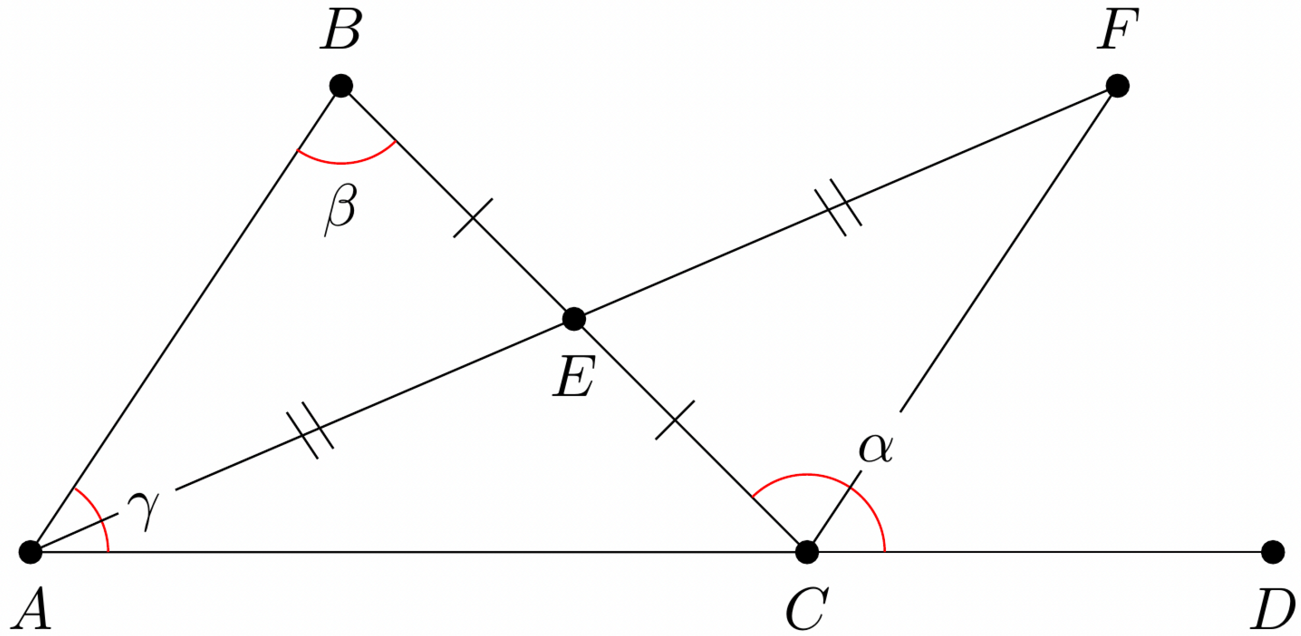
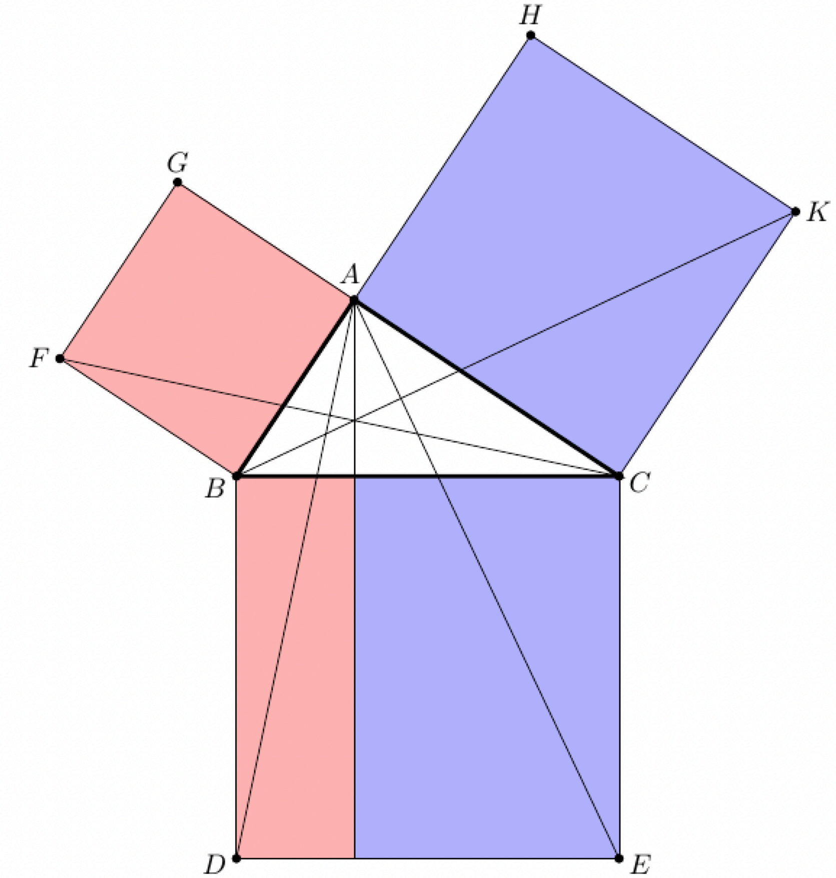
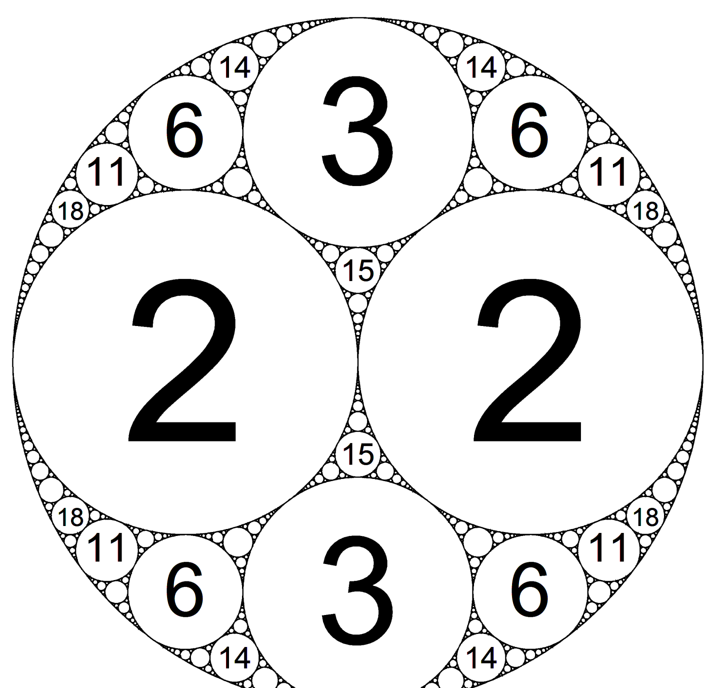




Chapter 4: Euclidean Geometry



Greek Math

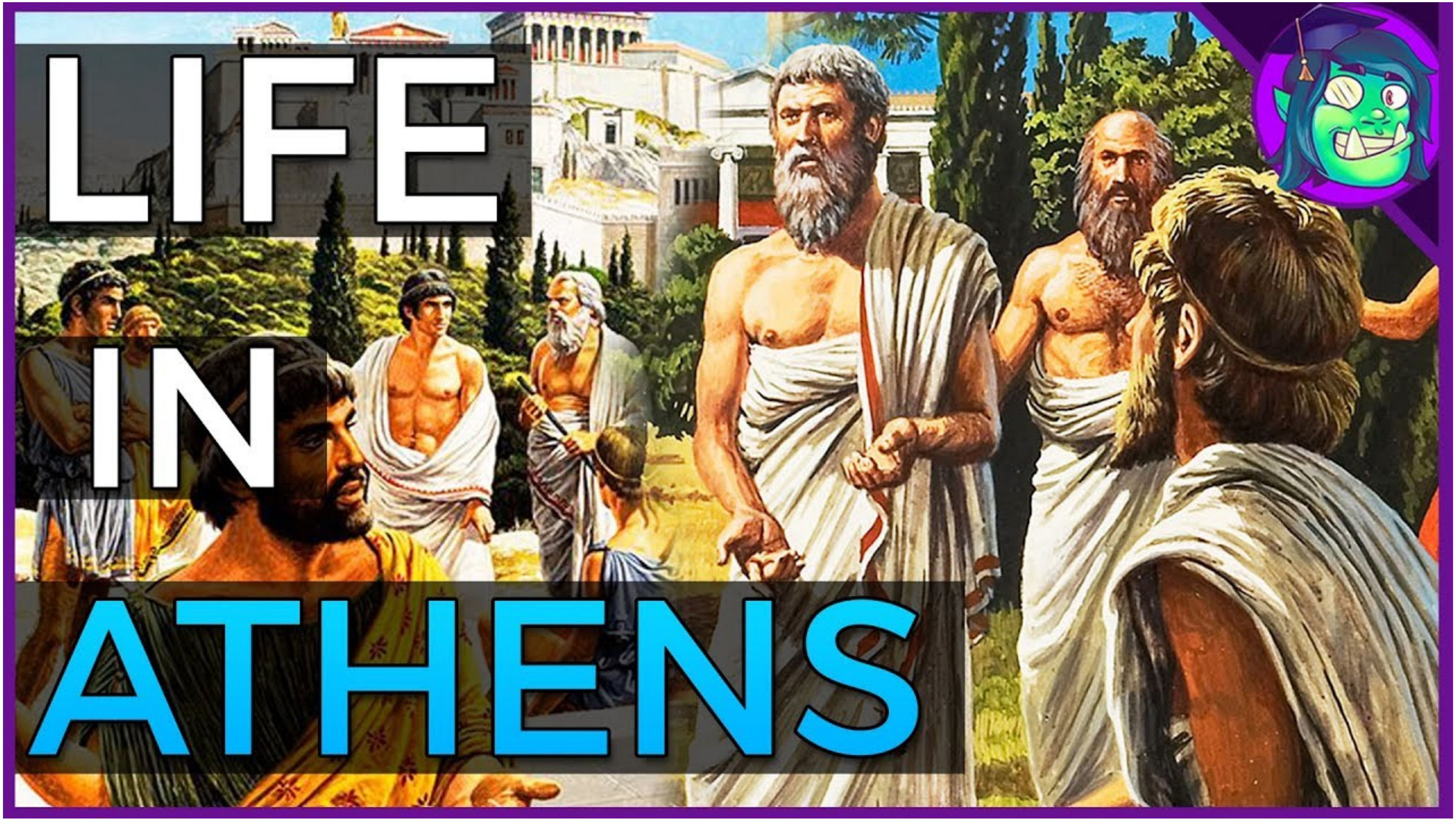
- Pythagoras died ~ 500BC. Euclid born ~325BC. Between them lived Socrates, Plato and Alexander the Great.

- Plato established the Academy in Athens ~387BC, giving scholars time, resources and collaborators. Legend says that above the entrance was "Let no man ignorant of geometry enter here."

- Many great Greek mathematicians studied at the Academy. Greatest of them was Eudoxus of Cnidus.

Academy in Athens

Life in Ancient Athens



The Elements

- Euclid wrote the *Elements*, which is the most important book in math history. Much of it is not his original work—it is compiled from other sources.
- The major sources were likely the Pythagoreans, Hippocrates and Eudoxus. Many of the proofs are believed to be original to Euclid.

- Its success is due to its logical and axiomatic presentation. This deductive style is the central approach to mathematics today.

The Elements

- The text was the primary math textbook for *two thousand years*.

- It is divided into 13 “books” on topics from geometry, algebra and number theory.

- Book I is devoted to propositions from planar geometry.

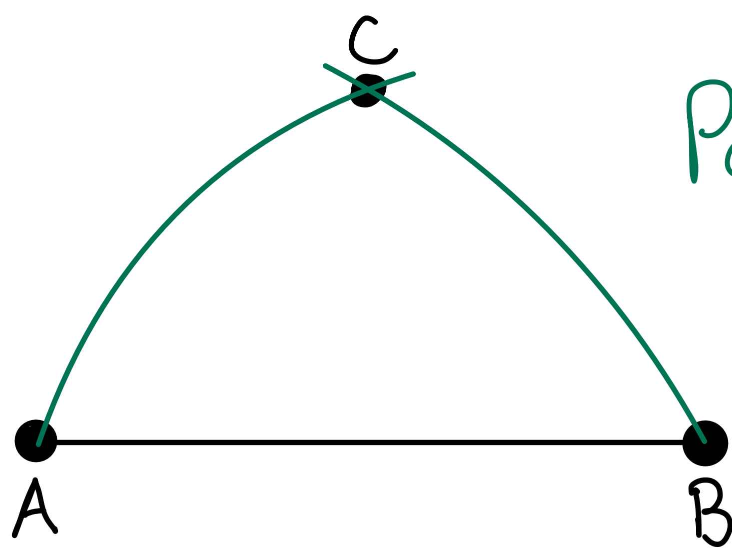
- Euclid’s postulates allow him to use a straightedge and compass, which play a central role.

Definitions, Common Notions and Postulates

Proposition Book 1 - postulate 1

On a given finite line, one can construct an equilateral triangle.

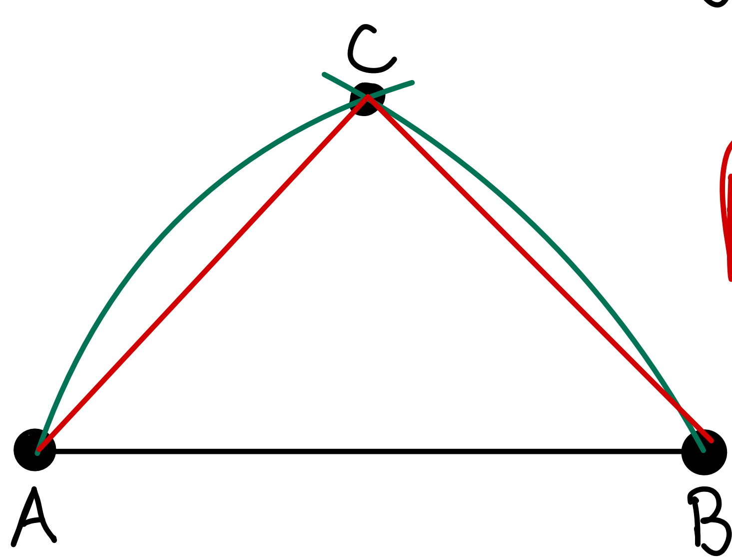
Proof



Postulate 3
(Twice)

Postulate 3 (twice)

[It is possible] to describe a circle with any center and distance



Postulate 1

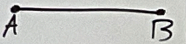
Postulate 1

[It is possible] to draw a straight line from any point to any [other] point

Therefore, $\overline{AB} = \overline{AC}$

re line,

n



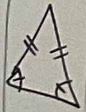
ostulate 3
(twice)

$\overline{AB} = \overline{AC}$ and $\overline{AB} = \overline{BC}$ by definition of a circle. $\overline{AC} = \overline{BC}$ by Common Notion 1. So, ΔABC is equilateral by definition.

Q.E.D.

Proposition I.4

SAS triangle Congruence



Propositions I.5 and I.6

A triangle is isosceles if and only if its base angles are congruent.

Proposition I.8

SSS triangle Congruence.

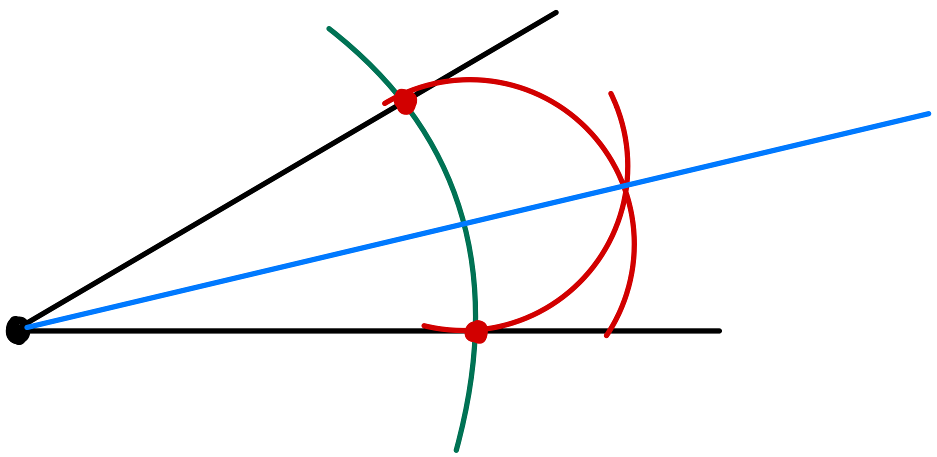
Proposition Book 1 - postulate 4: SAS triangle congruence

Proposition Book 1 - postulate 5 and 6: A triangle is isosceles if and only if its base angles are congruent.

Proposition Book 1 - postulate 8: SSS triangle congruence

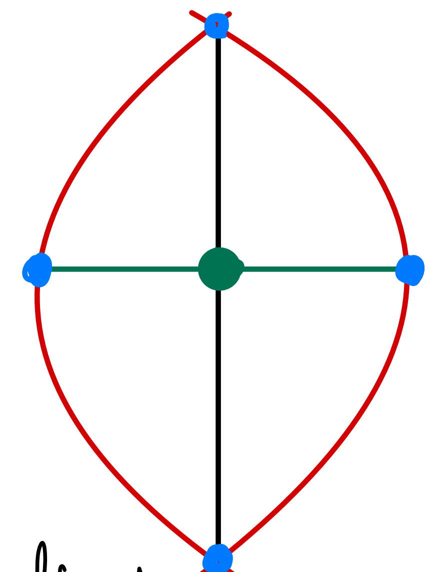
4 constructions

Book I.9

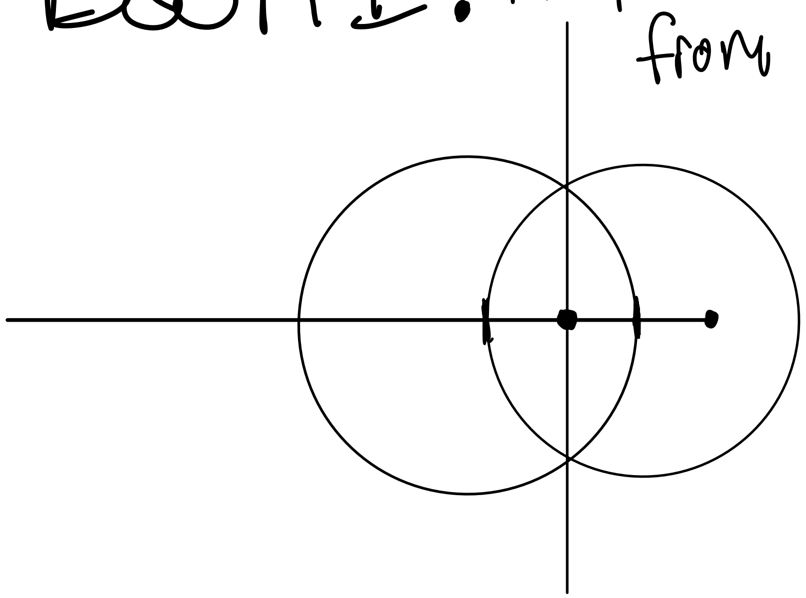


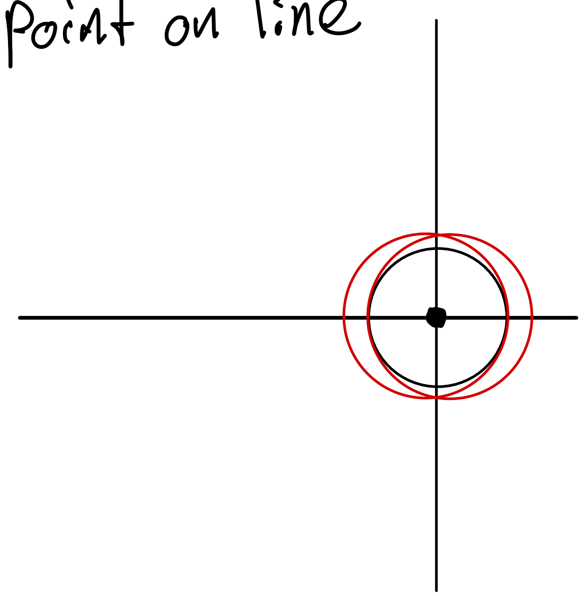
Book I.10

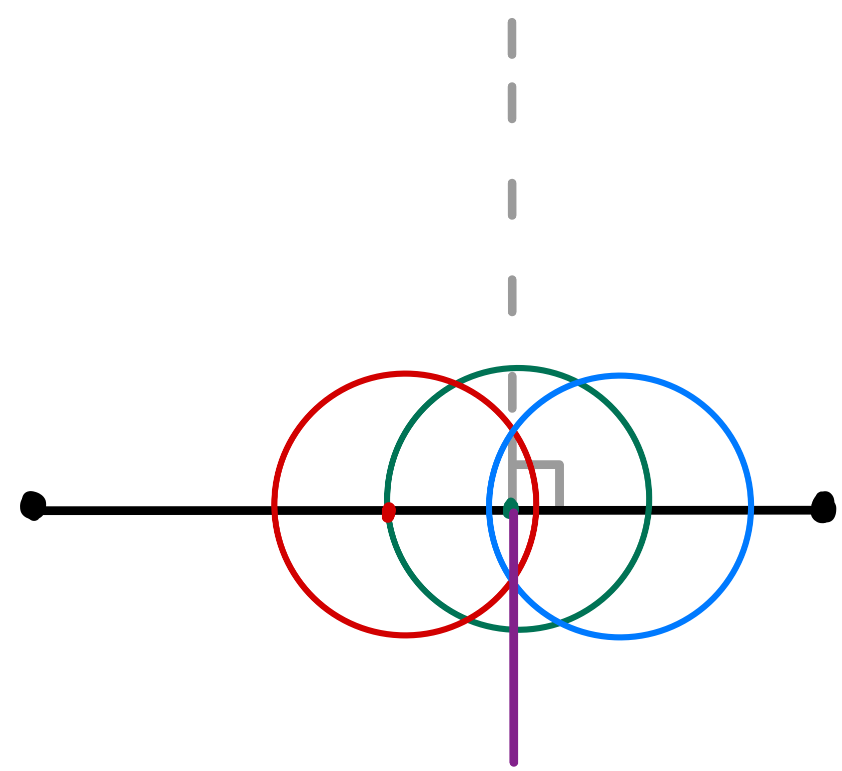
Bisect a line



Book I. Proposition 2 Perpendicular from point on line



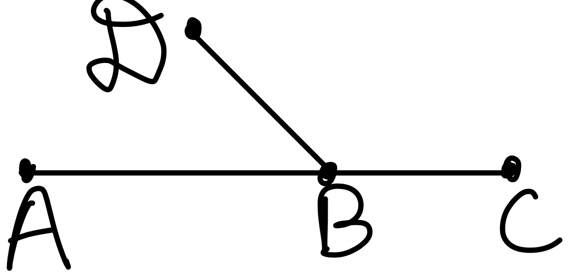




Propositions I.13 + I.14

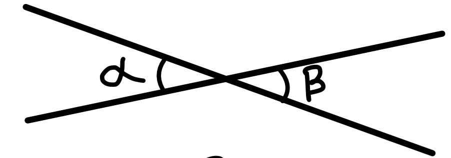
ABC is a straight line if and only if

$$\angle ABC + \angle DBC = \text{Two right angles (i.e. } 180^\circ)$$



Proposition I.15

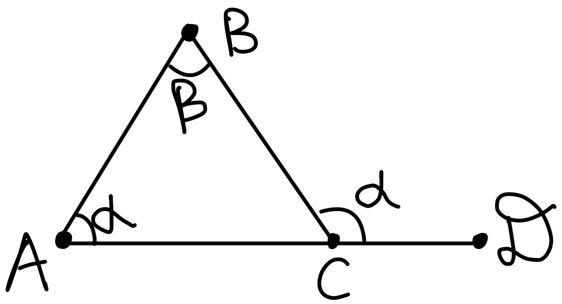
Vertical angles are equal



$\alpha = \beta$

Proposition I.16 (Exterior angle theorem)

An exterior angle of any triangle is greater than either of the opposite angles.



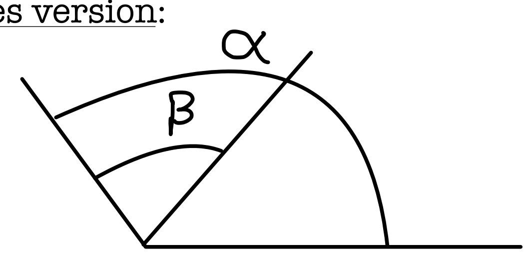
Claim: $\alpha > \beta$

$\alpha > \gamma$

Proof Idea. Common Notion 5:

The whole is greater than the part.

Angles version:



$\Rightarrow \alpha > \beta$

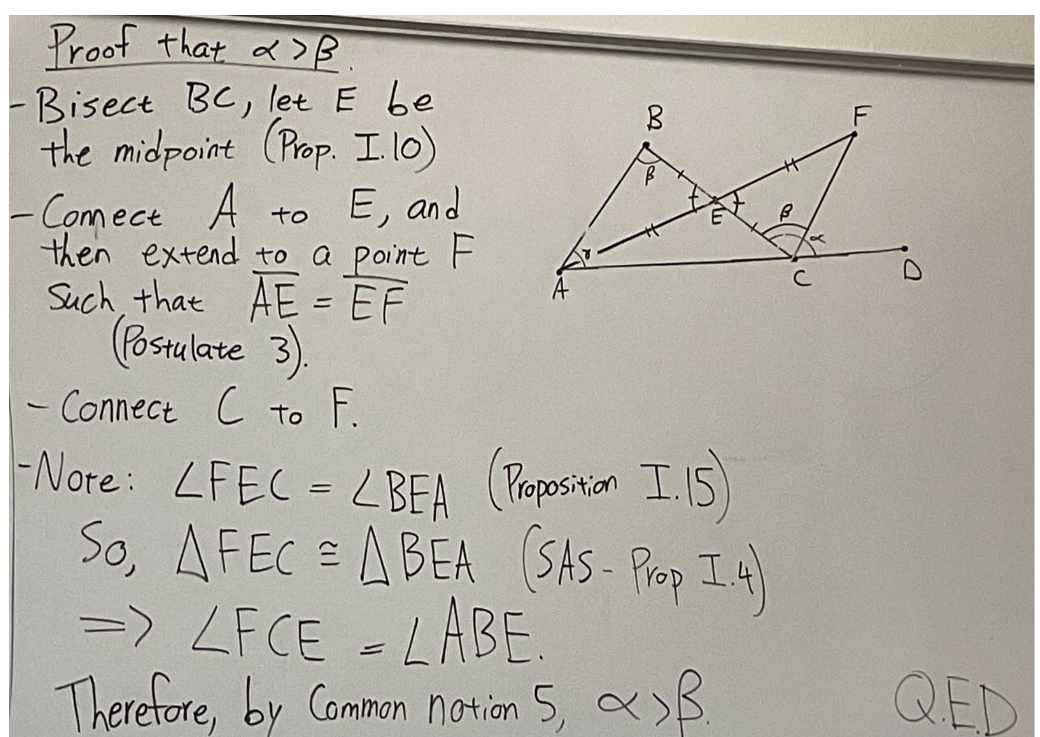
Proof that alpha > beta

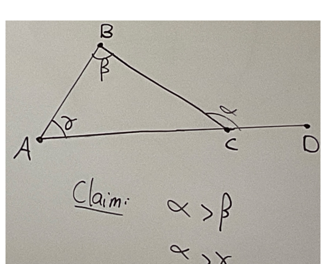
- Bisect BC, let E be the midpoint (prop. I.10)
- Connect A to E, and then extend to a point F such that AE = EF (Postulate 3)
- Connect C to F.
- Note: <FEC = <BEA (Proposition I.15).

So, ΔFEC ~= ΔBEA (SAS - Prop I.4)

=> <FCE = <ABE.

Therefore by common notion 5, alpha > beta





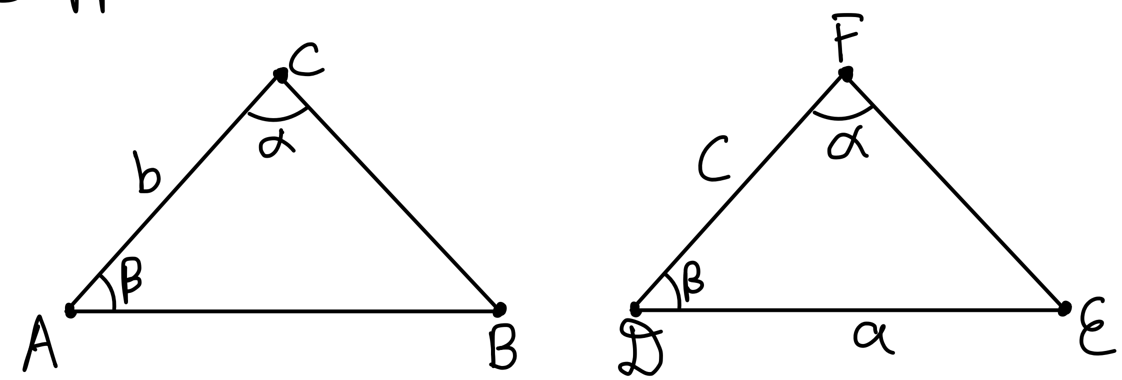
Prop I.20

Shortest distance between points is a straight line

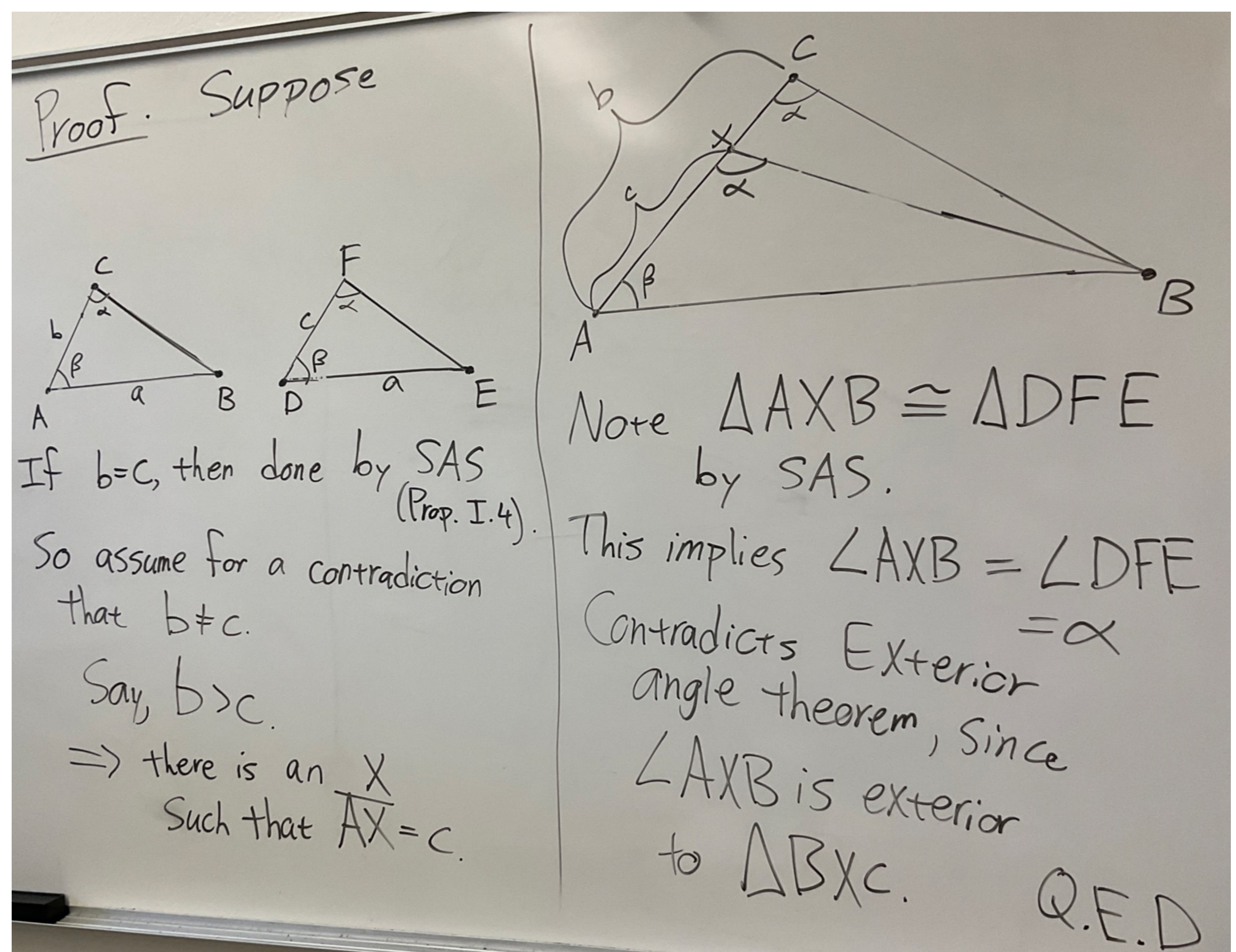
Prop I.26

AAS triangle congruence.

Proof. Suppose



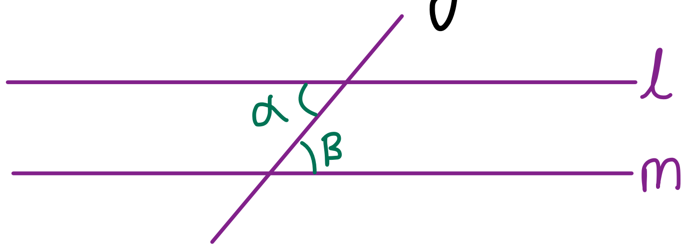
If b = c, then done by SAS (Prop. I.4)
So assume for a contradiction than b ≠ c.
Say, b > c.
=> there is a x such that AX = C



Parallelism:

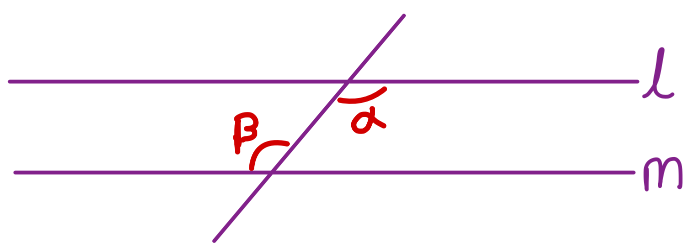
Proposition I.27 and I.29

In the diagram



$\alpha = \beta$ if and only if l is parallel to m.

And likewise



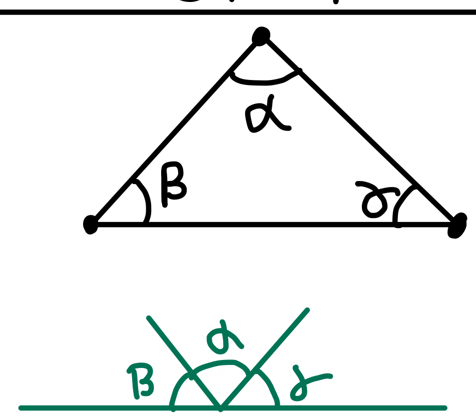
Proposition I.31

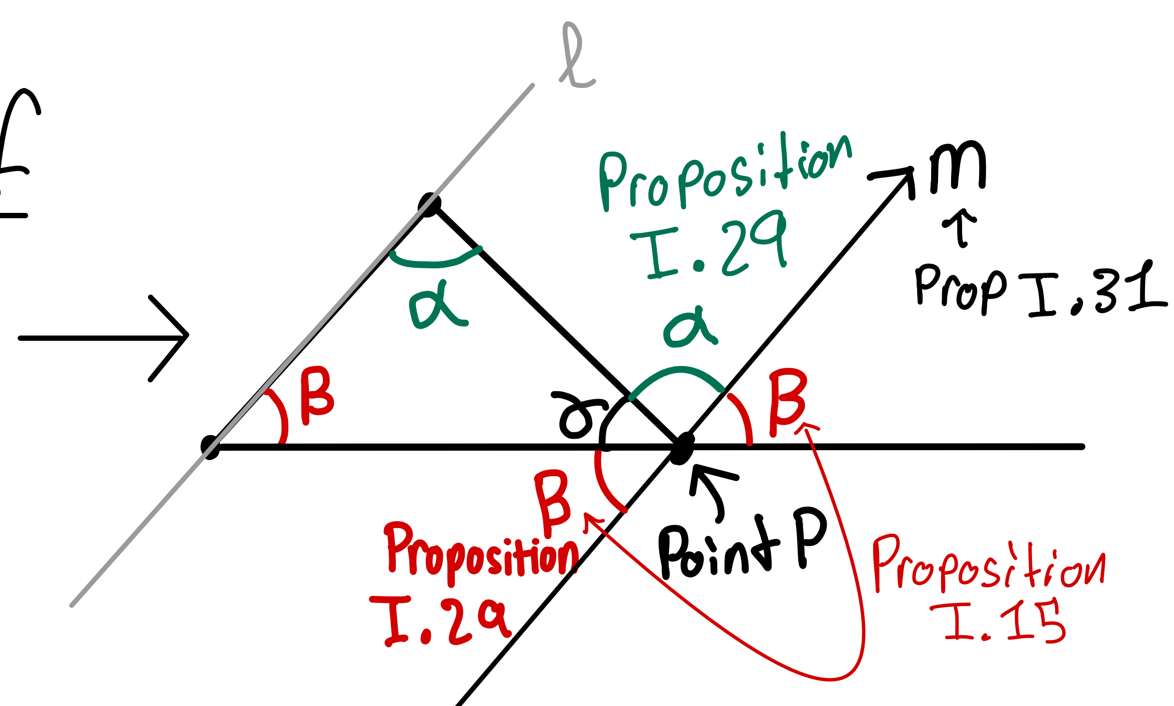
Given a line l and a point P not on l, one can construct a line m that is parallel to l and passes through P

Proposition I.32

The three interior angles of a triangle sum to $\underbrace{\text{two right angles}}_{\text{ie. } 180^\circ}$

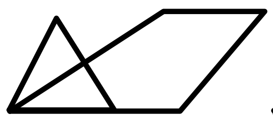
Sketch Proof





Parallelism:

Prop I.41

If , then

$2(\text{area of triangle}) = \text{area of parallelogram}$

Prop I.46

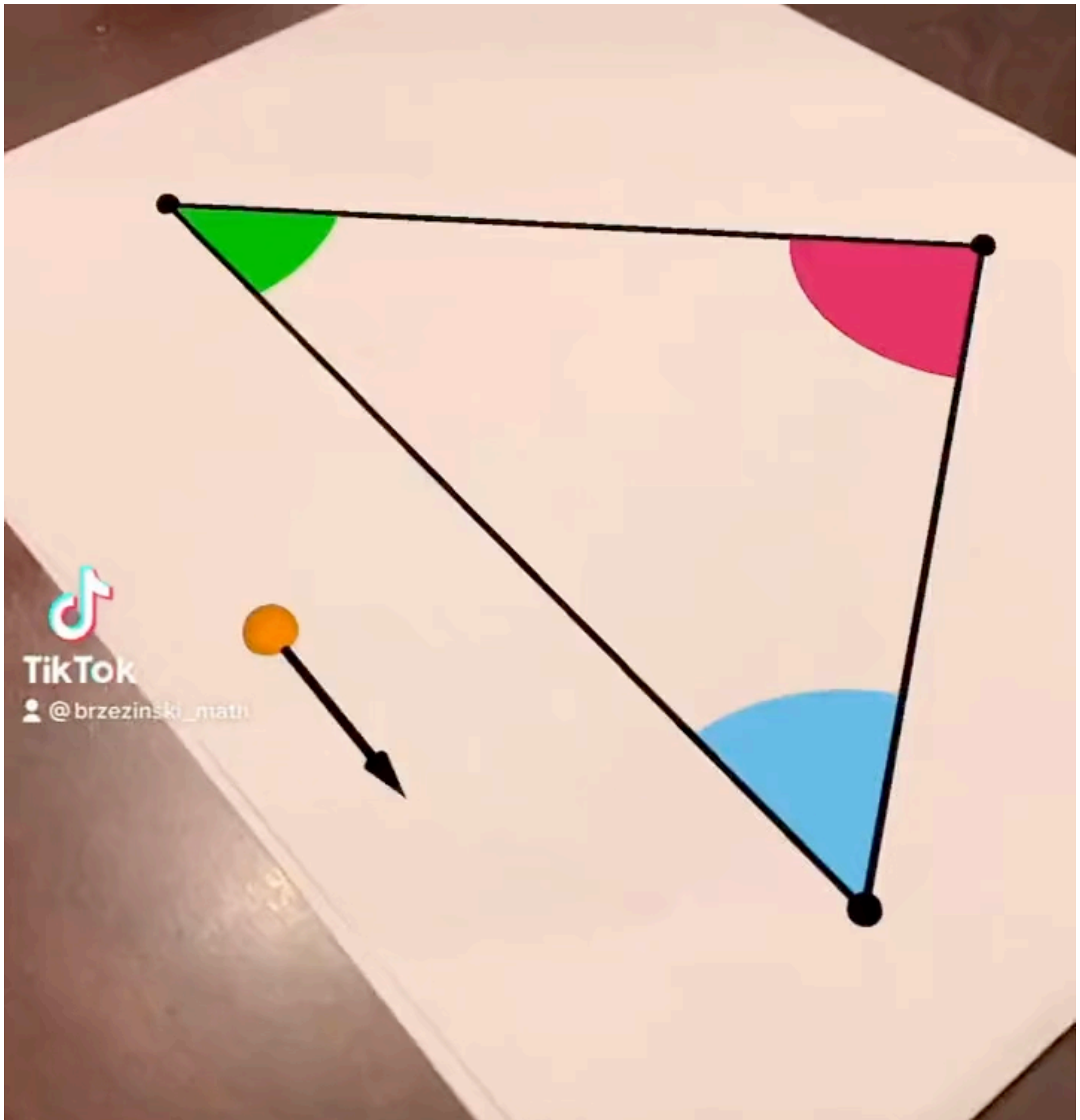
Construct of a square

Prop I.47

Pythagorean theorem

Proving Propositions

Triangle Angle Sum



[Link](https://www.geogebra.org/m/atx3asep)

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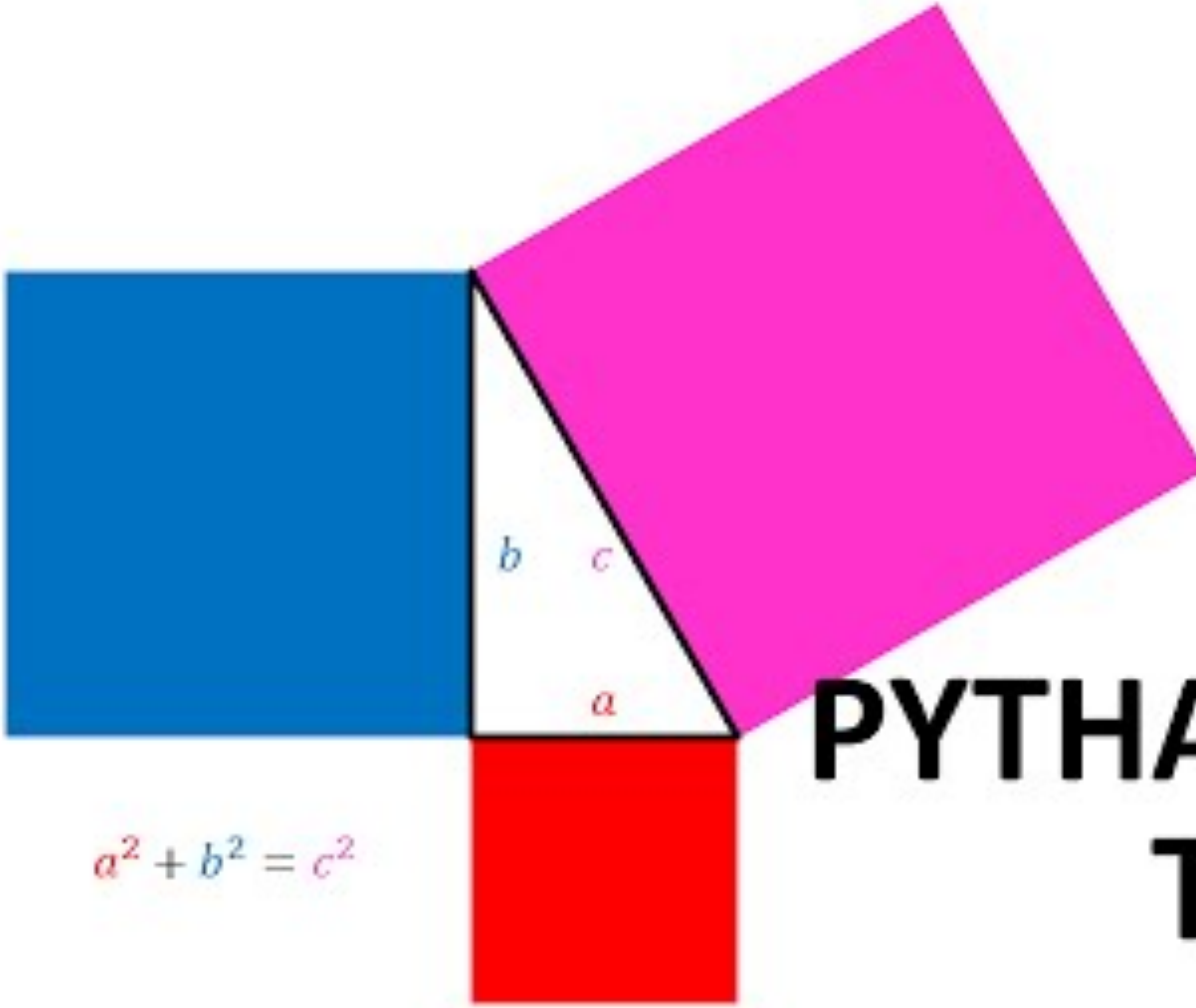


PYTHAGOREAN THEOREM

The Aftermath

- Book I showed how to construct an equilateral triangle and a square.

- Ancient Greeks also know how to construct a regular pentagon, and given a regular polygon with m sides, they could construct a regular polygon with $2m$ sides.

- Major open question: For which values of n is the regular n-gon constructible with a straightedge and compass?

The Aftermath

- There was no progress made until 1796, which a 19-year-old named Carl Friedrich Gauss used some beautiful algebra to find a construction of the regular 17-gon.

- Then, at 24, he entirely classified which regular polygons are constructible.

- He did not prove one direction of his theorem, though. That was done by Pierre Wantzel.

The Aftermath

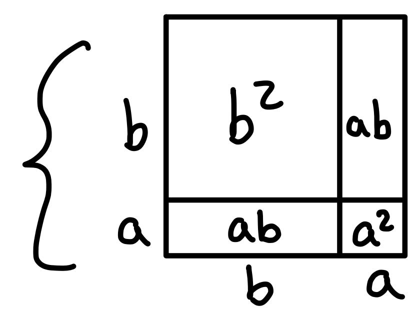
- The Gauss-Wantzel theorem says this:

- A regular n-gon can be constructed with straightedge and compass if and only if n is a power of 2 or the product of a power of 2 and any number of distinct Fermat primes.

- A Fermat prime is a prime of the form $2^{2^k} + 1$.

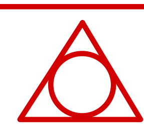
The Elements

- Book I: Planar geometry
- Book II: Algebraic geometry



Law of squares $\Rightarrow (a+b)^2$ area

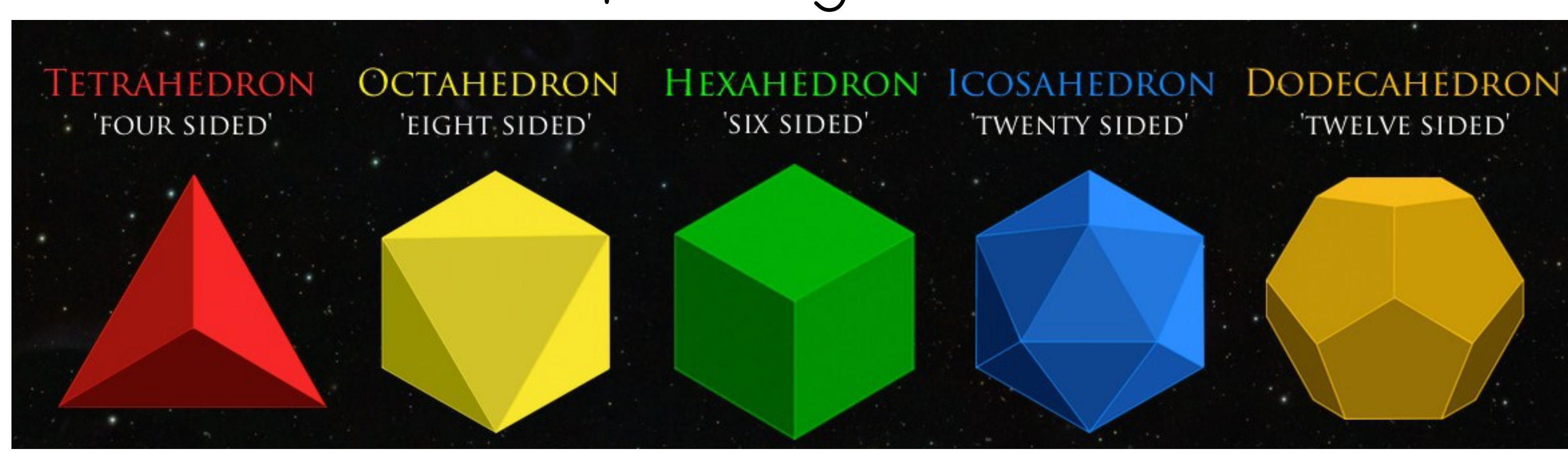
- Book III: Circles
- Book IV: Inscribing and circumscribing



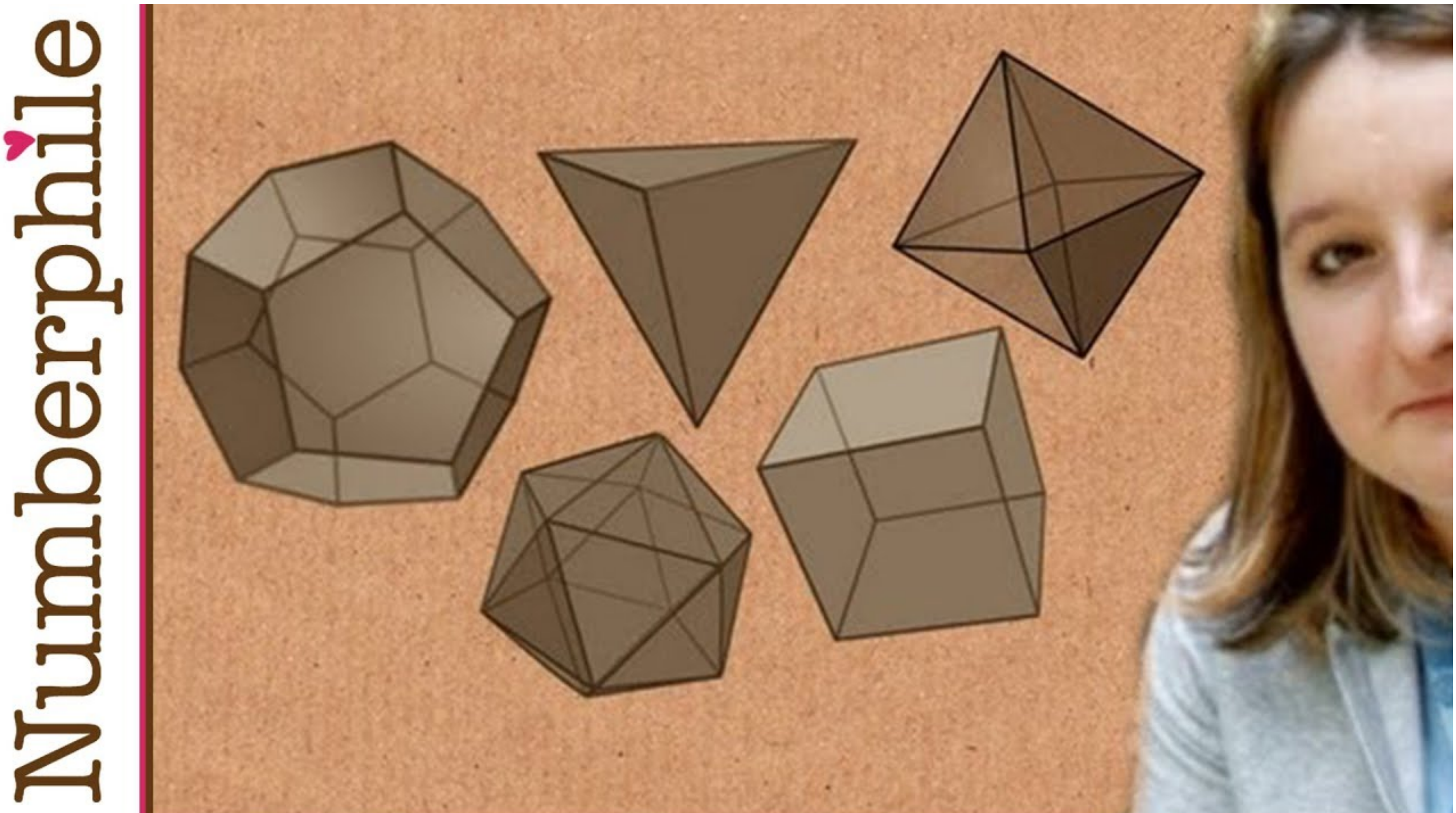
- Book V: Magnitudes, ratios and rules of arithmetic
- Book VI: Proportions between lines and shapes

The Elements

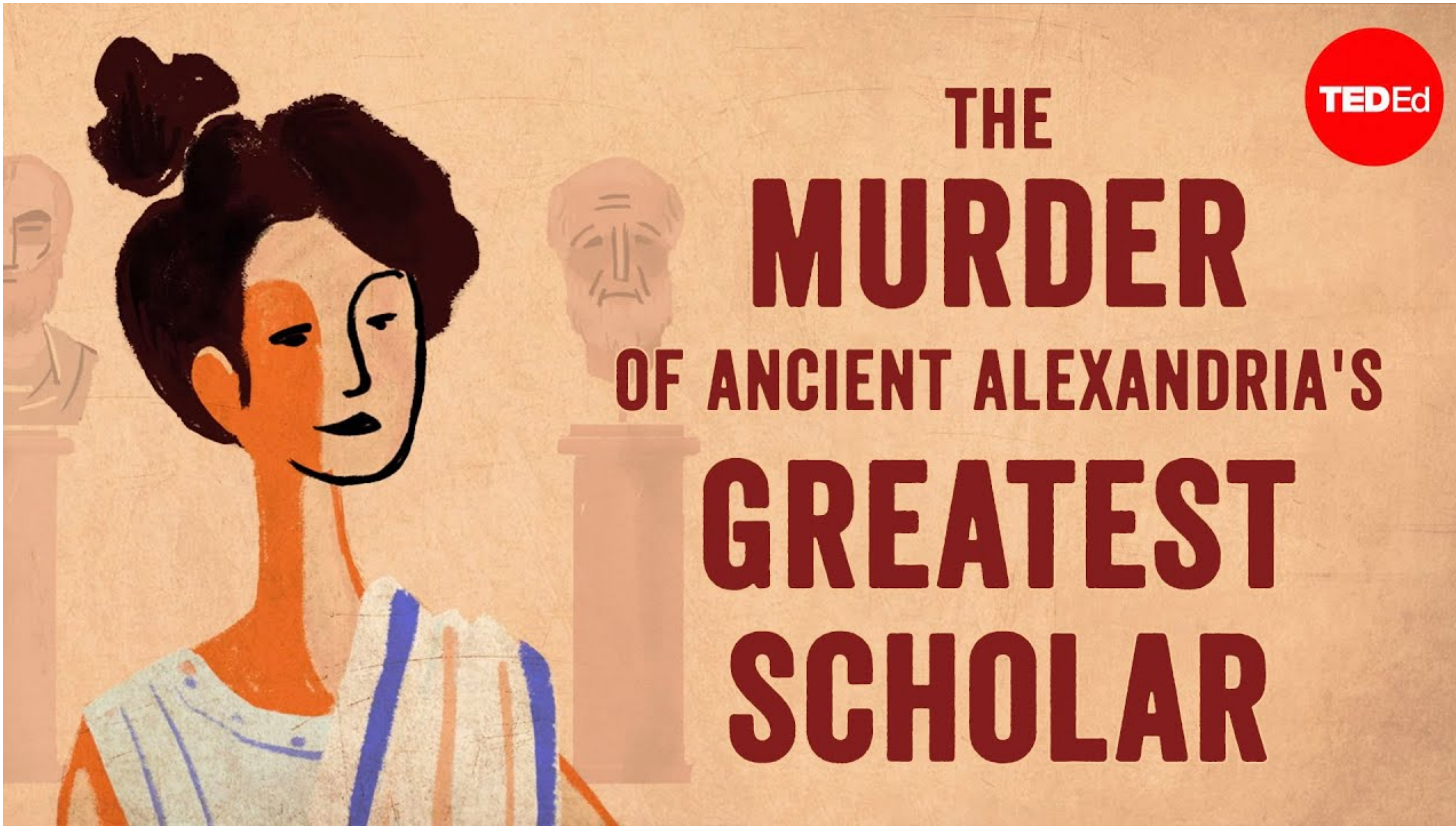
- Book VII, VIII and IX: Number theory
- Book X: Square roots
- Book XI: Solid figures
- Book XII: Volume
- Book XIII: Platonic solids - 3d Shapes - generalization



Platonic solids



Hypatia



Hypatia

- Hypatia is the first woman in math history that we know a lot about.

- She was a first-class thinker and highly respected philosopher, mathematician and teacher.

- Her life ended in tragedy, as she was brutally murdered for defending religious freedom.

The Legacy of Axiomatic Thinking

self-evident or unquestionable

Lincoln – movie

He read Euclid's Elements in his off hours

The Aftermath

- Recall that Euclidean geometry was built from 5 axioms, the final of which was *the parallel postulate*. This one was slightly controversial.

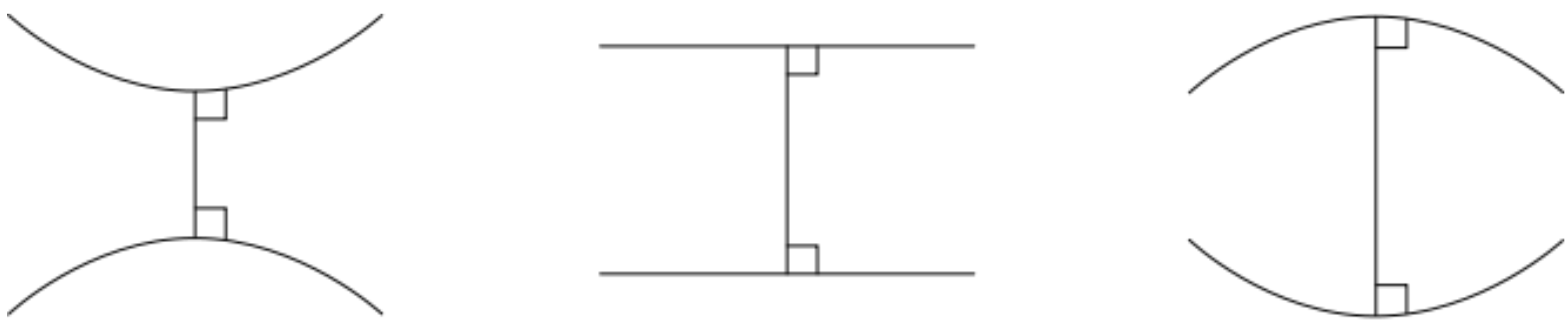
- Can that postulate be deduced from the other four? Unfortunately, no. 🥲

- In this way, the parallel postulate is *undecidable* if your axioms are only the first four.

The Aftermath

To make parallel lines

- This implies that there exist non-Euclidean geometries. Instead of the parallel postulate, you can have an axiom saying that if ℓ is a line and P is a point off of ℓ, then there are 2+ lines through P that are parallel to ℓ. Or no lines that are parallel.



Hyperbolic — Euclidean — Elliptic

Multiple lines — Zero ways

The Aftermath

- Our set theory today is built from the *ZFC axioms*. These are 8 completely basic axioms about sets, plus the slightly-controversial *axiom of choice*.

- From these, not only is set theory built, but number theory, too.

- Kurt Gödel proved that every set of axioms that leads to number theory must have undecidable statements. There must be statements that can neither be proven nor disproven.

Approximating $\sqrt{2}$

- This chapter is on geometry and the next chapter is on number theory. A bridge between these topics is using geometry to approximate $\sqrt{2}$.

- We previously learned about the Indian geometry from their *sulbasutras*. These also contained an approximation:

$$\sqrt{2} \approx 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34}.$$

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$$1.414213\ldots$$

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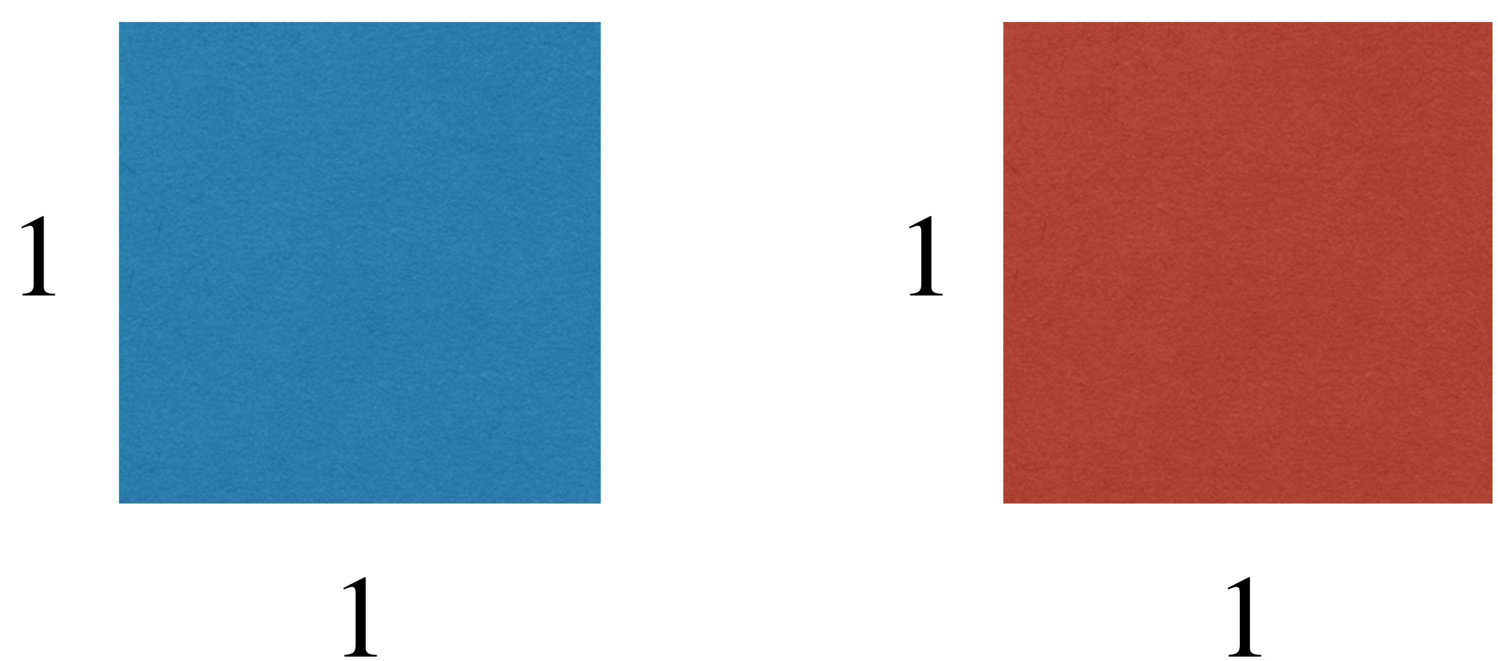
Approximating $\sqrt{2}$

- Goal:

$$\sqrt{2} \approx 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34}.$$

Suppose you have two 1×1 squares



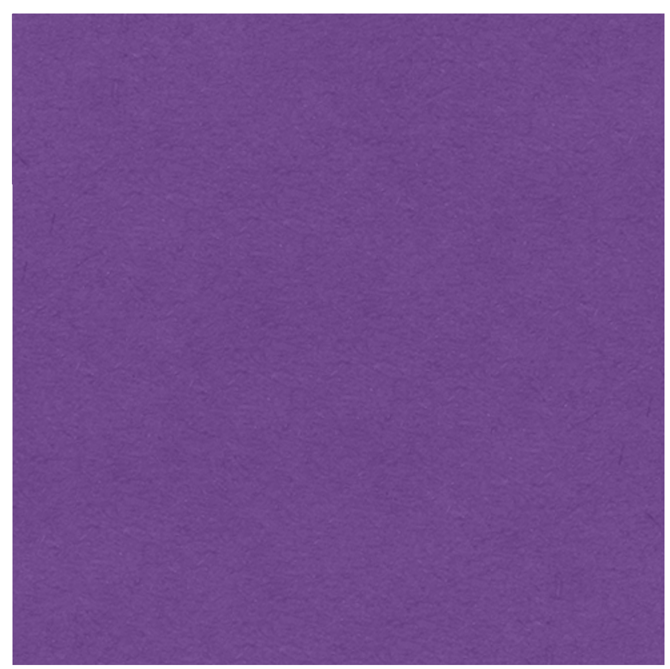
Approximating $\sqrt{2}$

- Goal:

$$\sqrt{2} \approx 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34}.$$

If you could combine them into a single square, it would be a $\sqrt{2} \times \sqrt{2}$ square

$\sqrt{2}$



$\sqrt{2}$

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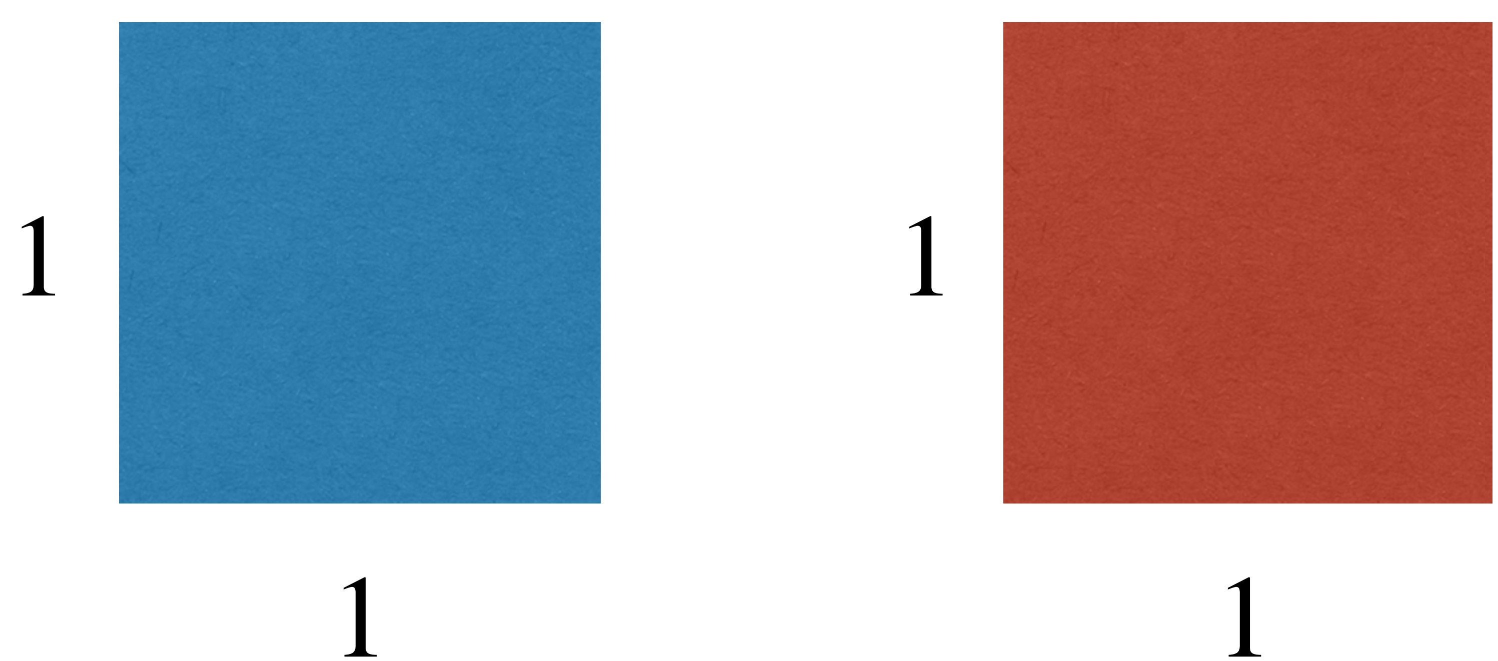

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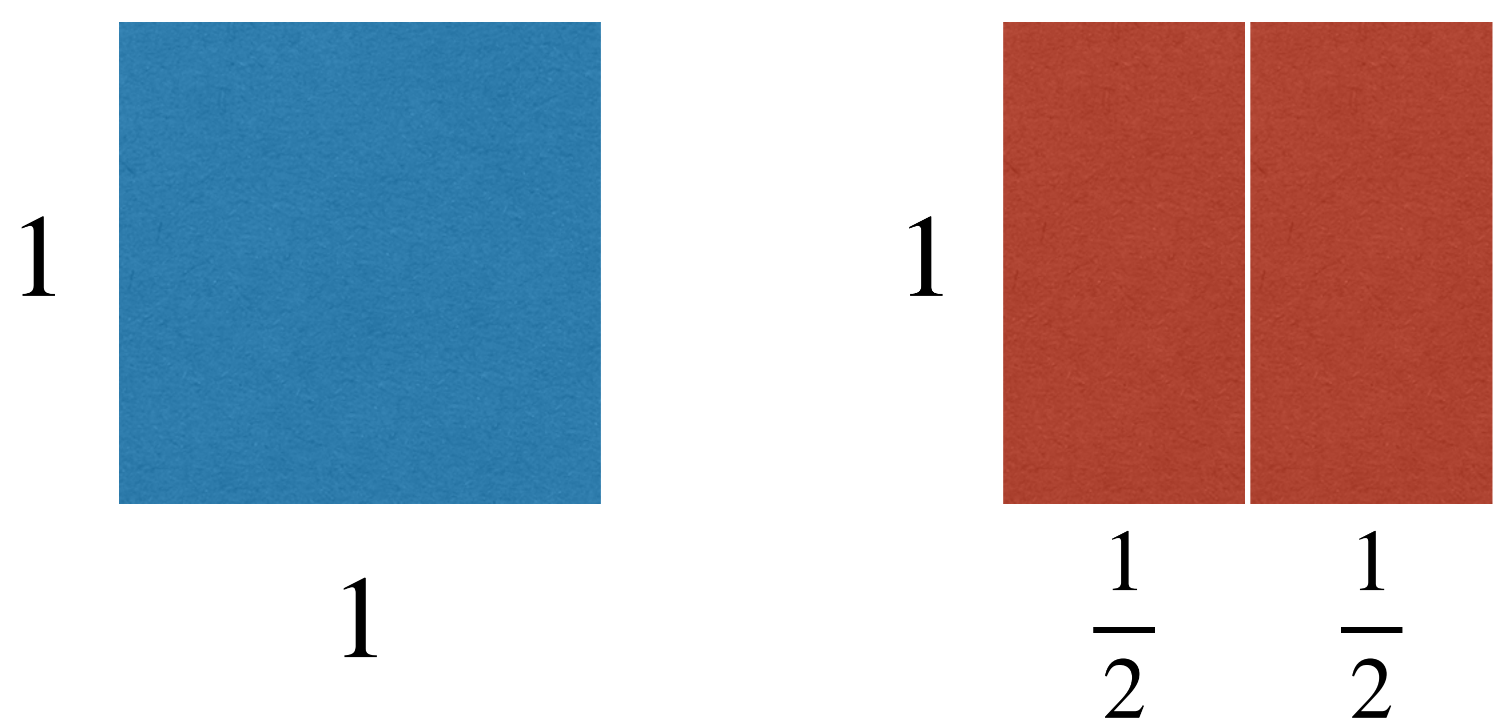

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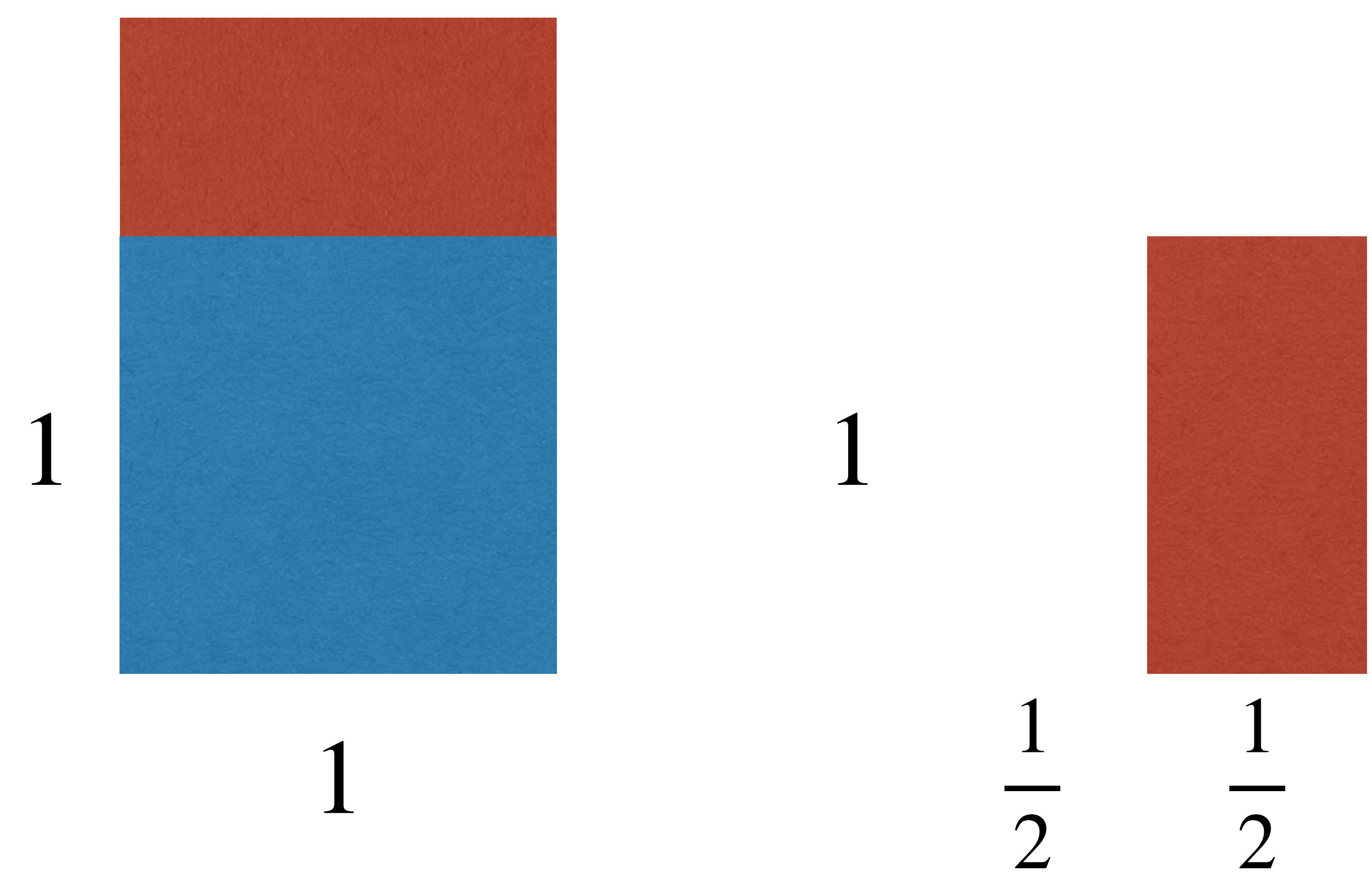

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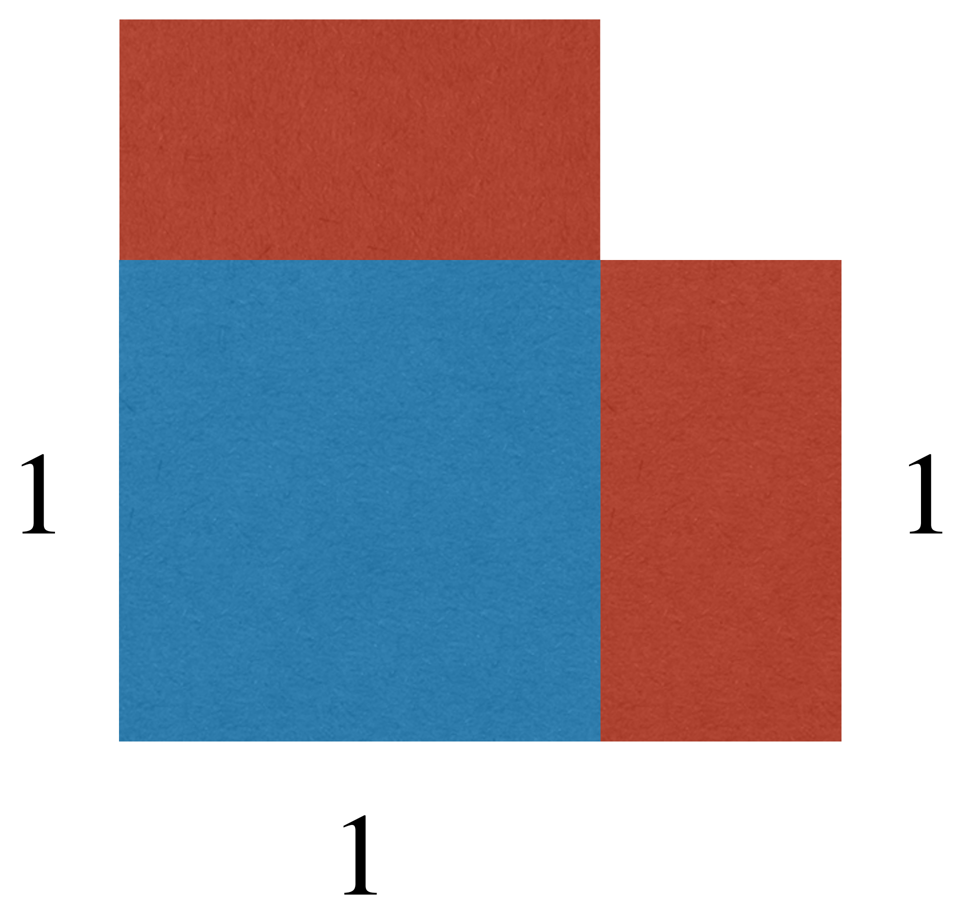

$\frac{1}{2}$ $\frac{1}{2}$

Approximating $\sqrt{2}$

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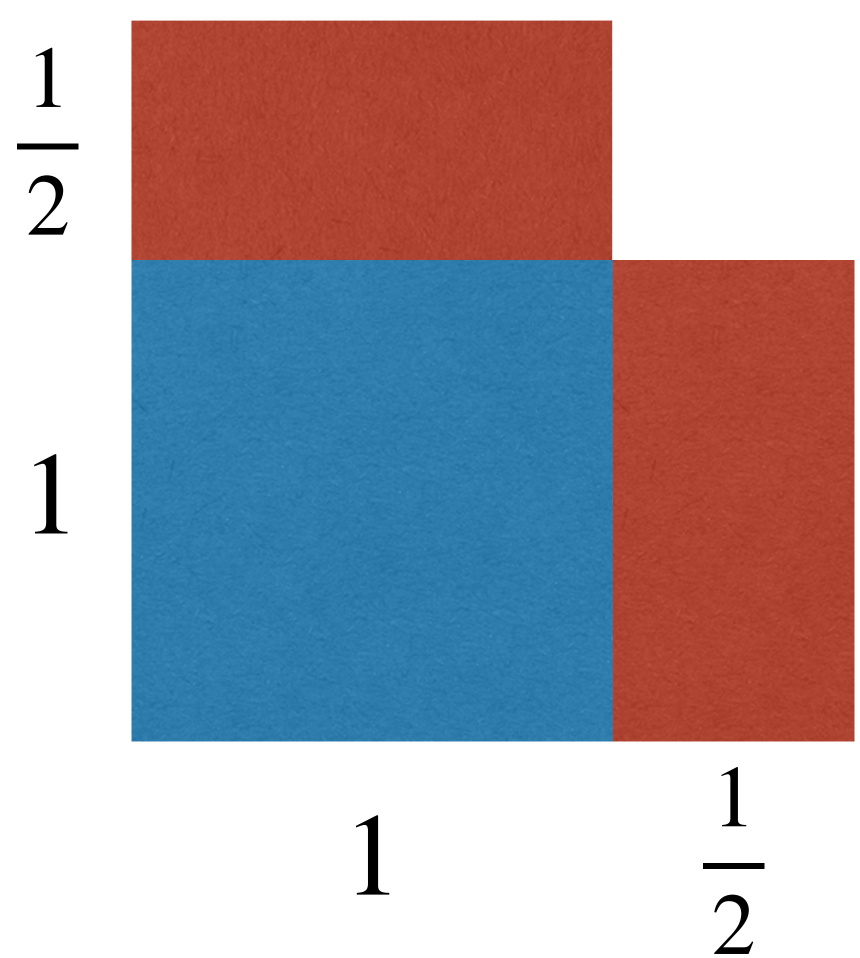

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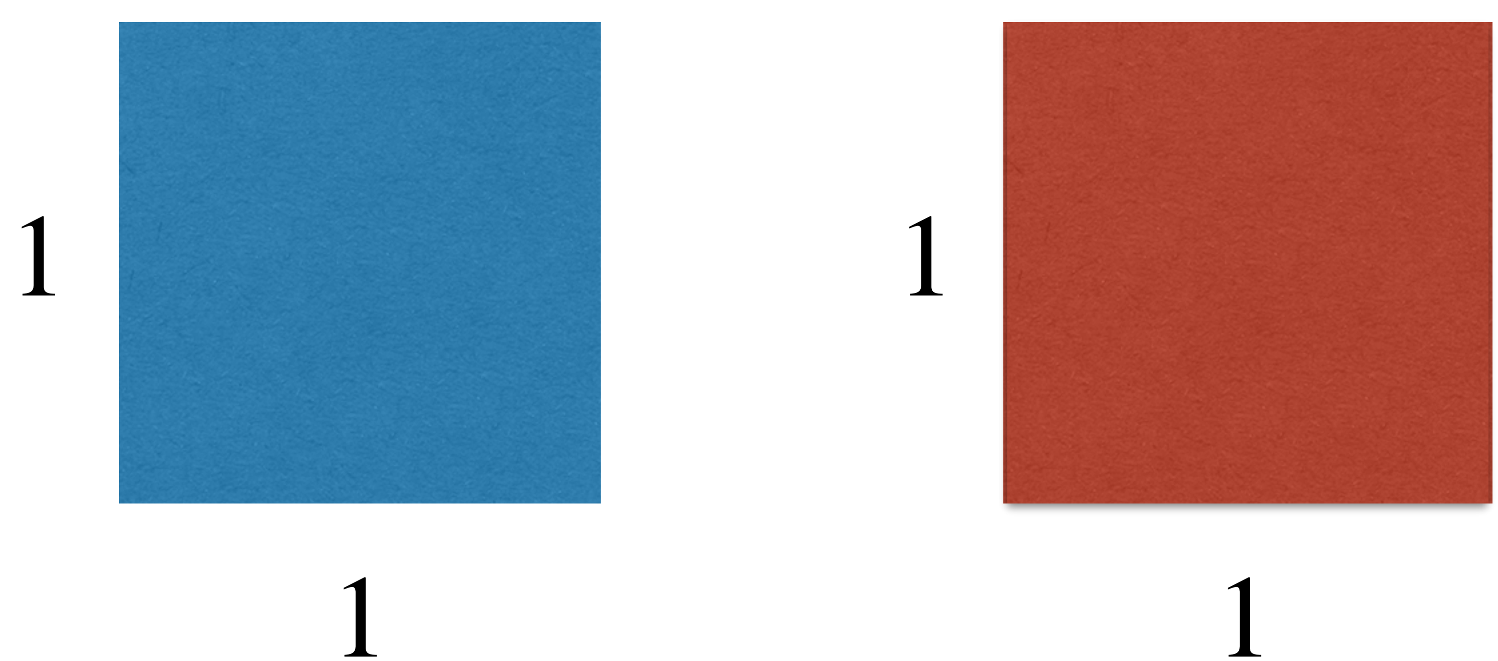

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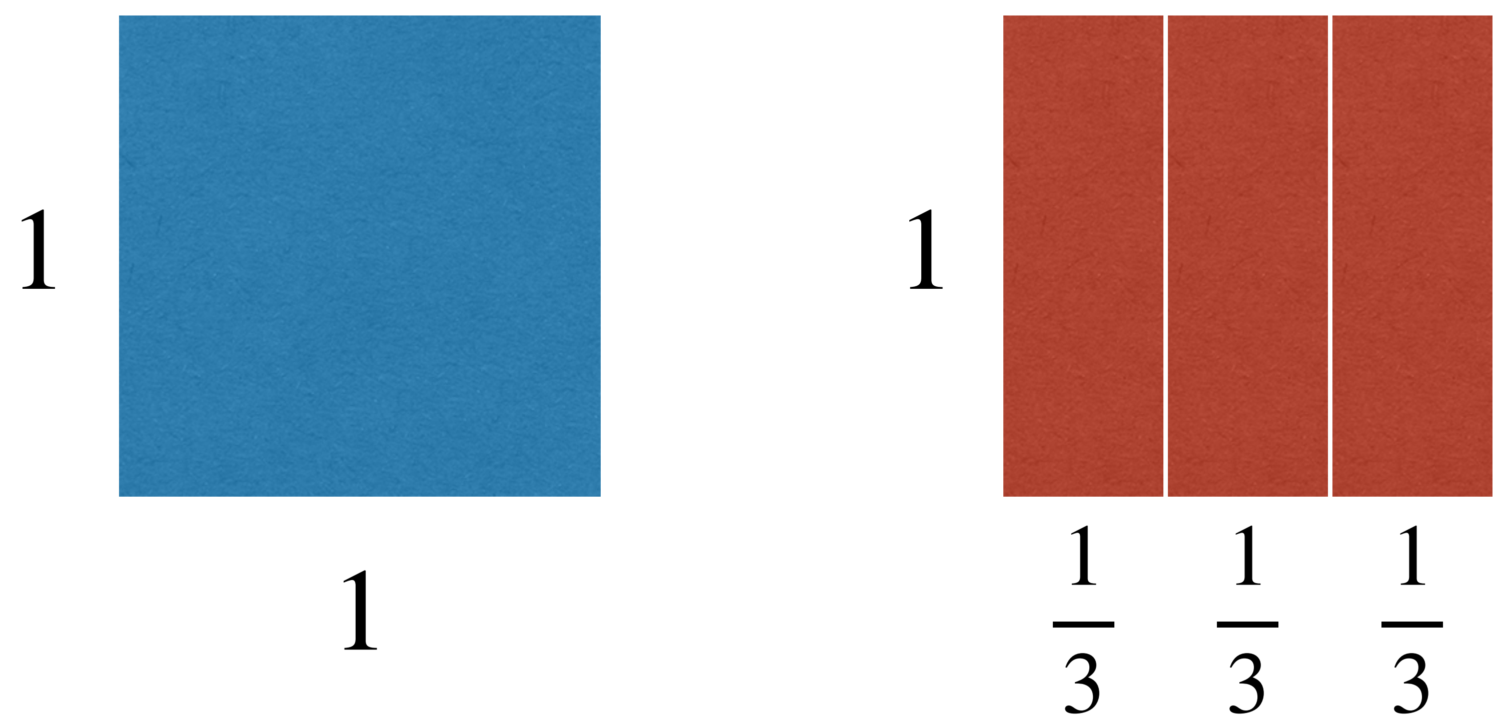

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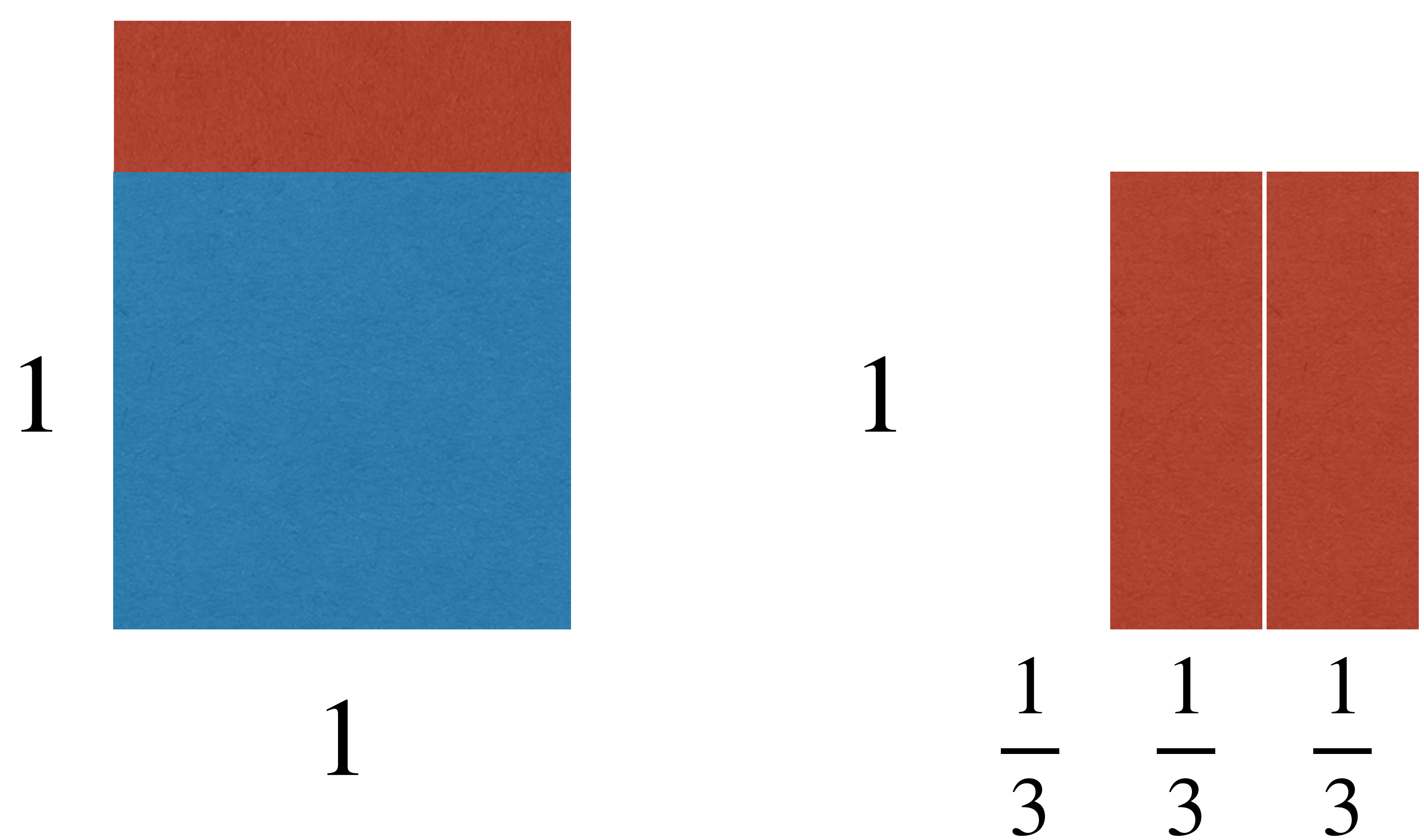

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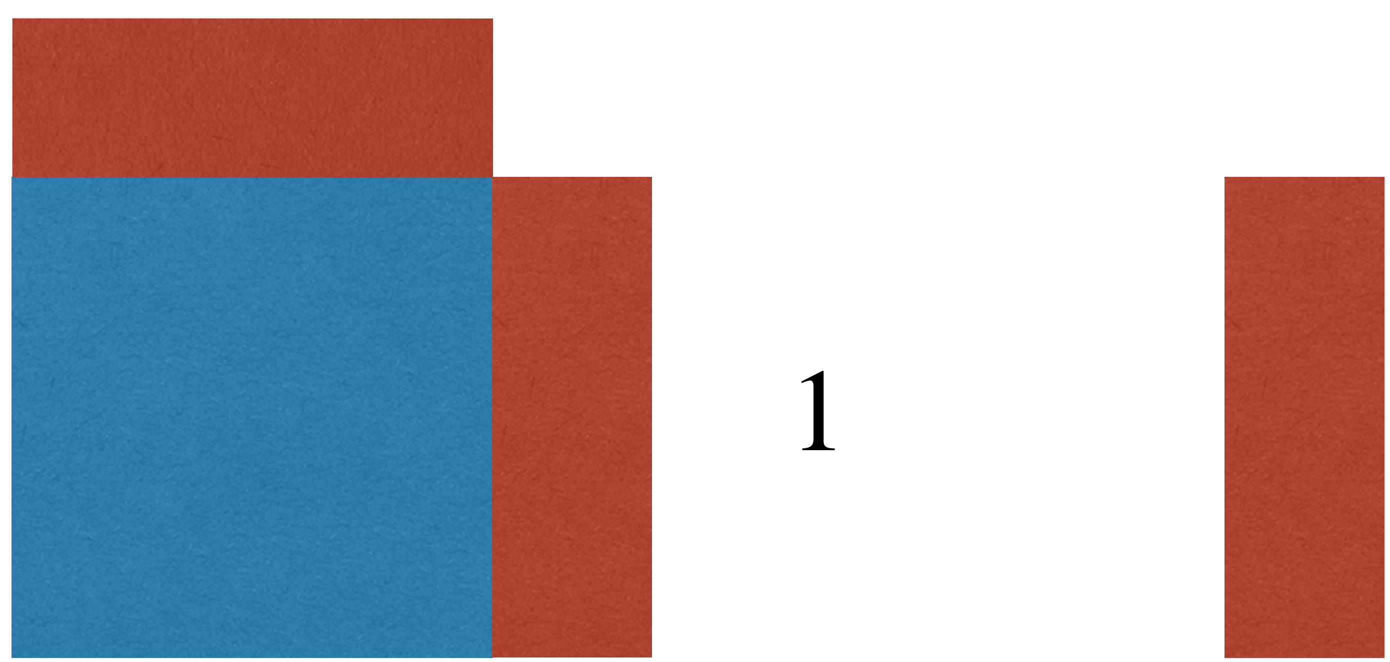

1 1

1 $\frac{1}{3}$ $\frac{1}{3}$ $\frac{1}{3}$

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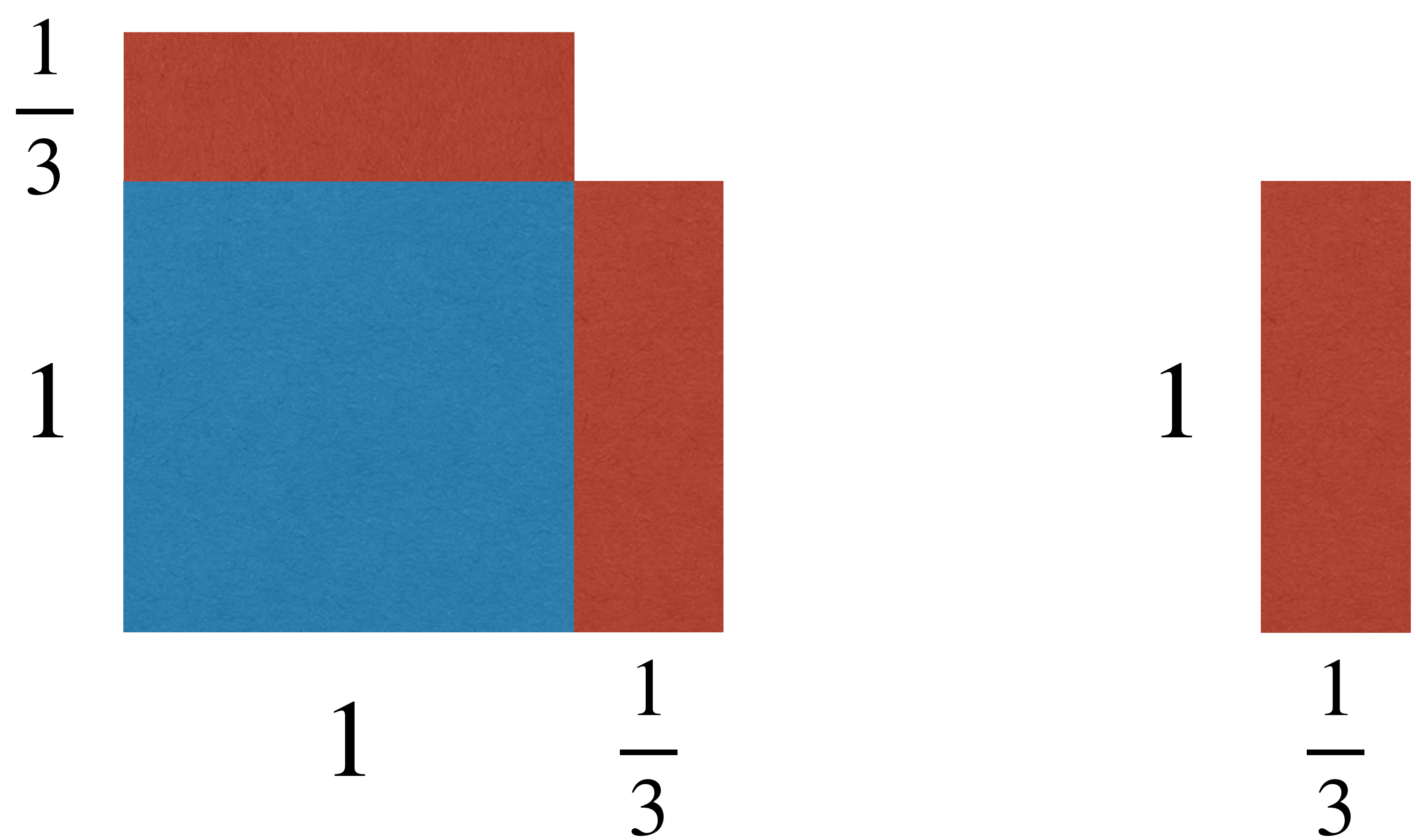

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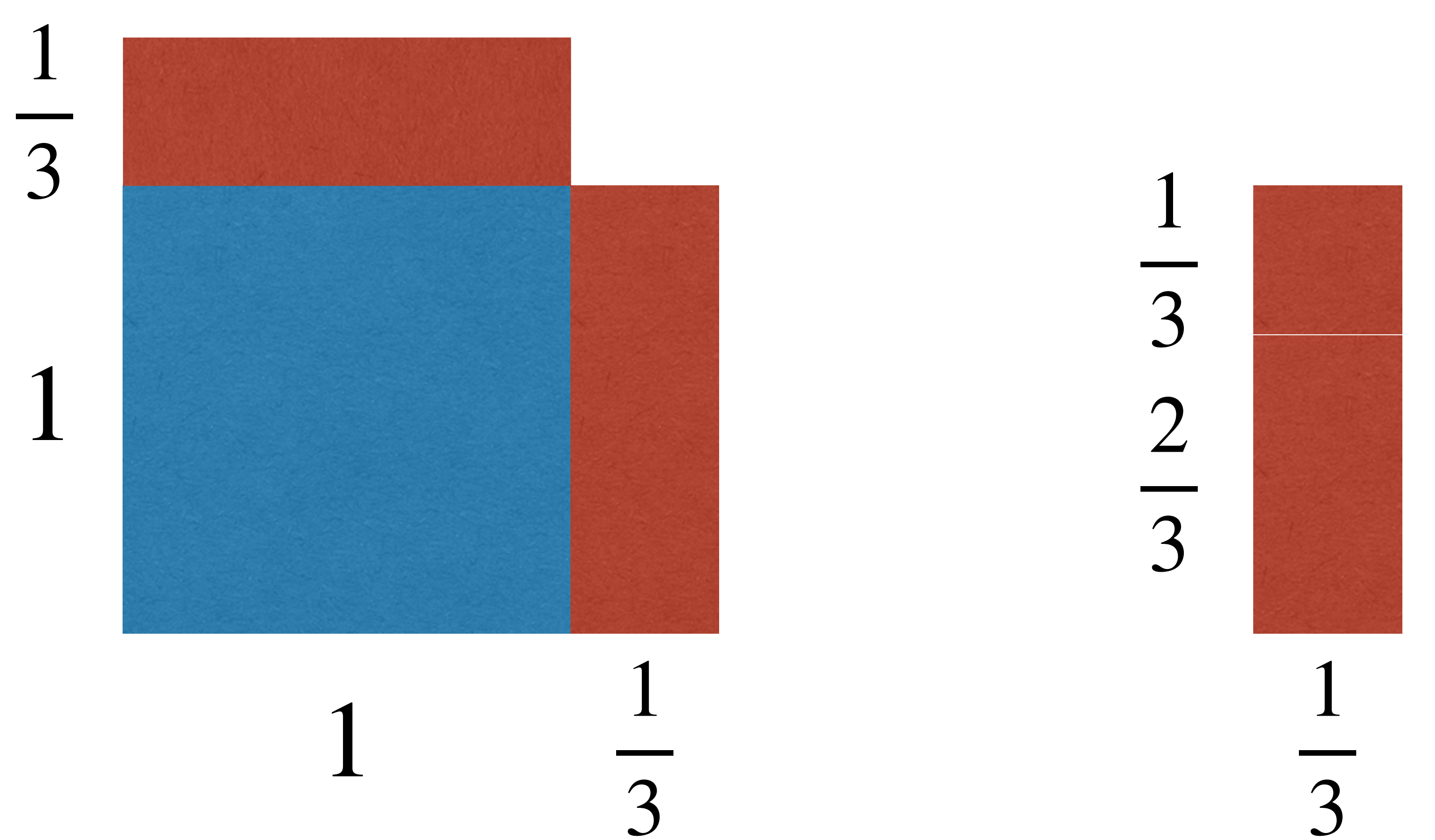
Suppose you have two 1×1 squares



Approximating $\sqrt{2}$

- Goal:

$$\sqrt{2} \approx 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34}.$$

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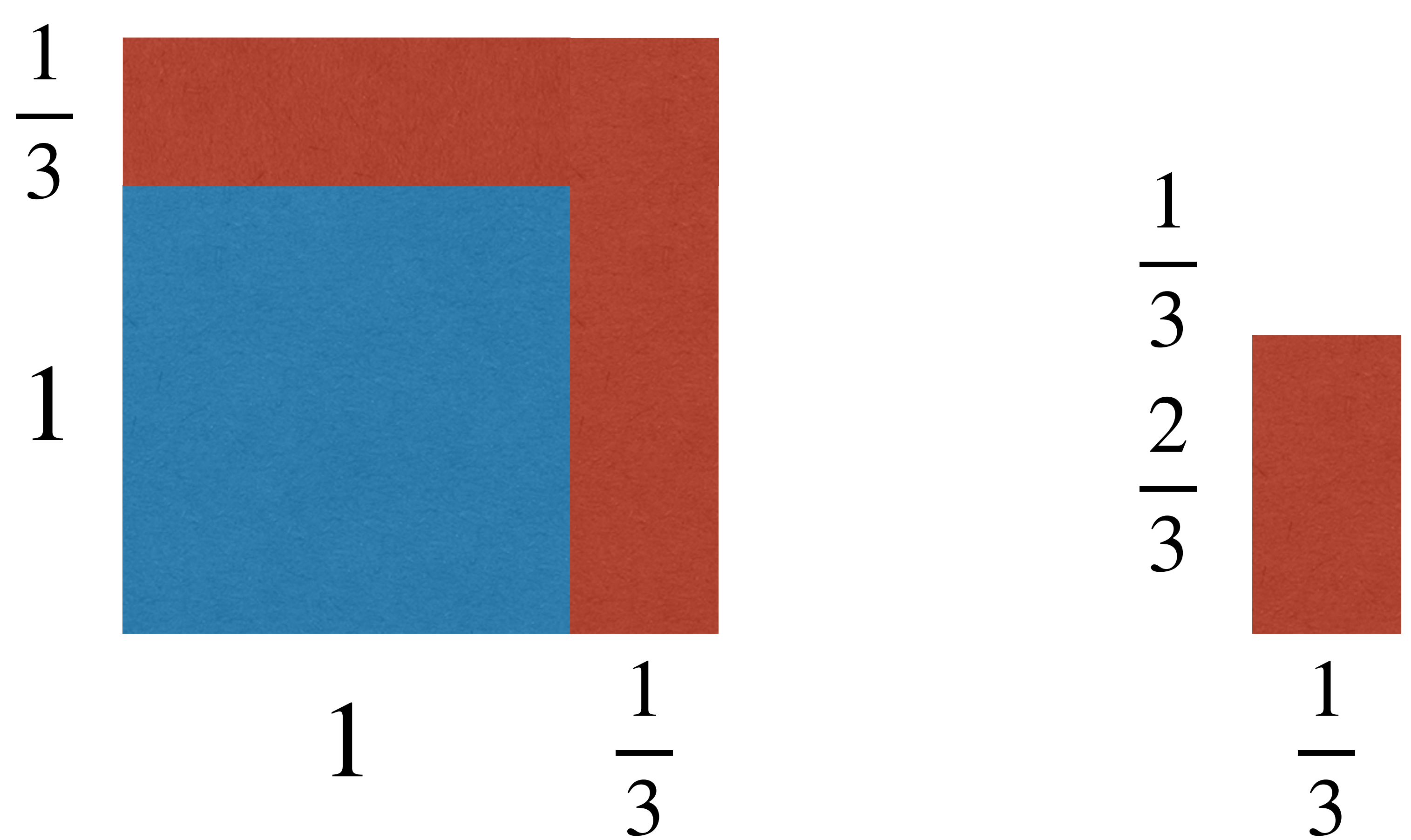

Approximating $\sqrt{2}$

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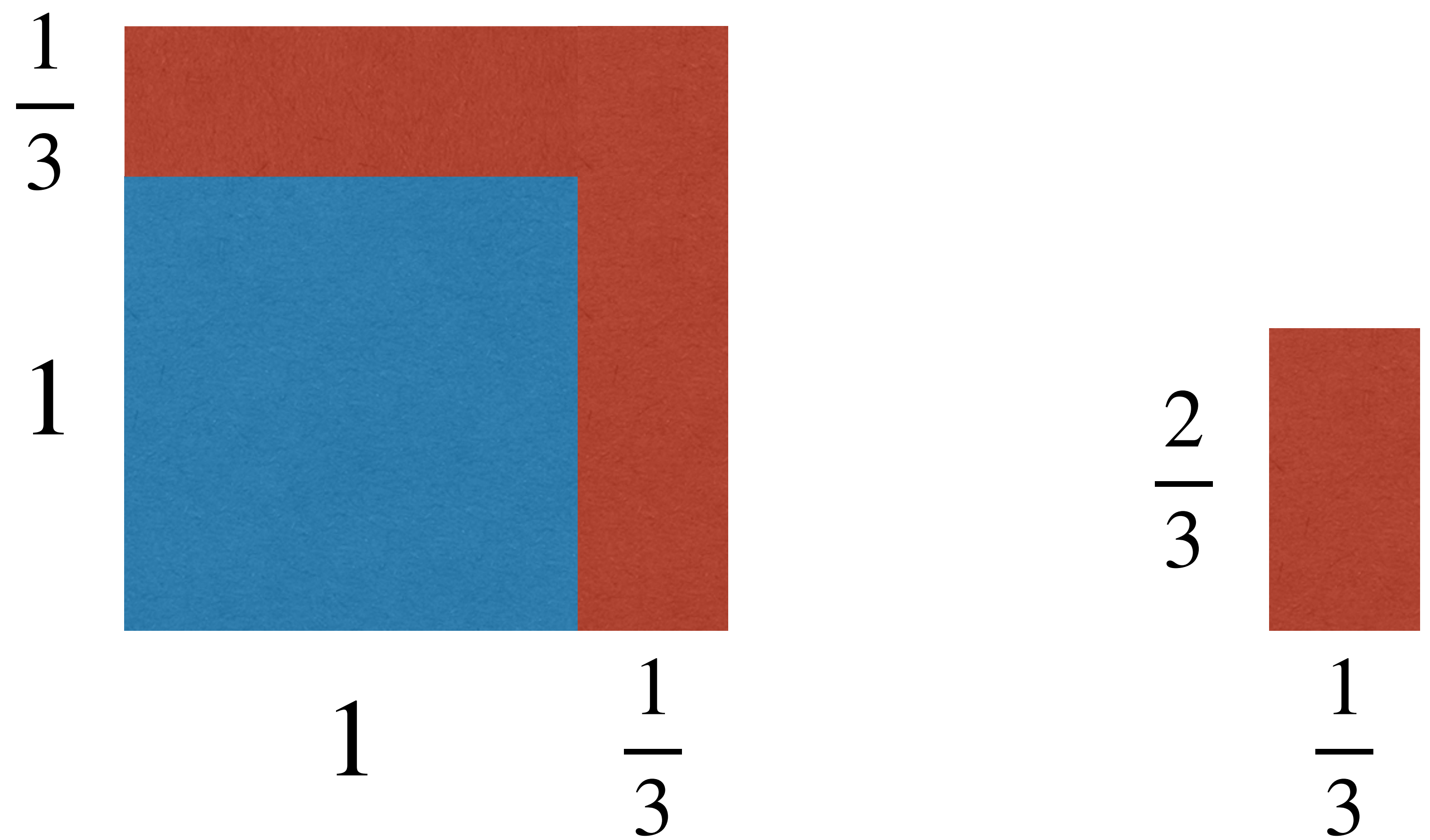
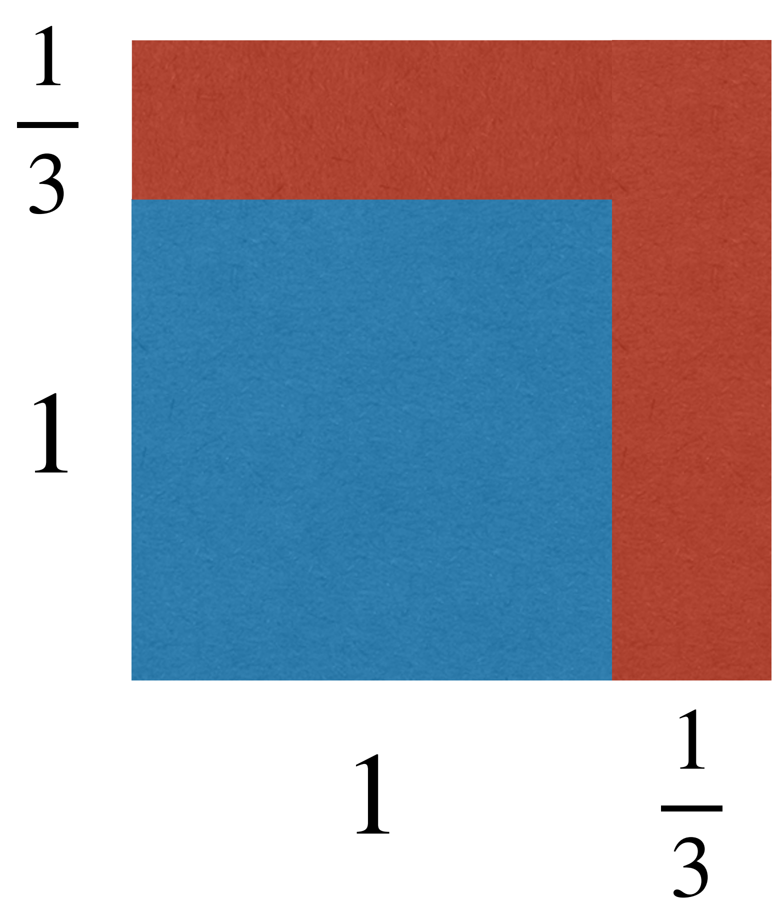


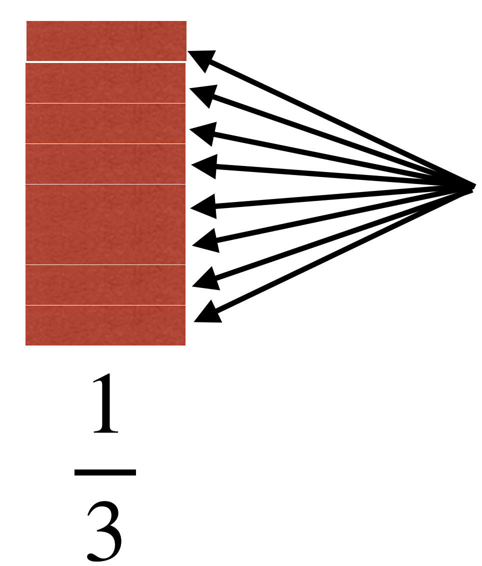
Approximating $\sqrt{2}$

- Goal:

$$\sqrt{2} \approx 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34}.$$

Divide into 8 pieces





$$\frac{1}{8} \cdot \frac{2}{3} = \frac{1}{3 \cdot 4}$$

Approximating $\sqrt{2}$

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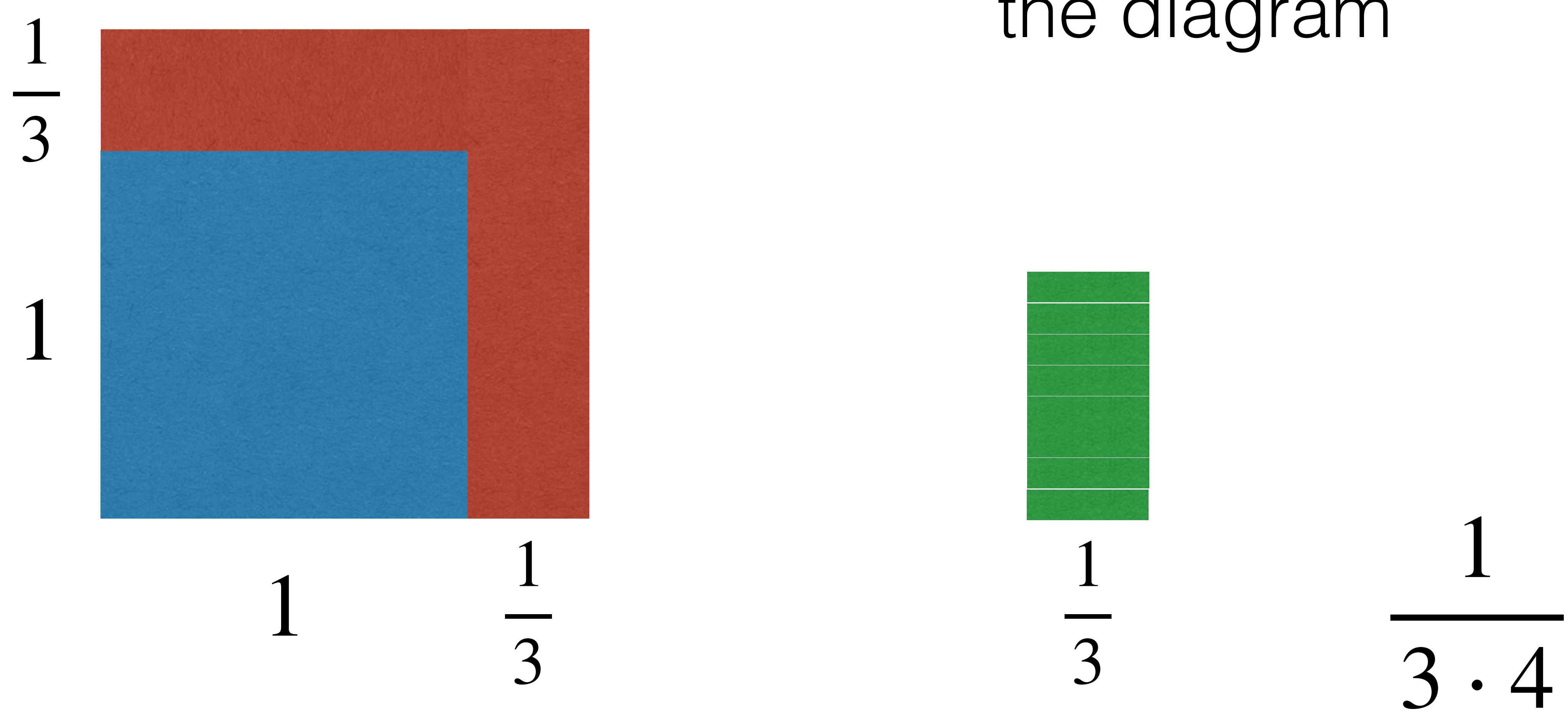
$$\sqrt{2} \approx 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34}.$$

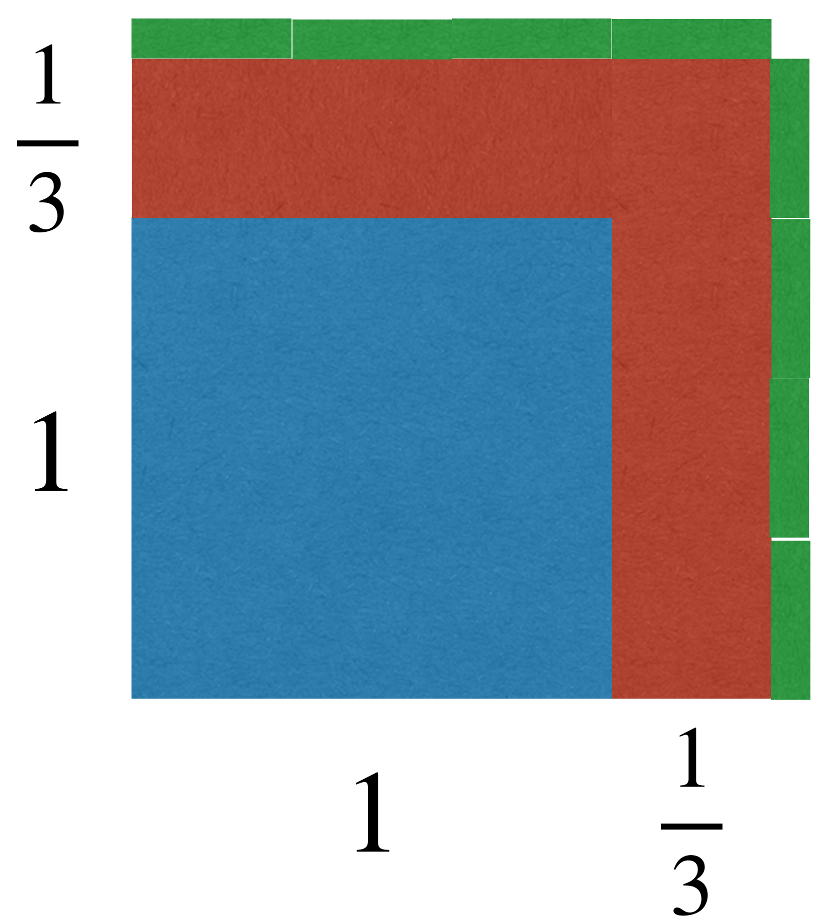
Place them on the diagram



Approximating $\sqrt{2}$

- Goal:

$$\sqrt{2} \approx 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34}.$$



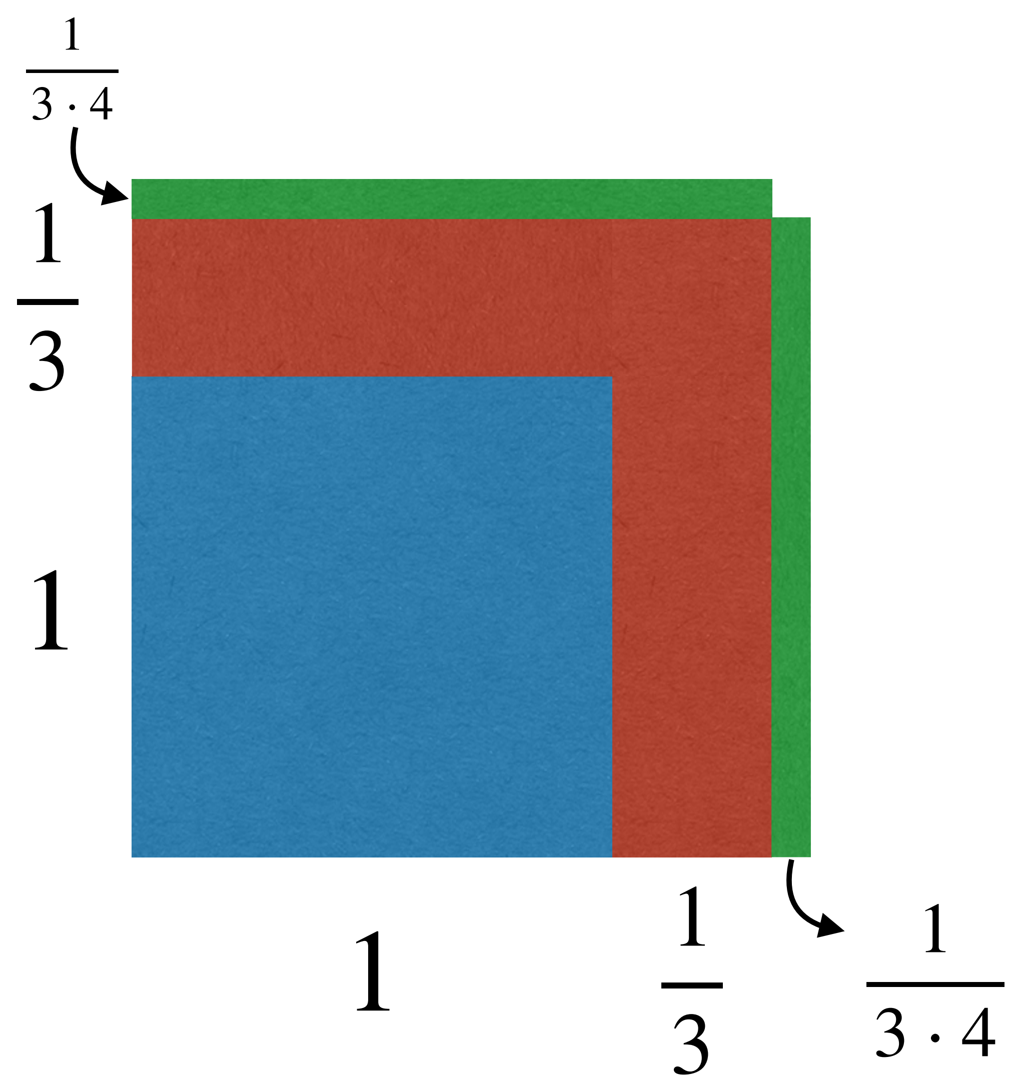
Place them on the diagram

$\frac{1}{3}$ $\qquad$ $\frac{1}{3 \cdot 4}$

Approximating $\sqrt{2}$

- Goal:

$$\sqrt{2} \approx 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34}.$$

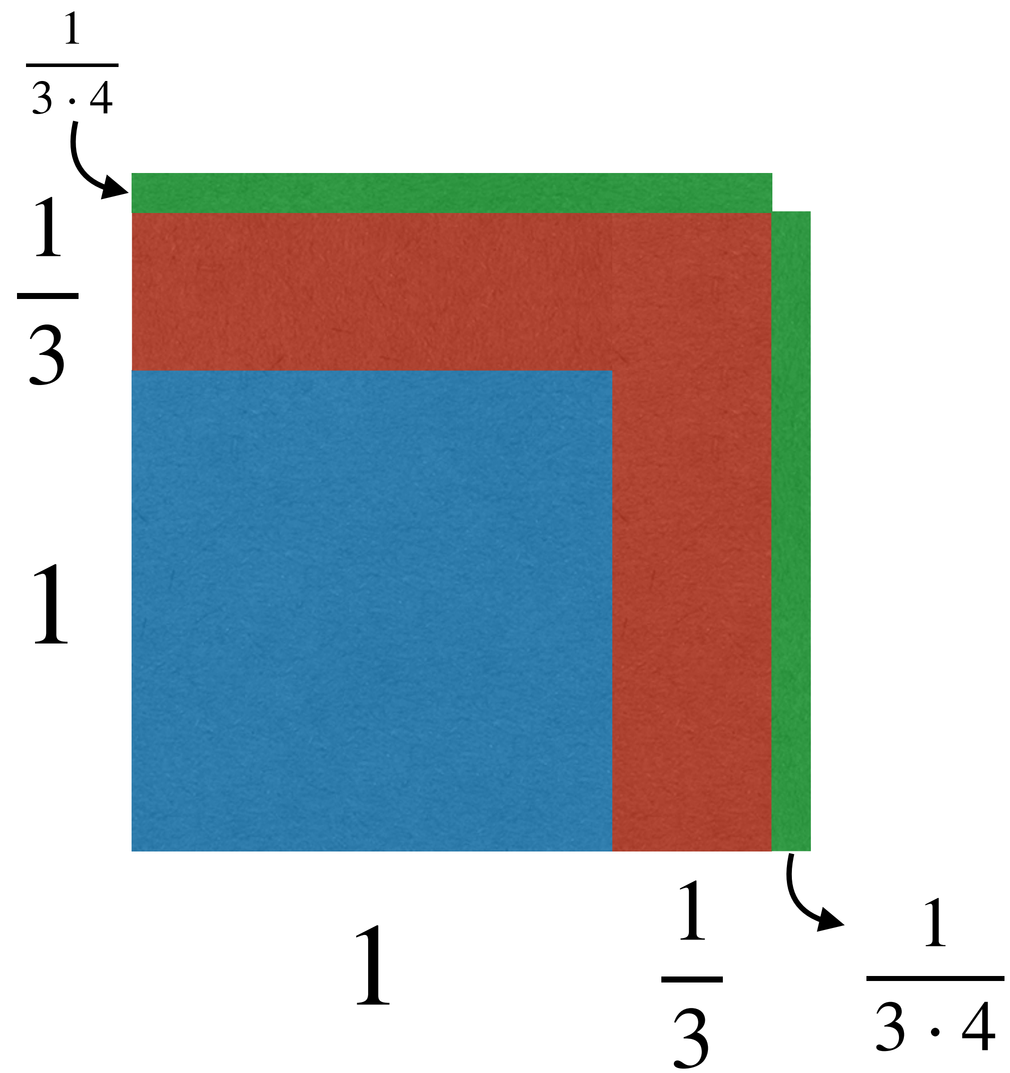


Place them on the diagram

Approximating $\sqrt{2}$

- Goal:

$$\sqrt{2} \approx 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34}.$$

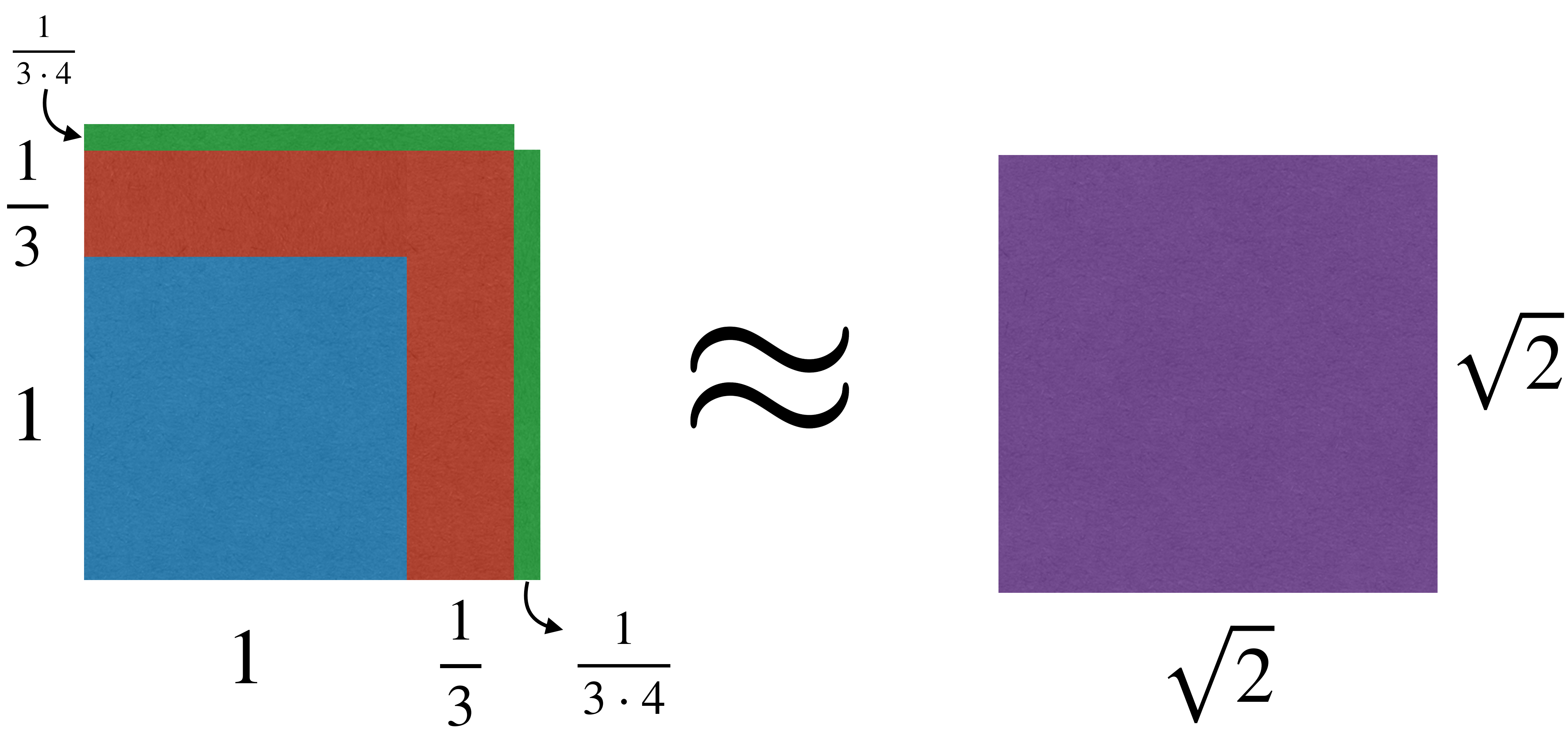


This is not a square, but it's close. And it does use all of the original two squares.

Approximating $\sqrt{2}$

- Goal:

$$\sqrt{2} \approx 1 + \frac{1}{3} + \frac{1}{3\cdot 4} - \frac{1}{3\cdot 4\cdot 34}.$$



Approximating $\sqrt{2}$

- Goal:

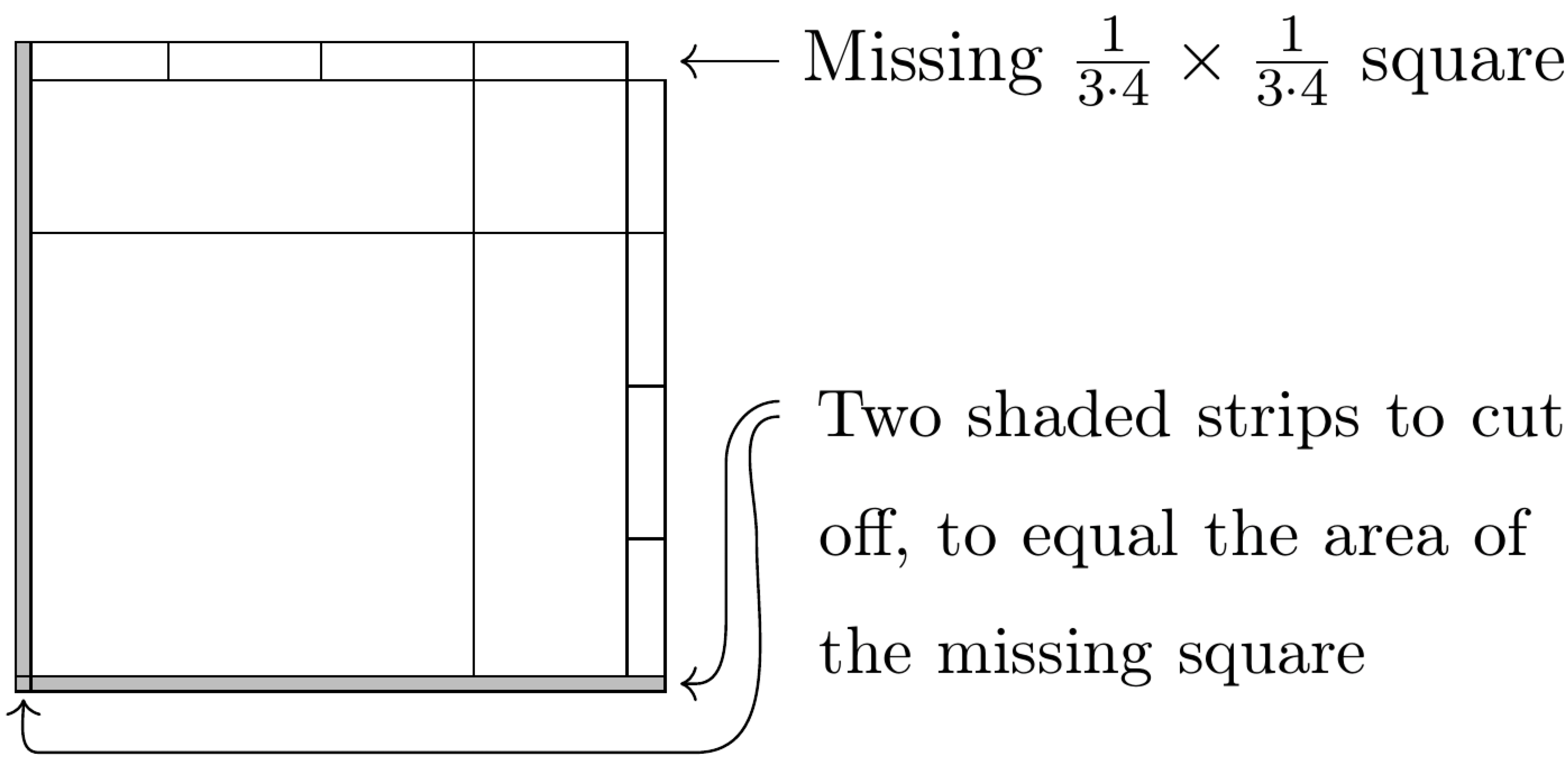
$$\sqrt{2} \approx 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34}\,.$$

$$1 + \frac{1}{3} + \frac{1}{3 \cdot 4} \approx \sqrt{2}$$

Approximating $\sqrt{2}$

- Goal:

$$\sqrt{2} \approx 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34}\,.$$



Approximating $\sqrt{2}$

- Goal:

$$\sqrt{2} \approx 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34}\,.$$

$$2 \cdot W \times \left(1 + \frac{1}{3} + \frac{1}{3 \cdot 4}\right) - W^2 = \frac{1}{3 \cdot 4} \times \frac{1}{3 \cdot 4}$$

Approximating $\sqrt{2}$

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$$2 \cdot W \times \left(1 + \frac{1}{3} + \frac{1}{3 \cdot 4}\right) = \frac{1}{3 \cdot 4} \times \frac{1}{3 \cdot 4}$$

Approximating $\sqrt{2}$

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$$2 \cdot W \times \left(1 + \frac{1}{3} + \frac{1}{3 \cdot 4}\right) = \frac{1}{3 \cdot 4} \times \frac{1}{3 \cdot 4}$$

$$\frac{2W(12 + 4 + 1)}{3 \cdot 4} = \frac{1}{3 \cdot 4 \cdot 3 \cdot 4}$$

Approximating $\sqrt{2}$

- Goal:

$$\sqrt{2} \approx 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34}.$$

$$2 \cdot W \times \left(1 + \frac{1}{3} + \frac{1}{3 \cdot 4}\right) = \frac{1}{3 \cdot 4} \times \frac{1}{3 \cdot 4}$$

$$\frac{2W(12+4+1)}{3 \cdot 4} = \frac{1}{3 \cdot 4 \cdot 3 \cdot 4}$$

$$34W = \frac{1}{3 \cdot 4}$$

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$$2 \cdot W \times \left(1 + \frac{1}{3} + \frac{1}{3 \cdot 4}\right) = \frac{1}{3 \cdot 4} \times \frac{1}{3 \cdot 4}$$

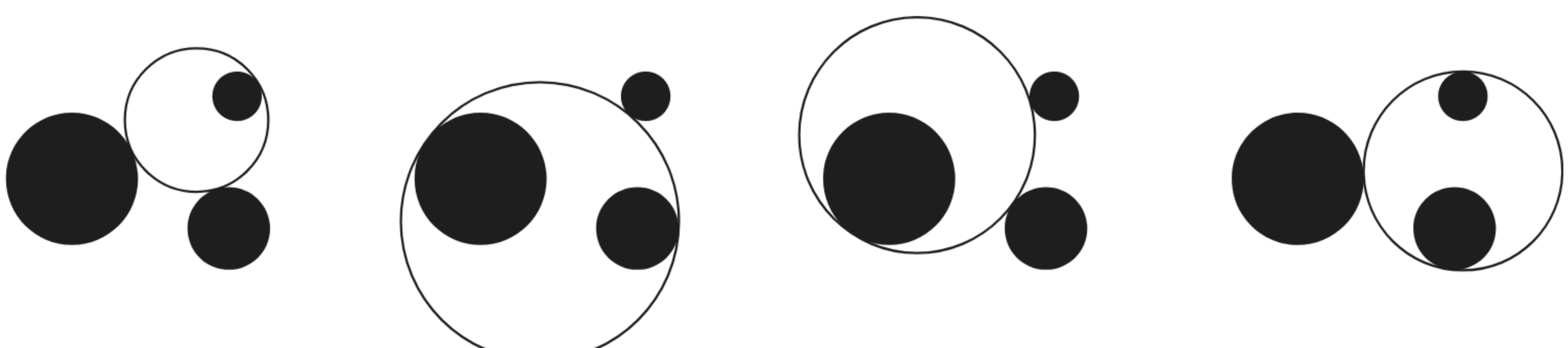
$$\frac{2W(12 + 4 + 1)}{3 \cdot 4} = \frac{1}{3 \cdot 4 \cdot 3 \cdot 4}$$

$$34W = \frac{1}{3 \cdot 4}$$

$$W = \frac{1}{3 \cdot 4 \cdot 34}. \quad \checkmark$$

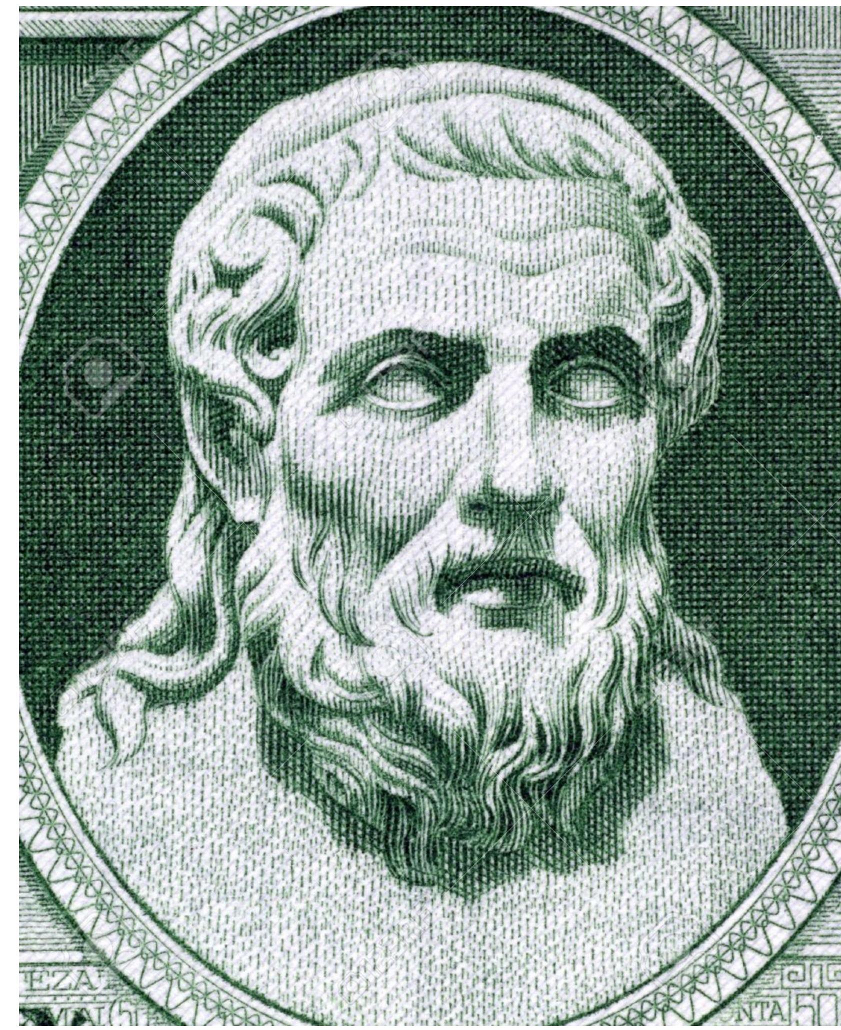
Shout-outs!

- Apollonius lived ~262-190 BC, modern-day Turkey.
- Wrote the text *Conic*. A major study in ellipses, parabolas, hyperbolas, and tangent lines.
- Apollonius problem: Given 3 circles, find a 4th circle tangent to all 3. (Solved in 1956.)

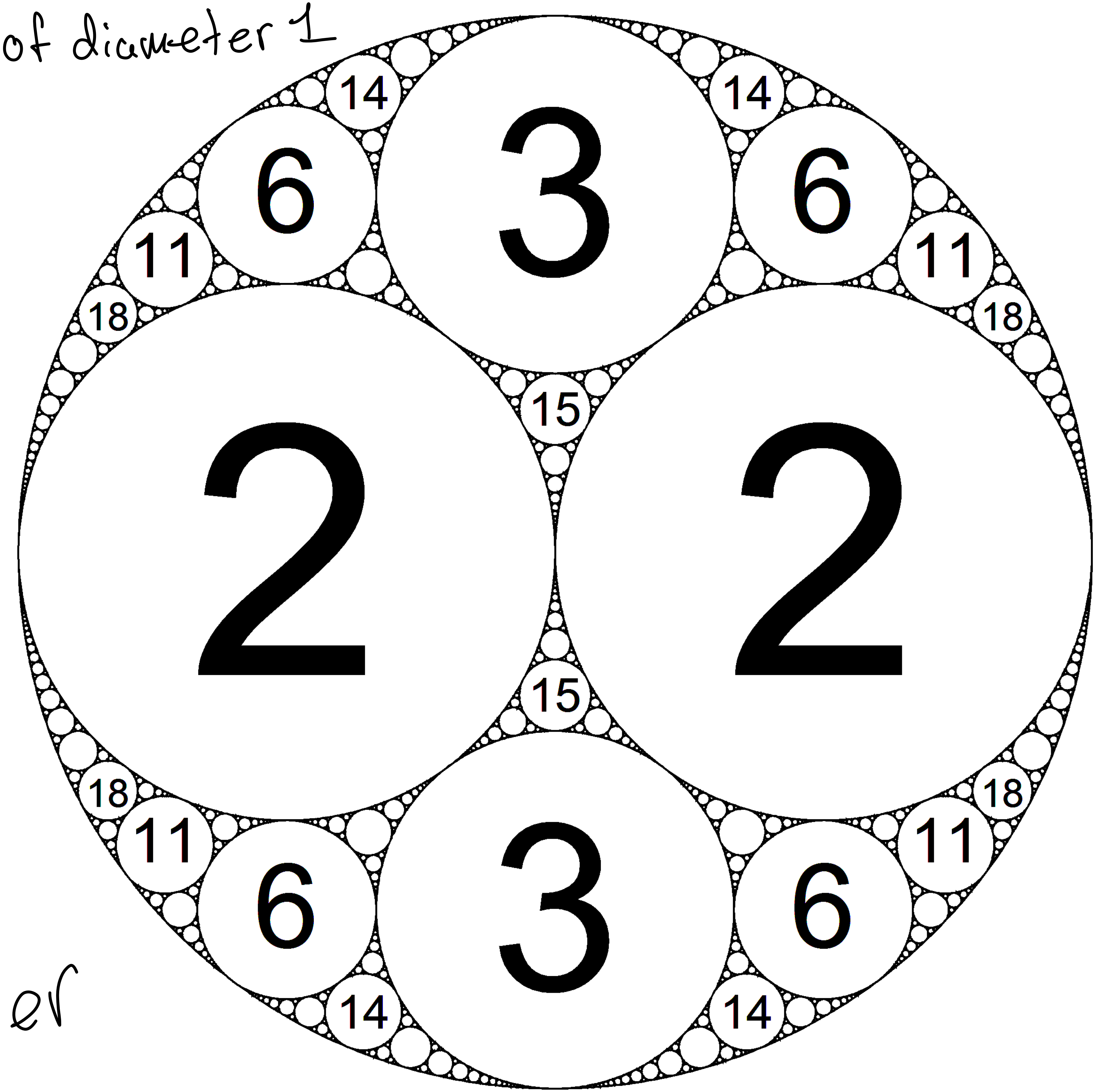


Shout-outs!

- A related problem is one of my favorite in all of math.



Circle of diameter 1



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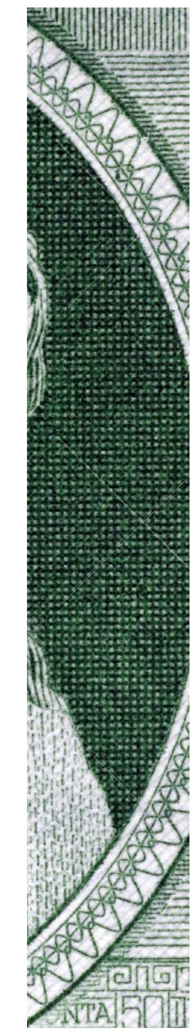

$\frac{1}{\text{integer}}$

$\frac{1}{\text{integer}}$

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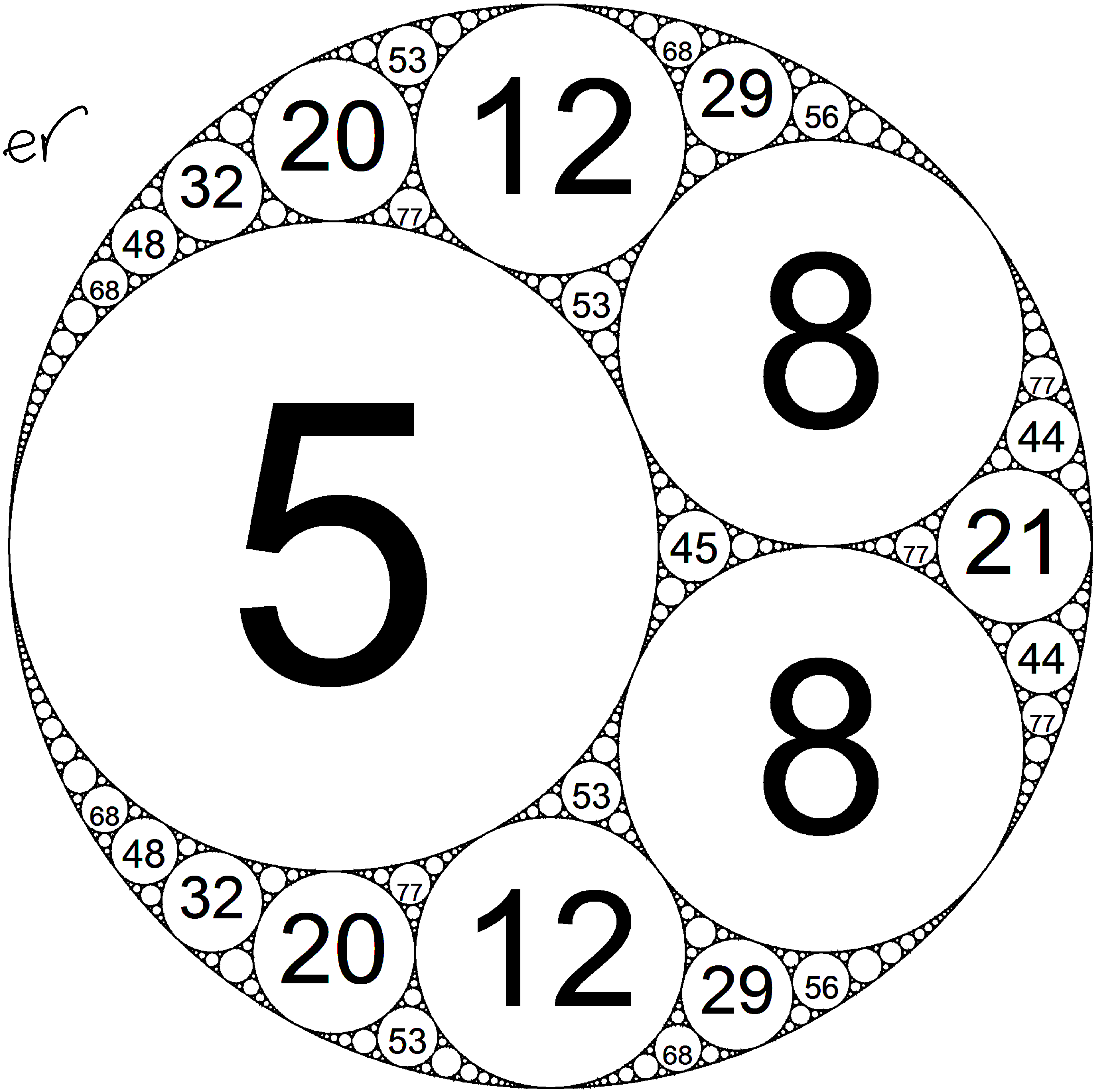

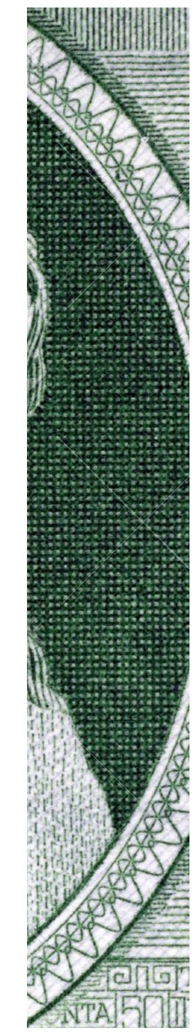

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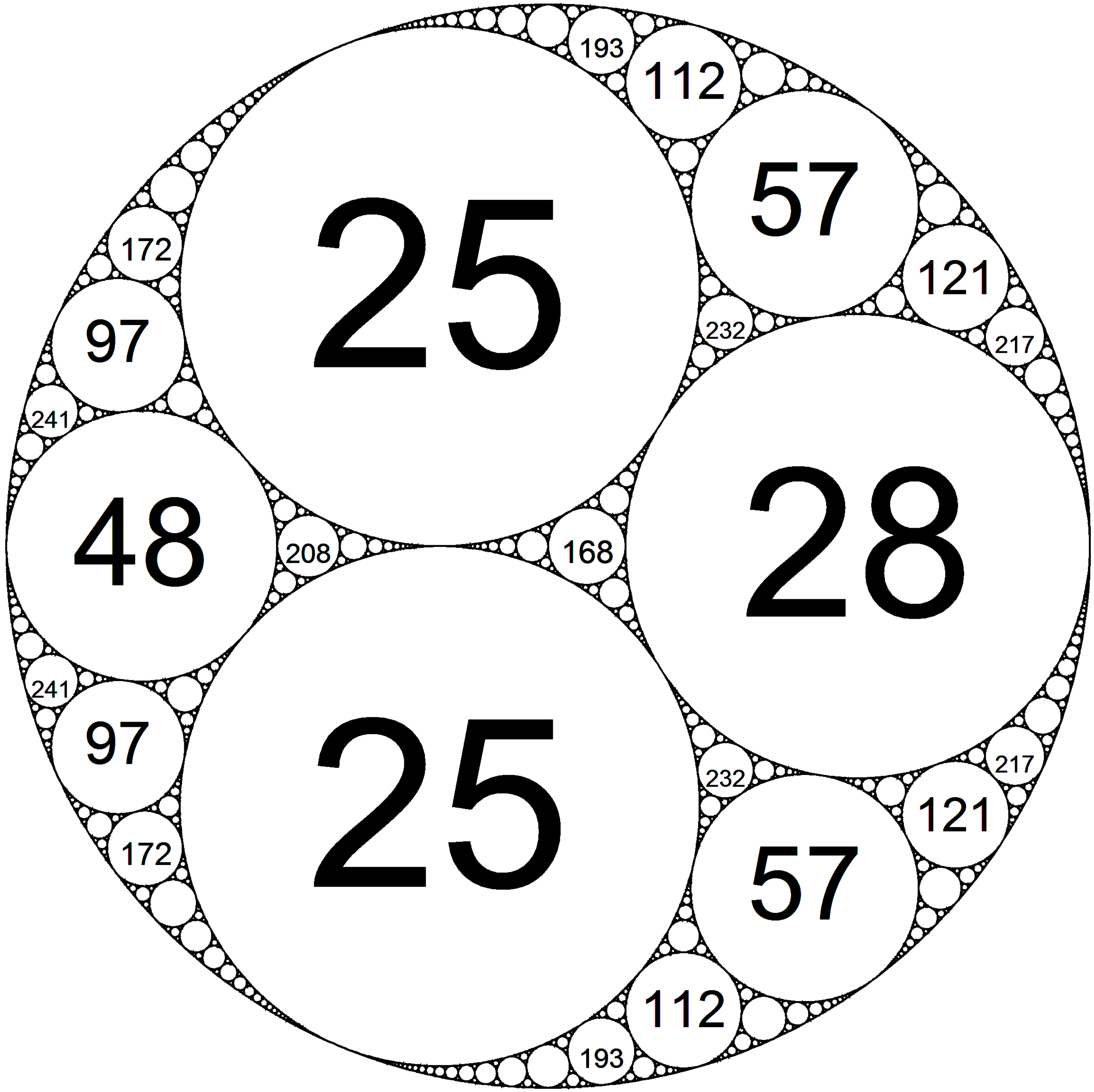
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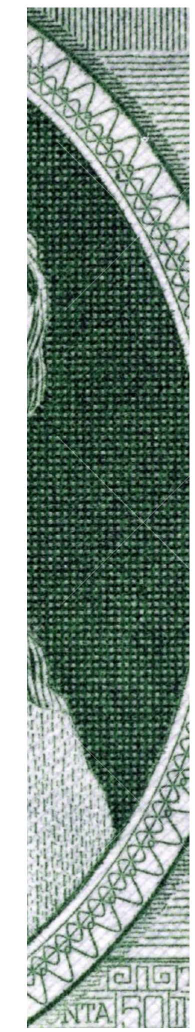

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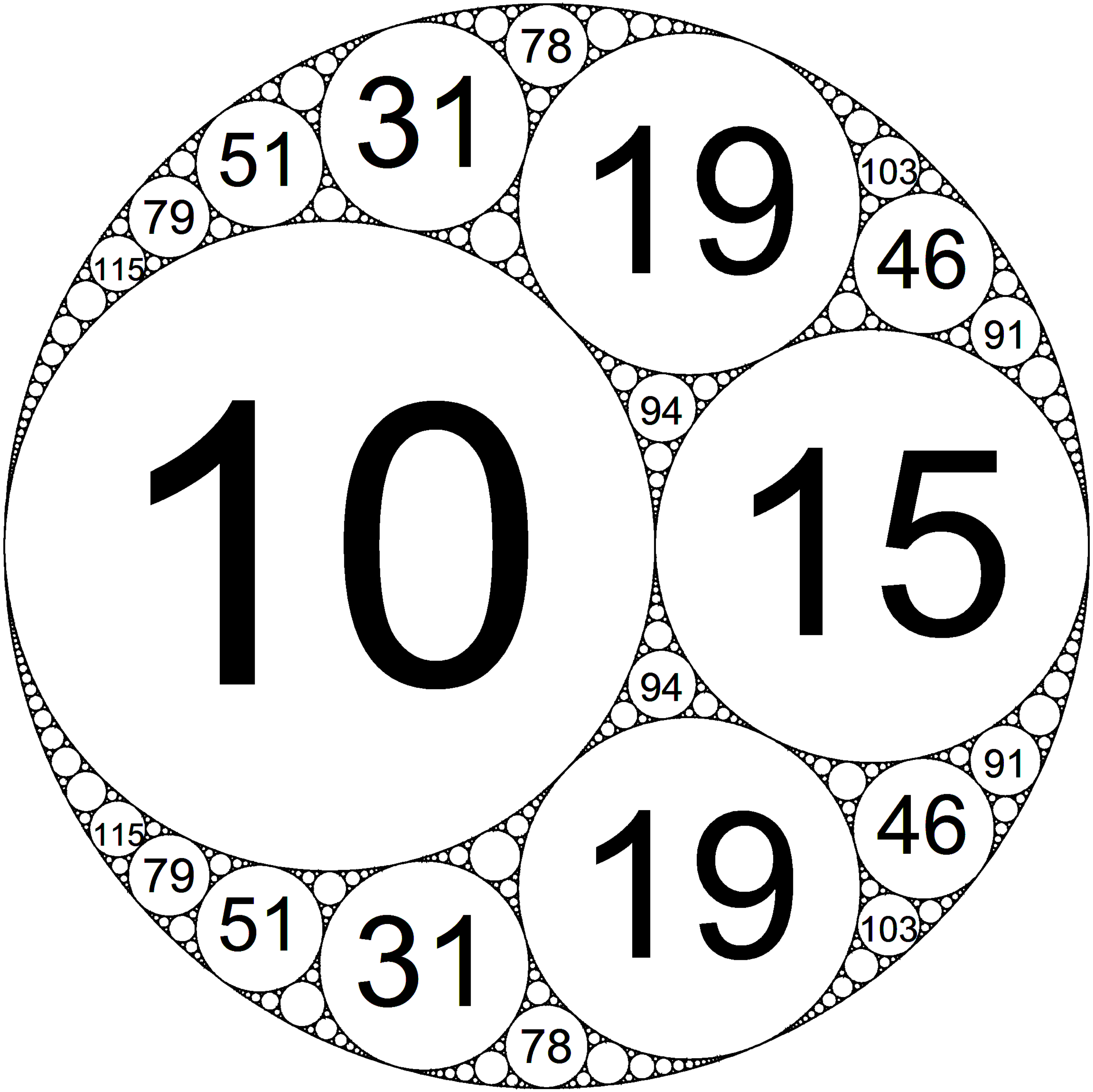

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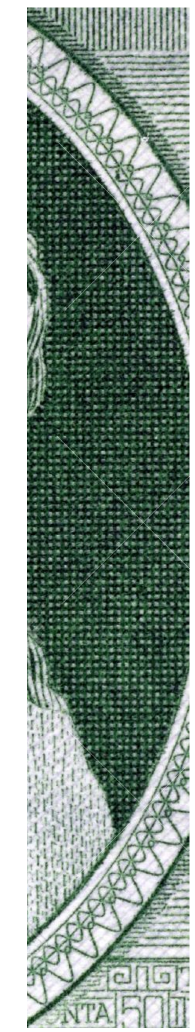

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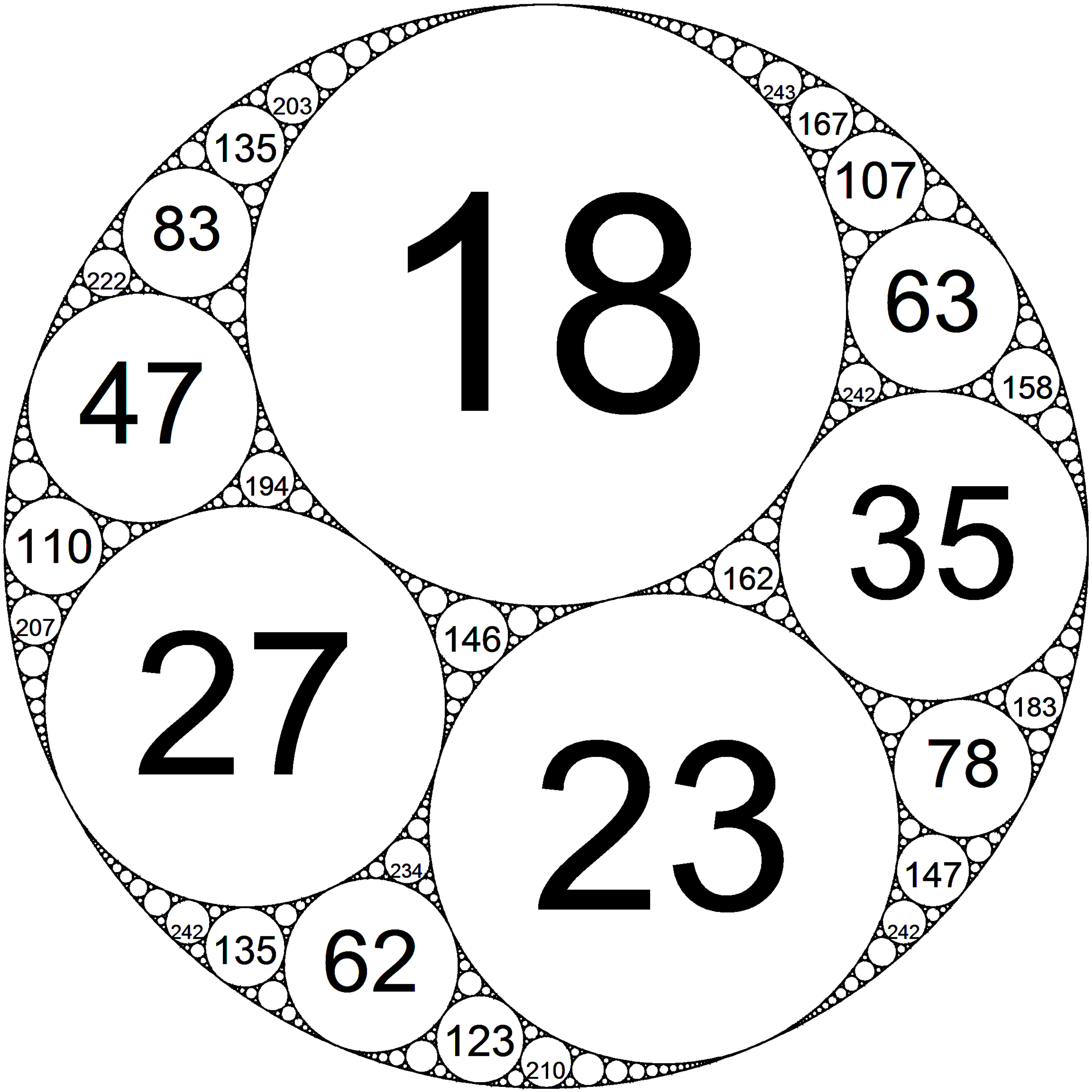
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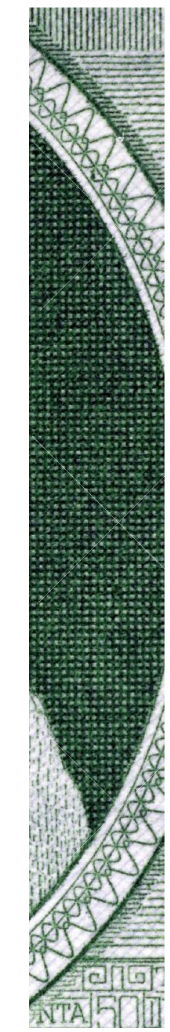
NTA

Shout-outs!

- Packing problems are still important areas of study.
- In the 17th century, Johannes Kepler conjectured that the most efficient way to pack spheres will result in about 75% of the volume being filled.
- This was proved in 1998 by Thomas Hales.

Shout-outs!

- In 2017, Maryna Viazovska generalized this to higher dimensions.
- Example: in 8 dimensions, only 25% of your space can be filled with hyperspheres. In 24 dimensions, only 0.1% of your space can be filled with hyperspheres.

- For her work, Viazovska won the 2022 Fields medal. She was the second woman in history to win it.

School of Athens

People's History

People's History of Geometries