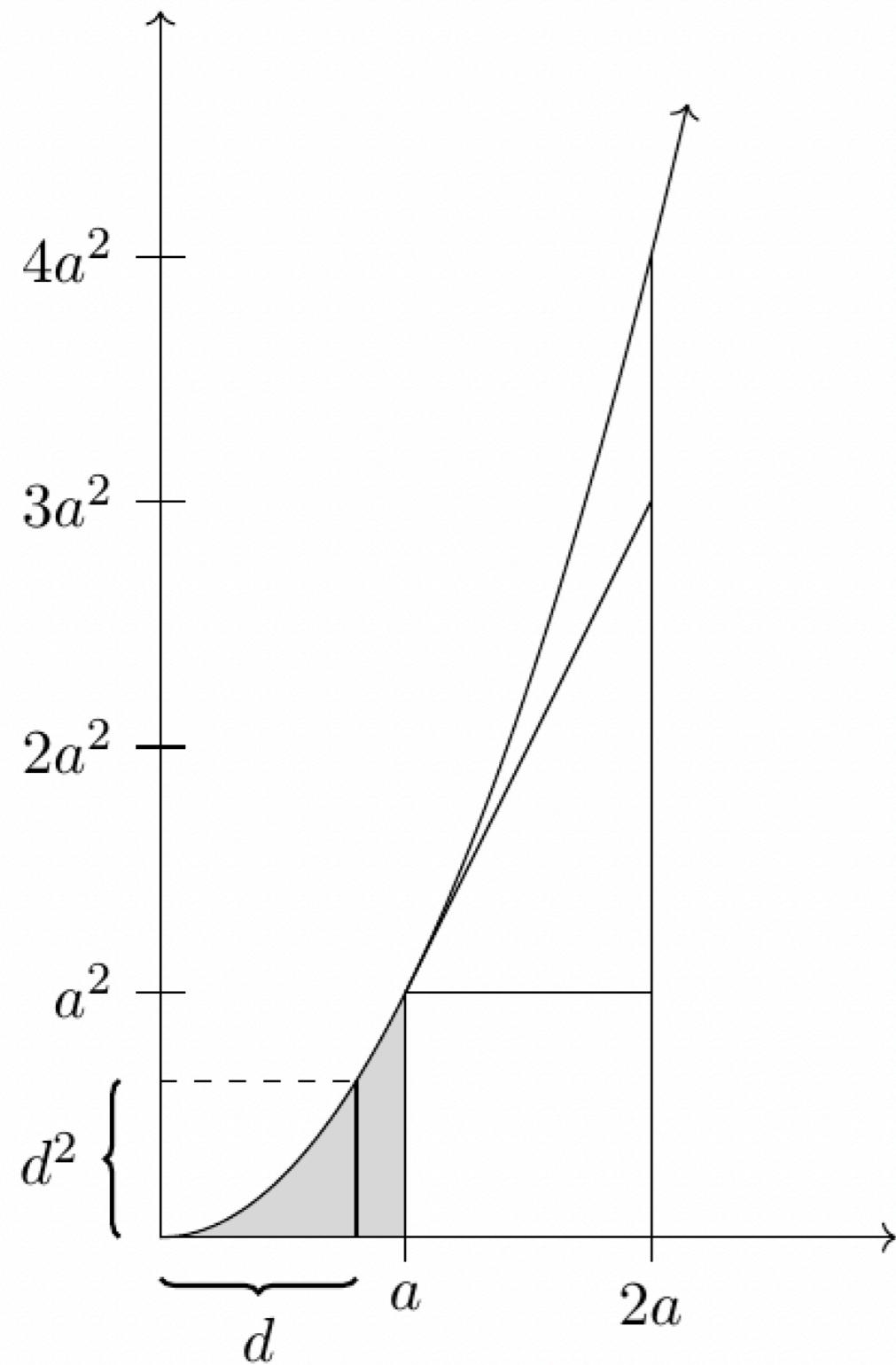
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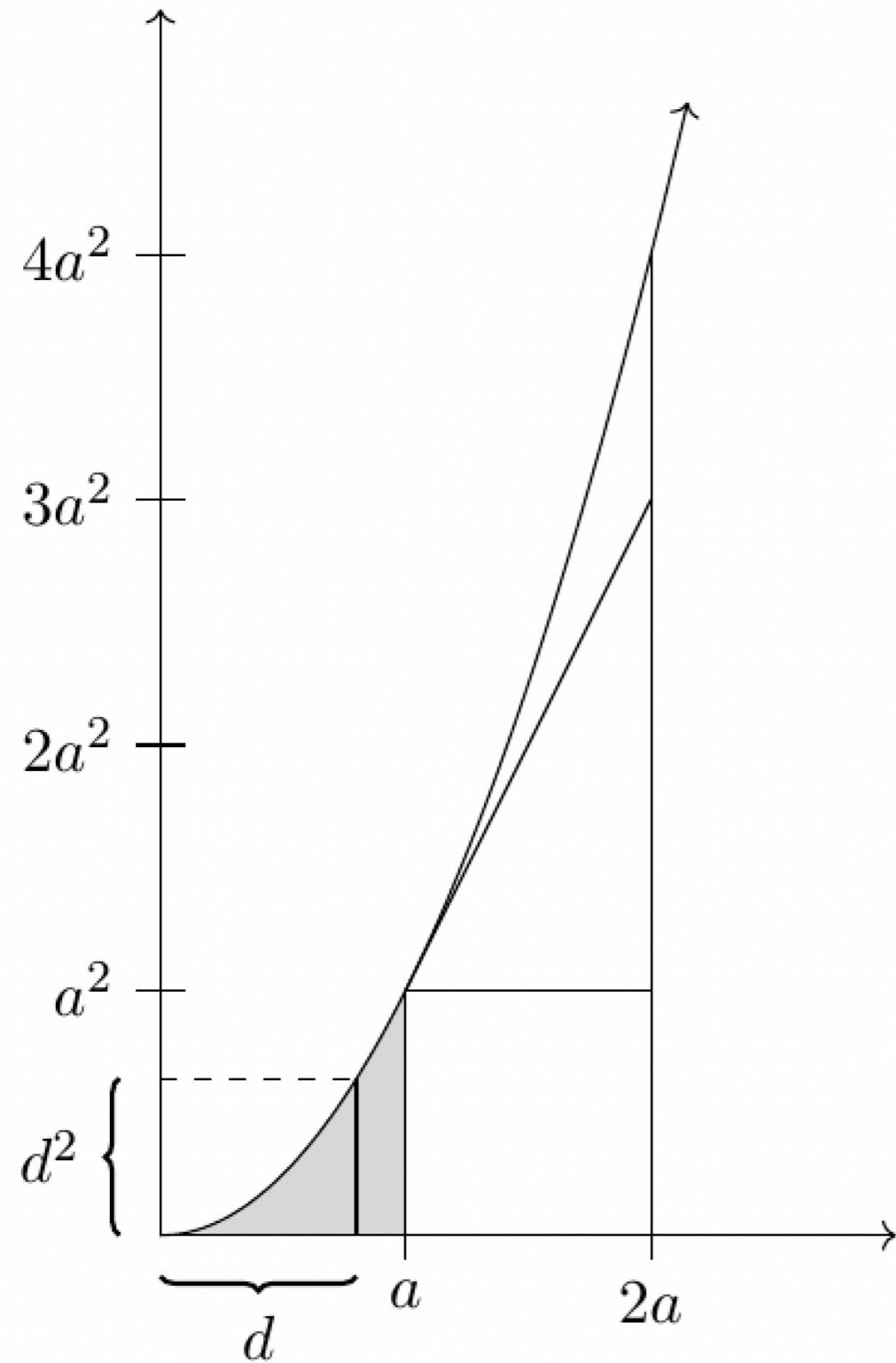
By Cavalieri's principle,
they have the same area!



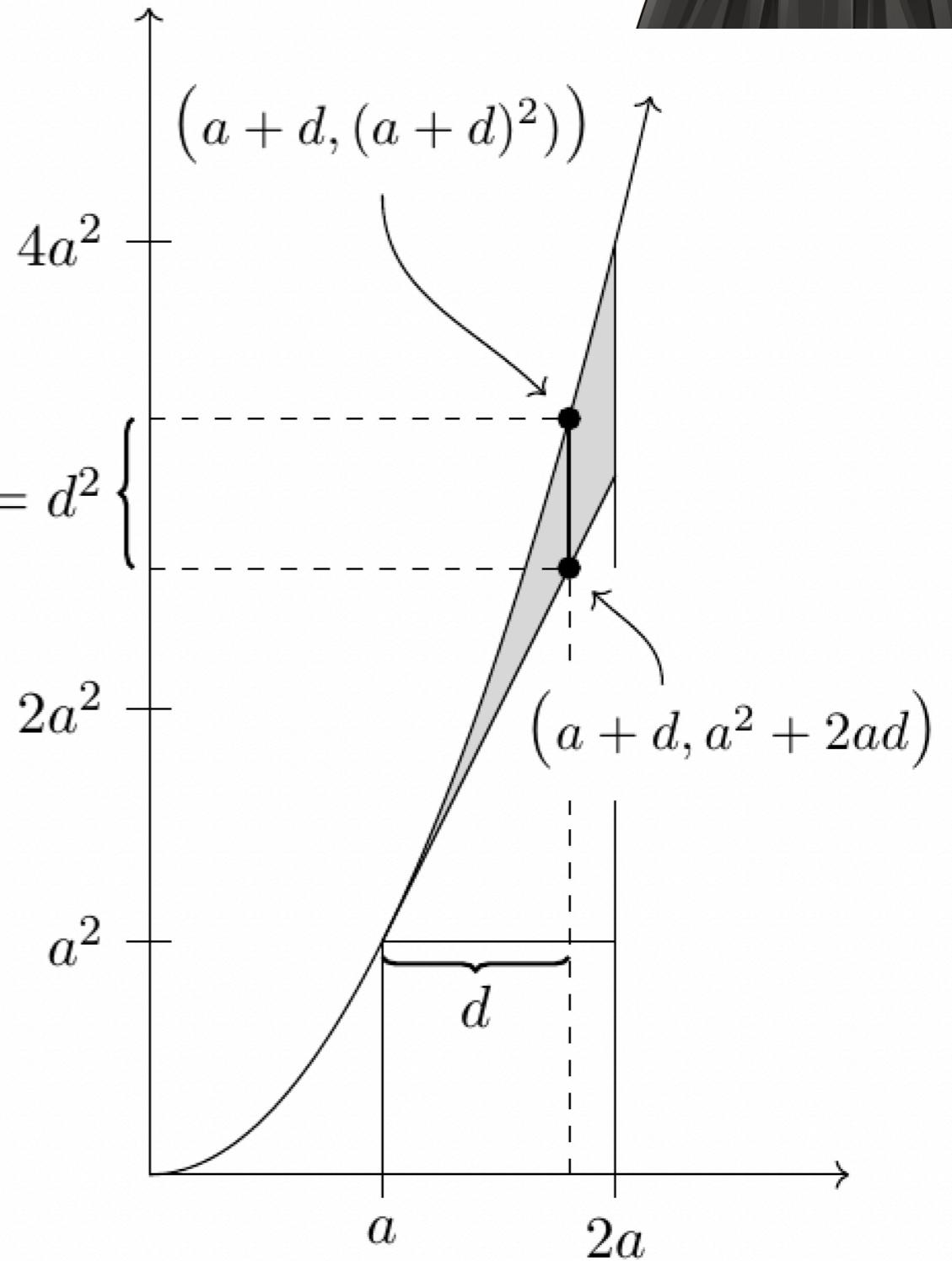
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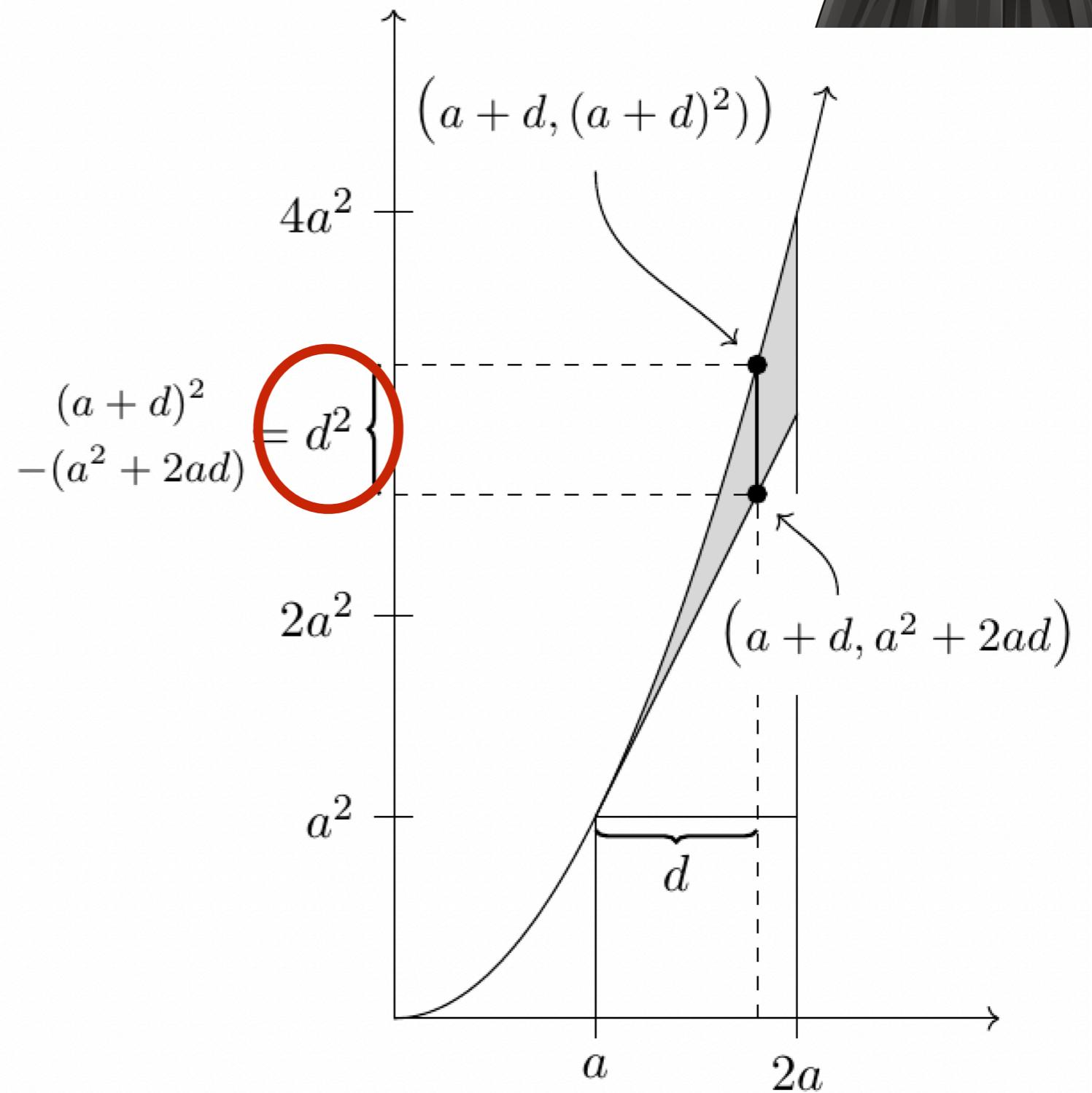
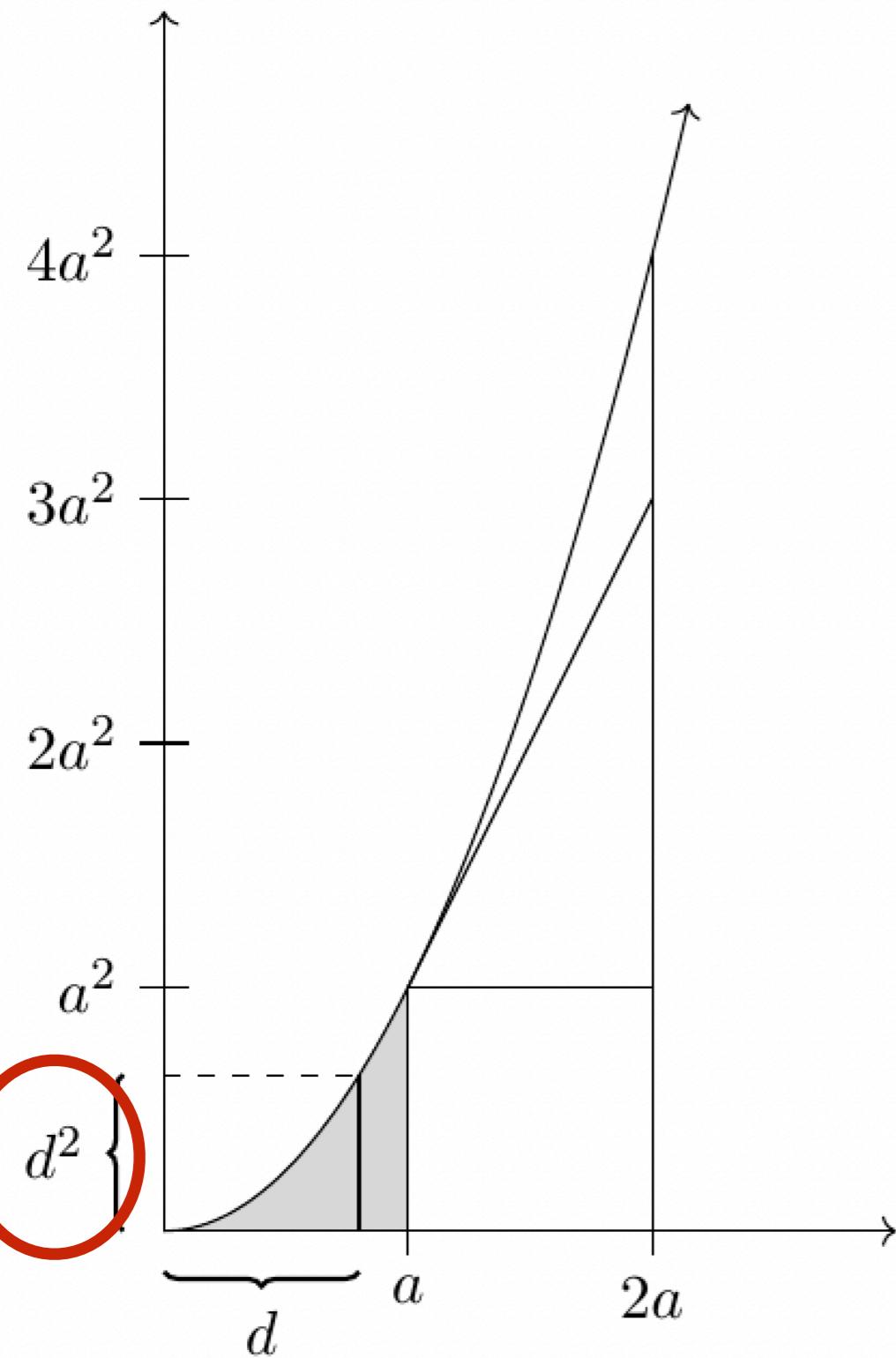
Cavalieri's Principle



$$\frac{(a+d)^2}{-(a^2 + 2ad)} = d^2 \quad \{$$



Cavalieri's Principle



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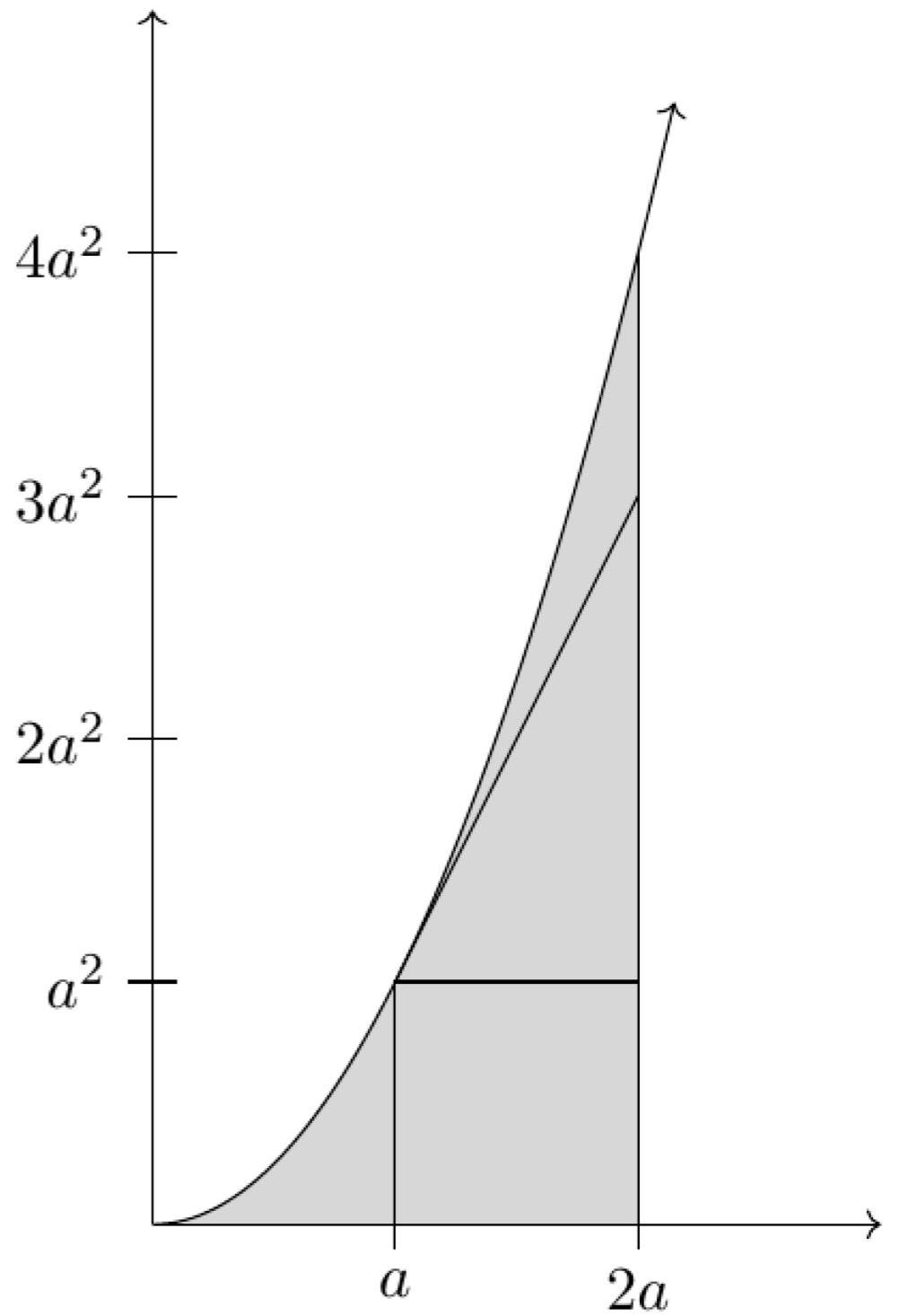


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Cavalieri's Principle



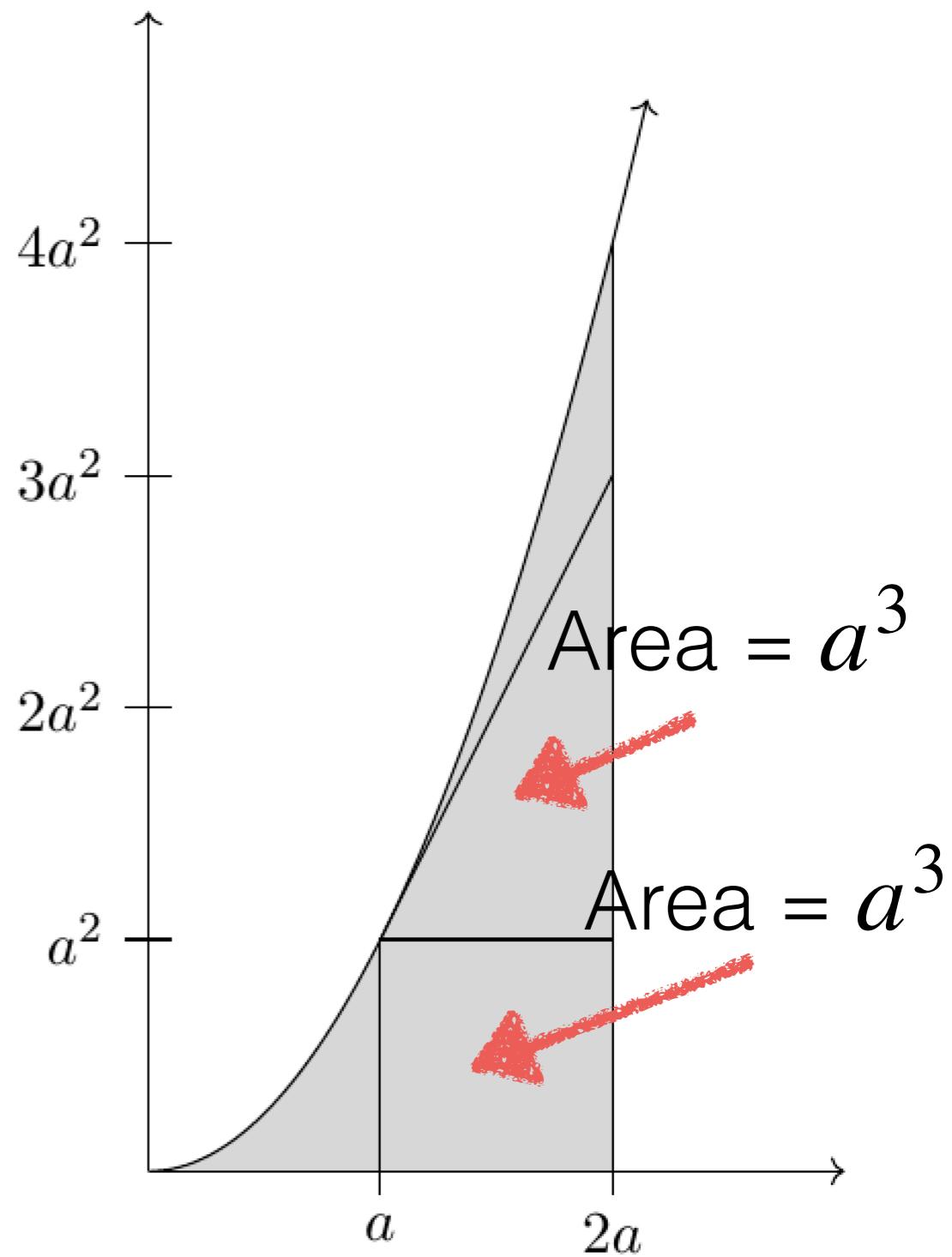
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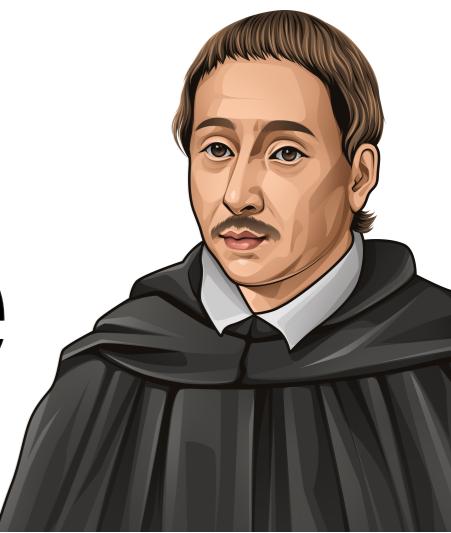
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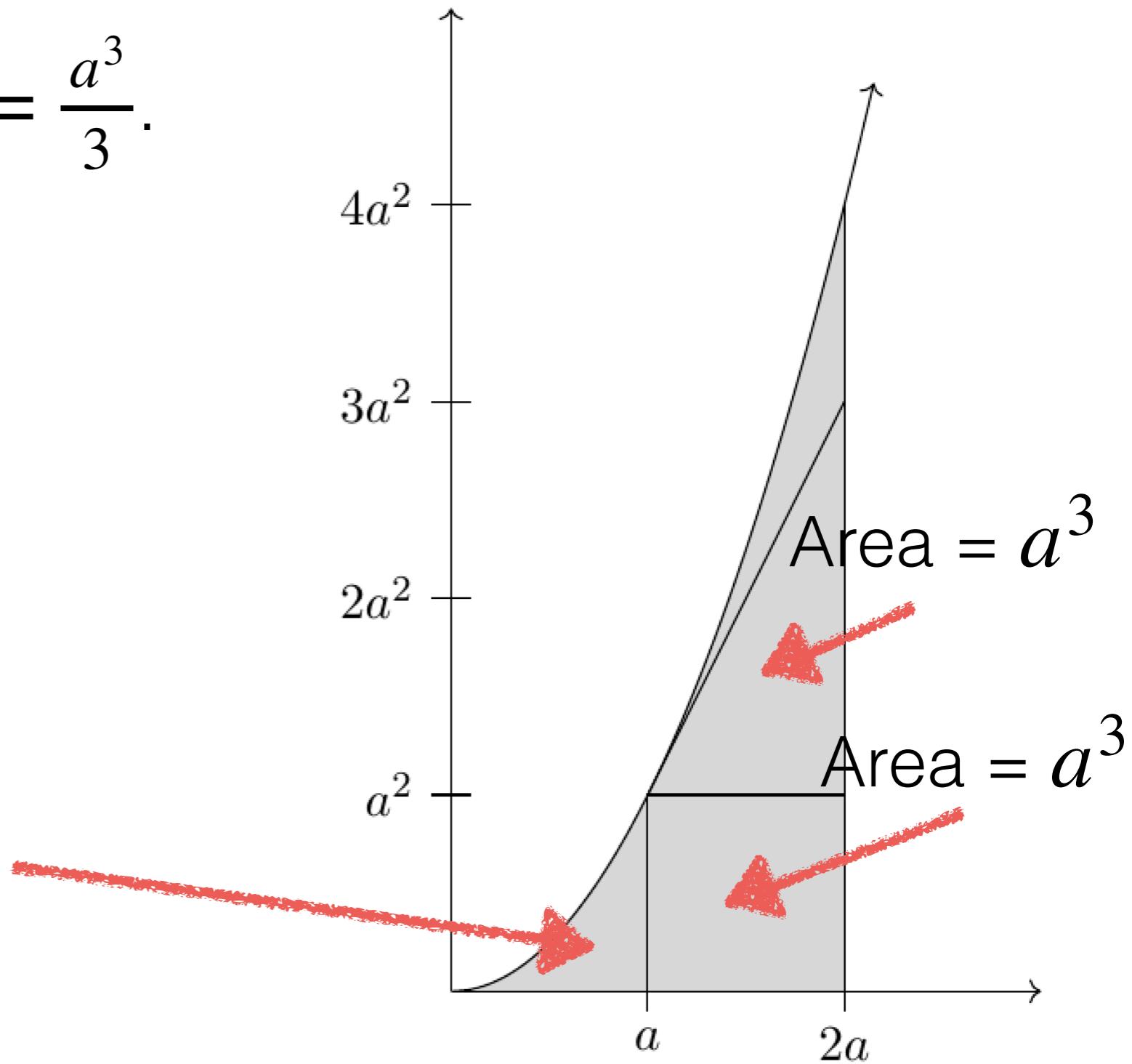


Cavalieri's Principle



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$$\text{Area} = \int_0^a x^2 \, dx$$



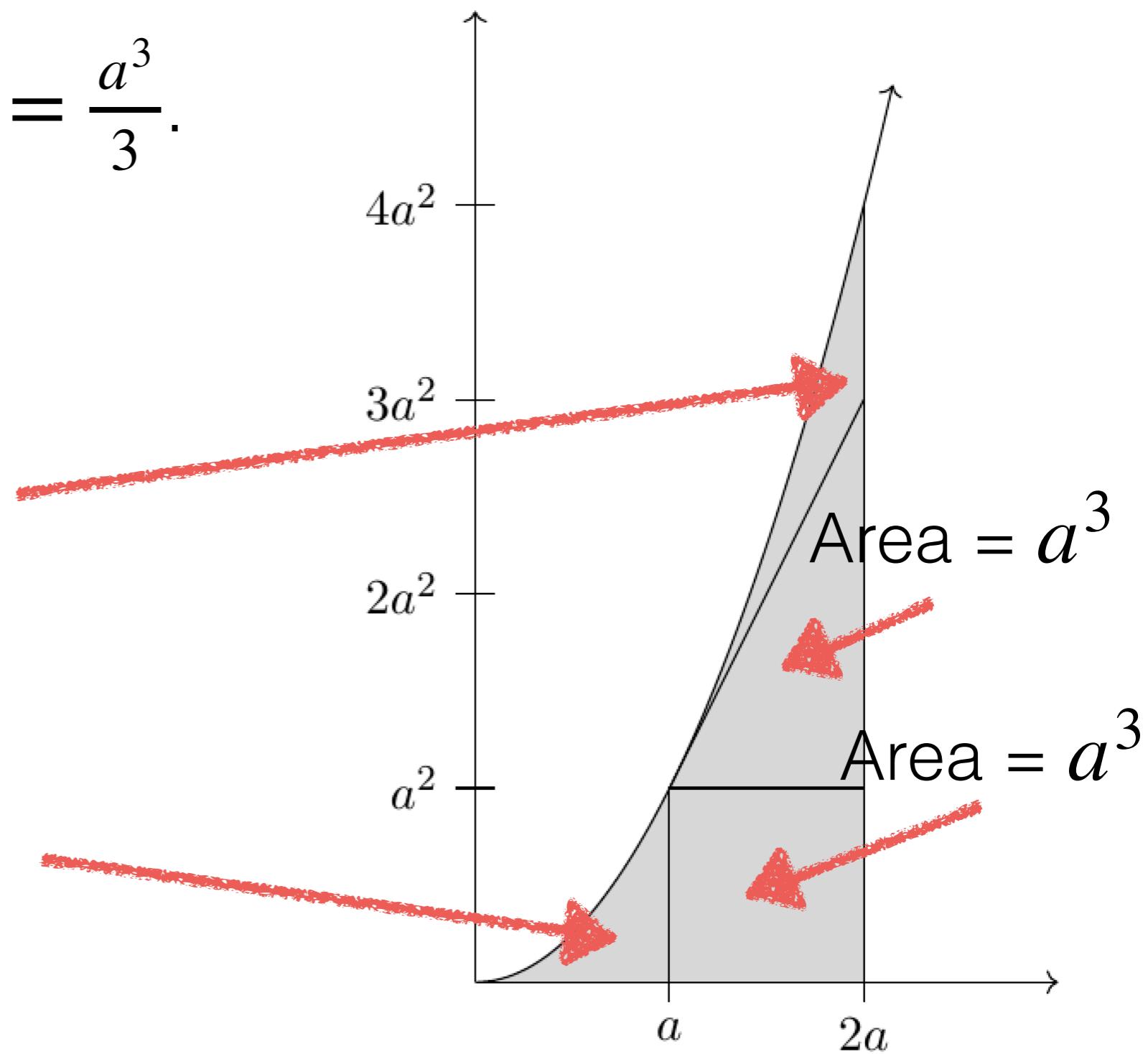
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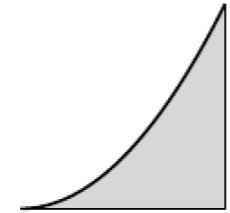
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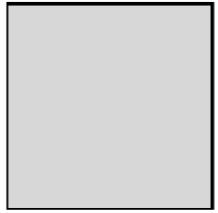


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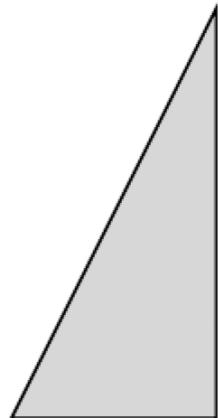
$$\int_0^{2a} x^2 dx =$$



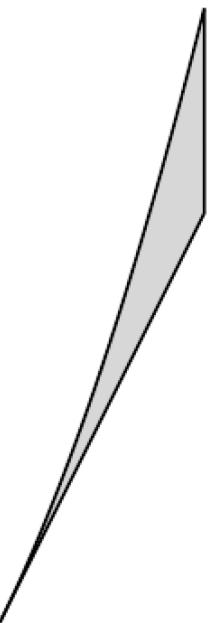
+



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$$\begin{aligned}\int_0^{2a} x^2 dx &= \text{Area under } y=x^2 \text{ from } x=0 \text{ to } x=2a \\&= \text{Area under } y=x^2 \text{ from } x=0 \text{ to } x=a + \text{Area of a square of side } a + \text{Area under } y=x^2 \text{ from } x=a \text{ to } x=2a \\&= \int_0^a x^2 dx + a^3 + a^3 + \int_0^a x^2 dx.\end{aligned}$$

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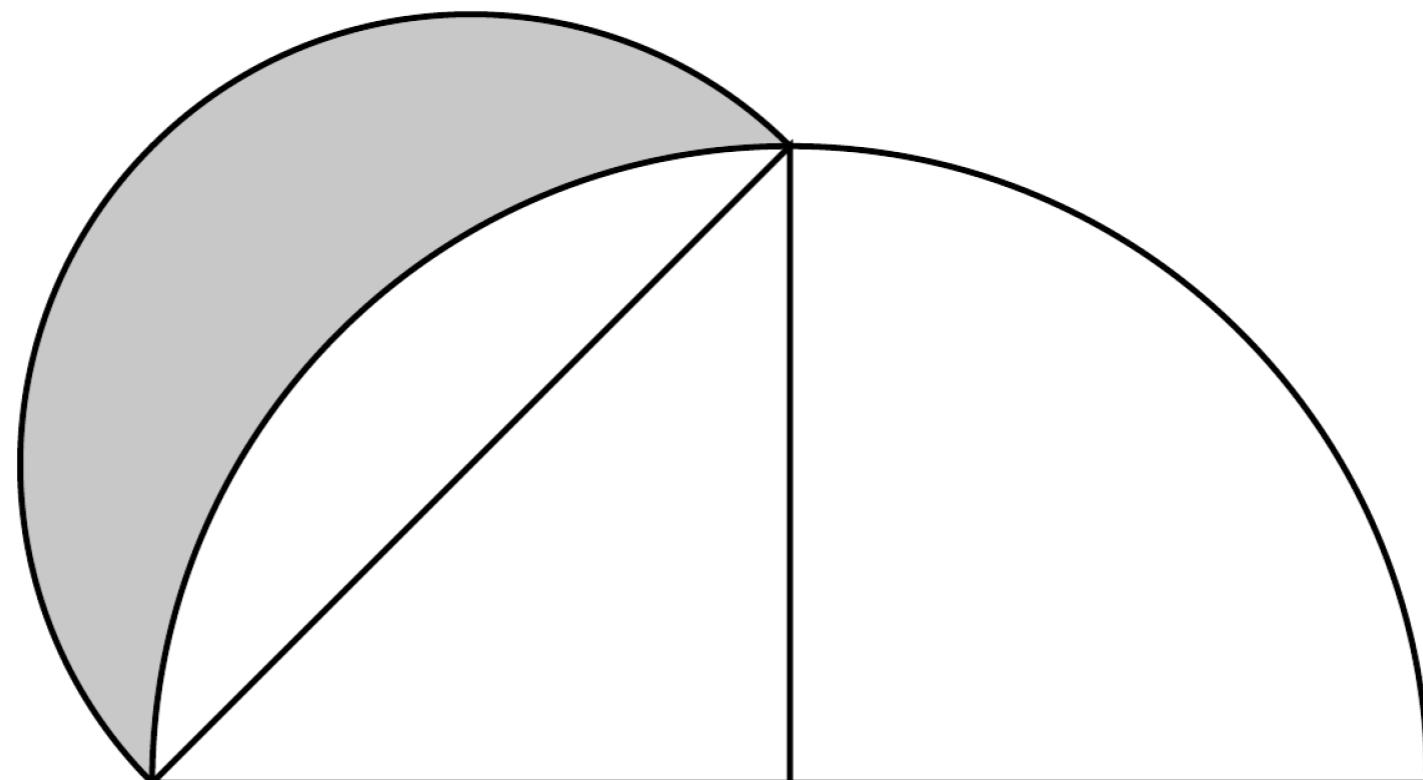
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Q.E.D.

Shout-outs!

- The first known person to determine the exact area of a region with curved sides was the Greek Hippocrates of Chio. In the mid-fifth century BC, he found the exact area of a *lune*, which is made with two semicircles as shown below.



Shout-outs!

- On the topic of analytic paradoxes and slicing:
- The Banach-Tarski paradox is perhaps the most important today.
- It says that one can take a solid ball, slice it up into five parts, move around those parts, and then put them back together to get two identical copies of the original ball.
- The slices are *very* weird. Not “normal” slices.