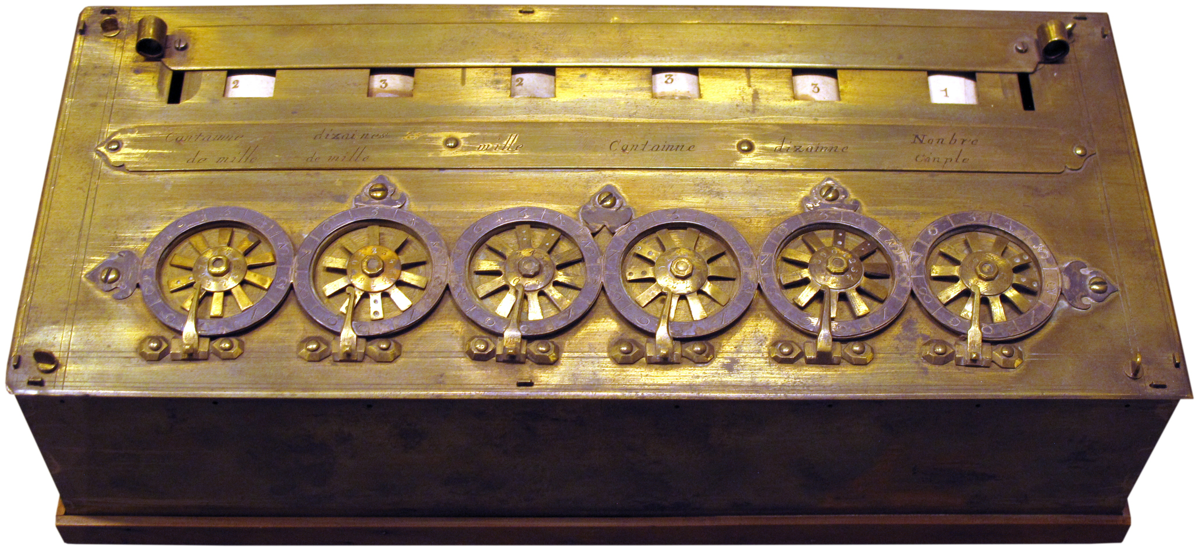
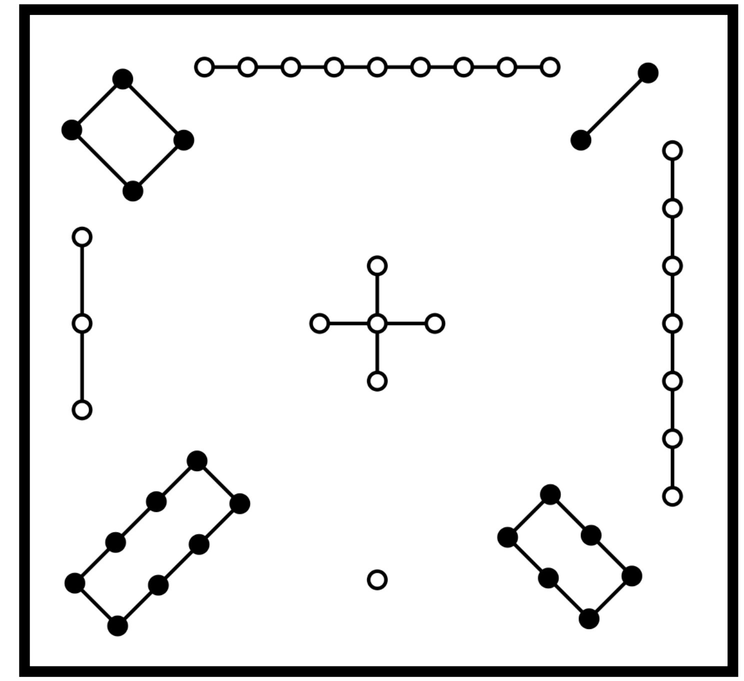
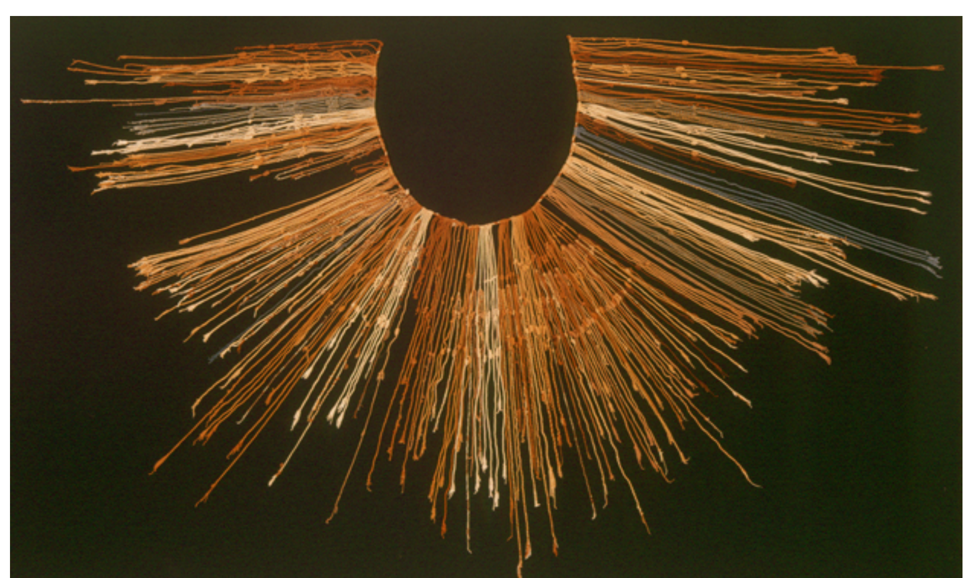
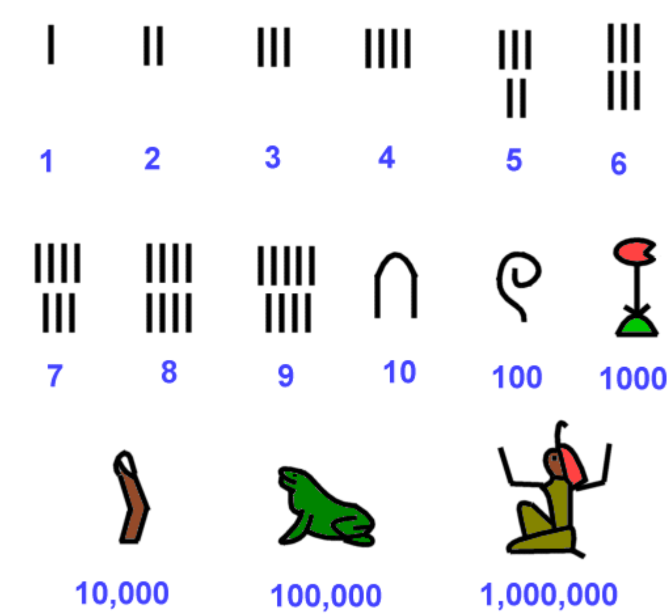




The History of Mathematics

Math 190, Professor Jay Cummings





Syllabus Key Points

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- Name: Jay Cummings

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- Name: Jay Cummings
- Email: Jay.Cummings@csus.edu

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- Name: Jay Cummings
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- Office: Shasta 253

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- Name: Jay Cummings
- Email: Jay.Cummings@csus.edu
- Office: Shasta 253
- Office Hours: Monday 10–11:30, Thursday 12-1:20

Syllabus Key Points

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  - Written Homeworks (25%). Due on Thursdays about every other week

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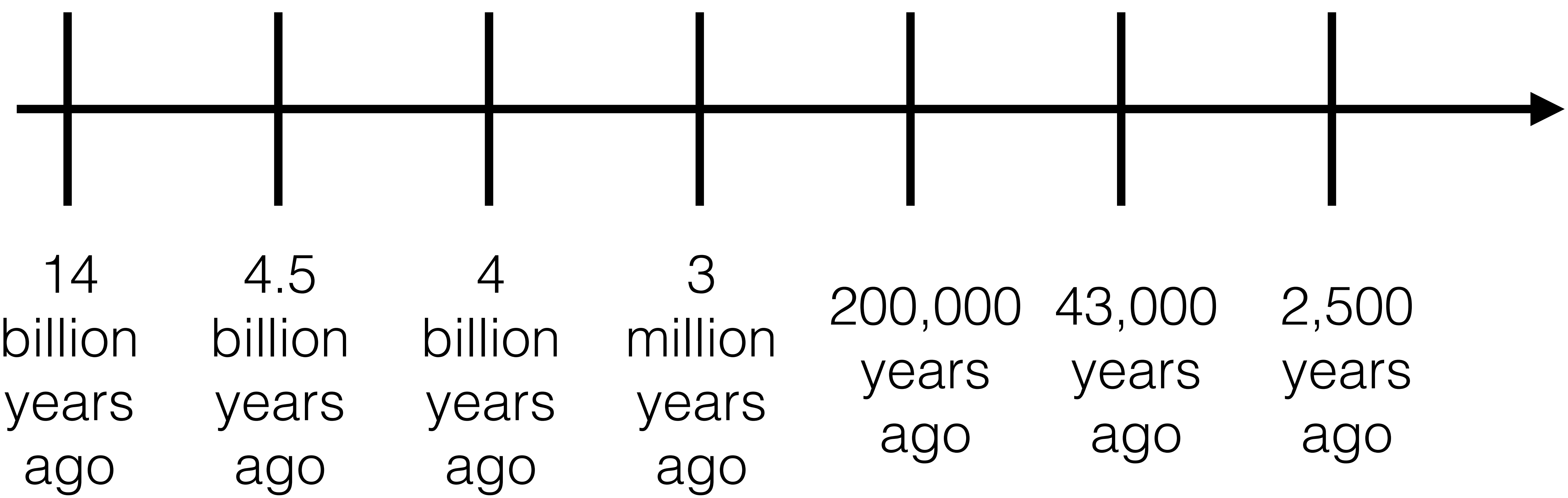
    - Final Exam (20%), comprehensive final exam
    - Essay-based, get questions 1 week ahead of time
- Attendance/Participation (10%). Mostly graded on showing up and paying attention.

When to Begin?

When to Begin?

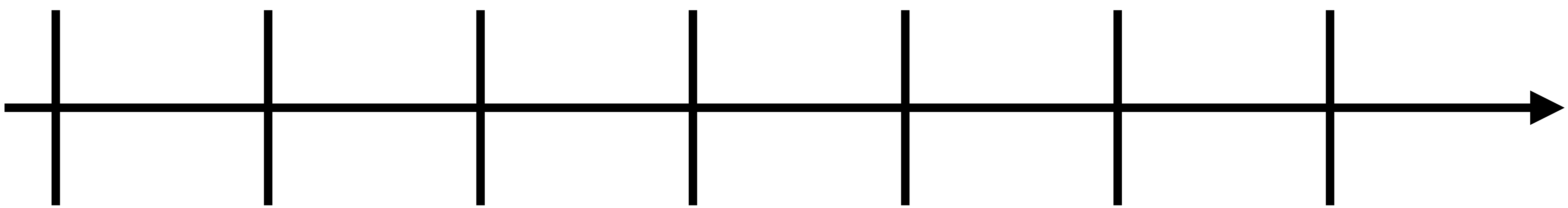
- Talk to a classmate about when you think the history of math began.

A Brief History of Time



A Brief History of Time

Universe born



14 billion years ago

4.5 billion years ago

4 billion years ago

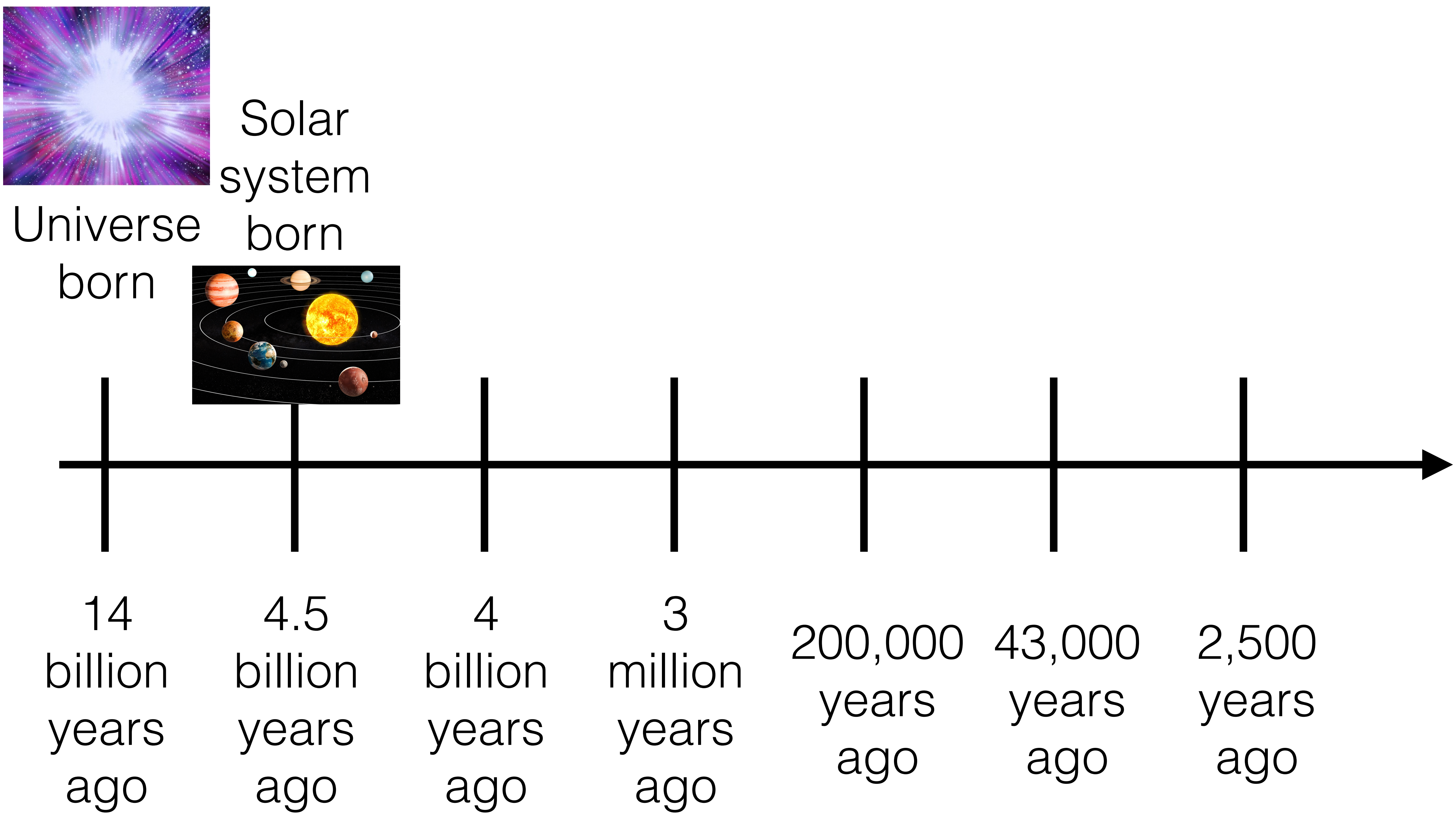
3 million years ago

200,000 years ago

43,000 years ago

2,500 years ago

A Brief History of Time



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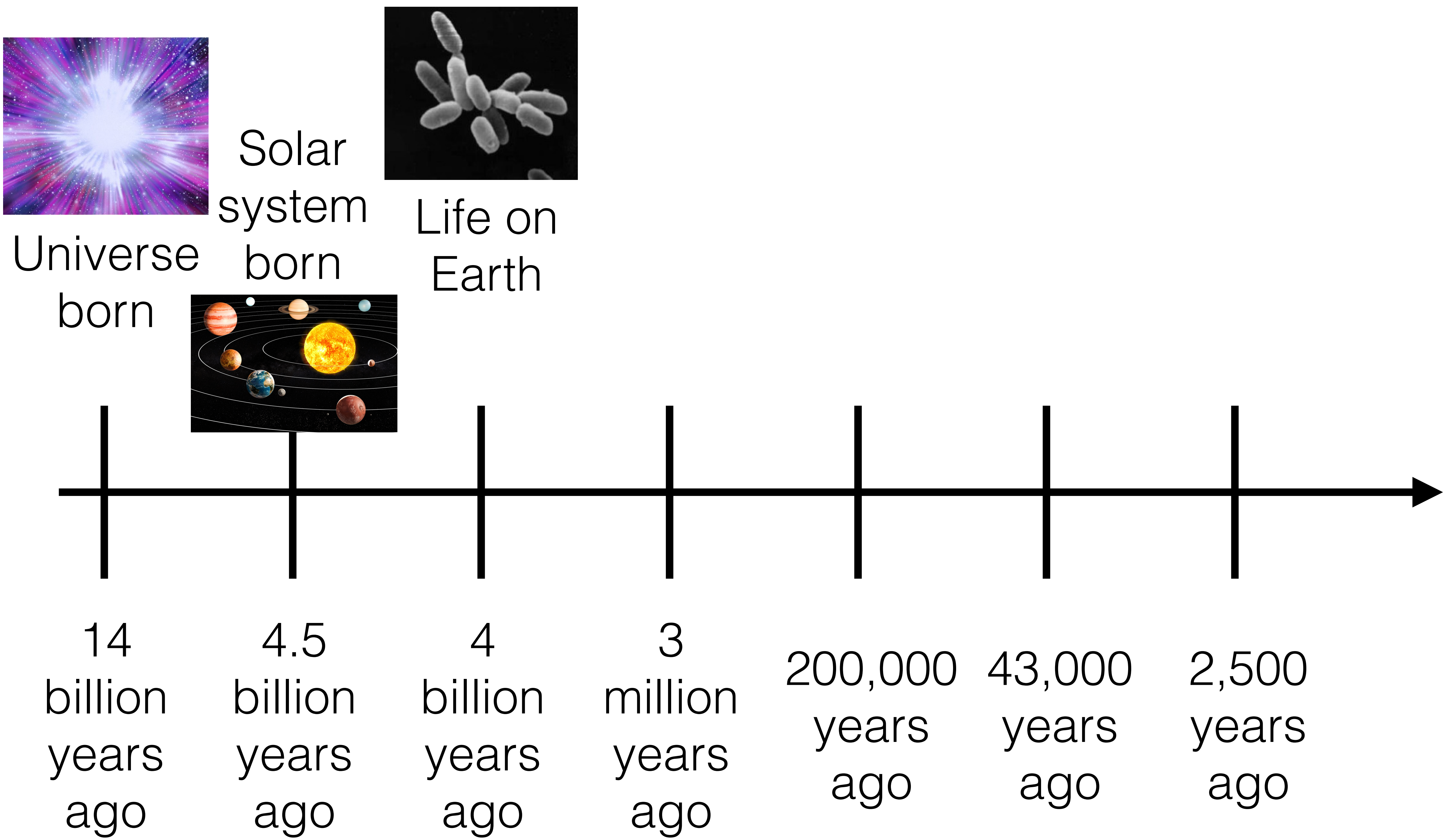

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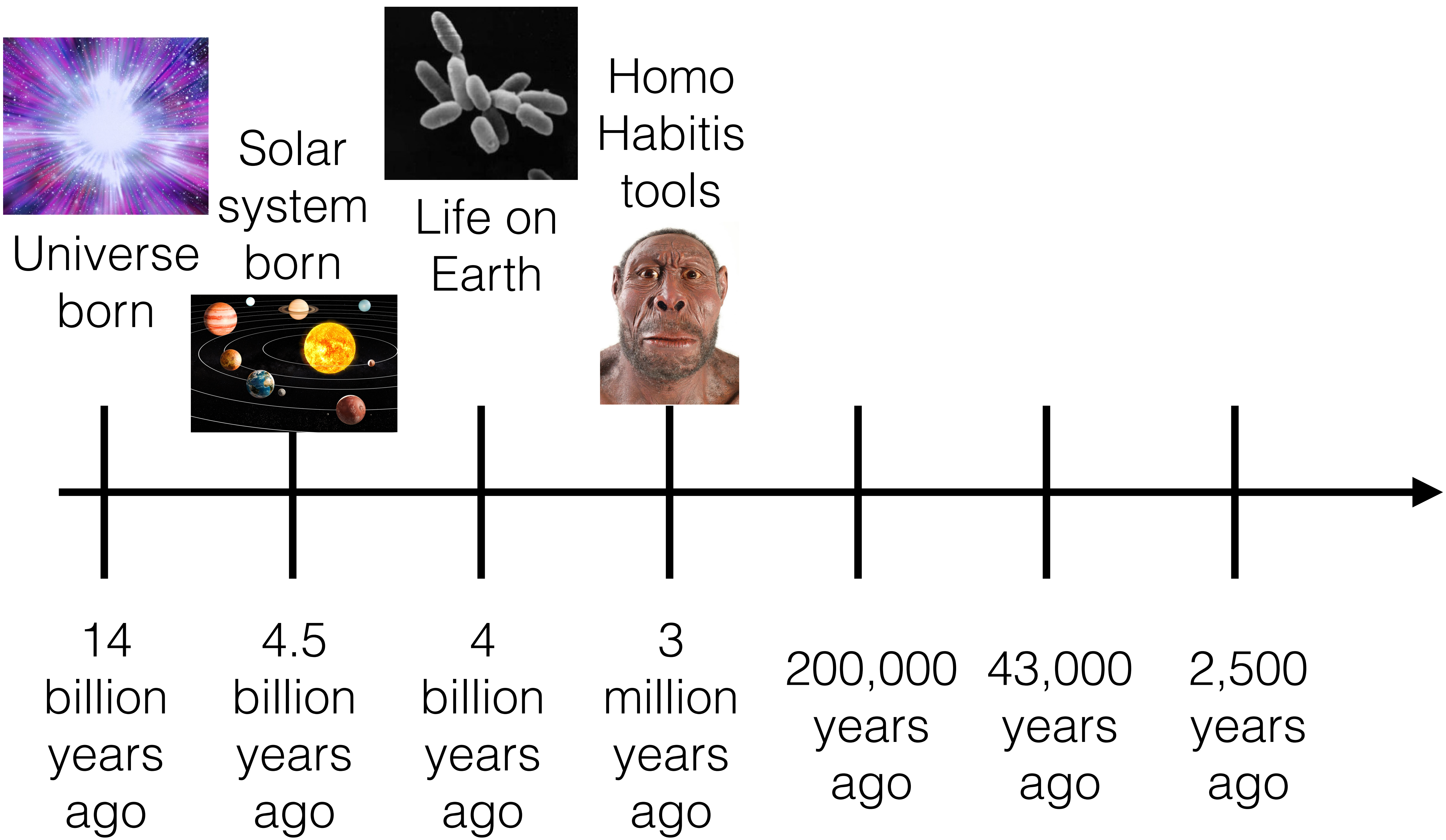

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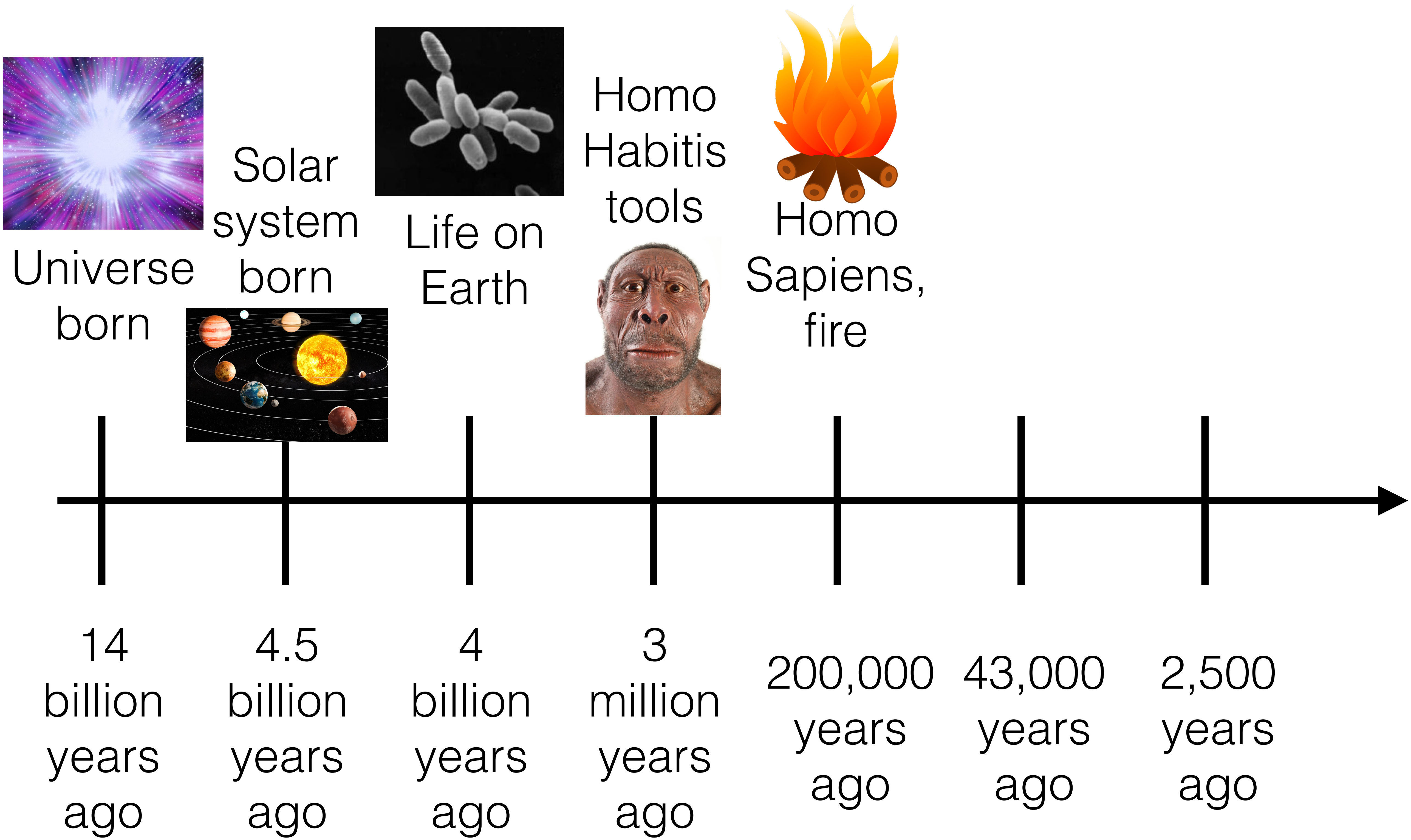

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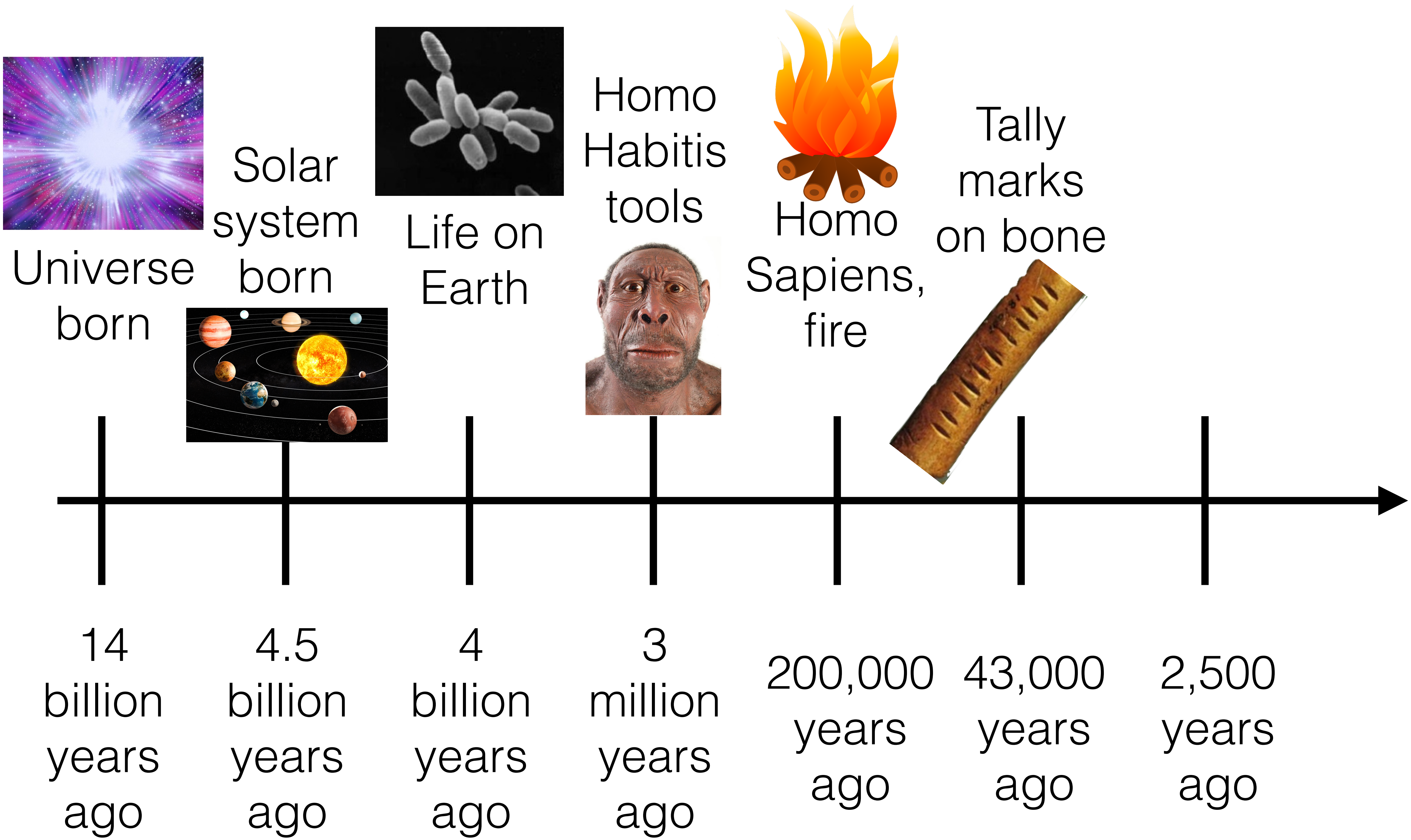

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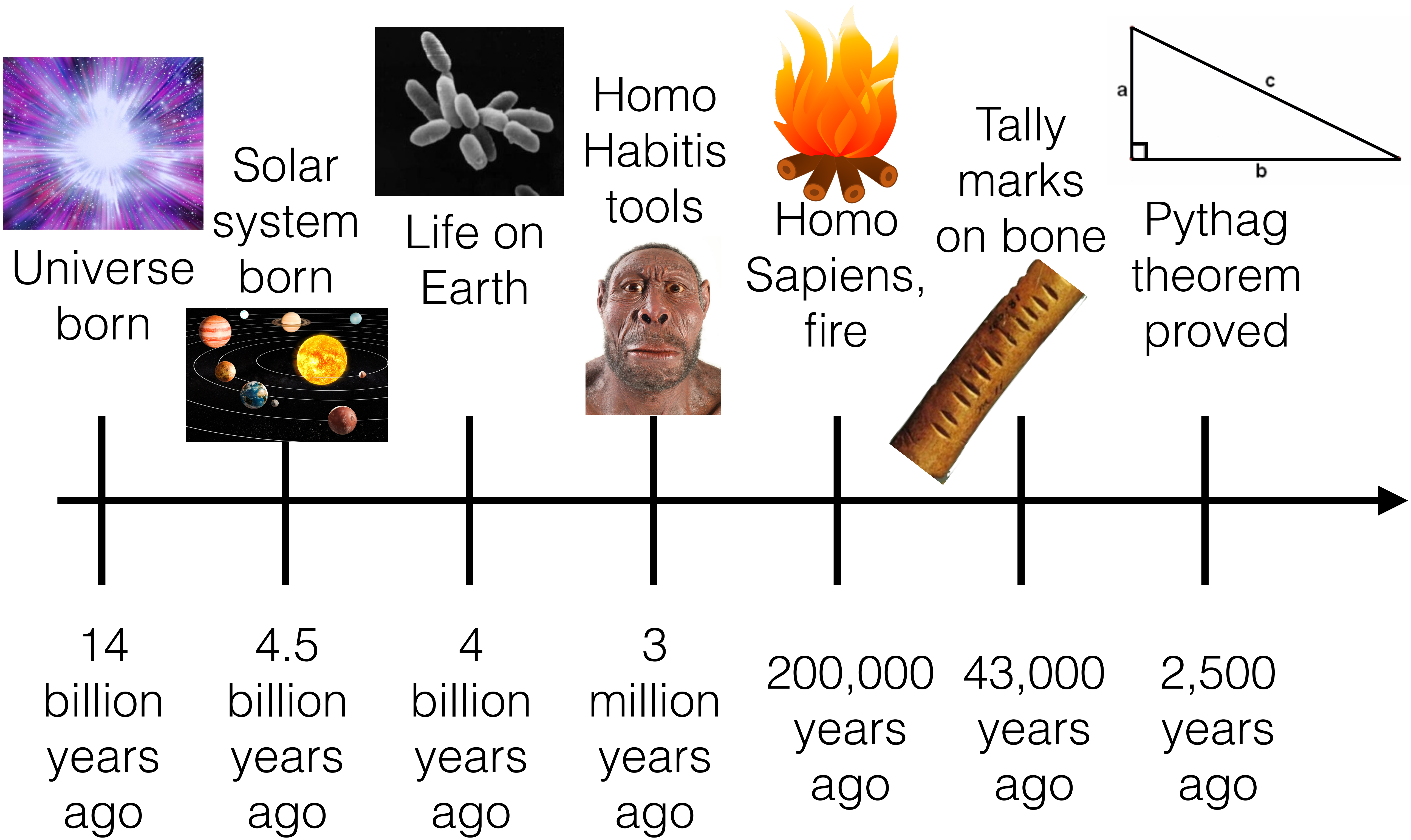

Lebombo Bone

Lebombo Bone

- From ~41,000 BC.
- 29 clear notches.

Lebombo Bone

Ishango Bone

Ishango Bone

- From ~18,000 BC.

Ishango Bone

- From ~18,000 BC.
- Has 3 columns, with 48, 60 and 60 notches.

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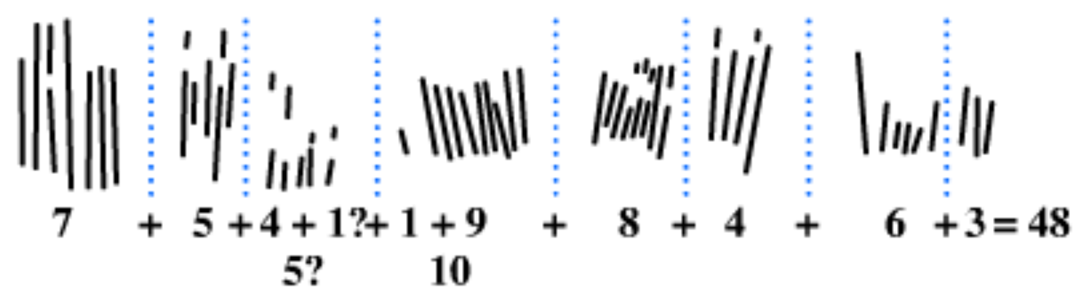

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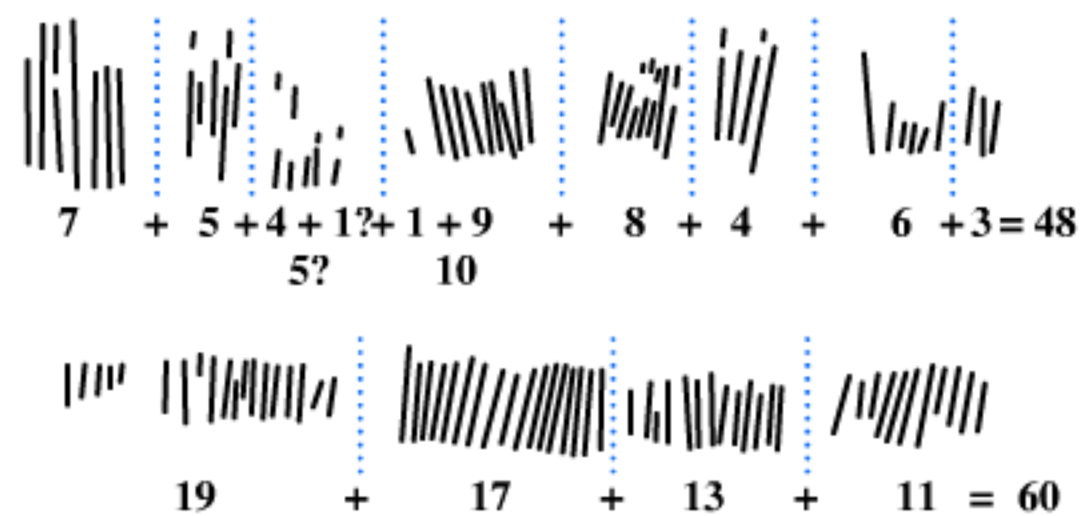

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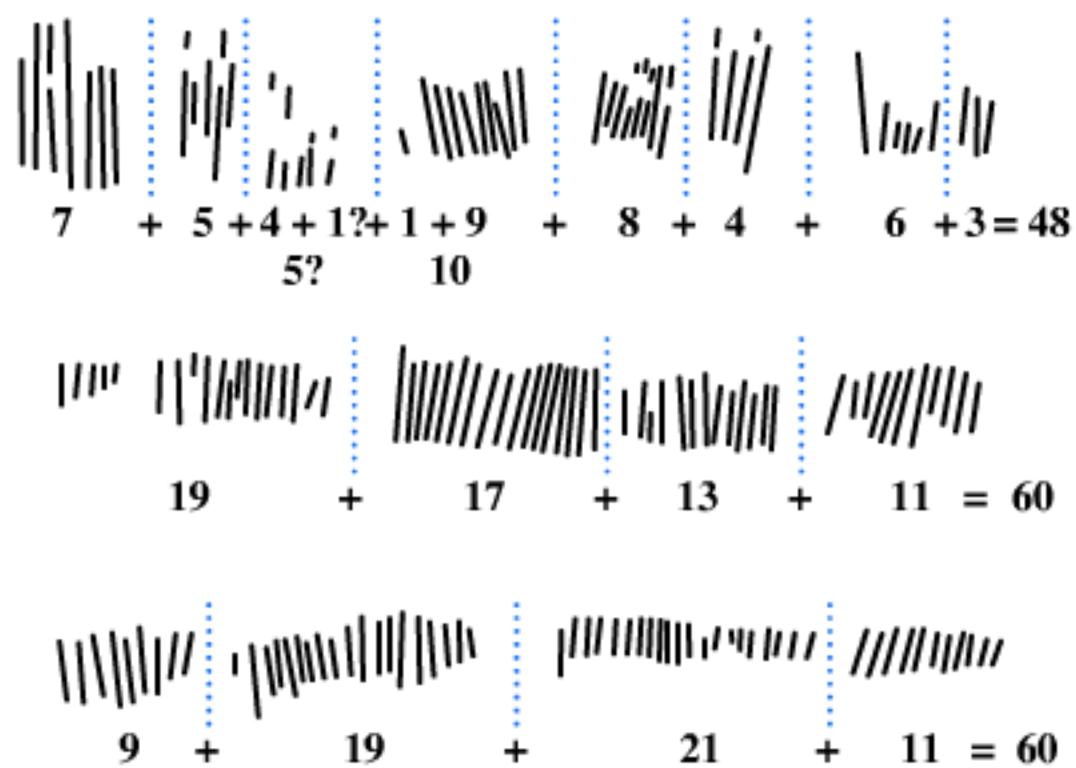

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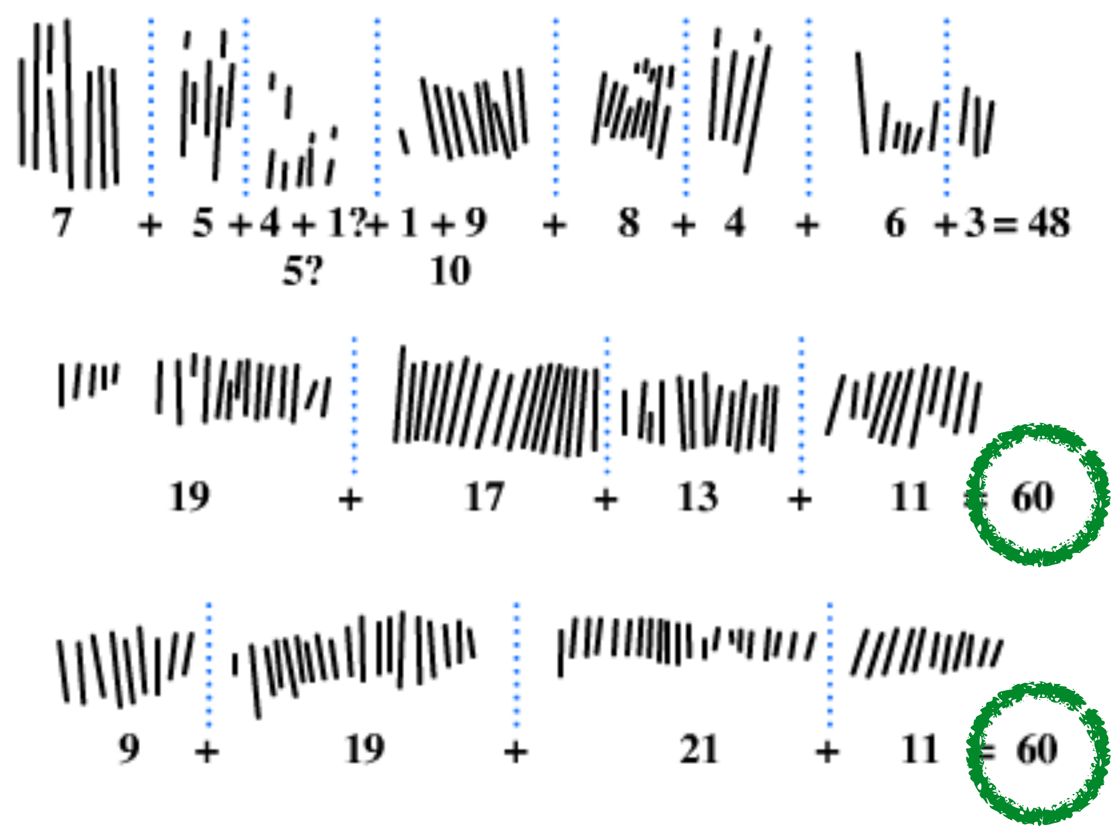

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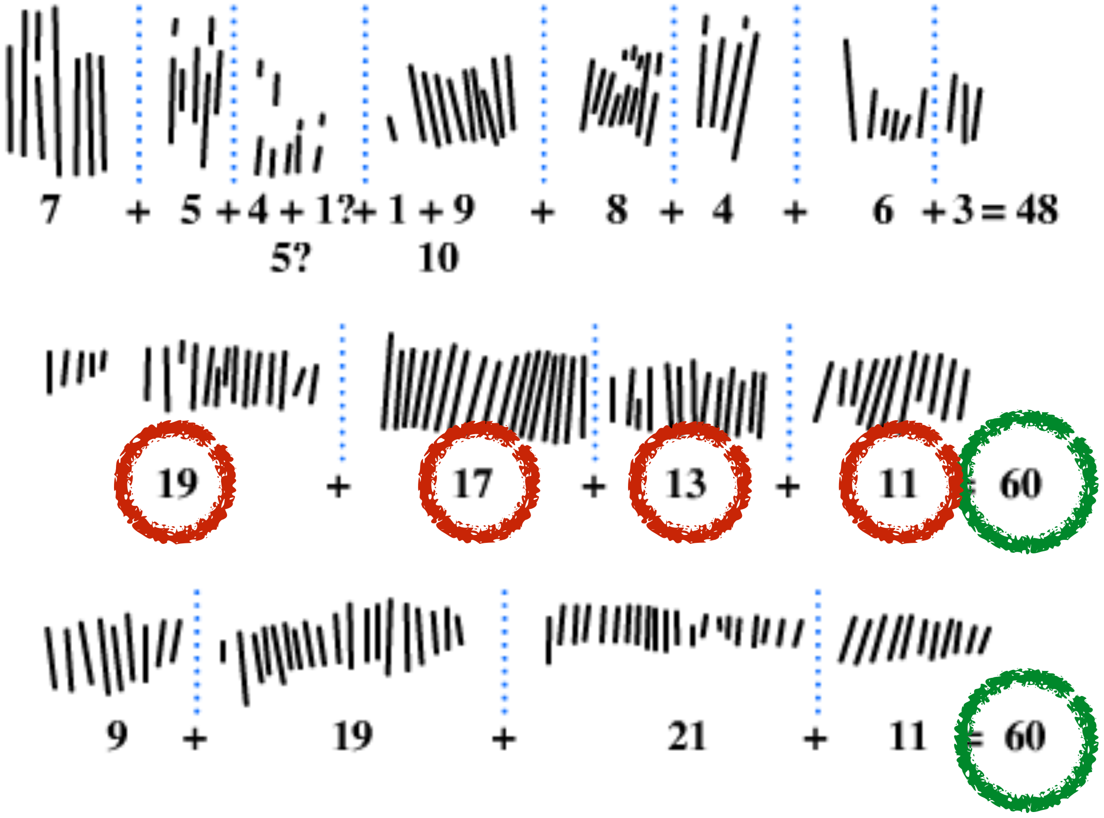

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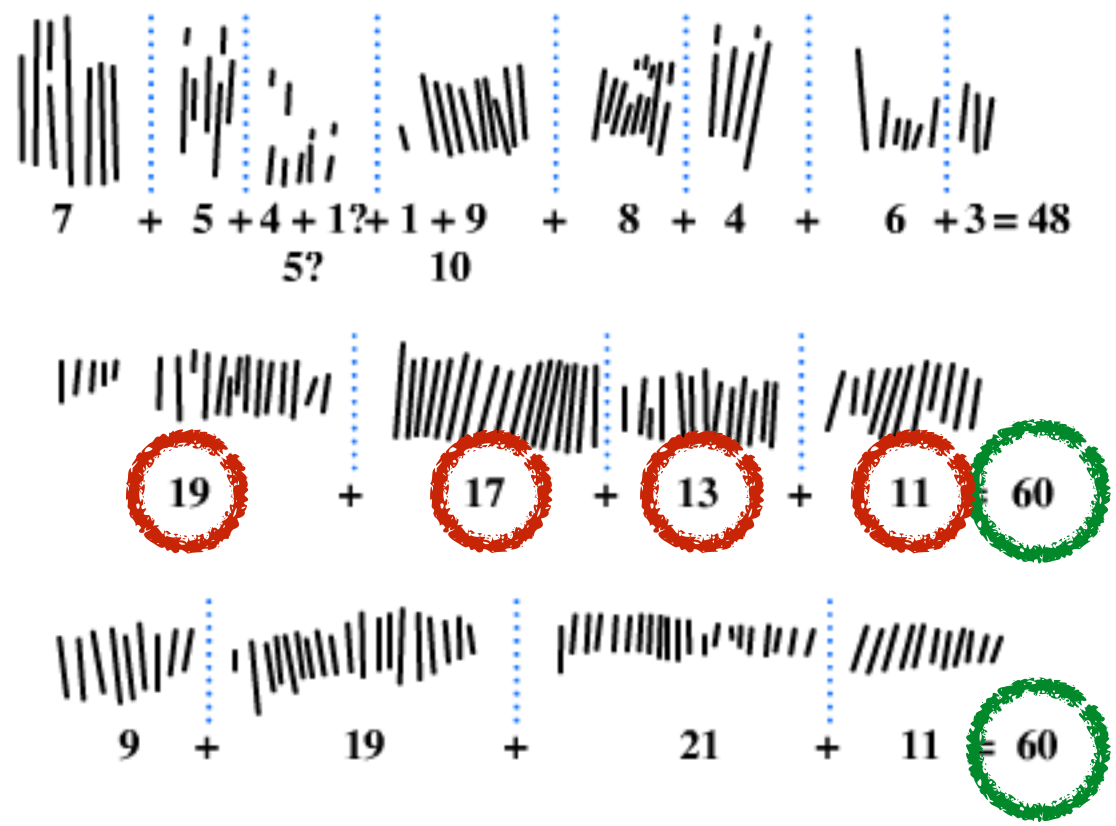

Both bones are the fibula of baboons!

Ishango Bone

How Humans Count

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- Knots in rope.   (Peru, China)
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<u>Question:</u> If counting is the start of math history, does it have to be humans counting?

Number Bases

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- Base 2 (binary system): Computers

Number Base Notation

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$$\begin{aligned} &3 \cdot 1{,}000 + 8 \cdot 100 + 3 \cdot 10 + 5 \\ &= 3 \cdot 10^3 + 8 \cdot 10^2 + 3 \cdot 10^1 + 5 \cdot 10^0 . \end{aligned}$$

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$$3 \cdot 1{,}000 + 8 \cdot 100 + 3 \cdot 10 + 5$$
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Notation. A number in base b is expressed as

$$(a_k a_{k-1} \ldots a_2 a_1 a_0)_b,$$

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Converting the number $(a_k a_{k-1} \ldots a_2 a_1 a_0)_b$ to base 10 gives

$$a_k b^k + a_{k-1} b^{k-1} + \ldots + a_2 b^2 + a_1 b^1 + a_0 \cdot b^0.$$

Practice

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Example. $(123)_4$ is a base 4 number.

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$(11010)_2$ is a base 2 number.

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$(11010)_2$ is a base 2 number.

$(11010)_3$ is a base 3 number.

Practice

Example. Write $(123)_4$ in base 10.

Practice

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Solution.

$$1 \cdot 4^2 + 2 \cdot 4^1 + 3 \cdot 4^0$$

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$$= 16 + 8 + 3$$

Practice

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Solution.

$$1 \cdot 4^2 + 2 \cdot 4^1 + 3 \cdot 4^0$$
$$= 16 + 8 + 3$$
$$= 27$$

Practice

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$$(34)_{10} = (1021)_3 \, .$$

4 Ways People Write Numbers

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- Multiplicative Grouping System: Have a character for each power of 10

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  Example: Chinese-Japanese numeral system

4 Ways People Write Numbers

$1 =$ 一	$10 =$ 十	$100 =$ 一百	$1{,}000 =$ 一千
$2 =$ 二	$20 =$ 二十	$200 =$ 二百	$2{,}000 =$ 二千
$3 =$ 三	$30 =$ 三十	$300 =$ 三百	$3{,}000 =$ 三千
$4 =$ 四	$40 =$ 四十	$400 =$ 四百	$4{,}000 =$ 四千
$5 =$ 五	$50 =$ 五十	$500 =$ 五百	$5{,}000 =$ 五千
$6 =$ 六	$60 =$ 六十	$600 =$ 六百	$6{,}000 =$ 六千
$7 =$ 七	$70 =$ 七十	$700 =$ 七百	$7{,}000 =$ 七千
$8 =$ 八	$80 =$ 八十	$800 =$ 八百	$8{,}000 =$ 八千
$9 =$ 九	$90 =$ 九十	$900 =$ 九百	$9{,}000 =$ 九千
$10 =$ 十	$100 =$ 一百	$1{,}000 =$ 一千	$10{,}000 =$ 一万

Example: Chinese-Japanese numeral system

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1 = 一	10 = 十	100 = 一百	1,000 = 一千
2 = 二	20 = 二十	200 = 二百	2,000 = 二千
3 = 三	30 = 三十	300 = 三百	3,000 = 三千
4 = 四	40 = 四十	400 = 四百	4,000 = 四千
5 = 五	50 = 五十	500 = 五百	5,000 = 五千
6 = 六	60 = 六十	600 = 六百	6,000 = 六千
7 = 七	70 = 七十	700 = 七百	7,000 = 七千
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$$5,062 \ = \ 5 \cdot 1,000 + 6 \cdot 10 + 2$$

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2 = 二	20 = 二十	200 = 二百	2,000 = 二千
3 = 三	30 = 三十	300 = 三百	3,000 = 三千
4 = 四	40 = 四十	400 = 四百	4,000 = 四千
5 = 五	50 = 五十	500 = 五百	5,000 = 五千
6 = 六	60 = 六十	600 = 六百	6,000 = 六千
7 = 七	70 = 七十	700 = 七百	7,000 = 七千
8 = 八	80 = 八十	800 = 八百	8,000 = 八千
9 = 九	90 = 九十	900 = 九百	9,000 = 九千
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Example: Chinese-Japanese numeral system

$$5,062 \;=\; 5 \cdot 1,000 + 6 \cdot 10 + 2$$

= 五千六十二

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  Example: Egyptian hieroglyphics

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$10,000 =$  , $100,000 =$  , $1,000,000 =$ 

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321

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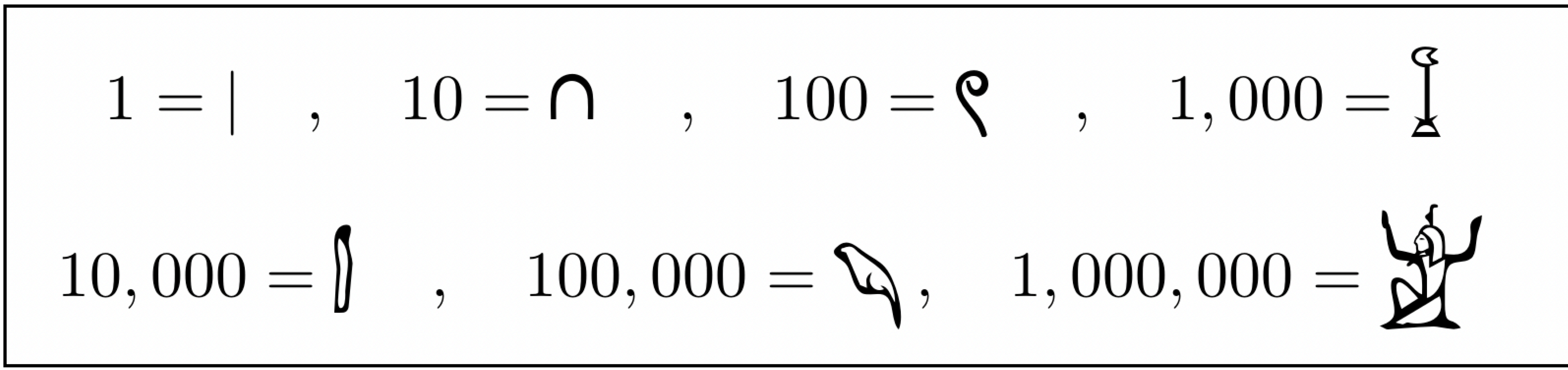

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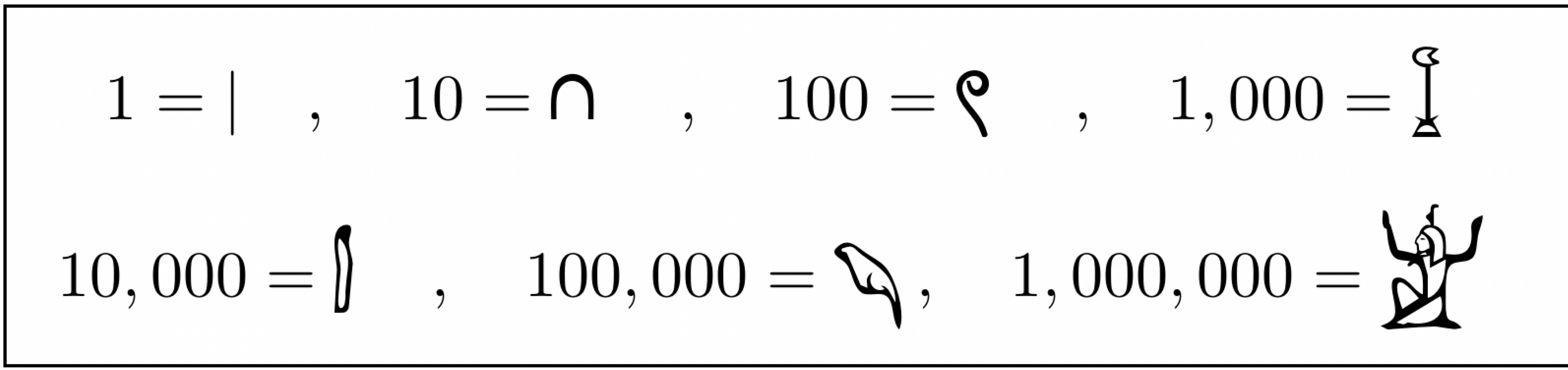

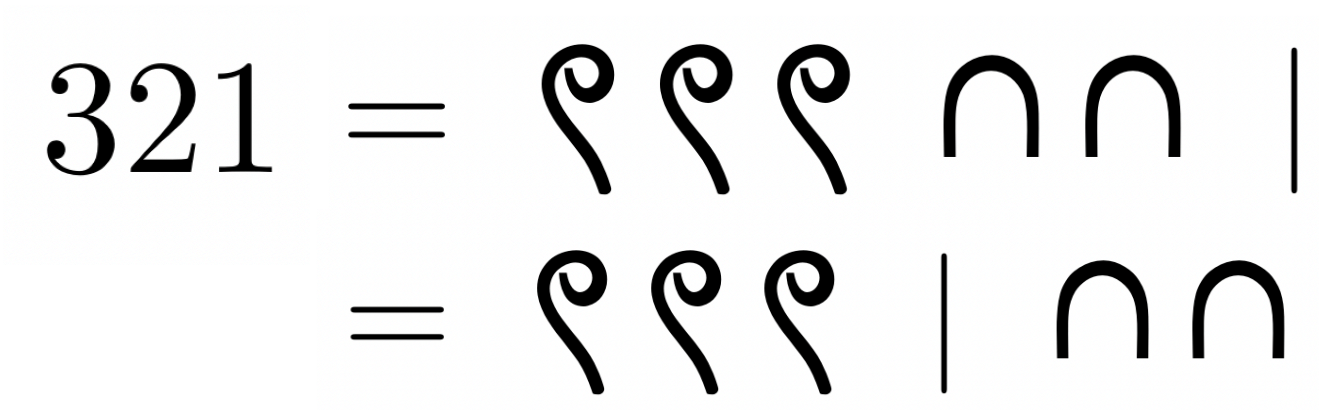

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1 = | , 10 = ∩ , 100 = 𓍢 , 1,000 = 𓆼

10,000 = 𓂭 , 100,000 = 𓆐 , 1,000,000 = 𓁨

321 = 𓍢𓍢𓍢 ∩∩ |

= 𓍢𓍢𓍢 | ∩∩

= 𓍢∩𓍢 | 𓍢∩

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$$
\begin{gathered}
1,2,\ldots,b-1 \\
b,2b,\ldots,(b-1)b \\
b^2,2b^2,\ldots,(b-1)b^2 \\
b^3,2b^3,\ldots,(b-1)b^3 \\
\vdots
\end{gathered}
$$

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$$1, 2, \ldots, b-1$$
$$b, 2b, \ldots, (b-1)b$$
$$b^2, 2b^2, \ldots, (b-1)b^2$$
$$b^3, 2b^3, \ldots, (b-1)b^3$$
$$\vdots$$

Example: Ionic Greek system

4 Ways People Write Numbers

$1 = \alpha$ (alpha)	$10 = \iota$ (iota)	$100 = \rho$ (rho)
$2 = \beta$ (beta)	$20 = \kappa$ (kappa)	$200 = \sigma$ (sigma)
$3 = \gamma$ (gamma)	$30 = \lambda$ (lambda)	$300 = \tau$ (tau)
$4 = \delta$ (delta)	$40 = \mu$ (mu)	$400 = \upsilon$ (upsilon)
$5 = \varepsilon$ (epsilon)	$50 = \nu$ (nu)	$500 = \phi$ (phi)
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$8 = \eta$ (eta)	$80 = \pi$ (pi)	$800 = \omega$ (omega)
$9 = \theta$ (theta)	$90 = \text{ϙ}$ (koppa)	$900 = \text{ϡ}$ (sampi)

Example: Ionic Greek system

Question: Which system is best?

Zero

Zero

The Indians and the Mayans were the two civilizations to discover zero and recognize its deep potential as:

Zero

The Indians and the Mayans were the two civilizations to discover zero and recognize its deep potential as:

- A symbol

Zero

The Indians and the Mayans were the two civilizations to discover zero and recognize its deep potential as:

- A symbol
- A number

Zero

The Indians and the Mayans were the two civilizations to discover zero and recognize its deep potential as:

- A symbol
- A number
- A magnitude

Zero

The Indians and the Mayans were the two civilizations to discover zero and recognize its deep potential as:

- A symbol
- A number
- A magnitude
- A direction separator

Zero

The Indians and the Mayans were the two civilizations to discover zero and recognize its deep potential as:

- A symbol
- A number
- A magnitude
- A direction separator
- A place-holder

Zero

The Indians and the Mayans were the two civilizations to discover zero and recognize its deep potential as:

- A symbol
- A number
- A magnitude
- A direction separator
- A place-holder
- An idea

Zero

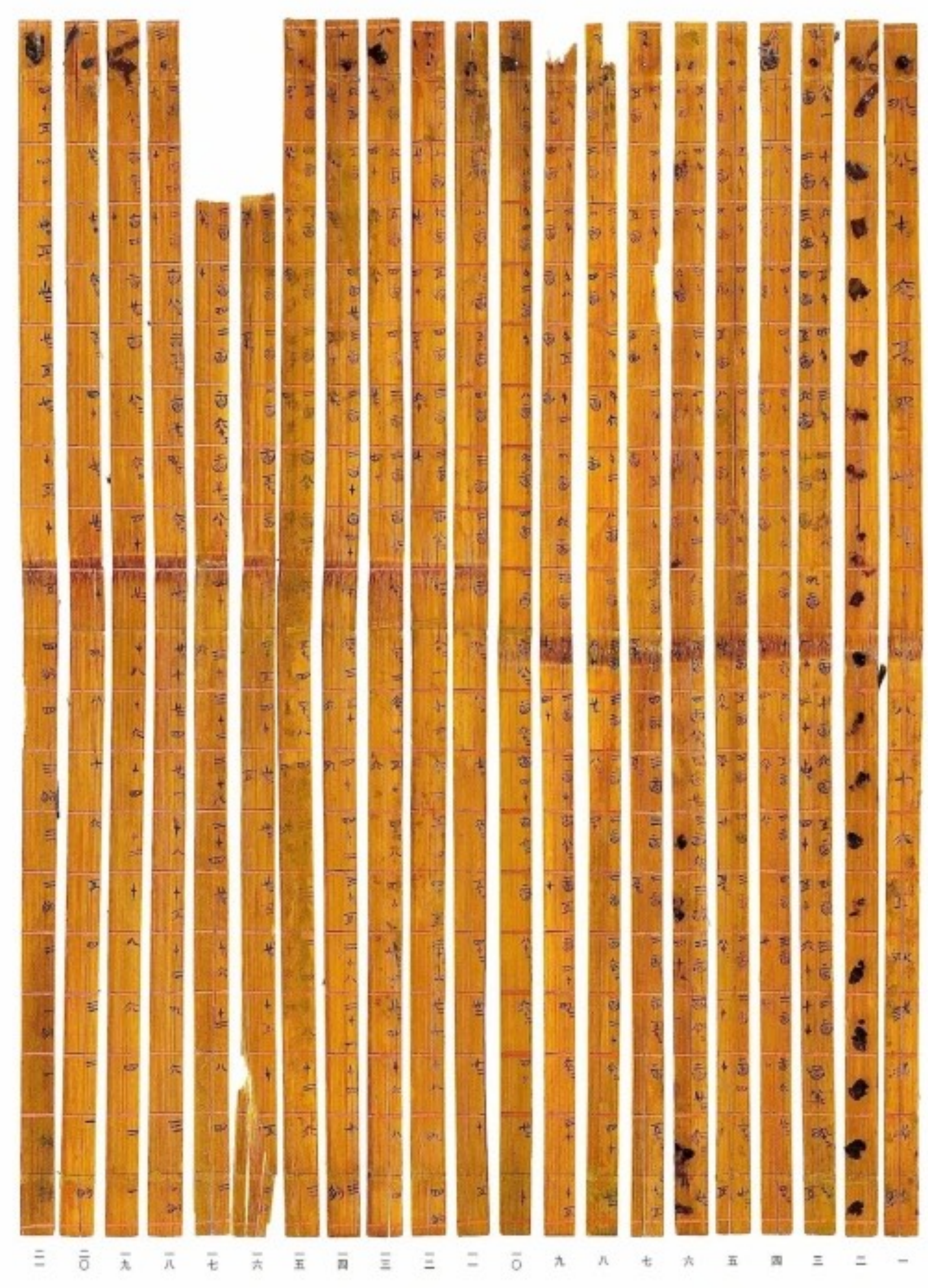
The Indians and the Mayans were the two civilizations to discover zero and recognize its deep potential as:

- A symbol
- A number
- A magnitude
- A direction separator
- A place-holder
- An idea

This was a big deal.

Think Like A
Math Historian

Think Like A Math Historian



Think Like A Math Historian

8	5	80	4		3	20	7	9	10	1/2	40	30	6	50	60	2	1	90	70
720	450	7200	360	90	270	1800	630	810	900	45	3600	2700	540	4500	5400	180	90	8100	6300
640	400	6400	320	80	240	1600	560	720	800	40	3200	2400	480	4000	4800	160	80	7200	5600
560	350	5600	280	70	210	1400	490	630	700	35	2800	2100	420	3500	4200	140	70	6300	4900
480	300	4800	240	60	180	1200	420	540	600	30	2400	1800	360	3000	3600	120	60	5400	4200
400	250	4000	200	50	150	1000	350	450	500	25	2000	1500	300	2500	3000	100	50	4500	3500
320	200	3200	160	40	120	800	280	360	400	20	1600	1200	240	2000	2400	80	40	3600	2800
240	150	2400	120	30	90	600	210	270	300	15	1200	900	180	1500	1800	60	30	2700	2100
160	100	1600	80	20	60	400	140	180	200	10	800	600	120	1000	1200	40	20	1800	1400
80	50	800	40	10	30	200	70	90	100	5	400	300	60	500	600	20	10	900	700
72	45	720	36	9	27	180	63	81	90	$4\frac{1}{2}$	360	270	54	450	540	18	9	810	630
64	40	640	32	8	24	160	56	72	80	4	320	240	48	400	480	16	8	720	560
56	35	560	28	7	21	140	49	63	70	$3\frac{1}{2}$	280	210	42	350	420	14	7	630	490
48	30	480	24	6	18	120	42	54	60	3	240	180	36	300	360	12	6	540	420
40	25	400	20	5	15	100	35	45	50	$2\frac{1}{2}$	200	150	30	250	300	10	5	450	350
32	20	320	16	4	12	80	28	36	40	2	160	120	24	200	240	8	4	360	280
24	15	240	12	3	9	60	21	27	30	$1\frac{1}{2}$	120	90	18	150	180	6	3	270	210
16	10	160	8	2	6	40	14	18	20	1	80	60	12	100	120	4	2	180	140
8	5	80	4	1	3	20	7	9	10	$\frac{1}{2}$	40	30	6	50	60	2	1	90	70
4	$2\frac{1}{2}$	40	2	$\frac{1}{2}$	$1\frac{1}{2}$	10	$3\frac{1}{2}$	$4\frac{1}{2}$	5	$\frac{1}{4}$	20	15	3	25	30	1	$\frac{1}{2}$	45	35

Think Like A Math Historian

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4^1/_2$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3^1/_2$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2^1/_2$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1^1/_2$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$^1/_2$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$^1/_4$	$^1/_2$	1	$1^1/_2$	2	$2^1/_2$	3	$3^1/_2$	4	$4^1/_2$	5	10	15	20	25	30	35	40	45	$^1/_2$

Think Like A Math Historian

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4\frac{1}{2}$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3\frac{1}{2}$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2\frac{1}{2}$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$

We will come back to this soon.

Arithmetic

Multiplication Tables

Multiplication Tables

- After counting comes arithmetic. Simple addition and subtraction could be done in head, on fingers, or with simple tools. Multiplication was probably much harder.

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- First known multiplication table is a 4,000 year old Babylonian (base 60) multiplication table.

Multiplication Tables

- After counting c. Simple addition and subtractic in head, on fingers, or with simple ion was probably much harder.

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MS 3866
Multiplication table for 1.12(=72).
Babylonia, 19th c. BC

Multiplication Tables

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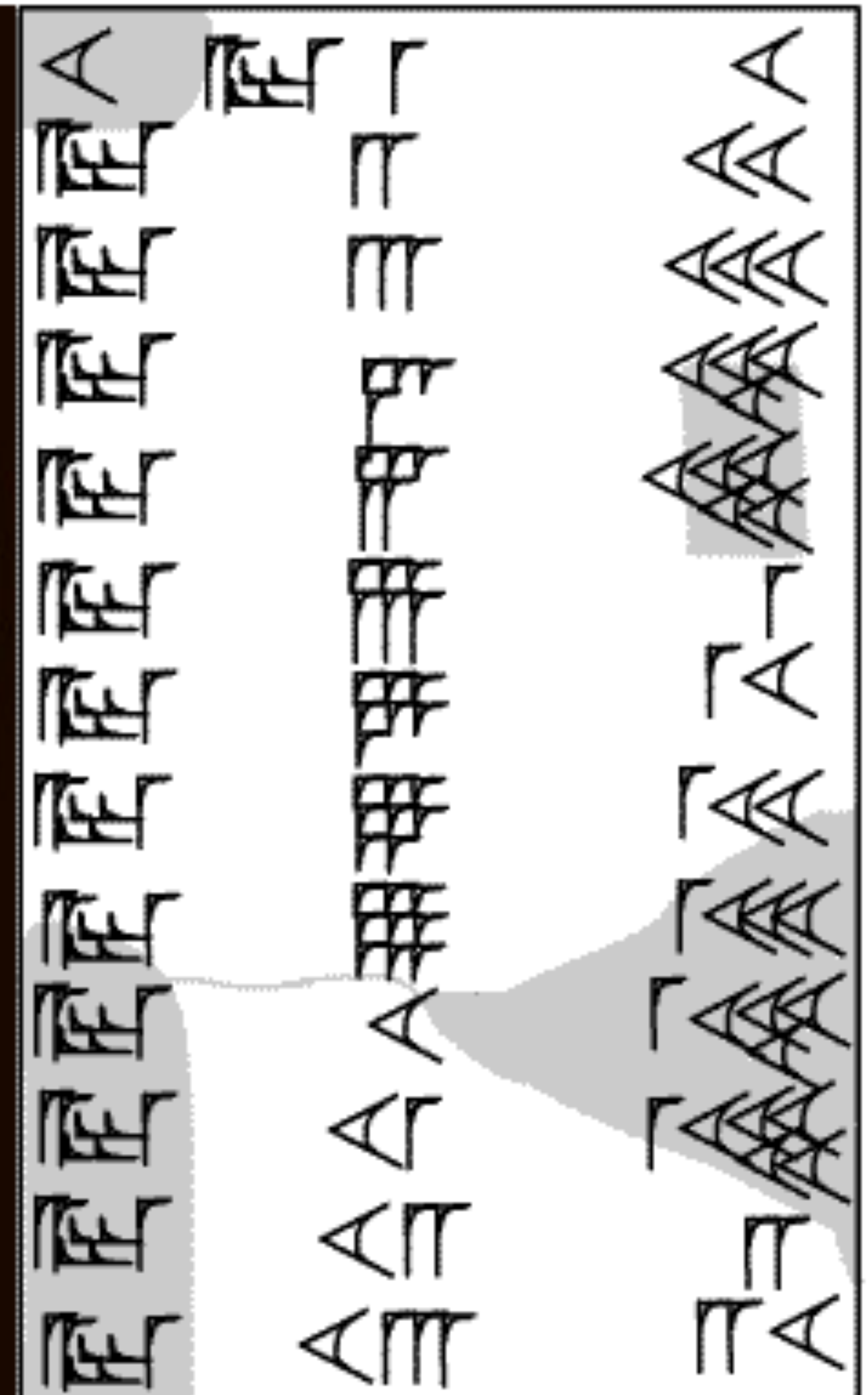
Multiplication Tables

- Aft … on
  an … gers,
  or … y
  mu
- Firs … old
  Ba



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- First known *decimal* multiplication table is a 2,300 year old Chinese multiplication table.

Multiplication Tables

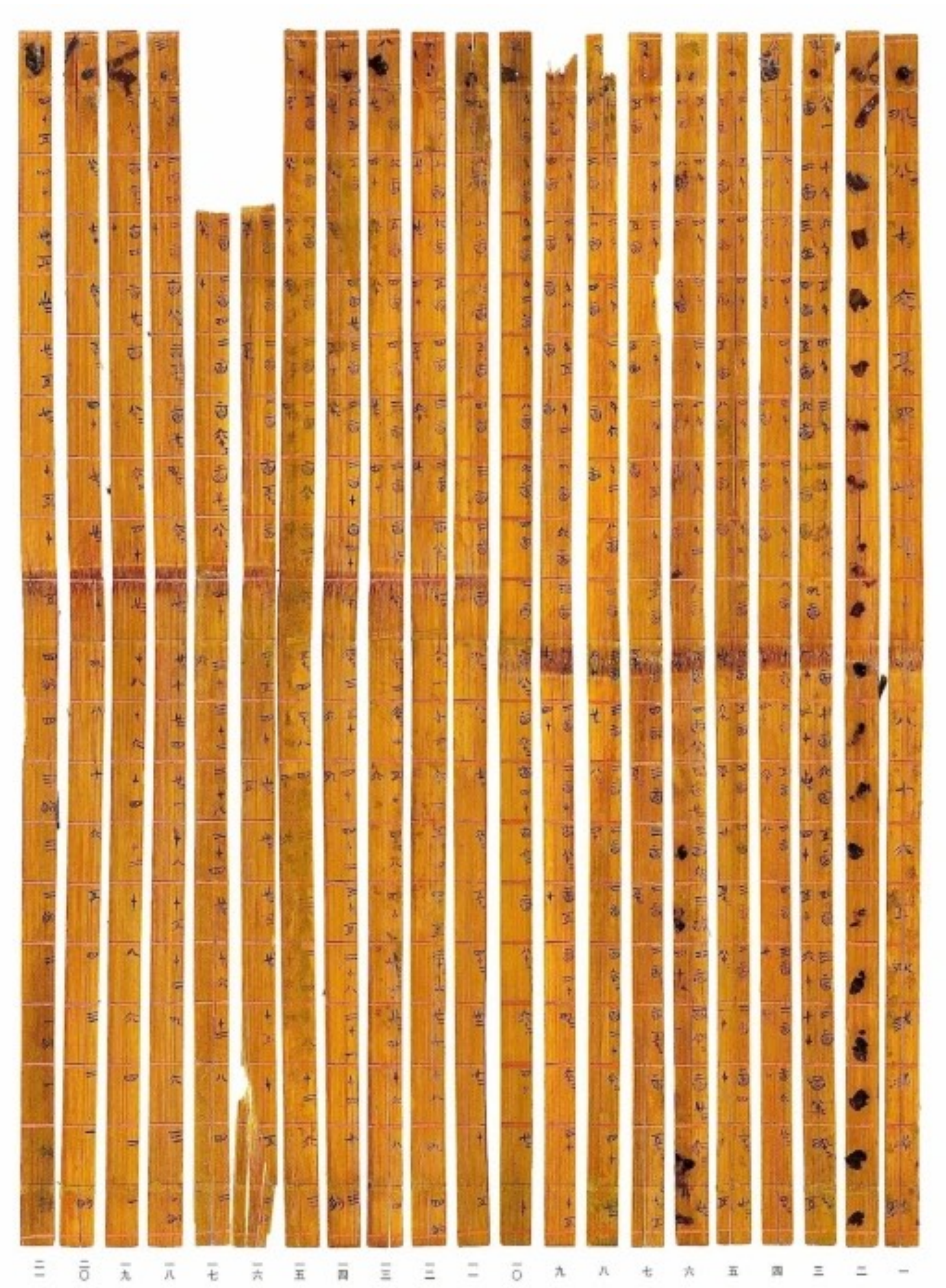
- After counting comes arithmetic. Simple addition and subtraction could be done in head, on fingers, or with simple tools. Multiplication was probably much harder.

- First known multiplication table is a 4,000 year old Babylonian (base 60) multiplication table.

- First known *decimal* multiplication table is a 2,300 year old Chinese multiplication table.

- It was discovered in 2009!

Tsinghua Bamboo Strips



Tsinghua Bamboo Slips

- An ancient table, written on bamboo strips and found in China, contains what is essentially a multiplication table.

- With a little extra work, one can use it to compute the product of any two integers or half integers from 0.5 to 99.5.
  That is, you can multiply together any two numbers from this list: 0.5, 1, 1.5, 2, 2.5, … , 99.5.

- This is the first known decimal multiplication table.

Tsinghua Bamboo Slips

- This multiplication table can compute the product of any two numbers from the set {0.5,1,1.5,2,2.5,3,3.5,…,99.5}. How?

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4\frac{1}{2}$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3\frac{1}{2}$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2\frac{1}{2}$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$

Think Like A
Math Historian

Tsinghua Bamboo Slips

- This multiplication table can compute the product of any two numbers from the set {0.5,1,1.5,2,2.5,3,3.5,…,99.5}. How?

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
4½	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
3½	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
2½	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
1½	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
½	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
¼	½	1	1½	2	2½	3	3½	4	4½	5	10	15	20	25	30	35	40	45	½

Tsinghua Bamboo Slips

- Example: $24 \cdot 36.5 = ?$

$1/2$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4\frac{1}{2}$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3\frac{1}{2}$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2\frac{1}{2}$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$

Tsinghua Bamboo Slips

- Example: $24 \cdot 36.5 = ?$

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
4½	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
3½	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
2½	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
1½	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
½	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
¼	½	1	1½	2	2½	3	3½	4	4½	5	10	15	20	25	30	35	40	45	½

Tsinghua Bamboo Slips

- Example: $24 \cdot 36.5 = ?$

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4\frac{1}{2}$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3\frac{1}{2}$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2\frac{1}{2}$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$

Tsinghua Bamboo Slips

- Example: $24 \cdot 36.5 = ?$

$1/2$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4\frac{1}{2}$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3\frac{1}{2}$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2\frac{1}{2}$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$

Tsinghua Bamboo Slips

- Example: $24 \cdot 36.5 = ?$

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
4 1/2	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
3 1/2	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
2 1/2	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
1 1/2	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
1/4	1/2	1	1 1/2	2	2 1/2	3	3 1/2	4	4 1/2	5	10	15	20	25	30	35	40	45	1/2

Tsinghua Bamboo Slips

- Example: $24 \cdot 36.5 = ?$

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4\frac{1}{2}$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3\frac{1}{2}$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2\frac{1}{2}$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$

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- Example: $24 \cdot 36.5 = ?$

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4\frac{1}{2}$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3\frac{1}{2}$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2\frac{1}{2}$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$

$$24 \cdot 36.5 = (20 + 4) \cdot (30 + 6 + 0.5)$$

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- Example: $24 \cdot 36.5 = ?$

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4\frac{1}{2}$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3\frac{1}{2}$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2\frac{1}{2}$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$

$$24 \cdot 36.5 = (20 + 4) \cdot (30 + 6 + 0.5)$$

Tsinghua Bamboo Slips

- Example: $24 \cdot 36.5 = ?$

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4^1/_2$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3^1/_2$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2^1/_2$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1^1/_2$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$^1/_2$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$^1/_4$	$^1/_2$	1	$1^1/_2$	2	$2^1/_2$	3	$3^1/_2$	4	$4^1/_2$	5	10	15	20	25	30	35	40	45	$^1/_2$

$$24 \cdot 36.5 = (20 + 4) \cdot (30 + 6 + 0.5)$$

Tsinghua Bamboo Slips

- Example: $24 \cdot 36.5 = ?$

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4\frac{1}{2}$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3\frac{1}{2}$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2\frac{1}{2}$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$

$$24 \cdot 36.5 = (20 + 4) \cdot (30 + 6 + 0.5)$$

Tsinghua Bamboo Slips

- Example: $24 \cdot 36.5 = ?$

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4\frac{1}{2}$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3\frac{1}{2}$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2\frac{1}{2}$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$

$$24 \cdot 36.5 = (20 + 4) \cdot (30 + 6 + 0.5)$$

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- Example: $24 \cdot 36.5 = ?$

$1/2$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4\frac{1}{2}$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3\frac{1}{2}$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2\frac{1}{2}$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$

$$24 \cdot 36.5 = (20 + 4) \cdot (30 + 6 + 0.5)$$

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- Example: $24 \cdot 36.5 = ?$

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4\frac{1}{2}$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3\frac{1}{2}$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2\frac{1}{2}$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$

$$24 \cdot 36.5 = (20 + 4) \cdot (30 + 6 + 0.5)$$
$$= 20 \cdot 30 + 20 \cdot 6 + 20 \cdot 0.5 + 4 \cdot 30 + 4 \cdot 6 + 4 \cdot 0.5$$

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- Example: $24 \cdot 36.5 = ?$

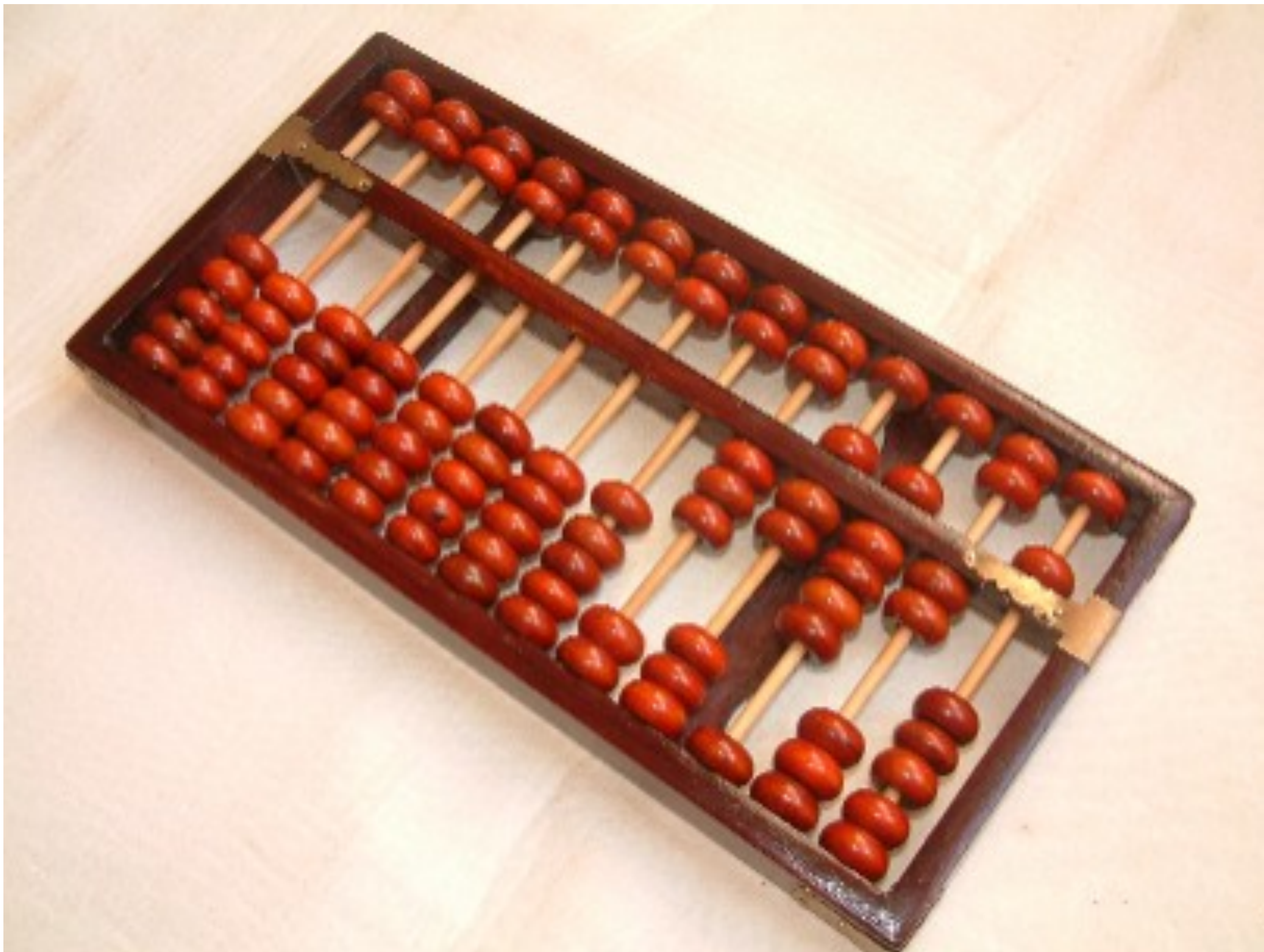
1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4\frac{1}{2}$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3\frac{1}{2}$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2\frac{1}{2}$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$

$$24 \cdot 36.5 = (20 + 4) \cdot (30 + 6 + 0.5)$$
$$= 20 \cdot 30 + 20 \cdot 6 + 20 \cdot 0.5 + 4 \cdot 30 + 4 \cdot 6 + 4 \cdot 0.5$$

= Look up on table, then add together

Ancient Calculators

Abacus



The Incan Empire

The Incan Empire

The Incan Empire

The Incan Empire

The Incan Empire

- The Incans were a great but relatively short-lived empire.

The Incan Empire

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- They did not have a written language.

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- How do you record numbers and do math without writing?

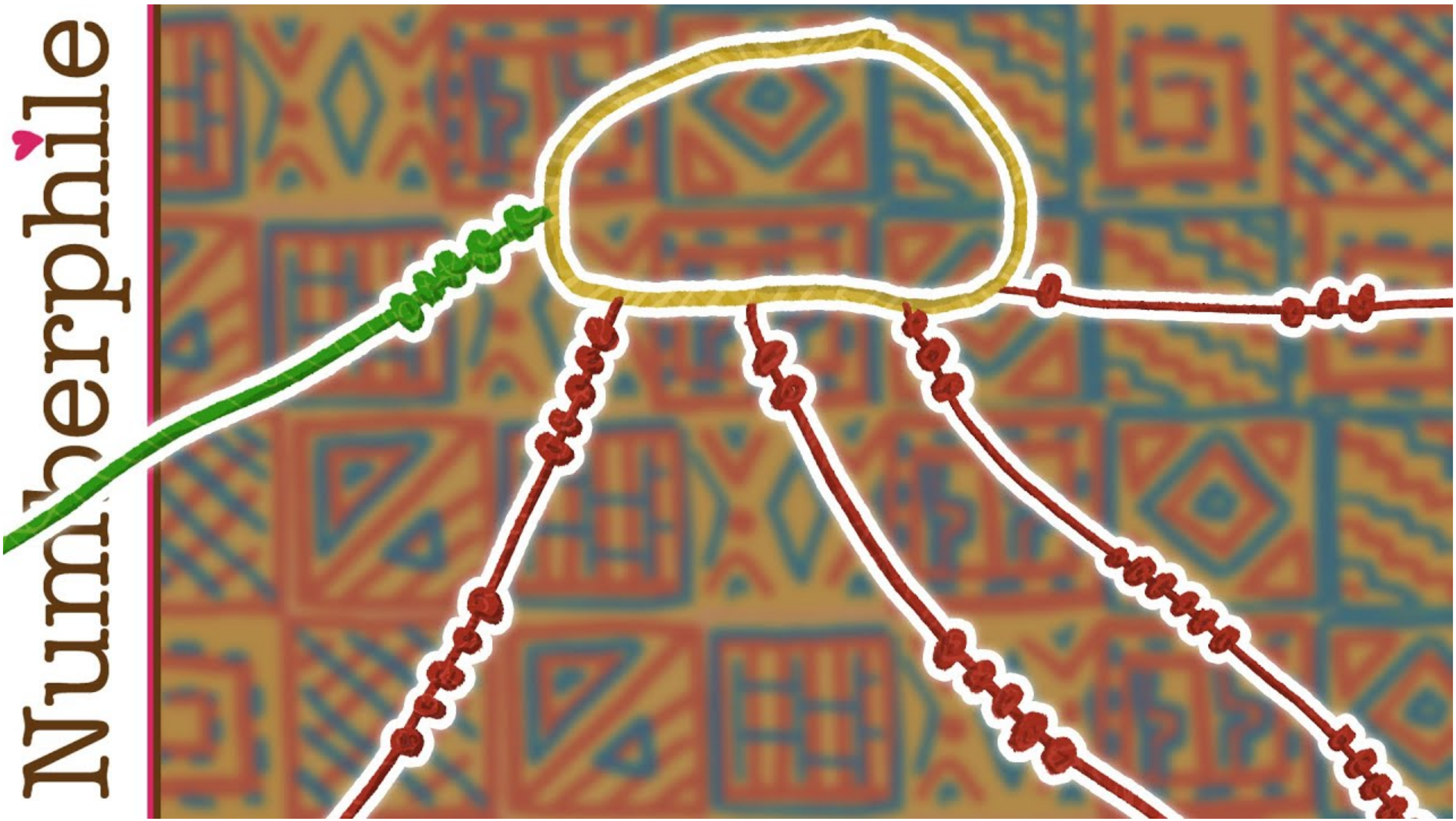
The Incan Empire

- The Incans were a great but relatively short-lived empire.
- They did not have a written language.
- How do you record numbers and do math without writing?
- The Incans used what is called a *quipu*.

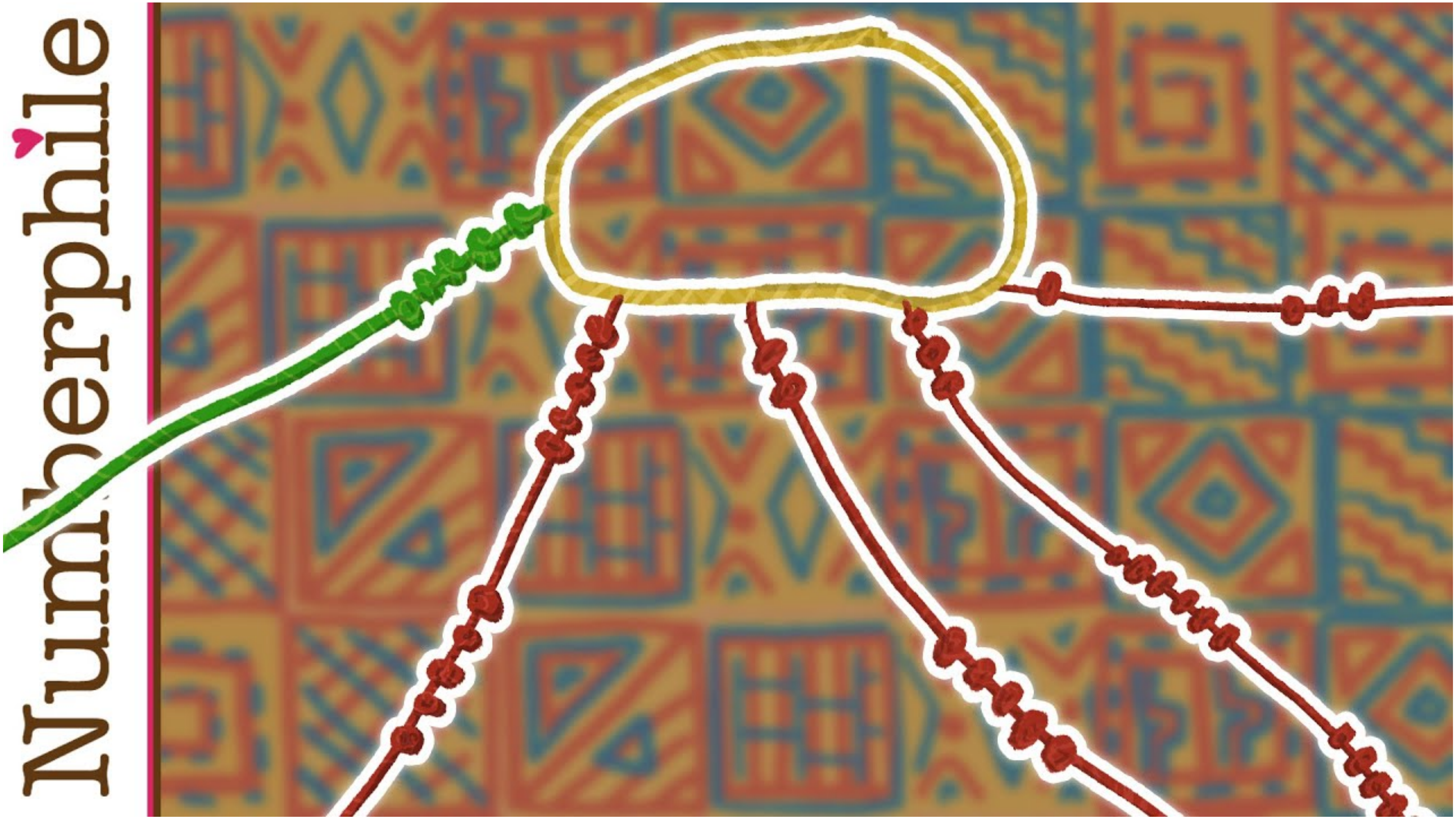
Devices

Devices

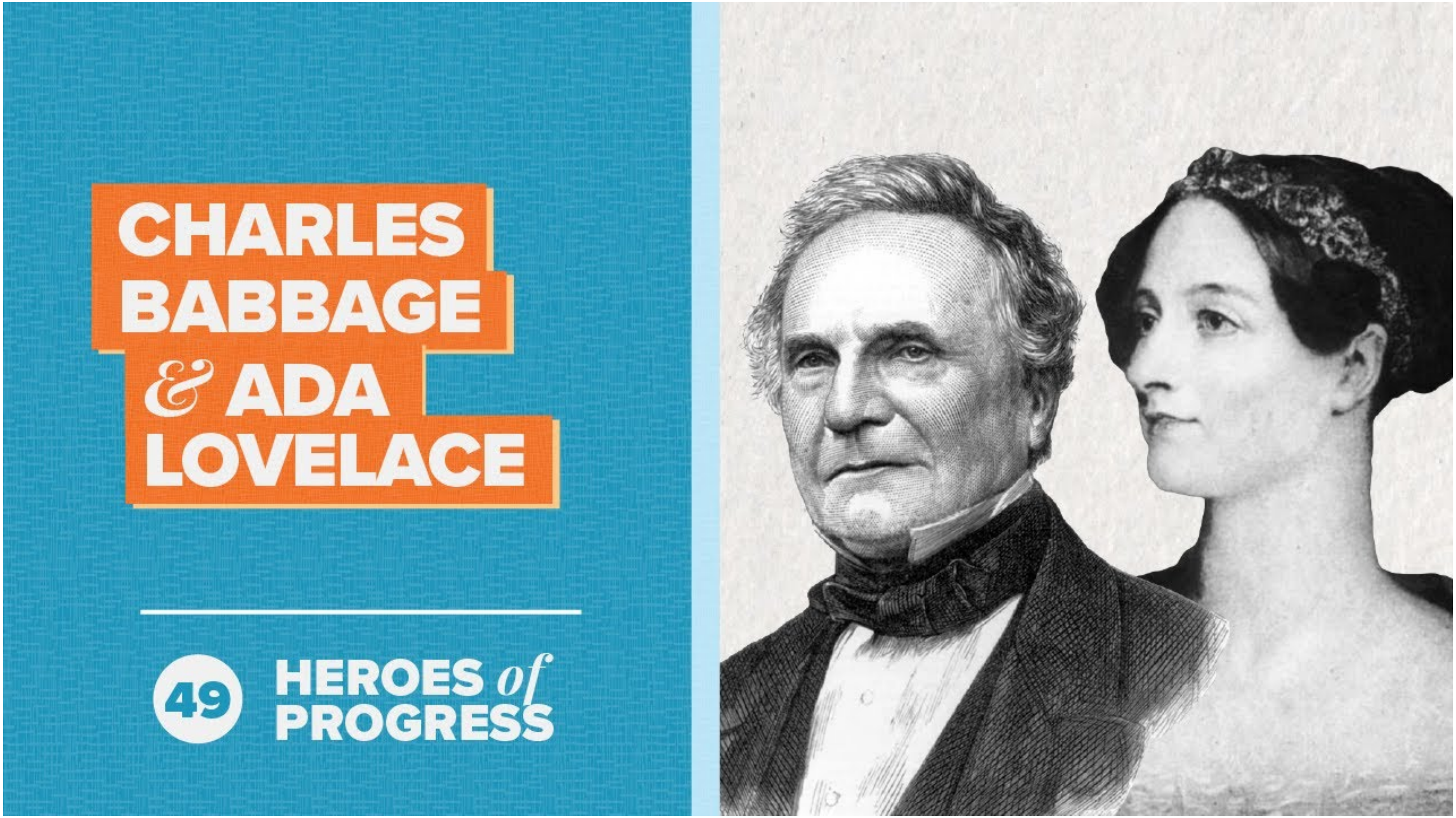
Devices



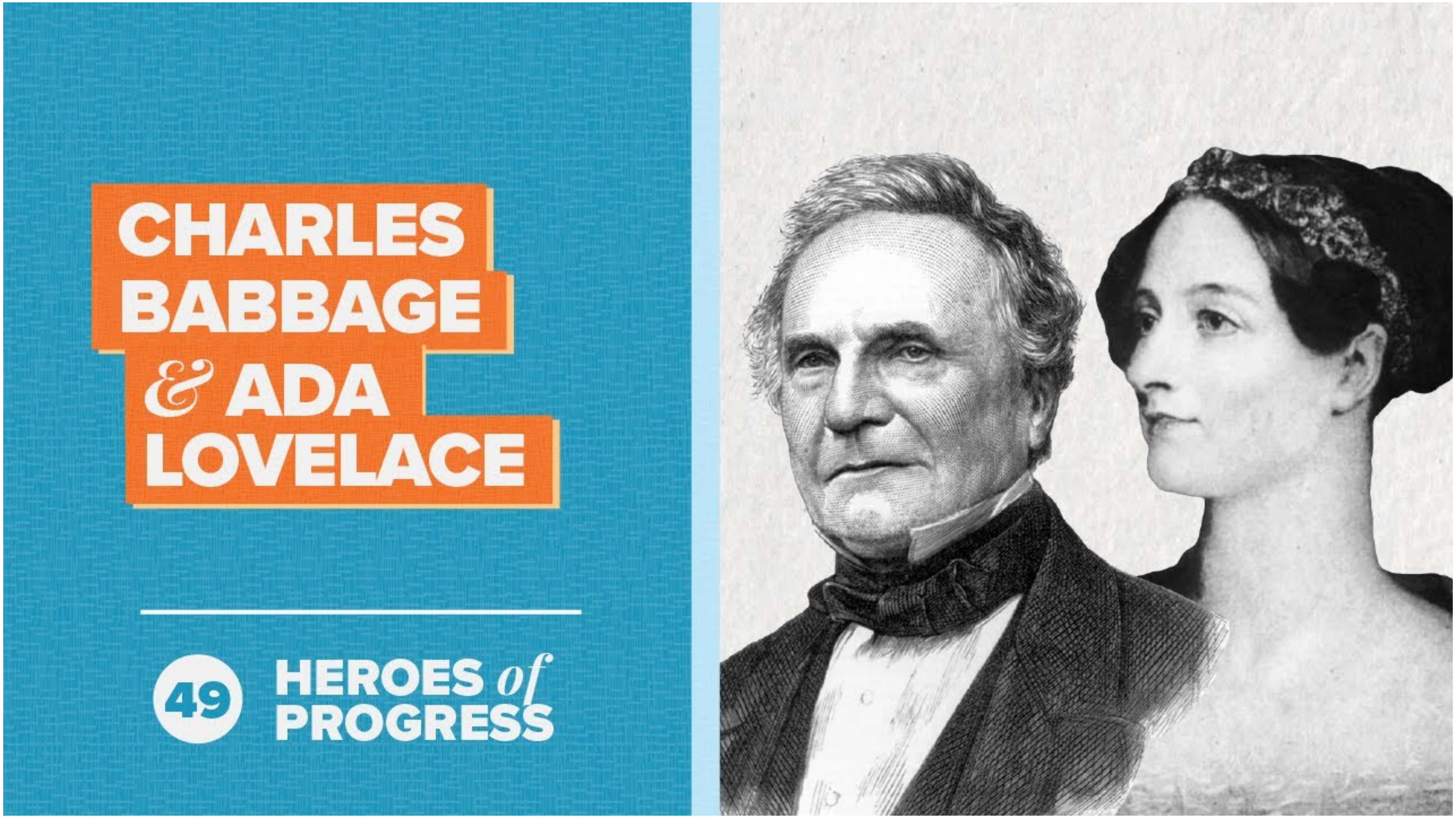
Devices



Aftermath: Computers



Aftermath: Computers

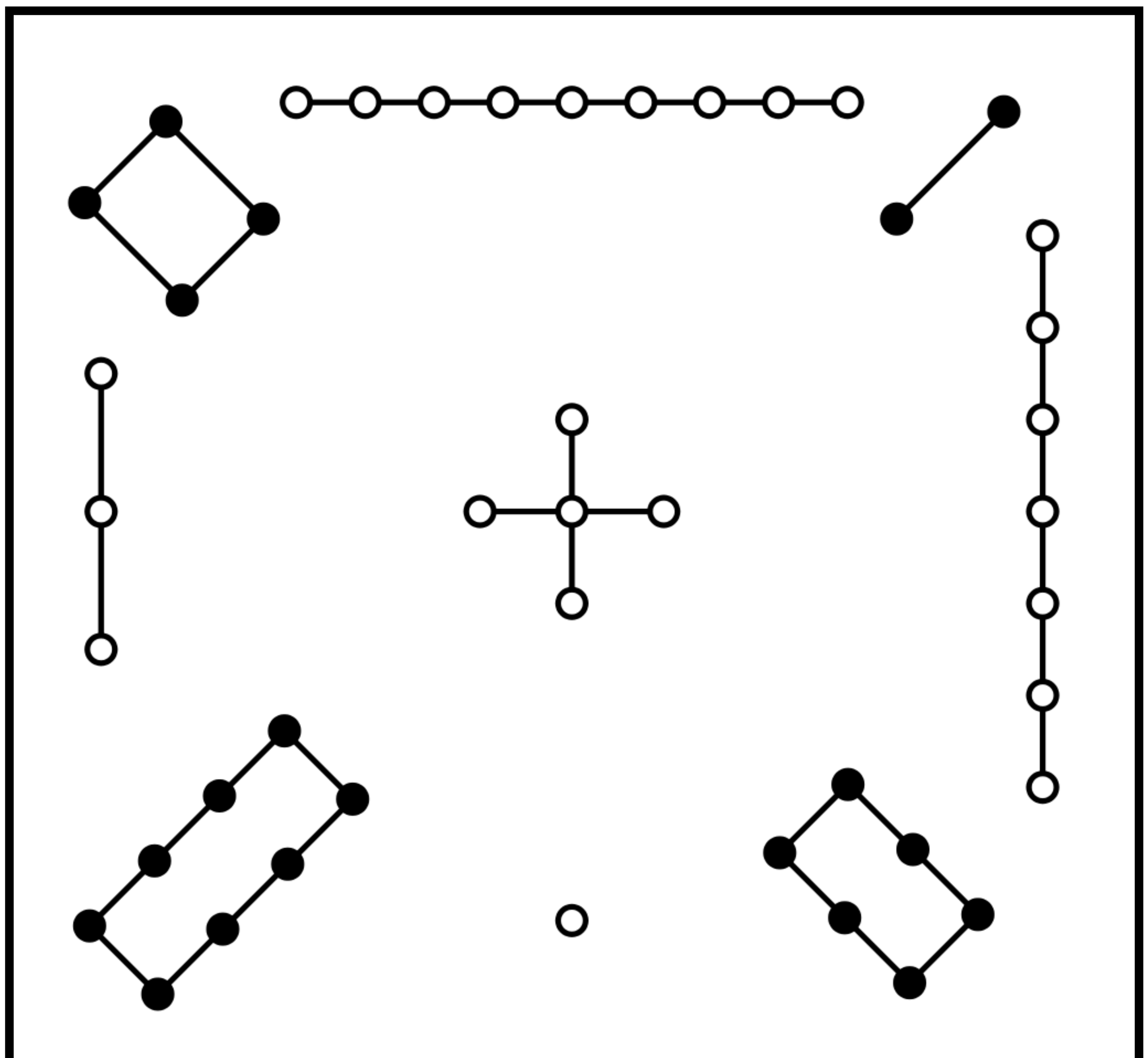


Think Like A Math Historian

Think Like a Math Historian

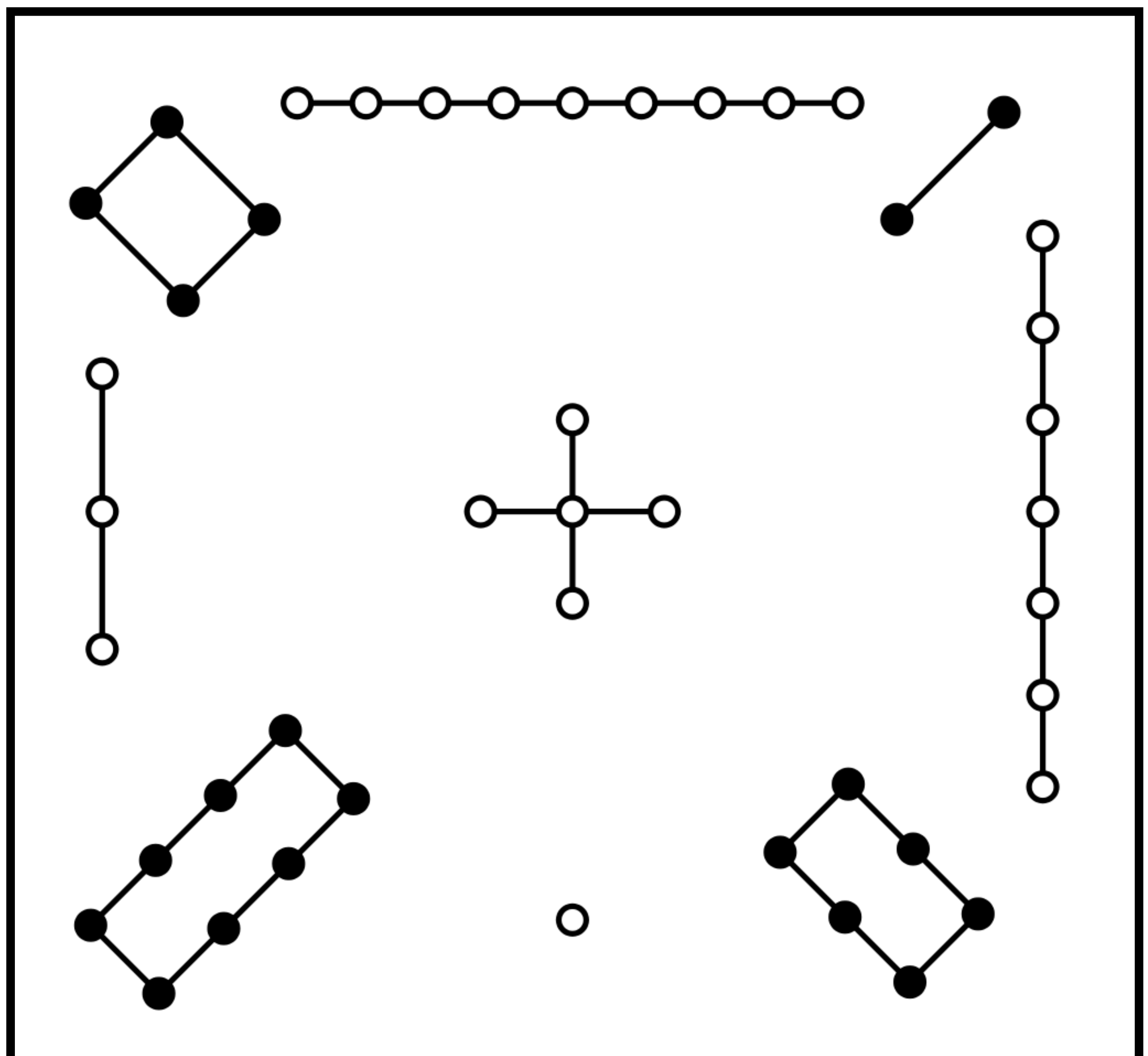
Think Like a Math Historian

- What does this mean?



Think Like a Math Historian

- Hint: Focus on how many dots are in each group.



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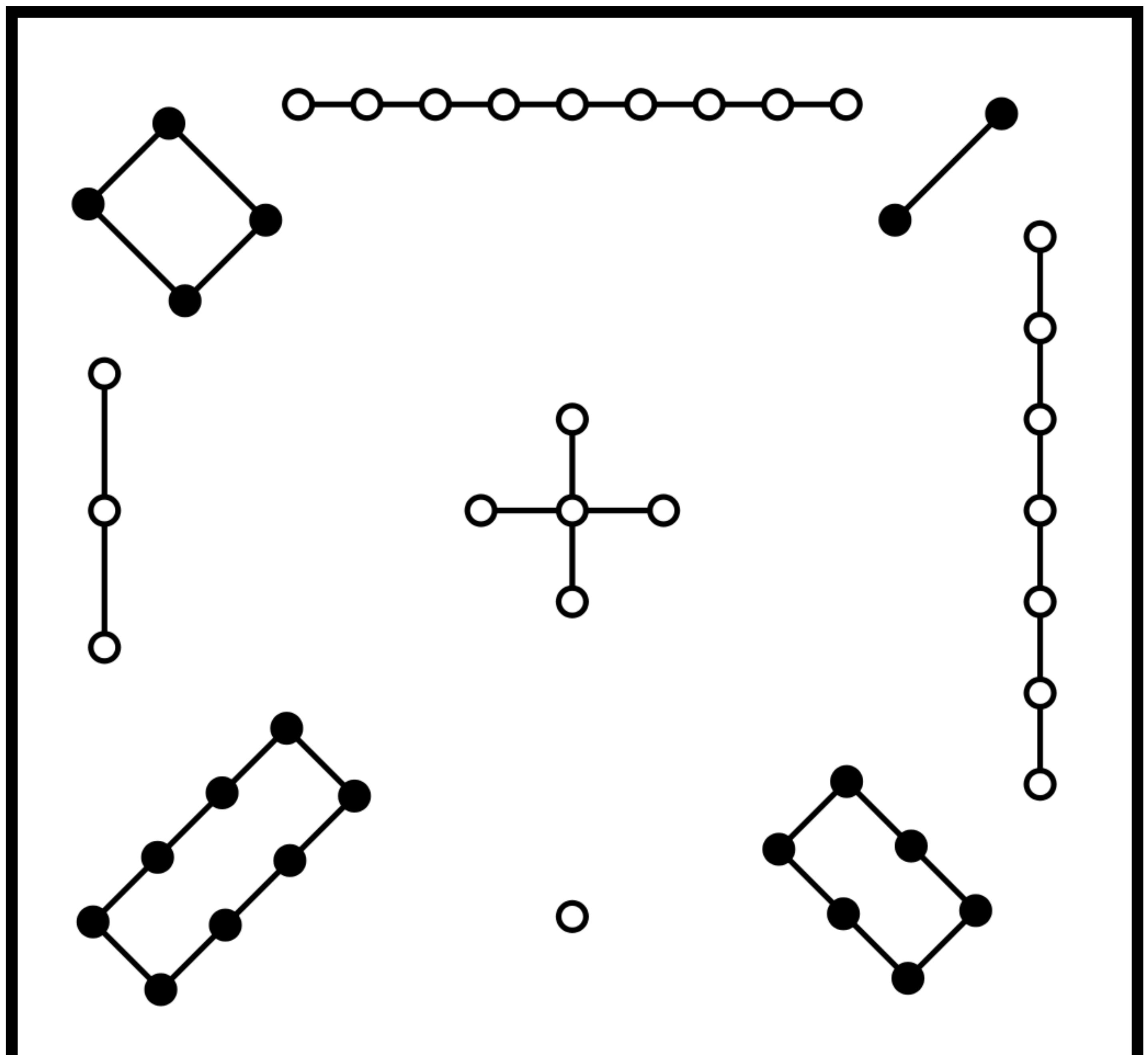

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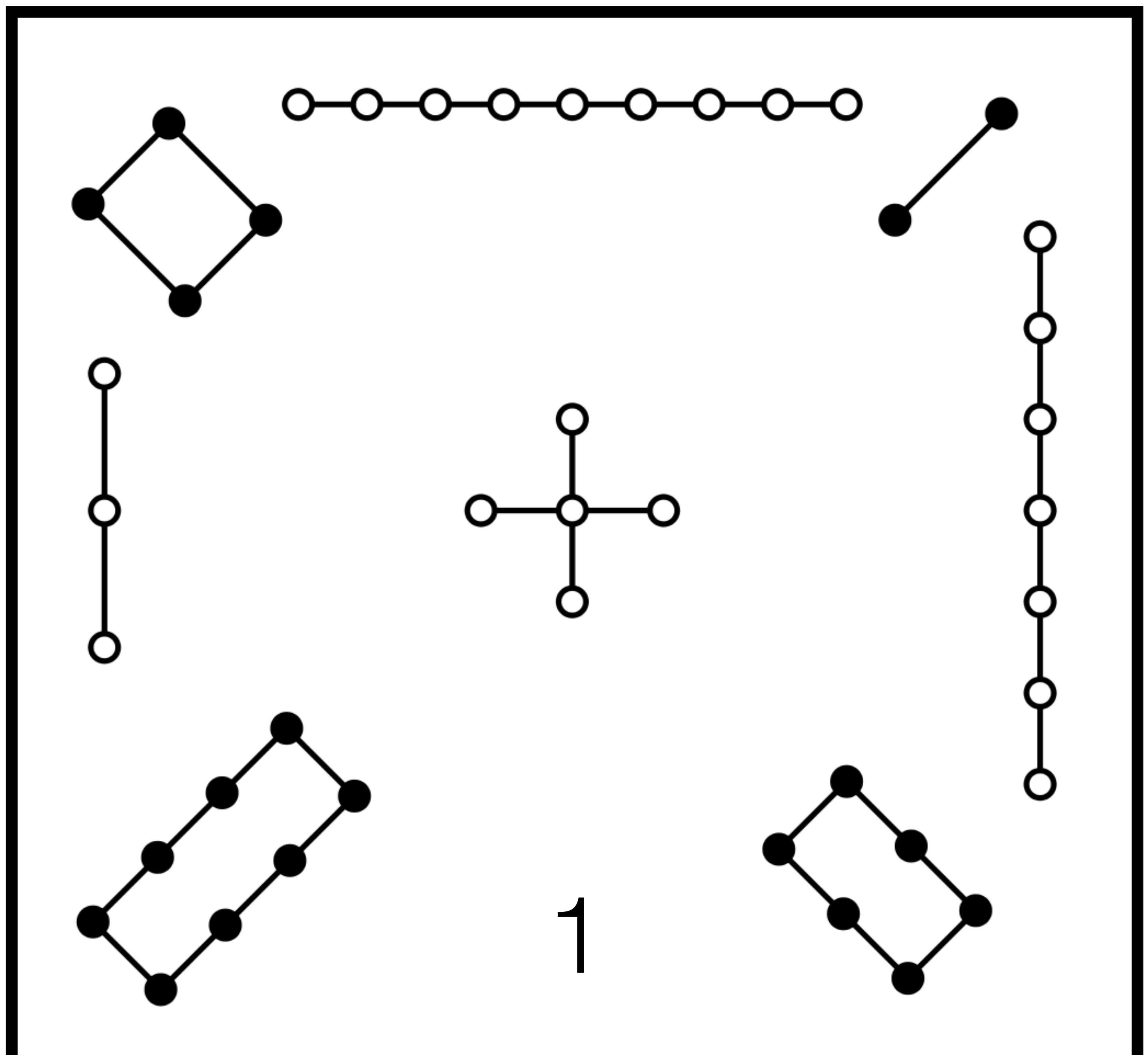
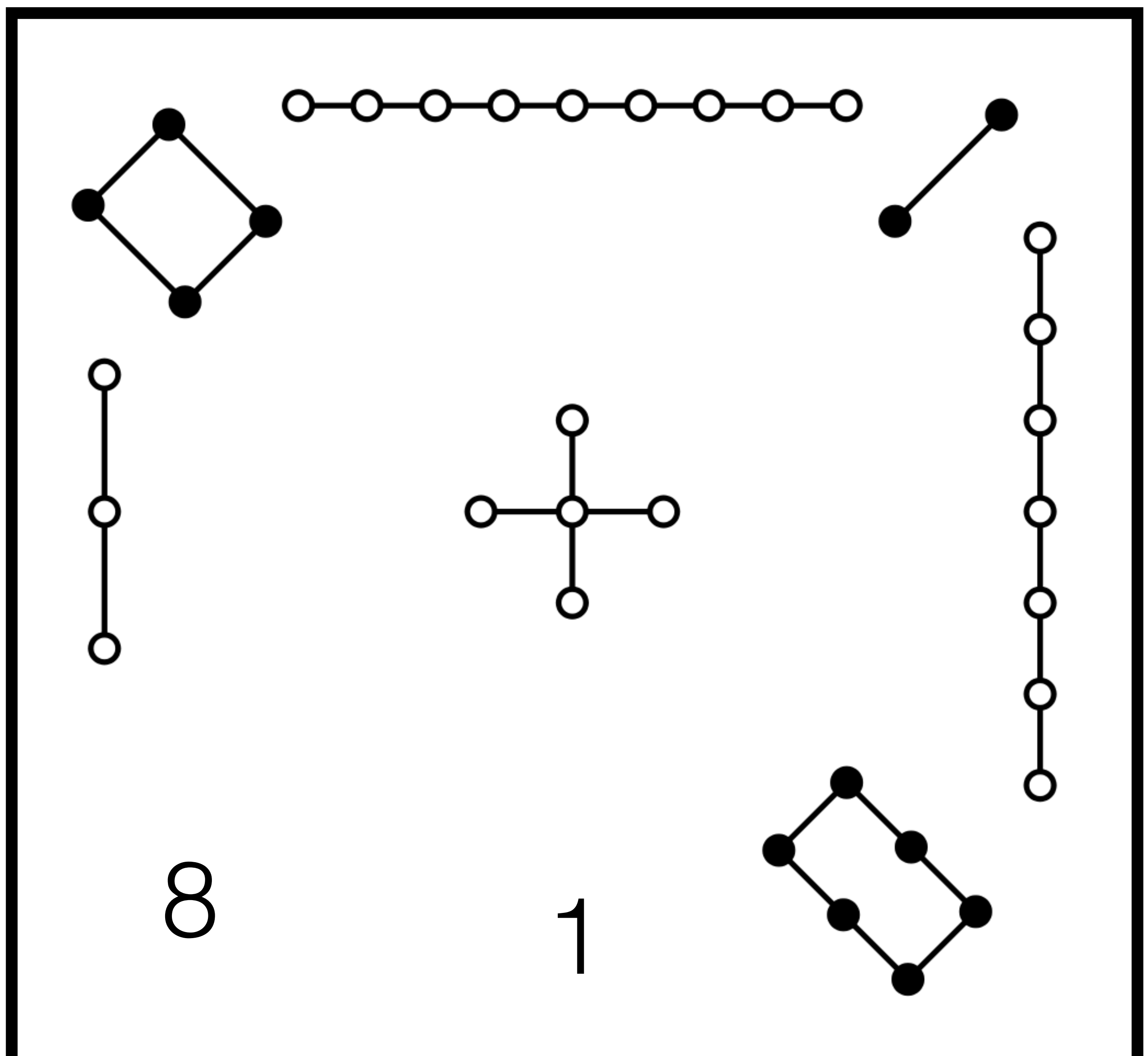


Think Like a Math Historian

- Hint: Focus on how many dots are in each group.



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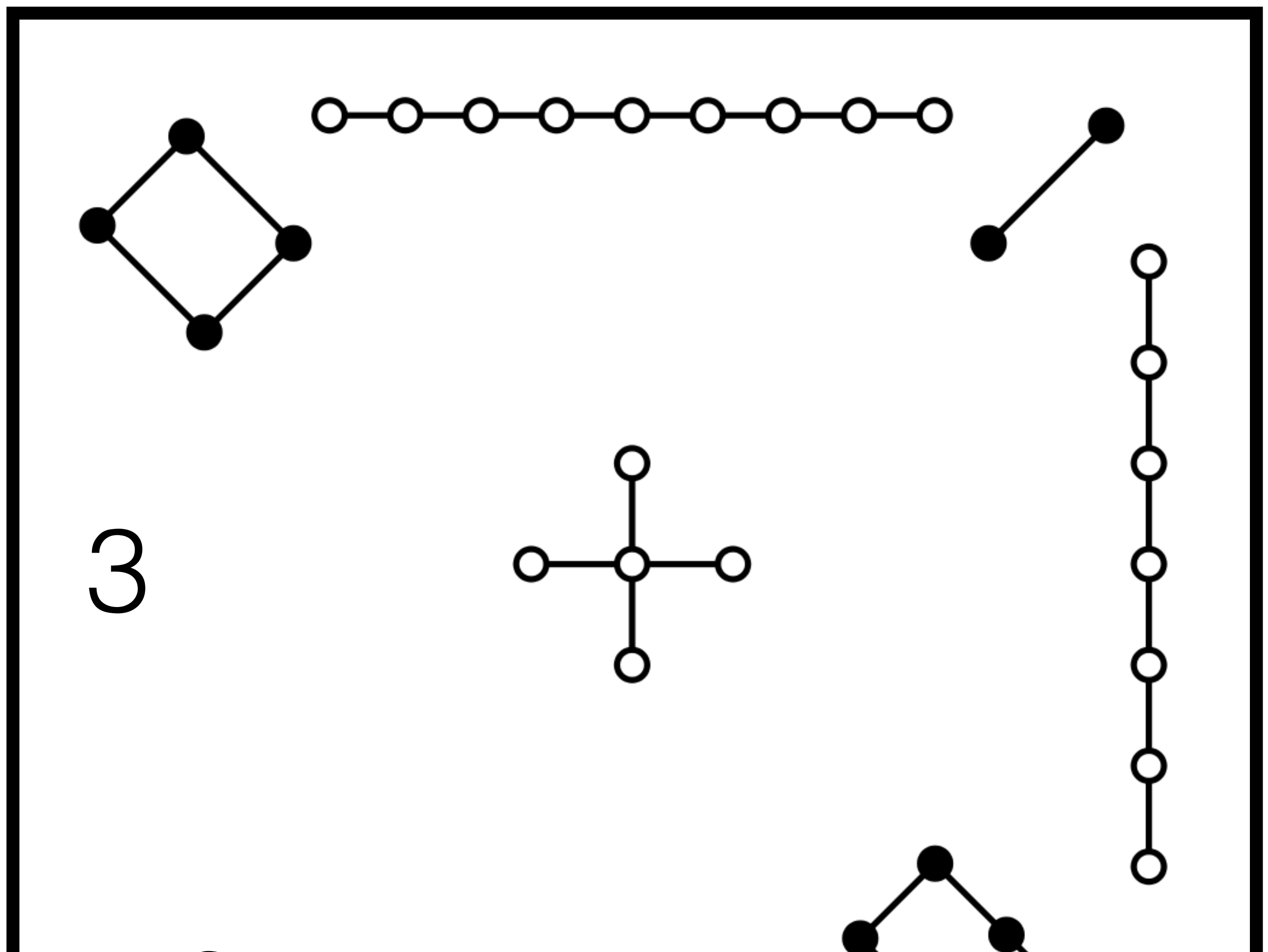

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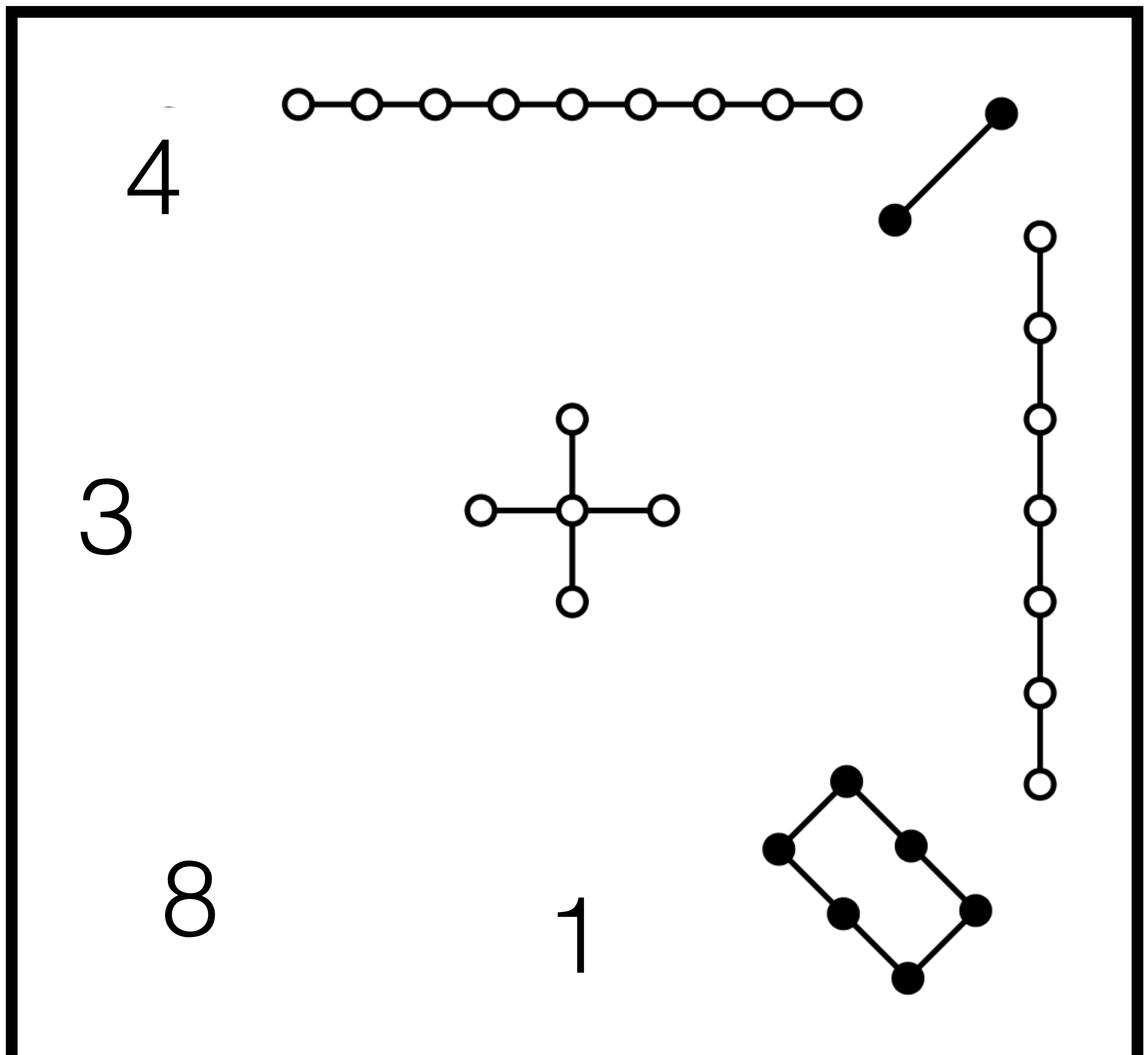

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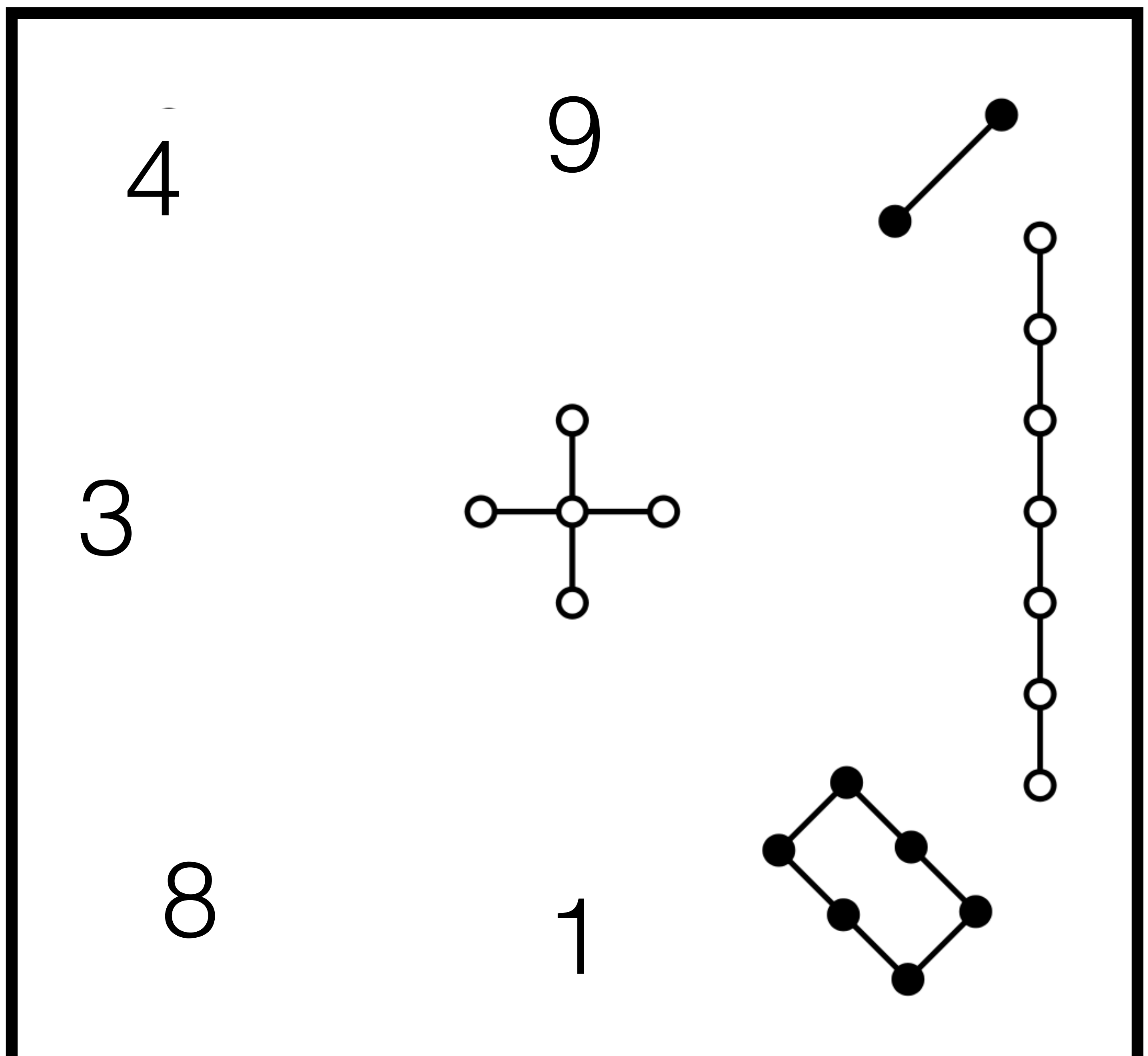

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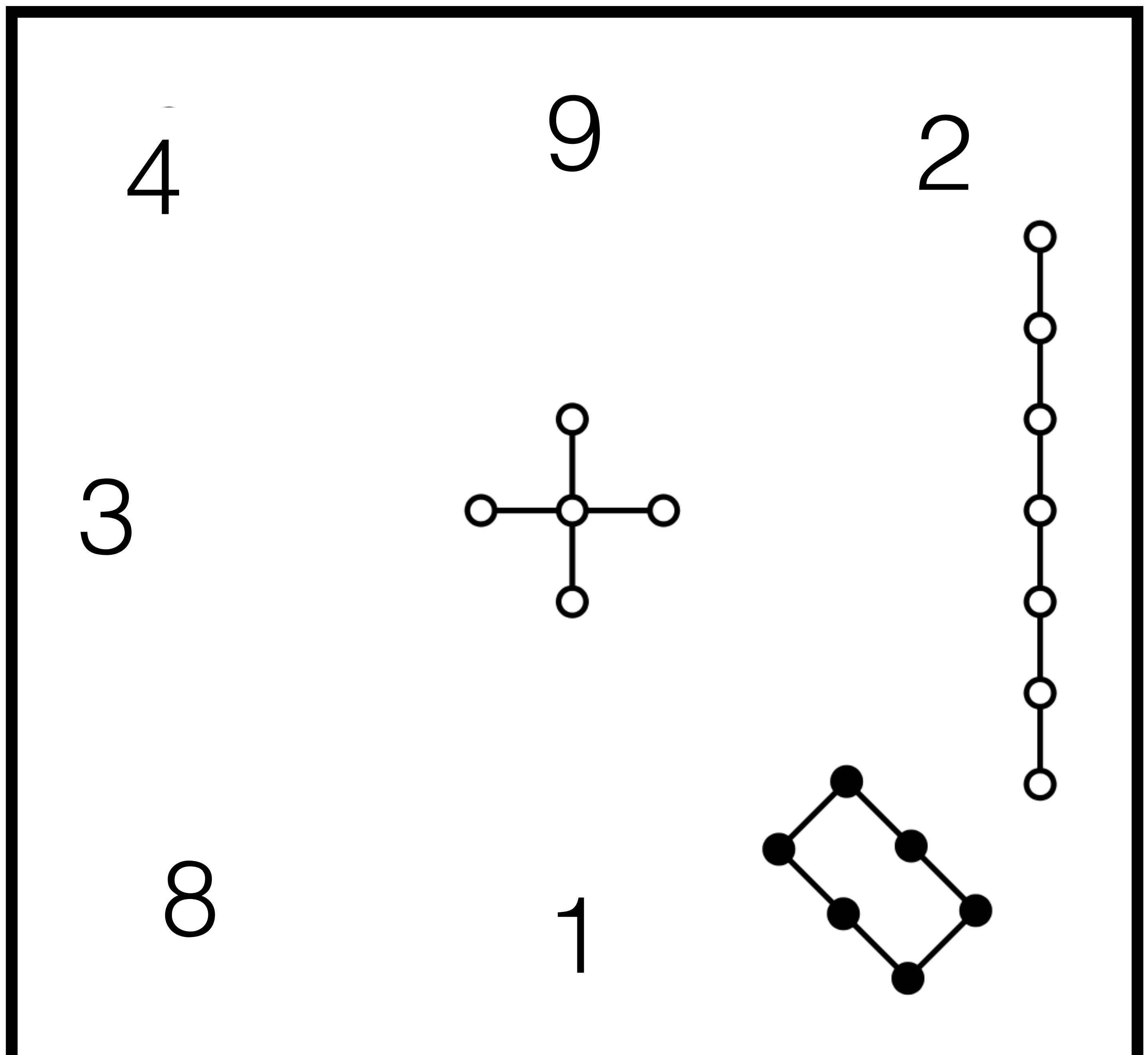

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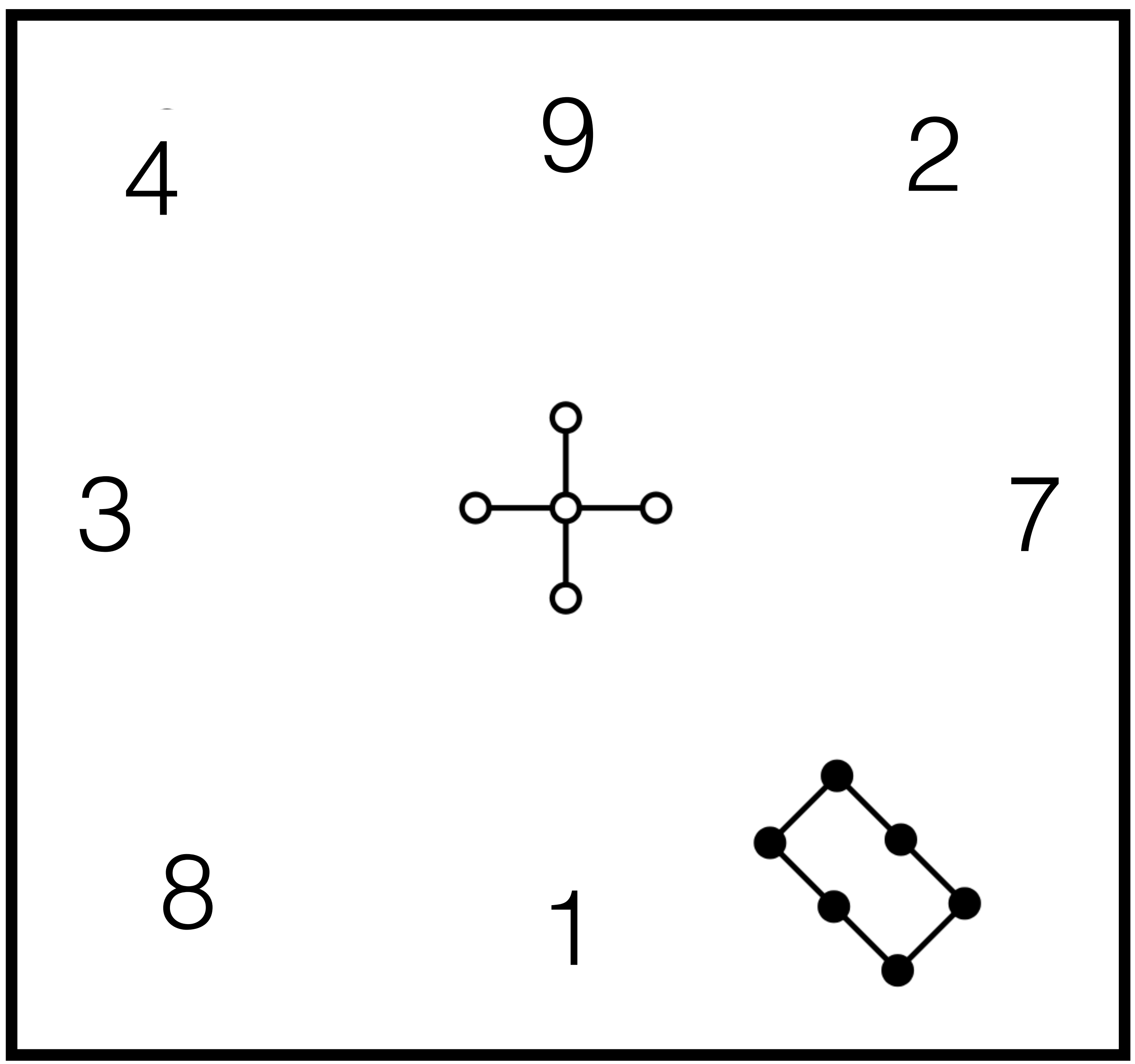

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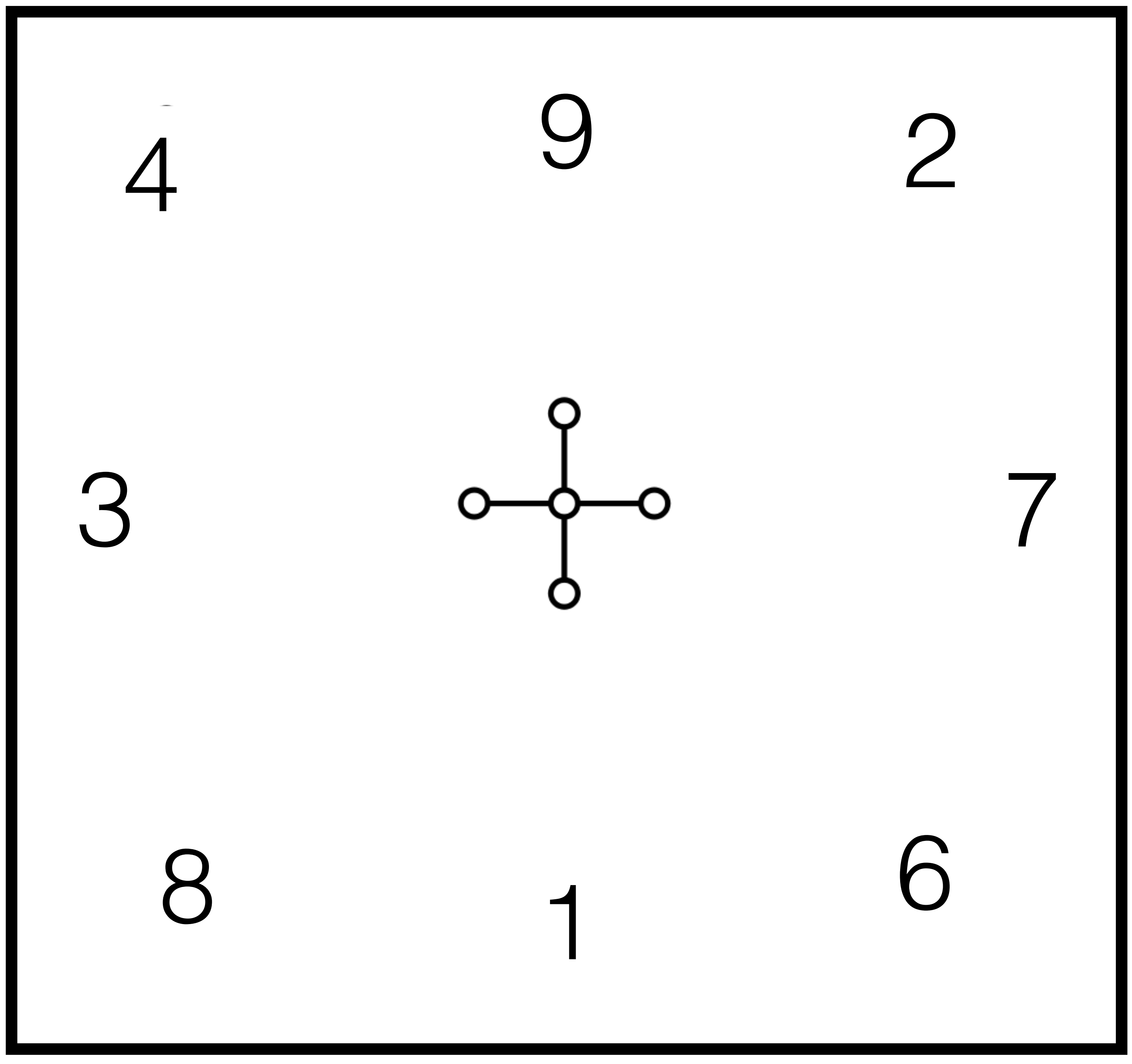
- Hint: Focus on how many dots are in each group.



Think Like a Math Historian

- Hint: Focus on how many dots are in each group.

4	9	2
3	5	7
8	1	6

Think Like a Math Historian

- Magic Square

4	9	2
3	5	7
8	1	6

Think Like a Math Historian

- Magic Square

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3	5	7
8	1	6

Think Like a Math Historian

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3	5	7
8	1	6

Think Like a Math Historian

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The Aftermath

The Aftermath

- Yang Hui, a much more modern Chinese mathematician (~1238-1298 AD) had a 3-step procedure to create this square.

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<u>Step 1:</u> Write numbers in a diamond.

The Aftermath

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<u>Step 1:</u> Write numbers in a diamond.

		1		
	4		2	
7		5		3
	8		6	
		9		

The Aftermath

- Yang Hui, a much more modern Chinese mathematician (~1238-1298 AD) had a 3-step procedure to create this square.

<u>Step 2:</u> Swap corners.

		1		
	4		2	
7		5		3
	8		6	
		9		

The Aftermath

- Yang Hui, a much more modern Chinese mathematician (~1238-1298 AD) had a 3-step procedure to create this square.

Step 2: Swap corners.

9

4 2

7 5 3

8 6

1

The Aftermath

- Yang Hui, a much more modern Chinese mathematician (~1238-1298 AD) had a 3-step procedure to create this square.

<u>Step 2:</u> Swap corners.

		9		
	4		2	
3		5		7
	8		6	
		1		

The Aftermath

- Yang Hui, a much more modern Chinese mathematician (~1238-1298 AD) had a 3-step procedure to create this square.

<u>Step 3:</u> Scrunch together.

		9		
	4		2	
3		5		7
	8		6	
		1		

The Aftermath

- Yang Hui, a much more modern Chinese mathematician (~1238-1298 AD) had a 3-step procedure to create this square.

<u>Step 3:</u> Scrunch together.

4	9	2
3	5	7
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<u>Step 3:</u> Scrunch together.

4	9	2
3	5	7
8	1	6

Hui also came up with a similar algorithm to construct a 4x4 magic square.

The Aftermath

- Yang Hui constructed a magic square of every size up to 10x10.

The Aftermath

- Yang Hui constructed a magic square of every size up to 10x10.

一	二十	二一	四十	四一	六十	六一	八十	八一	一百
九九	八二	七九	六二	五九	四二	三九	二二	十九	二
三	十八	二三	三八	四三	五八	六三	七八	八三	九八
九七	八四	七七	六四	五七	四四	三七	二四	十七	四
五	十六	二五	三六	四五	五六	六五	七六	八五	九六
九五	八六	七五	六六	五五	四六	三五	二六	十五	六
十四	七	三四	二七	五四	四七	七四	六七	九四	八七
八八	九三	六八	七三	四八	五三	二八	三三	八	十三
十二	九	三二	二九	五二	四九	七二	六九	九二	八九
九一	九十	七一	七十	五一	五十	三一	三十	十一	十

The Aftermath

- Yang Hui constructed a magic square of every size up to 10x10.

一	二十	二一	四十	四一	六十	六一	八十	八一	一百
九九	八二	七九	六二	五九	四二	三九	二二	十九	二
三	十八	二三	三八	四三	五八	六三	七八	八三	九八
九七	八四	七七	六四	五七	四四	三七	二四	十七	四
五	十六	二五	三六	四五	五六	六五	七六	八五	九六
九五	八六	七五	六六	五五	四六	三五	二六	十五	六
十四	七	三四	二七	五四	四七	七四	六七	九四	八七
八八	九三	六八	七三	四八	五三	二八	三三	八	十三
十二	九	三二	二九	五二	四九	七二	六九	九二	八九
九一	九十	七一	七十	五一	五十	三一	三十	十一	十

1	20	21	40	41	60	61	80	81	100
99	82	79	62	59	42	39	22	19	2
3	18	23	38	43	58	63	78	83	98
97	84	77	64	57	44	37	24	17	4
5	16	25	36	45	56	65	76	85	96
95	86	75	66	55	46	35	26	15	6
14	7	34	27	54	47	74	67	94	87
88	93	68	73	48	53	28	33	8	13
12	9	32	29	52	49	72	69	92	89
91	90	71	70	51	50	31	30	11	10

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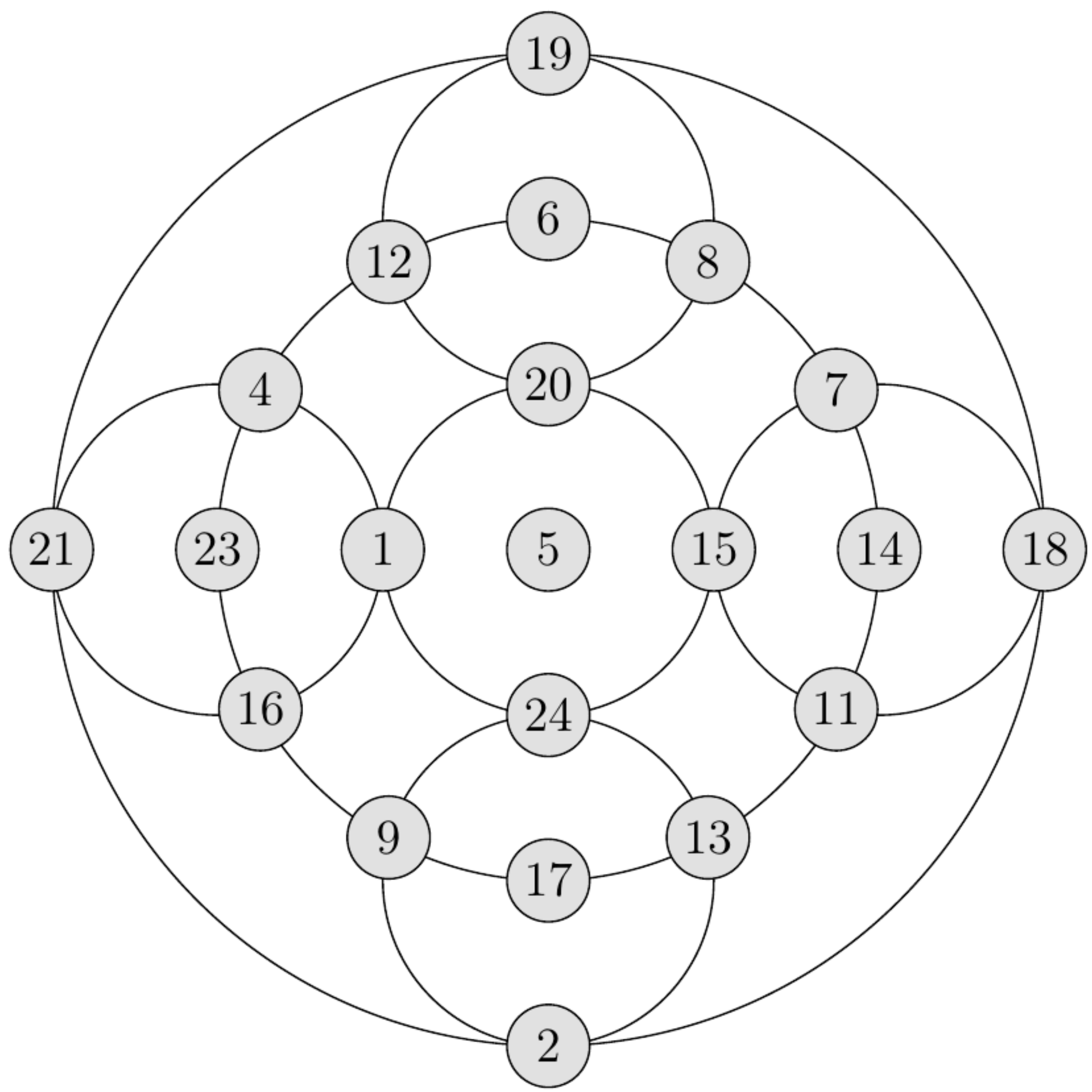

Algorithm to Construct Magic Squares

Shout-Outs!

- Crows

Counting Crows

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- Other animals may count too, to some extent. For example, crows.

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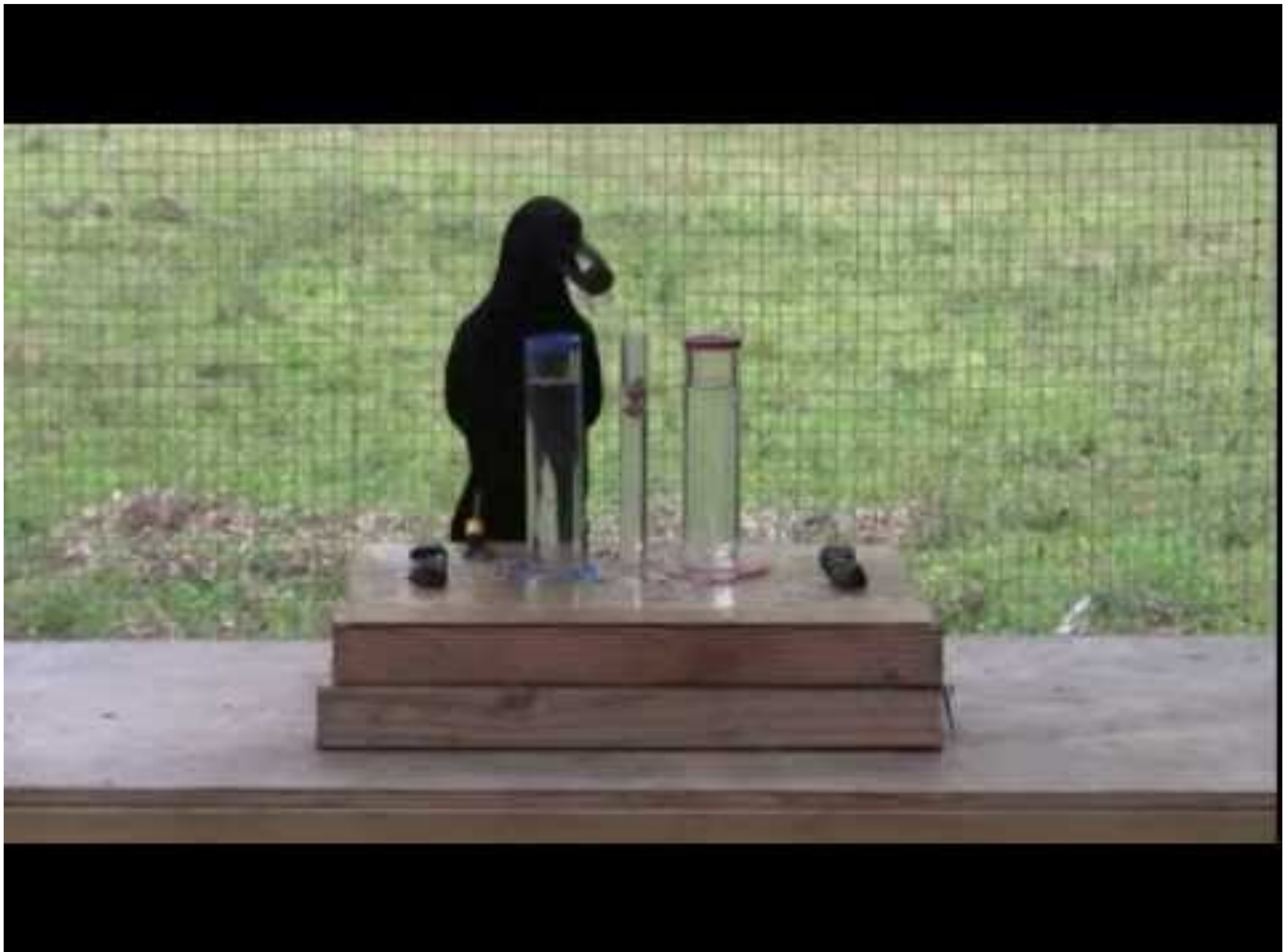
- Old story of a hunter hiding in a shelter.

Counting Crows



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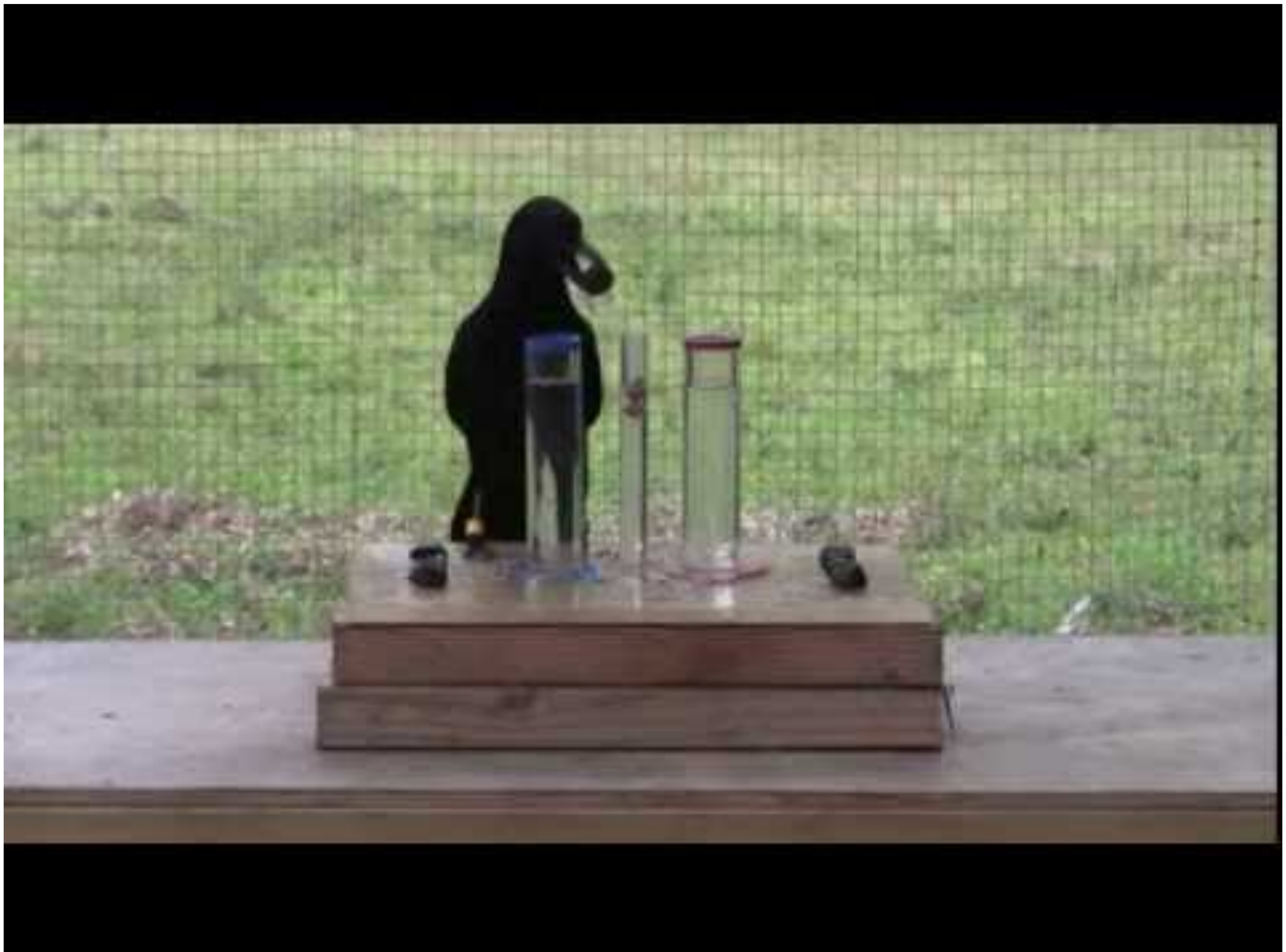

Shout-Outs!

- Napier’s Bones

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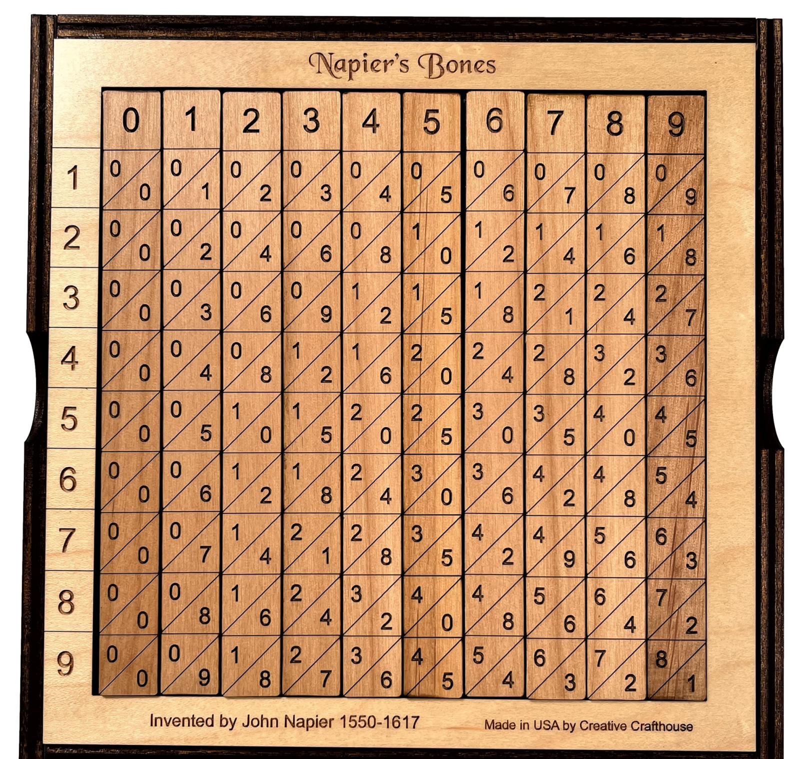

People's History

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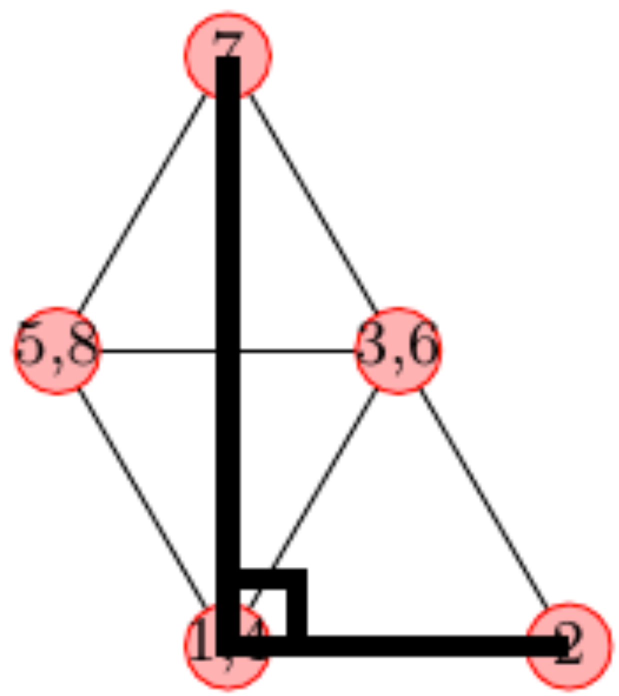
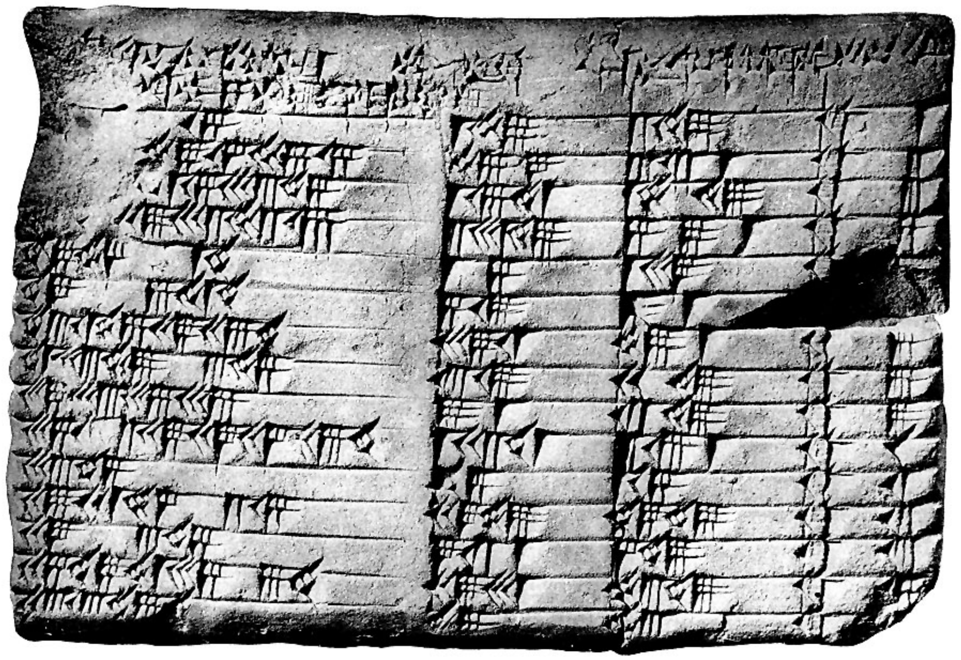
- Some Roman elites fought back in an effort to keep math opaque.



Chapter 2: Ancient Methods



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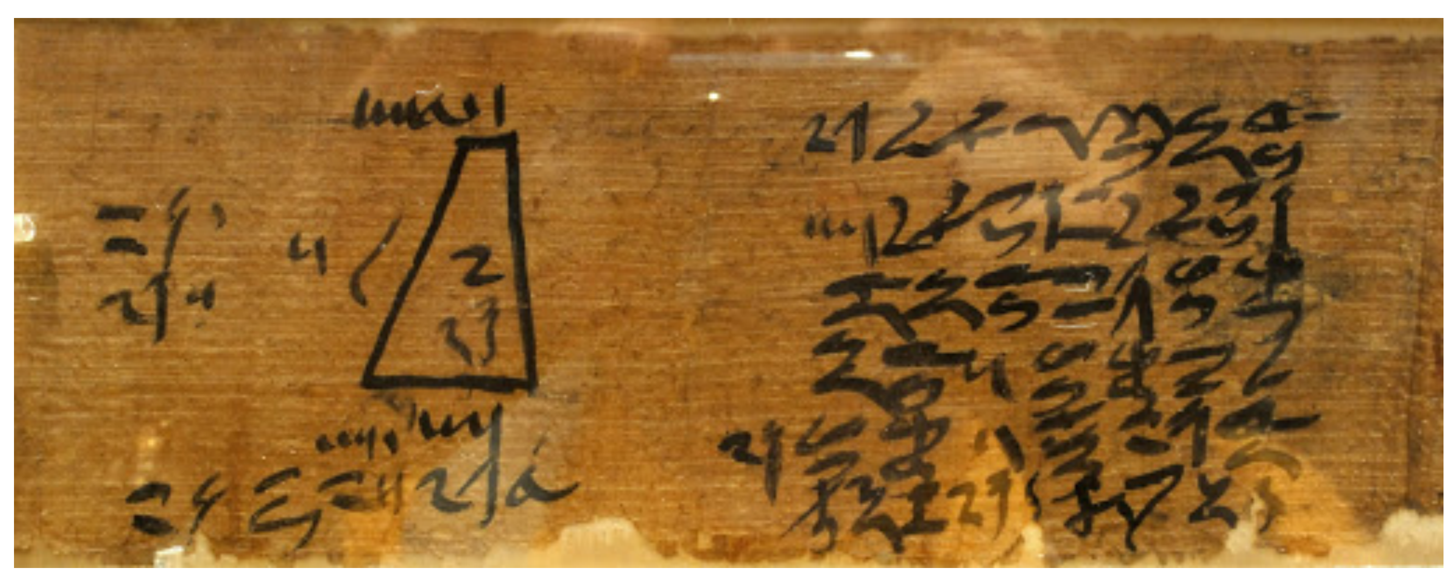
Egyptian and Babylonian Mathematics

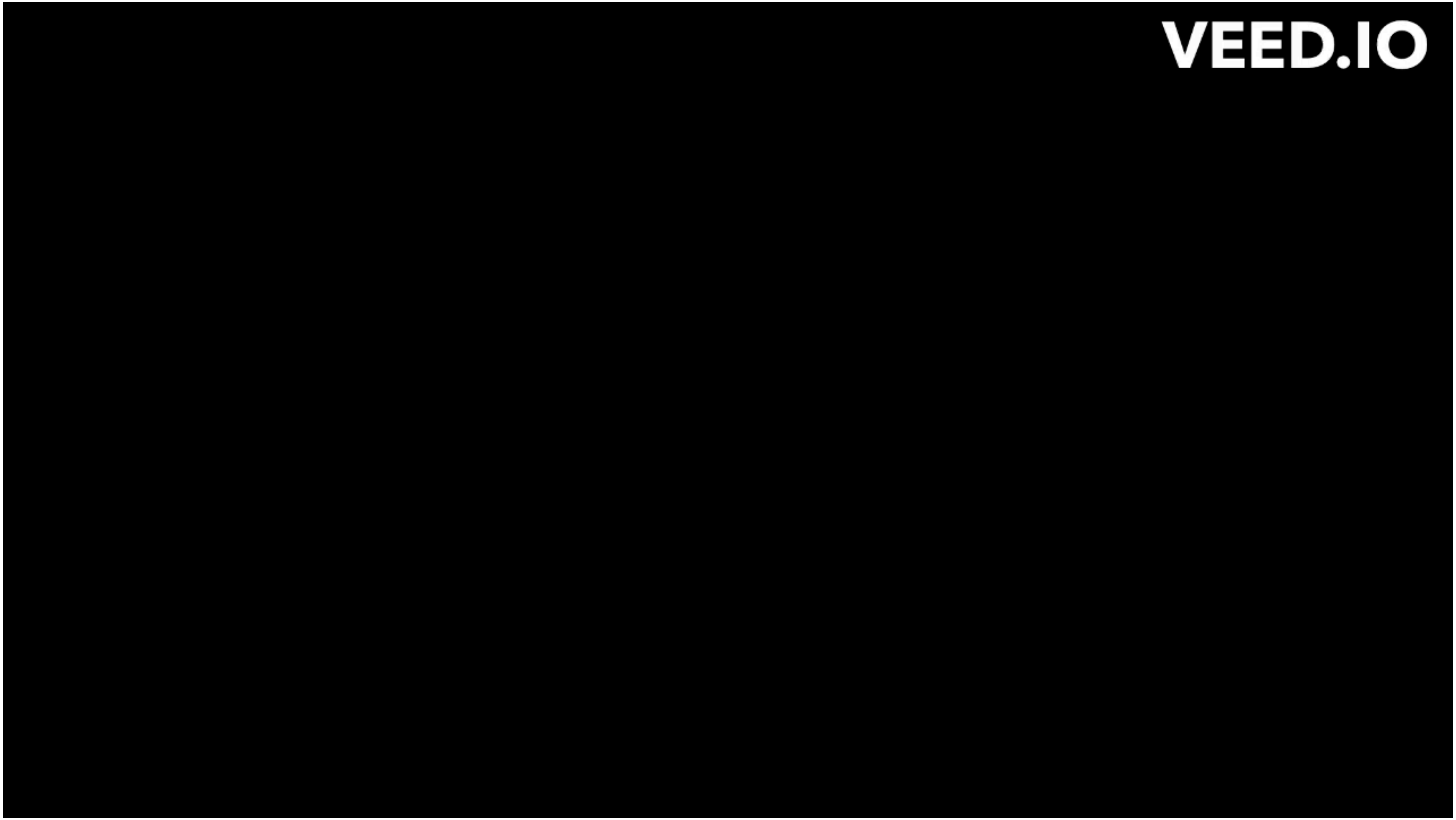


Ancient Egyptian Mathematics

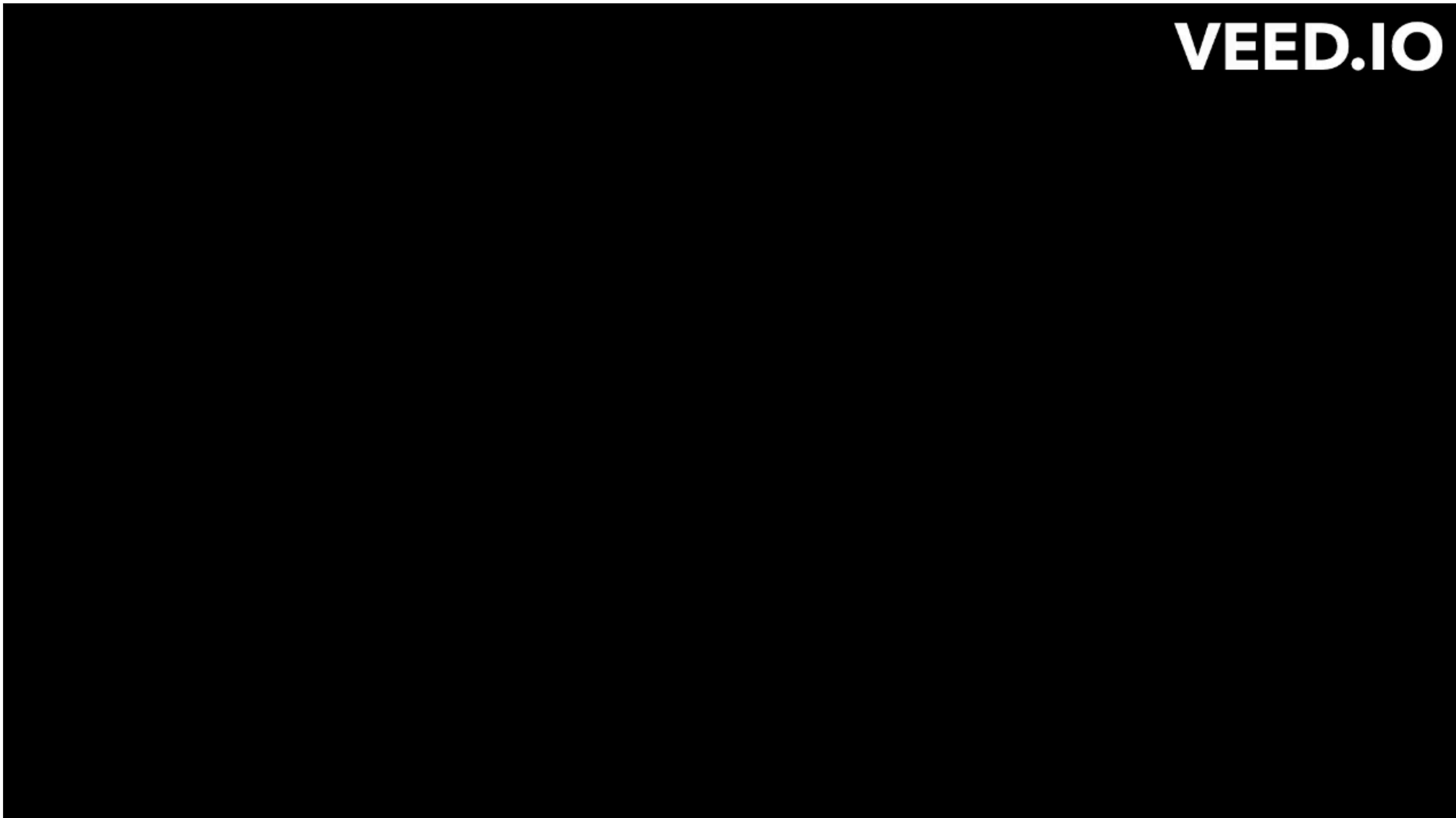
Egypt


Romania
Croatia
Serbia
Italy
Rome
Bulgaria
Black Sea
Georgia
Uzbekistan
drid
pain
Tyrrhenian Sea
Istanbul
Ankara
Azerbaijan
Greece
Turkey
Turkmenistan
Ta
Syria
Tunisia
Mediterranean Sea
Lebanon
Afghanistan
Iraq
Israel
Iran
Paki
Cairo
القاهرة
Algeria
Libya
Egypt
Persian Gulf
Riyadh
الرياض
United Arab
Emirates
Saudi Arabia
Jeddah
جدة
Red Sea
Oman
Mali
Niger
Sudan
Yemen
Chad
Eritrea
Burkina
Faso
Gulf of Aden
Arabian Sea
Djibouti
Benin
Nigeria

Egypt



Egypt



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- Important example: Constructing right angles.

Think Like A
Math Historian

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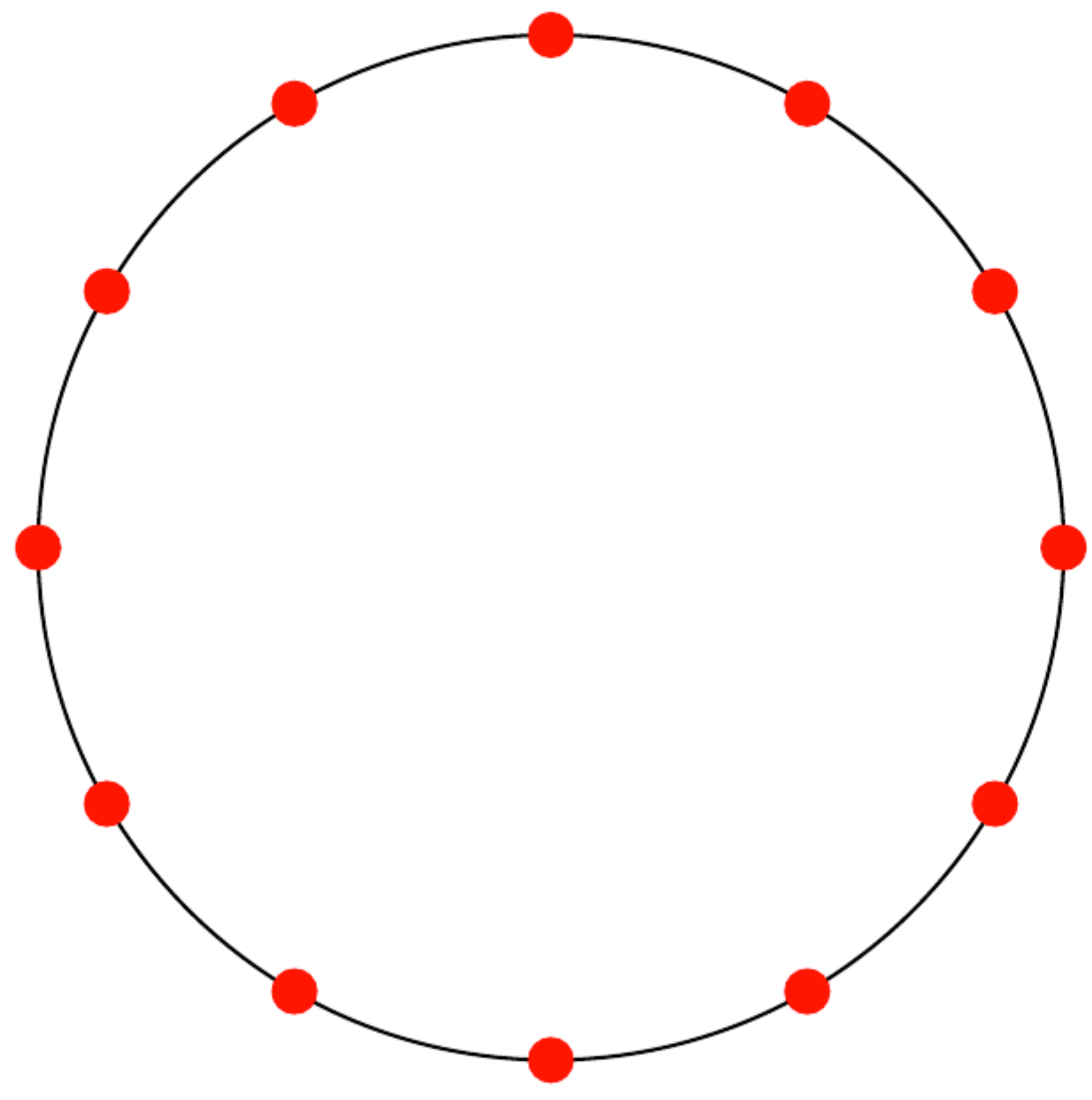

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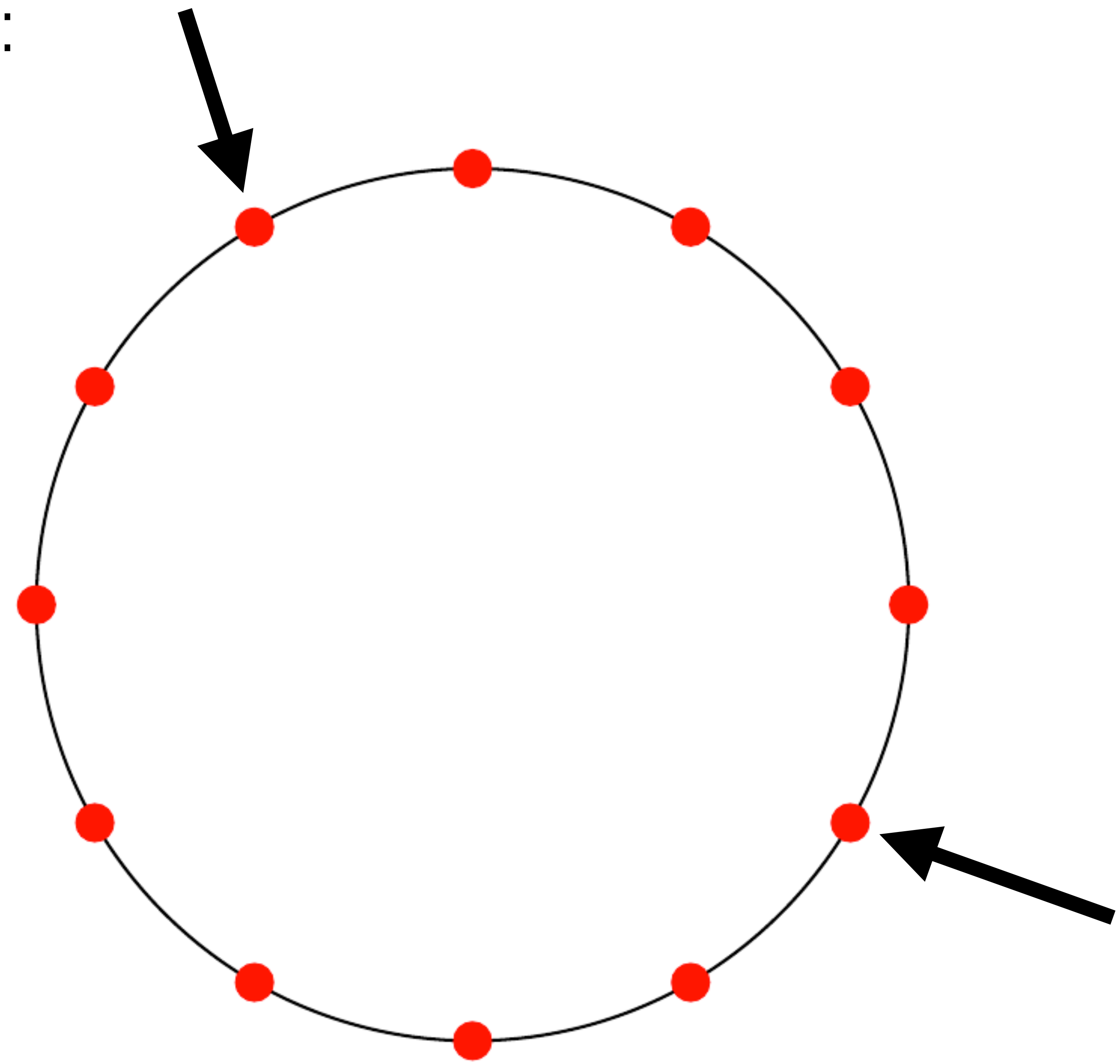

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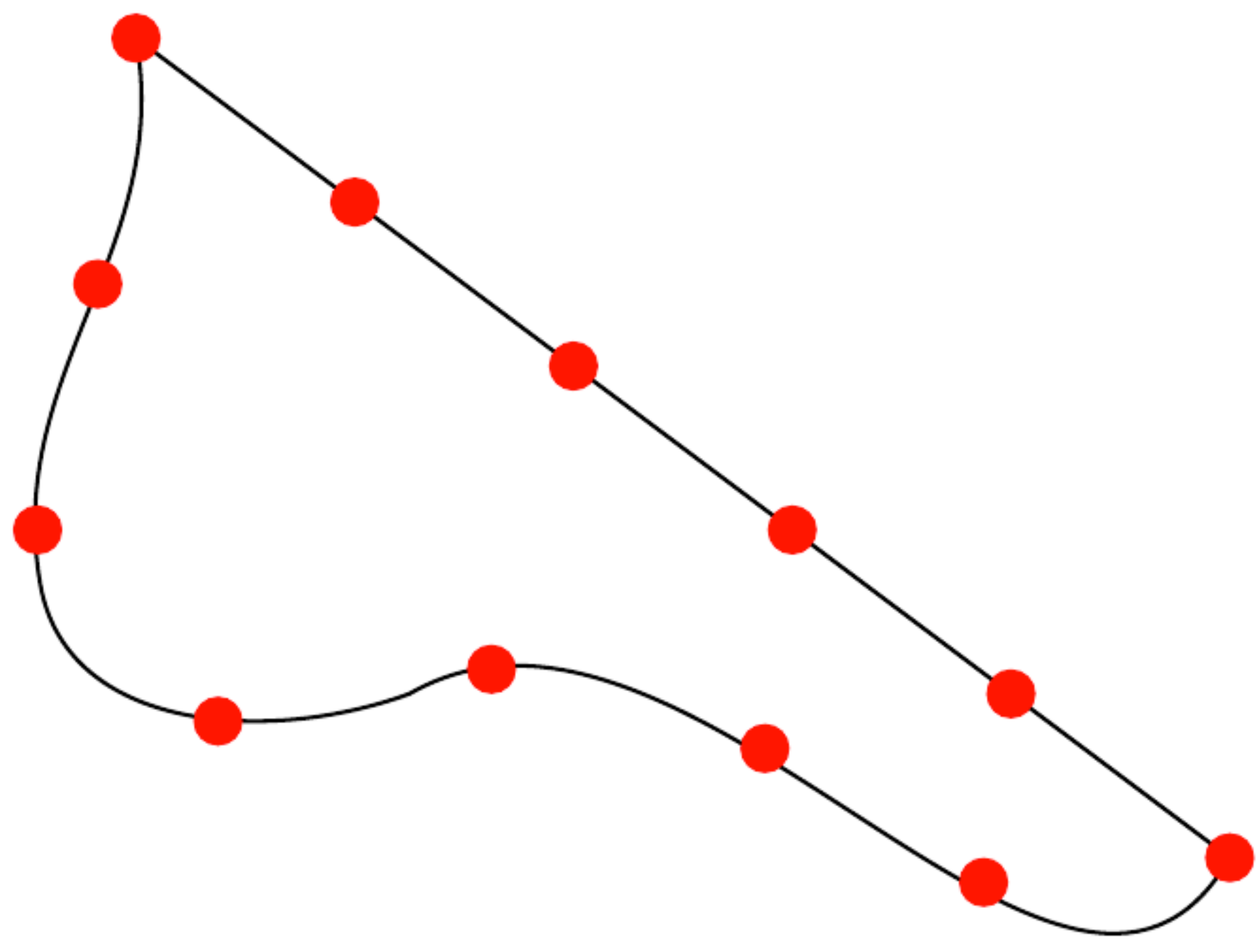

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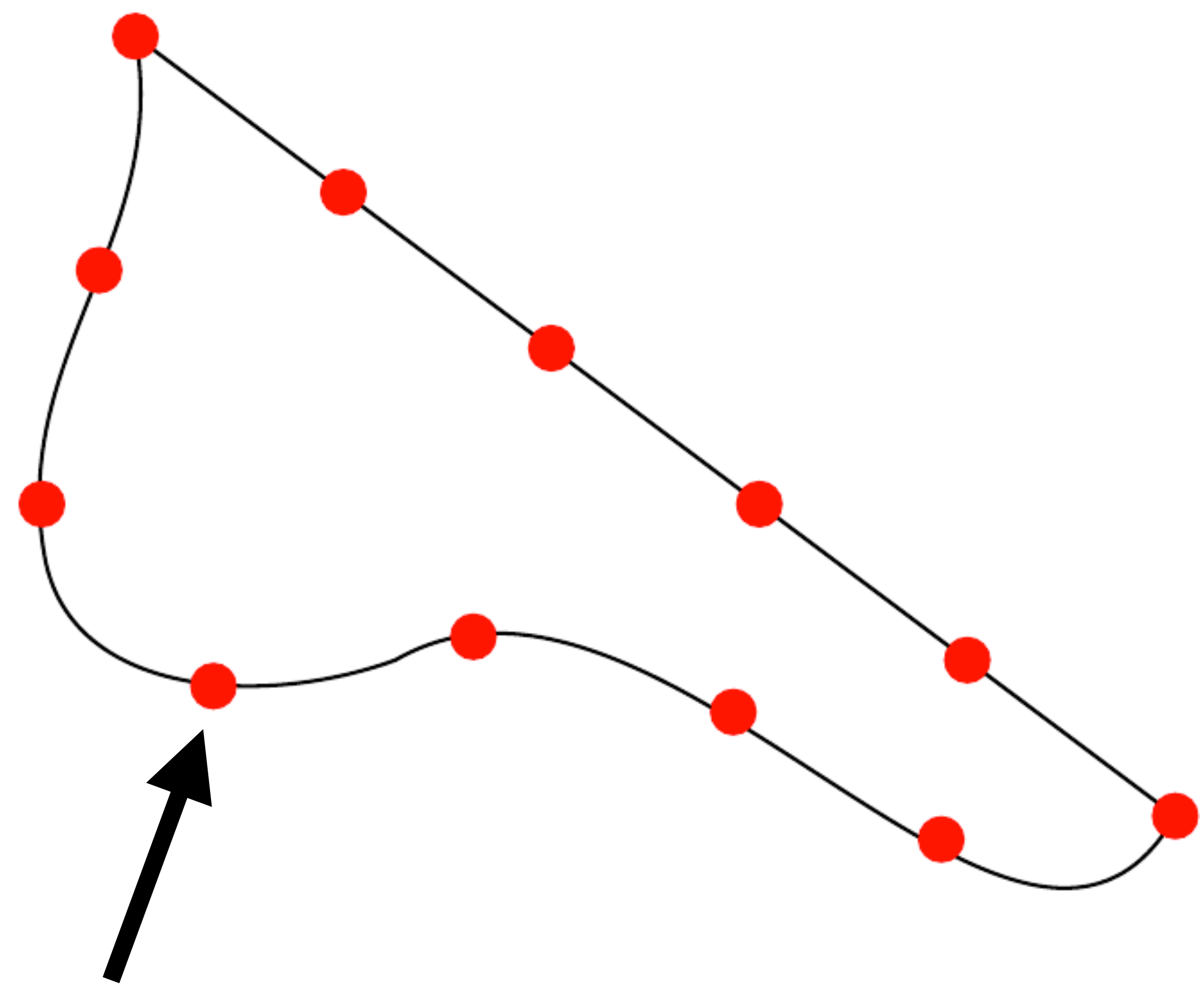

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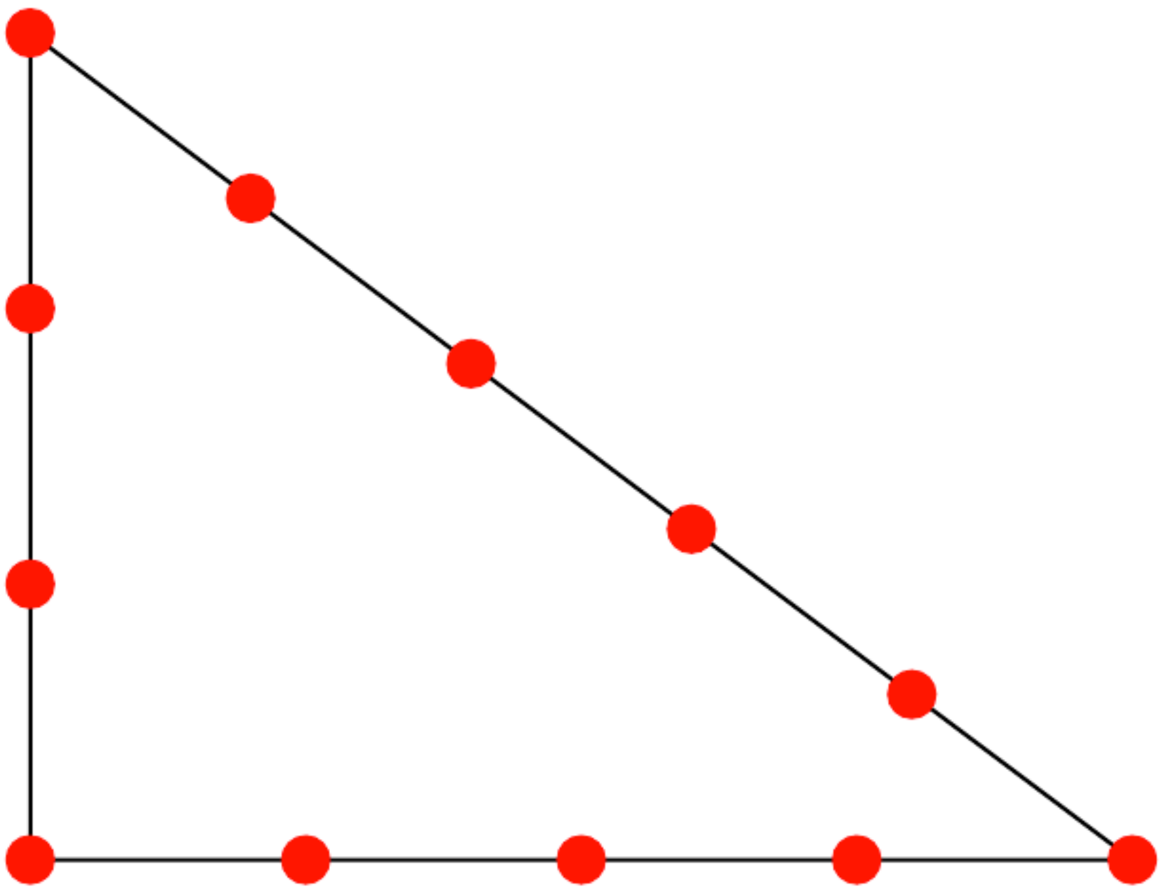

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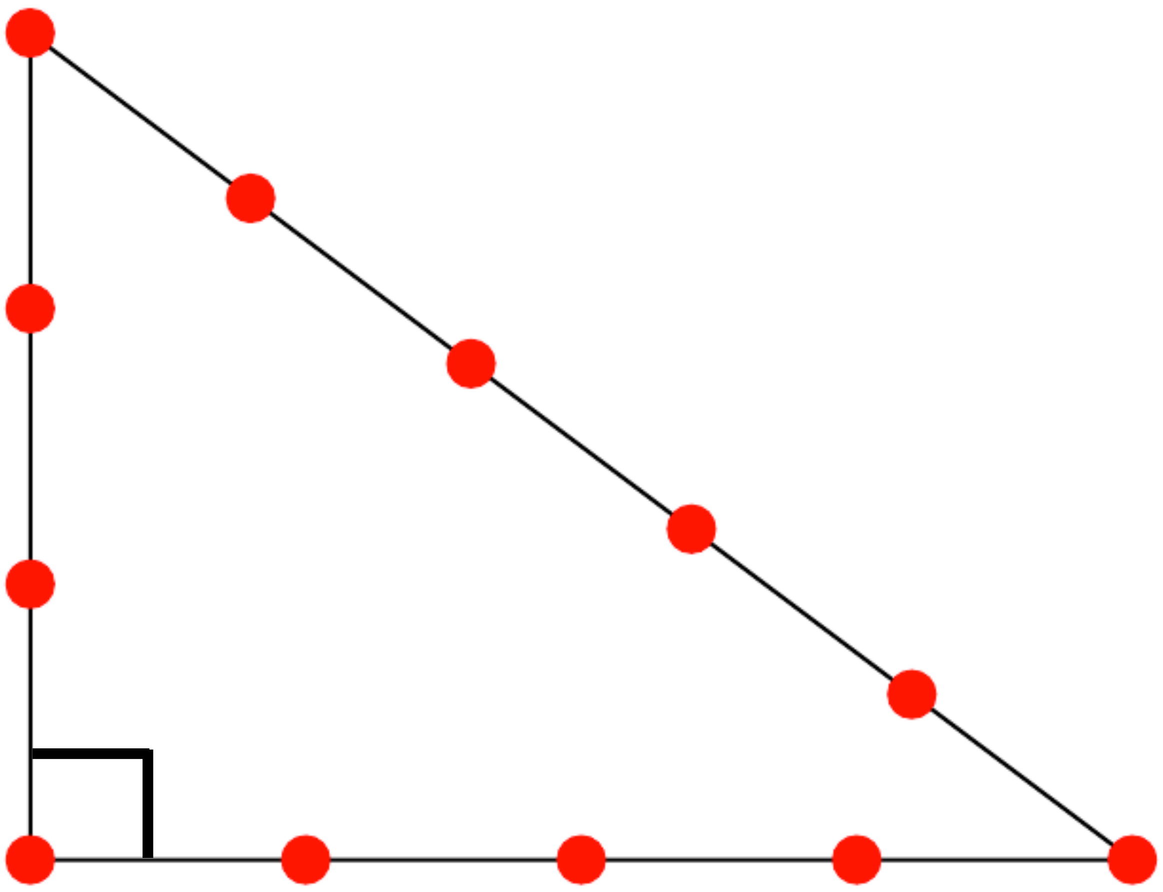

Egyptian Numerical Methods

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Egyptian Methods

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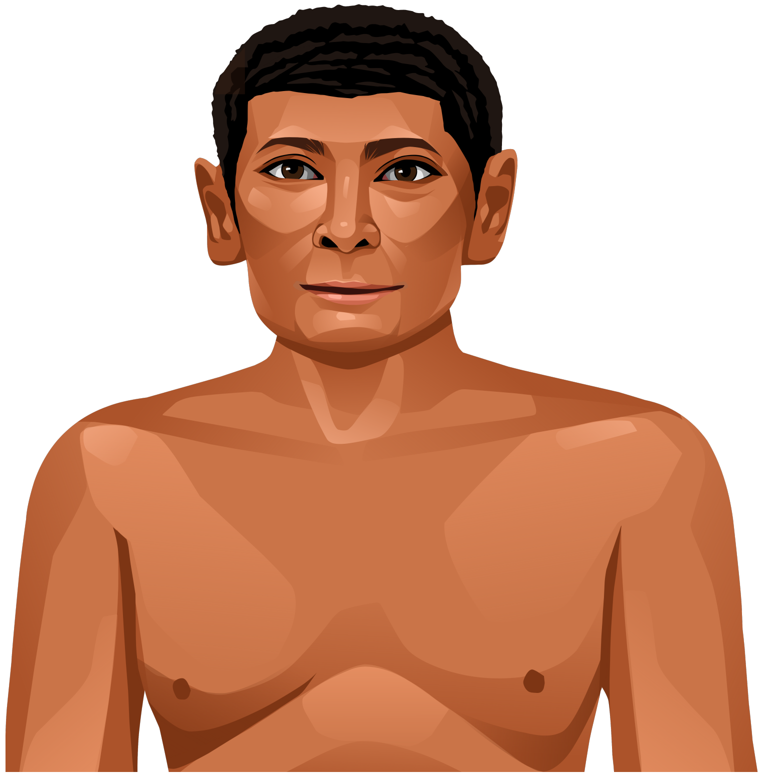

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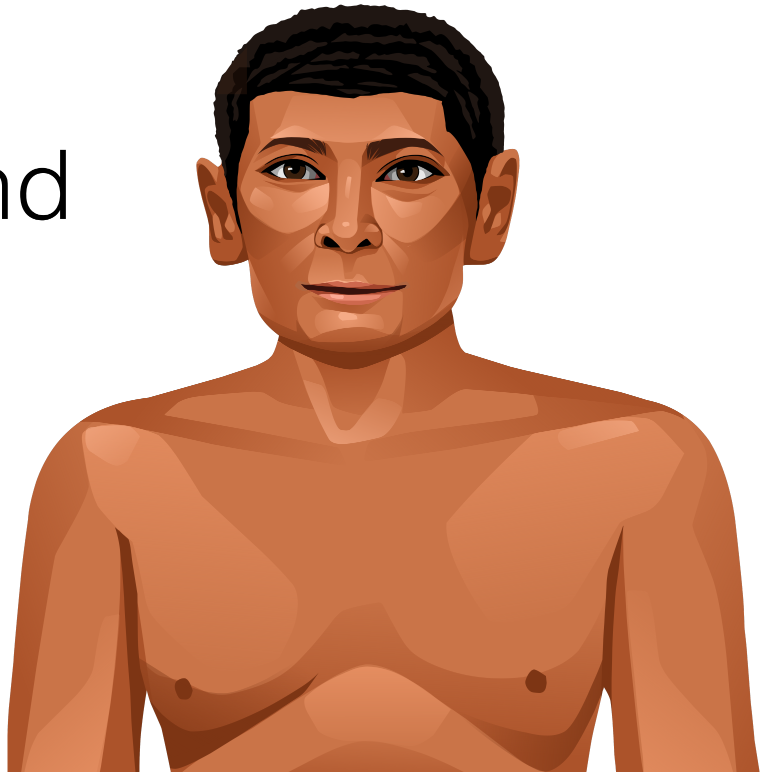

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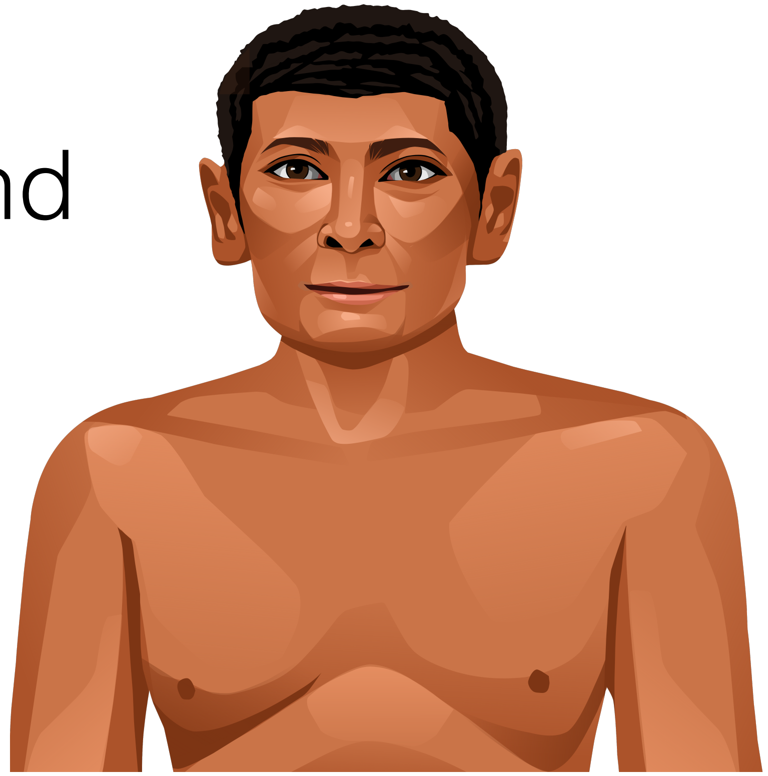

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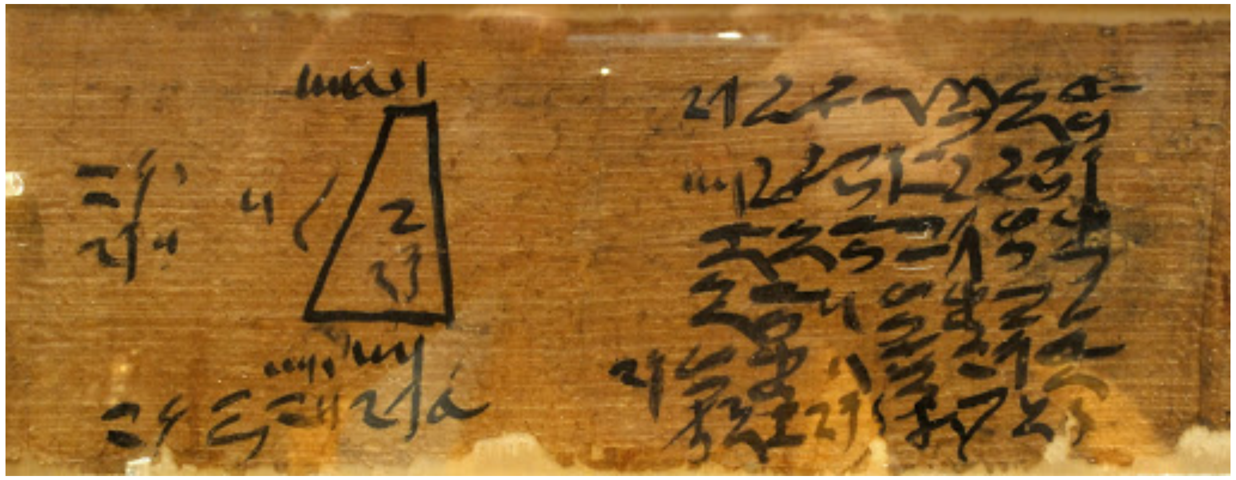
- Combined: 85 problems and solutions (all numerical) on arithmetic, algebra and geometry.



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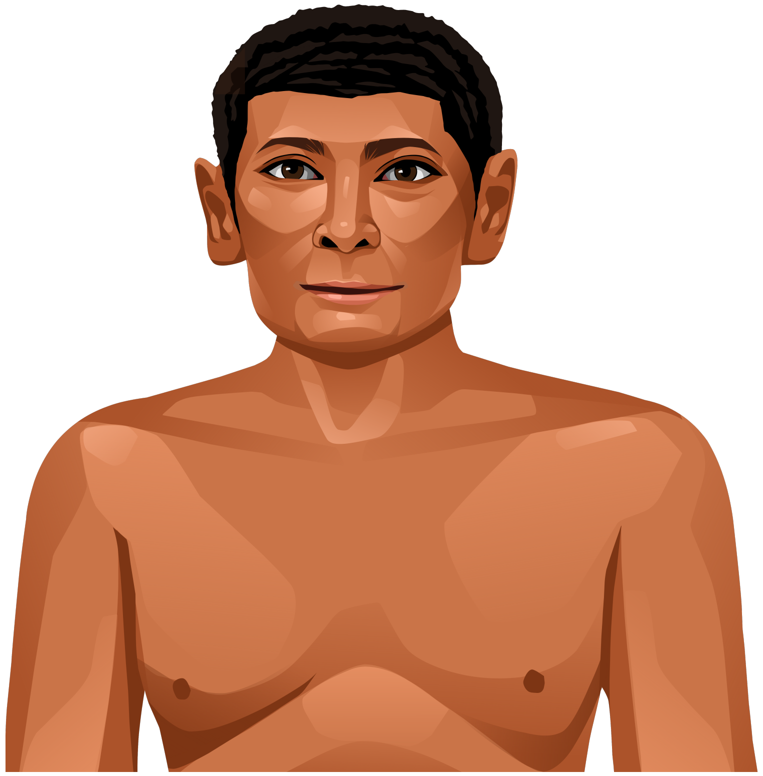
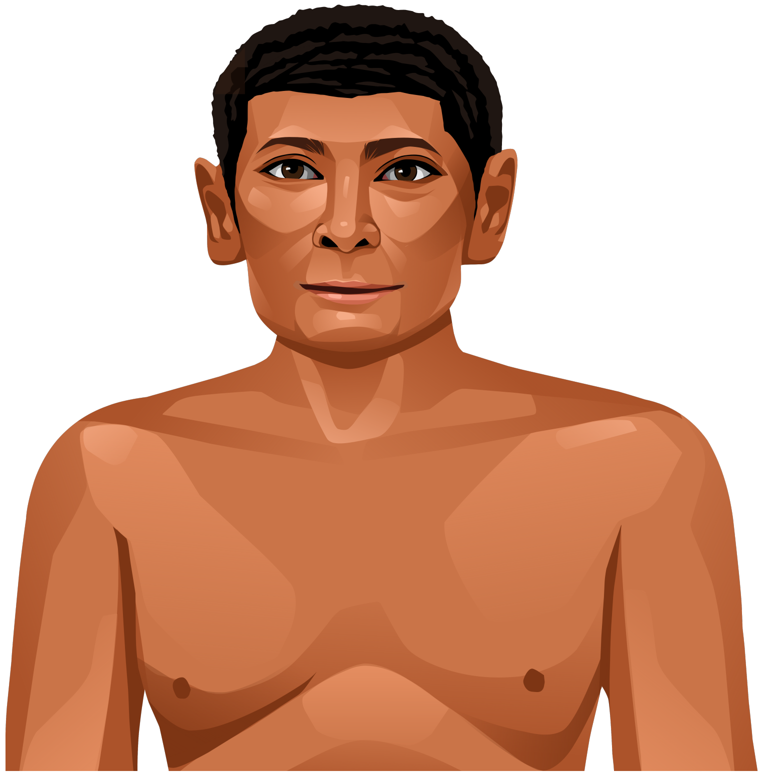


Egyptian Methods

- Rind pictures:

  https://www.britishmuseum.org/collection/object/Y_EA10058



Think Like A Math Historian

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$$1 \to 21$$
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$$16 \to 336$$
$$32 \to 672$$

$$\begin{array}{r} 21 \\ 84 \\ + \quad 168 \\ 672 \\ \hline 945 \end{array}$$

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Why does it work? You (might) tell me on your homework!

Egyptian Fractions

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- The Rhind papyrus noted that $\dfrac{2}{3n} = \dfrac{1}{2n} + \dfrac{1}{6n}$.
- This is a formula for how to take 2/3 of an Egyptian fraction: Apply this rule to each term.
- The Rhind also had tables of values. Example: how to write

$$\frac{1}{10}, \frac{2}{10}, \ \cdots \ , \frac{9}{10}$$

as Egyptian fractions.

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$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)}.$$

- Second, arithmetic is tough. Even simply multiplying by 2 can make a fraction unrecognizable.

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1. Find the largest unit fraction less than $\frac{m_1}{n_1}$; suppose it is $\frac{1}{k_1}$.
2. Subtract: $\frac{m_1}{n_1} - \frac{1}{k_1}$, and simplify this to a single new fraction, $\frac{m_2}{n_2}$.
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One can prove that this algorithm at some point terminates, producing a *finite* list sum of unit fractions equaling $\frac{m_1}{n_1}$:

$$\frac{m_1}{n_1} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_t}.$$

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- Supp … ople. How … ame amou …
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- Writing it with Egyptian fractions, $\frac{5}{6} = \frac{1}{2} + \frac{1}{3}$. So one way is to give everyone a half and a third!

Geometry

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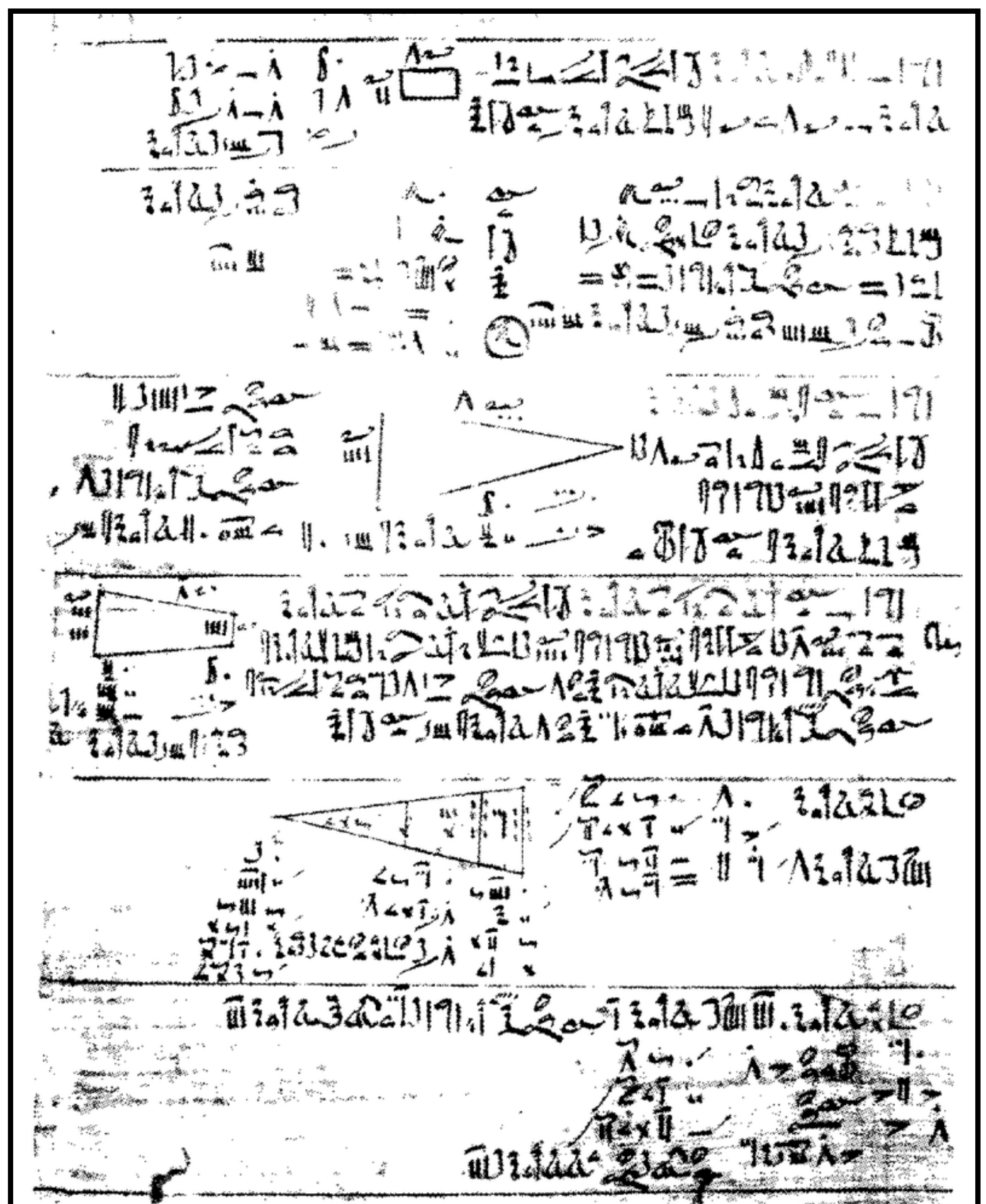

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- In Problem 41, Ahmes computes the volume of a cylindrical granary. Given a diameter of d and height of h, he says its volume is

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$$[(1 - 1/9)d]^2 h = \frac{256}{81} r^2 h\,.$$

- This essentially means they were using $\pi = \dfrac{256}{81} = 3.1604\ldots$ Which is wrong, but not too bad.

Babylonian Mathematics

Mesopotamia

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Babylonia

Romania
Croatia
Serbia
Italy
Rome
Bulgaria
Black Sea
Georgia
Uzbekistan
Istanbul
Ankara
Azerbaijan
Tyrrhenian Sea
Greece
Turkey
Turkmenistan
Syria
Lebanon
Mediterranean Sea
Tunisia
Afghanistan
Iraq
Israel
Iran
Cairo
القاهرة
Algeria
Libya
Egypt
Persian Gulf
Riyadh
الرياض
United Arab Emirates
Saudi Arabia
Jeddah
جدة
Red Sea
Oman
Mali
Niger
Sudan
Chad
Yemen
Eritrea
Burkina Faso
Gulf of Aden
Arabian Sea
Djibouti
Benin

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- Babylonian mathematics was generally more advanced than Ancient Egyptian mathematics.

Think Like A Math Historian

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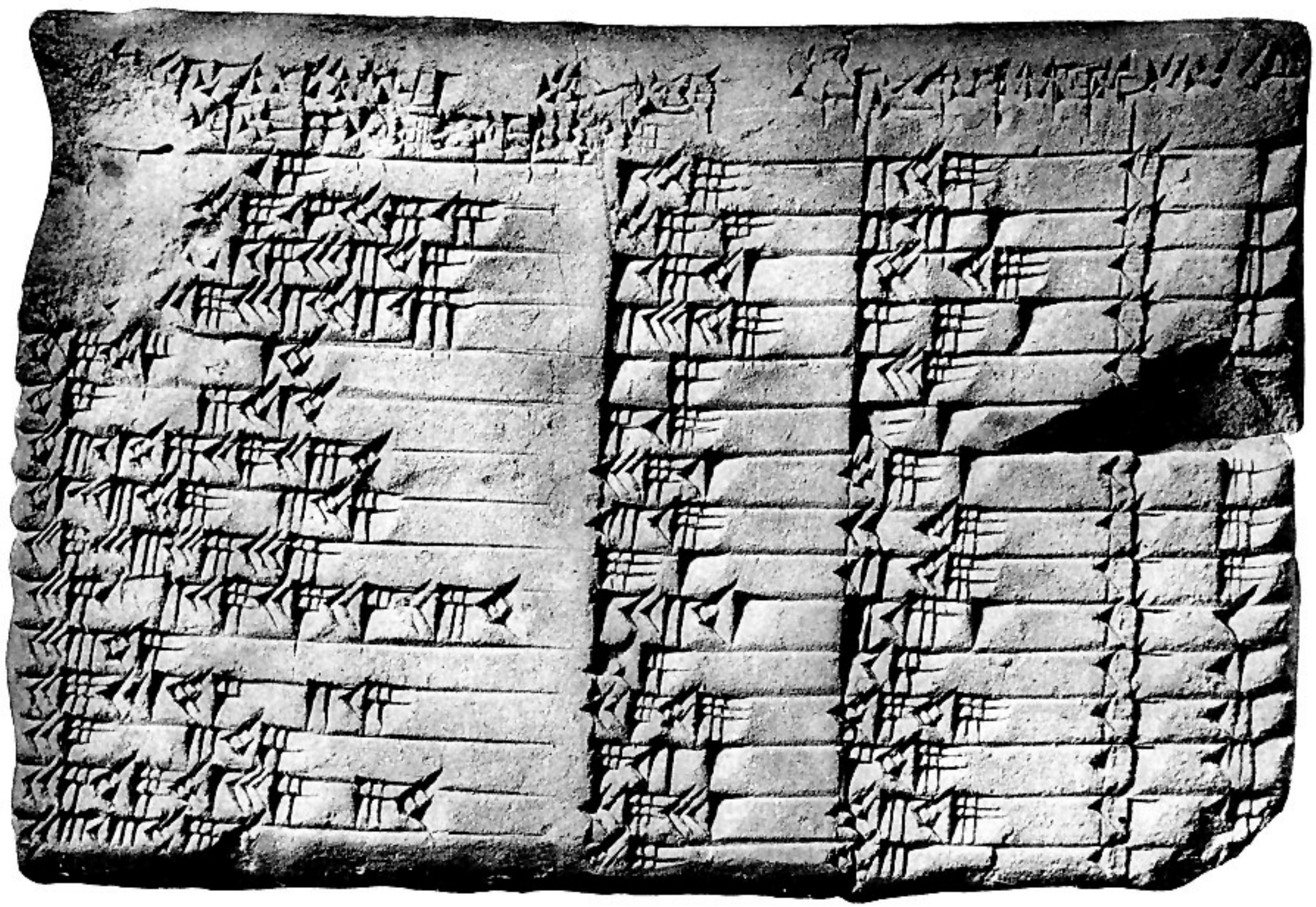

Babylonian Methods

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Ratio	Width	Diagonal	Row number
1; 59 0 15	1 59	2 49	1
1; 56 56 58 14 50 6 15	56 7	1 20 25*	2
1; 55 7 41 15 33 45	1 16 41	1 50 49	3
1; 53 10 29 32 52 16	3 31 49	5 9 1	4
1; 48 54 1 40	1 5	1 37	5
1; 47 6 41 40	5 19	8 1	6
1; 43 11 56 28 26 40	38 11	59 1	7
1; 41 33 45 14 3 45	13 19	20 49	8
1; 38 33 36 36	8 1*	12 49 9	9
1; 35 10 2 28 27 24 26 40	1 22 41	2 16 1	10
1; 33 45	45	1 15	11
1; 29 21 54 2 15	27 59	48 49	12
1; 27 0 3 45	2 41*	4 49	13
1; 25 48 51 35 6 40	29 31	53 49	14
1; 23 13 46 40	56	1 46	15

Babylonian Methods

- Plimpton 322 is a tablet from around 1800 BC.

Ratio	Width	Diagonal	Row number
1; 59 0 15	1 59	2 49	1
1; 56 56 58 14 50 6 15	56 7	1 20 25*	2
1; 55 7 41 15 33 45	1 16 41	1 50 49	3
1; 53 10 29 32 52 16	3 31 49	5 9 1	4
1; 48 54 1 40	1 5	1 37	5
1; 47 6 41 40	5 19	8 1	6
1; 43 11 56 28 26 40	38 11	59 1	7
1; 41 33 45 14 3 45	13 19	20 49	8
1; 38 33 36 36	8 1*	12 49 9	9
1; 35 10 2 28 27 24 26 40	1 22 41	2 16 1	10
1; 33 45	45	1 15	11
1; 29 21 54 2 15	27 59	48 49	12
1; 27 0 3 45	2 41*	4 49	13
1; 25 48 51 35 6 40	29 31	53 49	14
1; 23 13 46 40	56	1 46	15

Babylonian Methods

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1; 59 0 15	1 59	2 49	1
1; 56 56 58 14 50 6 15	56 7	1 20 25*	2
1; 55 7 41 15 33 45	1 16 41	1 50 49	3
1; 53 10 29 32 52 16	3 31 49	5 9 1	4
1; 48 54 1 40	1 5	1 37	5
1; 47 6 41 40	5 19	8 1	6
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1; 41 33 45 14 3 45	13 19	20 49	8
1; 38 33 36 36	8 1*	12 49 9	9
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1; 33 45	45	1 15	11
1; 29 21 54 2 15	27 59	48 49	12
1; 27 0 3 45	2 41*	4 49	13
1; 25 48 51 35 6 40	29 31	53 49	14
1; 23 13 46 40	56	1 46	15

Babylonian Methods

- Plimpton 322 is a tablet from around 1800 BC.

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1; 59 0 15	1 59	2 49	1
1; 56 56 58 14 50 6 15	56 7	1 20 25*	2
1; 55 7 41 15 33 45	1 16 41	1 50 49	3
1; 53 10 29 32 52 16	3 31 49	5 9 1	4
1; 48 54 1 40	1 5	1 37	5
1; 47 6 41 40	5 19	8 1	6
1; 43 11 56 28 26 40	38 11	59 1	7
1; 41 33 45 14 3 45	13 19	20 49	8
1; 38 33 36 36	8 1*	12 49 9	9
1; 35 10 2 28 27 24 26 40	1 22 41	2 16 1	10
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1; 29 21 54 2 15	27 59	48 49	12
1; 27 0 3 45	2 41*	4 49	13
1; 25 48 51 35 6 40	29 31	53 49	14
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$3 \cdot 60^2 + 31 \cdot 60 + 49$

Babylonian Methods

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1; 59 0 15	1 59	2 49	1
1; 56 56 58 14 50 6 15	56 7	1 20 25*	2
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1; 29 21 54 2 15	27 59	48 49	12
1; 27 0 3 45	2 41*	4 49	13
1; 25 48 51 35 6 40	29 31	53 49	14
1; 23 13 46 40	56	1 46	15

$$3 \cdot 60^2 + 31 \cdot 60 + 49 = 12{,}709$$

Babylonian Methods

- Plimpton 322 is a tablet from around 1800 BC.

Ratio	Width	Diagonal	Row number
$\frac{28{,}561}{14{,}400}$	119	169	1
$\frac{23{,}280{,}625}{11{,}943{,}936}$	3,367	4,825	2
$\frac{44{,}209{,}201}{23{,}040{,}000}$	4,601	6,649	3
$\frac{343{,}768{,}681}{182{,}250{,}000}$	12,709	18,541	4
$\frac{9{,}409}{5{,}184}$	65	97	5
$\frac{231{,}361}{129{,}600}$	319	481	6
$\frac{12{,}538{,}681}{7{,}290{,}000}$	2,291	3,541	7
$\frac{1{,}560{,}001}{921{,}600}$	799	1,249	8
$\frac{591{,}361}{360{,}000}$	481	46,149	9
$\frac{66{,}601{,}921}{41{,}990{,}400}$	4,961	8,161	10
$\frac{25}{16}$	45	75	11
$\frac{8{,}579{,}041}{5{,}760{,}000}$	1,679	2,929	12
$\frac{5{,}034{,}241}{34{,}560{,}00}$	161	289	13
$\frac{10{,}426{,}441}{7{,}290{,}000}$	1,771	3,229	14
$\frac{678{,}811}{486{,}000}$	56	106	15

Babylonian Methods

- Plimpton 322 is a tablet from around 1800 BC.

Ratio	Width	Diagonal	Row number	Height
$\frac{28{,}561}{14{,}400}$	119	169	1	120
$\frac{23{,}280{,}625}{11{,}943{,}936}$	3,367	4,825	2	3,456
$\frac{44{,}209{,}201}{23{,}040{,}000}$	4,601	6,649	3	4,800
$\frac{343{,}768{,}681}{182{,}250{,}000}$	12,709	18,541	4	13,500
$\frac{9{,}409}{5{,}184}$	65	97	5	72
$\frac{231{,}361}{129{,}600}$	319	481	6	360
$\frac{12{,}538{,}681}{7{,}290{,}000}$	2,291	3,541	7	2,700
$\frac{1{,}560{,}001}{921{,}600}$	799	1,249	8	960
$\frac{591{,}361}{360{,}000}$	481	46,149	9	600
$\frac{66{,}601{,}921}{41{,}990{,}400}$	4,961	8,161	10	6,480
$\frac{25}{16}$	45	75	11	60
$\frac{8{,}579{,}041}{5{,}760{,}000}$	1,679	2,929	12	2,400
$\frac{5{,}034{,}241}{34{,}560{,}00}$	161	289	13	240
$\frac{10{,}426{,}441}{7{,}290{,}000}$	1,771	3,229	14	2,700
$\frac{678{,}811}{486{,}000}$	56	106	15	90

Babylonian Methods

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$$45^2 + 60^2 = 75^2$$

Babylonian Methods

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$$(\text{Width})^2 + (\text{Height})^2 = (\text{Diagonal})^2$$

Babylonian Methods

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Questions:

Babylonian Methods

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$$(\text{Width})^2 + (\text{Height})^2 = (\text{Diagonal})^2$$

Questions:

1. Why these (large) Pythagorean triples?

Babylonian Methods

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$(\text{Width})^2 + (\text{Height})^2 = (\text{Diagonal})^2$

Questions:

1. Why these (large) Pythagorean triples?

2. What's happening in the first column?

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Ratio	Width	Diagonal	Row number	Height
$\frac{28,561}{14,400}$	119	169	1	120
$\frac{23,280,625}{11,943,936}$	3,367	4,825	2	3,456
$\frac{44,209,201}{23,040,000}$	4,601	6,649	3	4,800
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2. What’s happening in the first column?

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$\frac{28,561}{14,400}$	119	169	1	120
$\frac{23,280,625}{11,943,936}$	3,367	4,825	2	3,456
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2. What's happening in the first column?

$$\text{Ratio} = \frac{(\text{Diagonal})^2}{(\text{Height})^2}$$

Babylonian Methods

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2. What's happening in the first column?

$$\text{Ratio} = \frac{(\text{Diagonal})^2}{(\text{Height})^2}$$

$$\frac{(\text{Width})^2}{(\text{Height})^2} + 1 = \frac{(\text{Diagonal})^2}{(\text{Height})^2}$$

Babylonian Methods

- Plimpton 322 is a tablet from around 1800 BC.

Ratio	Width	Diagonal	Row number	Height
$\frac{28{,}561}{14{,}400}$	119	169	1	120
$\frac{23{,}280{,}625}{11{,}943{,}936}$	3,367	4,825	2	3,456
$\frac{44{,}209{,}201}{23{,}040{,}000}$	4,601	6,649	3	4,800
$\frac{343{,}768{,}681}{182{,}250{,}000}$	12,709	18,541	4	13,500
$\frac{9{,}409}{5{,}184}$	65	97	5	72
$\frac{231{,}361}{129{,}600}$	319	481	6	360
$\frac{12{,}538{,}681}{7{,}290{,}000}$	2,291	3,541	7	2,700
$\frac{1{,}560{,}001}{921{,}600}$	799	1,249	8	960
$\frac{591{,}361}{360{,}000}$	481	46,149	9	600
$\frac{66{,}601{,}921}{41{,}990{,}400}$	4,961	8,161	10	6,480
$\frac{25}{16}$	45	75	11	60
$\frac{8{,}579{,}041}{5{,}760{,}000}$	1,679	2,929	12	2,400
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$$(\text{Width})^2 + (\text{Height})^2 = (\text{Diagonal})^2$$

Questions:

1. Why these (large) Pythagorean triples?

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Babylonian Methods

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- Consider the sequence with

$$a_1 = 1 \qquad \text{and} \qquad a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right).$$

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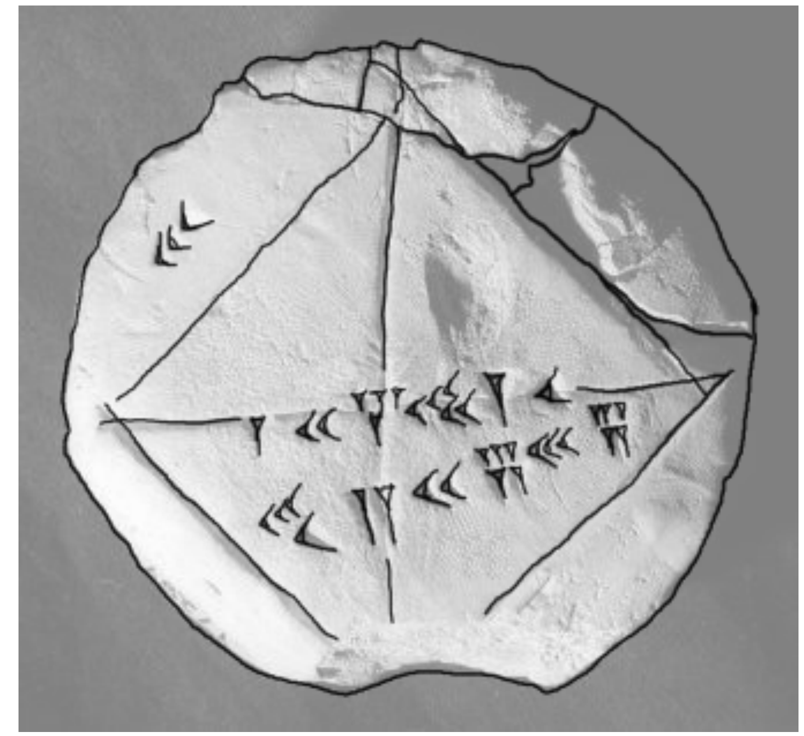
- Indeed, $a_1, a_2, a_3, \ldots$ *converges* to $\sqrt{2}$.

Babylonian Methods

Babylonian Methods



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- It’s so good that computers use it today. It’s quadratically convergent. It has been called “the greatest known computation of the ancient world.”

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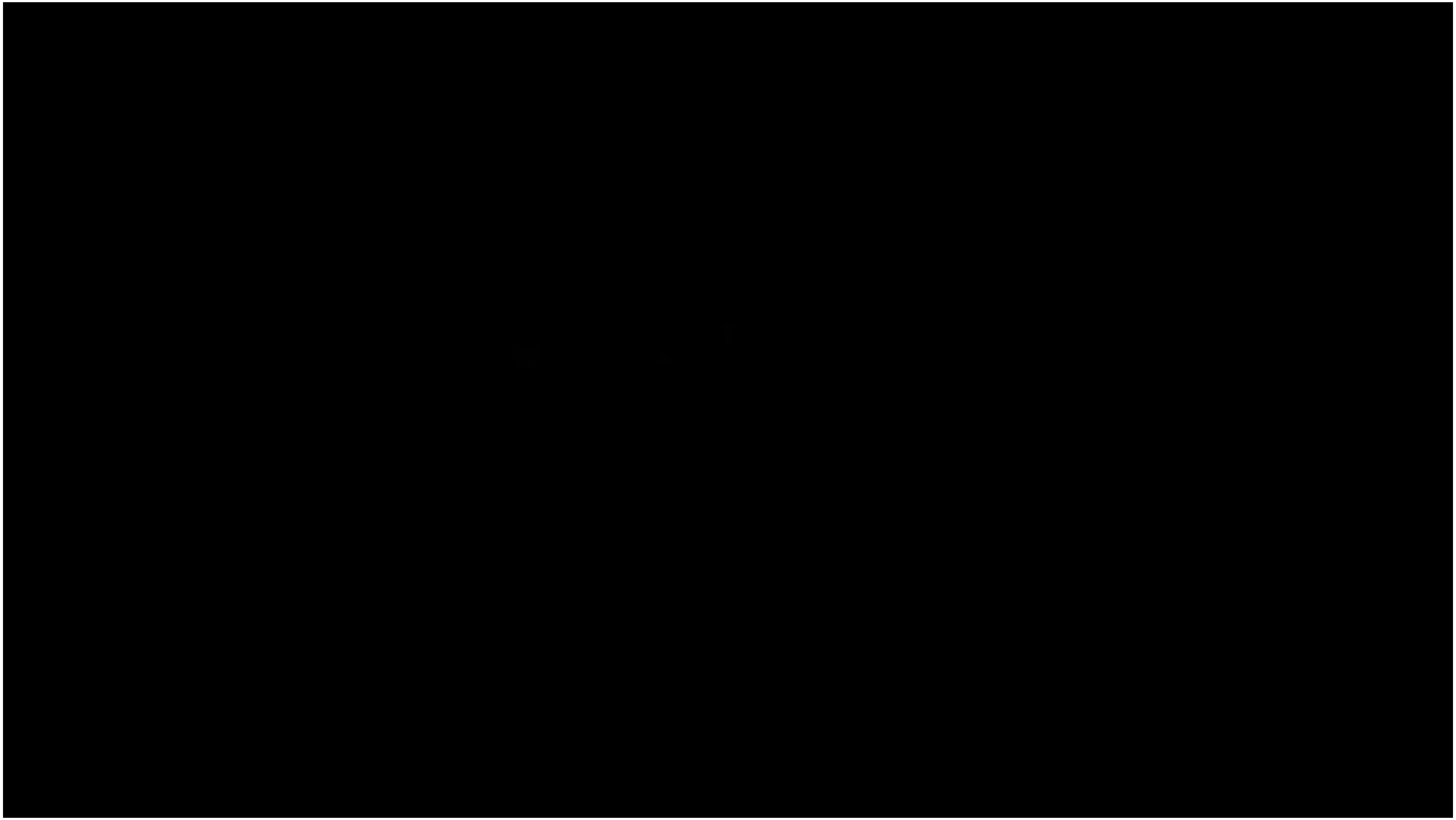
Vending Machine



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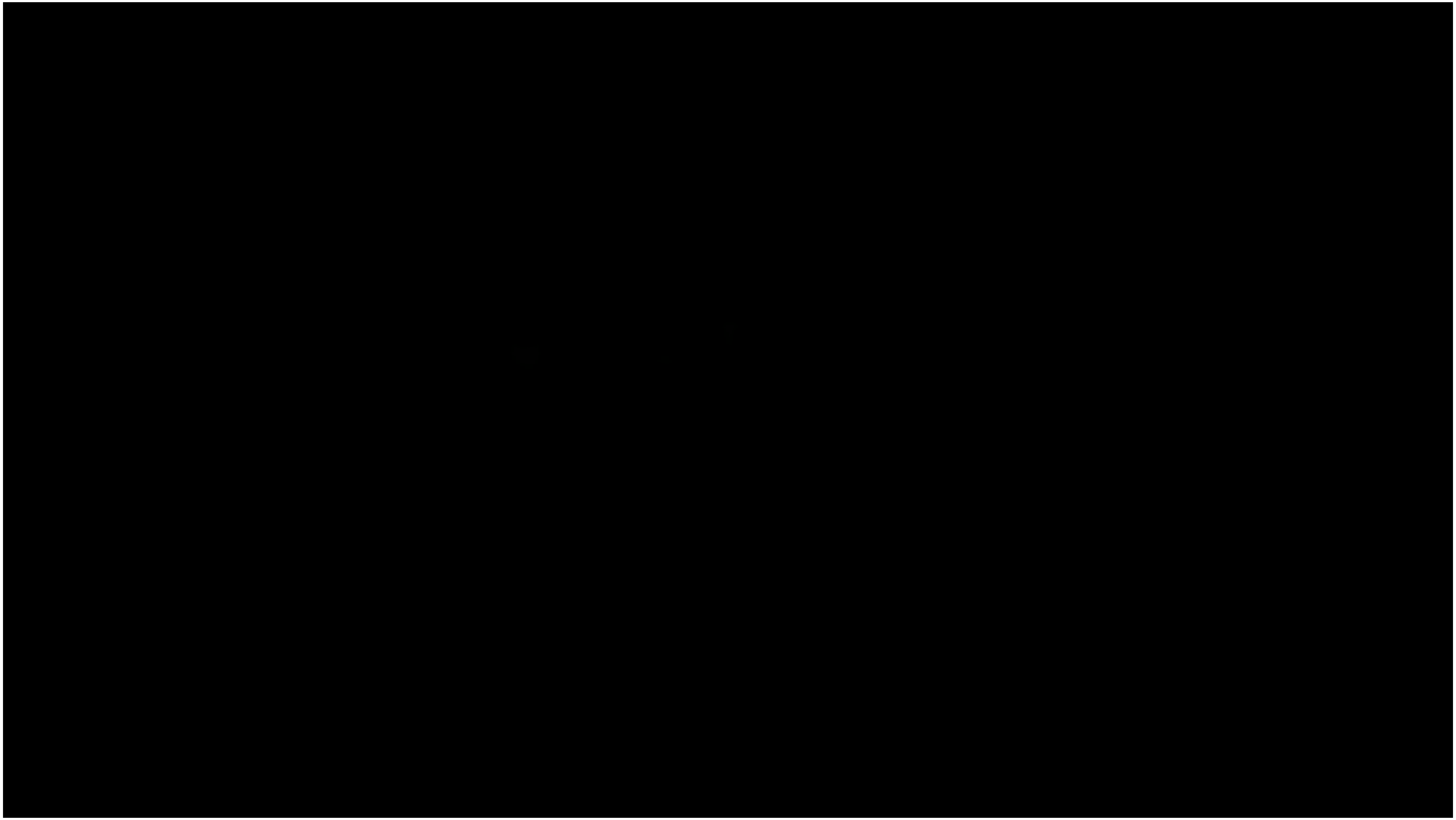

Ancient Chinese Mathematics

Chinese History

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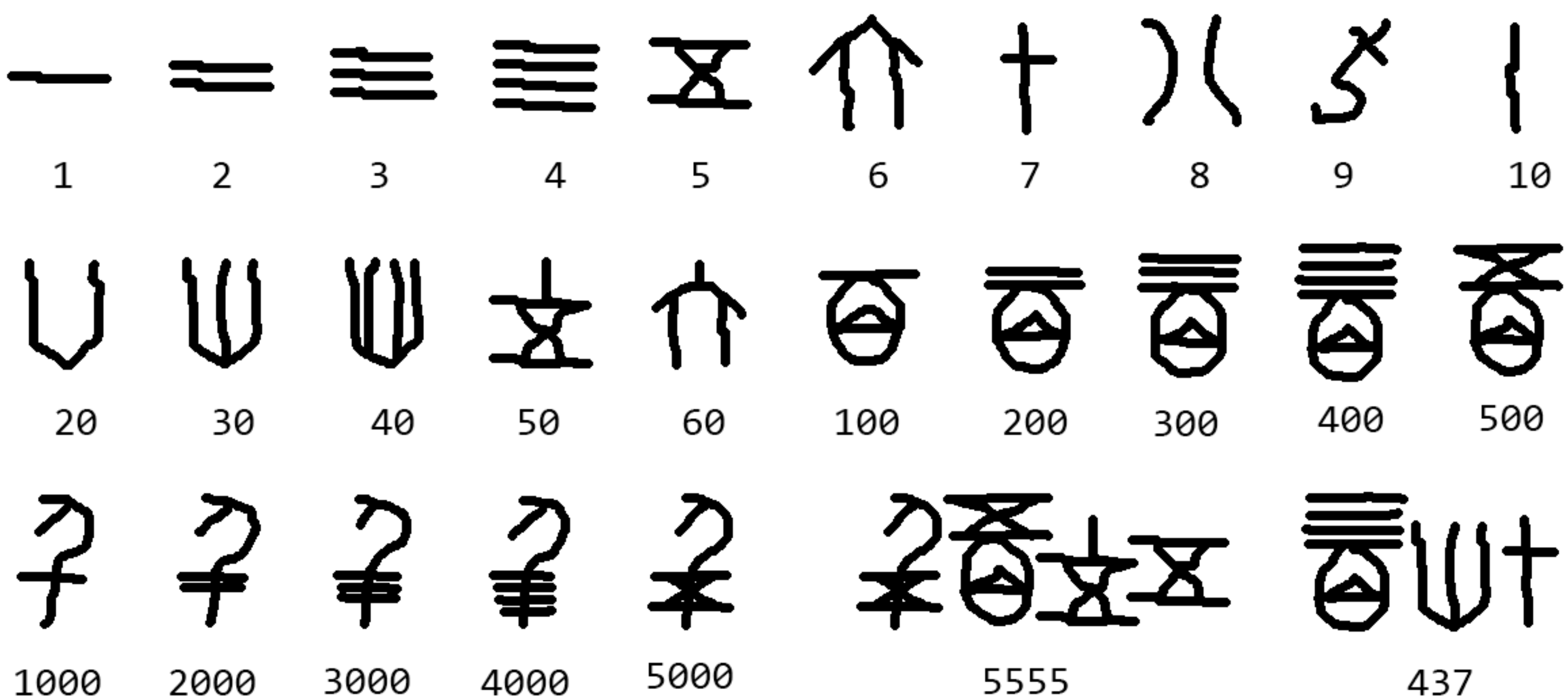

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- Qin joined remote stretches of wall into The Great Wall of China. Required many math techniques for materials/manpower needed, and its design.

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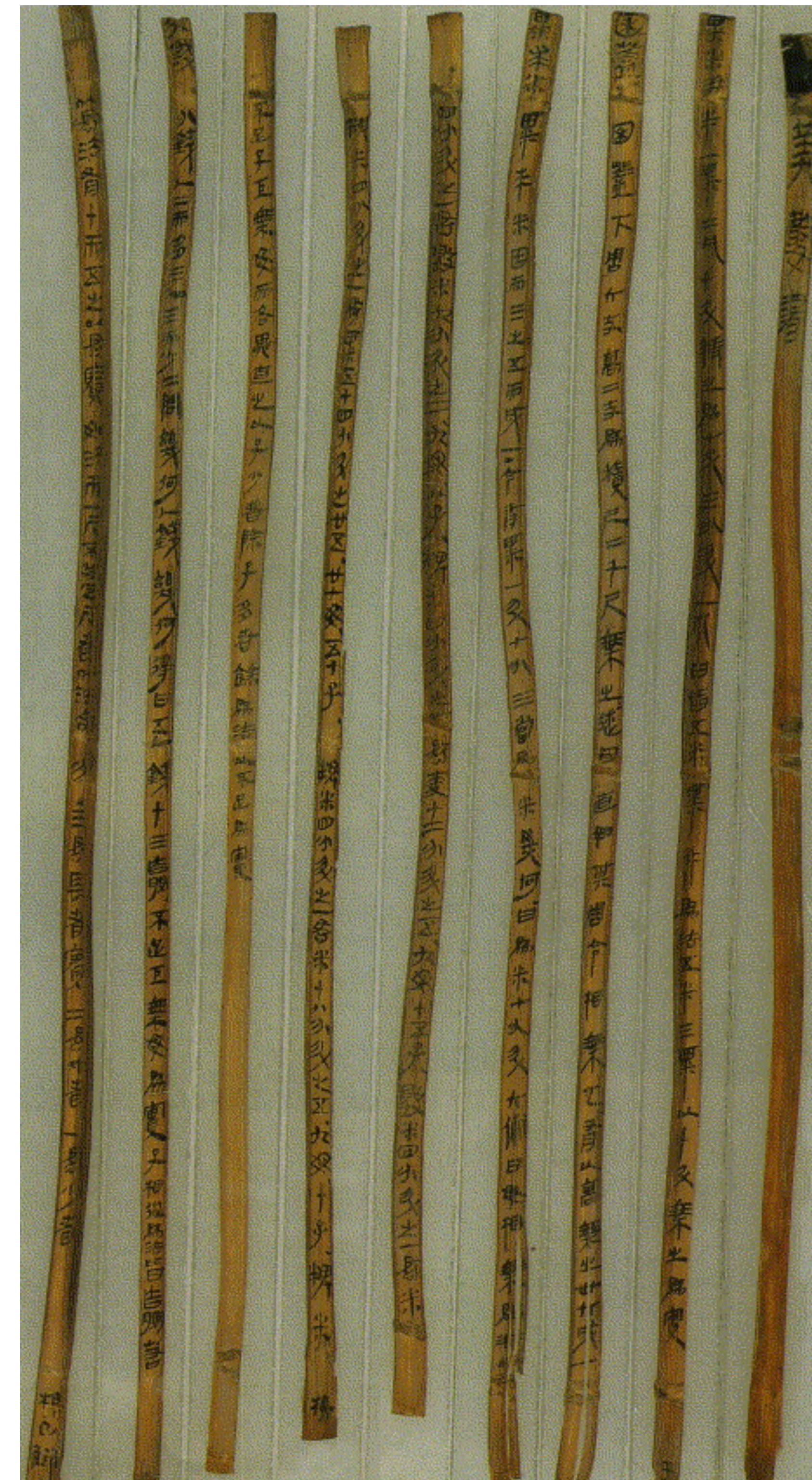

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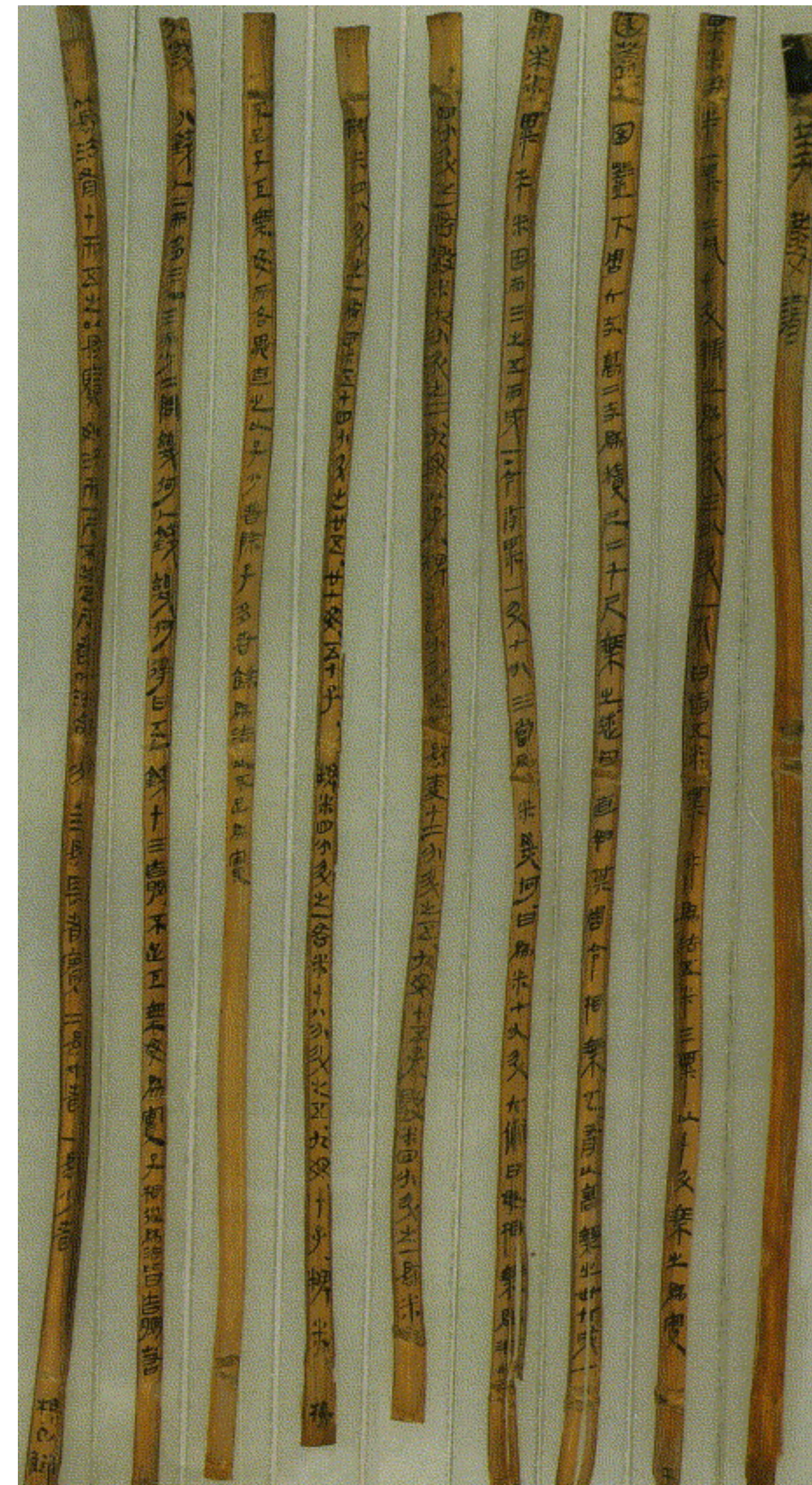
- Book contained 69 math questions, each followed by an answer and a method to solve it.



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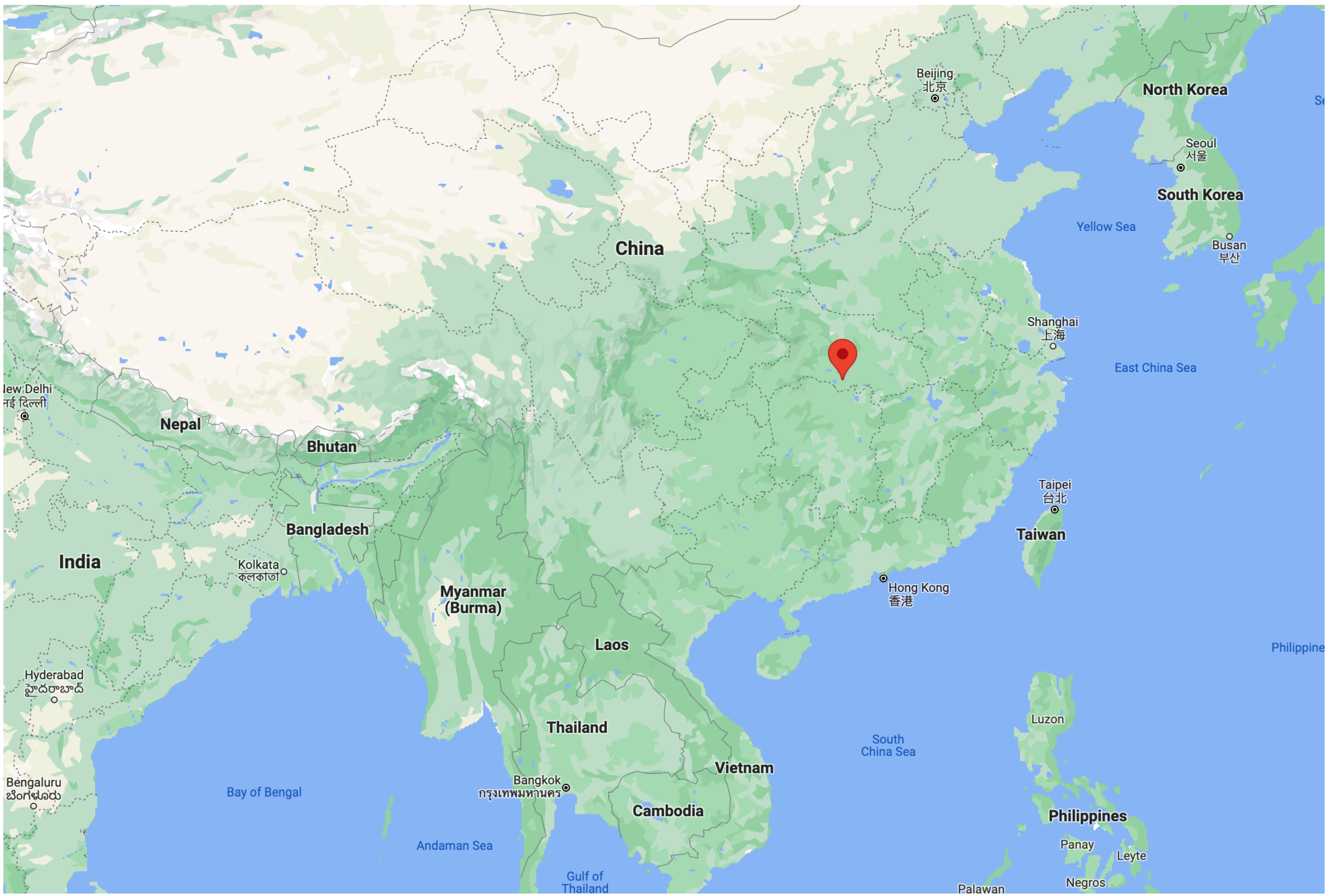

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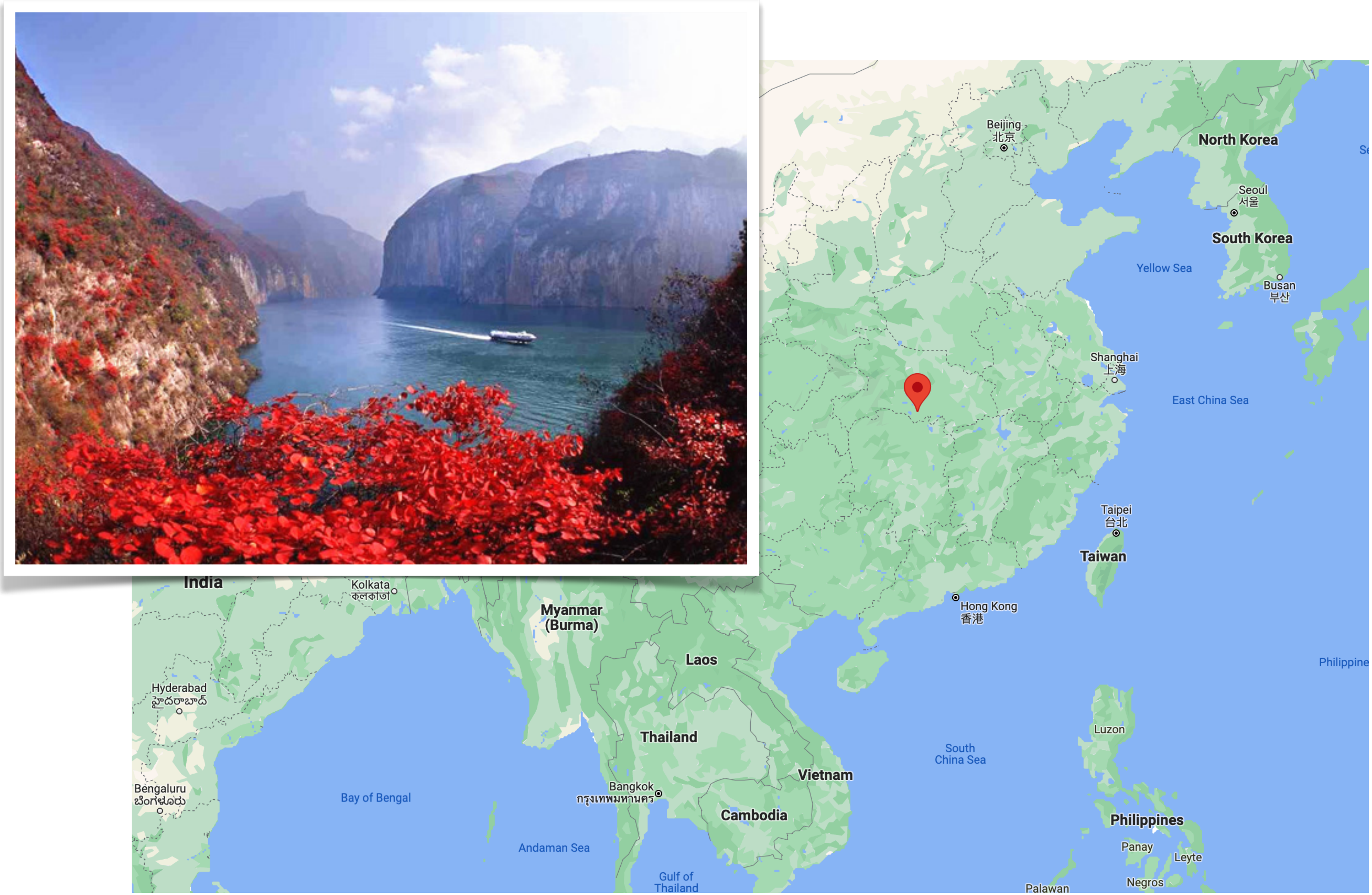

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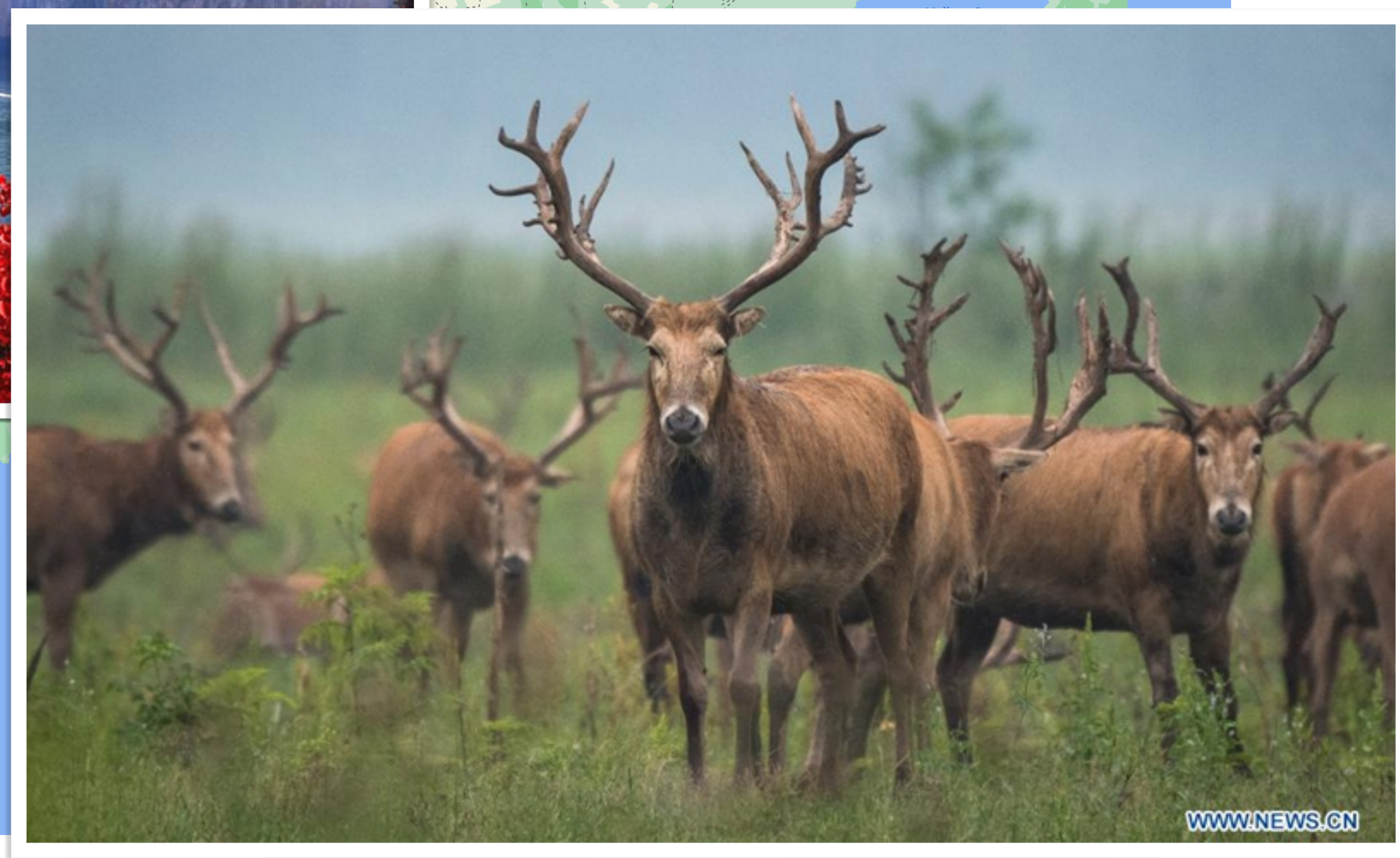

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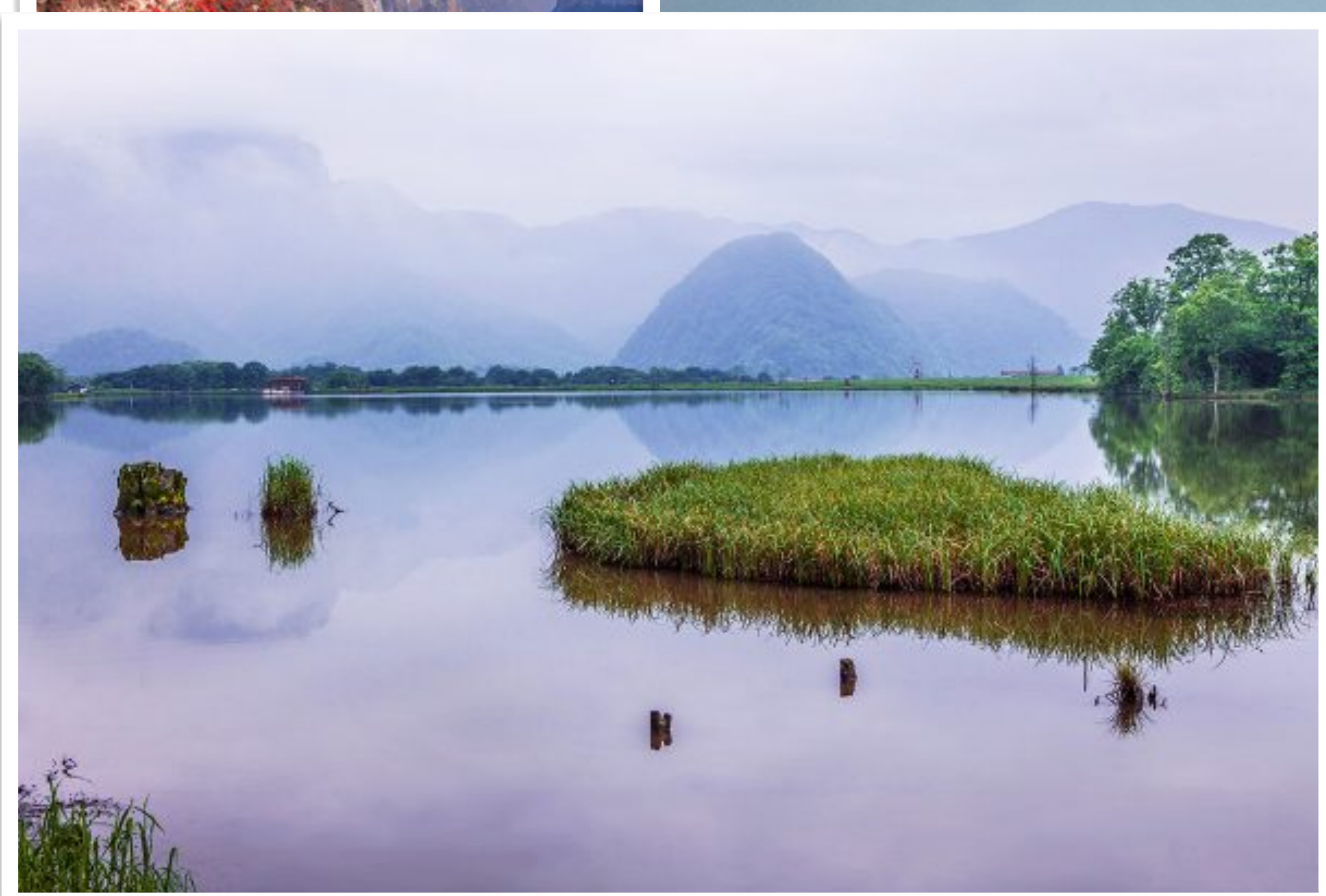

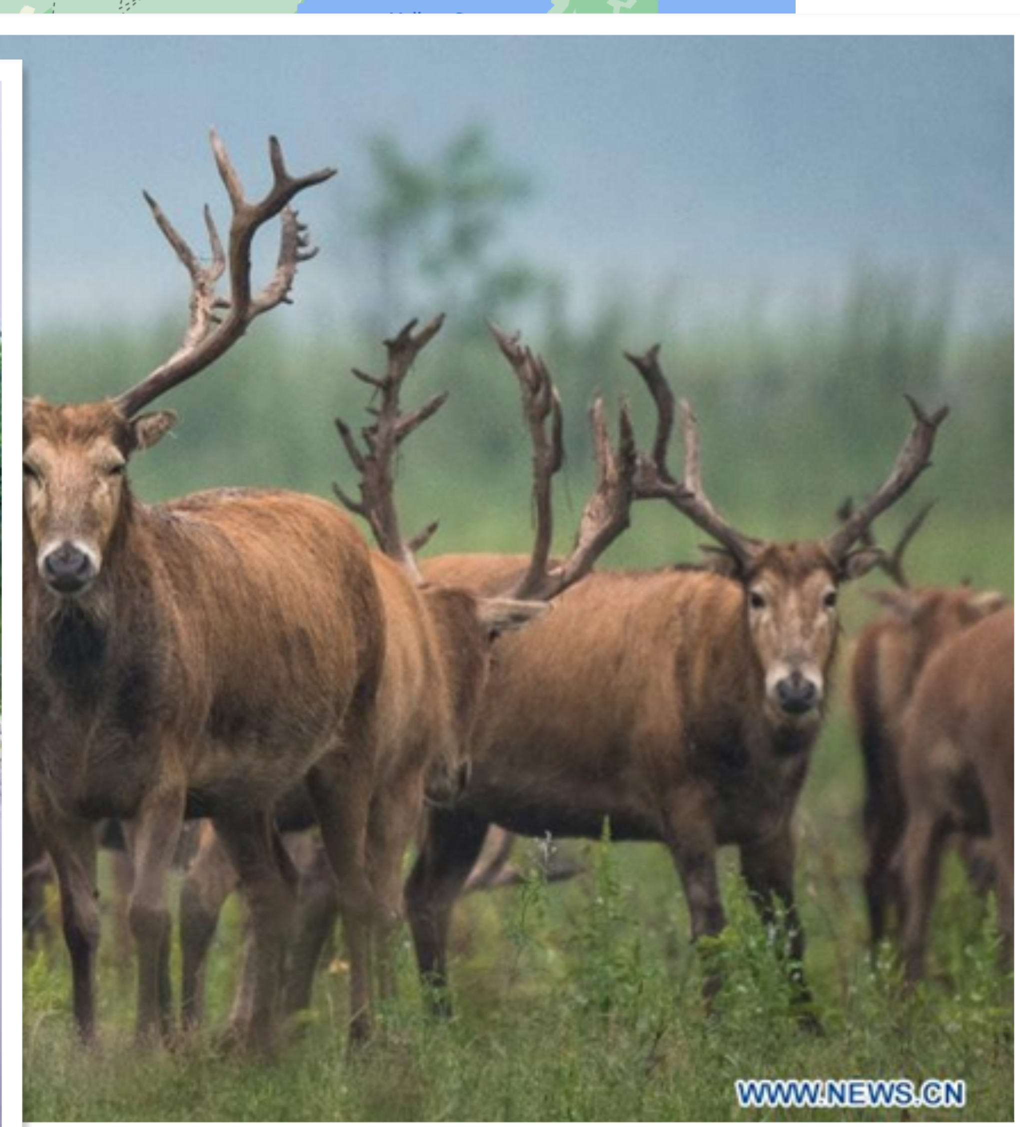

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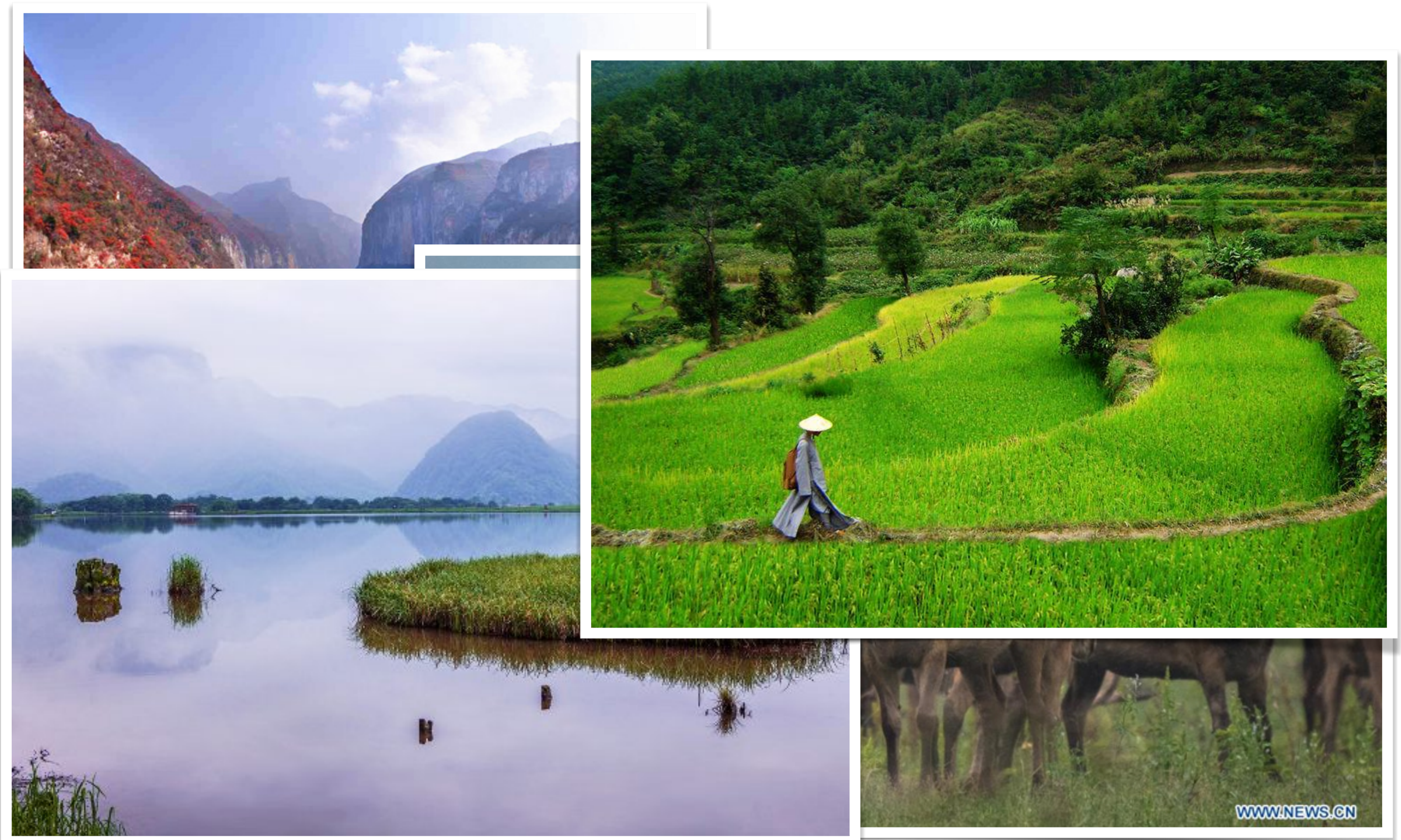

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Think Like A
Math Historian

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- This is the first example in history of an important area of math. What area of math is it?

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- In their particular example, $c - a = 1$. So if their example was supposed to only represent those cases, it was correct. Otherwise it had an error.

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- If you guess the number of people and coins, and once you are too high and once you are too low, here is a procedure to tell you how to combine your wrong guesses into a correct solution.

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- Modern notation: $x + \frac{1}{4}x = 15$. Solve for x. They give a method to solve this problem.

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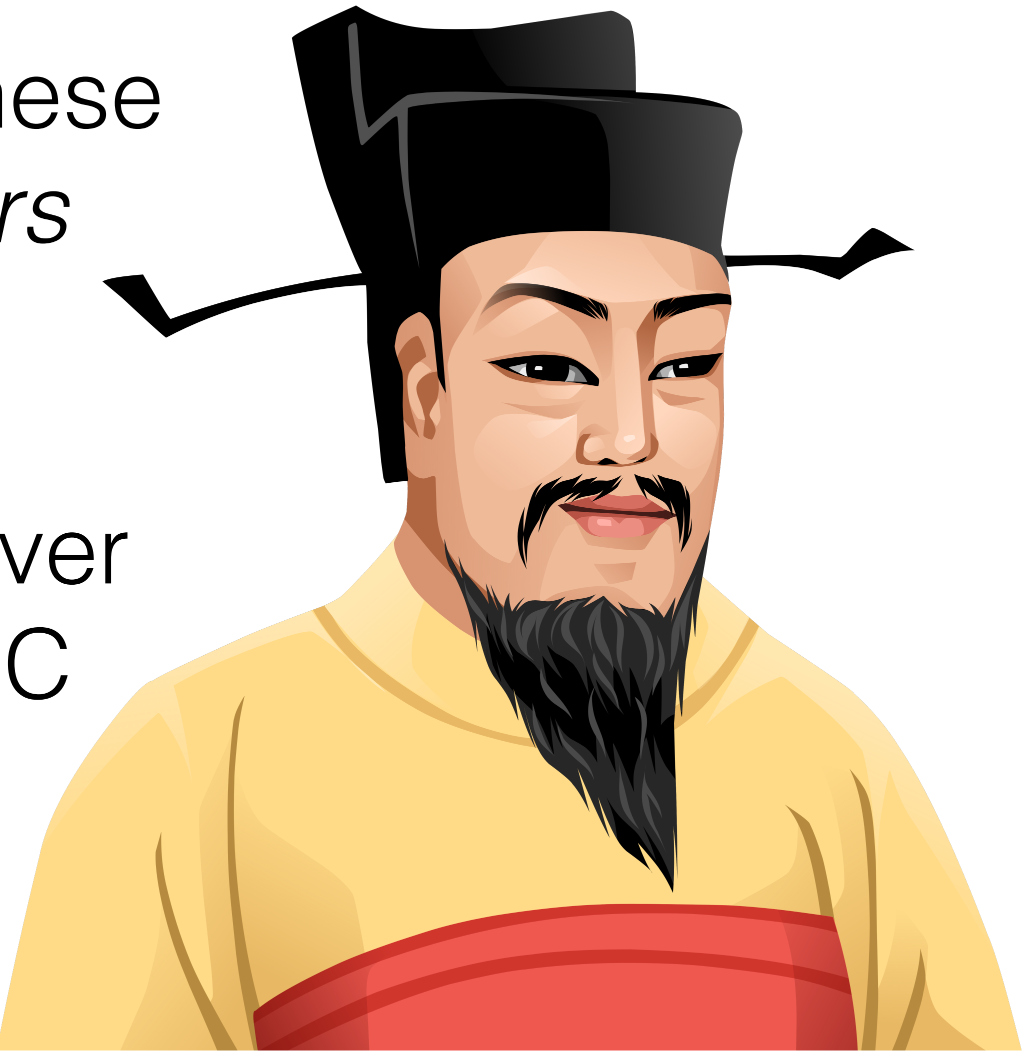

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- It was written by many authors over hundreds of years, from ~1000BC to ~100 BC. In 263 AD, Liu Hui published an annotated edition.

- It is organized like the *Book on Numbers and Computation*. That is, a problem is stated, an answer is given, and a procedure is presented for how to solve the problem and similar problems.

Show a copy of

Nine Chapters

Chinese Mathematics

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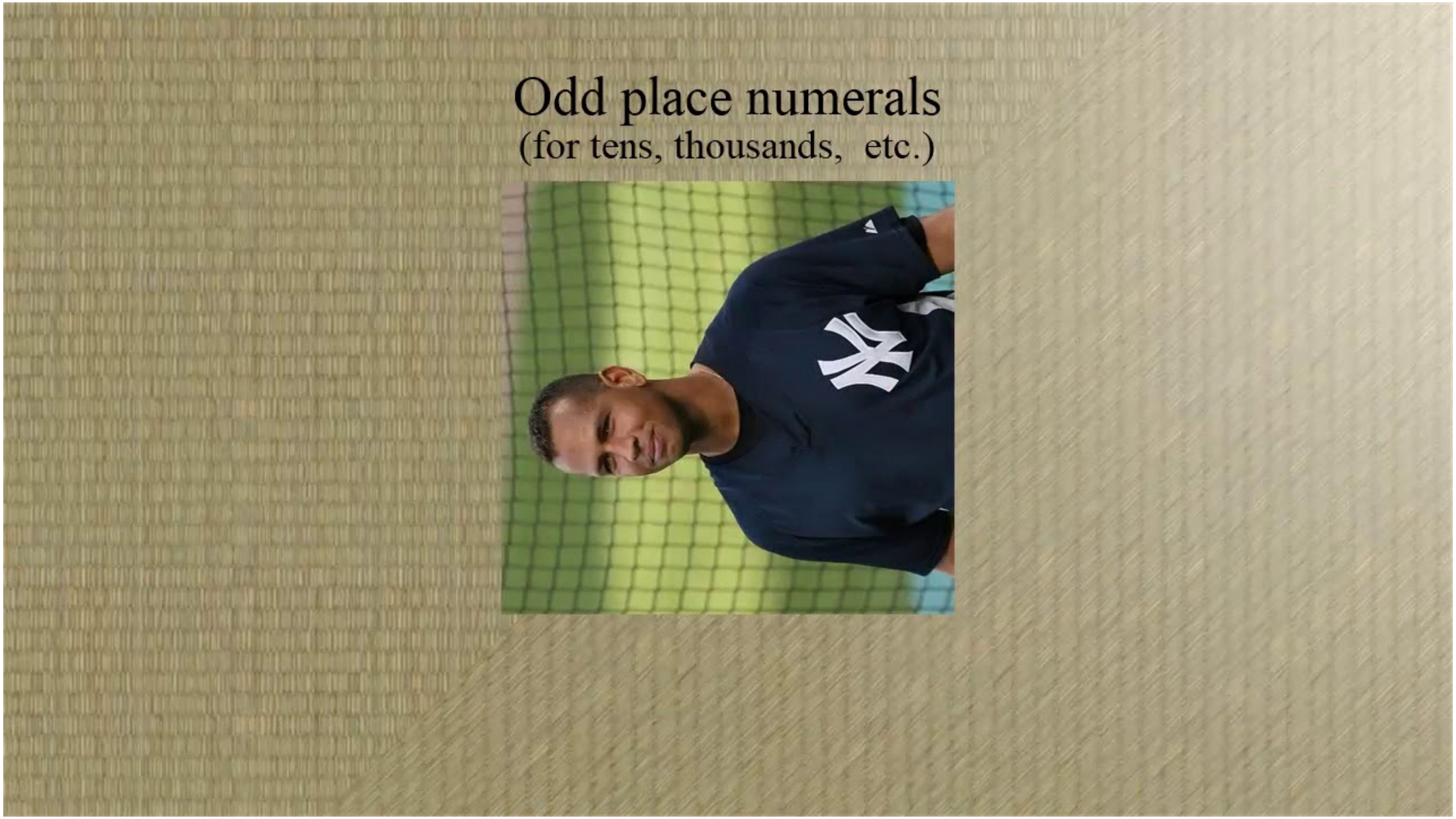
- *Fang cheng* is essentially Gaussian elimination (discovered 2,000 years before Gauss!)

Chinese Mathematics

- First example on *fang cheng* in the *Nine Chapters*:

> Given 3 bundles of top grade paddy, 2 bundles of medium grade paddy, and 1 bundle of low grade paddy, together they yield 39 *dou* of grain. Given 2 bundles of top grade paddy, 3 bundles of medium grade paddy, and 1 bundle of low grade paddy, together they yield 34 *dou* of grain. Given 1 bundle of top grade paddy, 2 bundles of medium grade paddy, and 3 bundles of low grade paddy, together they yield 26 *dou* of grain. Problem: how much grain does one bundle of high, medium and low grade paddy together yield? Answer: Top grade paddy yields $9\frac{1}{4}$ *dou* per bundle; medium grade paddy yields $4\frac{1}{4}$ *dou* per bundle; low grade paddy yields $2\frac{3}{4}$ *dou* per bundle.

Chinese Mathematics



Chinese Mathematics

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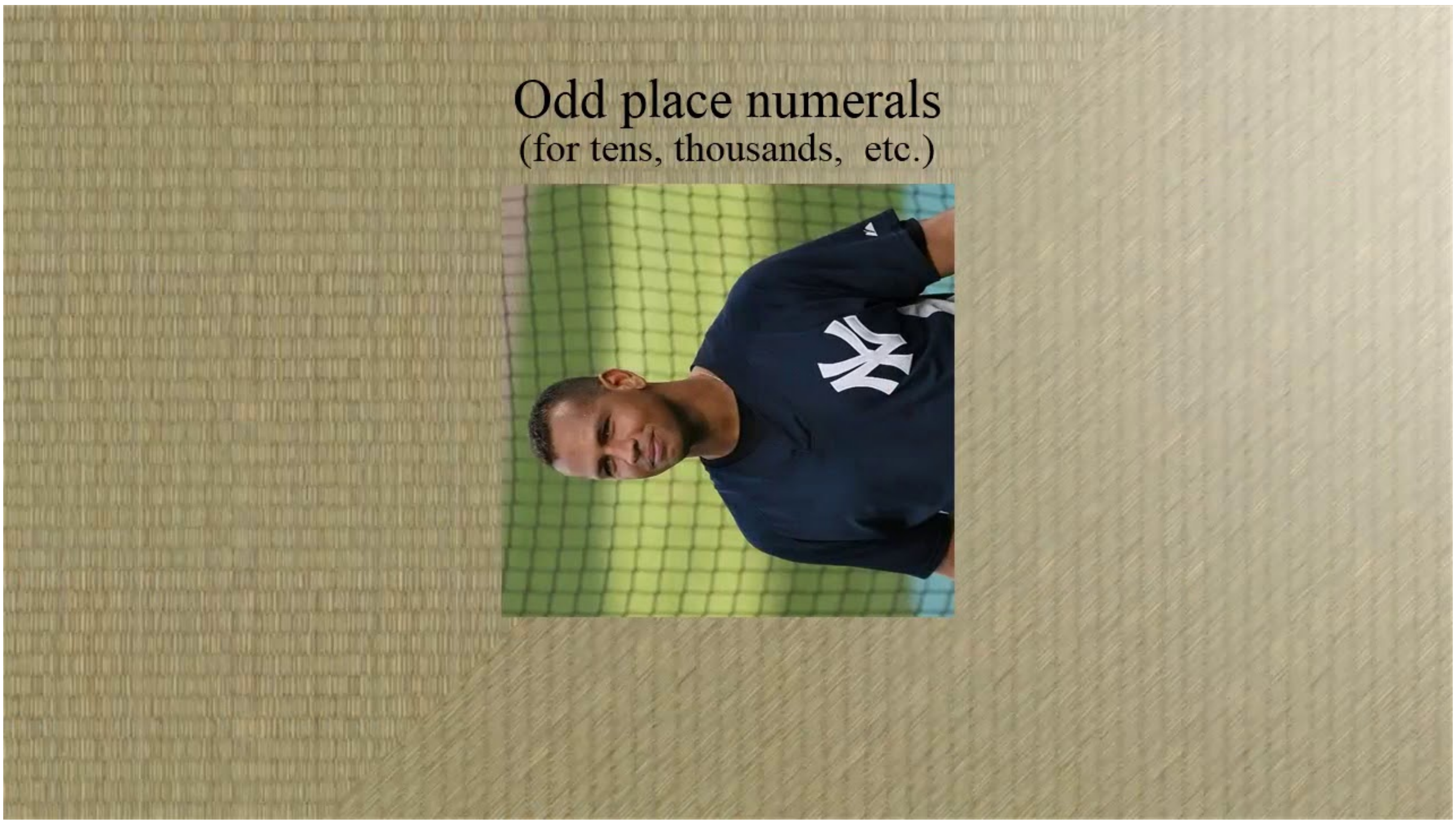

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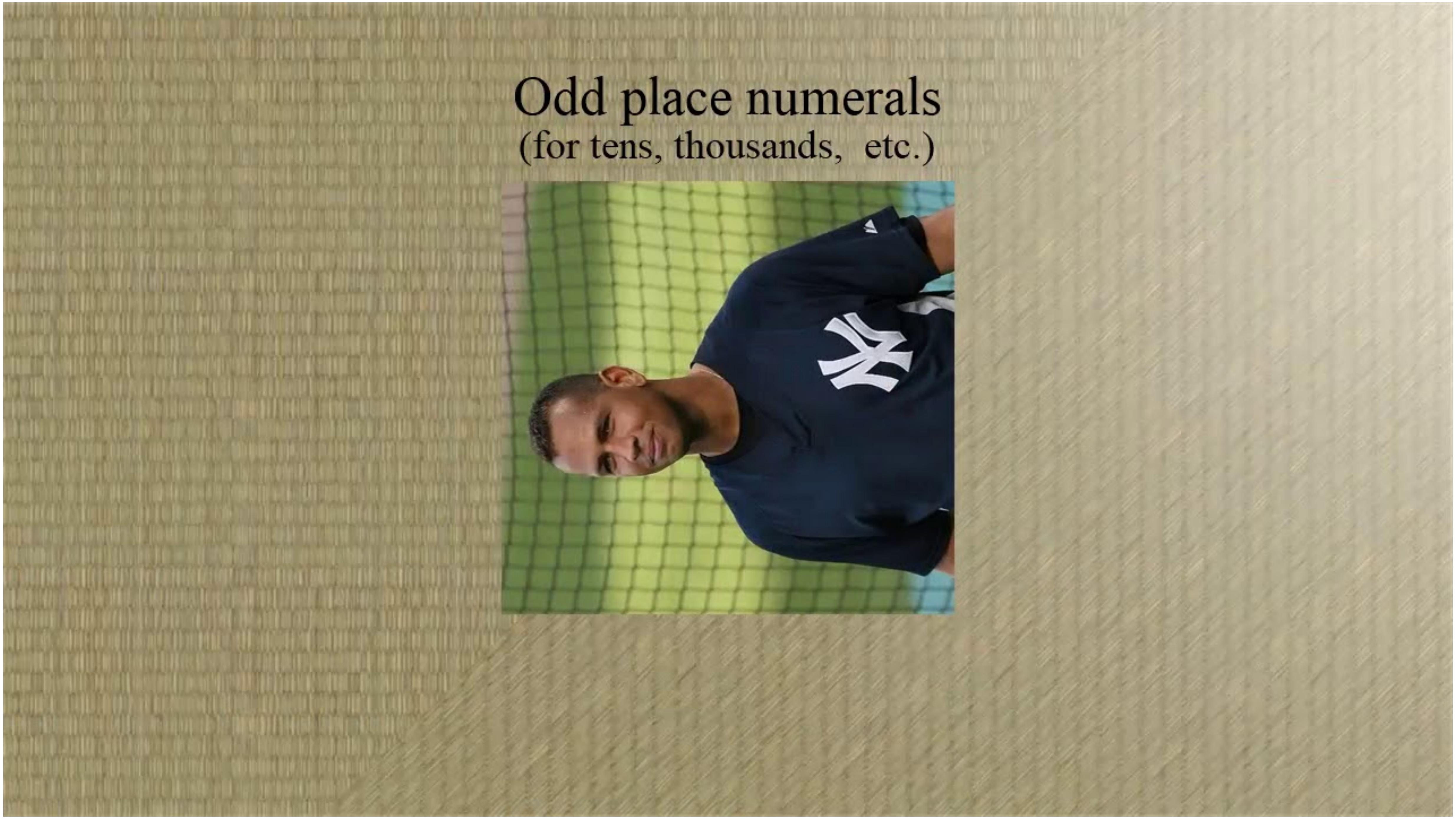

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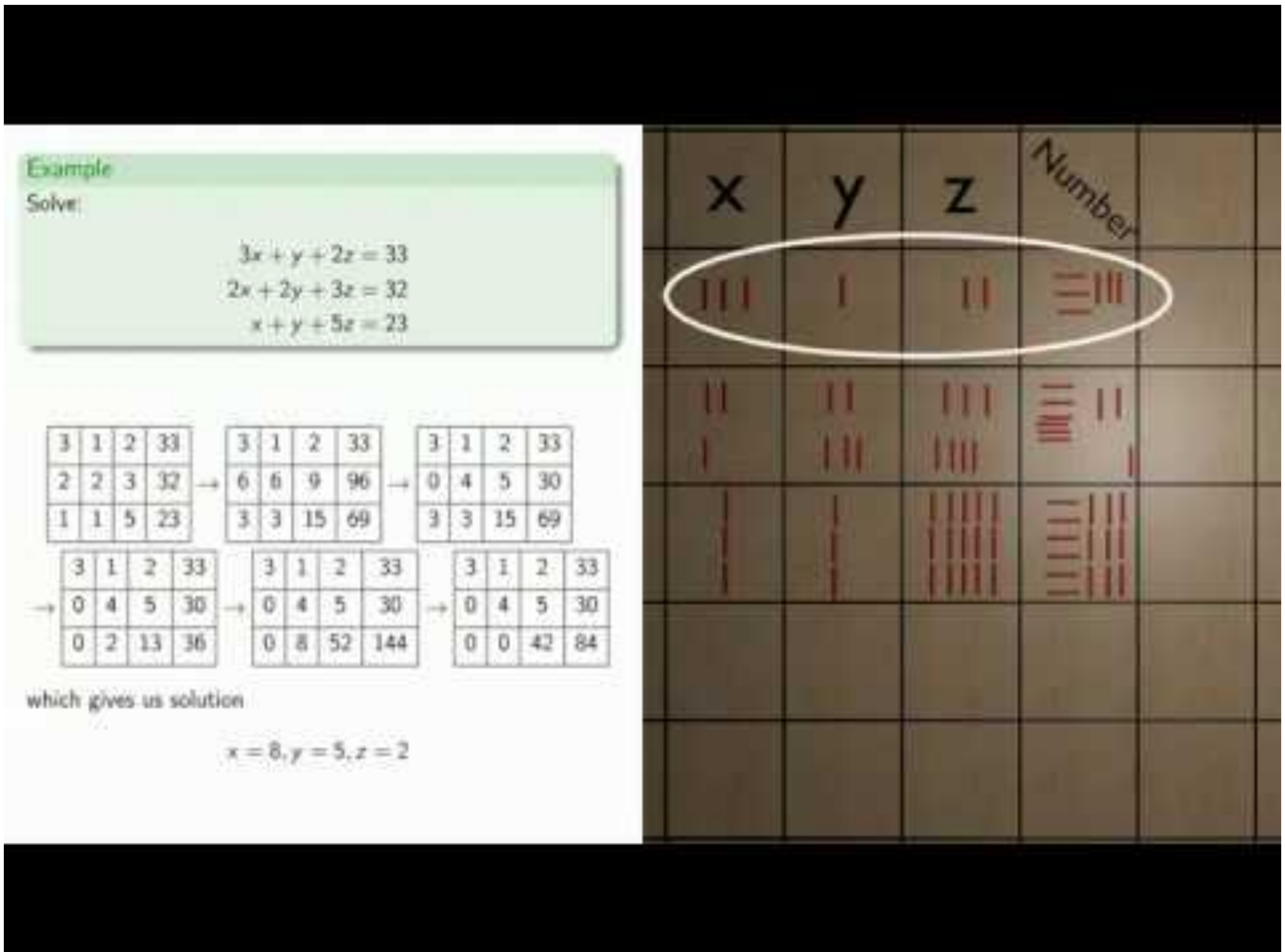

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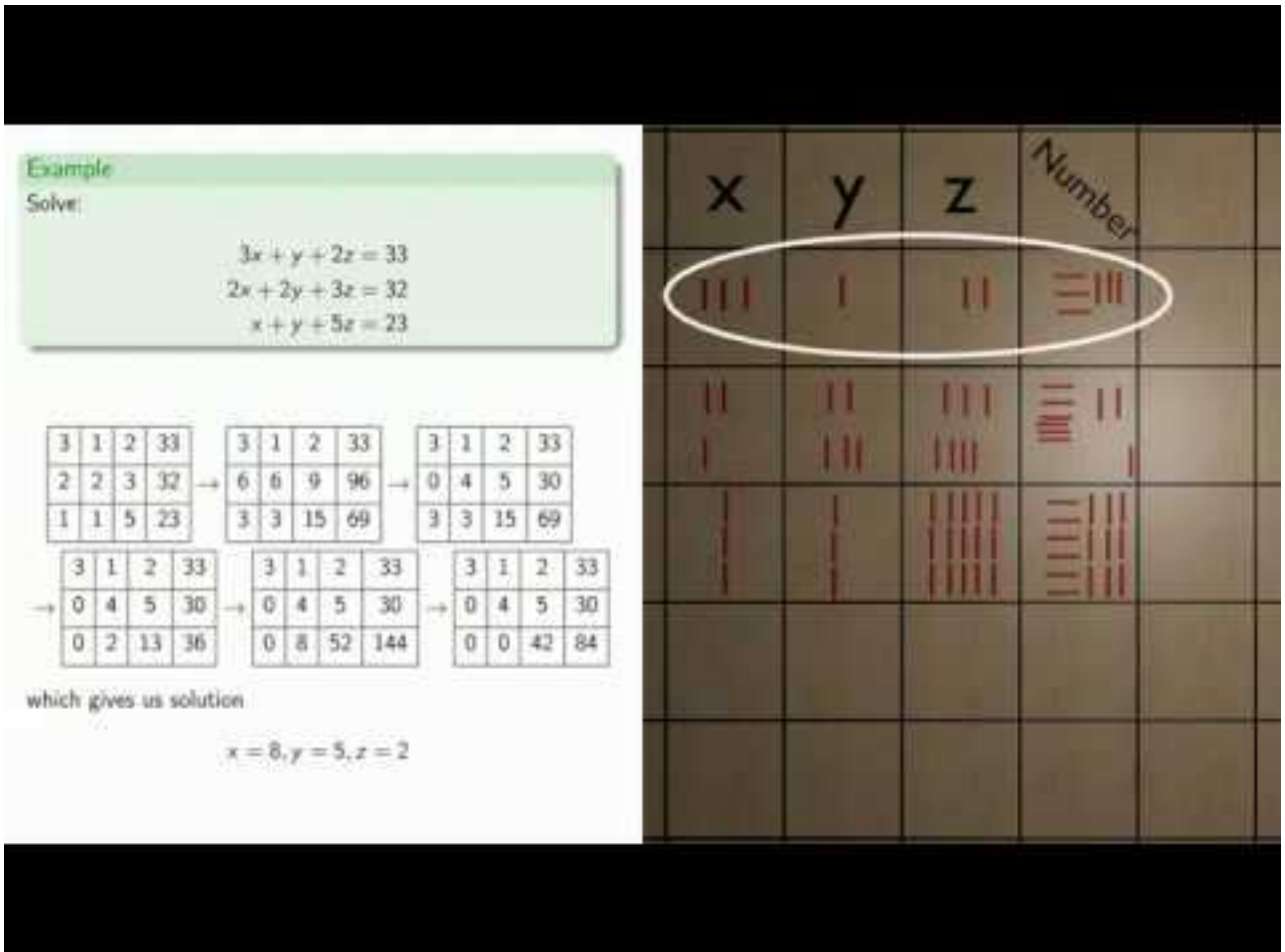

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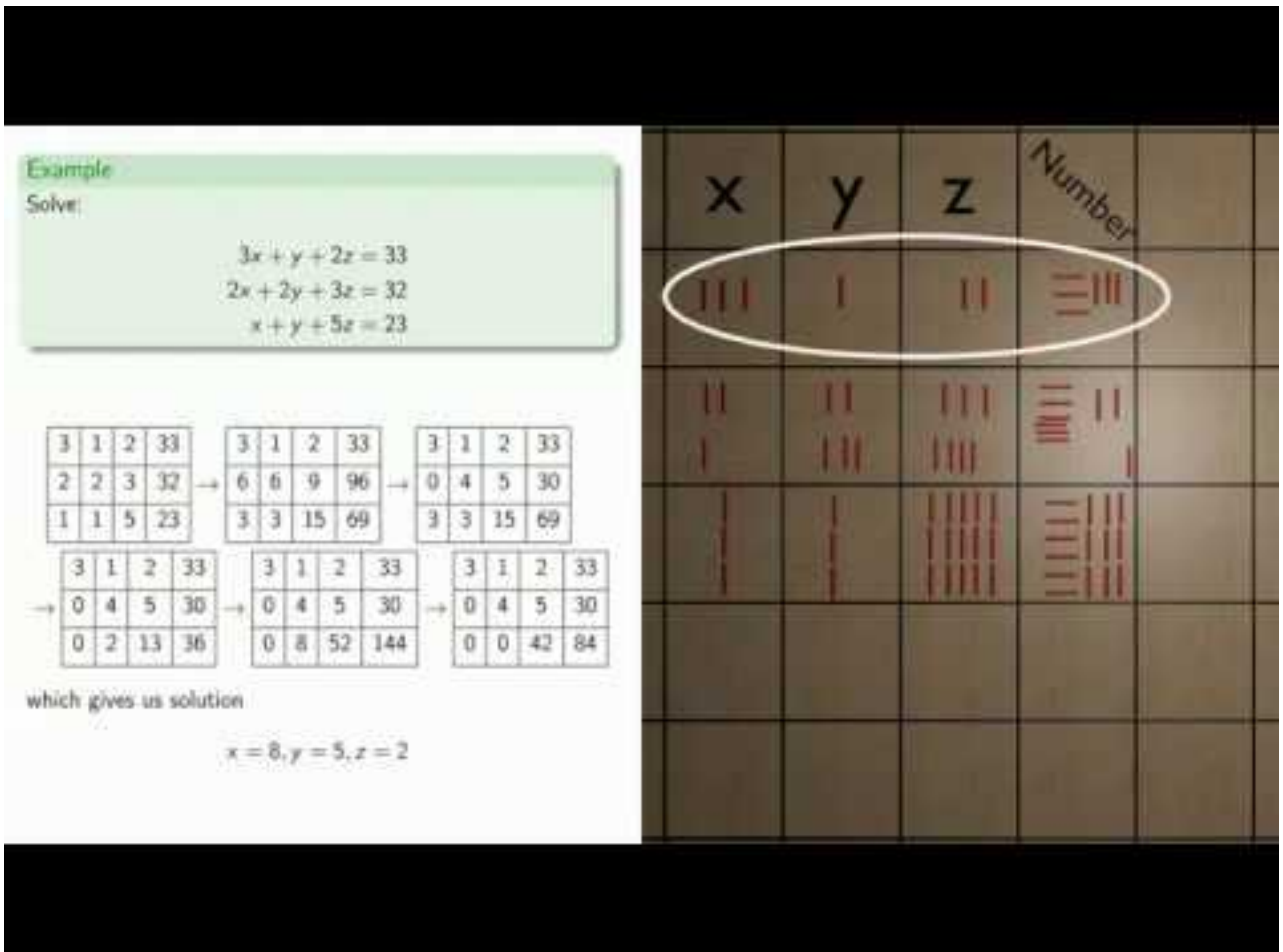

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- *Fang cheng* uses a more complicated approach so that there are only integers until the very end. This is advantageous when solving the problem using a counting board.

- Gaussian elimination is happy to introduce fractions as soon as back-substitution begins.

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Chapter 4: Finding dimensions of a shape given its area or volume. Finding the square and cube roots.

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Chapter 9: Problems using the Pythagorean theorem.

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- Issac Newton was the first in the West. He noticed that no Algebra book he knew of solved systems of linear equations, so he created Gaussian elimination.

> “And you are to know, that by each Æquation one unknown Quantity may be taken away, and consequently, when there are as many Æquations and unknown Quantities, all at length may be reduc’d into one, in which there shall be only one Quantity unknown.”

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- Calling it Gaussian elimination started in the 1950s based on a misreading of history.

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- Before universal education, it was really important to clearly communicate how to do arithmetic.

- One pioneer in this was Wang Zhenyi, from 18th century China. She wrote a five-volume text doing so.

People’s History

People's History of Trigonometry

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- The Mayans had a similar method.

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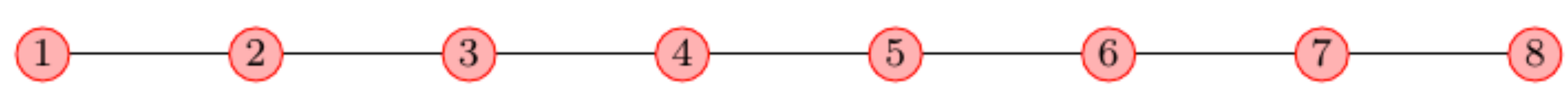

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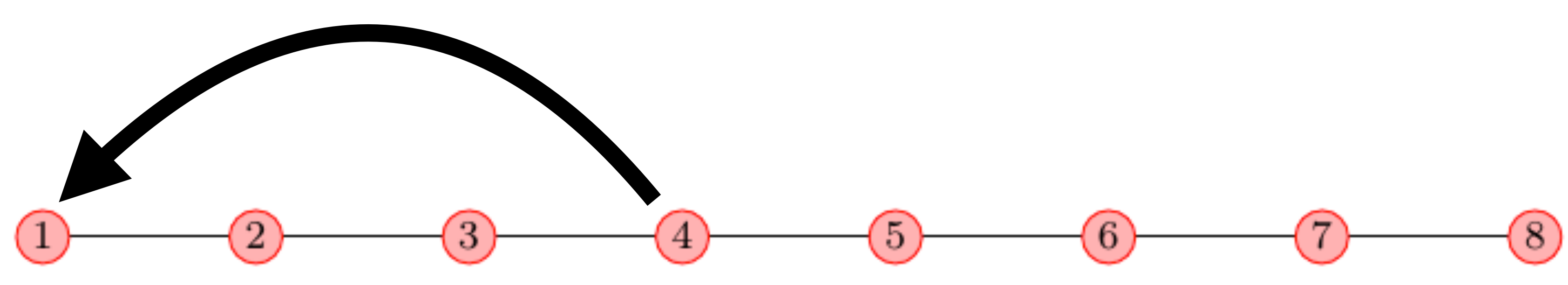

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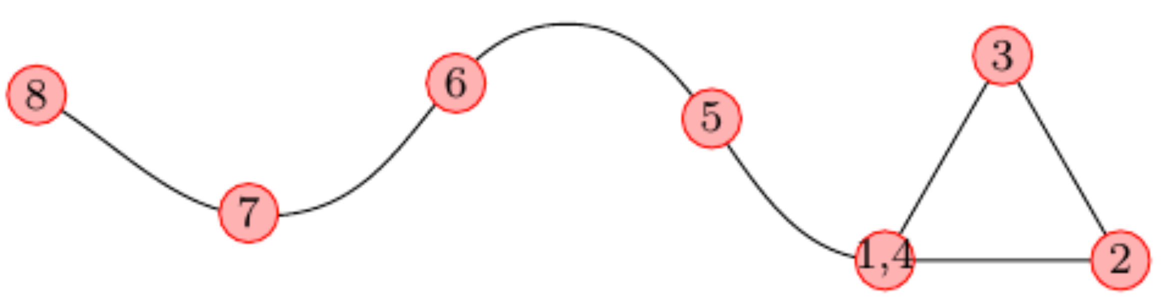

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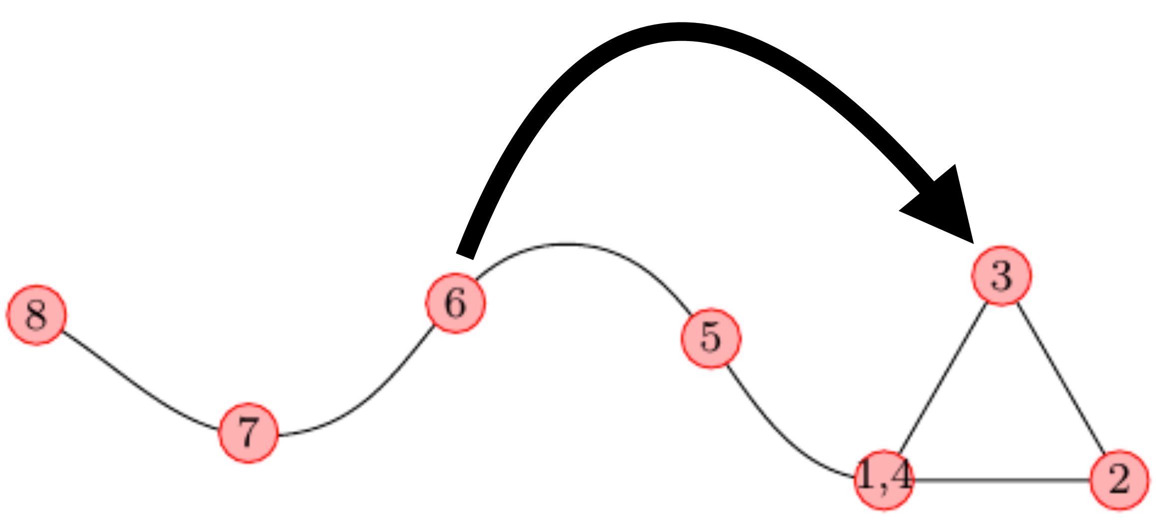

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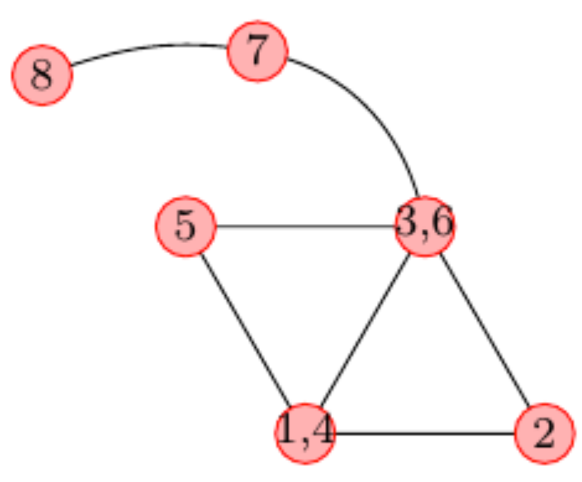

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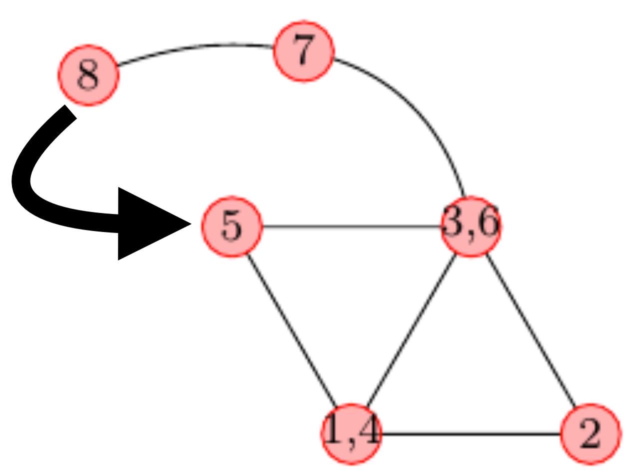

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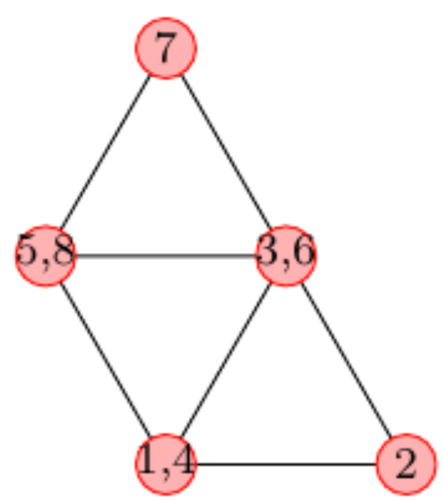

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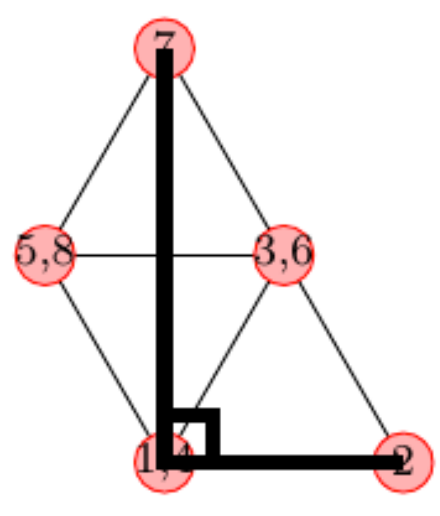

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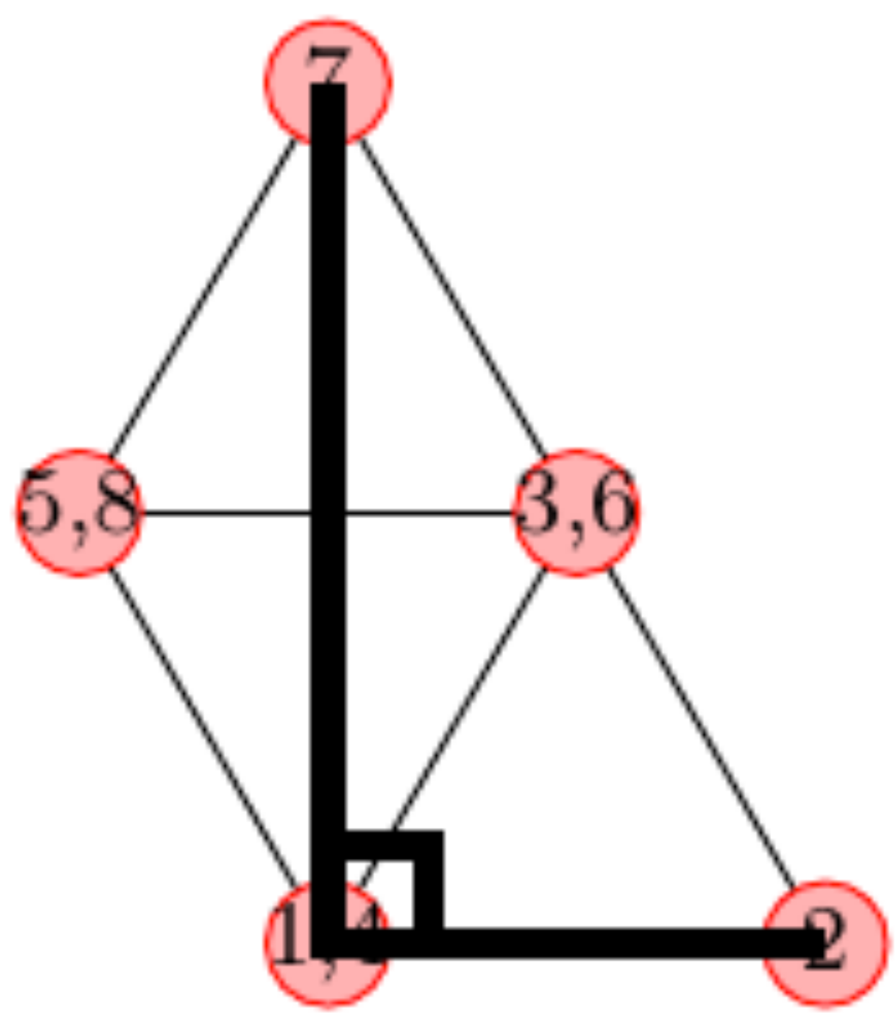

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- Roman architect Vitruvius recorded a clever approach using the Sun.

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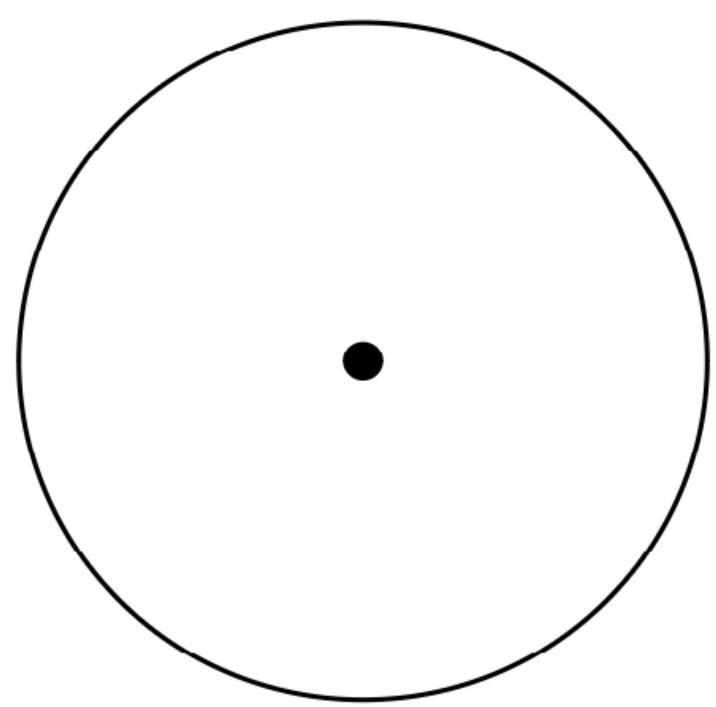

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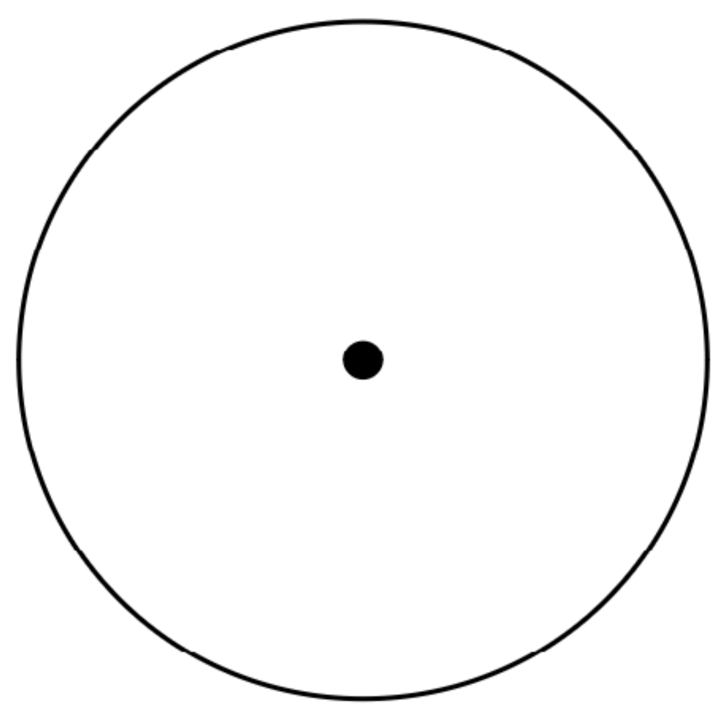

People's History of Trigonometry

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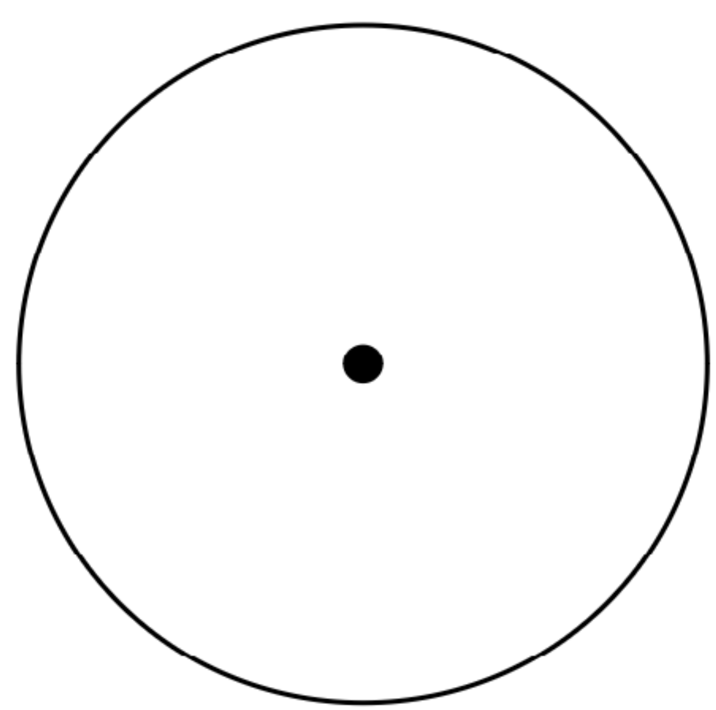

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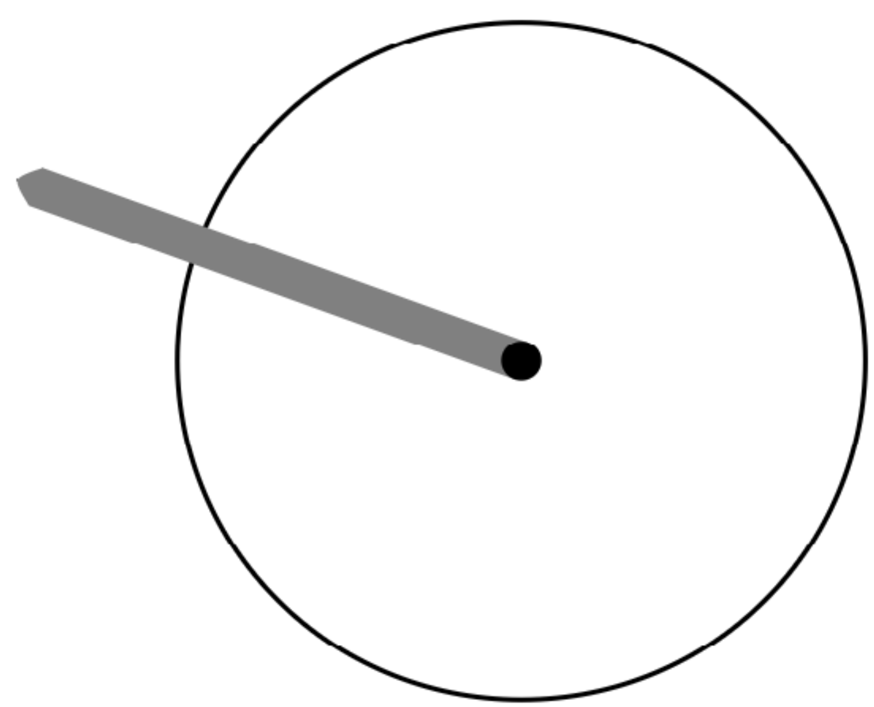

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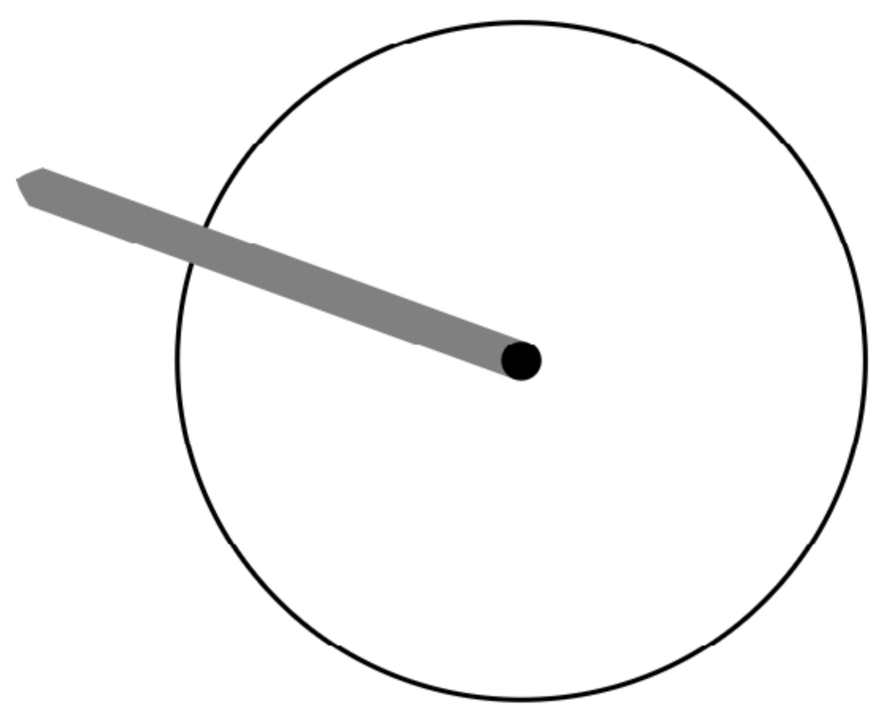

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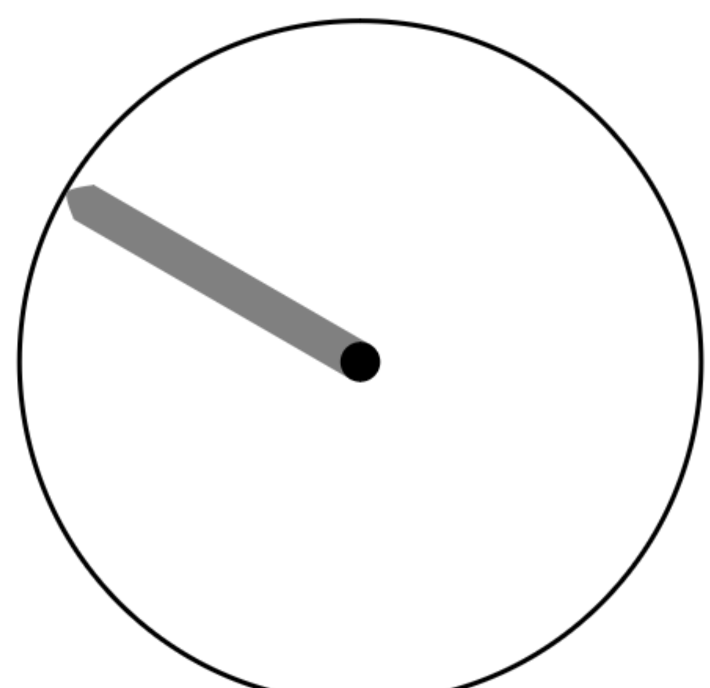

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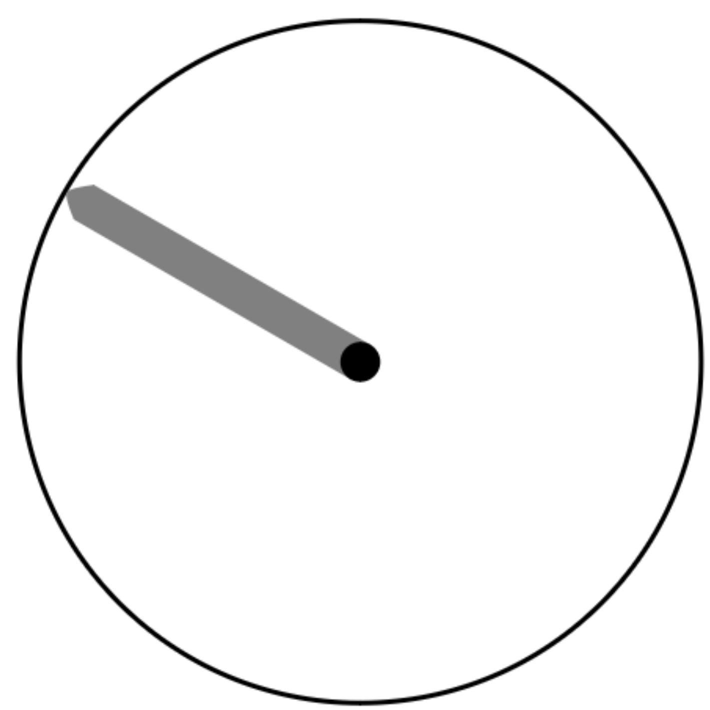

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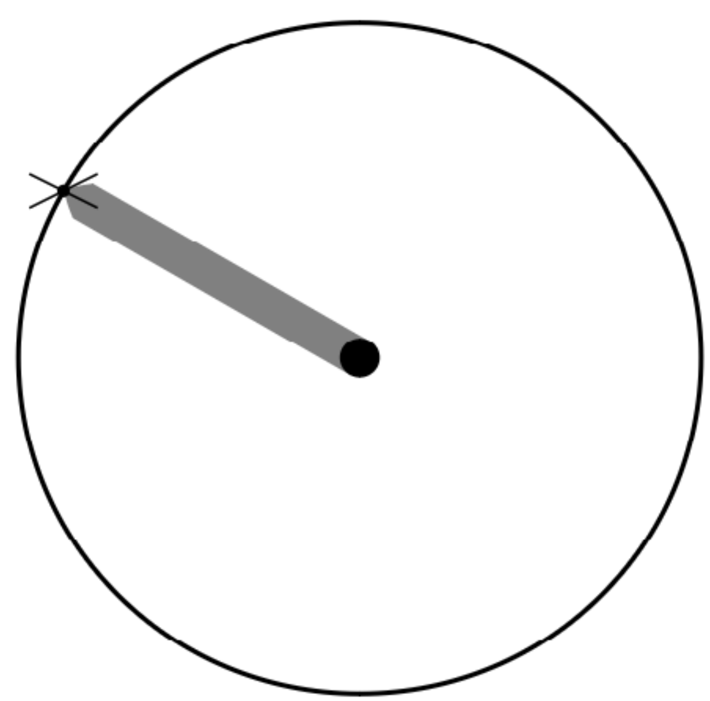

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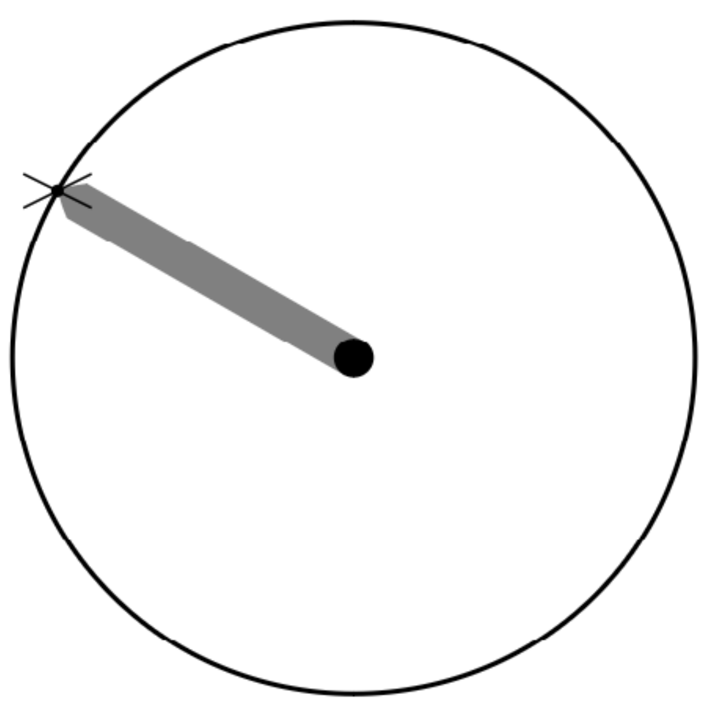

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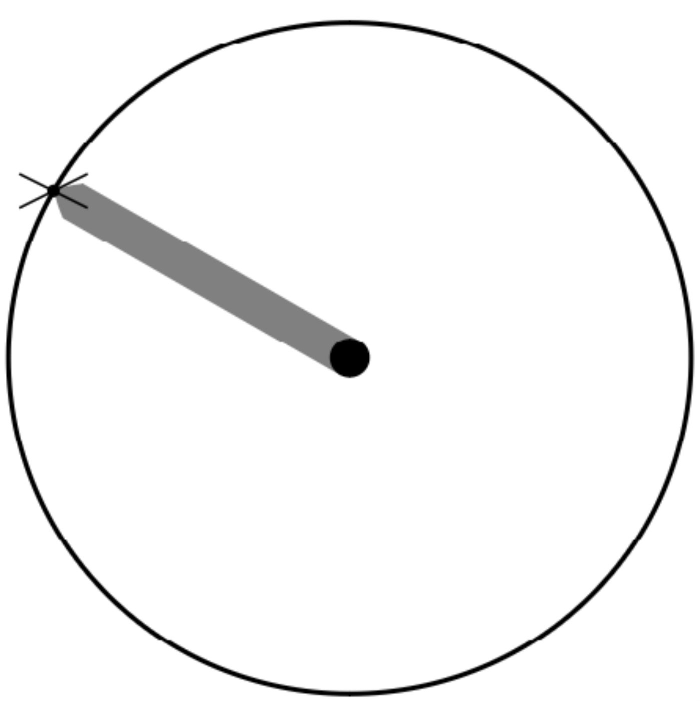

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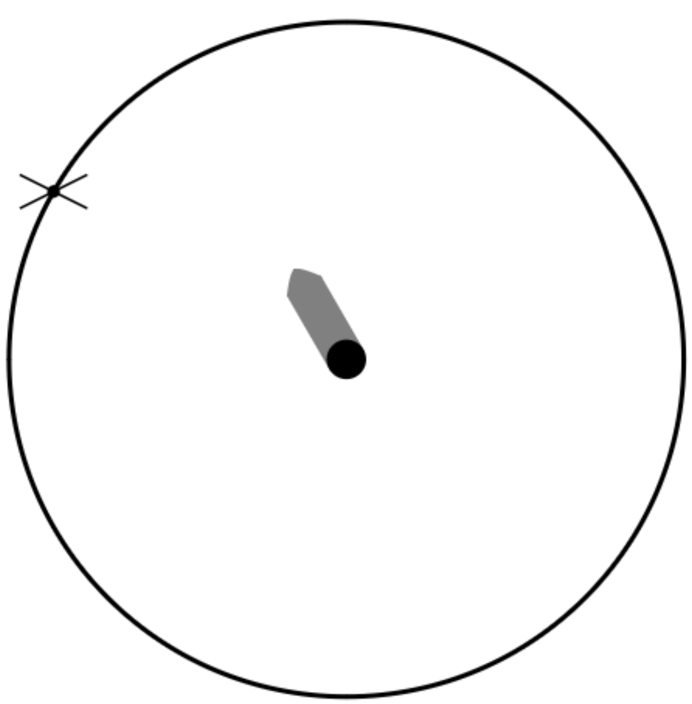

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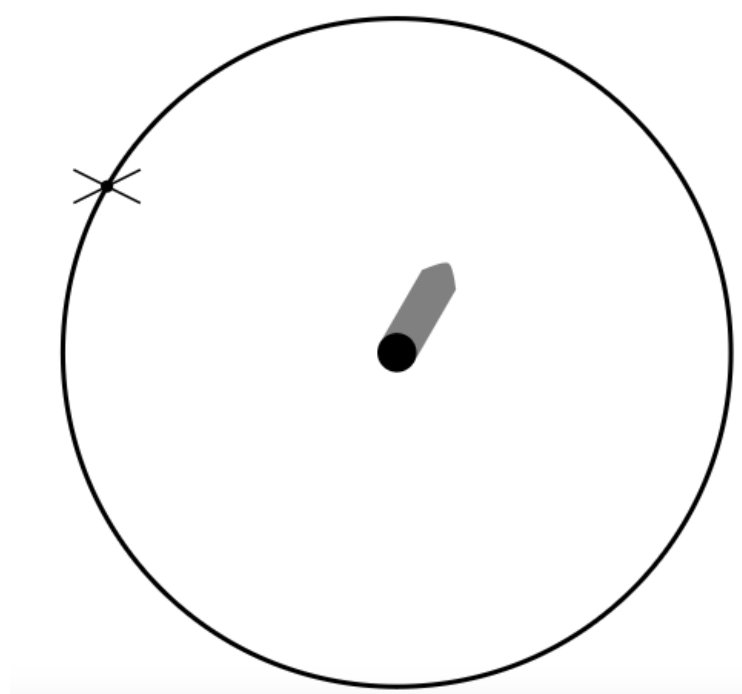

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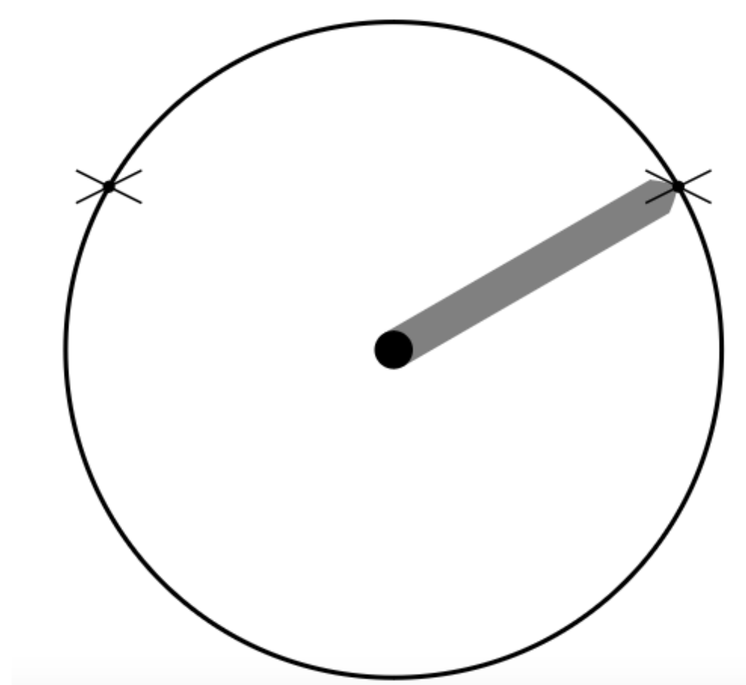

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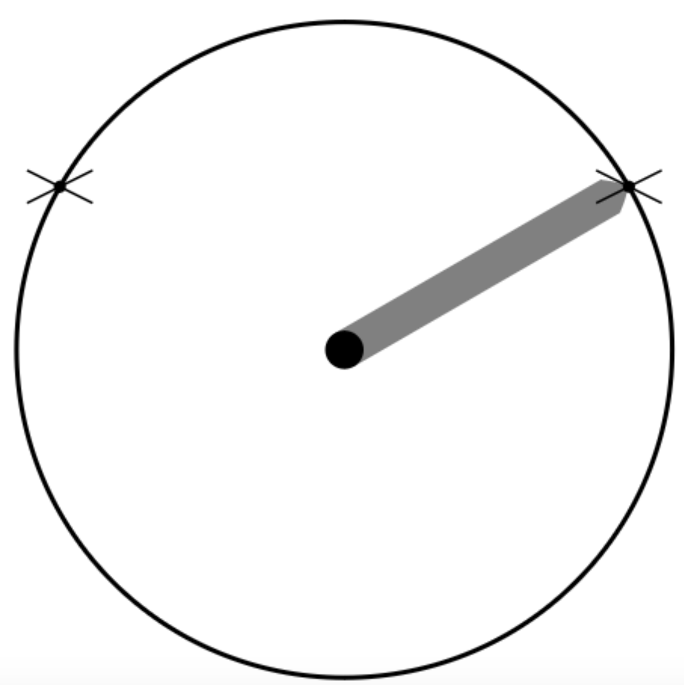

People’s History of Trigonometry

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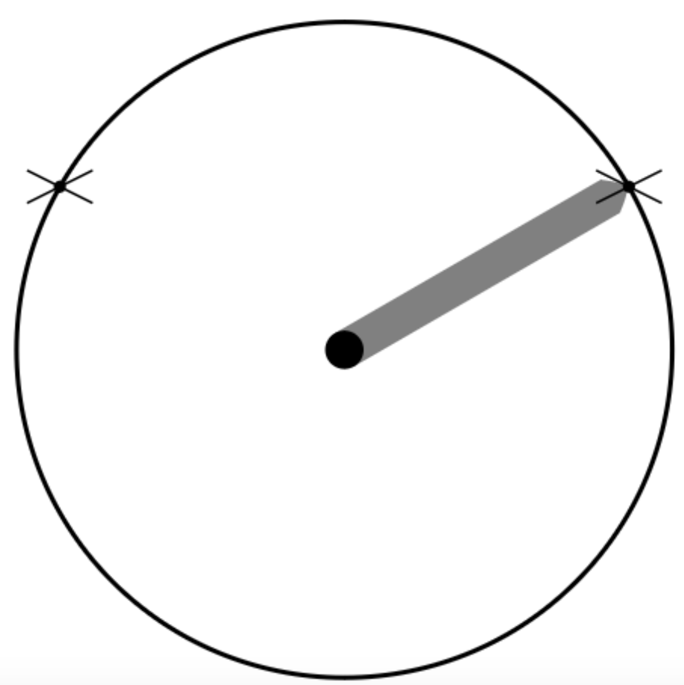

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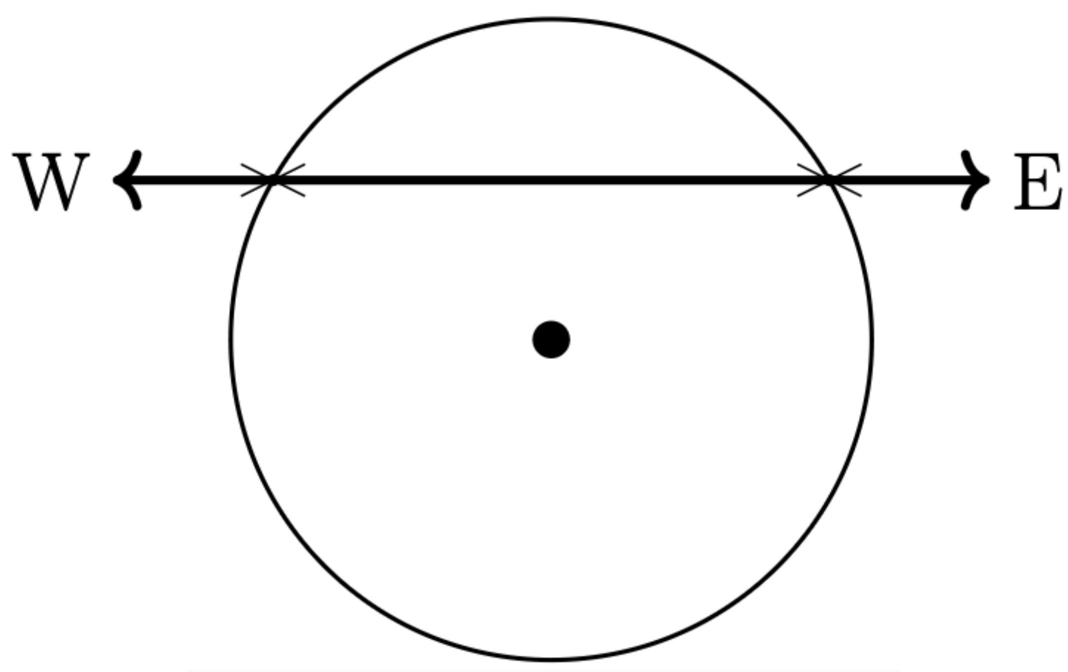

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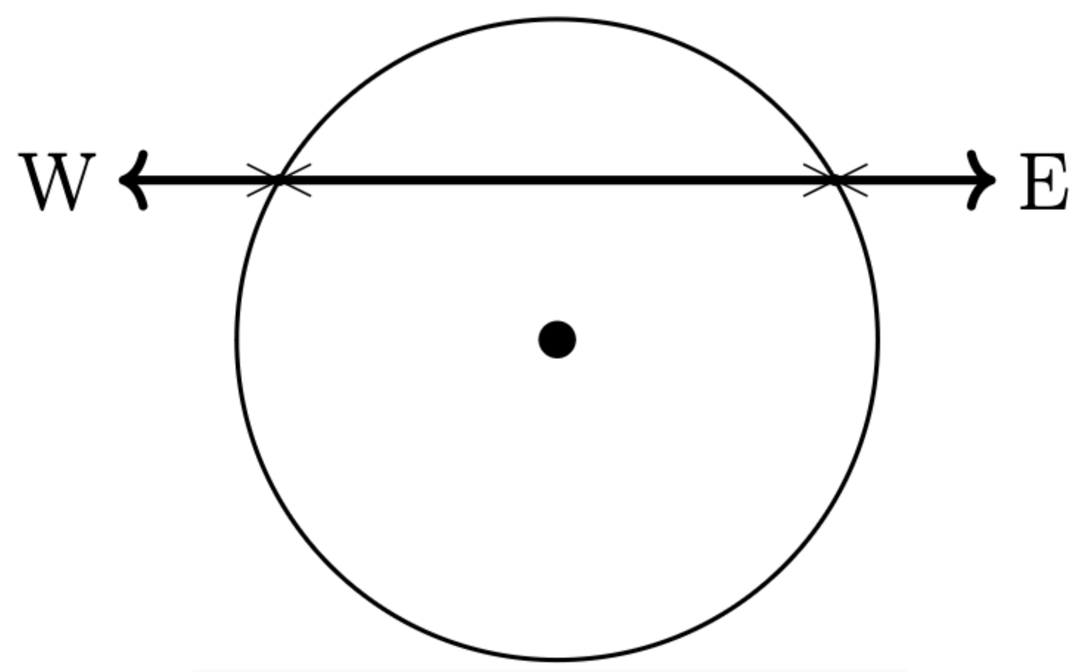

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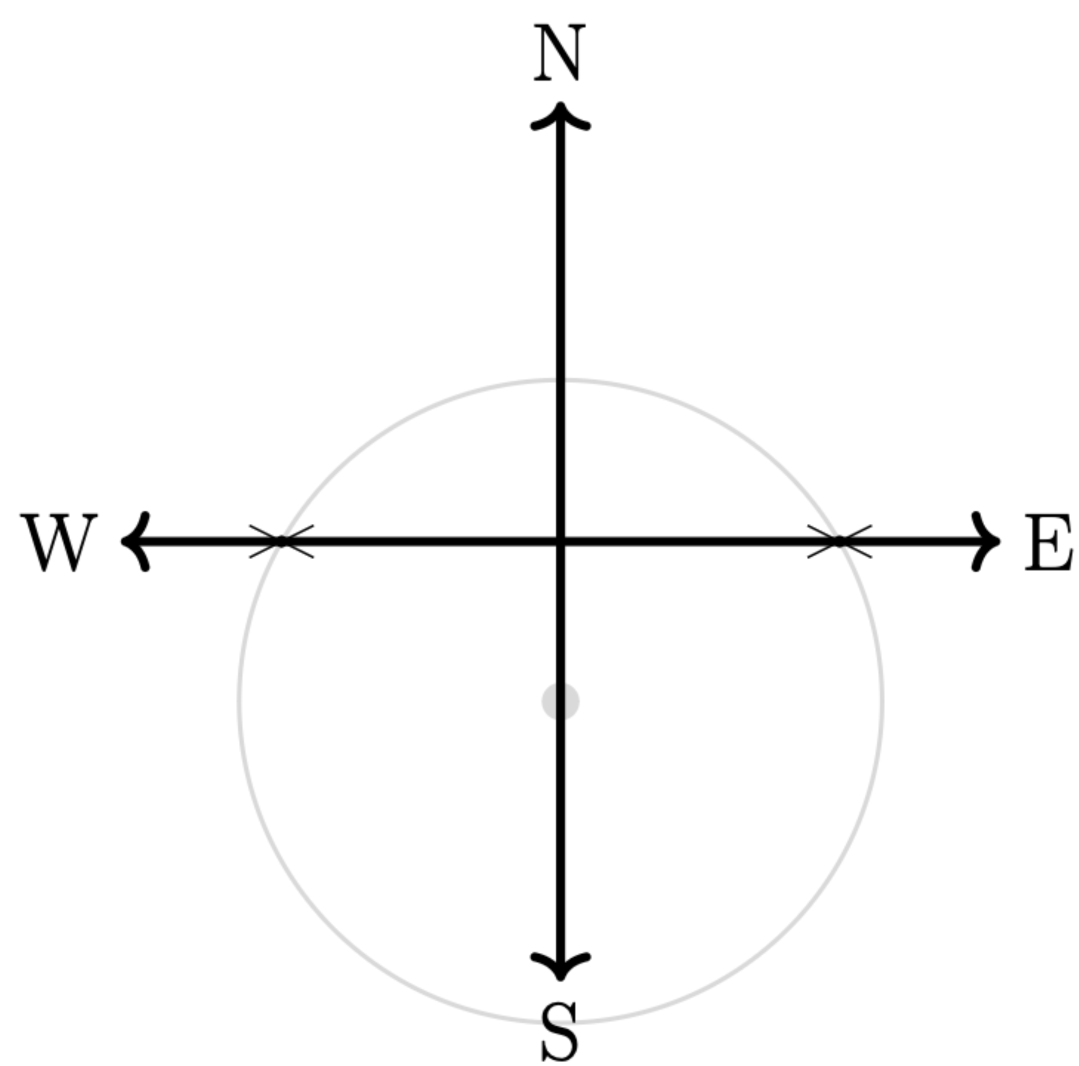

People’s History of Trigonometry

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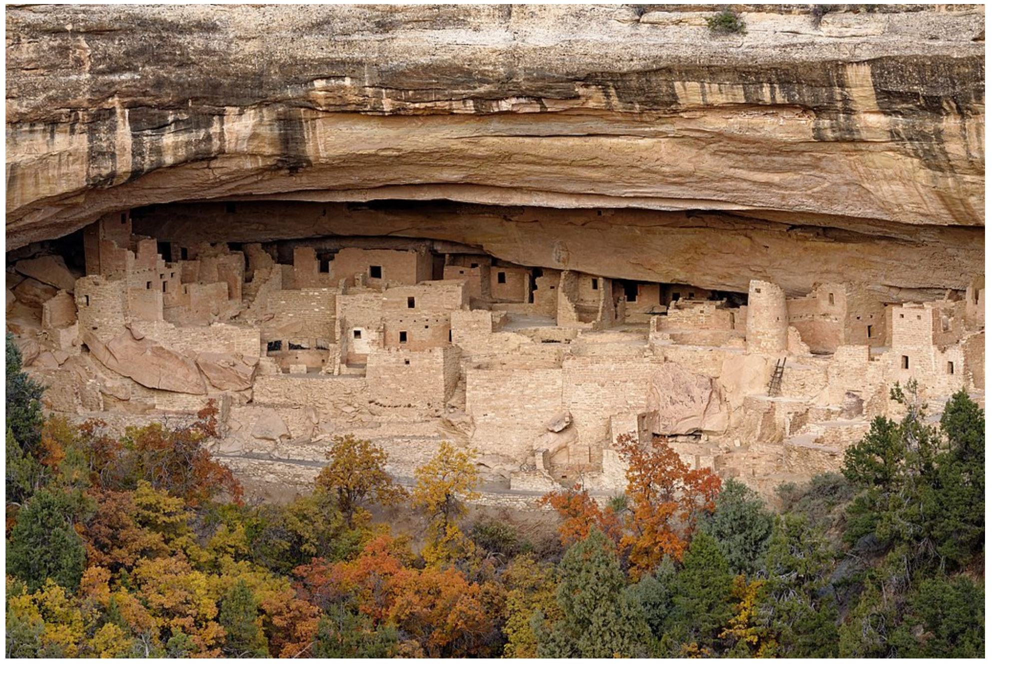
- Note: This same technique may have been known to the Ancestral Puebloans.



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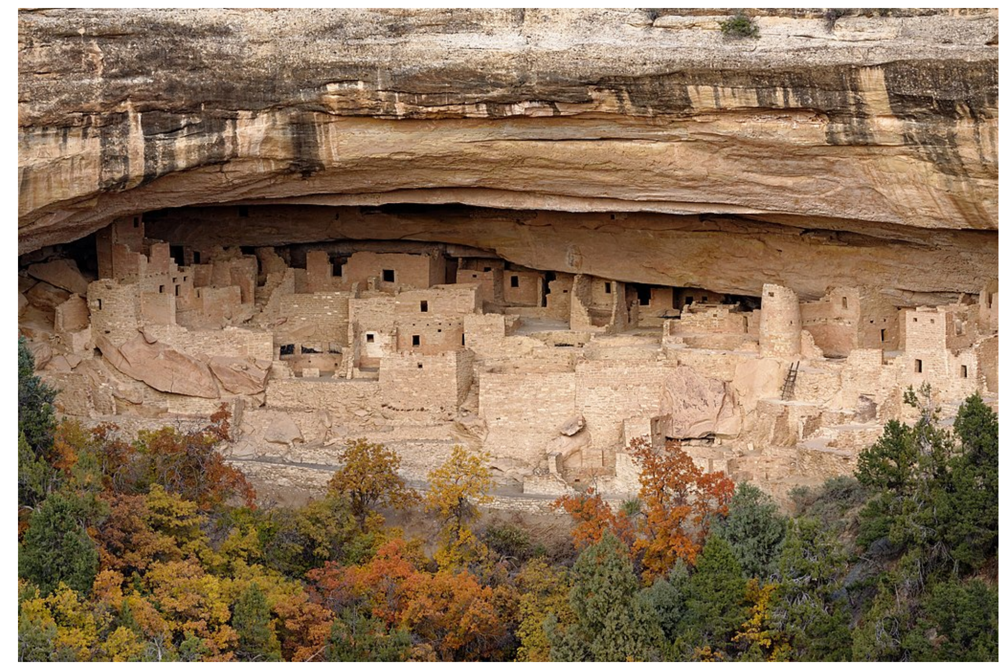
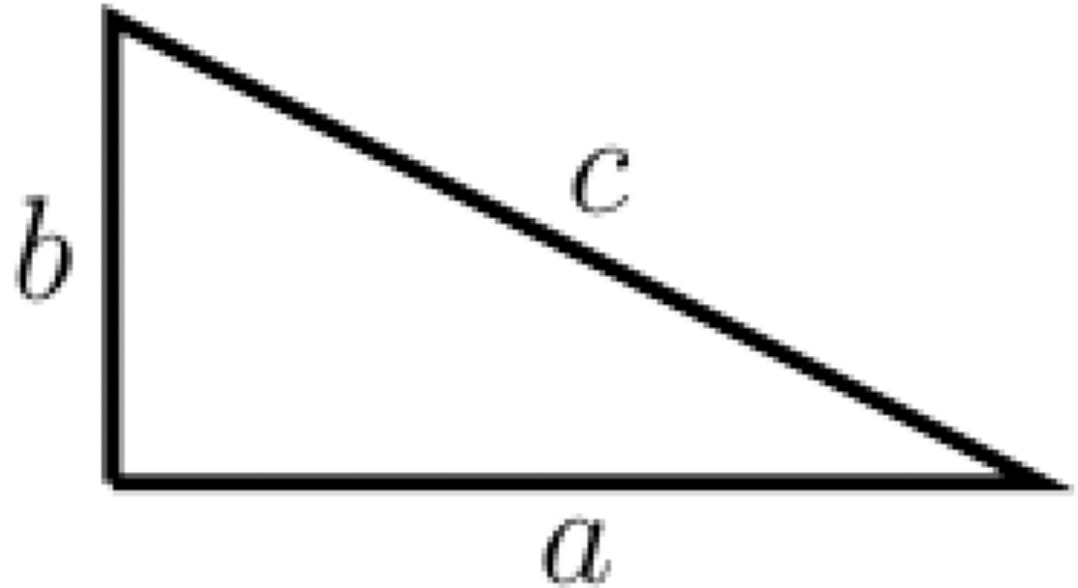
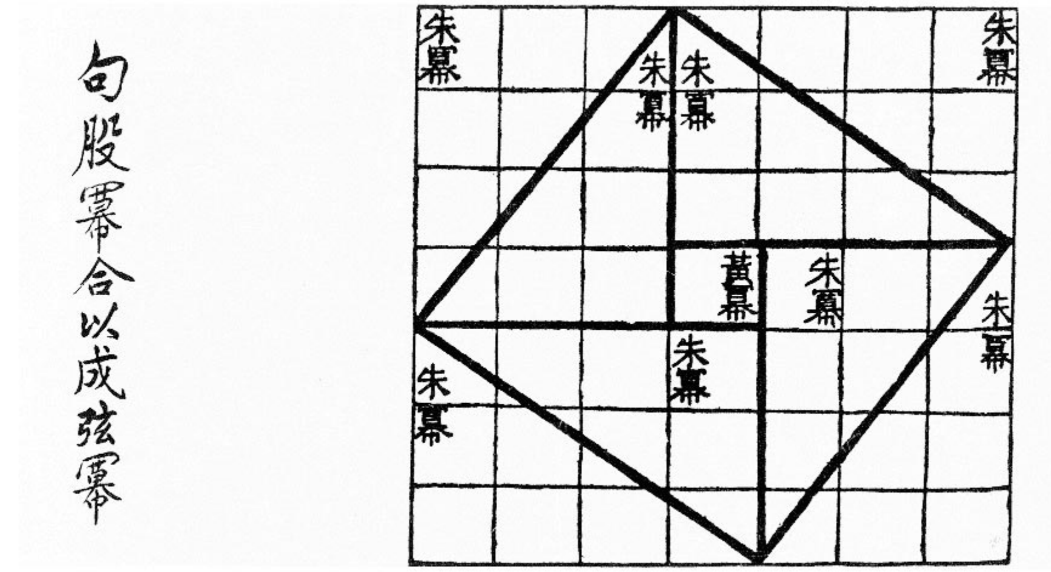
- But they left no records, so some speculation is required.







Chapter 3: Demonstrative Mathematics

Demonstrative Math

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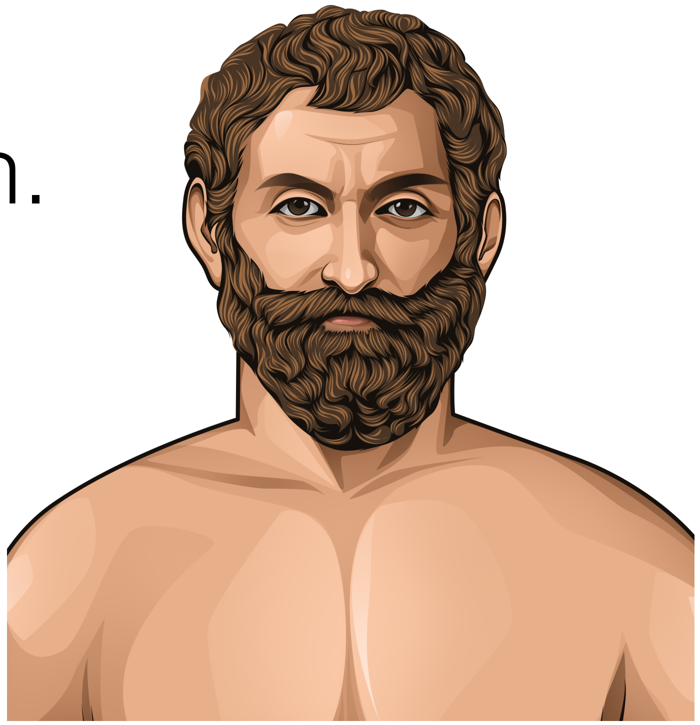

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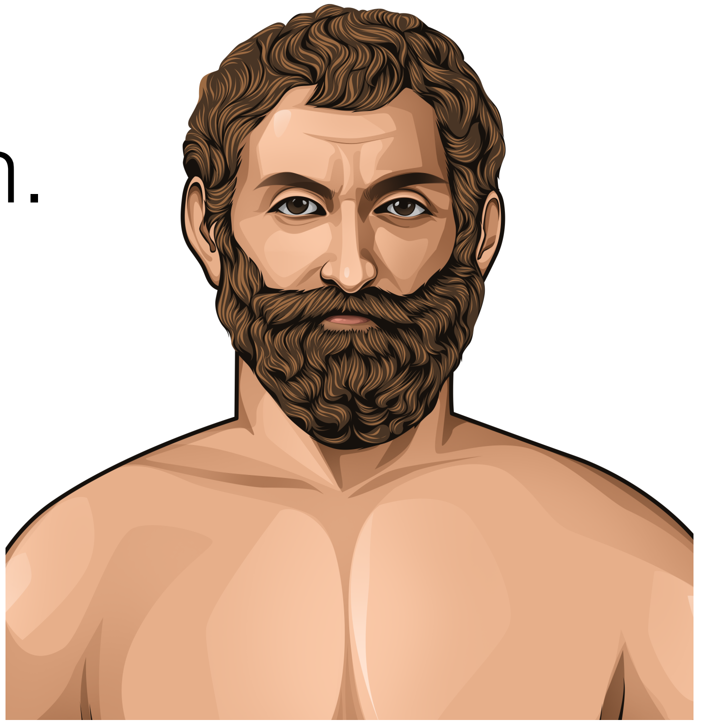

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- Aesop had two stories about him.

The First Proof

- Thales is the first person to whom a mathematical proof is attributed.
- He’s also called the Father of Science because he used evidence and theories, not mythology, to study and explain the world.
- Aesop had two stories about him.
- First absent-minded prof in history.

The First Proof

Theorem. The diameter of any circle divides that circle into two equal halves.

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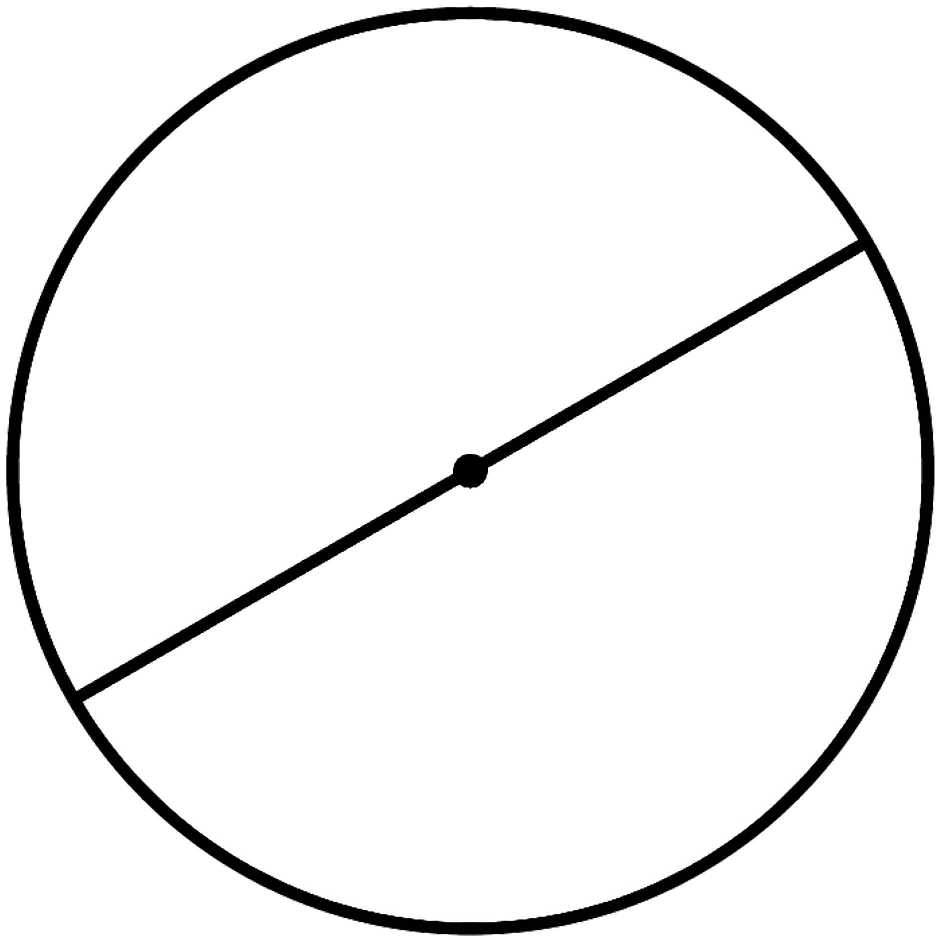
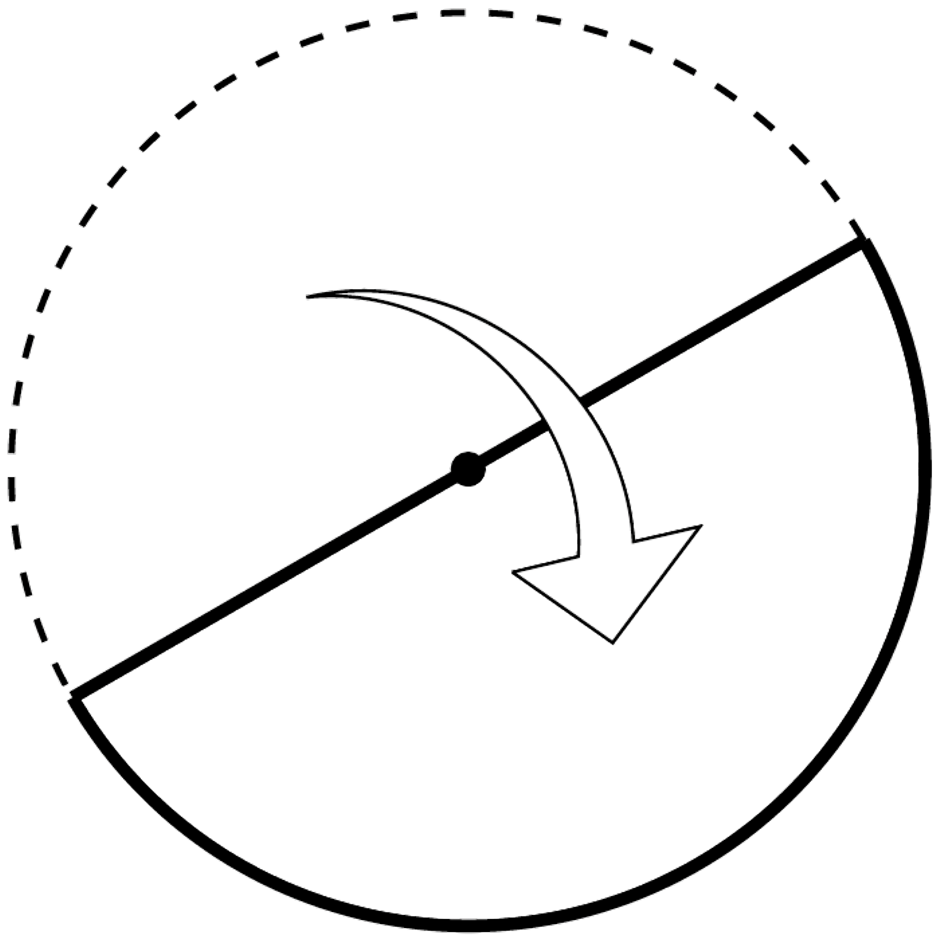
Proof. Suppose you have a circle and diameter



The First Proof

Theorem. The diameter of any circle divides that circle into two equal halves.

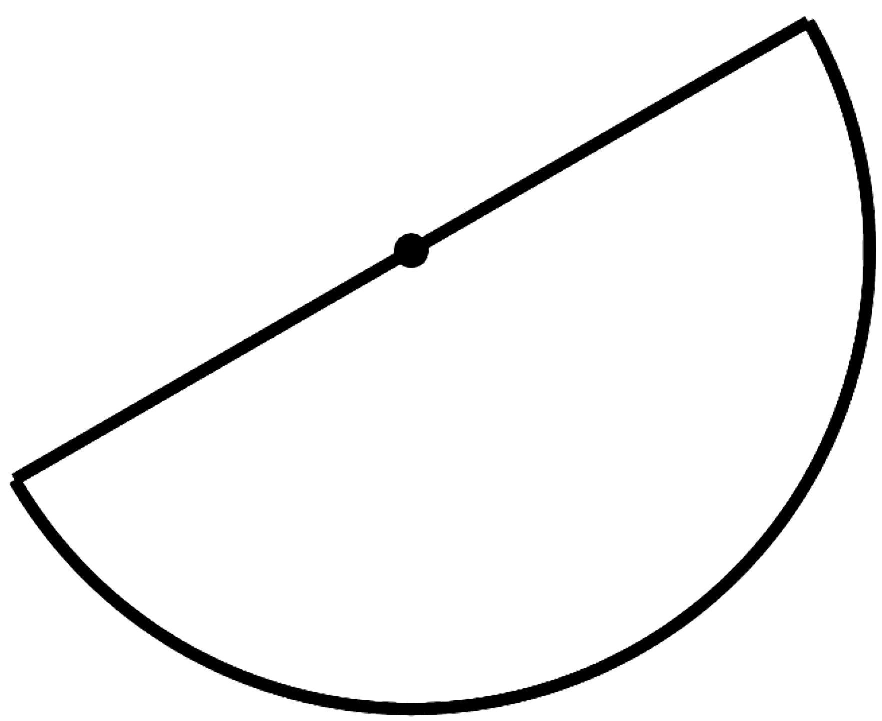
Proof. Take half the circle and flip it over the other half.



The First Proof

Theorem. The diameter of any circle divides that circle into two equal halves.

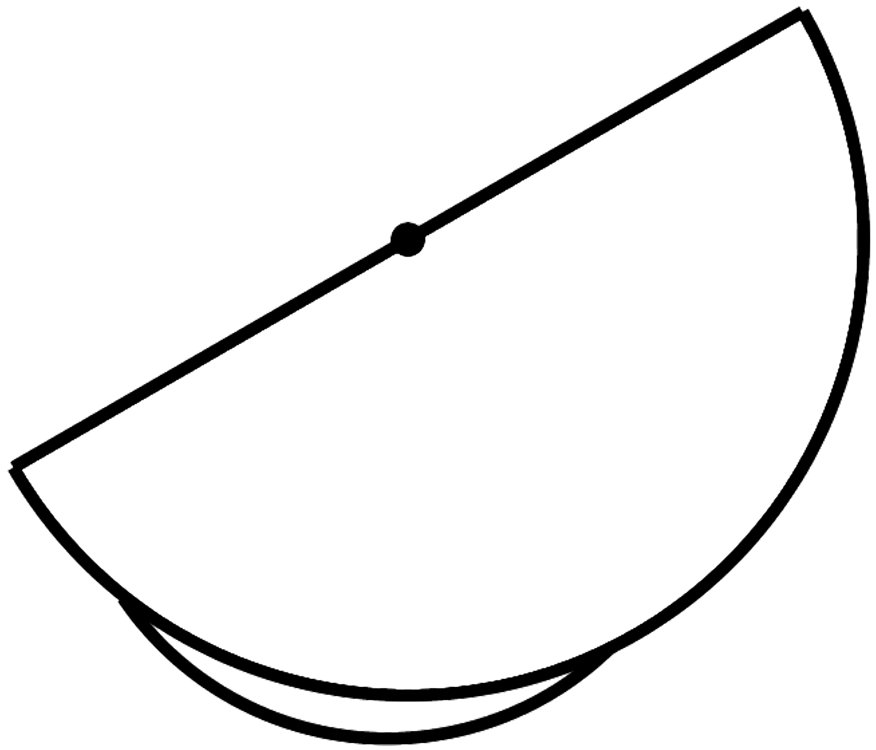
Proof. We want to show that the picture looks like this:



The First Proof

Theorem. The diameter of any circle divides that circle into two equal halves.

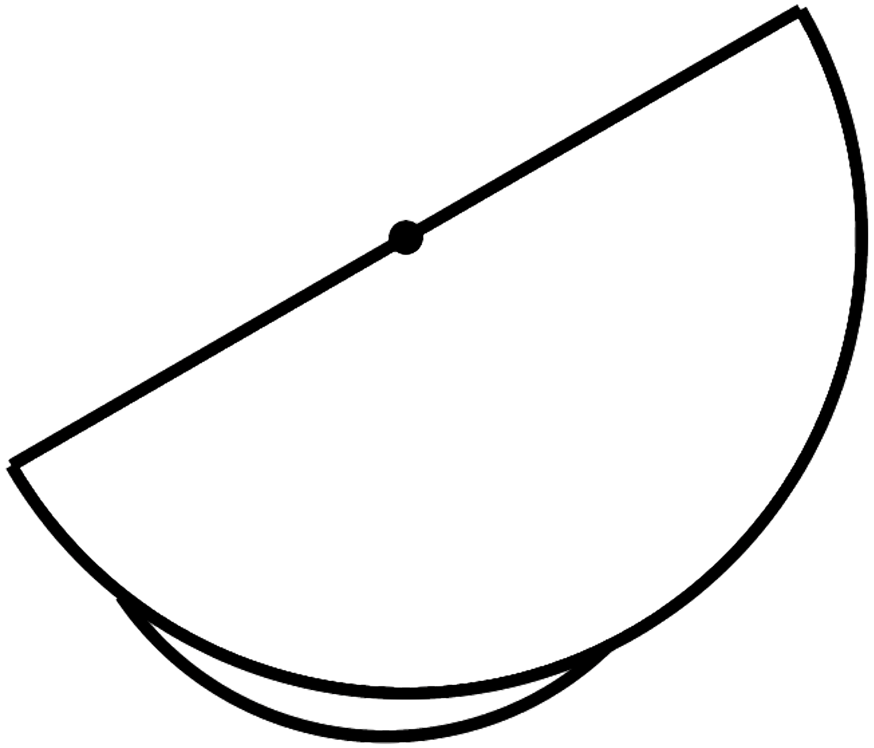
Proof. If it doesn't, then it looks something like this:



The First Proof

Theorem. The diameter of any circle divides that circle into two equal halves.

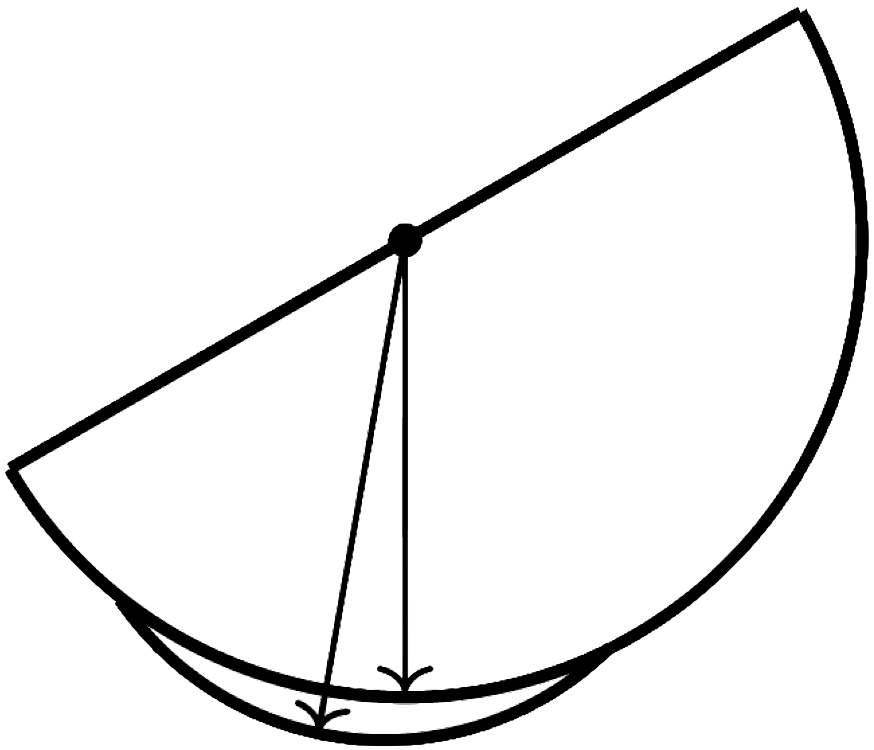
Proof. This gives us a contradiction. Why?



The First Proof

Theorem. The diameter of any circle divides that circle into two equal halves.

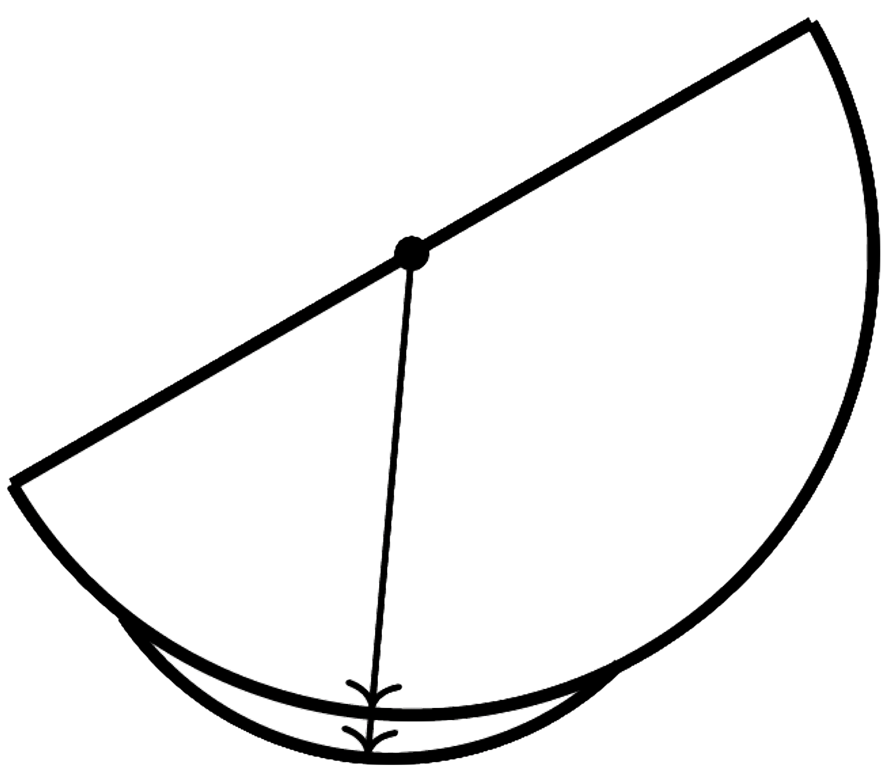
Proof. It's a contradiction because it produced two separate radii! But a circle has just one radius.



The First Proof

Theorem. The diameter of any circle divides that circle into two equal halves.

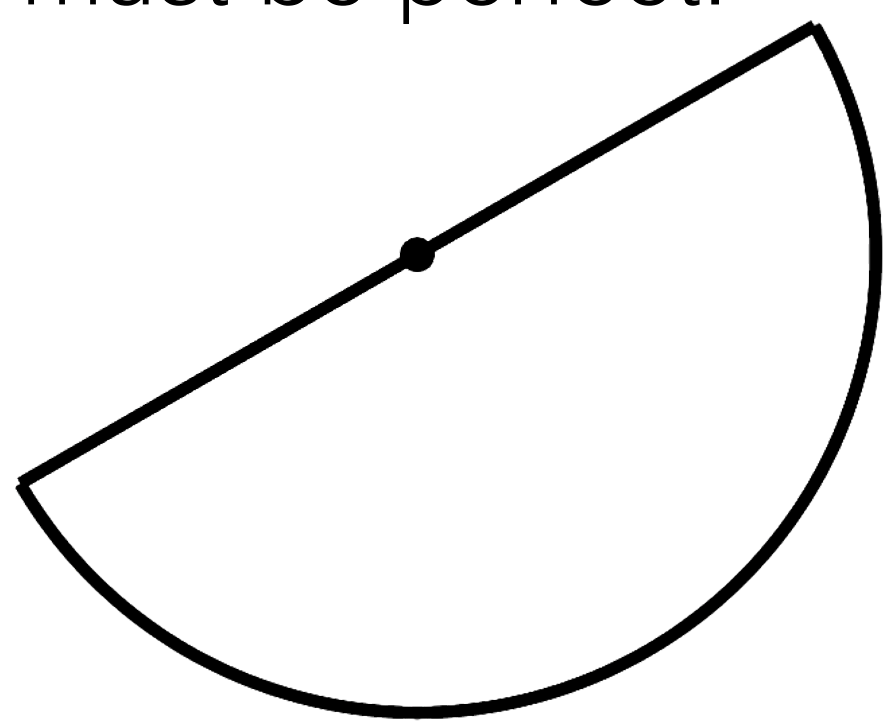
Proof. This fact is even clearer if you line up the two radii that appear.



The First Proof

Theorem. The diameter of any circle divides that circle into two equal halves.

Proof. Either way, the conclusion is the same. If the two parts did not perfectly overlap then we would get two radii, which is impossible. So the overlap must be perfect.



The First Proof

Other things Thales proved:

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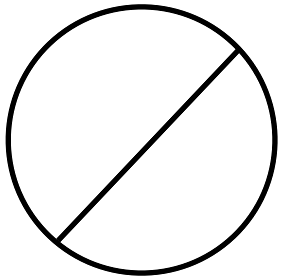

1. A circle is bisected by any diameter.

The First Proof

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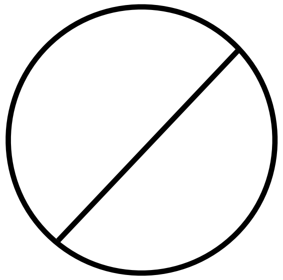
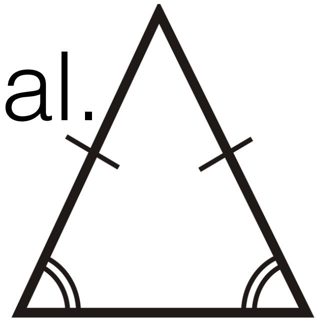


1. A circle is bisected by any diameter.
2. The base angles of an isosceles triangle are equal.



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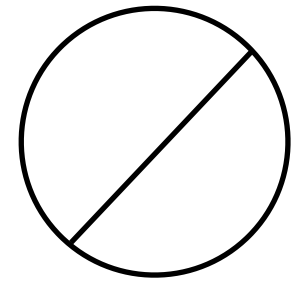

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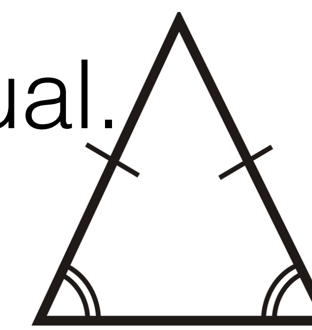
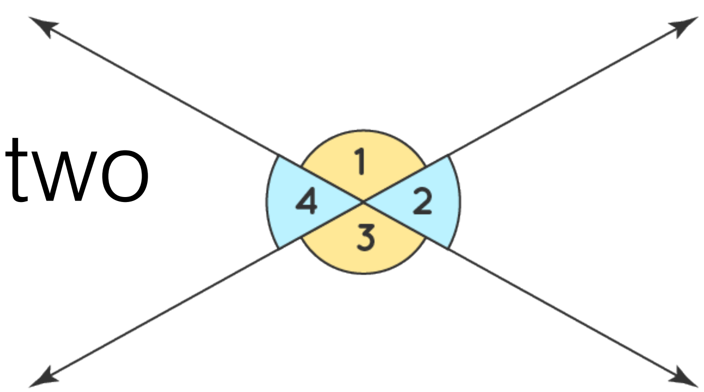


3. The vertical angles formed by two intersecting lines are equal.



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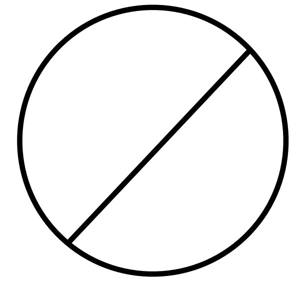

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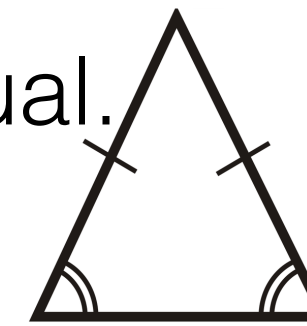

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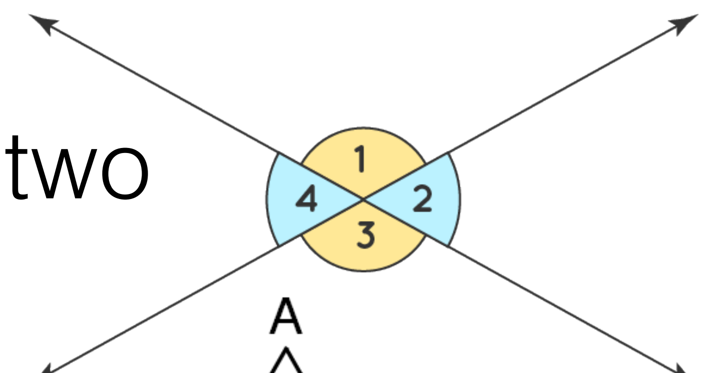
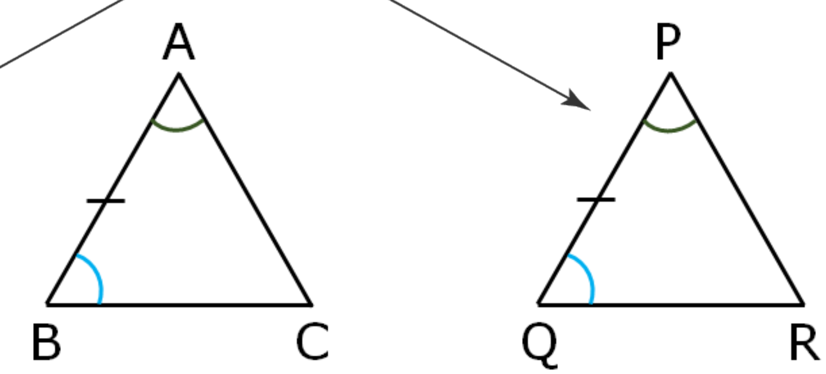


4. The ASA triangle congruence.



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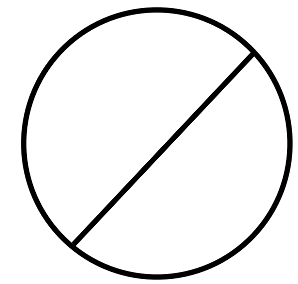

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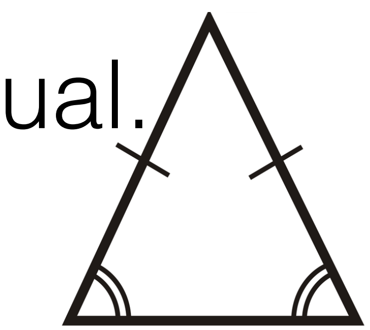

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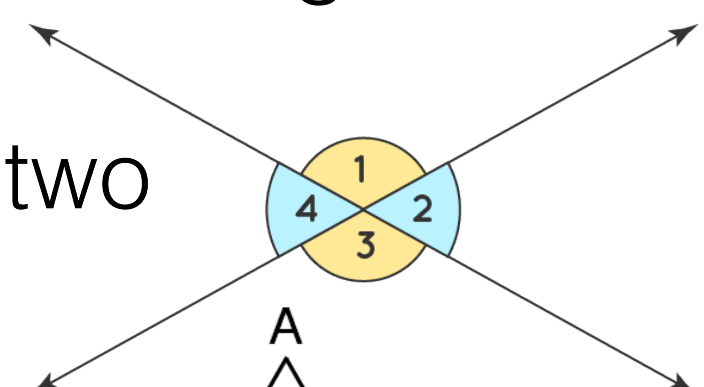

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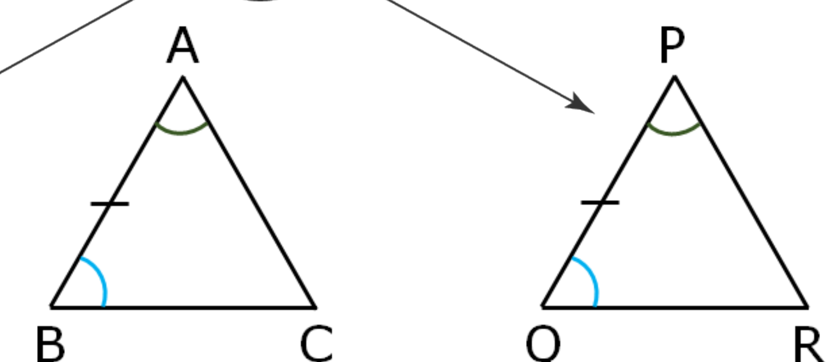
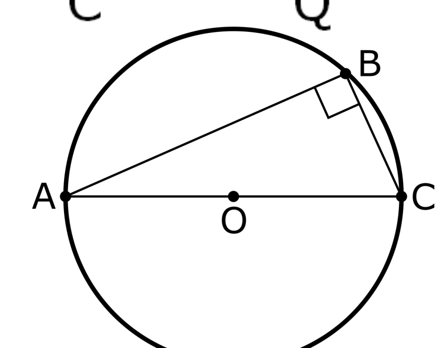


5. The angle inscribed in a semicircle is a right angle.



Historial Uncertainty

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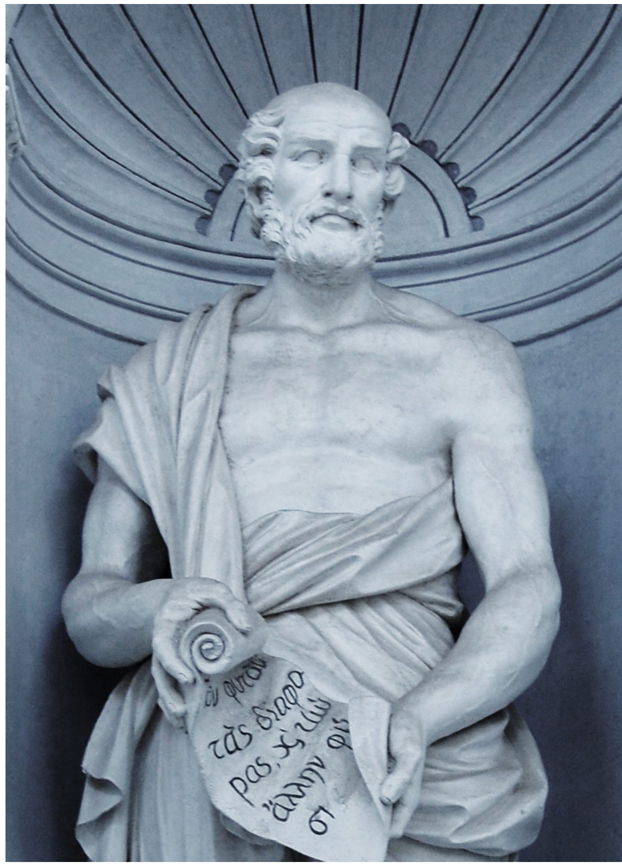
- Eudemus of Rhodes (~370 BC — ~300 BC) is considered the first historian of science. He also worked closely with Aristotle.



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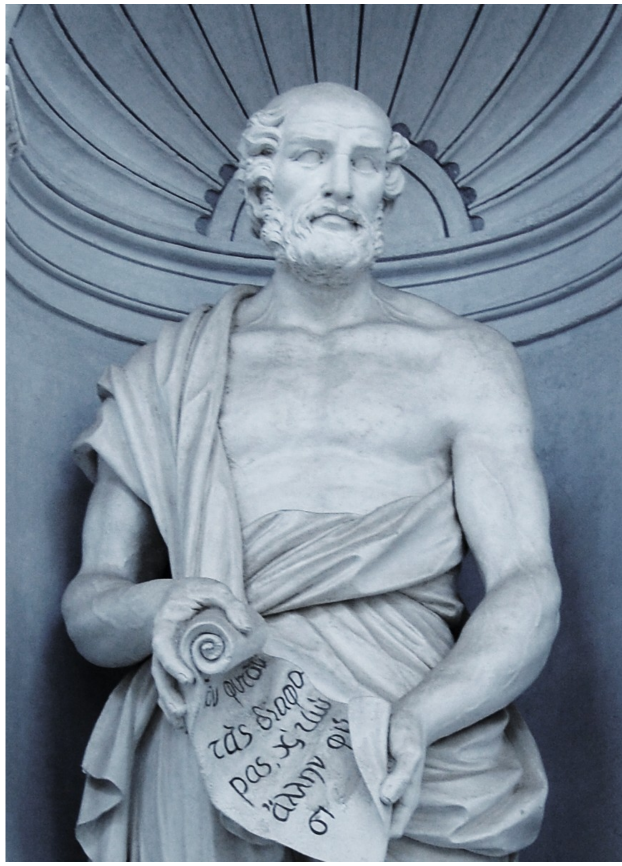
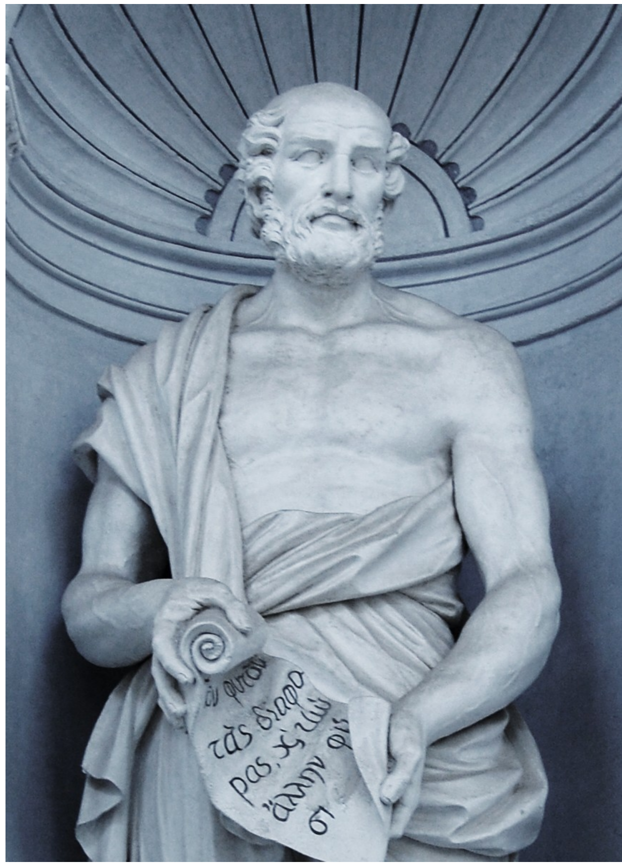


Historial Uncertainty

- Eudemus of Rhodes (~370 BC — ~300 BC) is considered the first historian of science. He also worked closely with Aristotle.
- He wrote *History of Geometry*, a book about all the geometry known to the Greeks. Sadly, no copy of this book exists today.



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Historial Uncertainty

- Proclus (412 AD — 485 AD) gave a short sketch of the history of geometry which seems to be based on Eudemus' book *History of Geometry*. He discusses Thales, Pythagoras and others.
- Stories about Pythagoras are often contradictory or clearly false. This casts doubt on the rest.
- Yet, from Proclus and a few others, there is (non-conclusive) evidence that Pythagoras made important contributions to math.

Pythagoras

Pythagoras

- Born in Samos around 572 BC

Pythagoras

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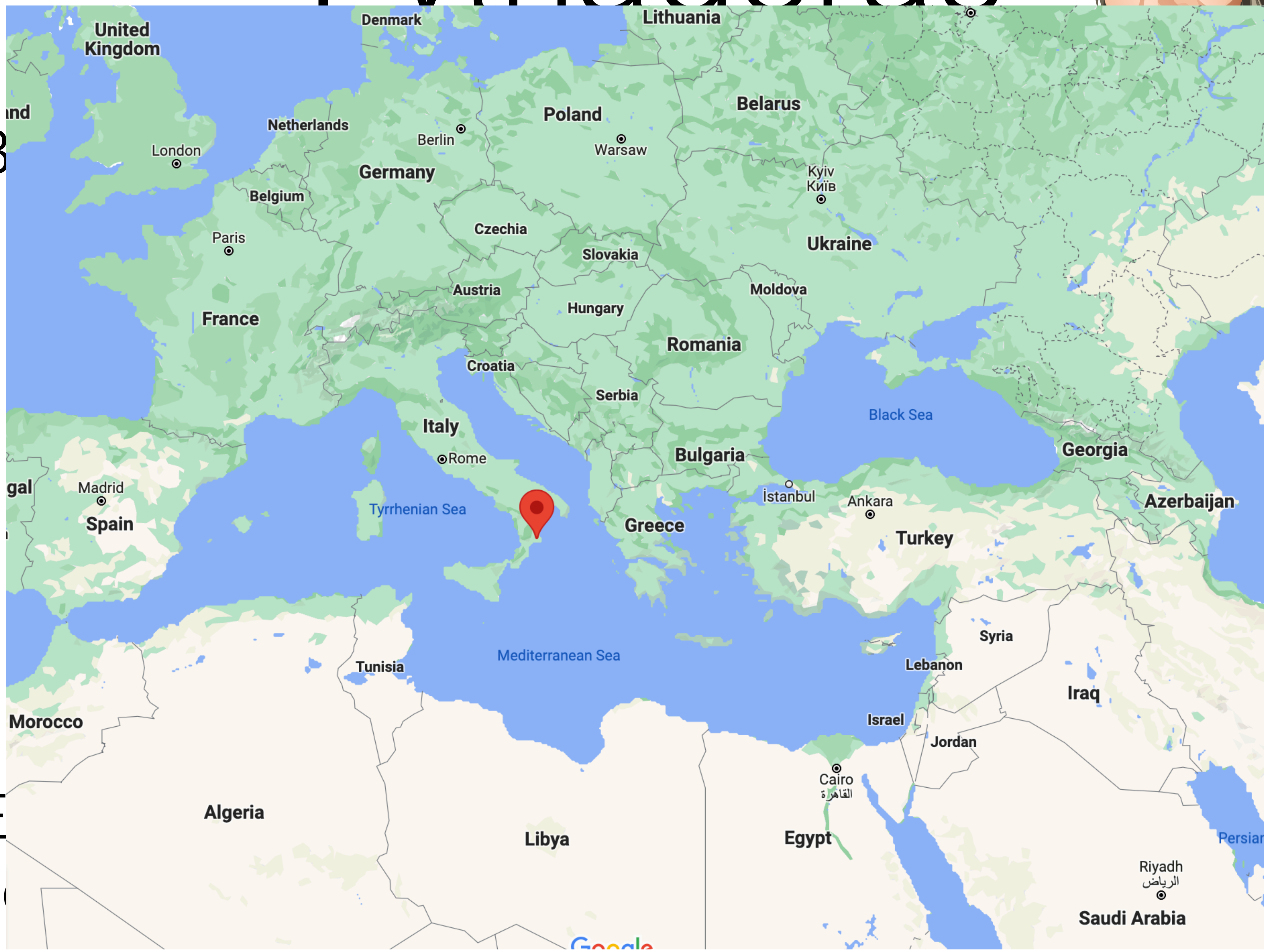
- Established his “Brotherhood” in Crotona, in southern Italy.

Pythagoras

- B



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  S

Pythagorean Brotherhood

- It was basically a cult, with Pythagoras at the helm. Special diet, exercise, activities, rituals. May have studied numerology and mathematics.

Pythagorean Brotherhood

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- If they did, then it was probably a "dissection proof."
- If it was, then it may have been the proof that I will show you in a moment.

Pythagorean Triples

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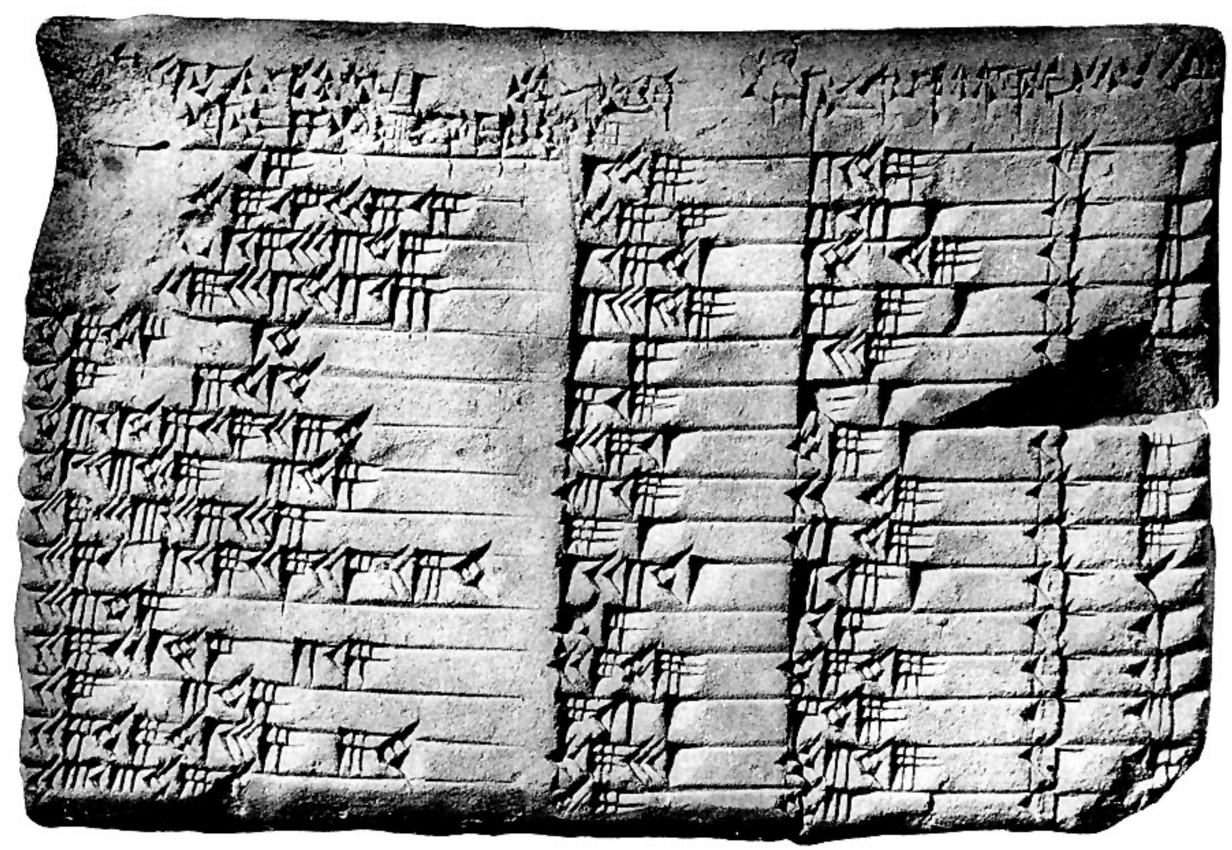
- Example: $3^2 + 4^2 = 5^2$, so $(3,4,5)$ is a Pythagorean triple.
- Recall: 15 PTs appeared in Plimpton 322 (~1800 BC).
- Nearly all ancient cultures found some Pythagorean triples.



Pythagorean Theorem

Pythagorean Theorem

- Warm-up: State the Pythagorean theorem.

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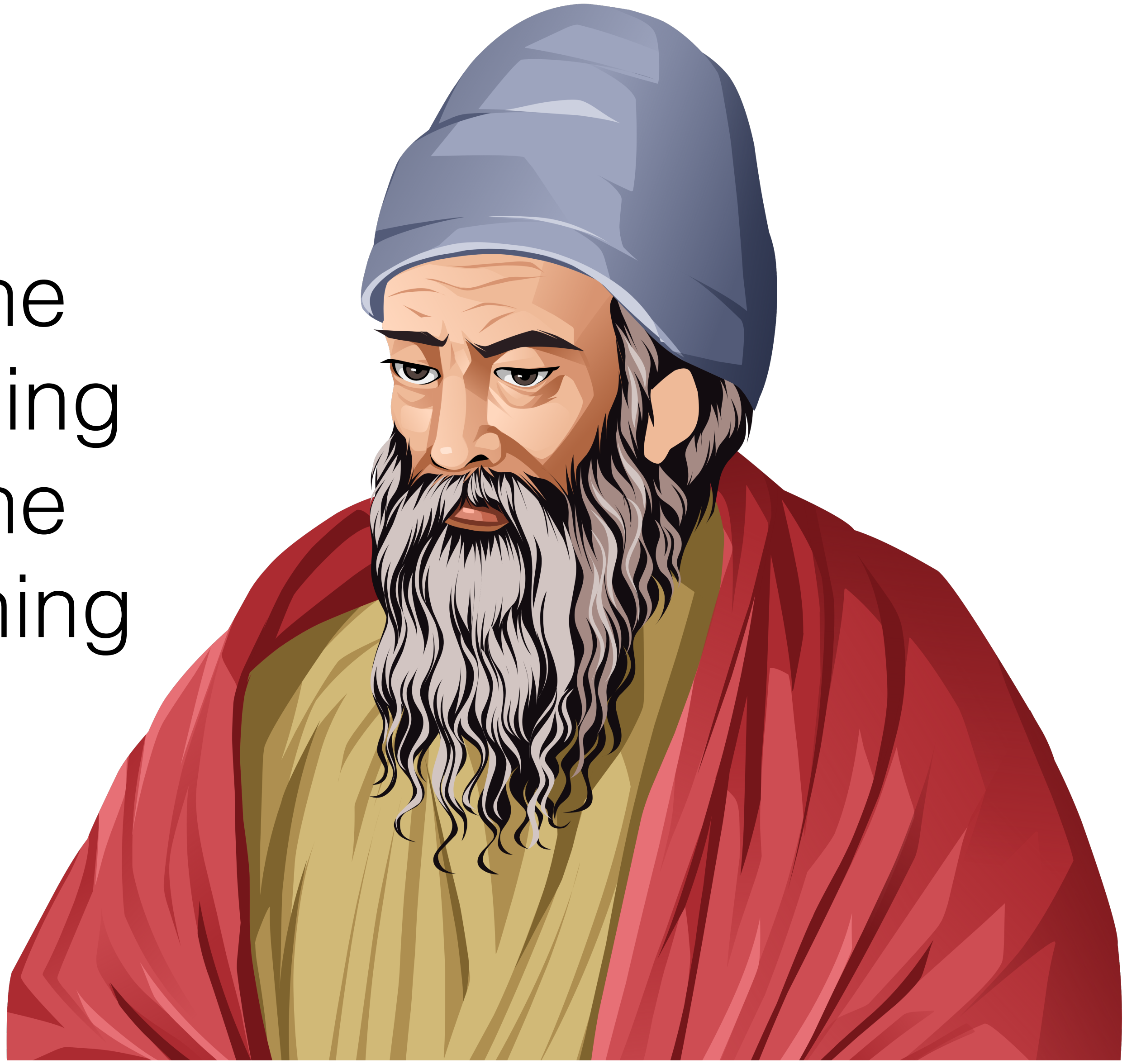
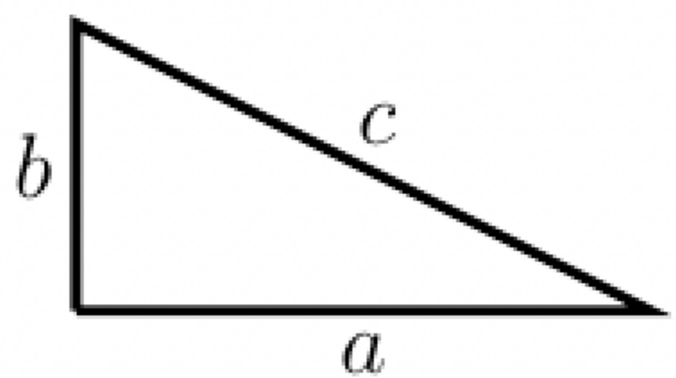
- Euclid:

  “In right-angled triangles, the square on the side subtending the right angle is equal to the squares on the side containing the right angle.”

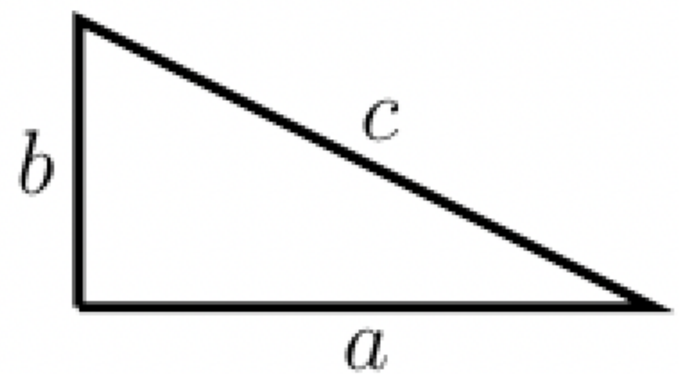


Pythagorean Theorem



$$a^2 + b^2 = c^2$$

Pythagorean Theorem



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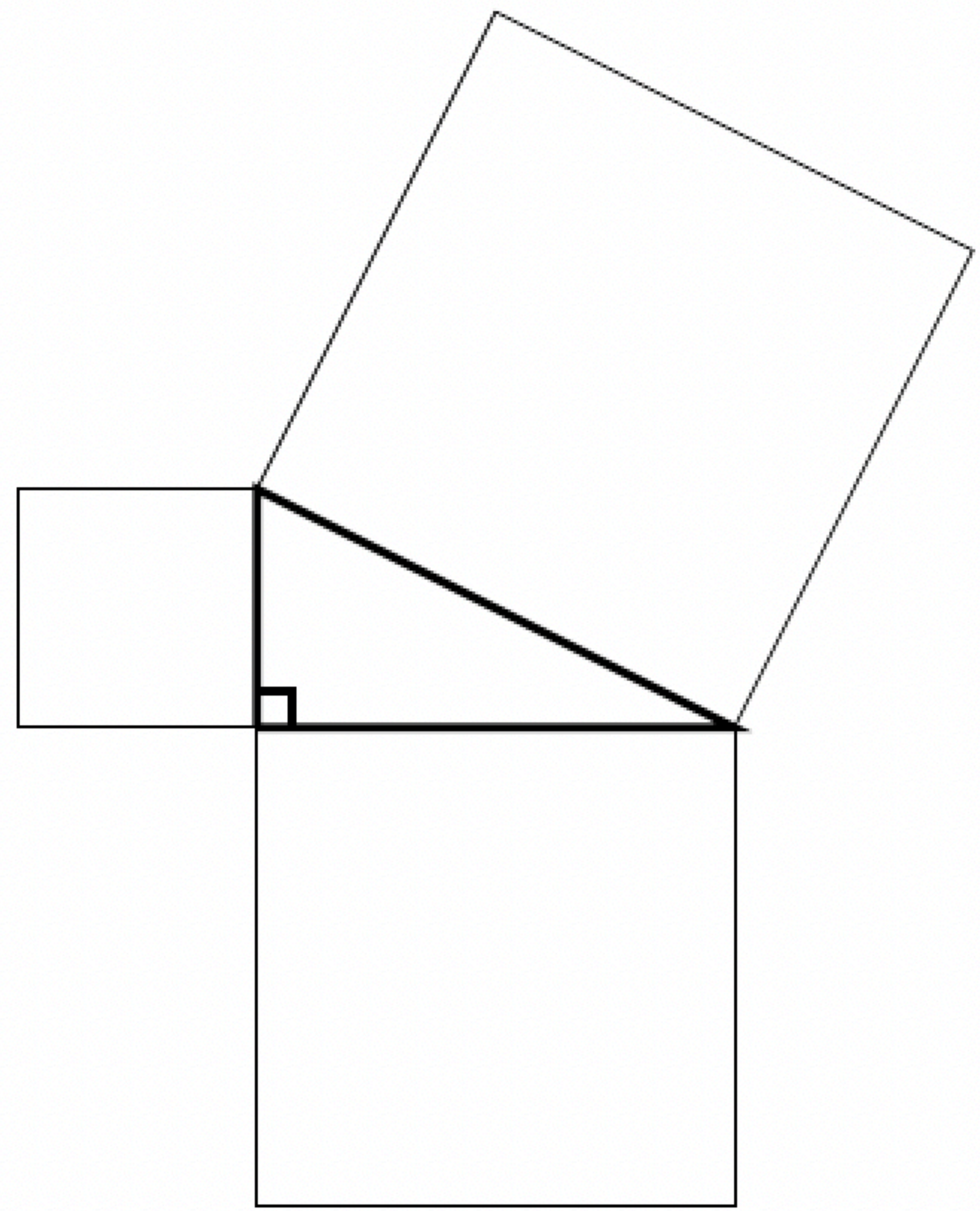

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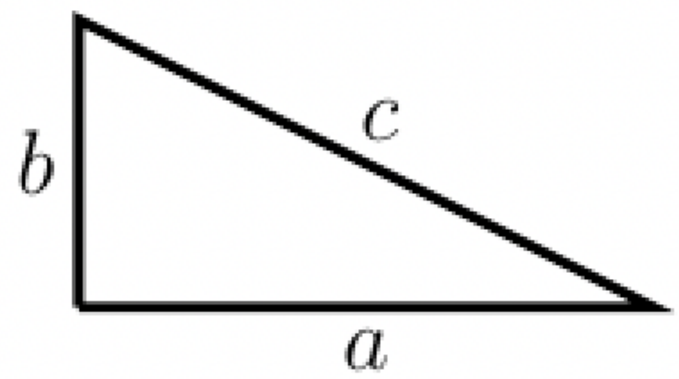

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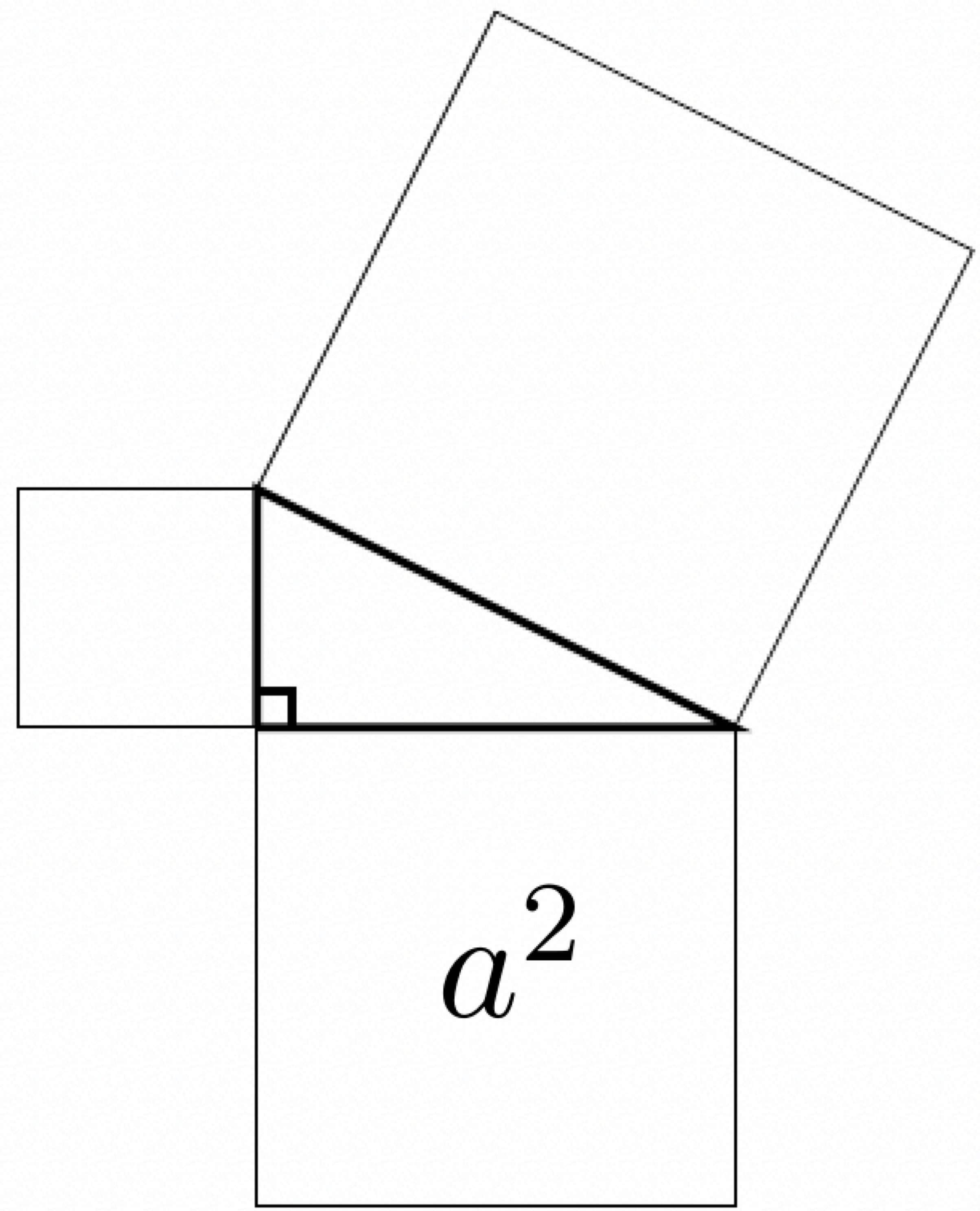

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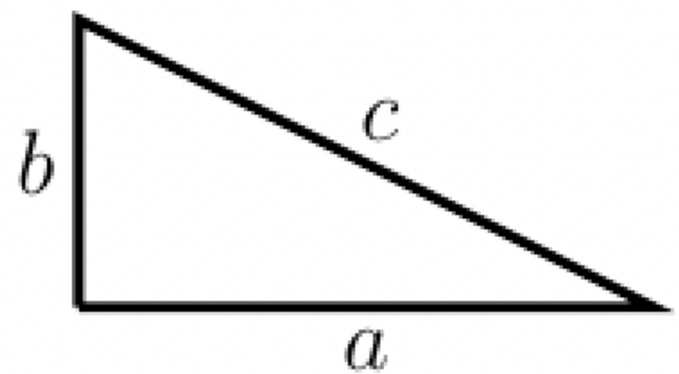

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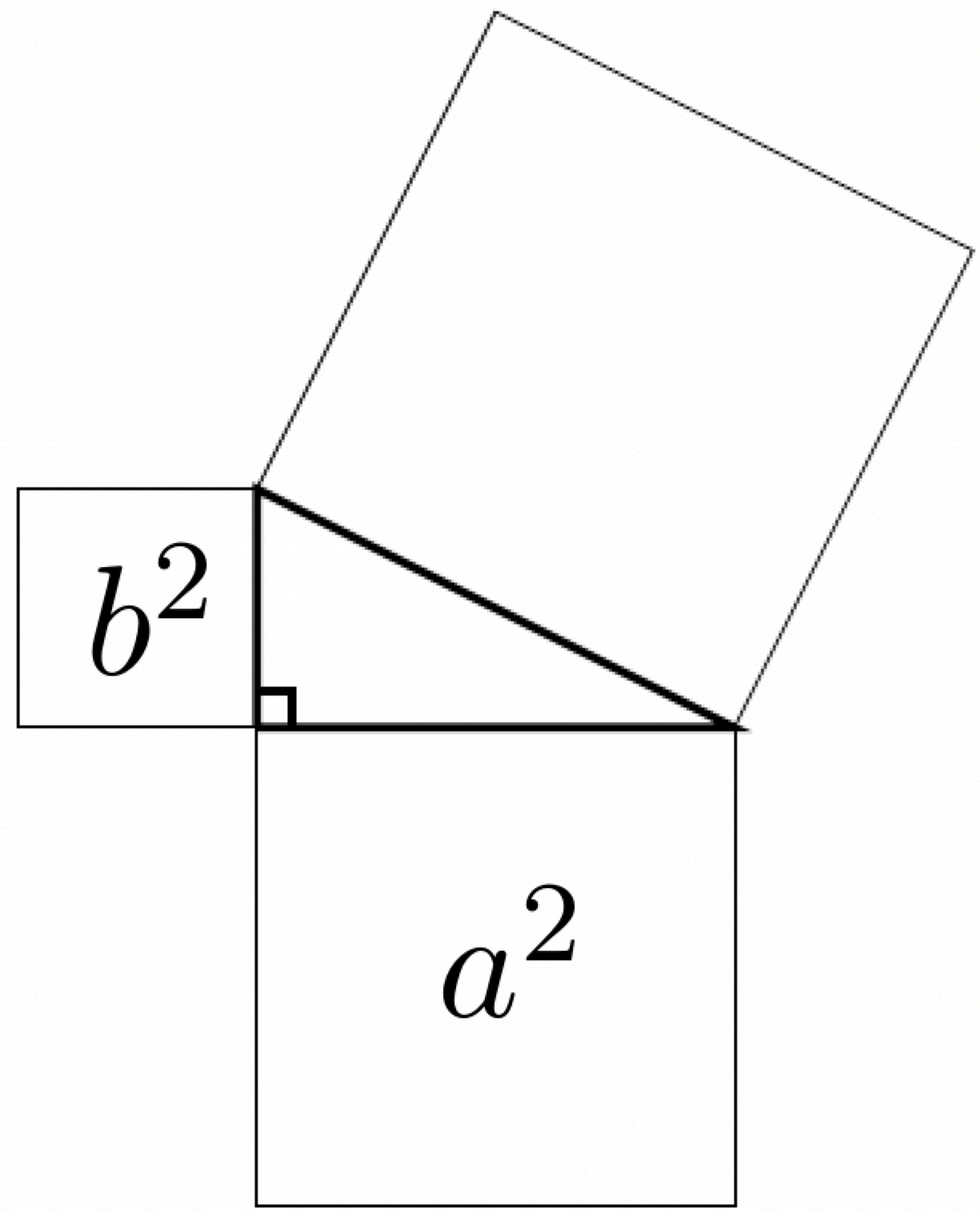

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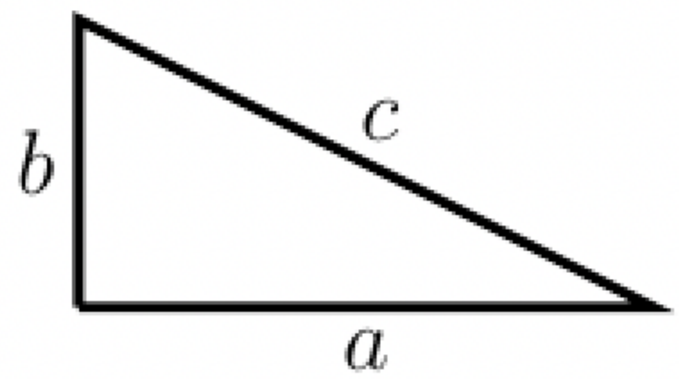

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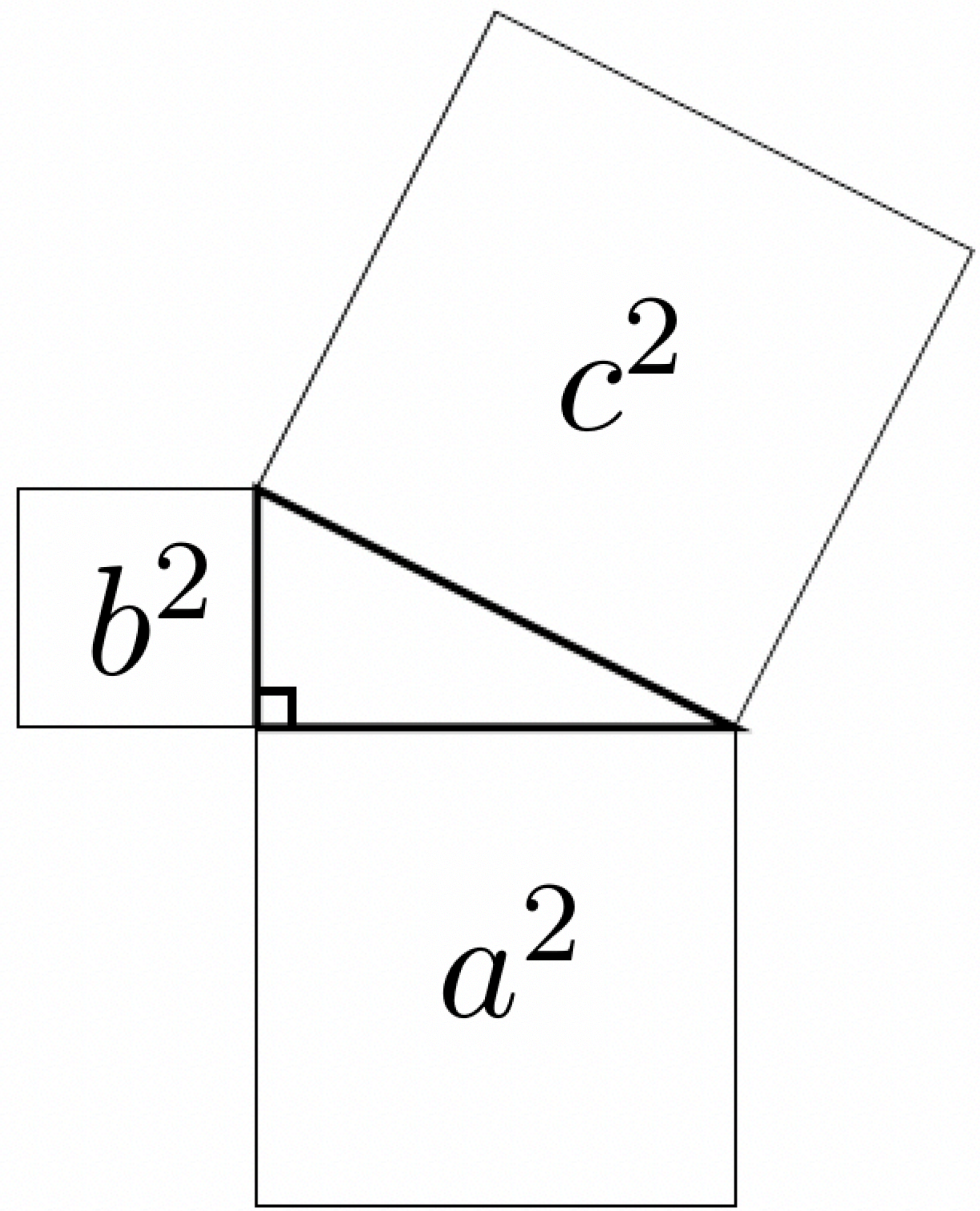
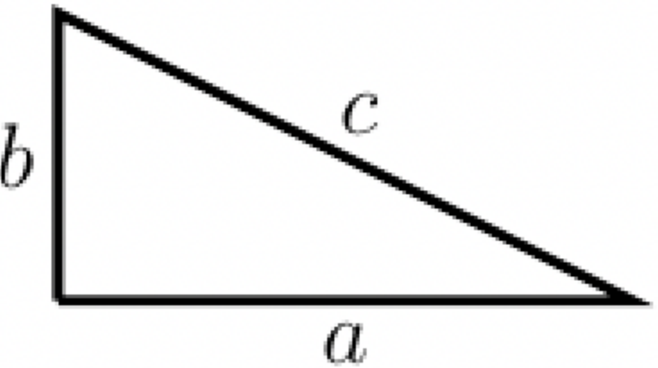
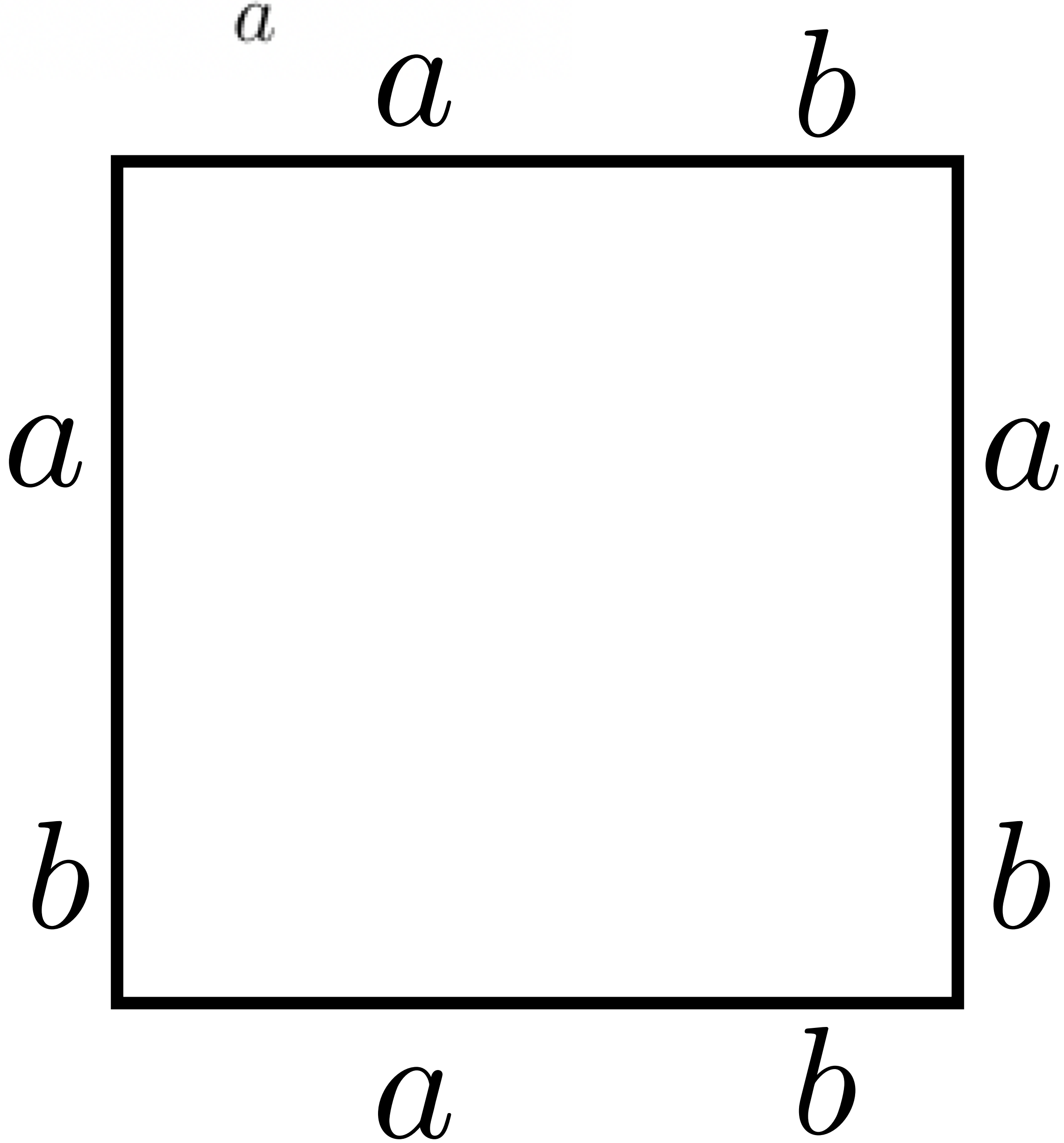
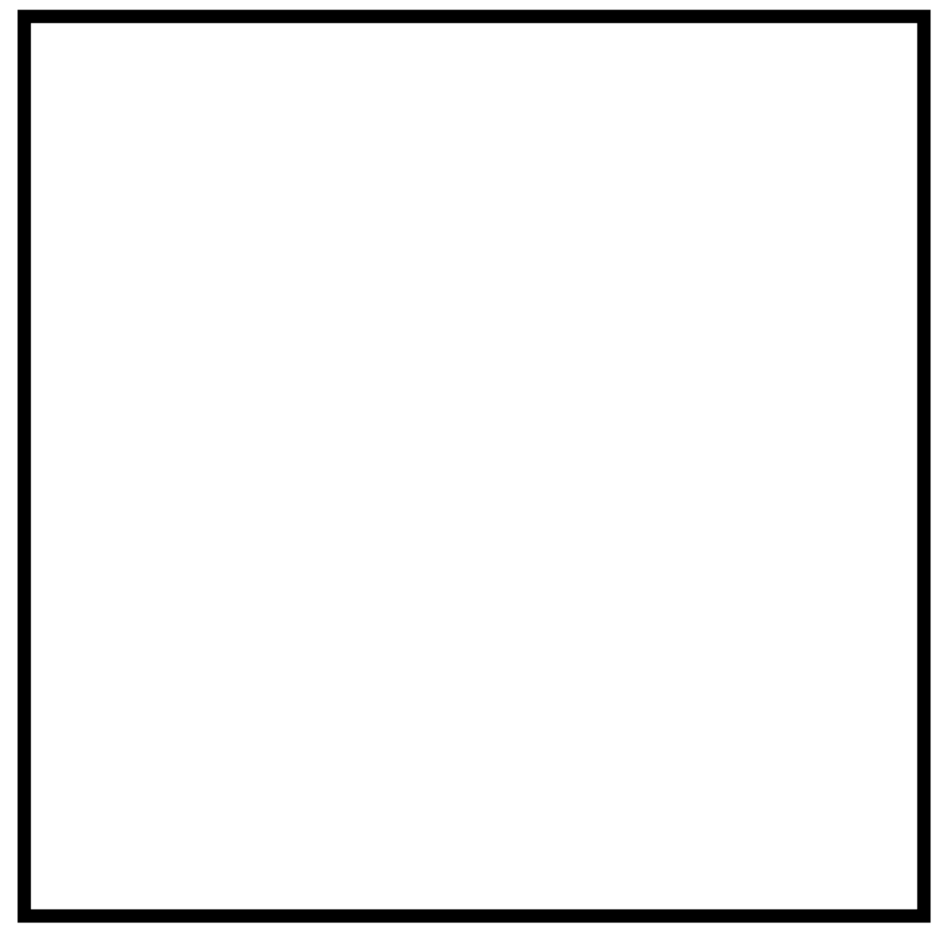


How They *Might* Have Proved It







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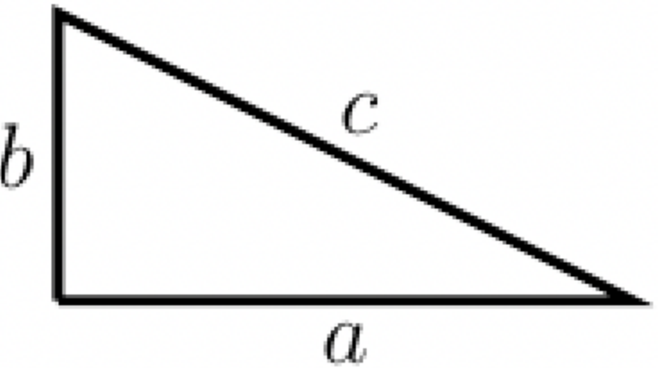

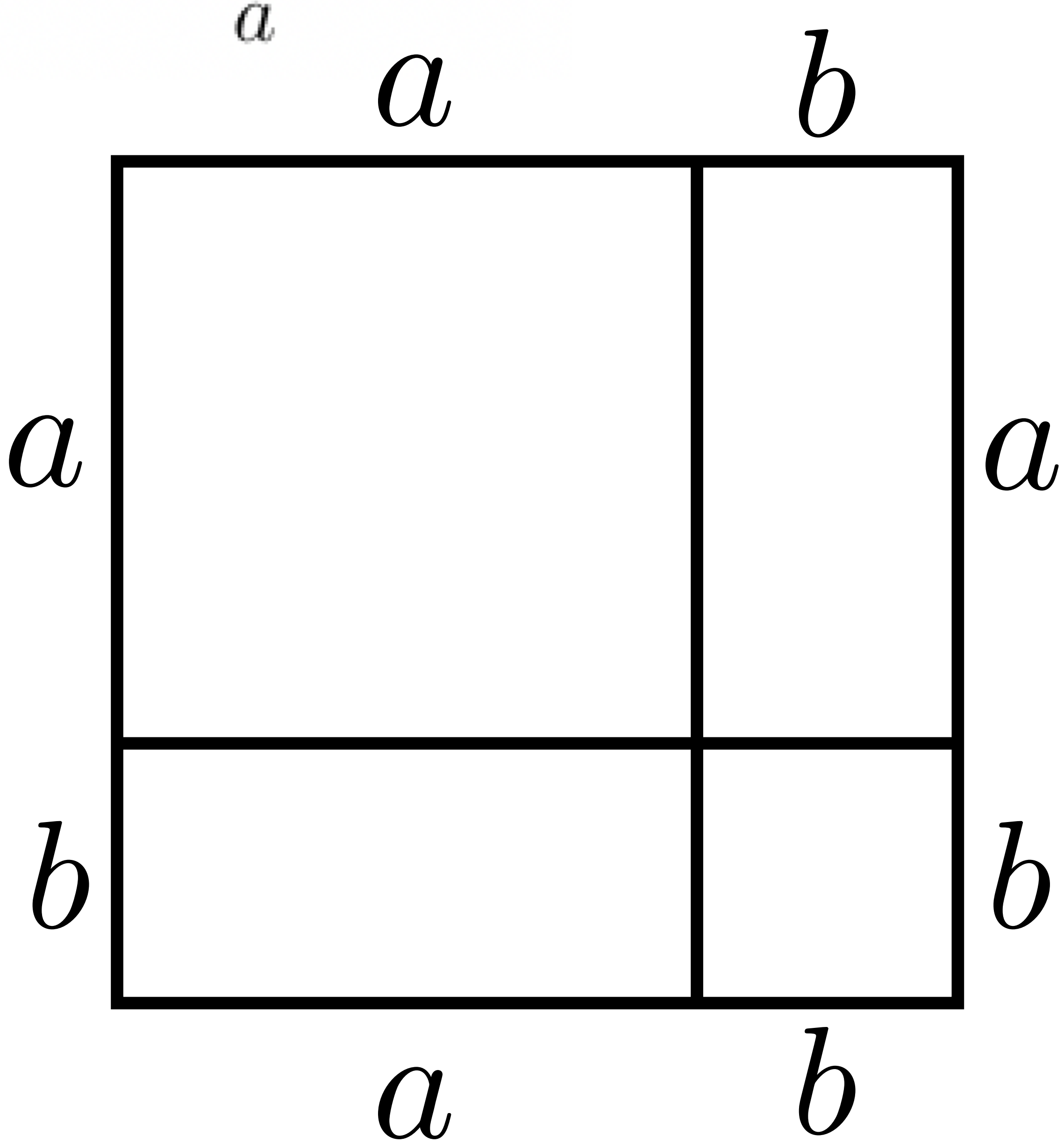

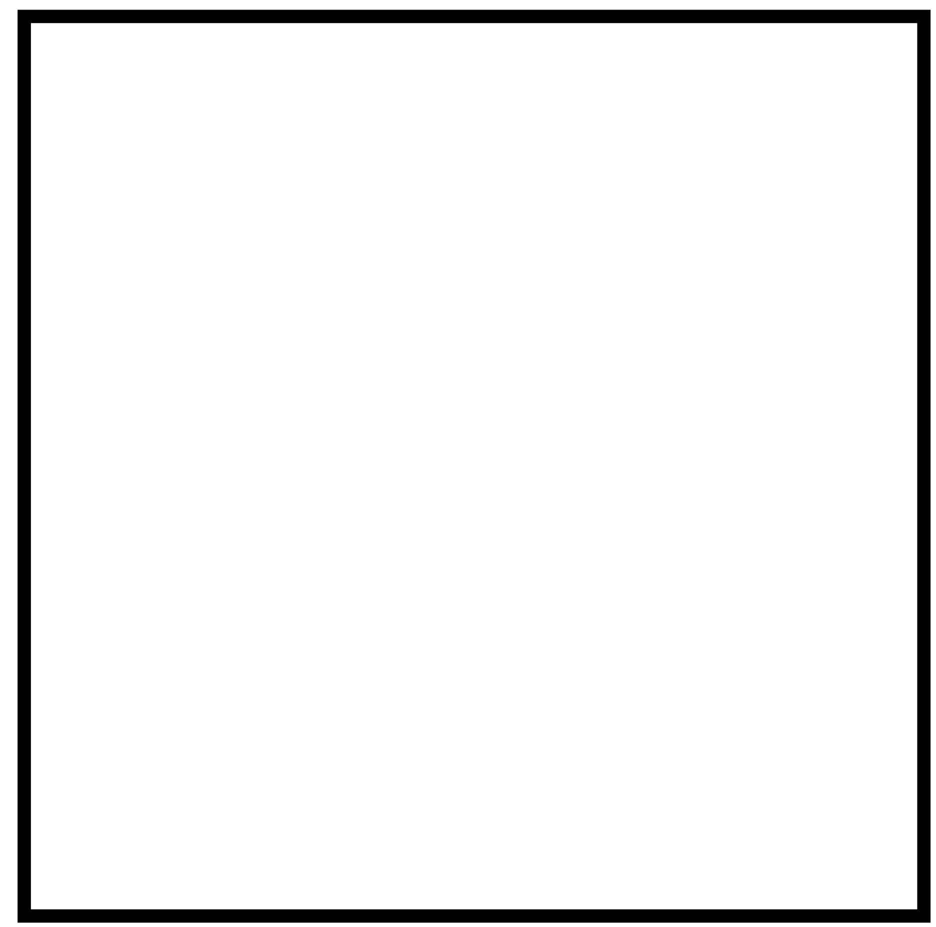

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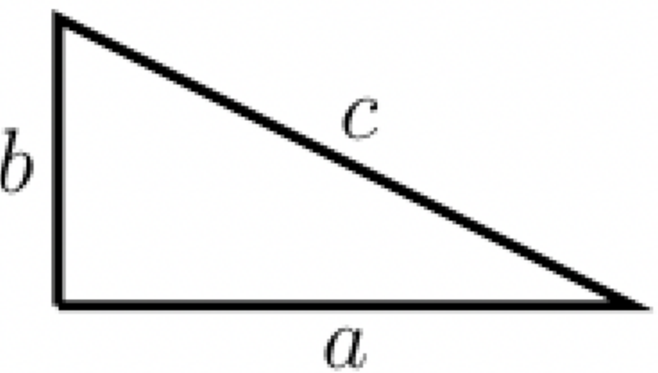

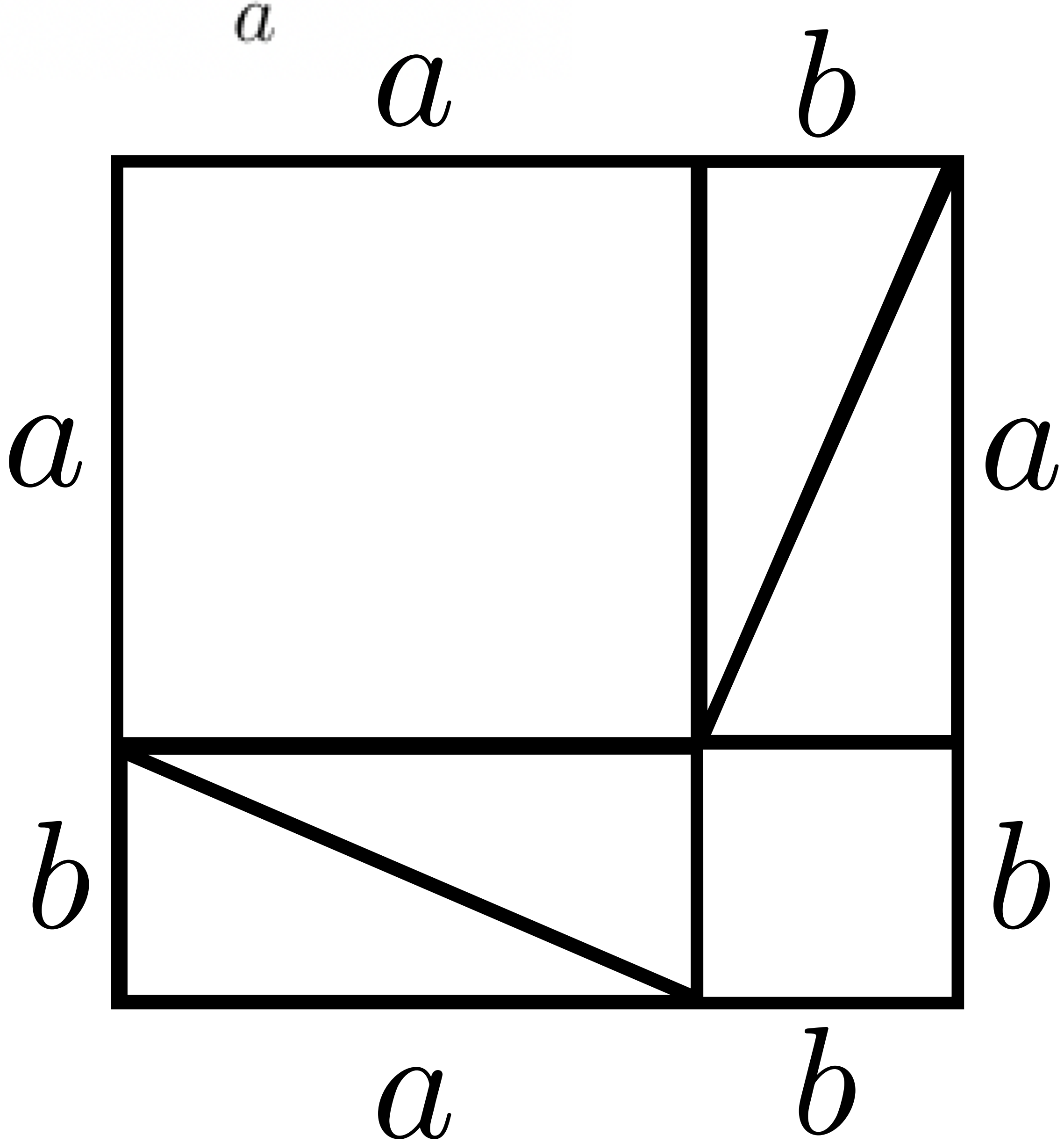

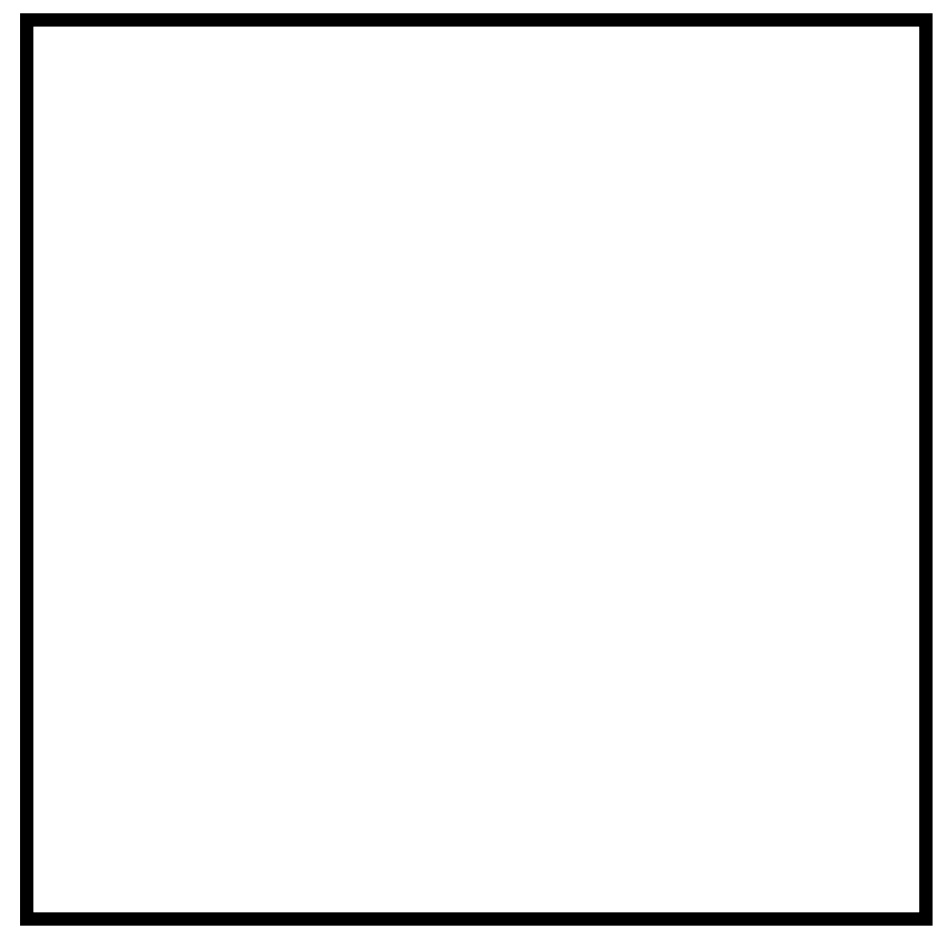

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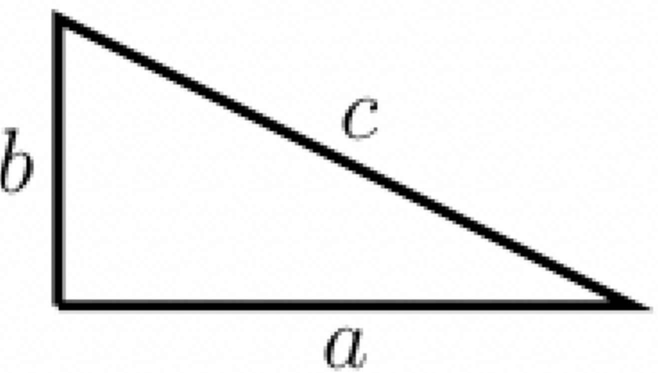

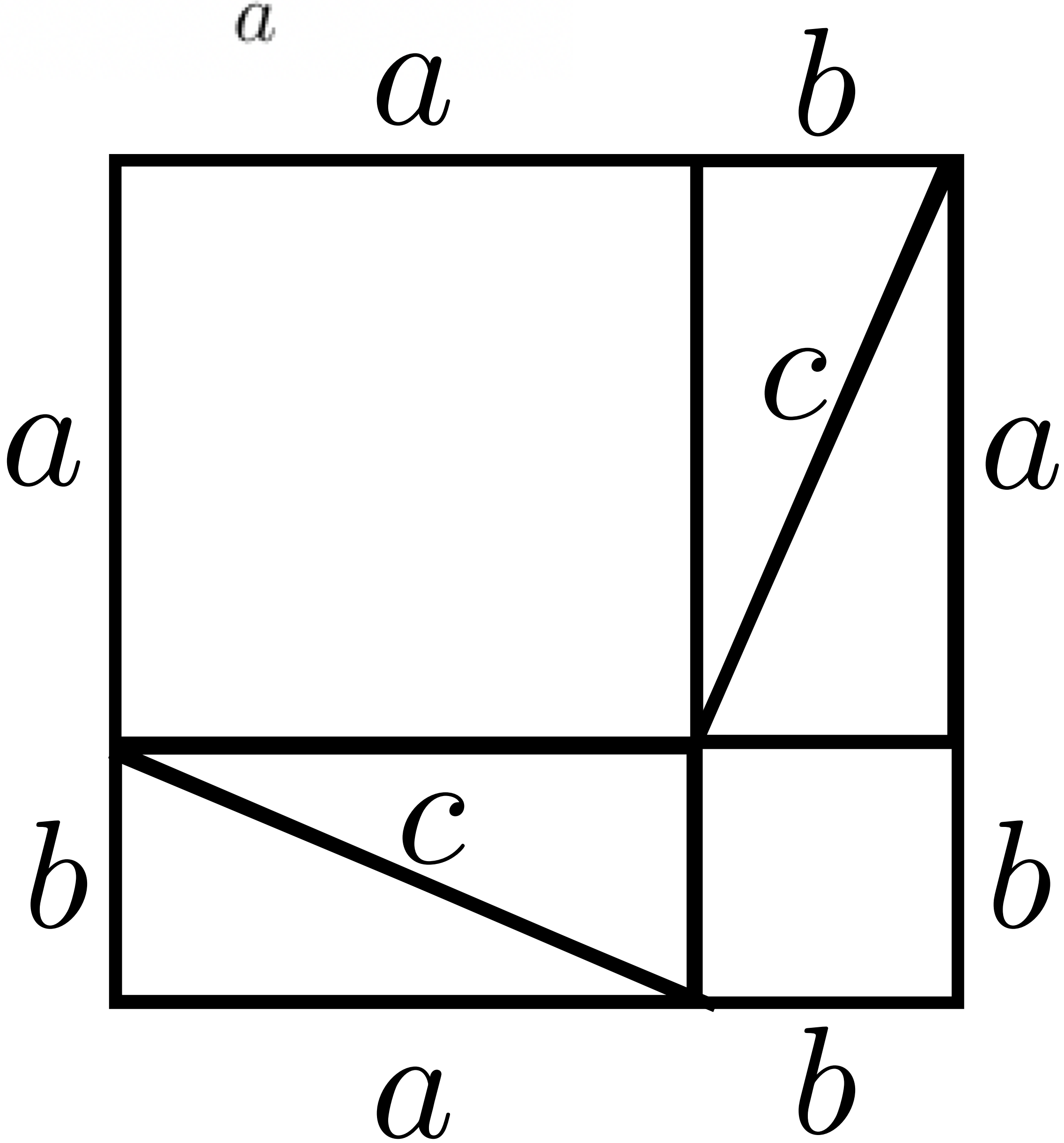

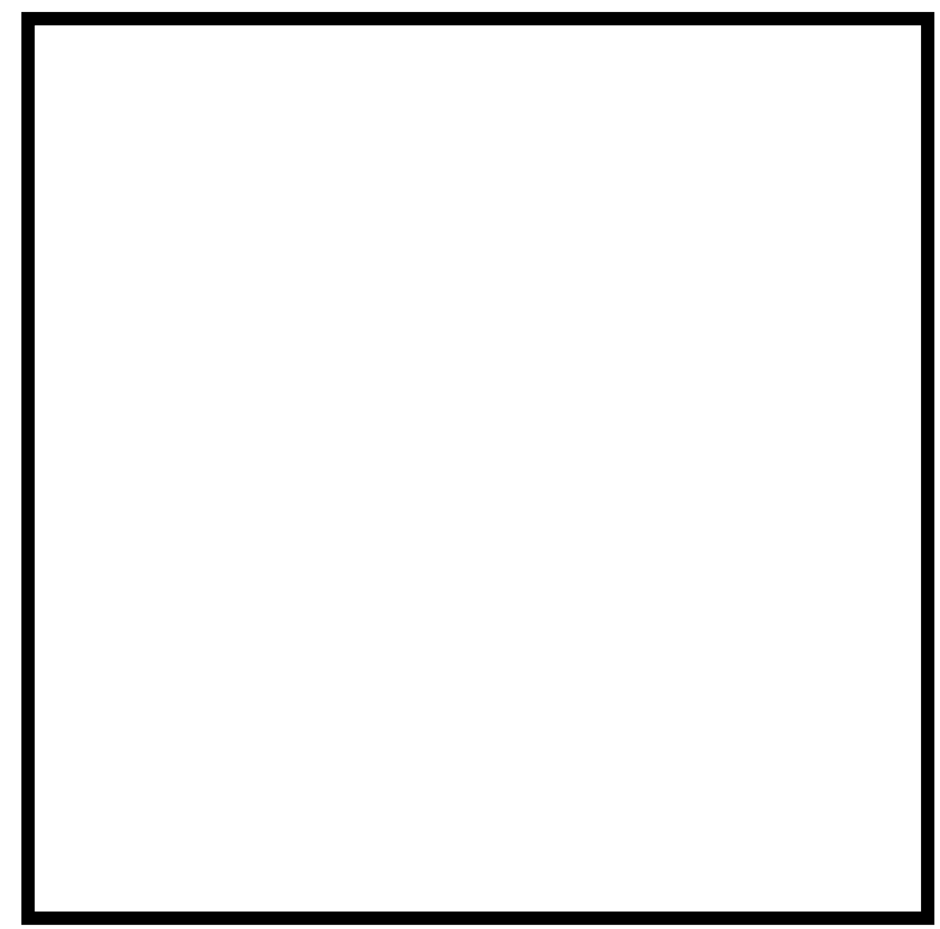

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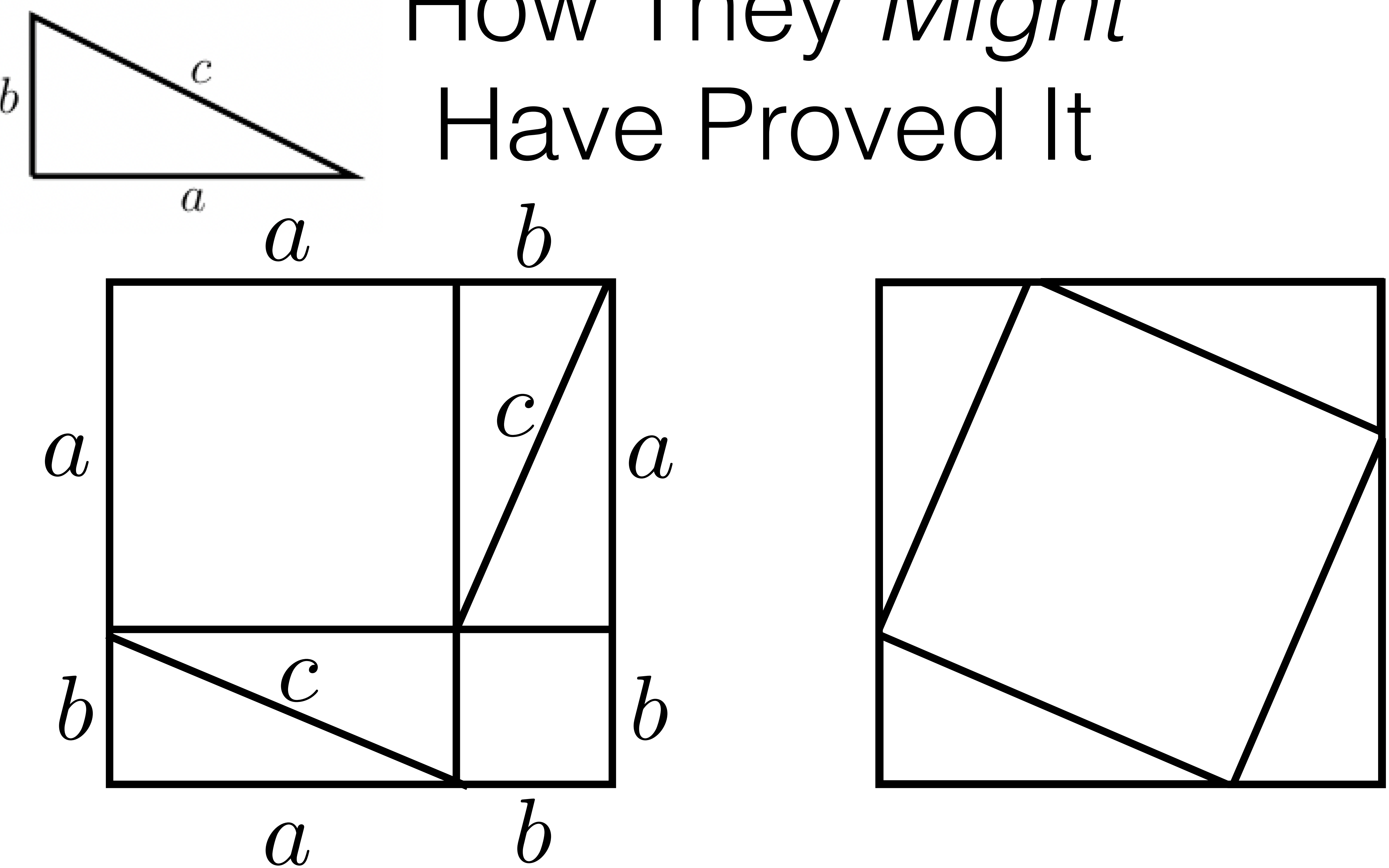

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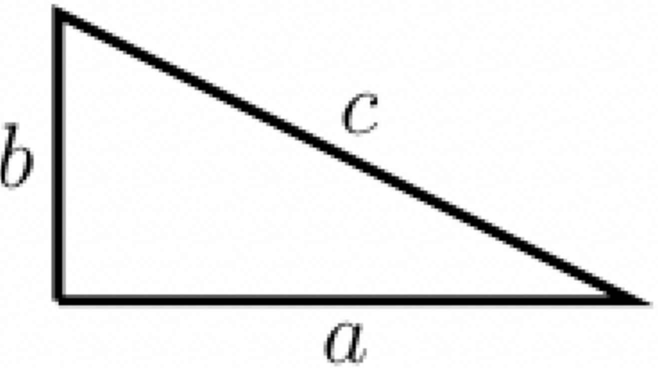

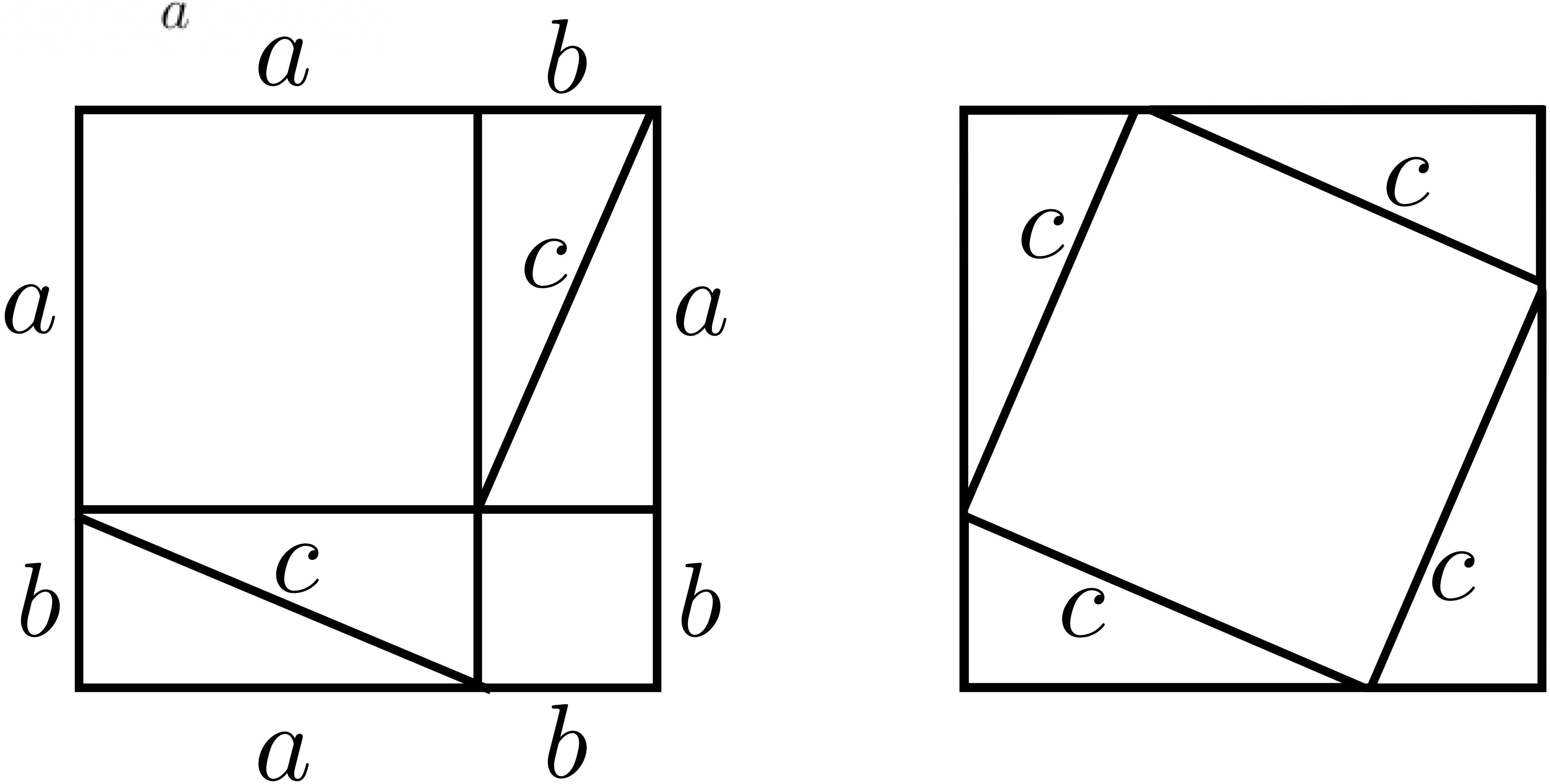

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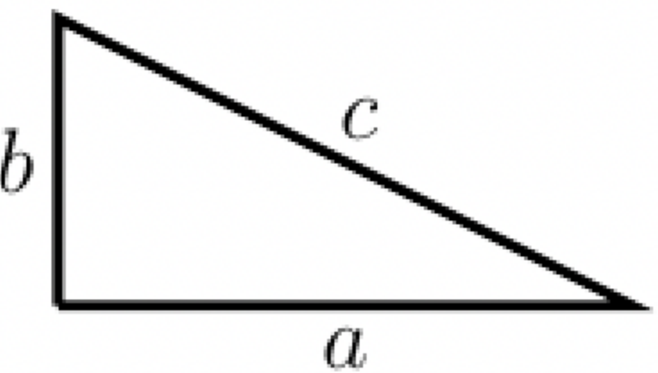

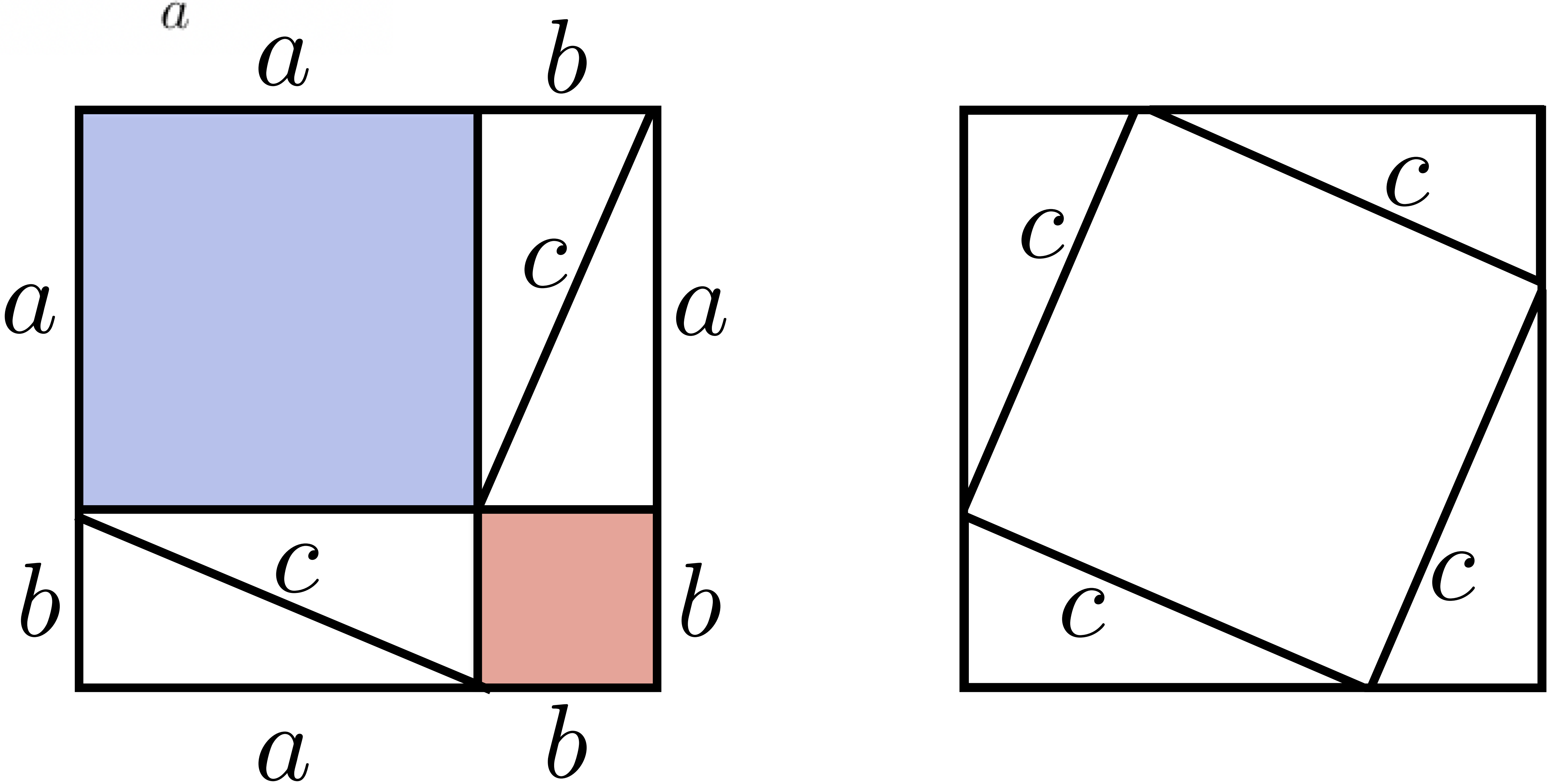

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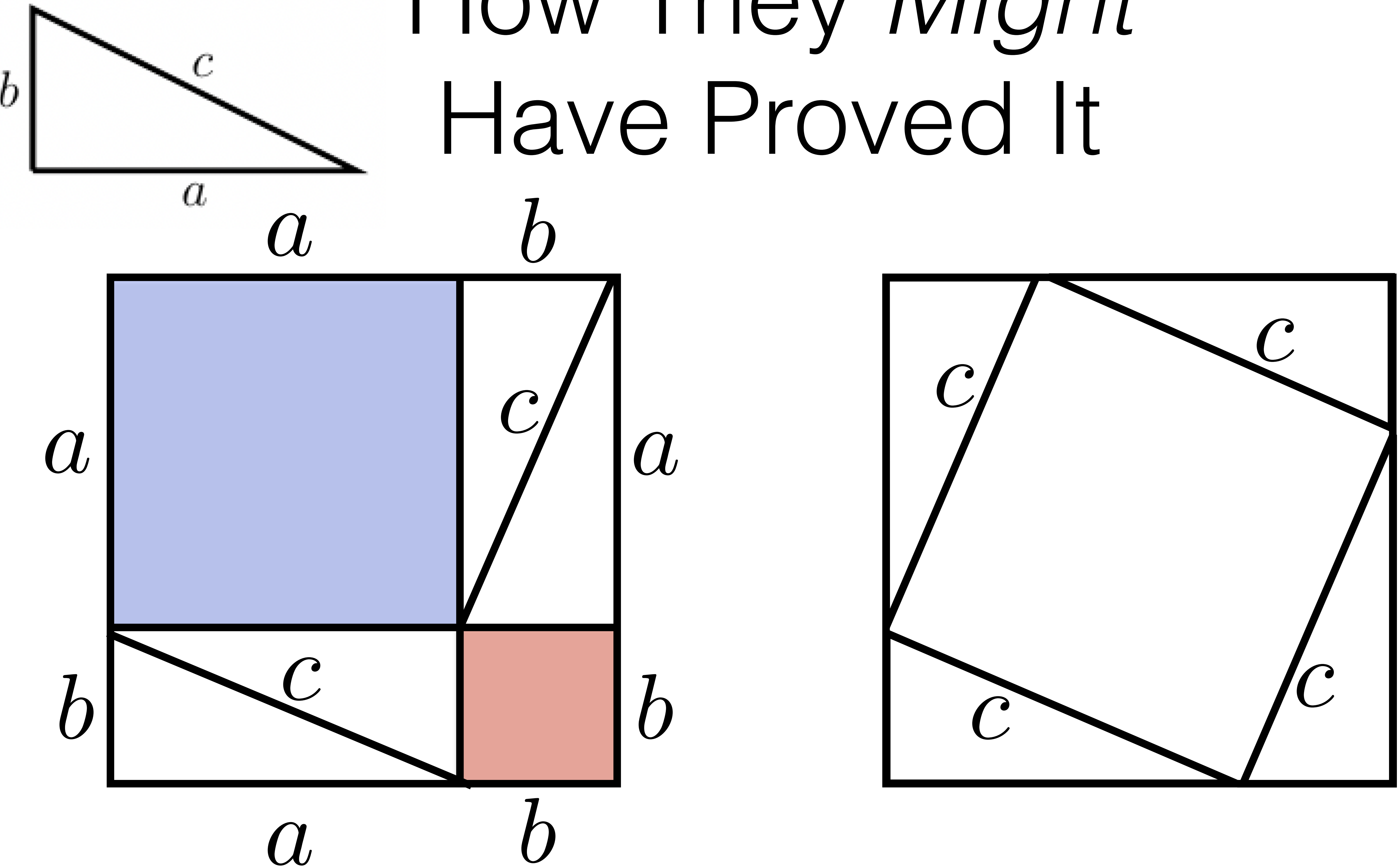

$a^2 + b^2$ + 4 triangles

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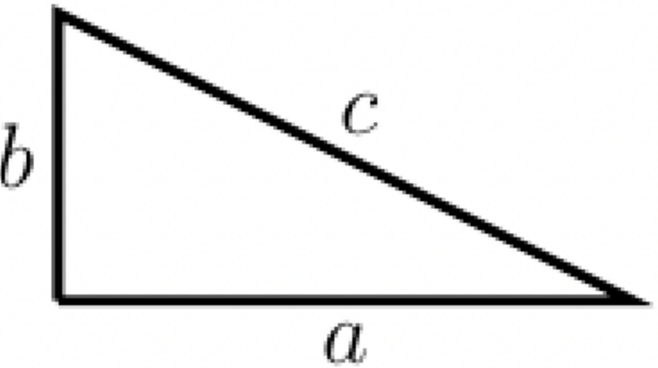
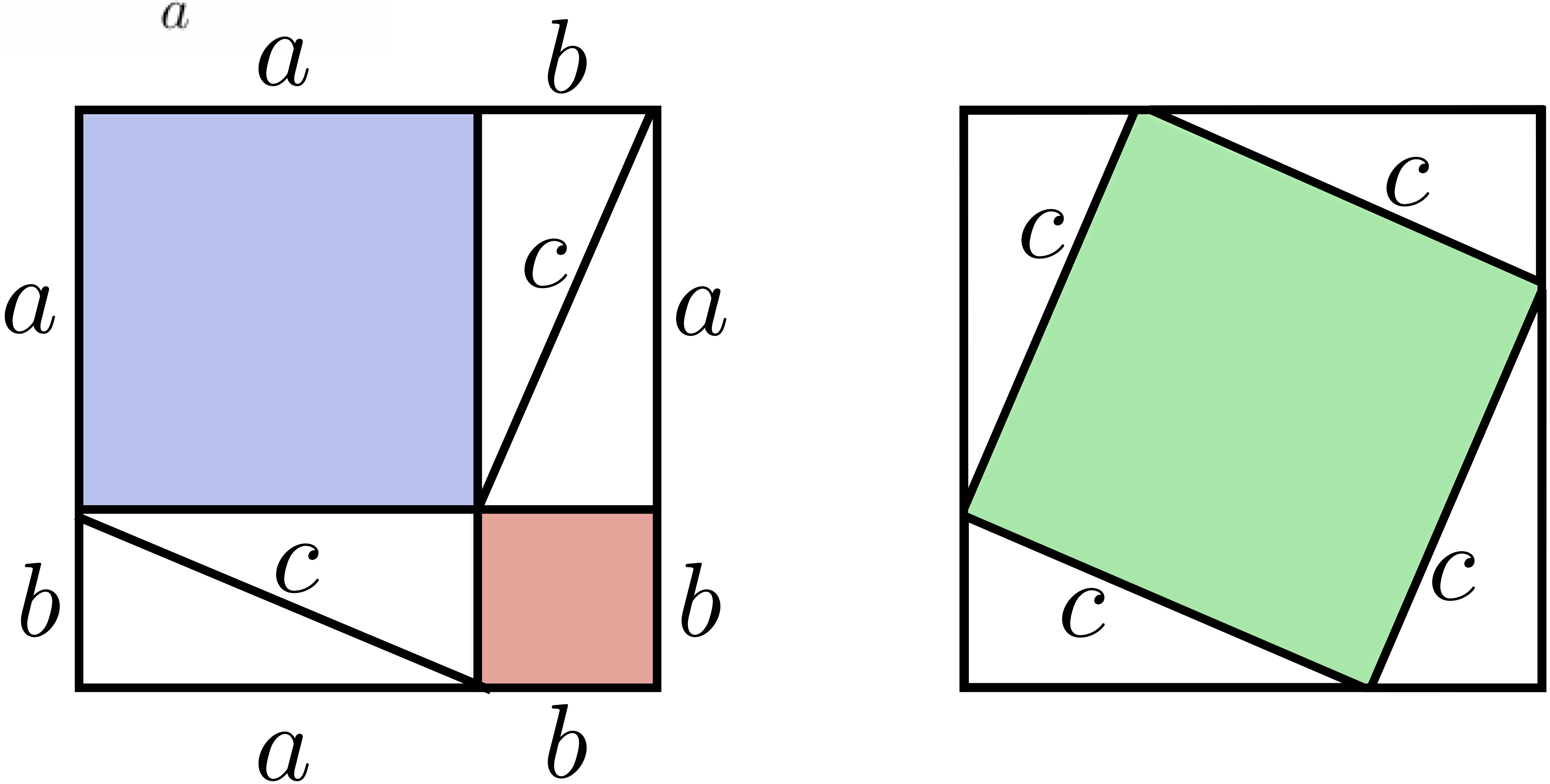
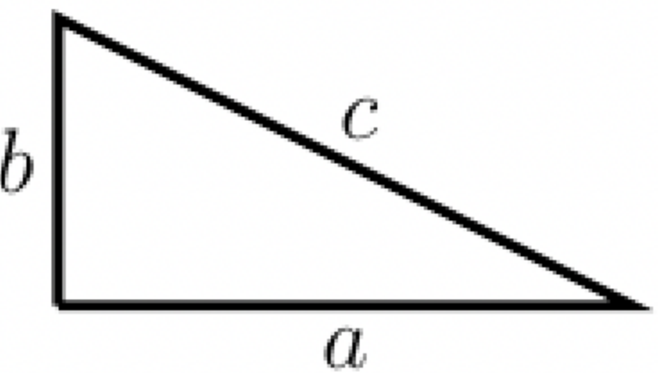
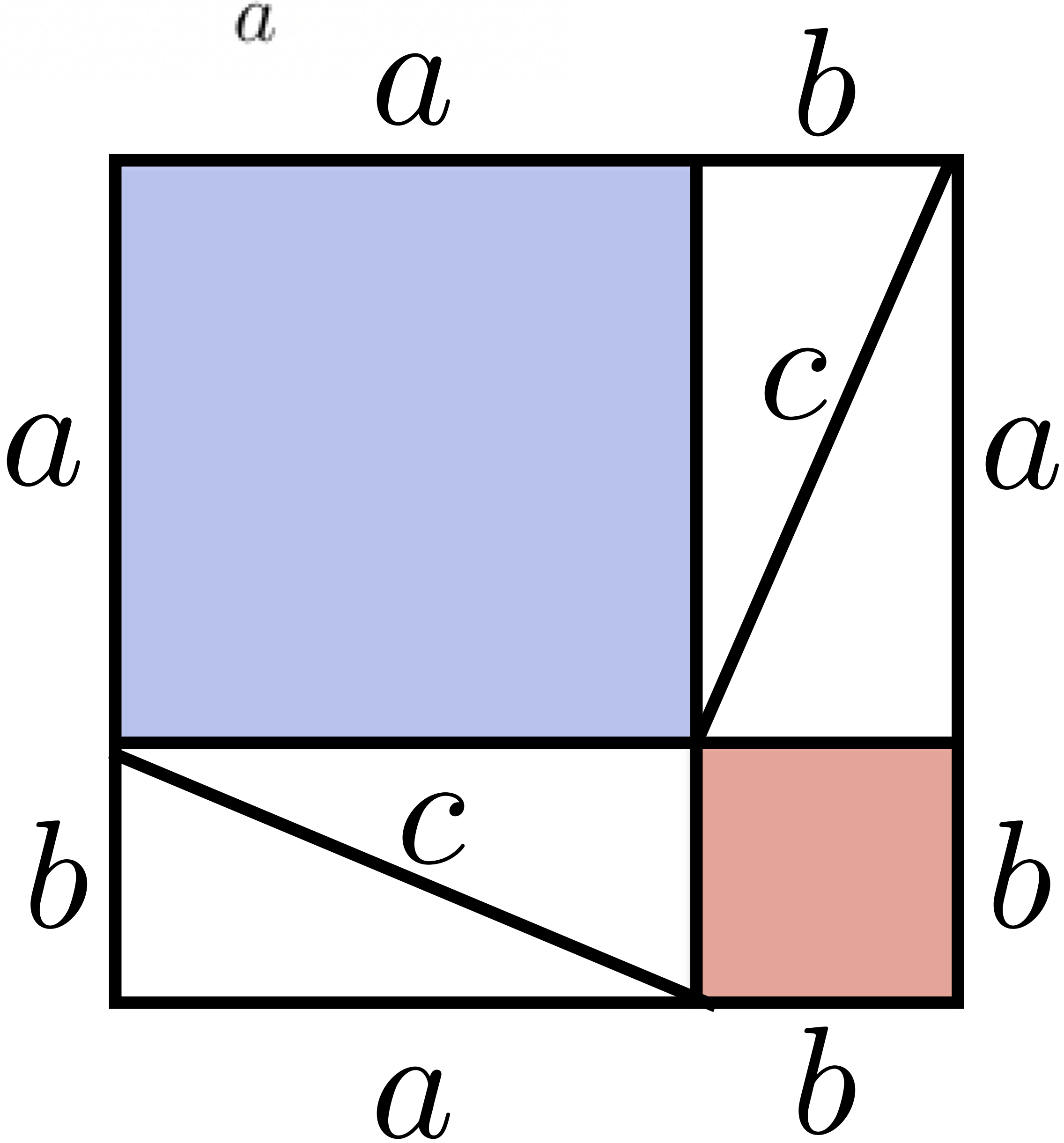


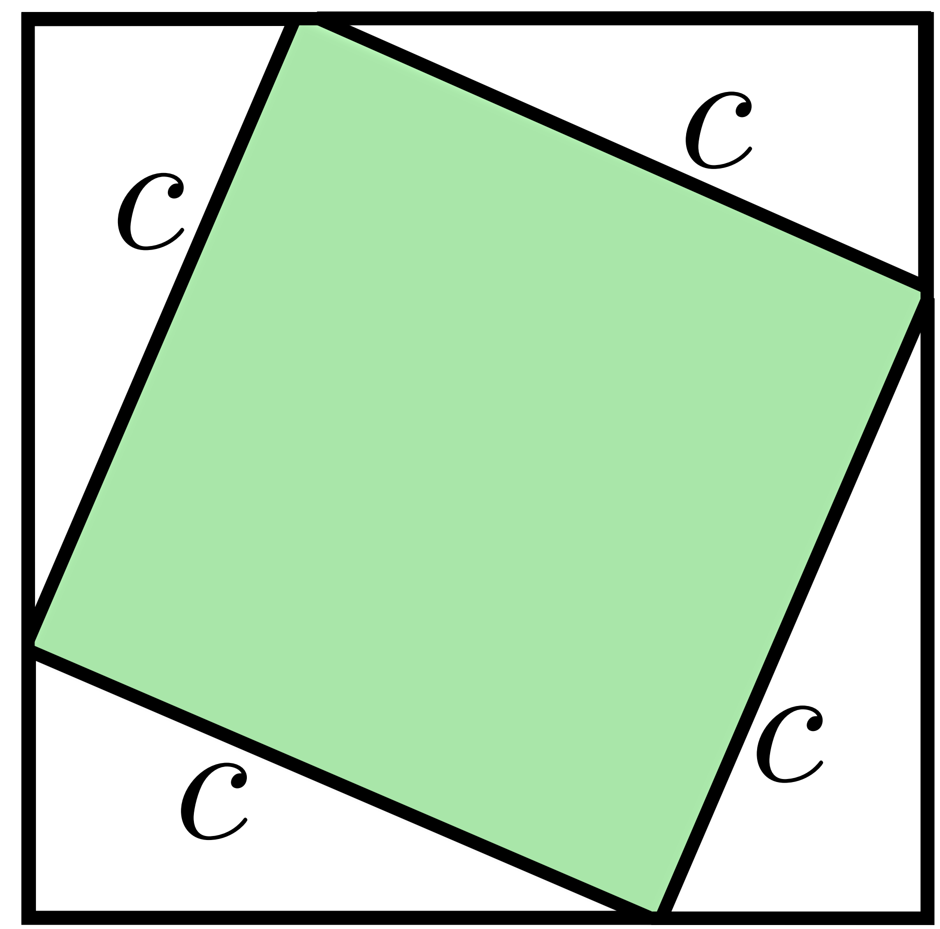


$a^2 + b^2$ + 4 triangles

How They *Might* Have Proved It







$a^2 + b^2$ + 4 triangles $=$ c^2 + 4 triangles

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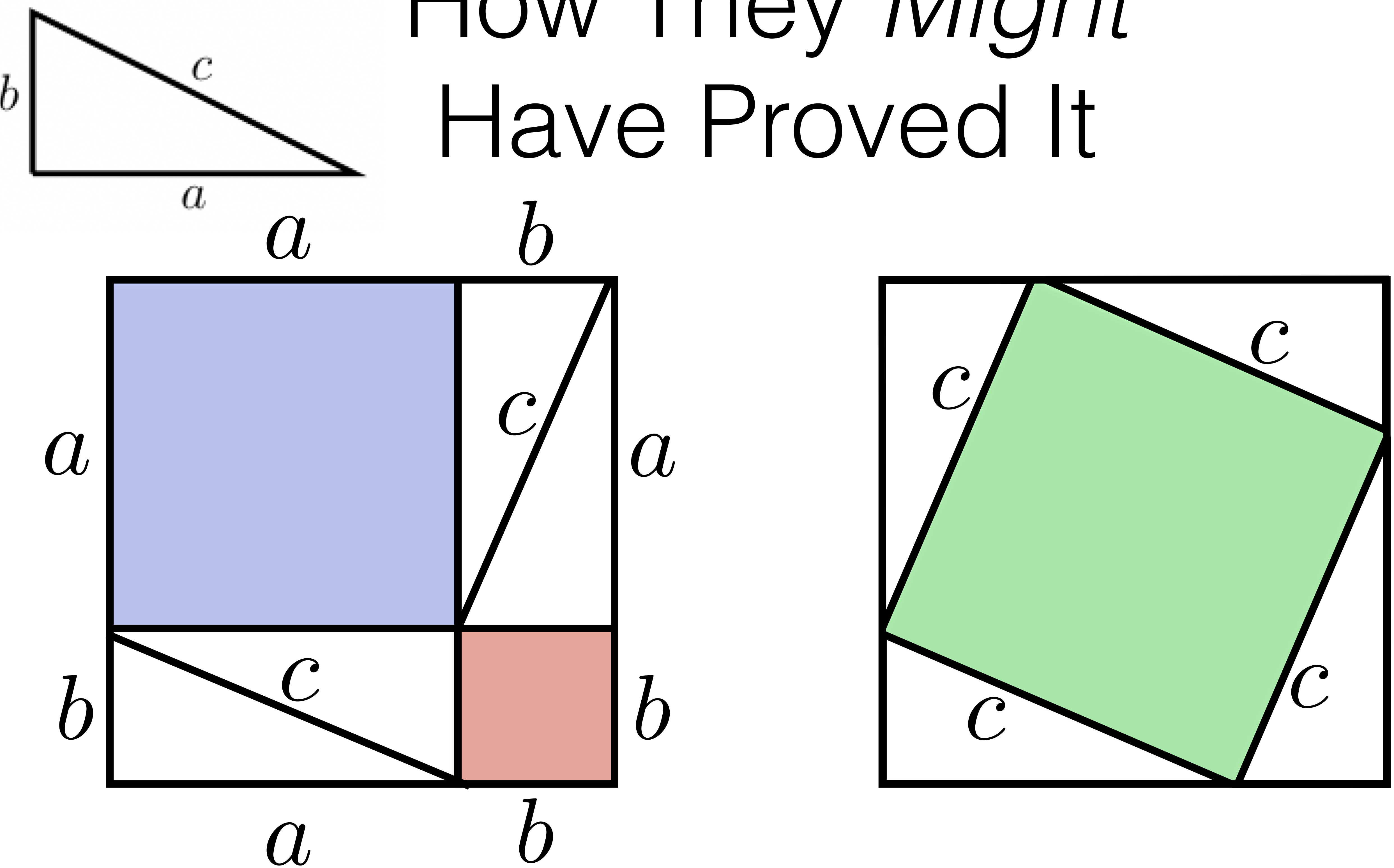

$$a^2 + b^2 + \cancel{4\text{ triangles}} = c^2 + \cancel{4\text{ triangles}}$$

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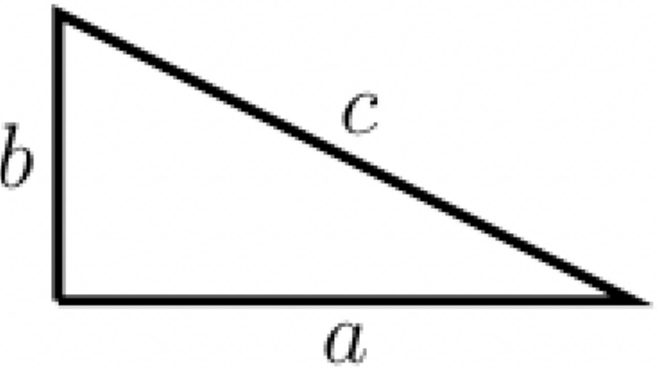

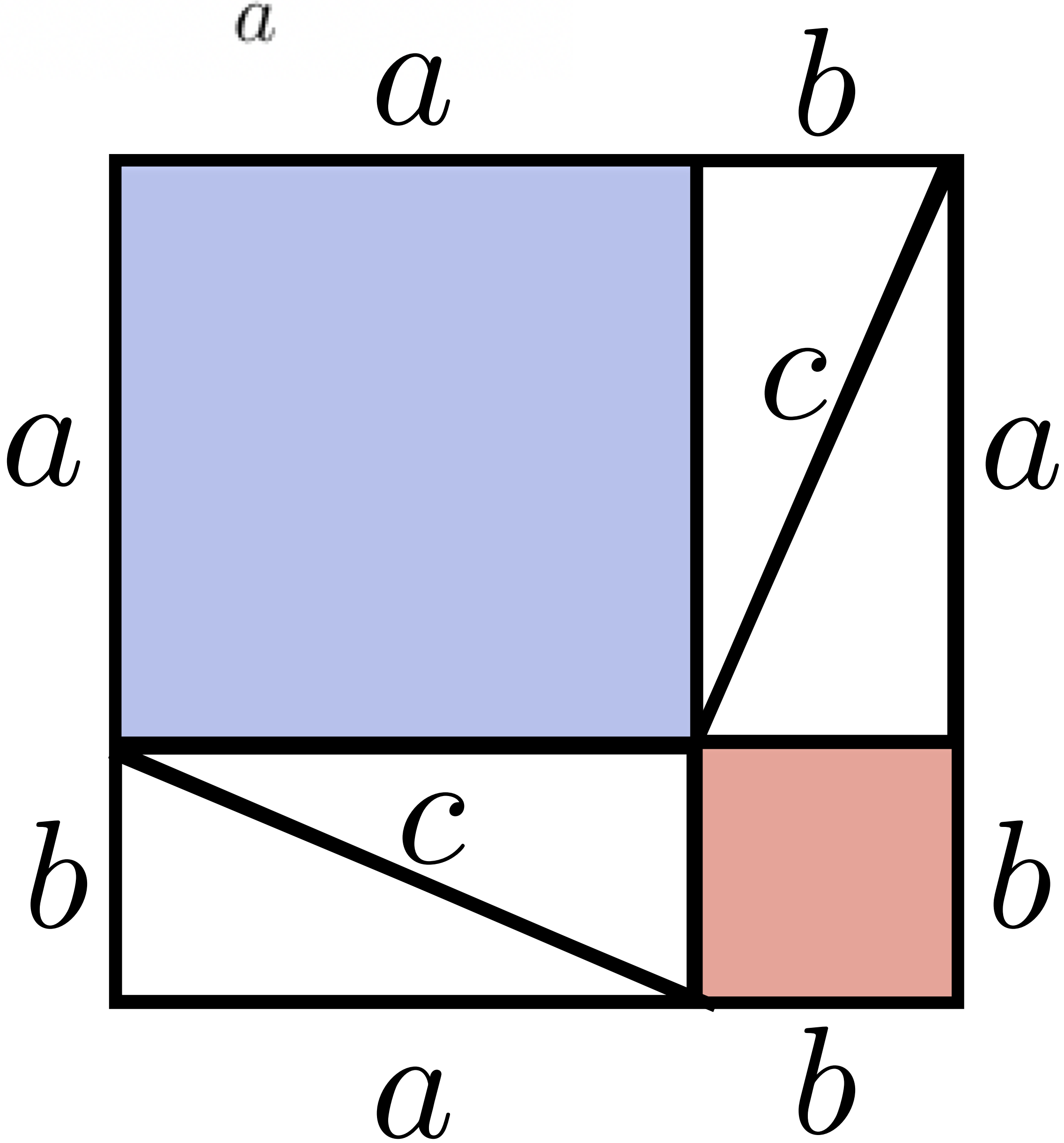

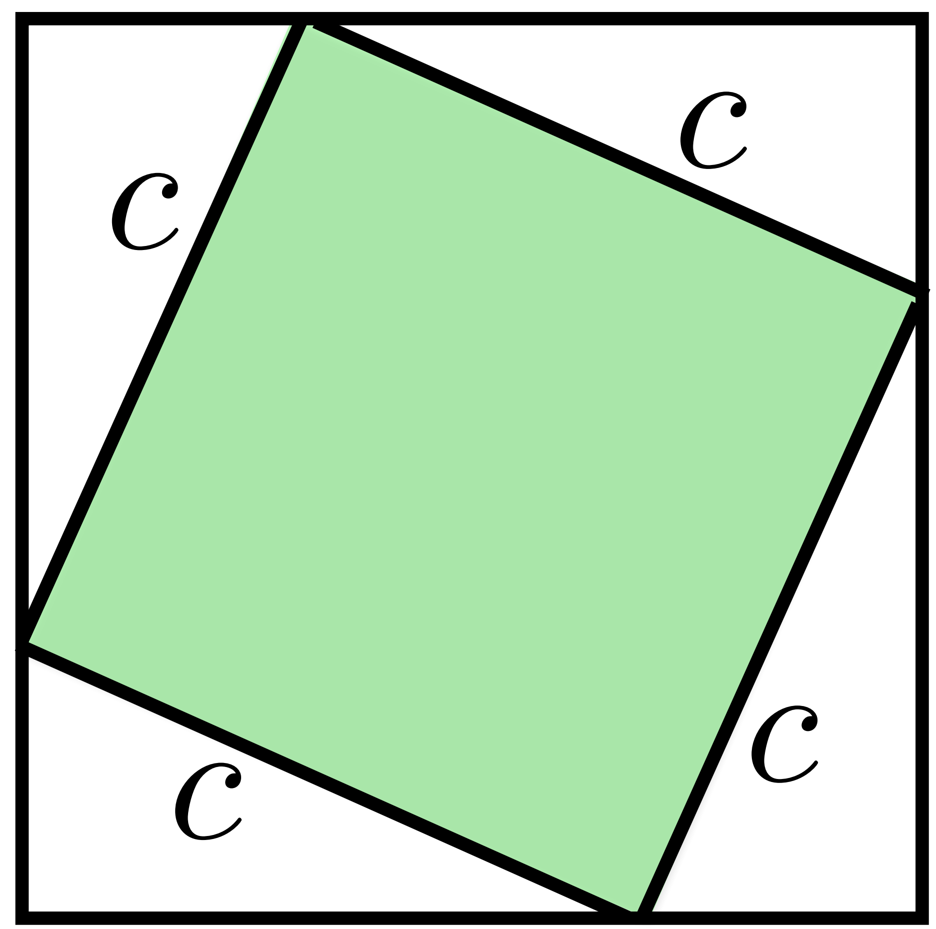

$$a^2 + b^2 = c^2$$

A *Possible* Proof

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- One of the oldest math texts in China is *Zhoubi Suanjing*. It was written around the time of Pythagoras.

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- Time to Think Like A Math Historian™!

Think Like A Math Historian

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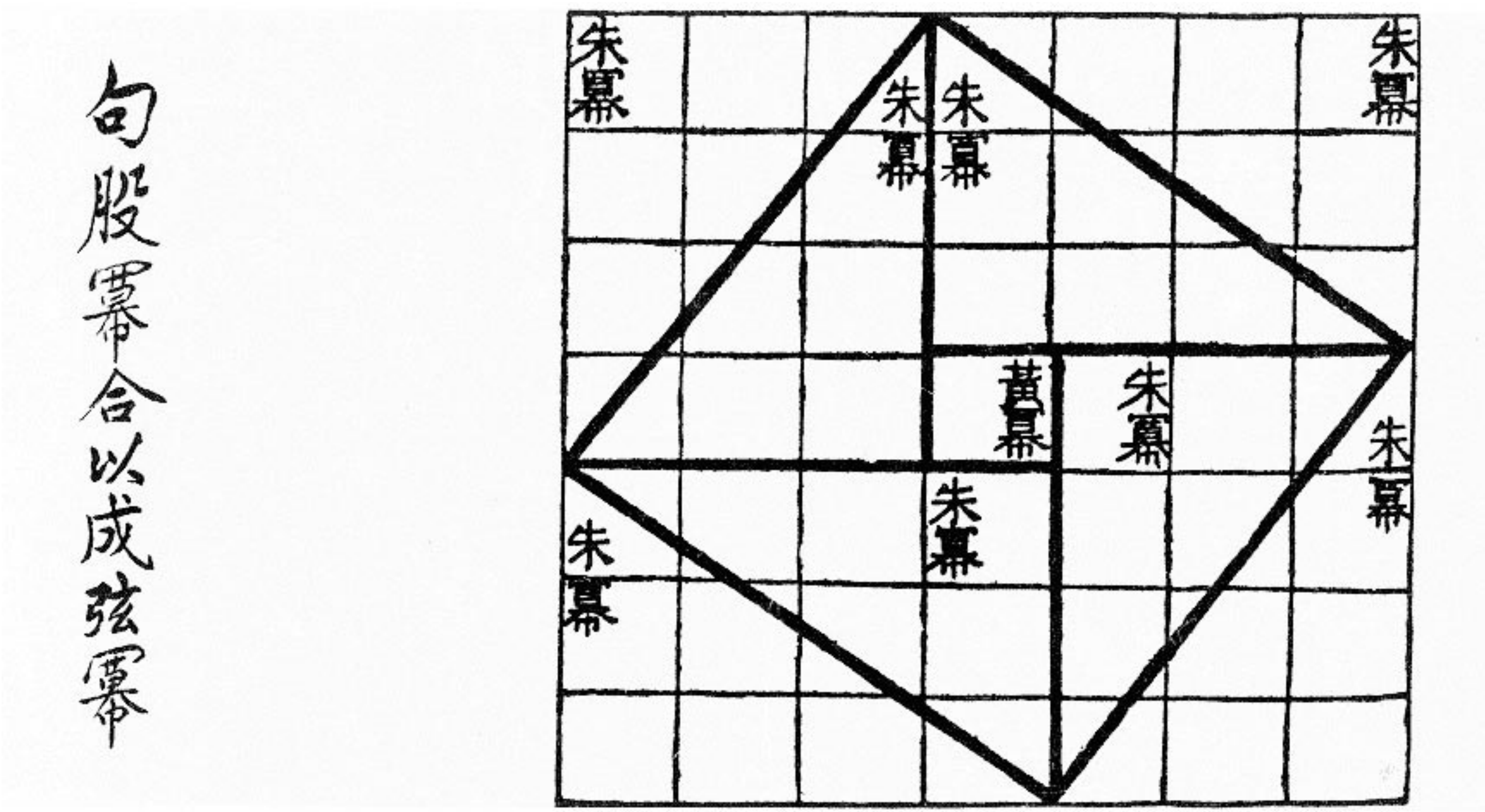

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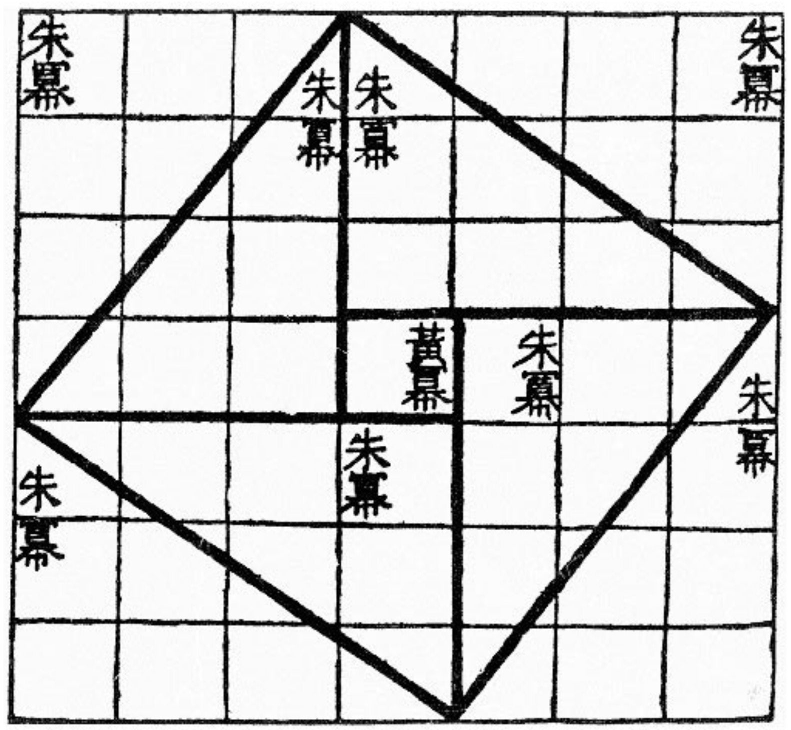

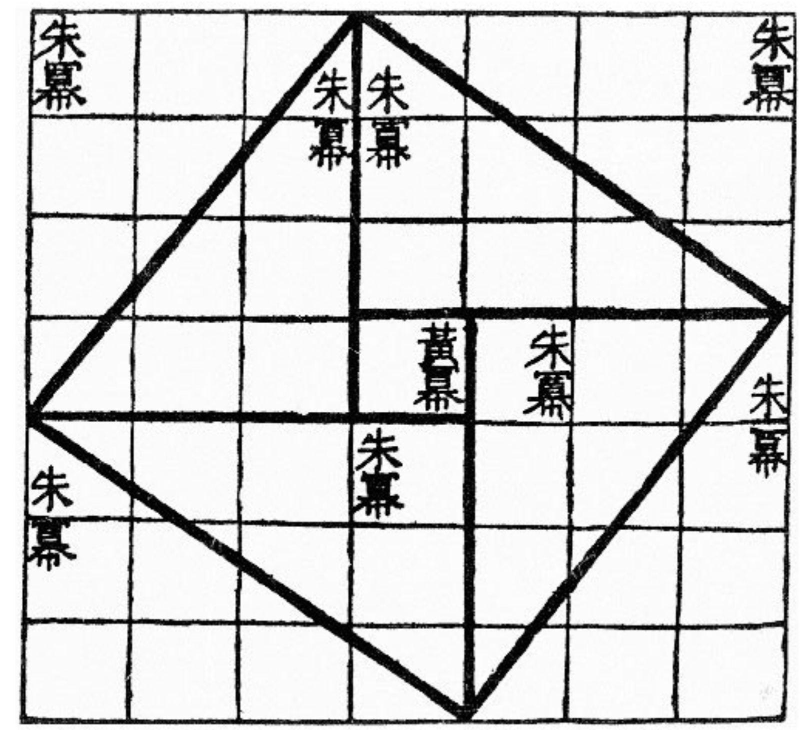

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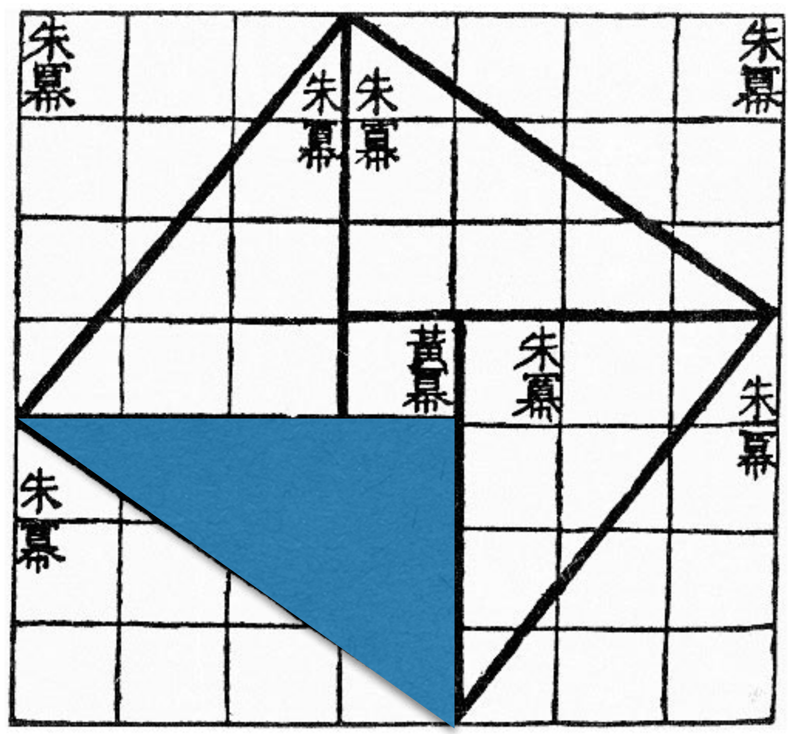

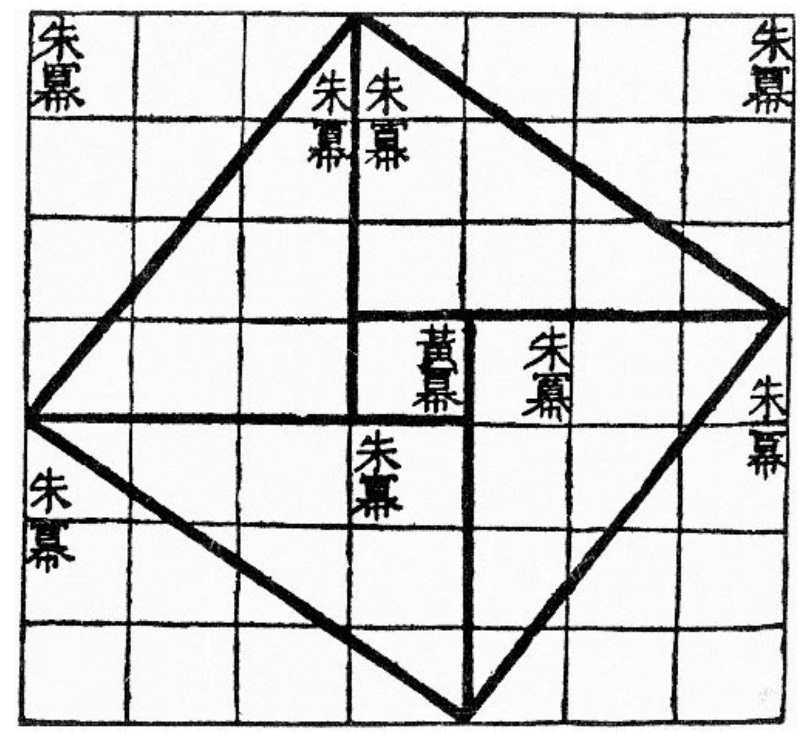

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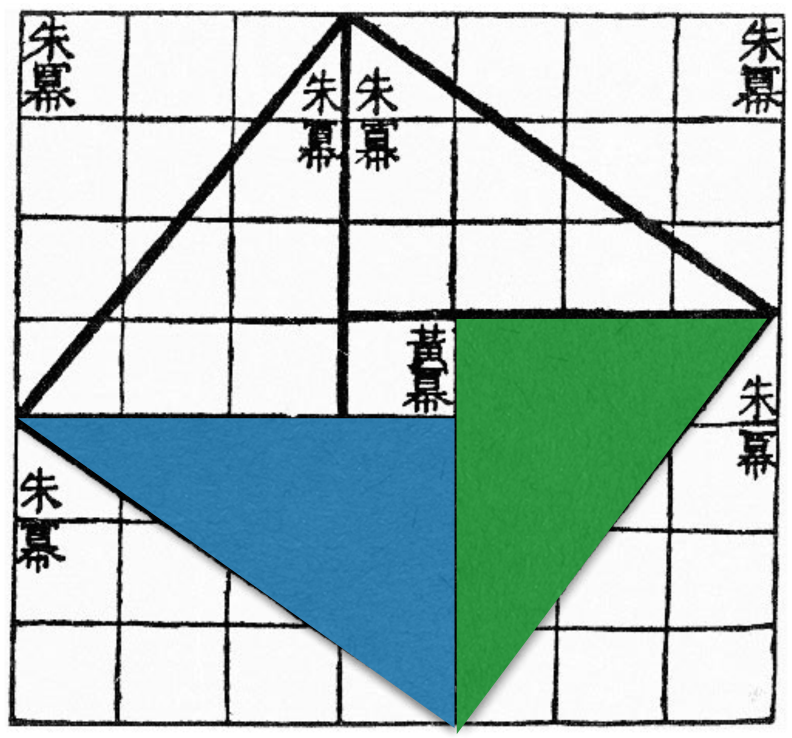

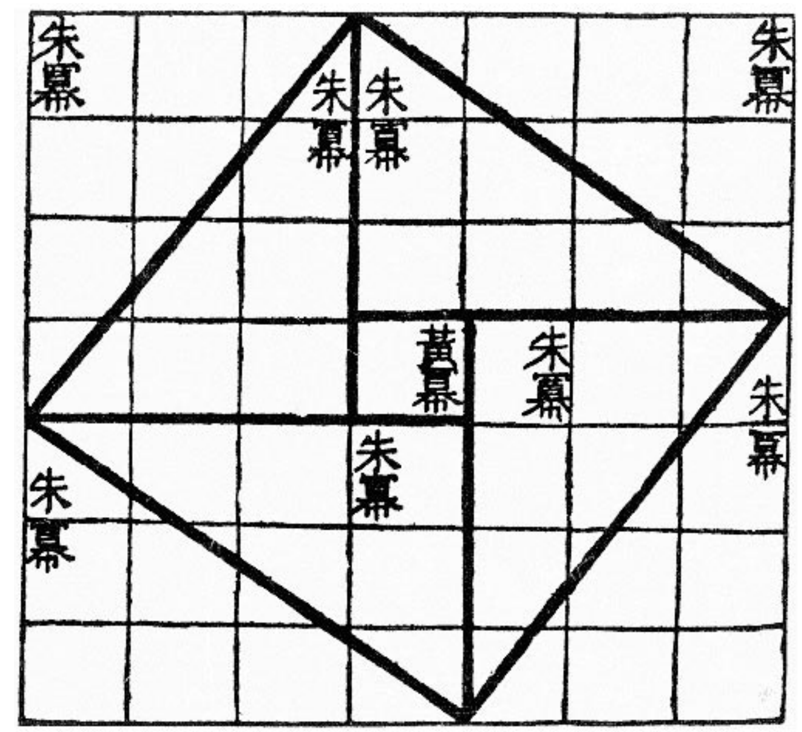

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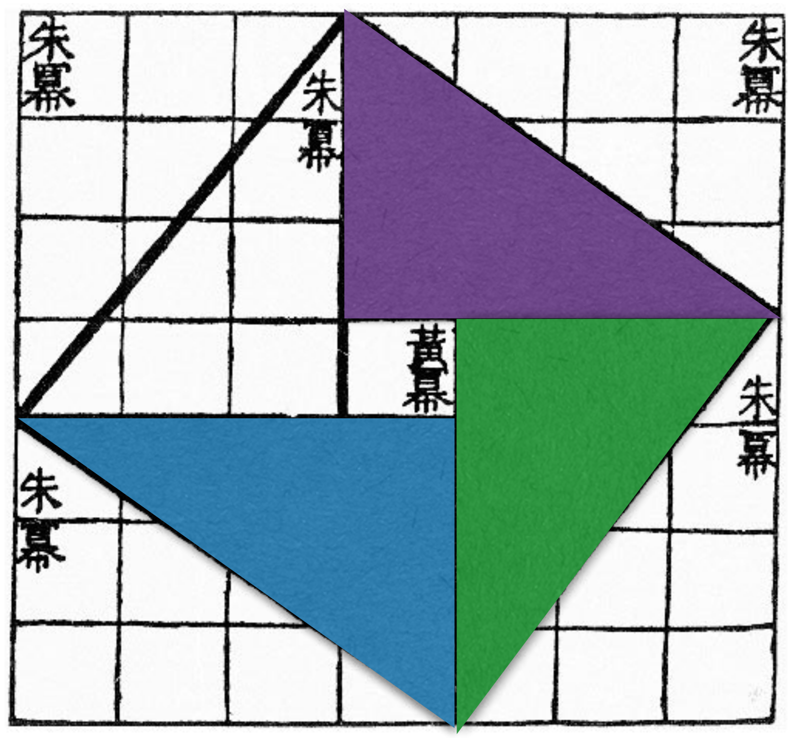

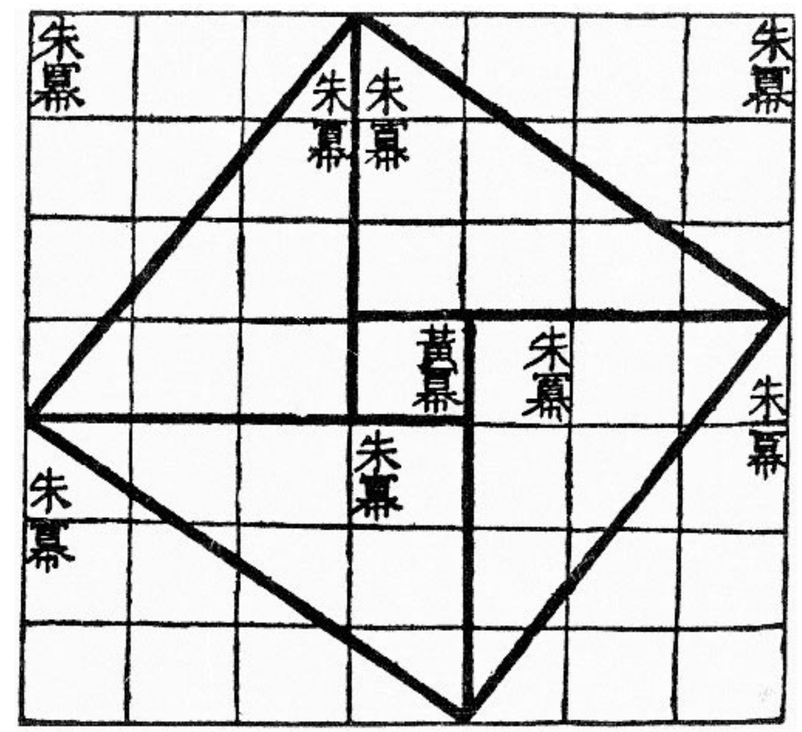

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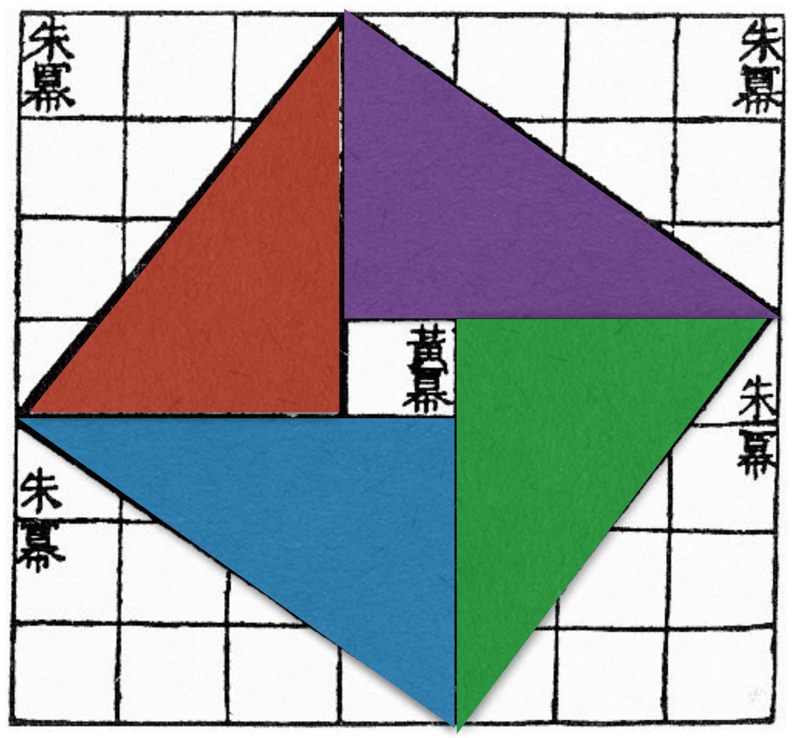

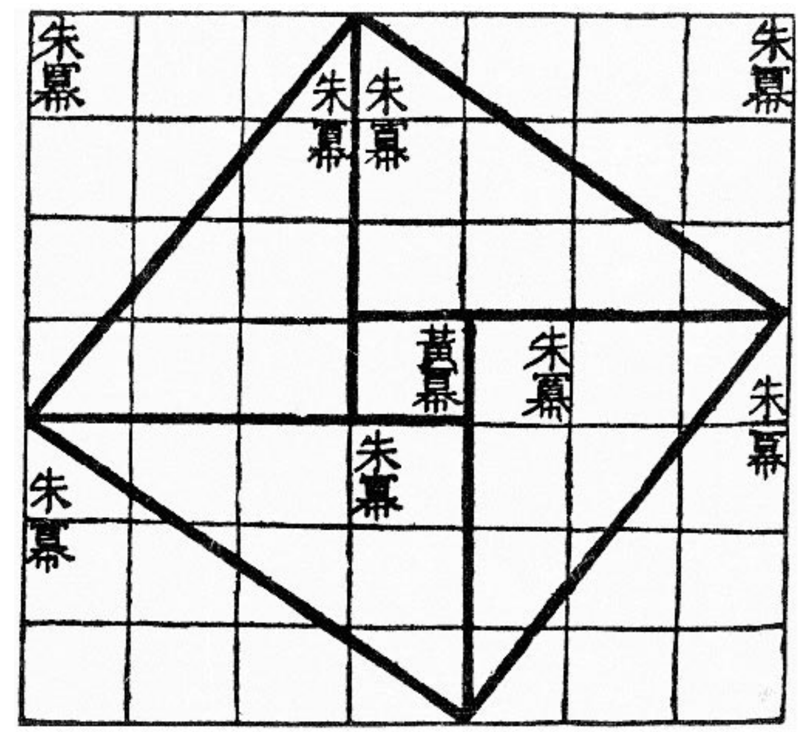

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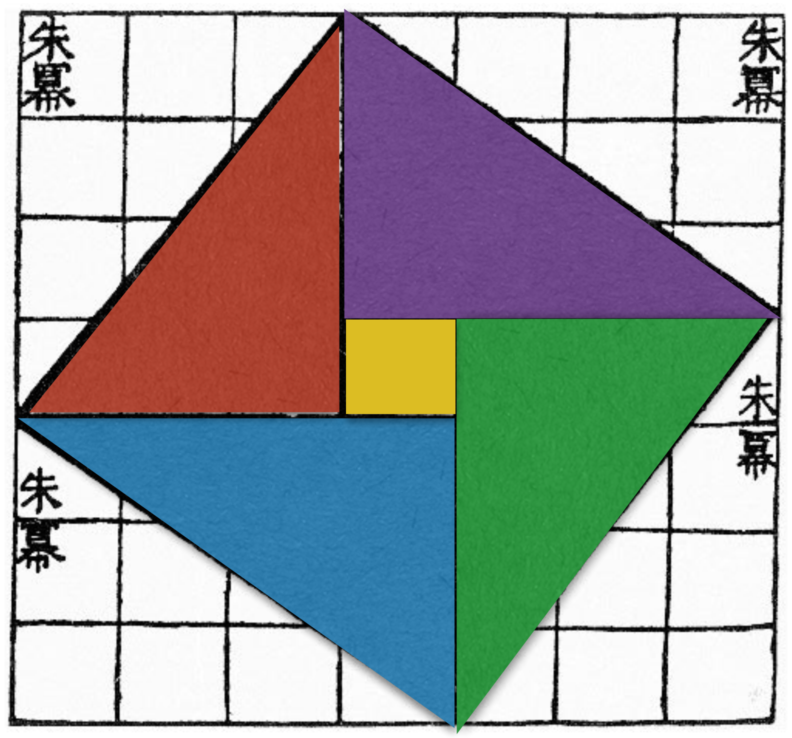

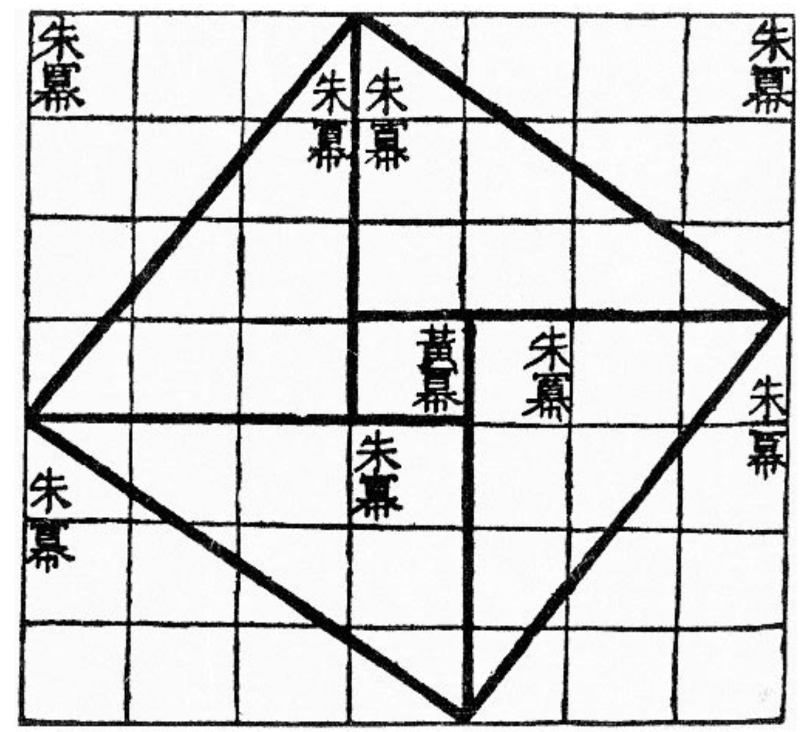

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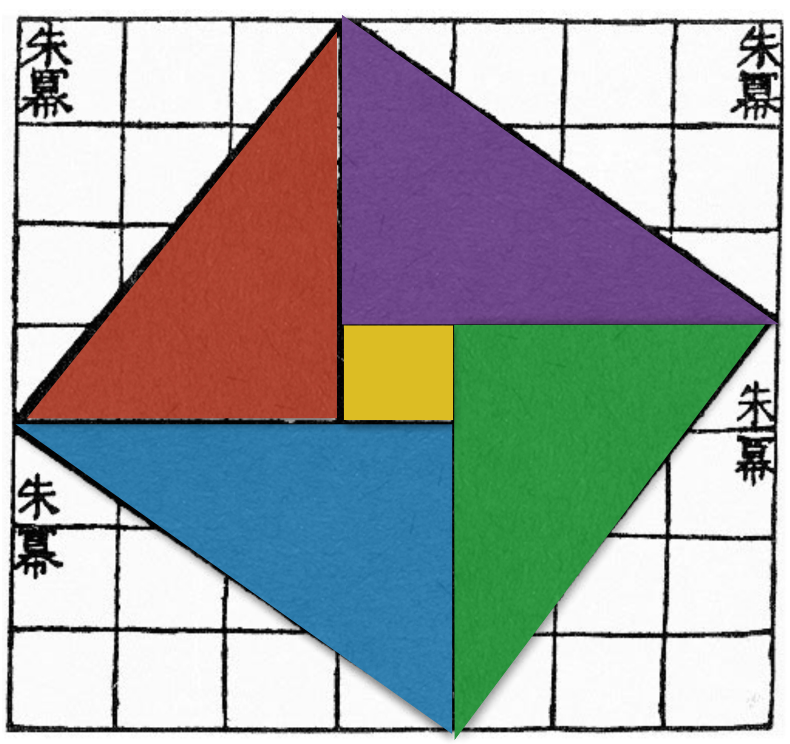

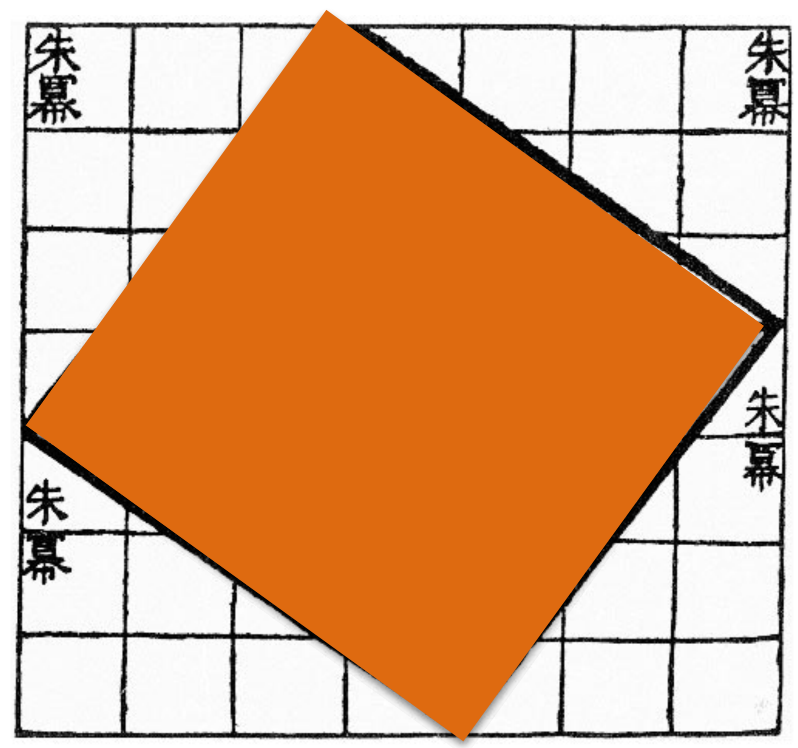

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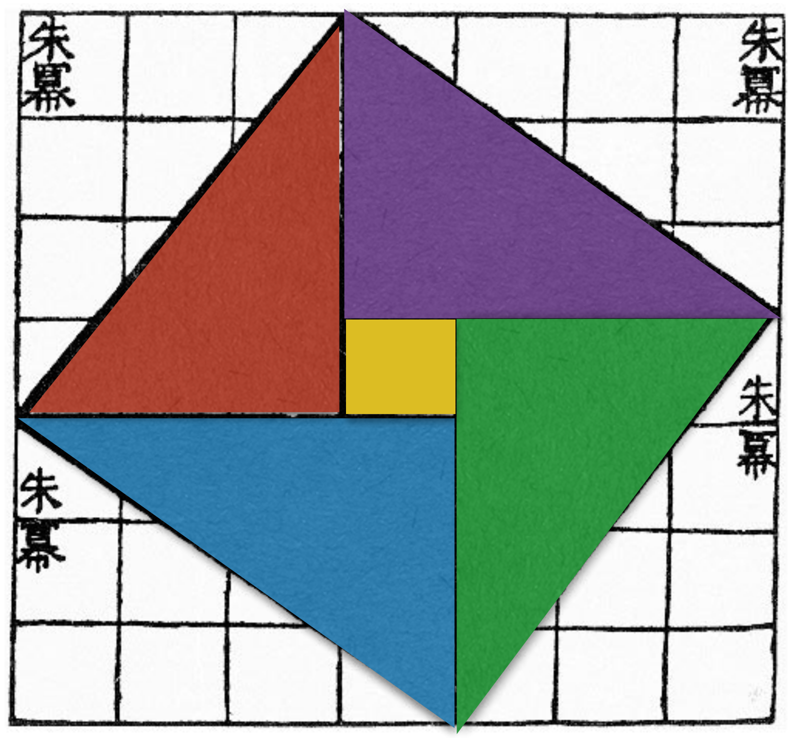

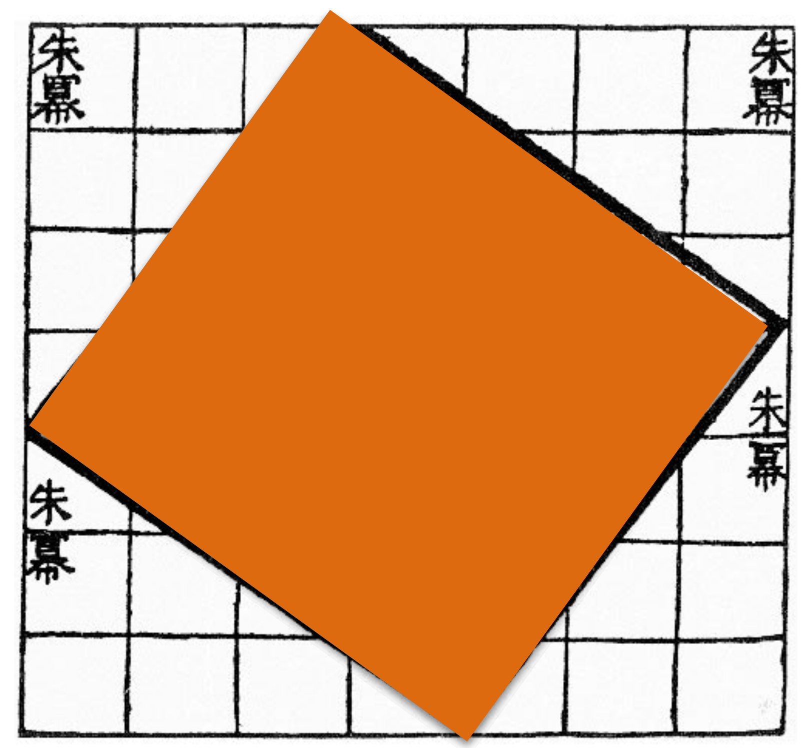

4 triangles + little square = Big square

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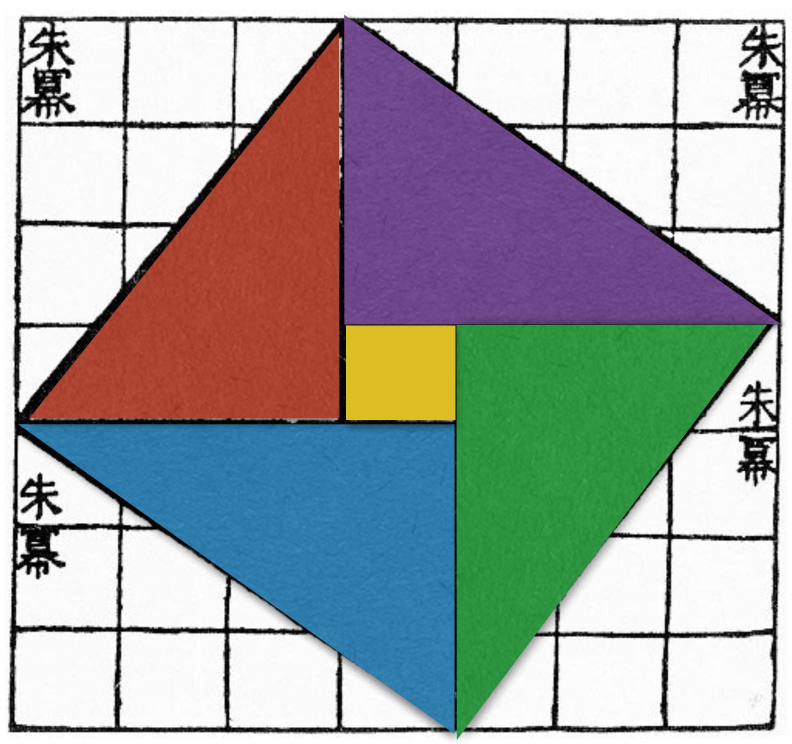

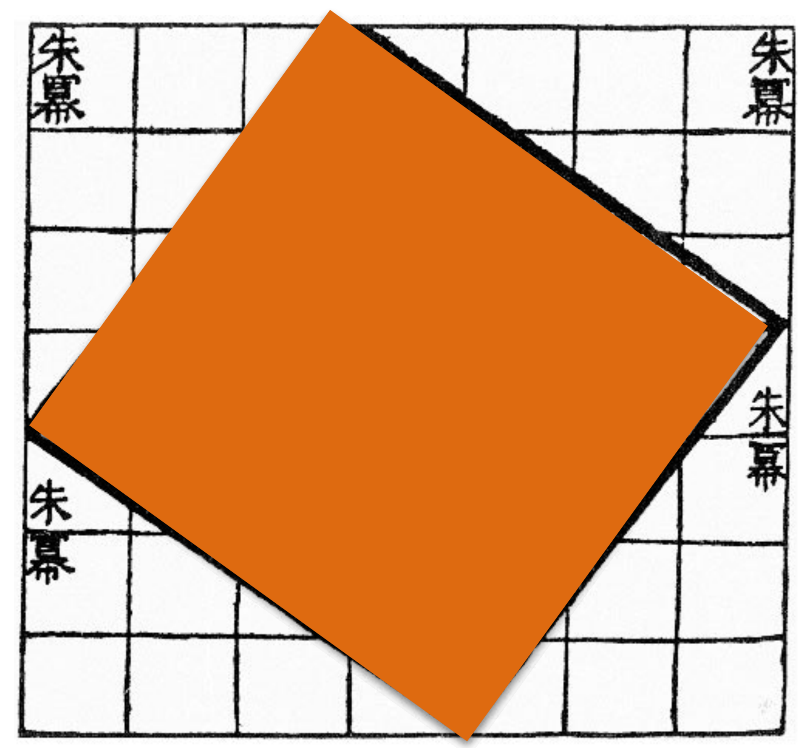

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$$4 \cdot \frac{1}{2}(3 \cdot 4)$$

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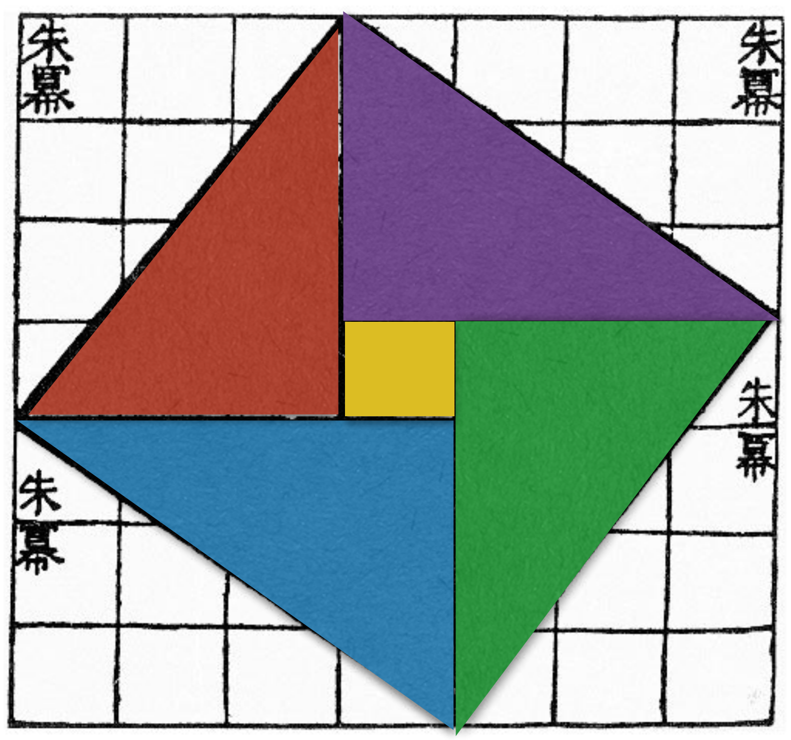

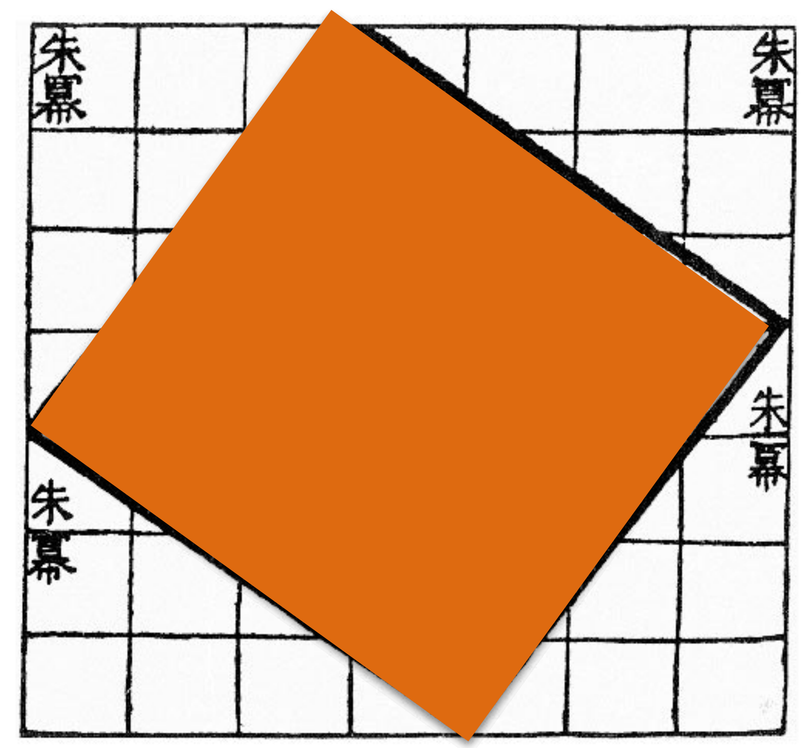

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$$4 \cdot \frac{1}{2}(3 \cdot 4) \quad + \quad 1 \cdot 1$$

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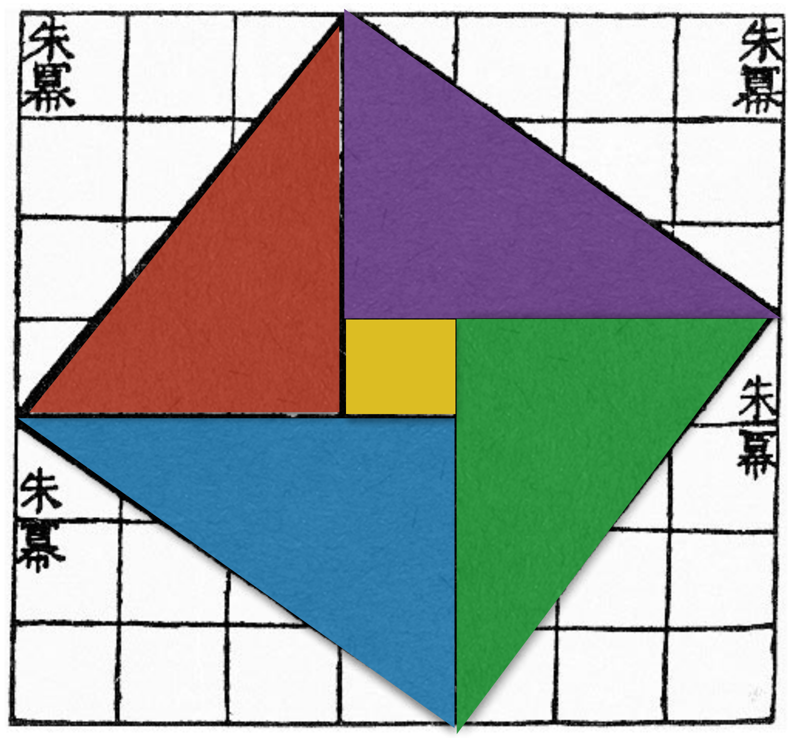
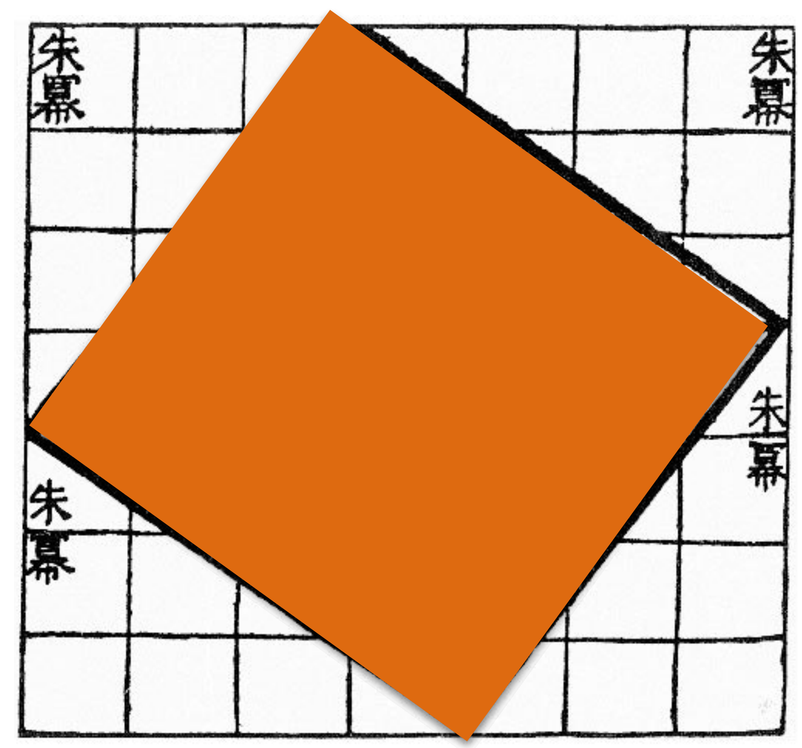
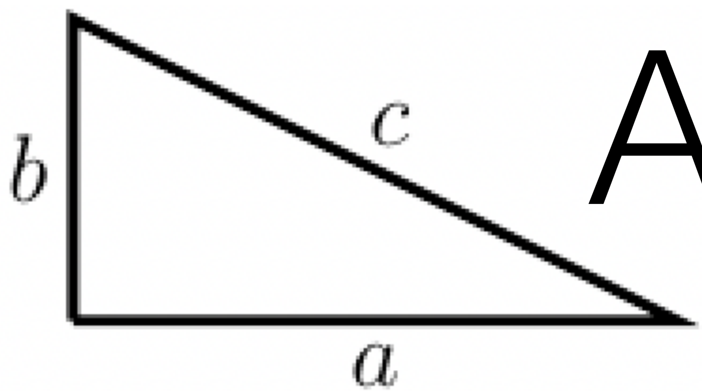




4 triangles + little square = Big square

$$4 \cdot \frac{1}{2}(3 \cdot 4) \quad + \quad 1 \cdot 1 \quad = \quad 5^2$$

A *Possible* Proof



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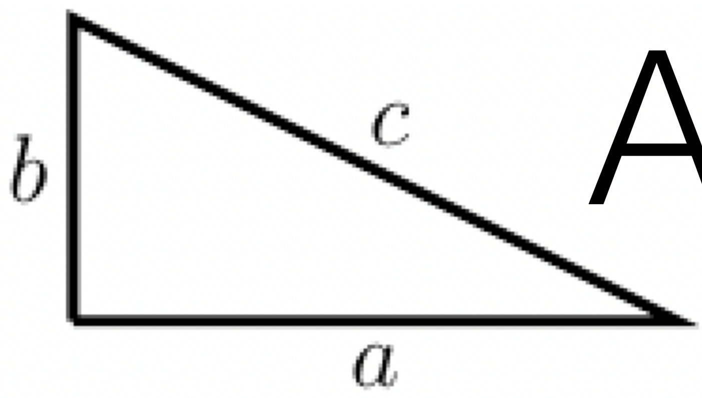

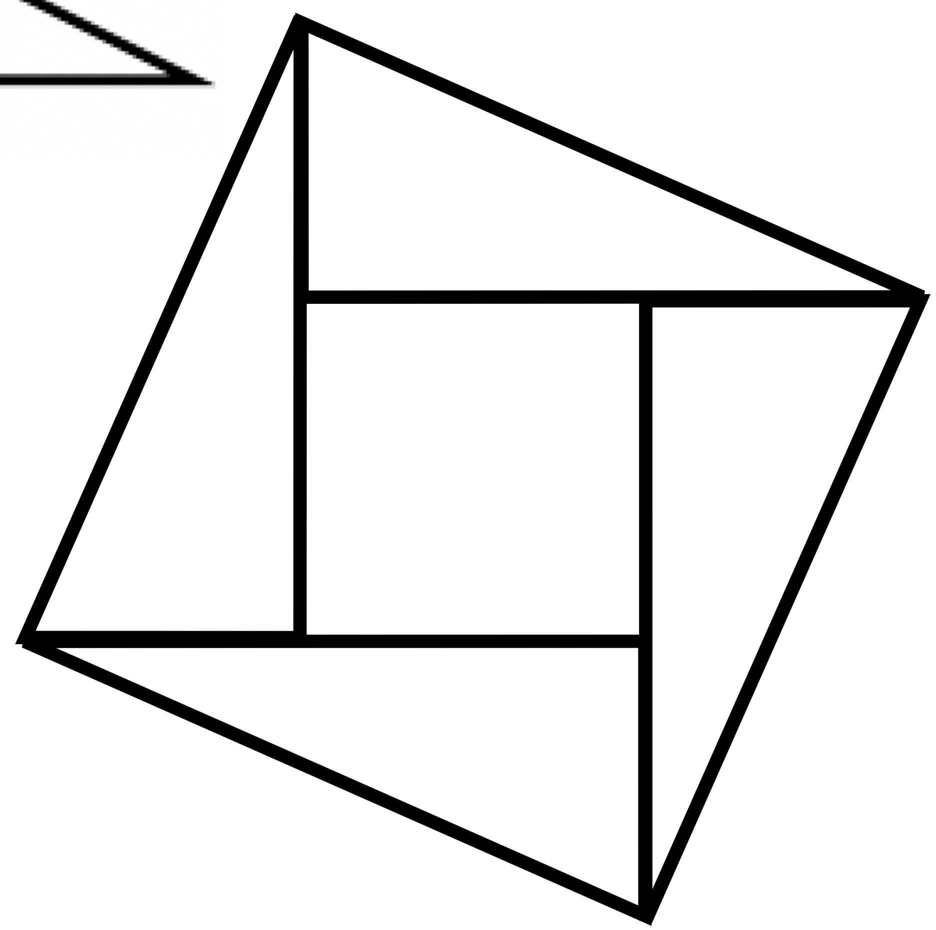

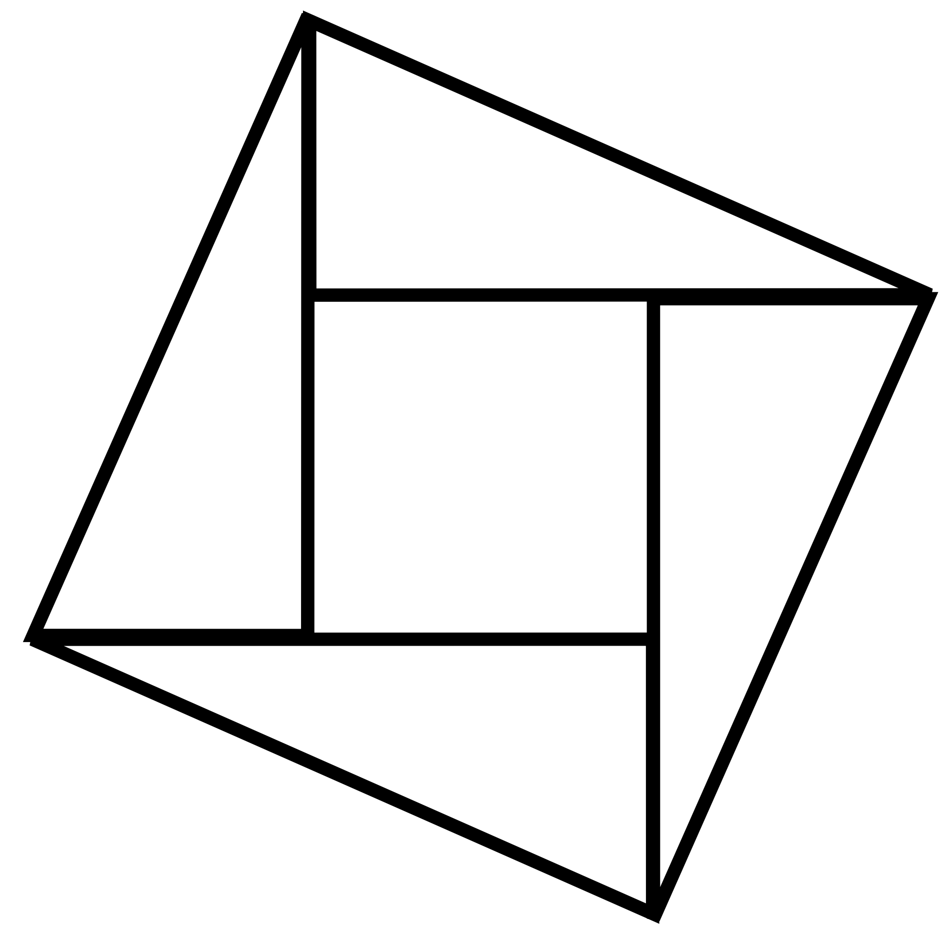

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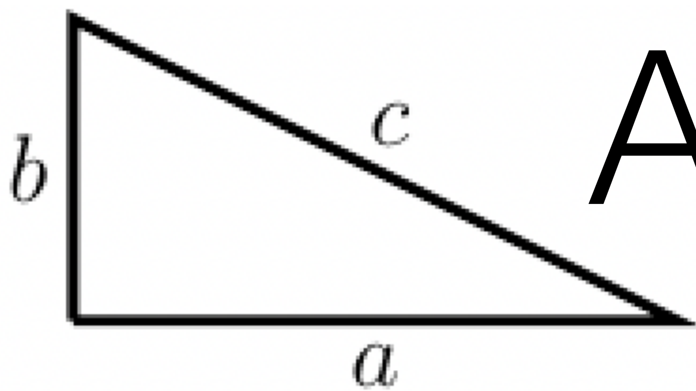

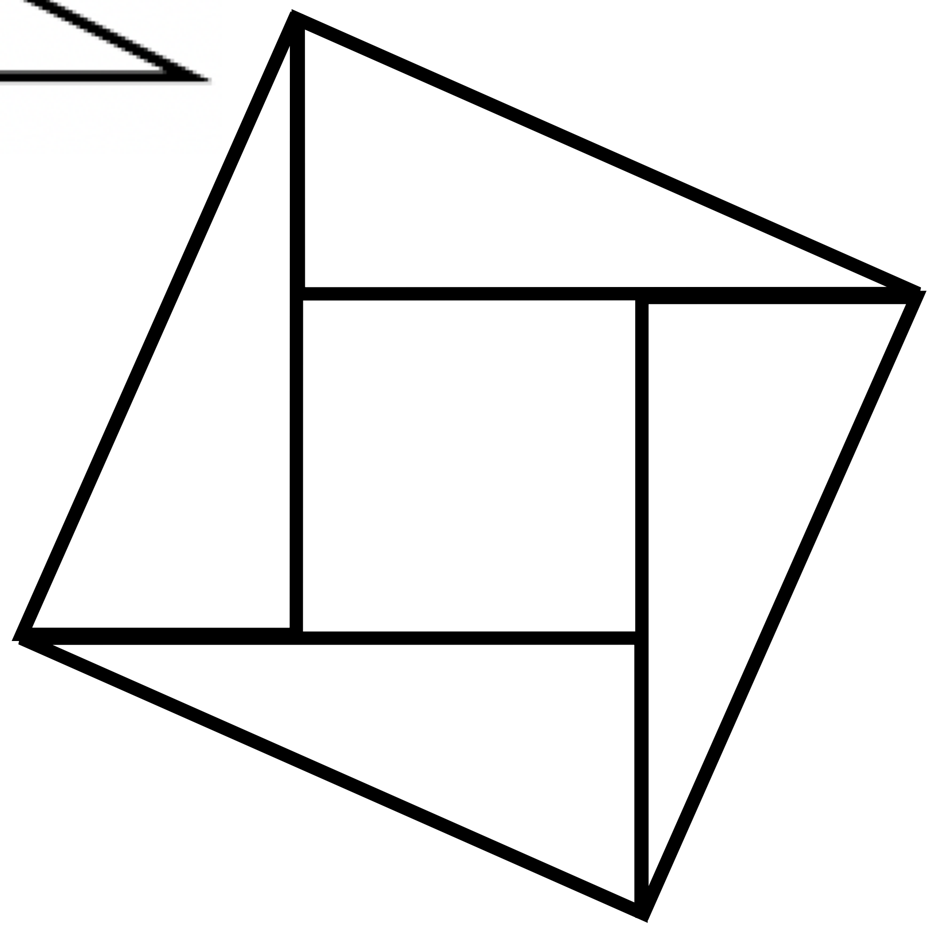

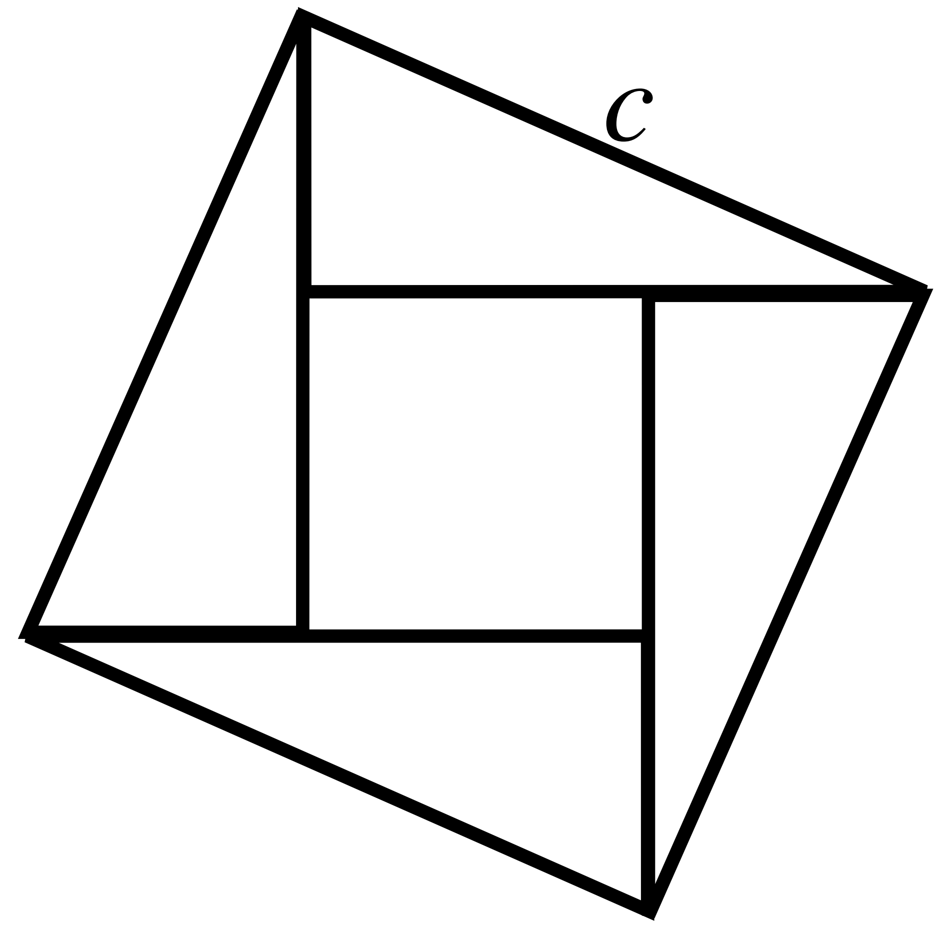

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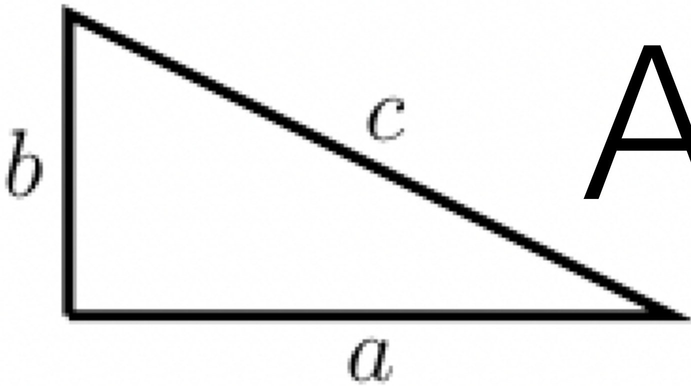

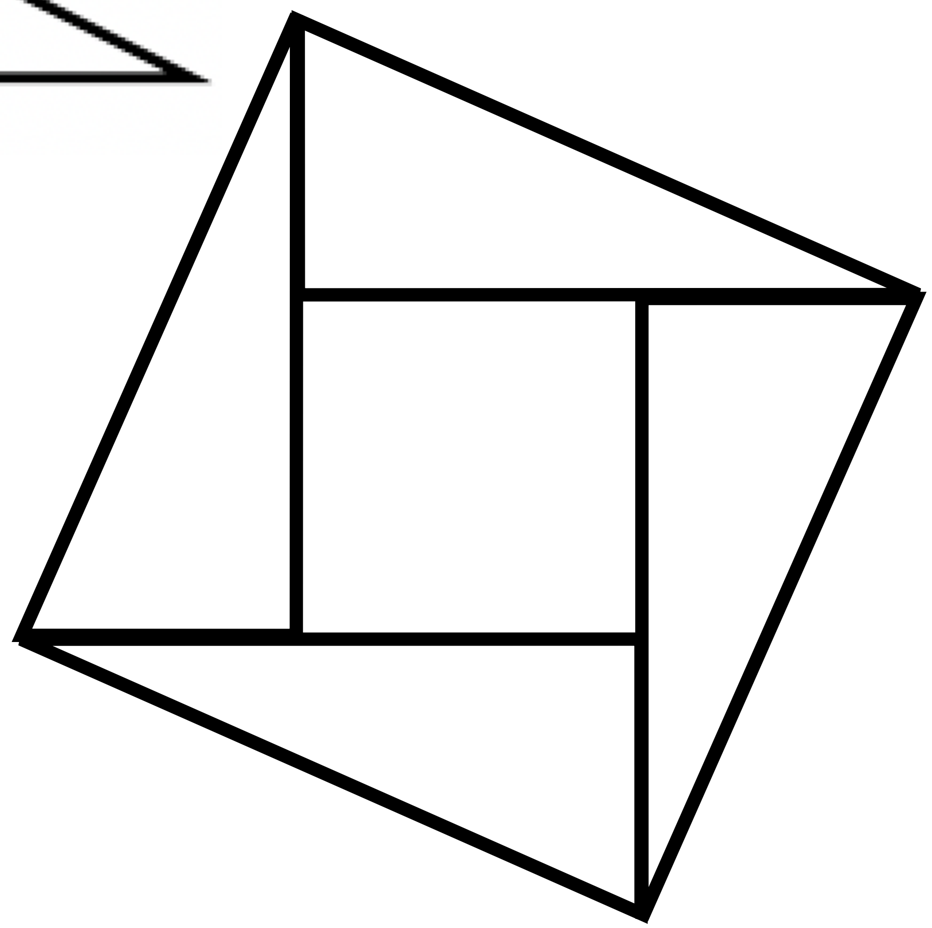

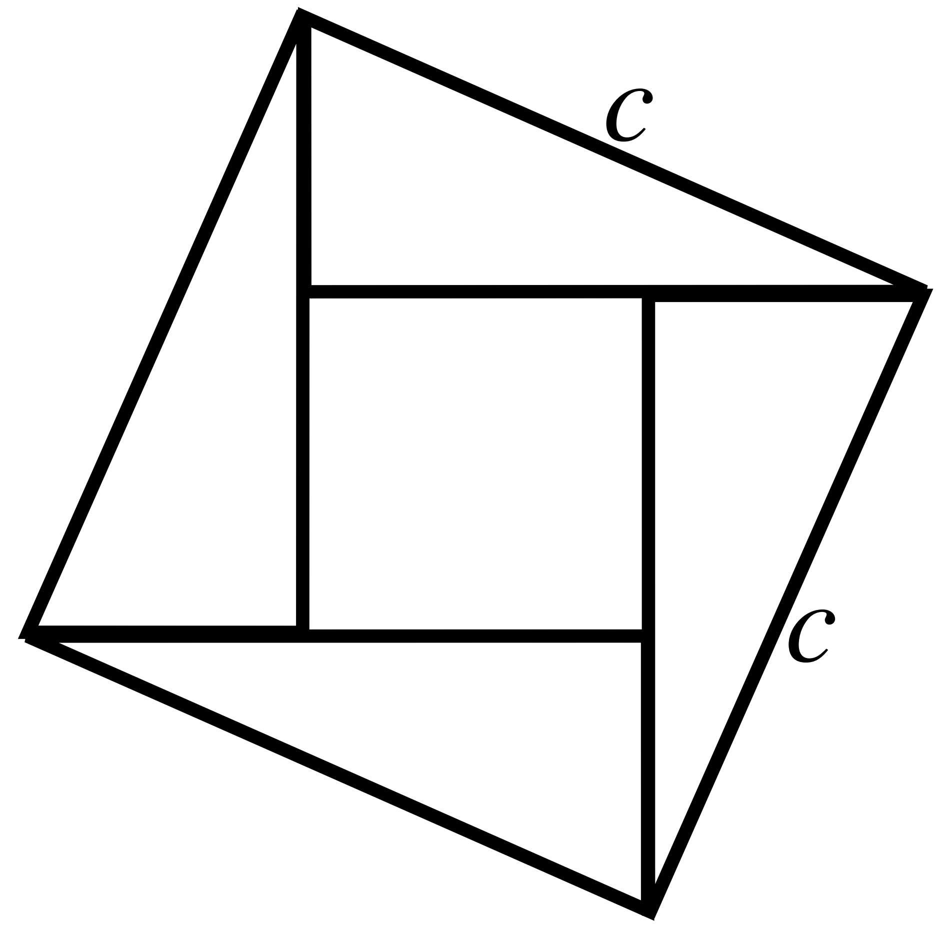

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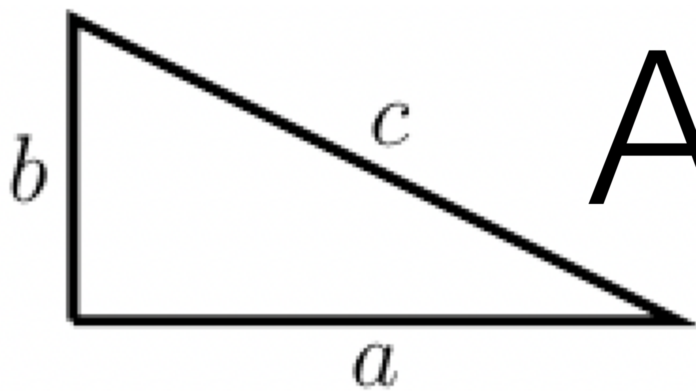

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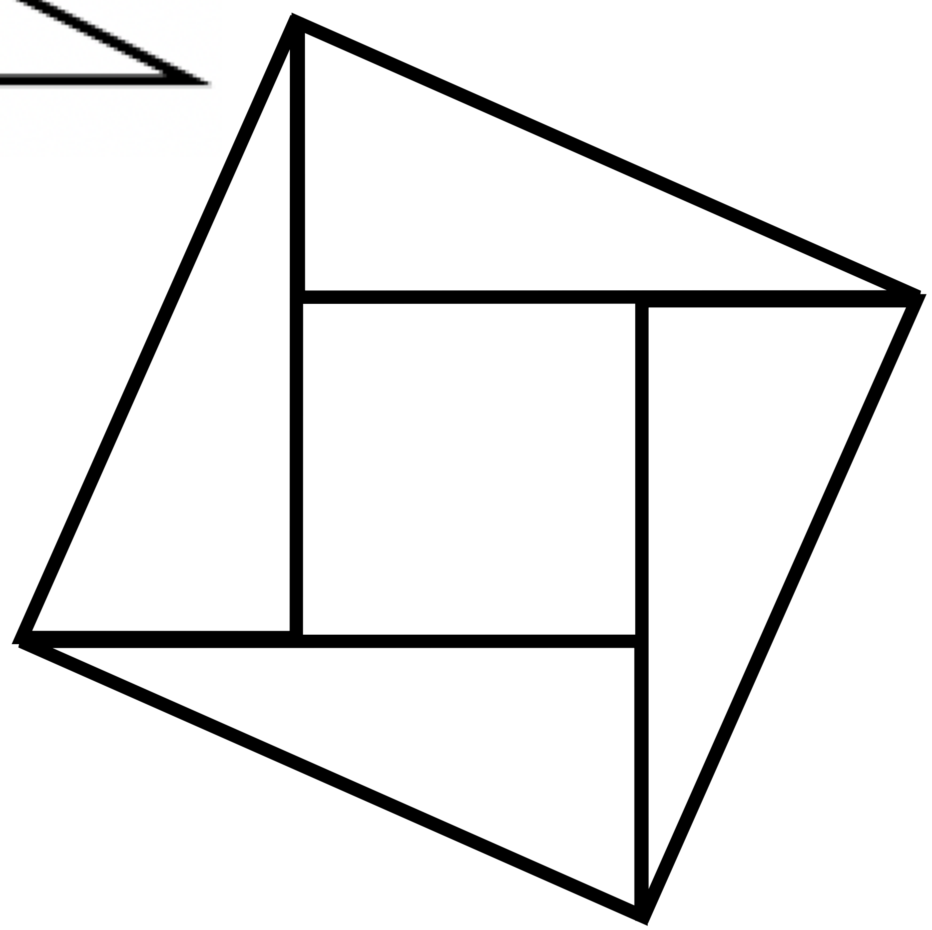

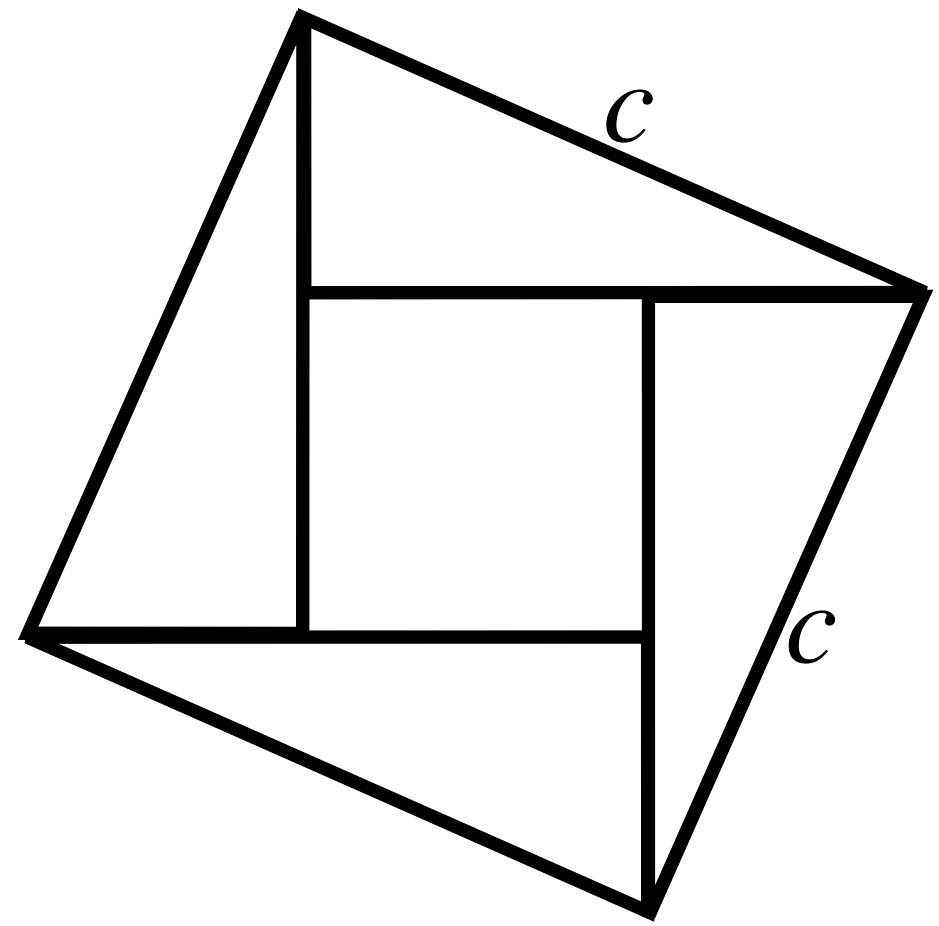

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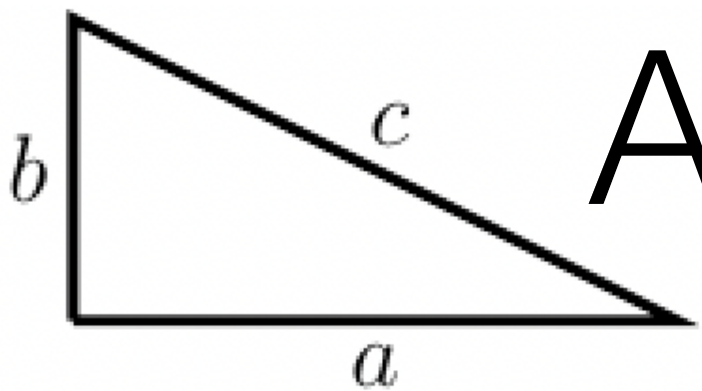

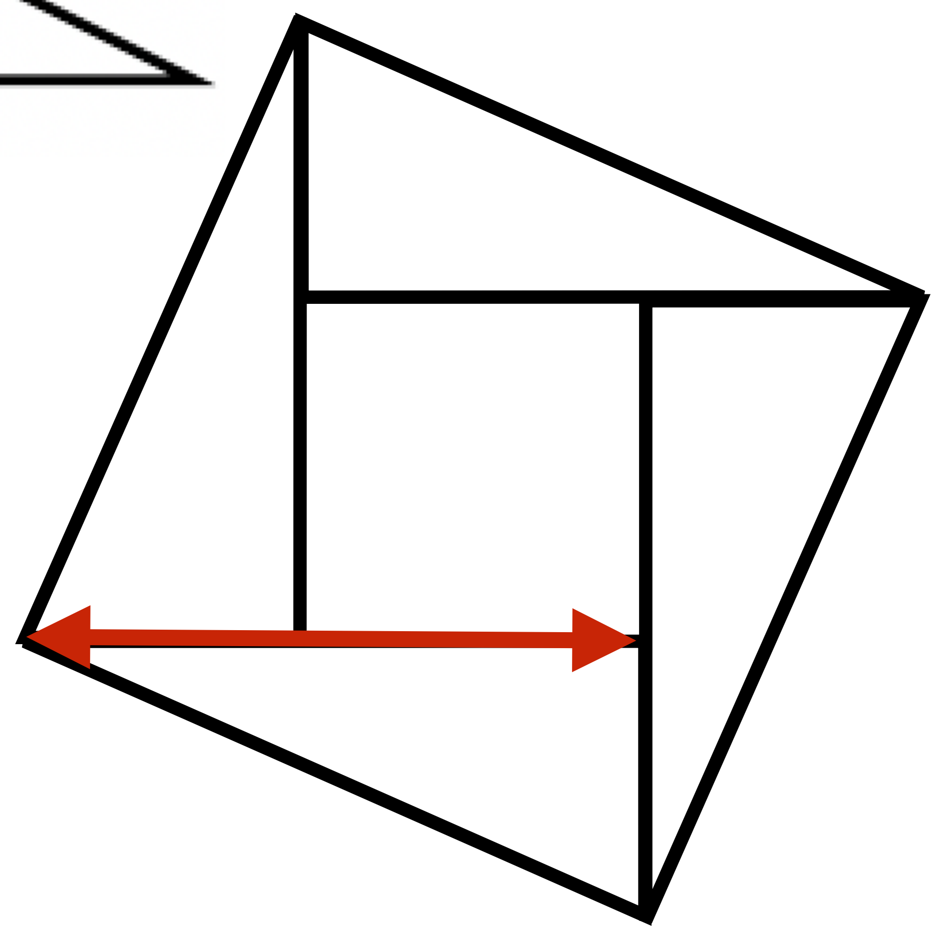

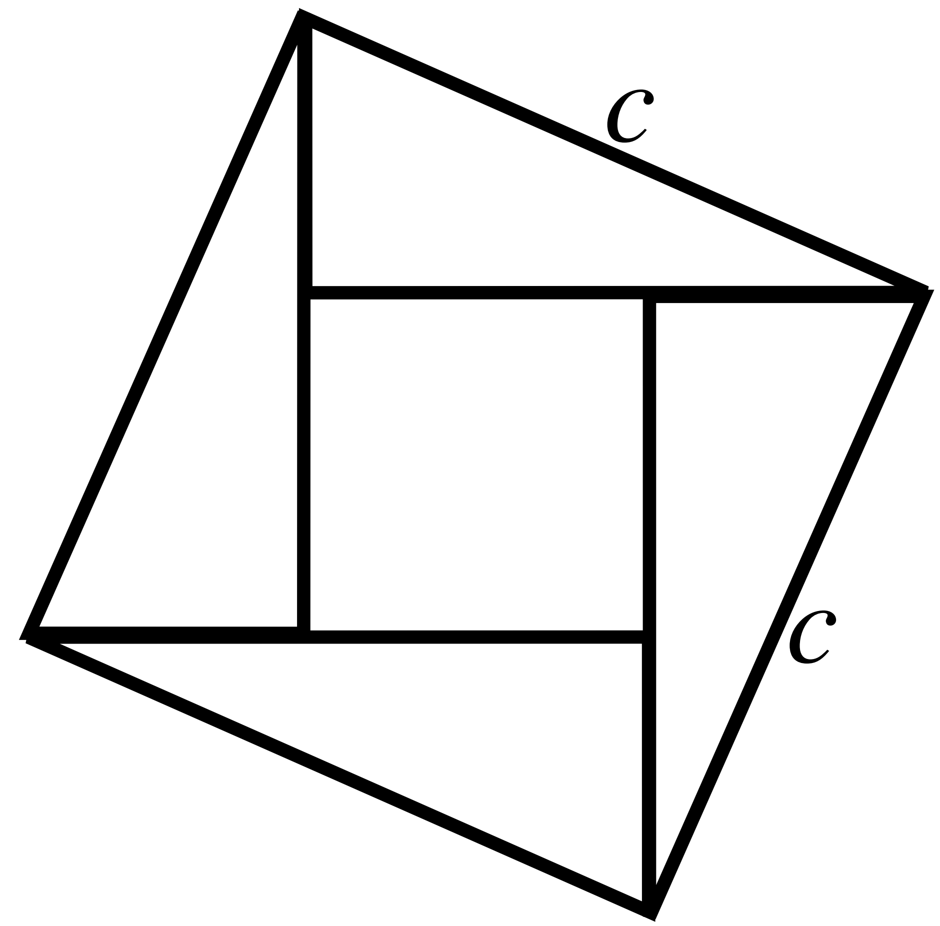

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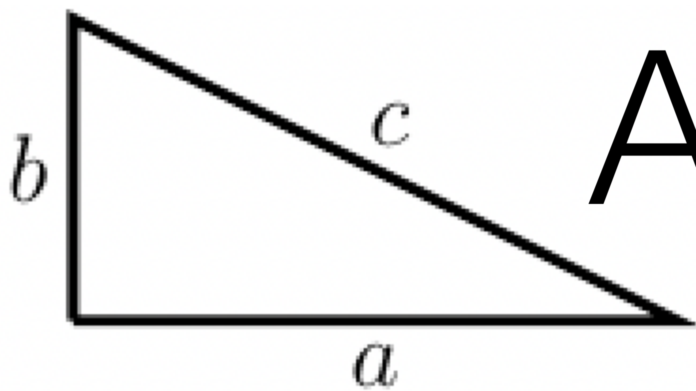

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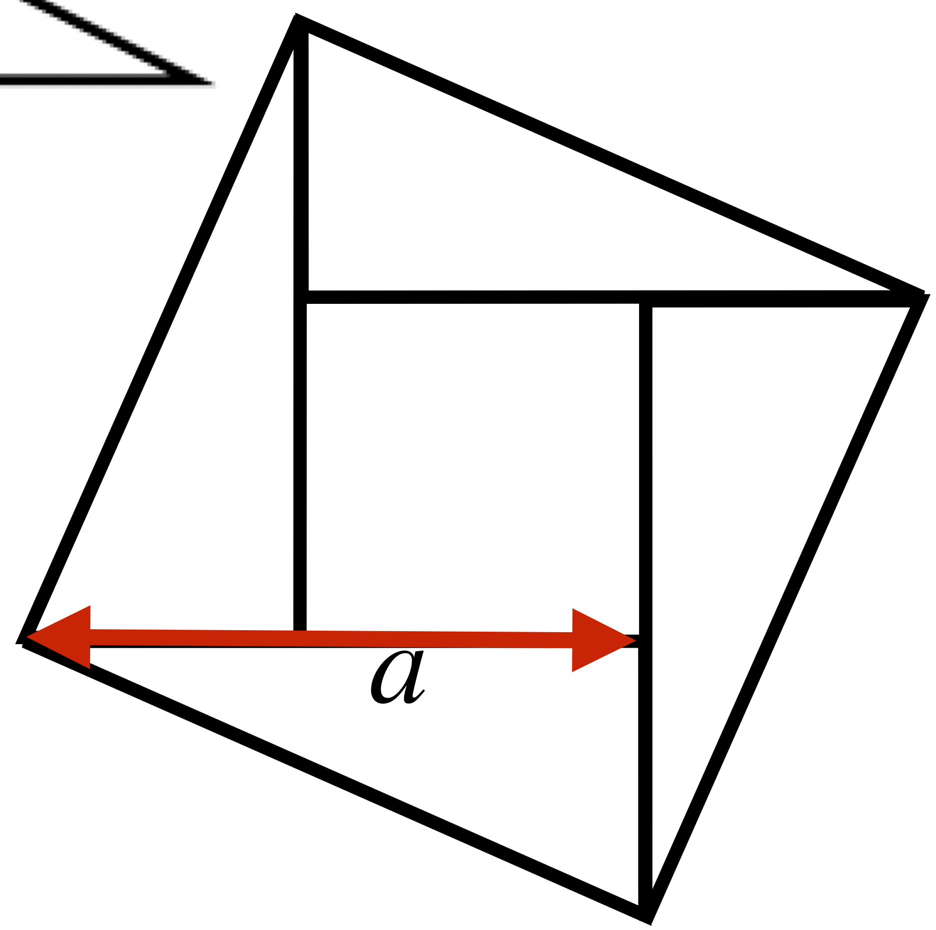

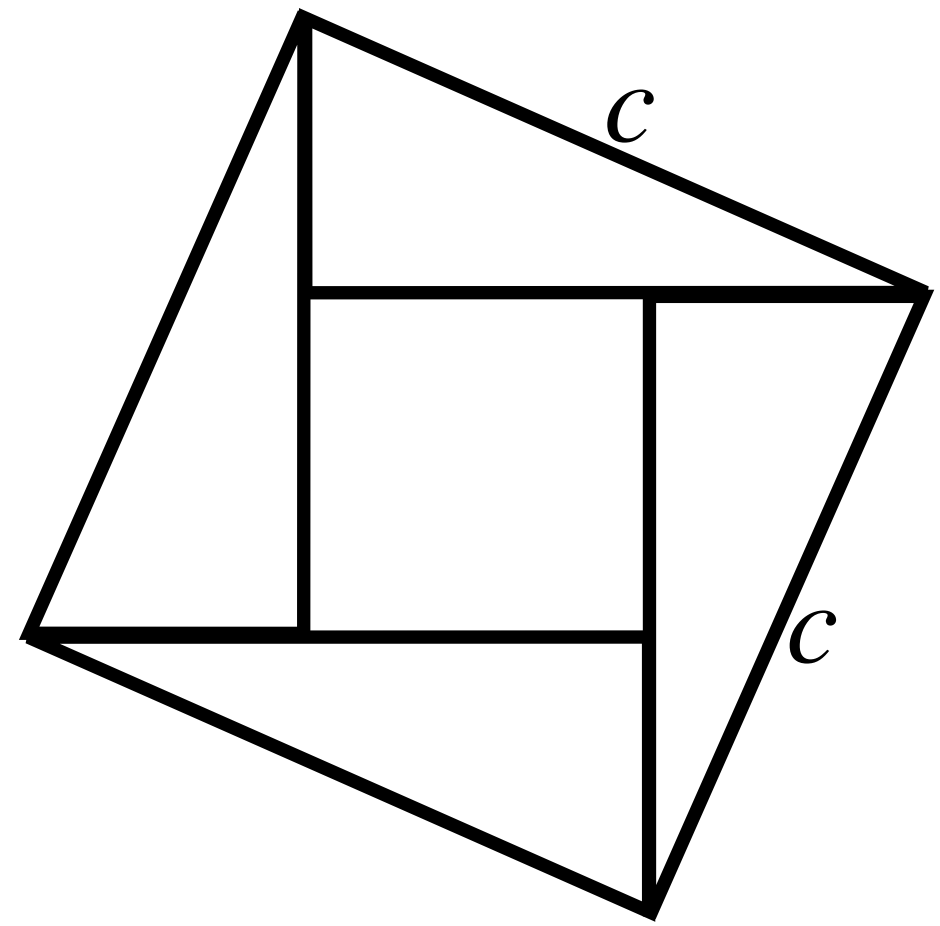

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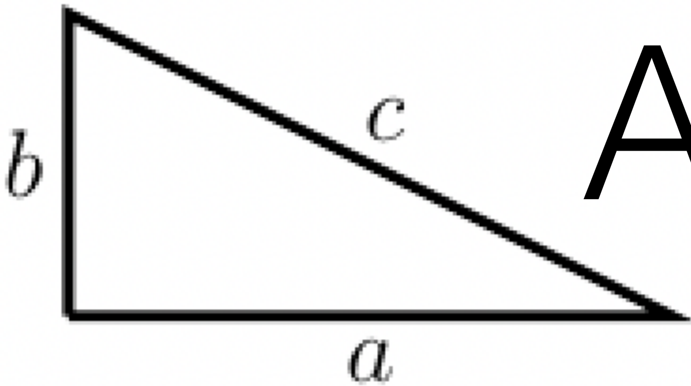

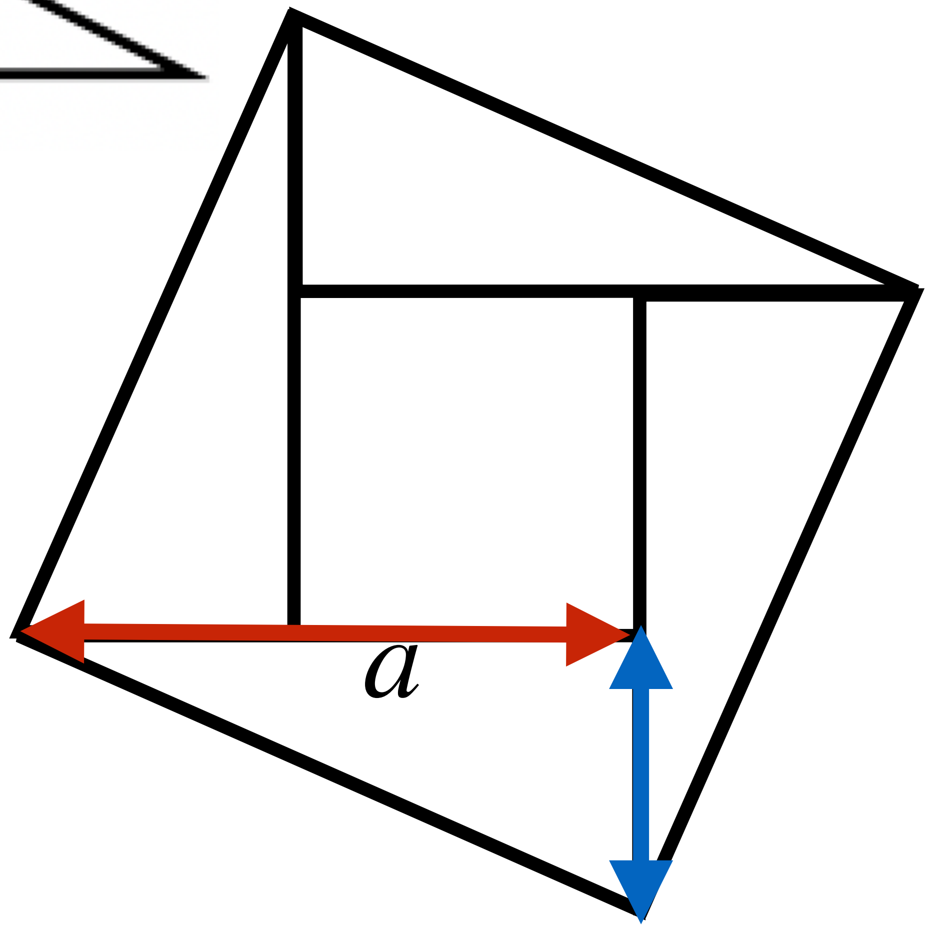

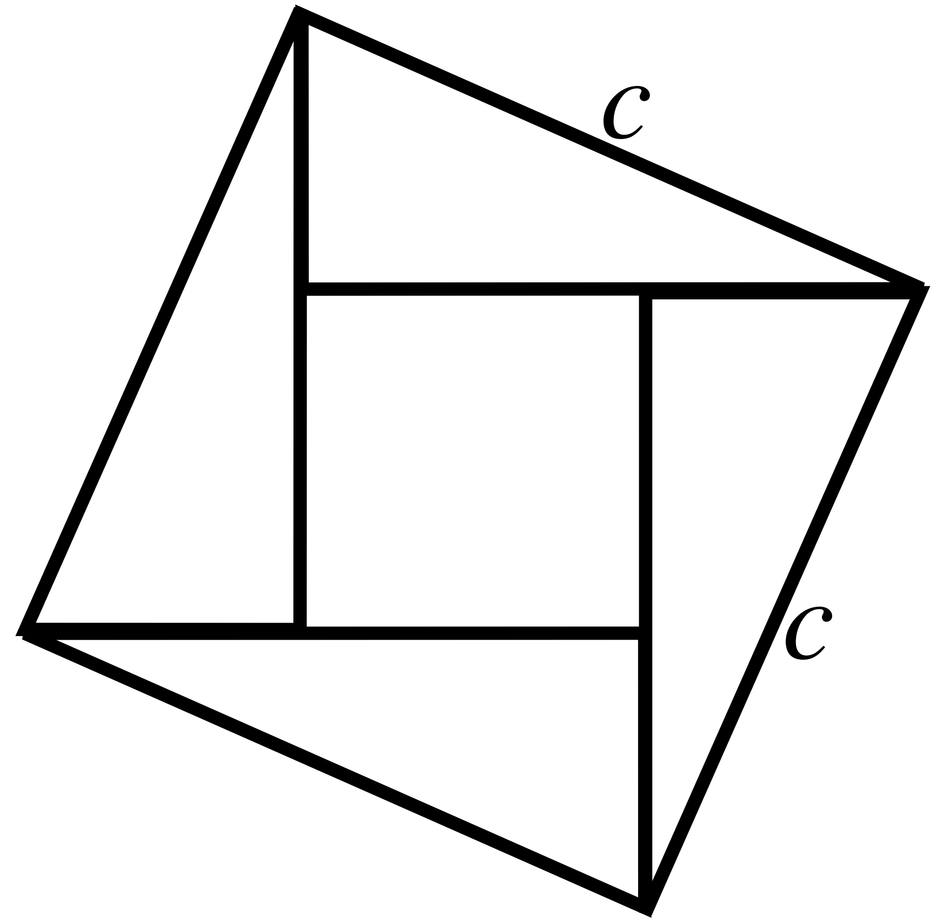

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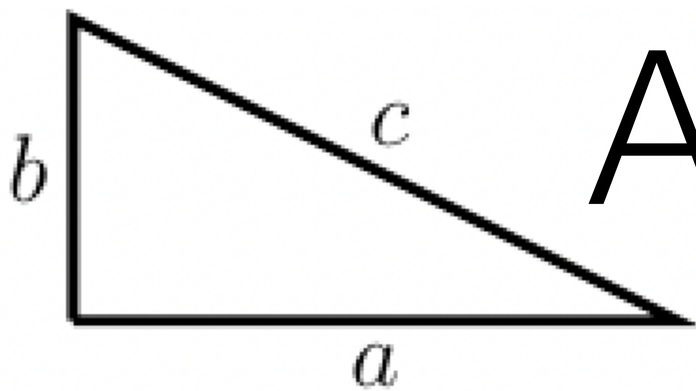

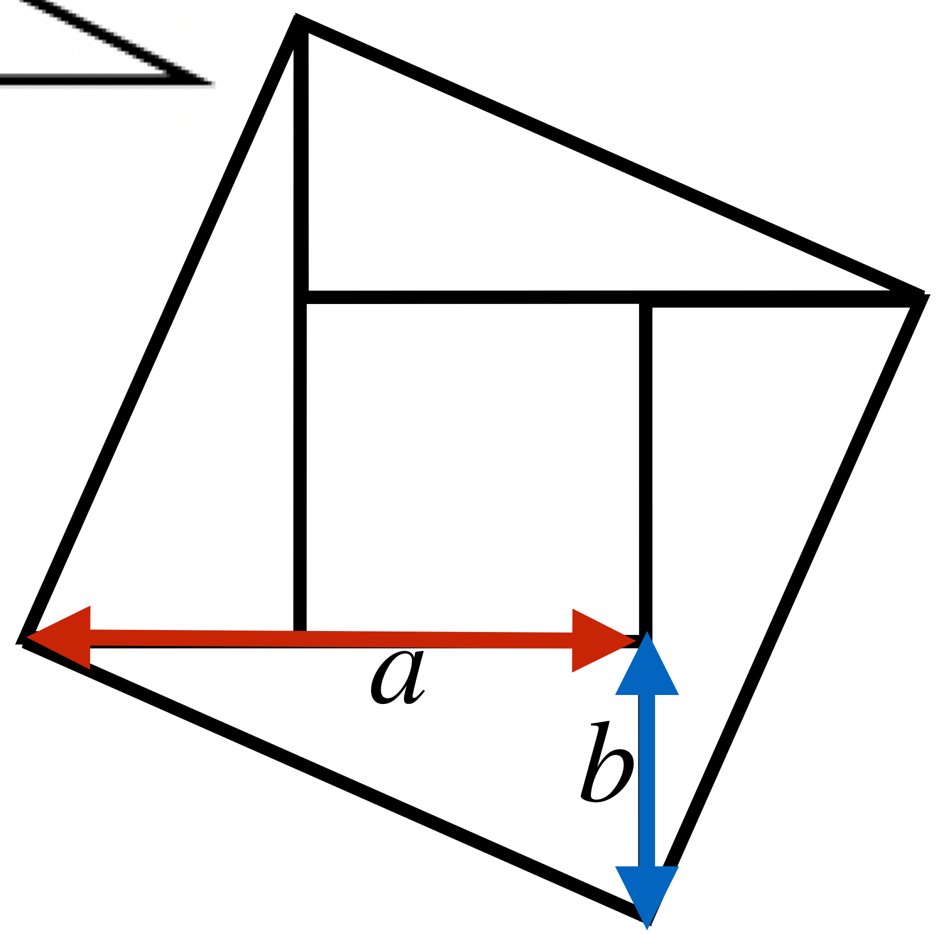

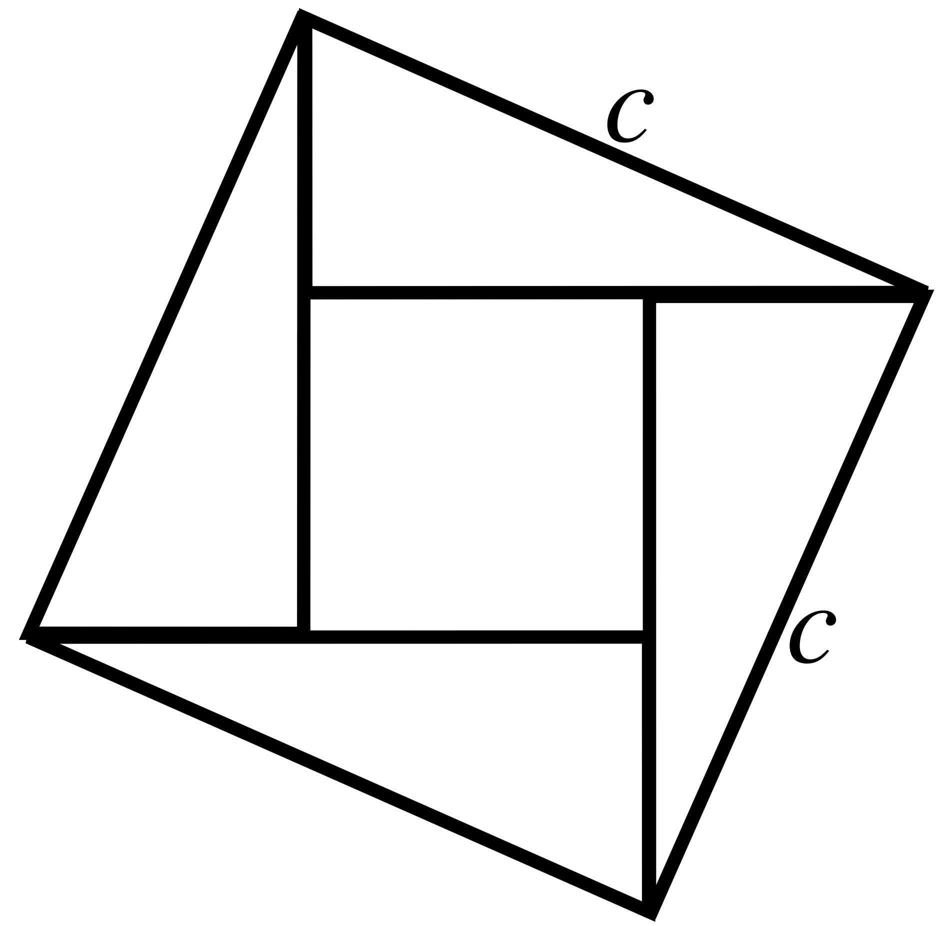

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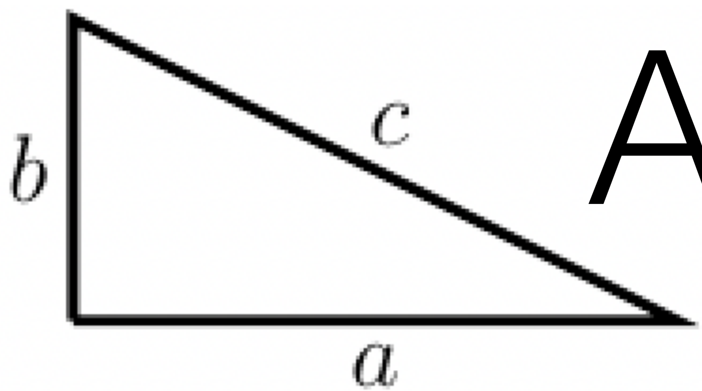

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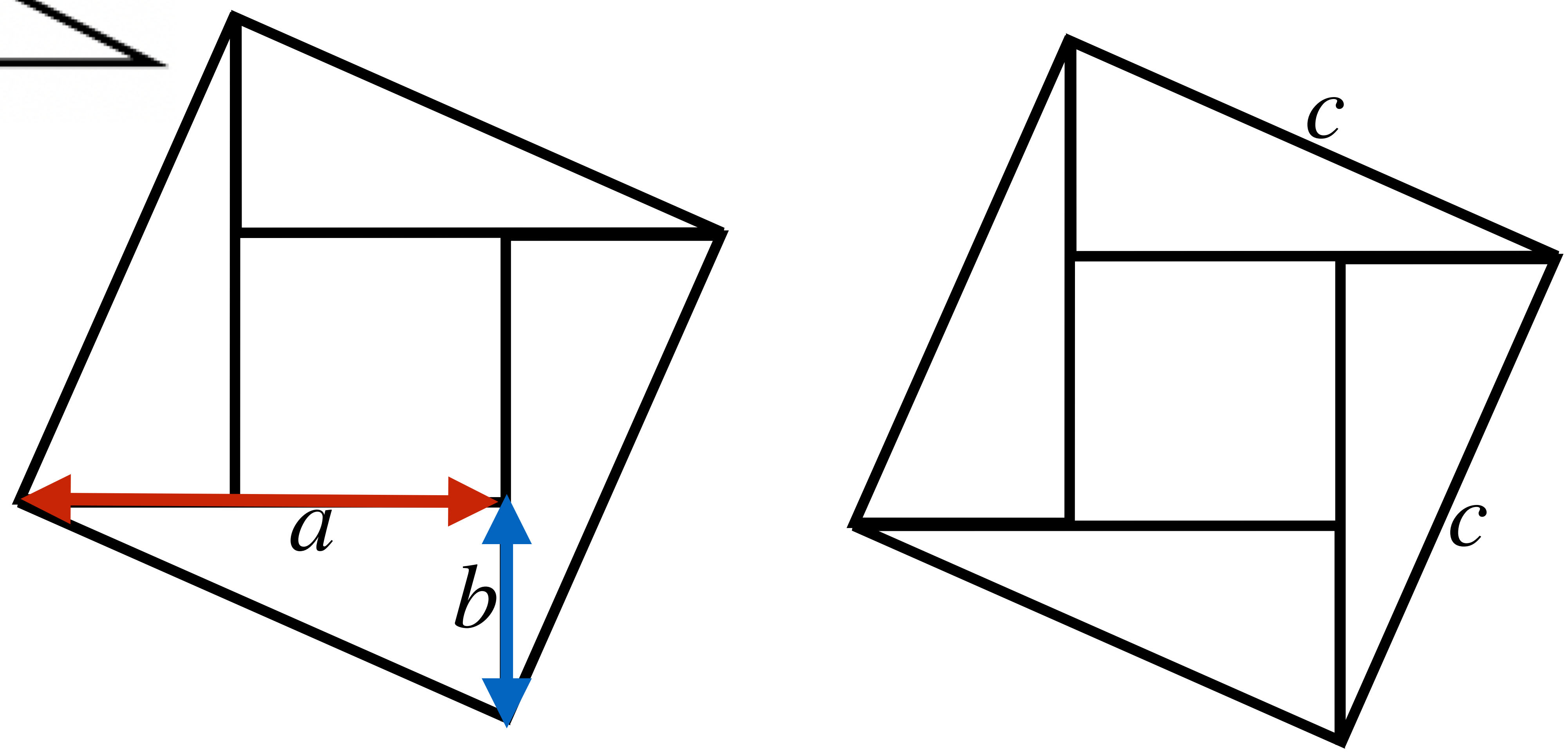
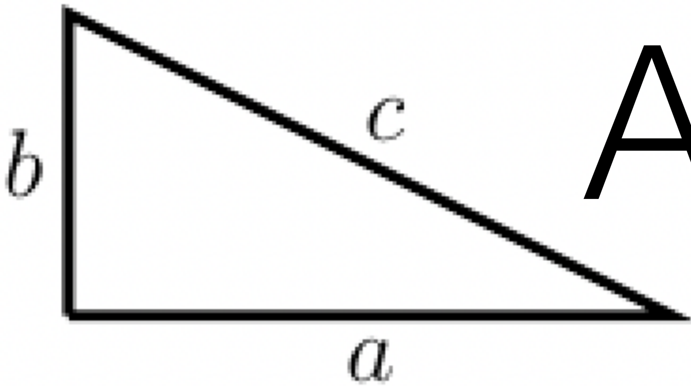


4 triangles + little square = Big square

$$4 \cdot \left(\frac{1}{2} a \cdot b\right) \quad + \qquad = \quad c^2$$



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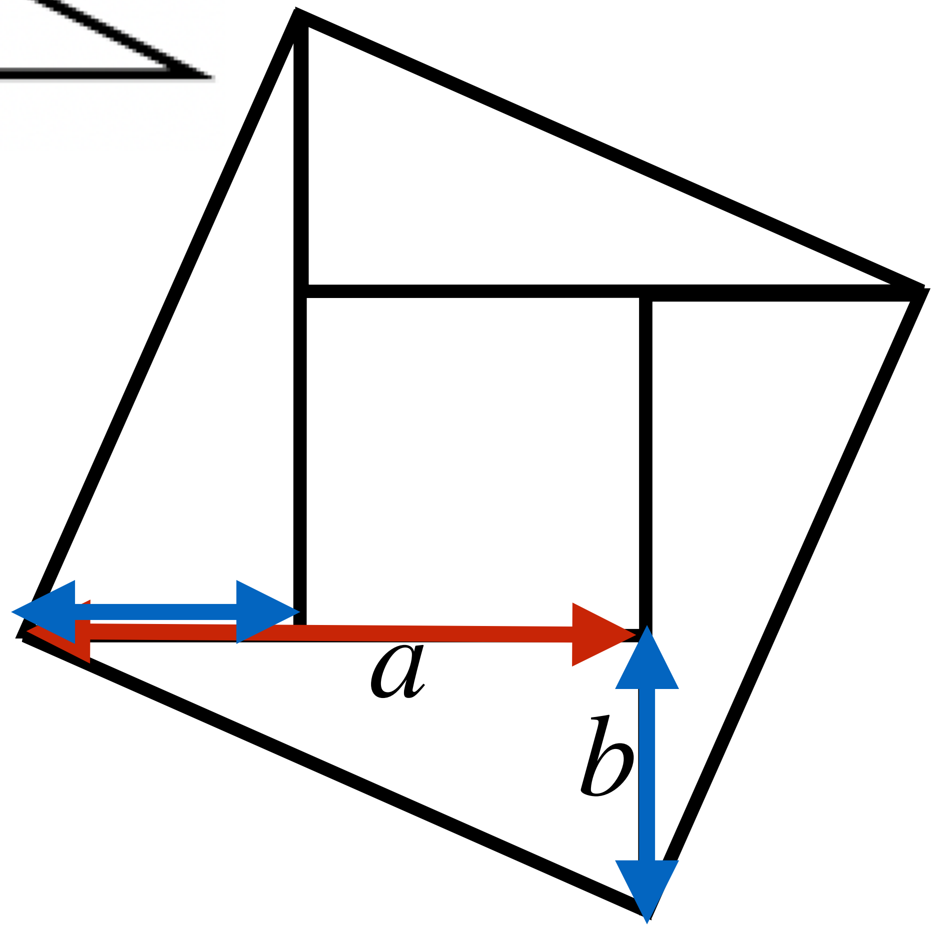

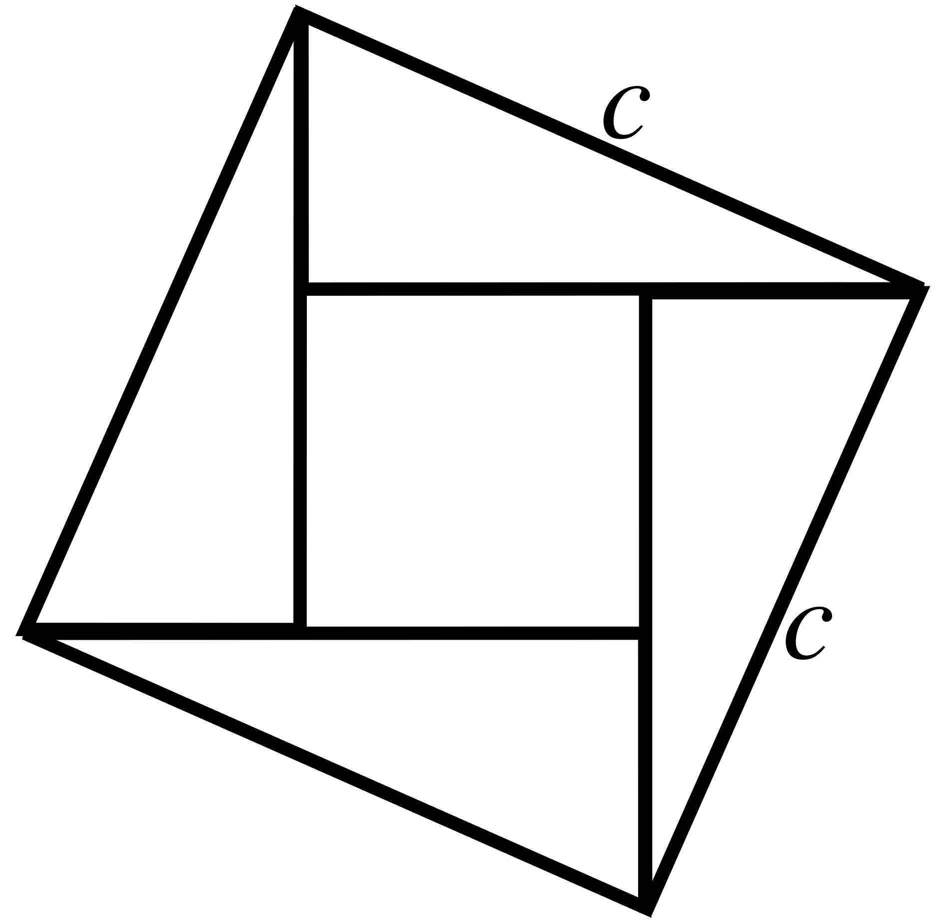

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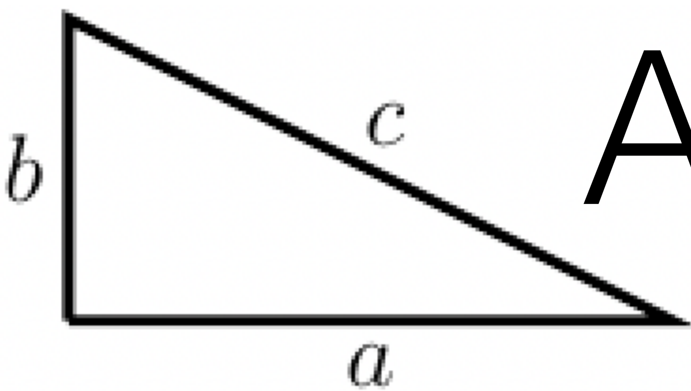

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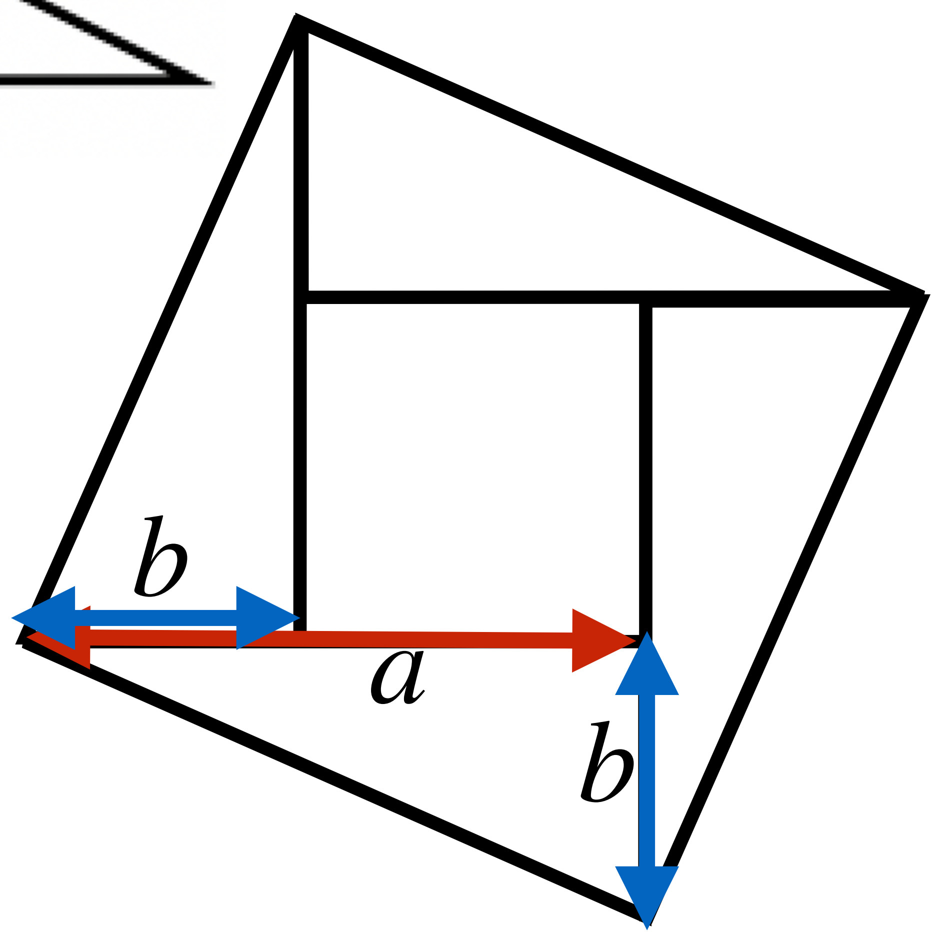

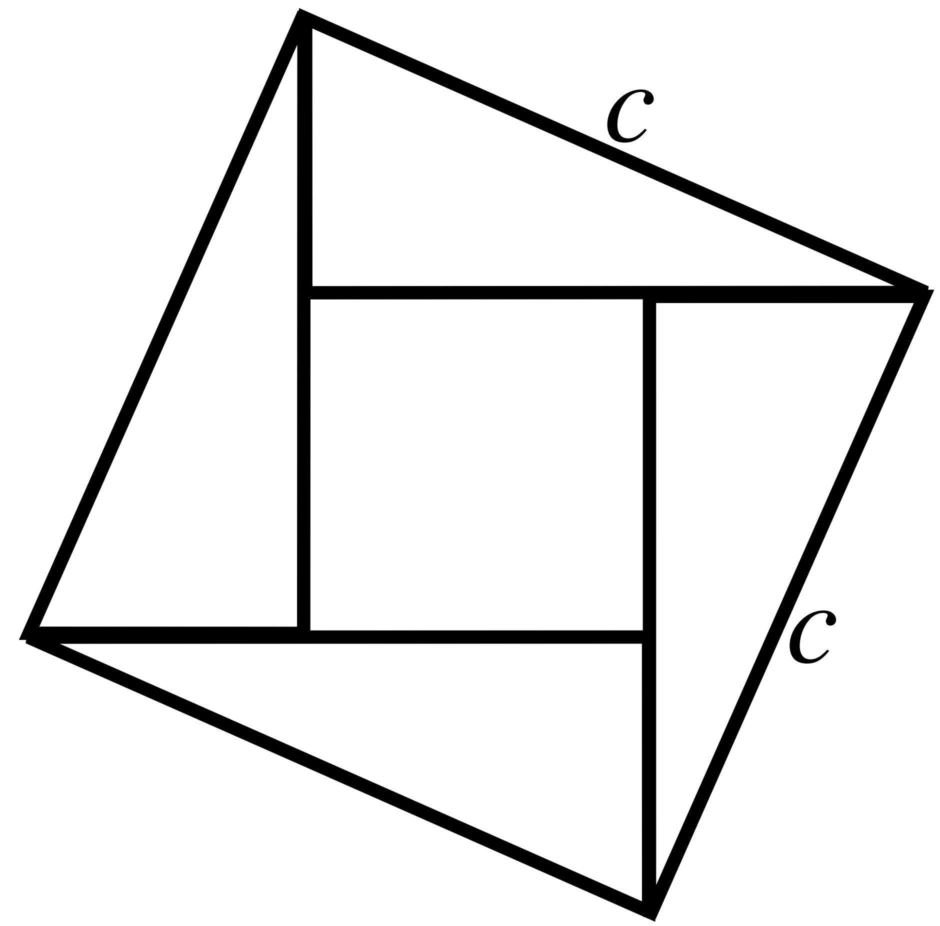

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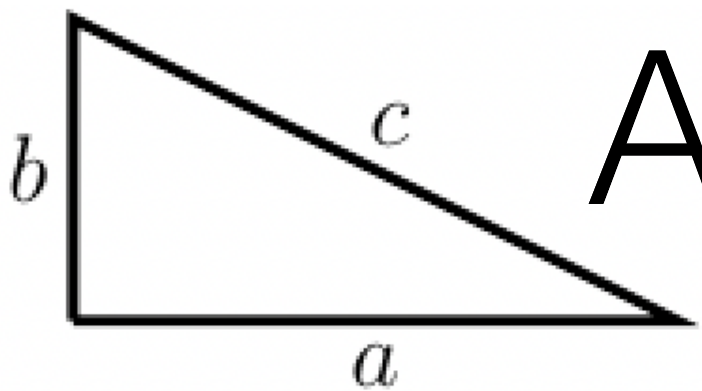

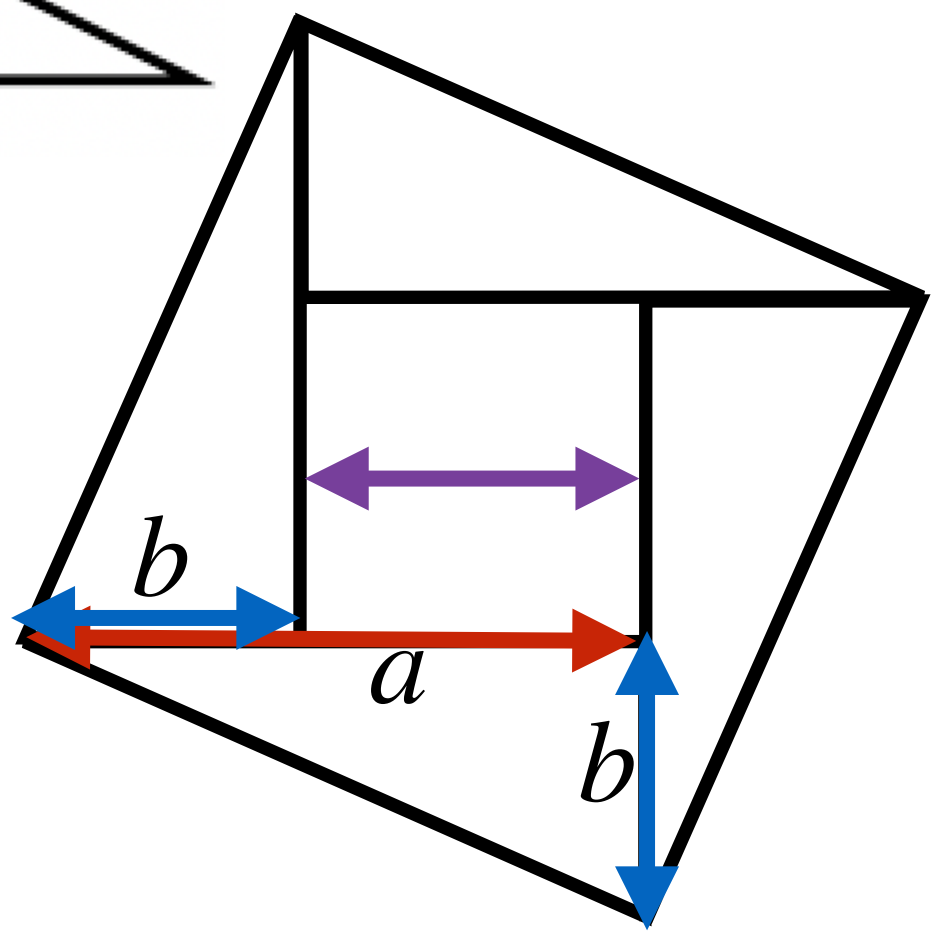

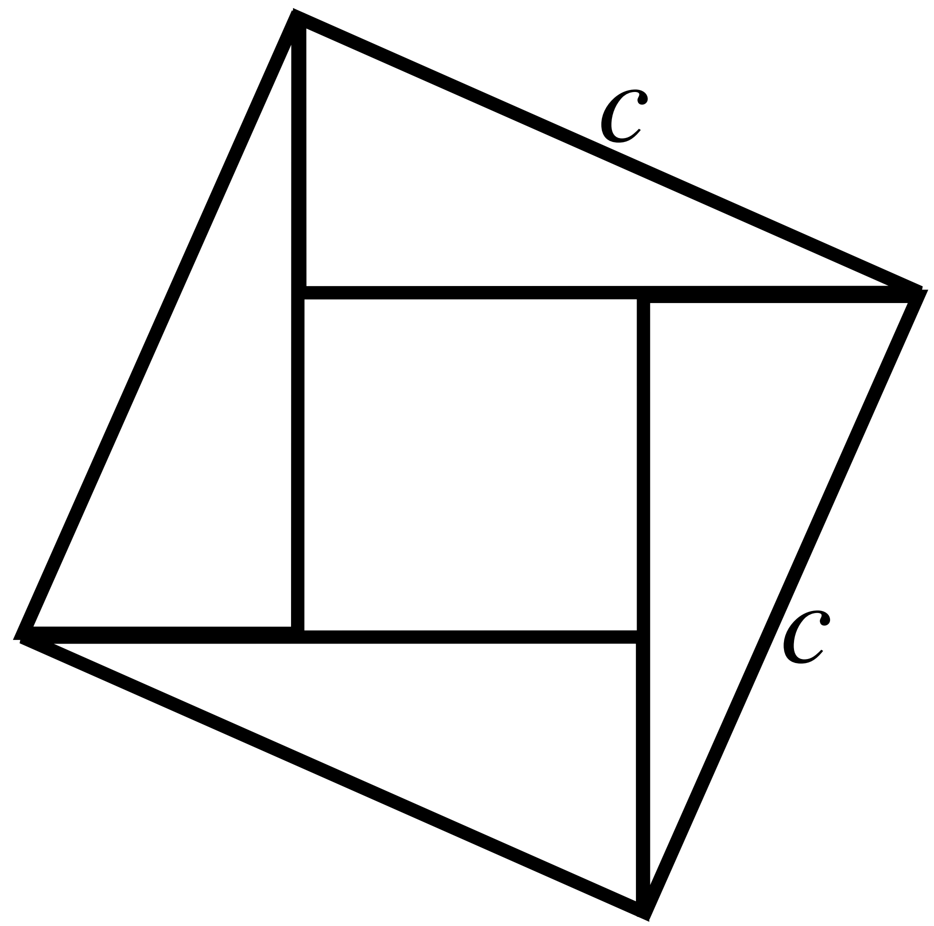

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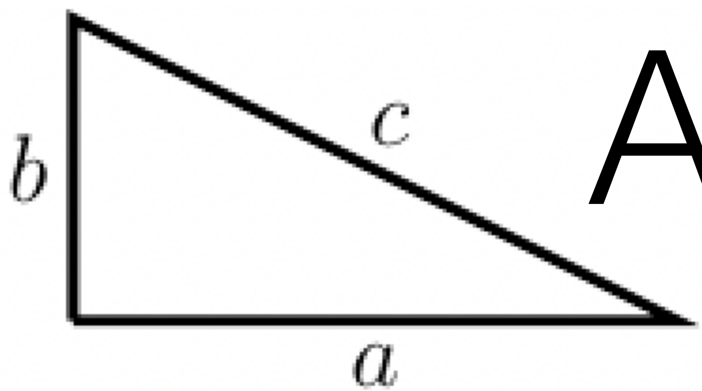

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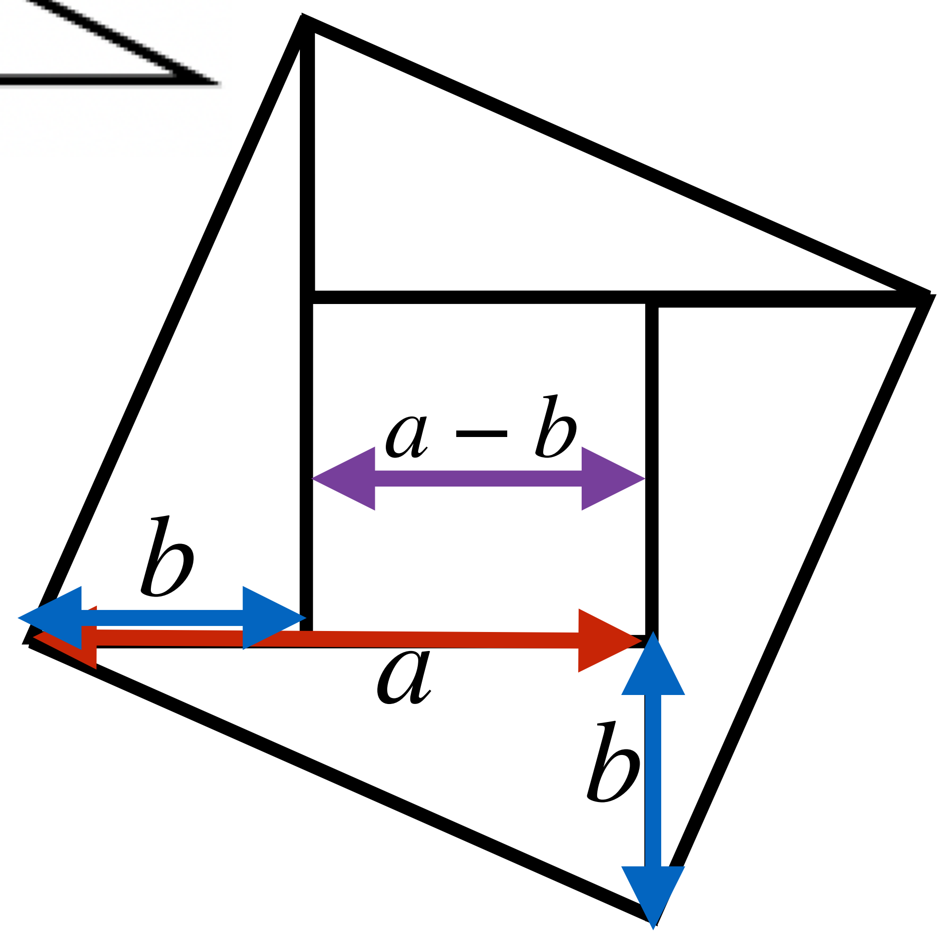
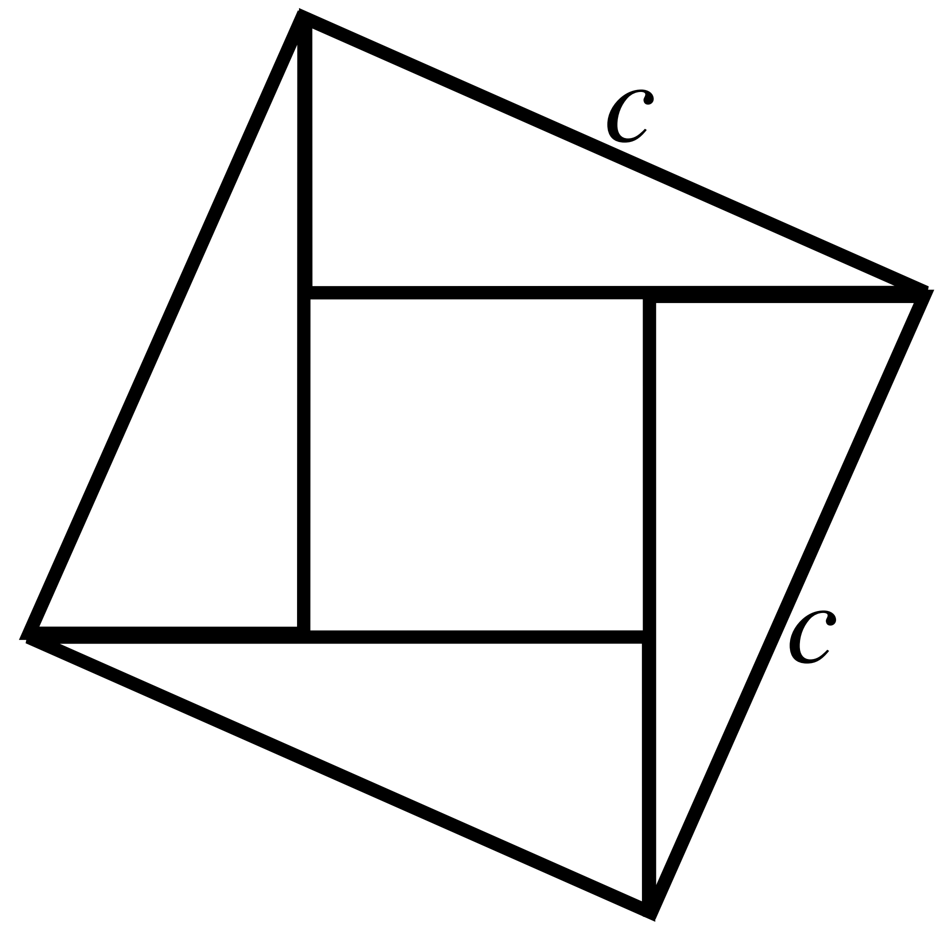
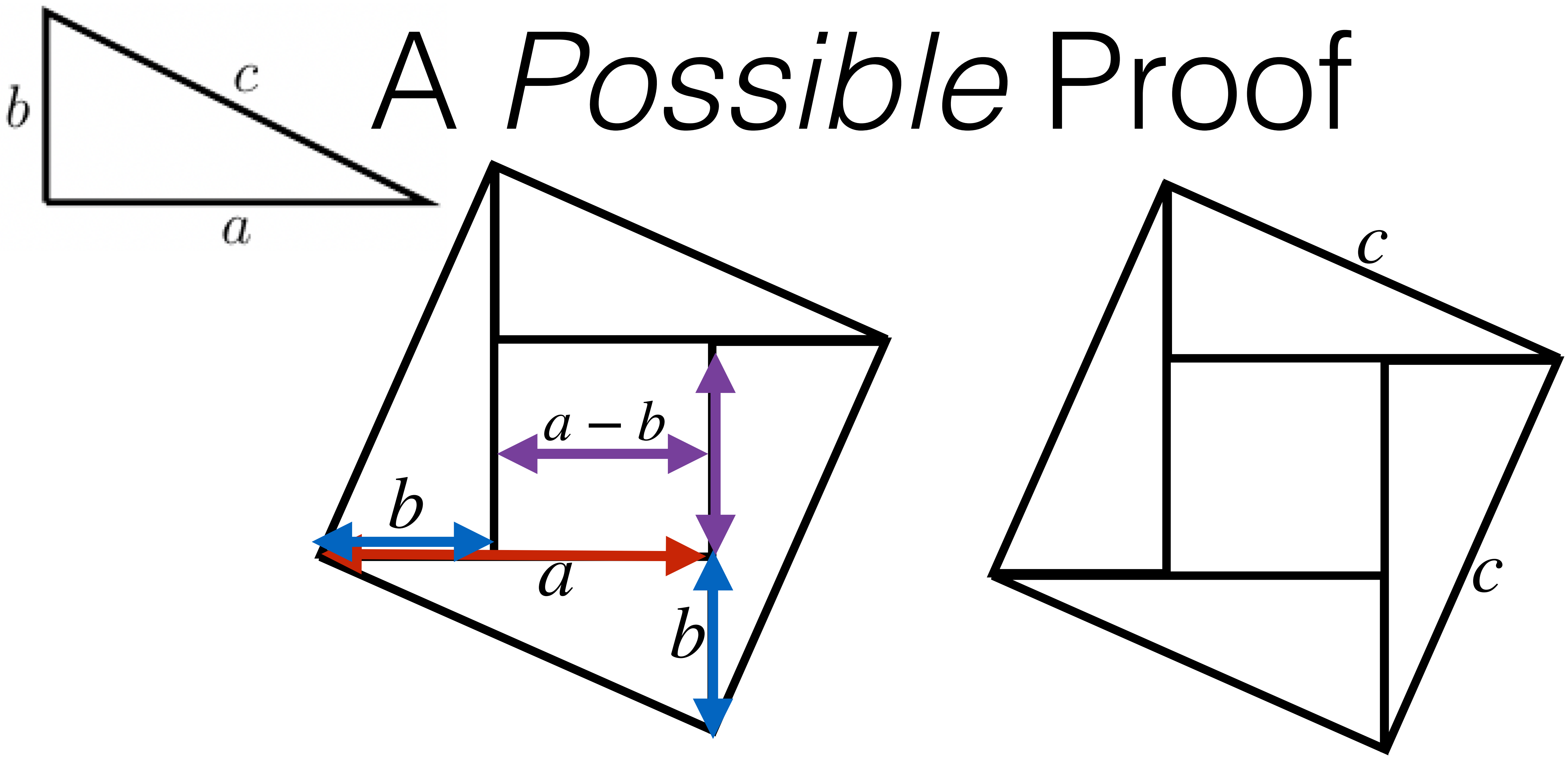




4 triangles + little square = Big square

$$4 \cdot \left(\frac{1}{2} a \cdot b\right) \quad + \qquad = \quad c^2$$

A *Possible* Proof



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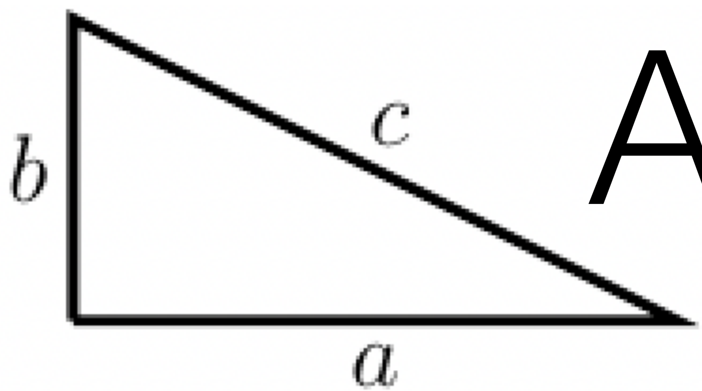

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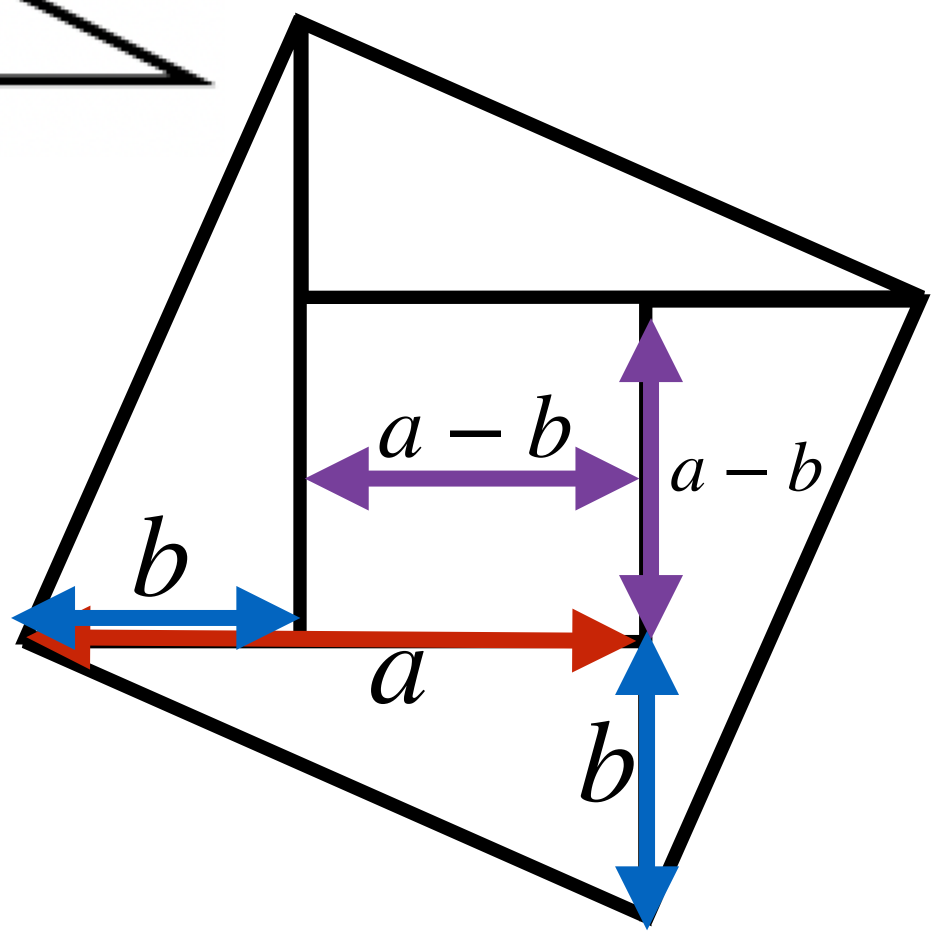

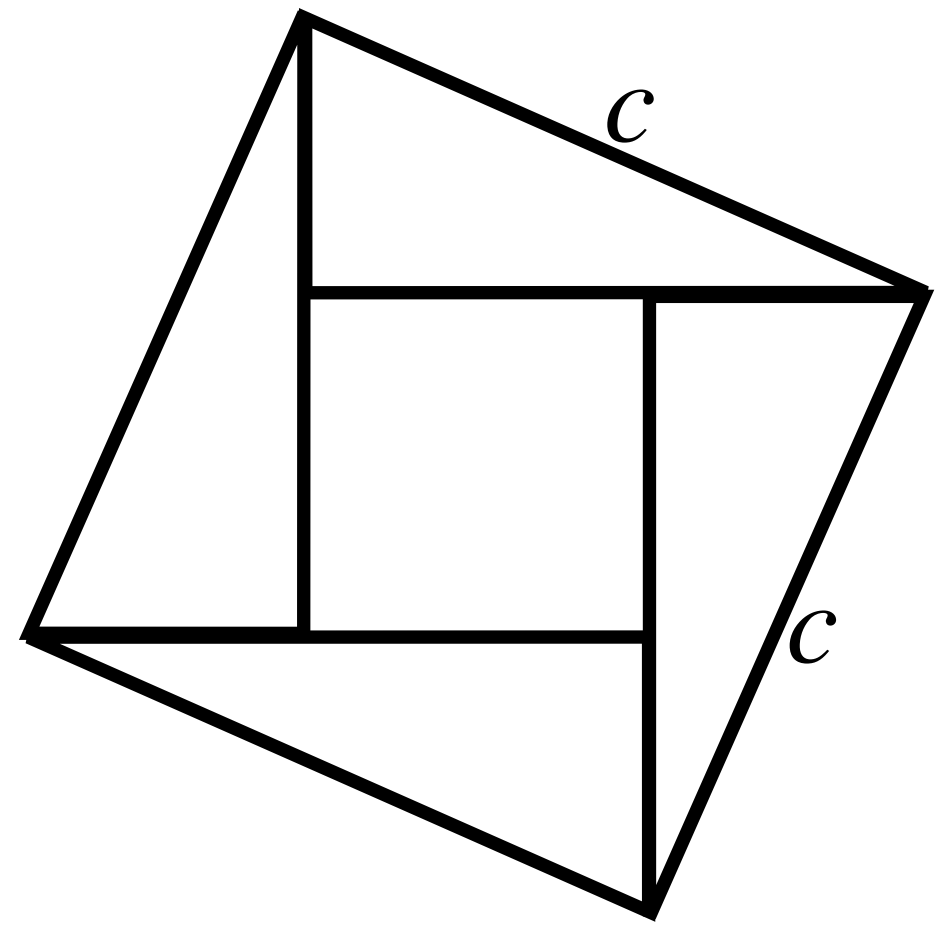

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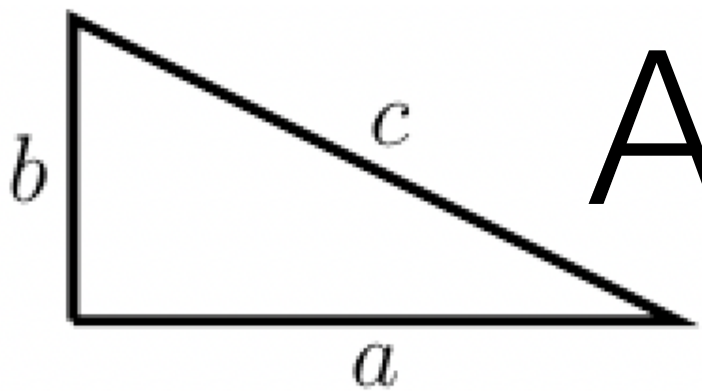
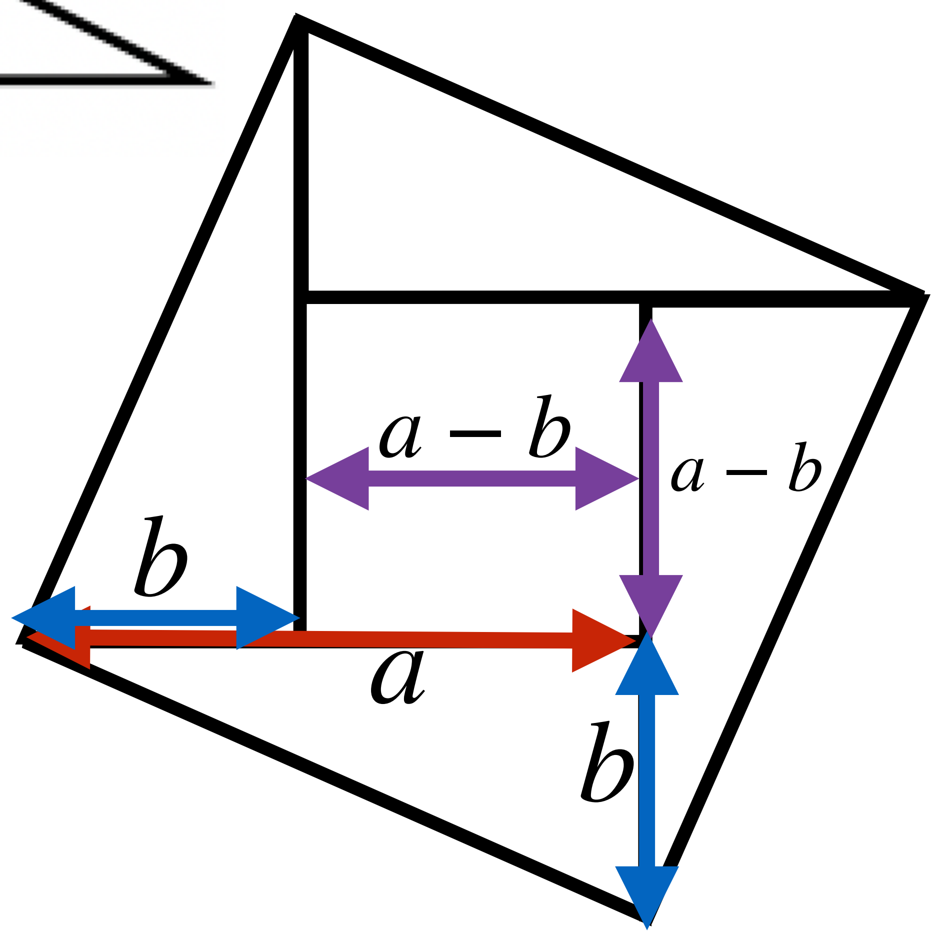
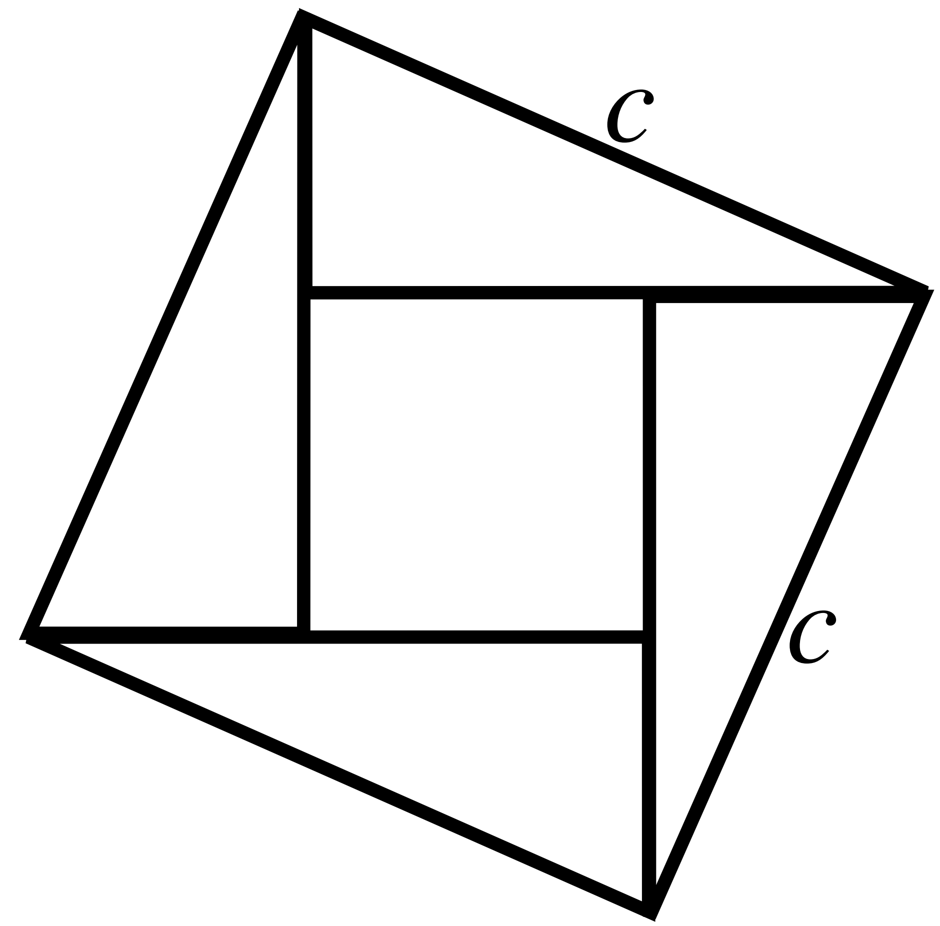


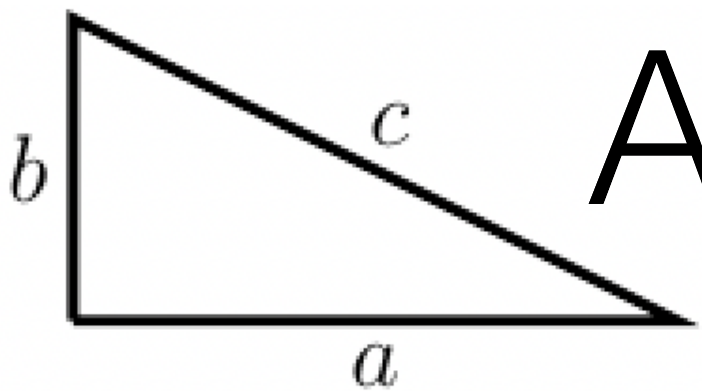


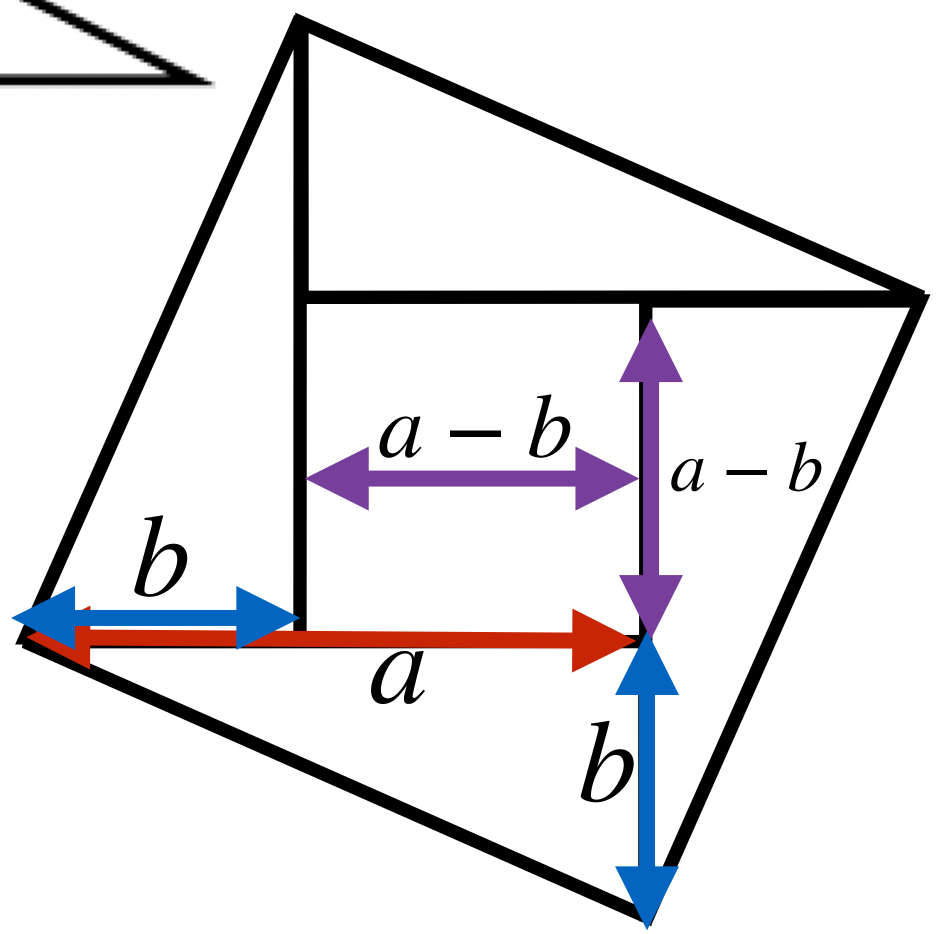


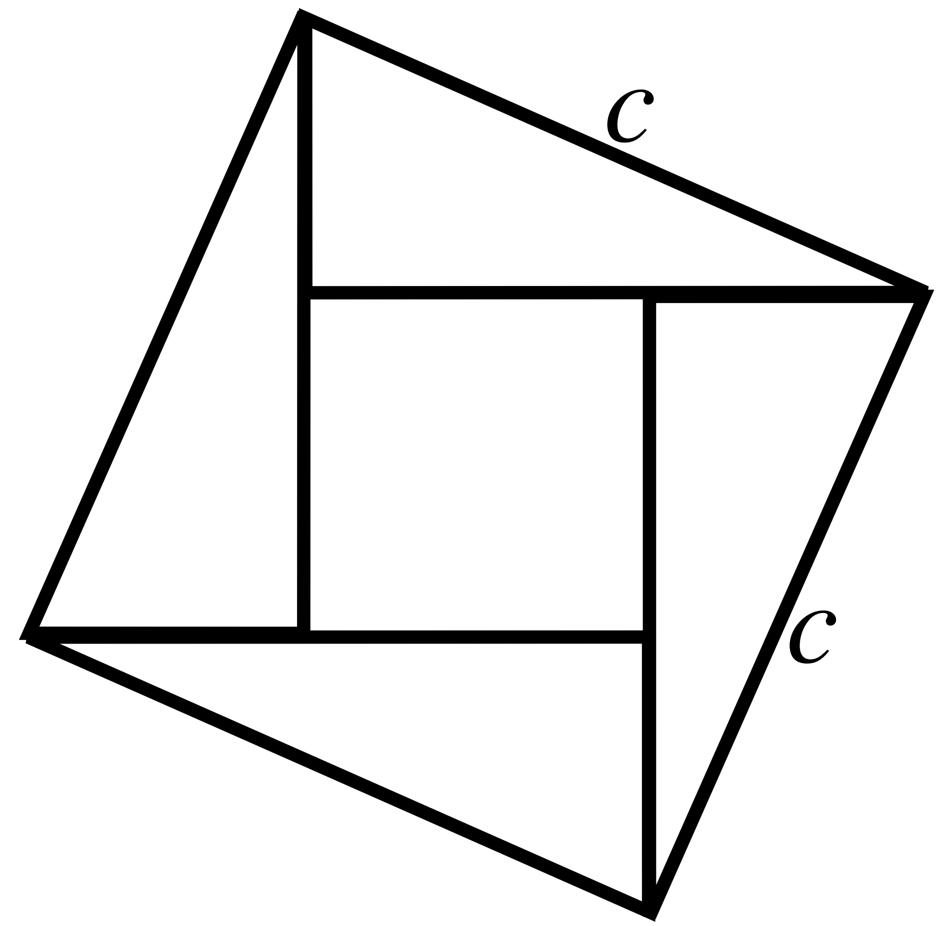
4 triangles + little square = Big square

$$4 \cdot \left(\frac{1}{2} a \cdot b\right) \quad + \quad (a-b)^2 \quad = \quad c^2$$

A *Possible* Proof





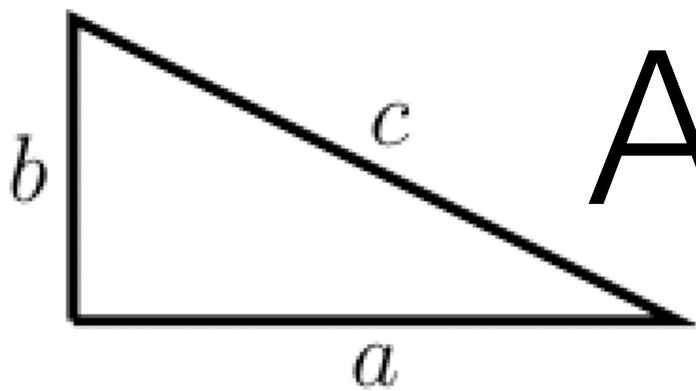


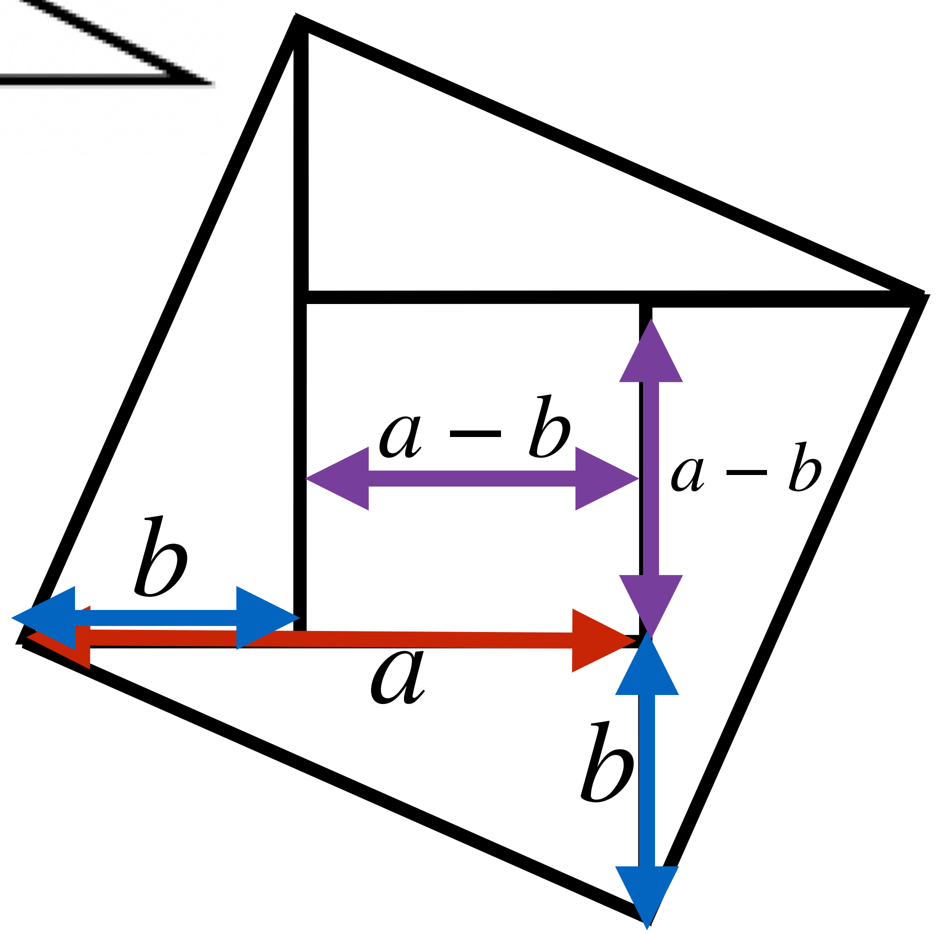
4 triangles + little square = Big square

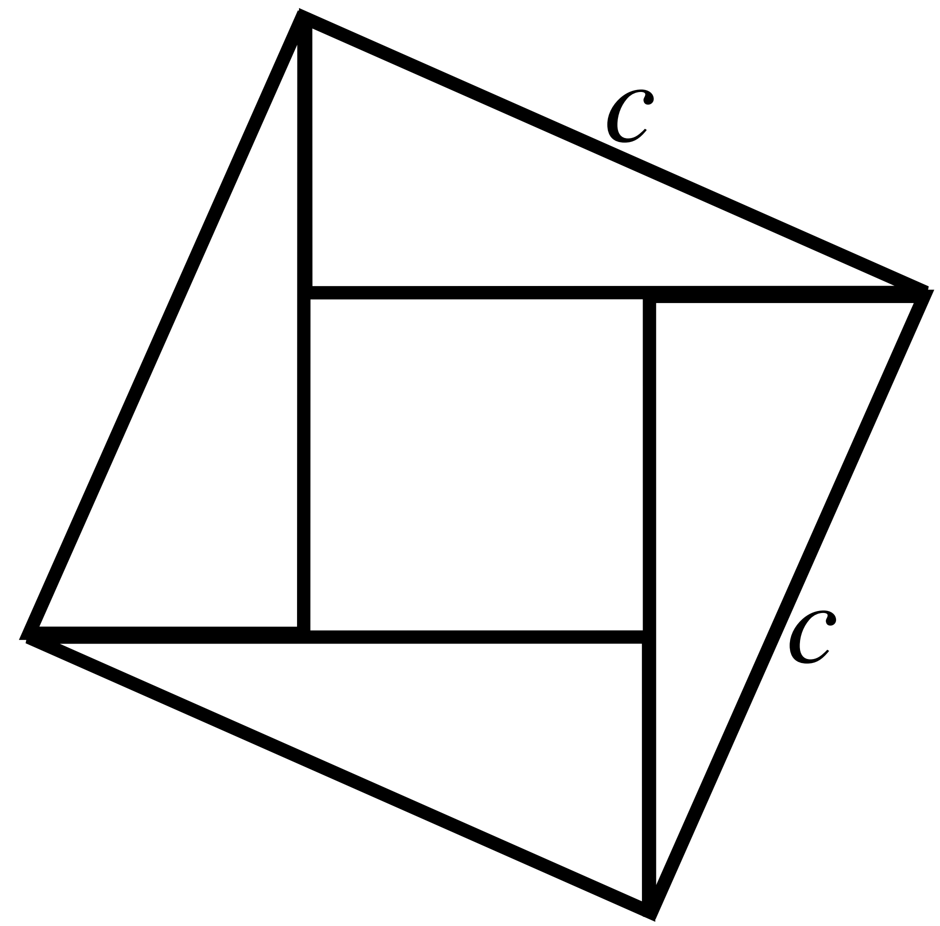
$$4 \cdot \left(\frac{1}{2} a \cdot b\right) \quad + \quad (a-b)^2 \quad = \quad c^2$$

$$2ab + a^2 + b^2 - 2ab \quad = \quad c^2$$

A *Possible* Proof



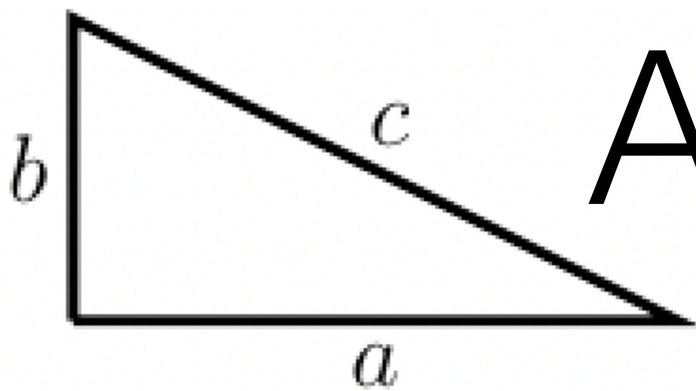




4 triangles + little square = Big square

$$4 \cdot \left(\frac{1}{2}a \cdot b\right) \quad + \quad (a-b)^2 \quad = \quad c^2$$

$$\cancel{2ab} + a^2 + b^2 - \cancel{2ab} \quad = \quad c^2$$



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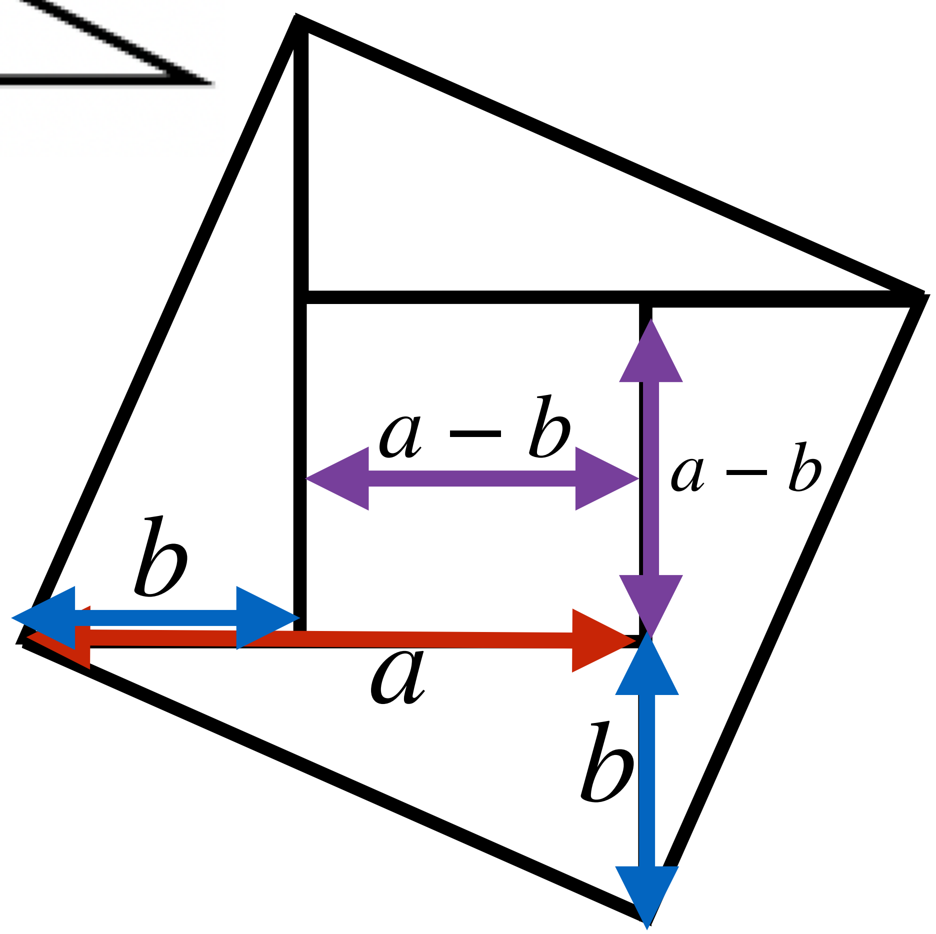

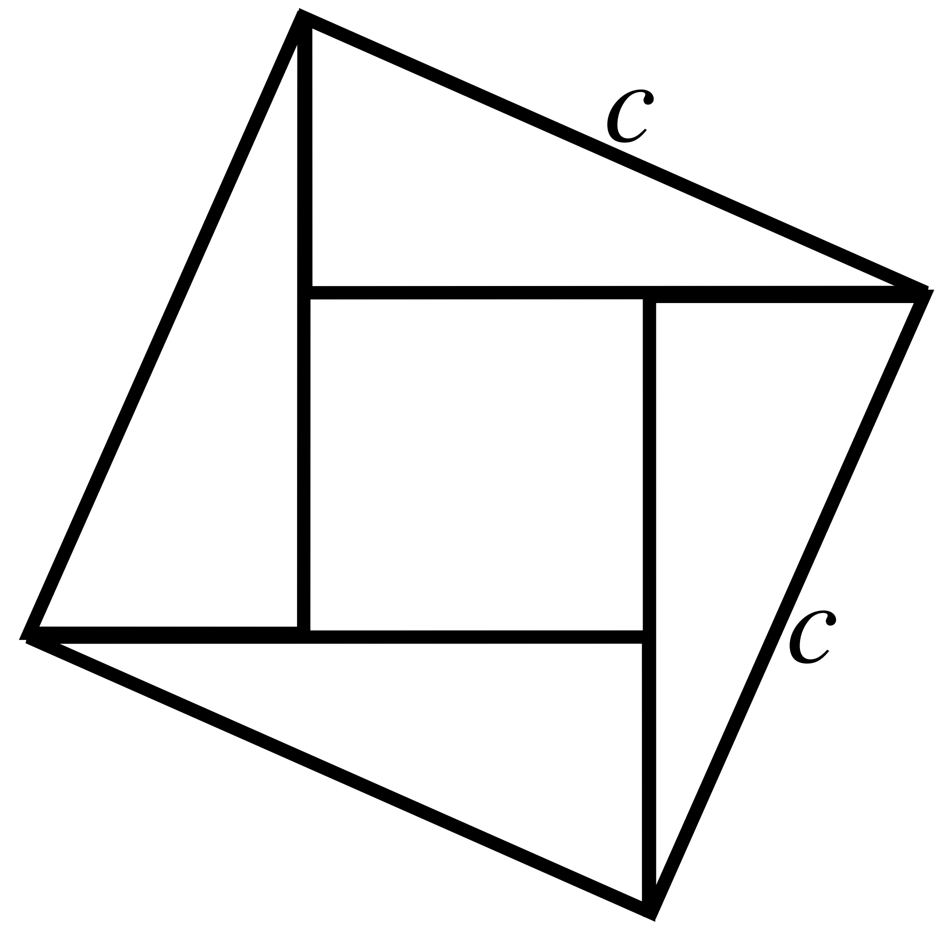

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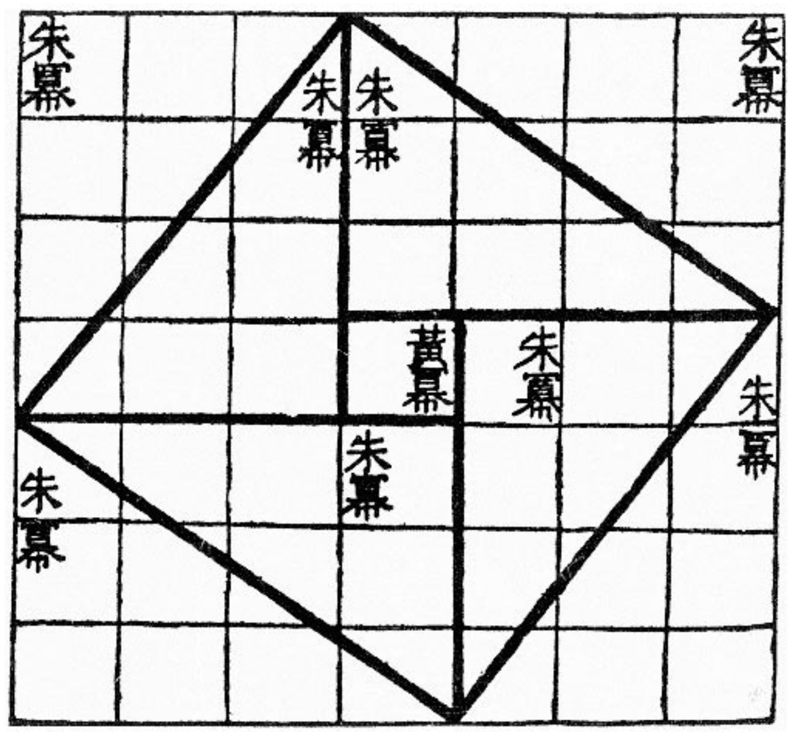
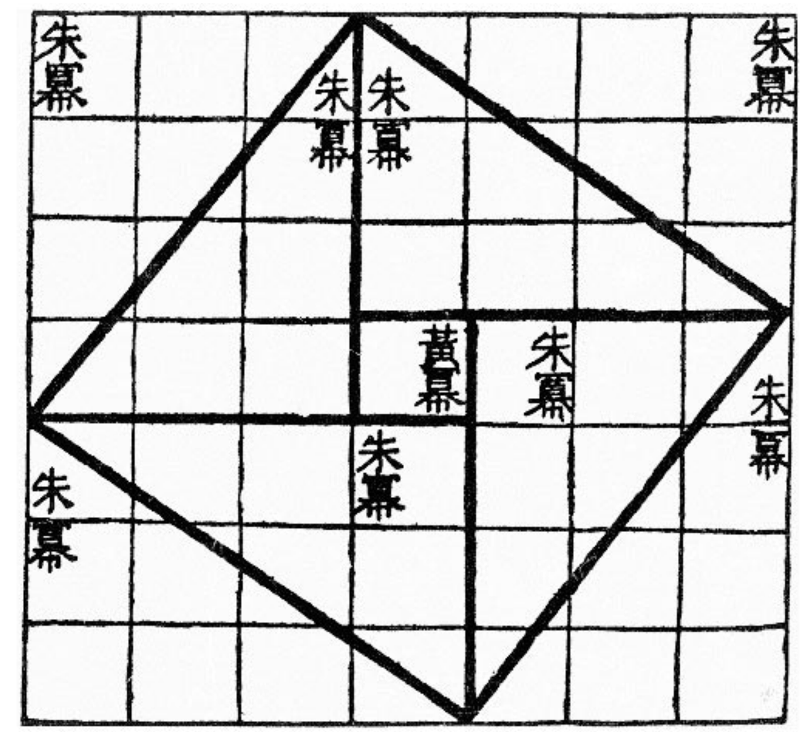
$$\cancel{2ab} + a^2 + b^2 - \cancel{2ab} \quad = \quad c^2 \qquad \boxed{a^2 + b^2 = c^2}$$

Second *Possible* Proof





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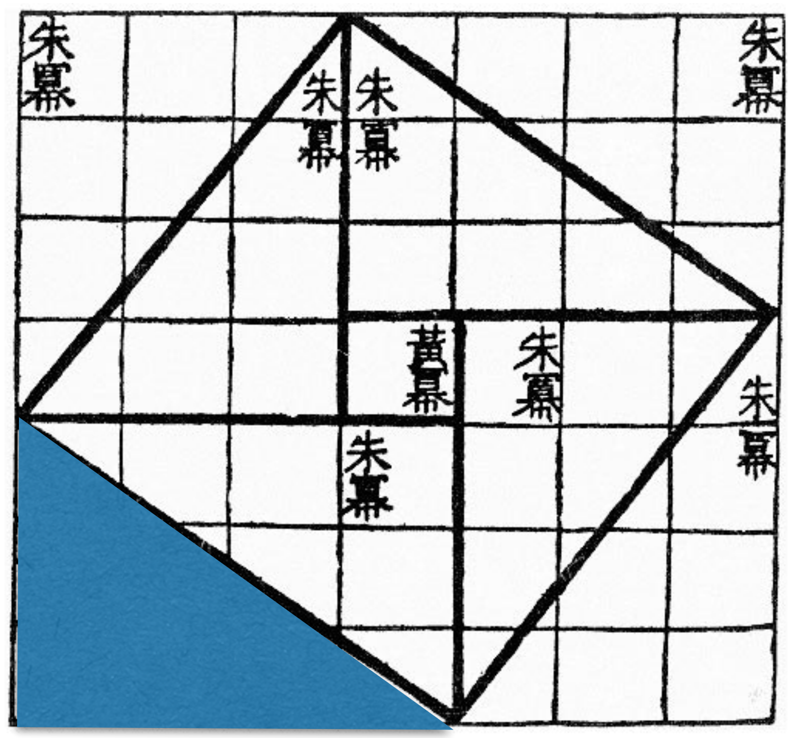

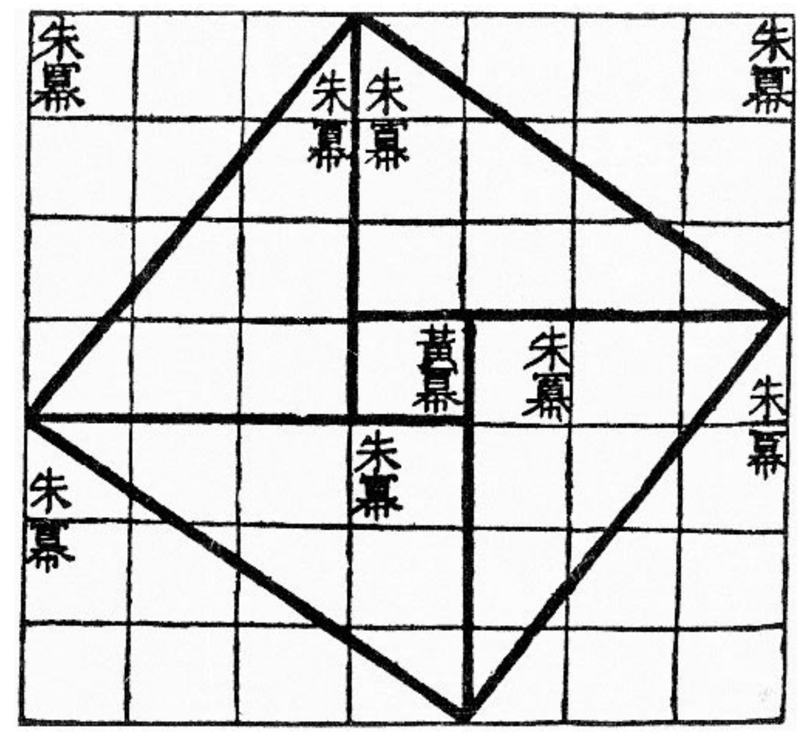

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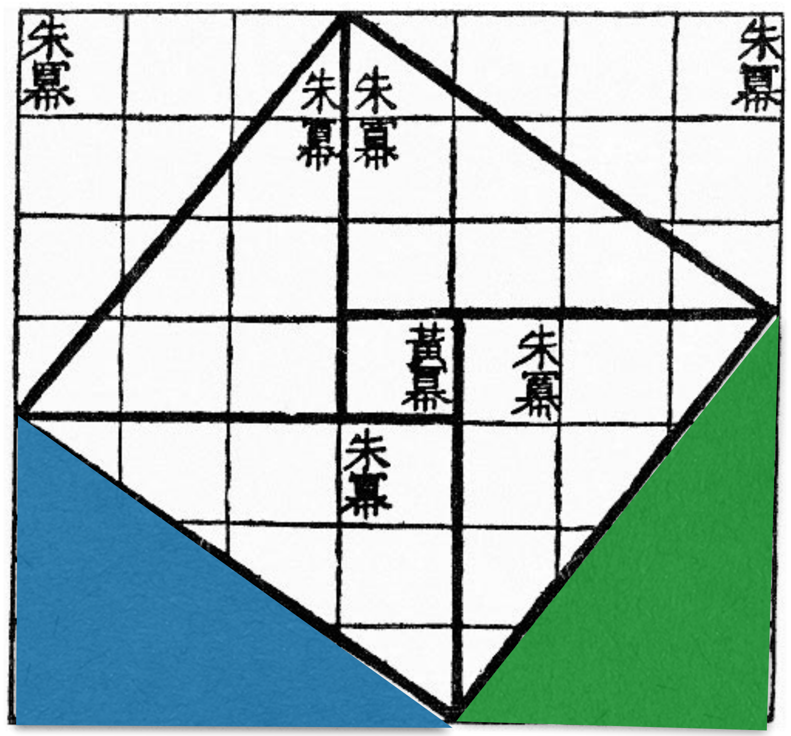

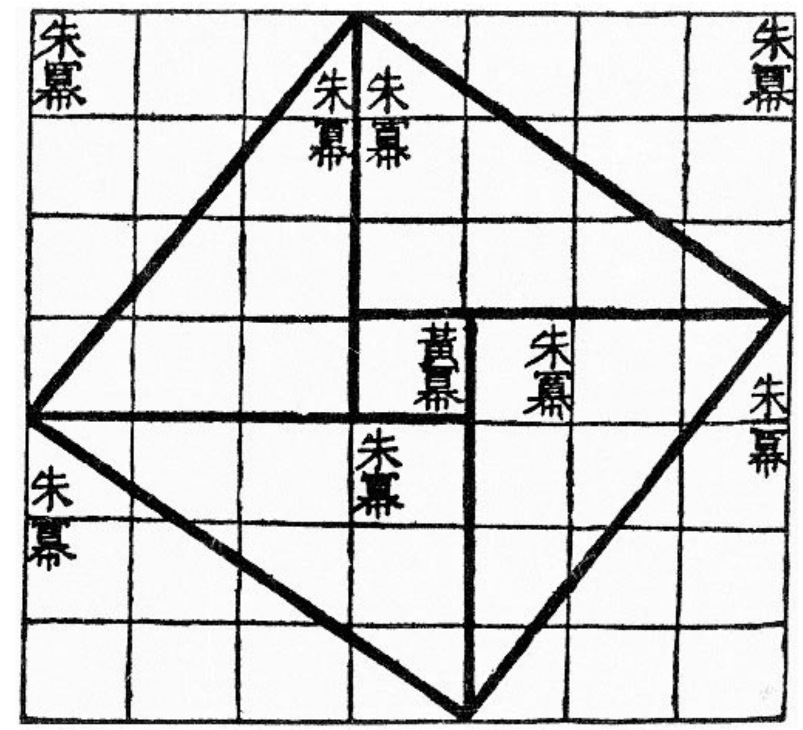

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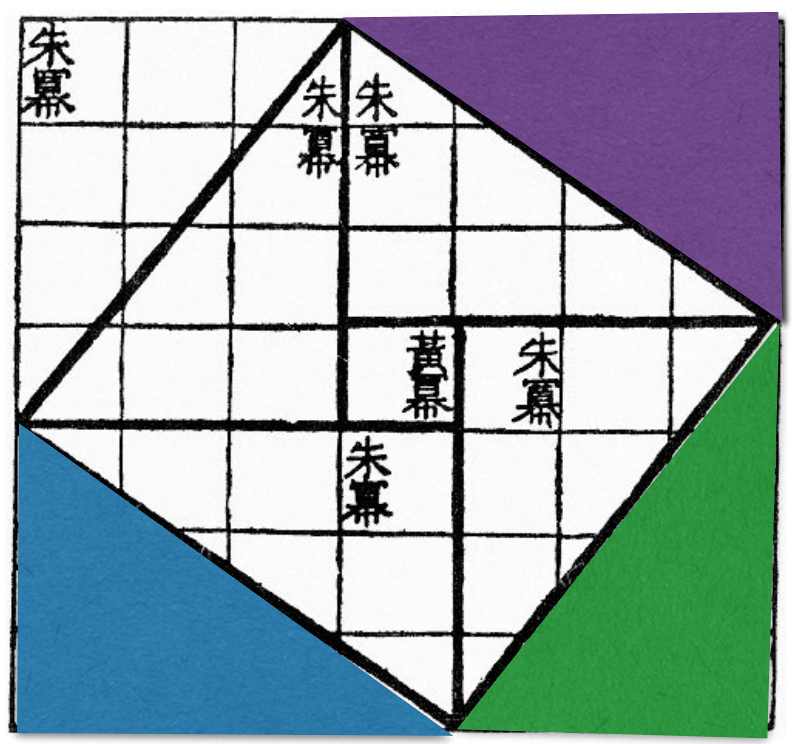

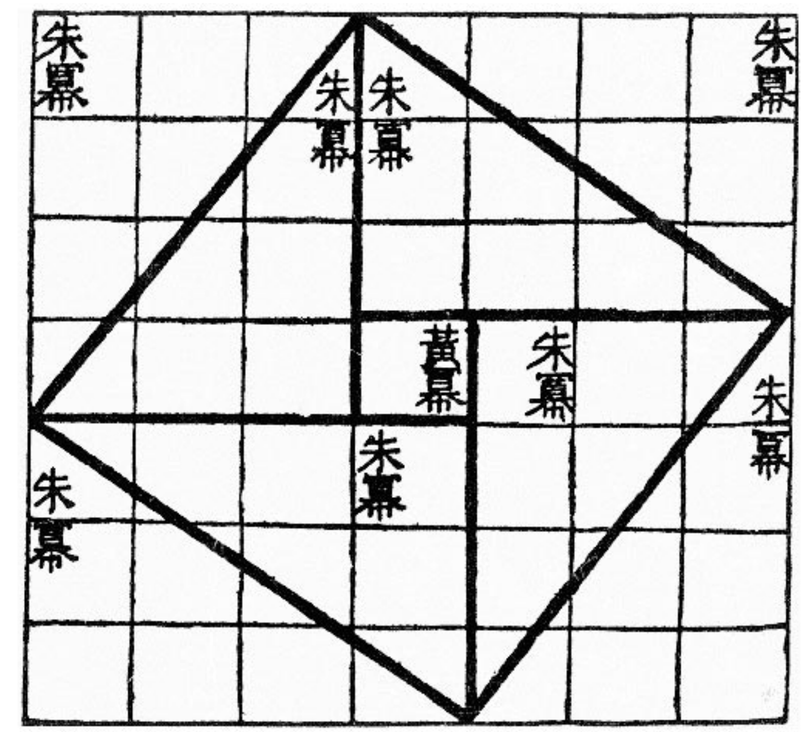

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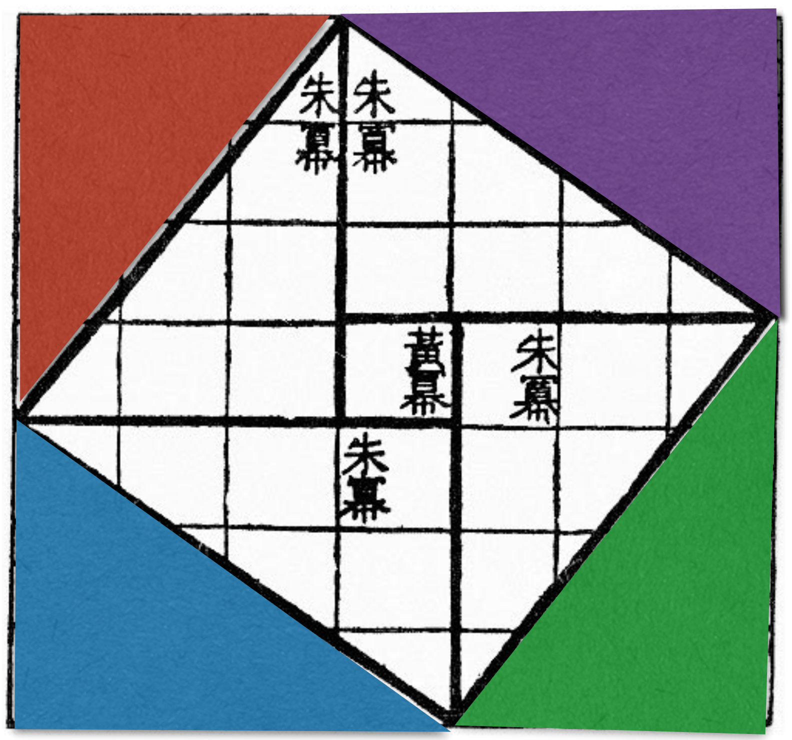

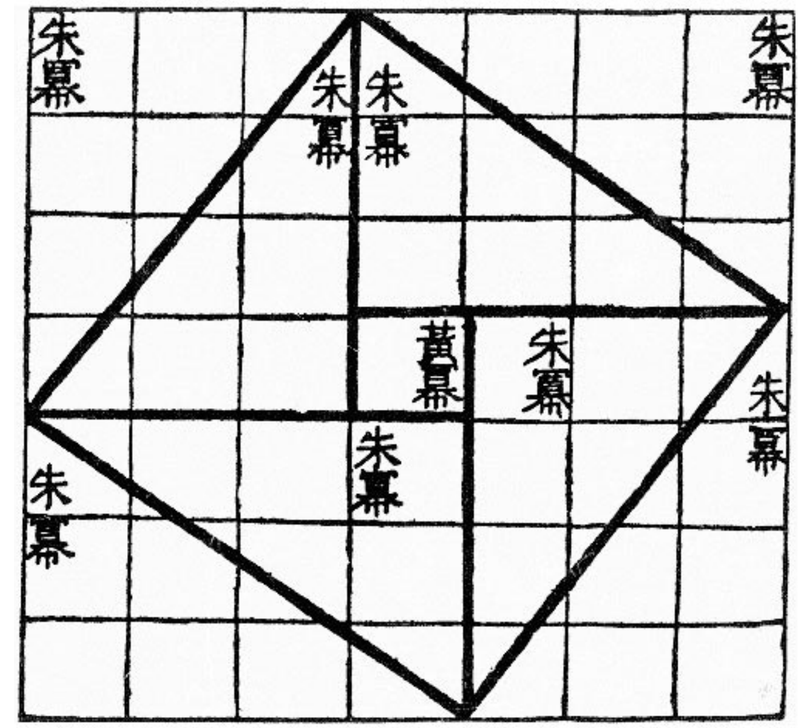

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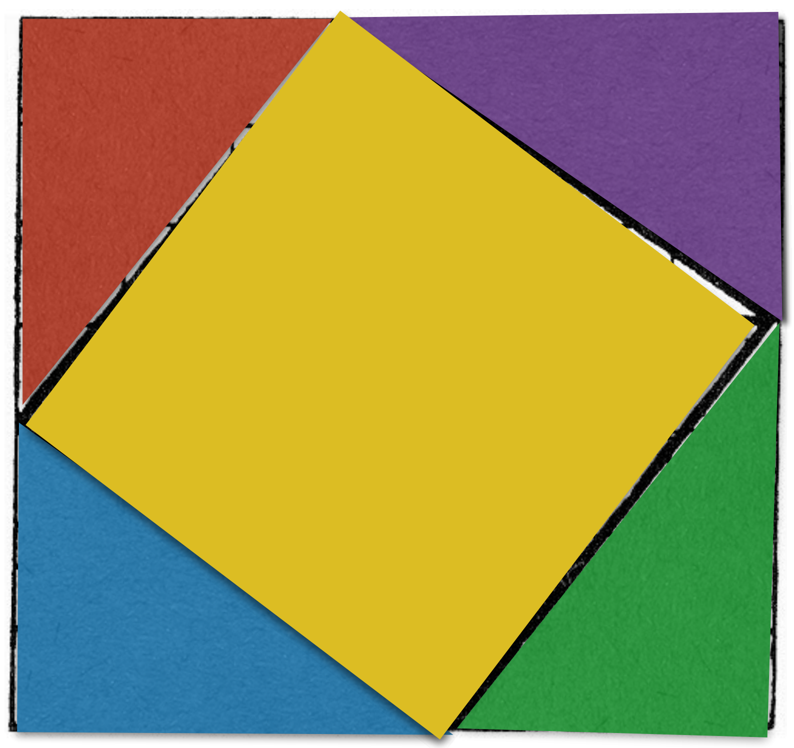

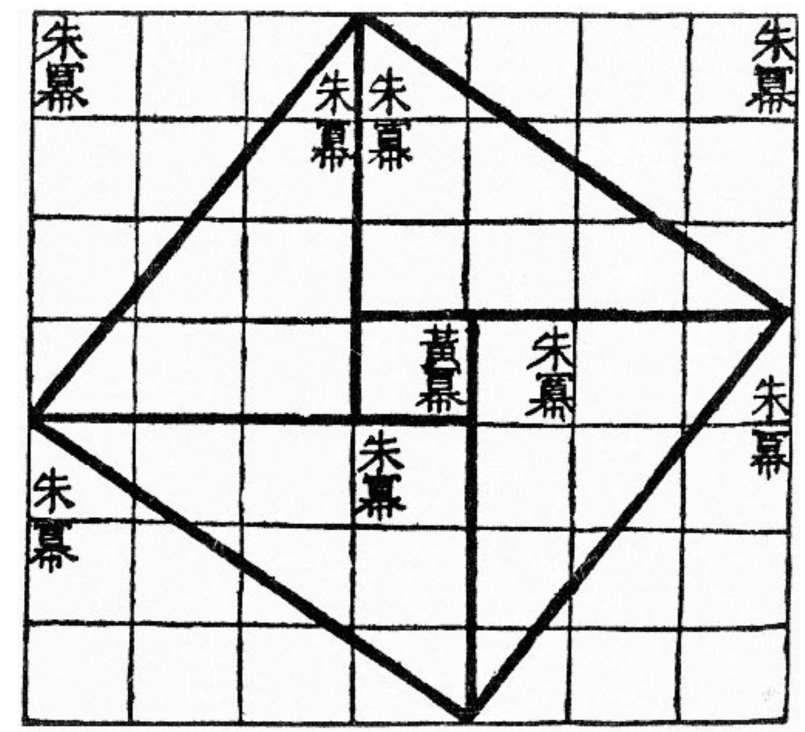

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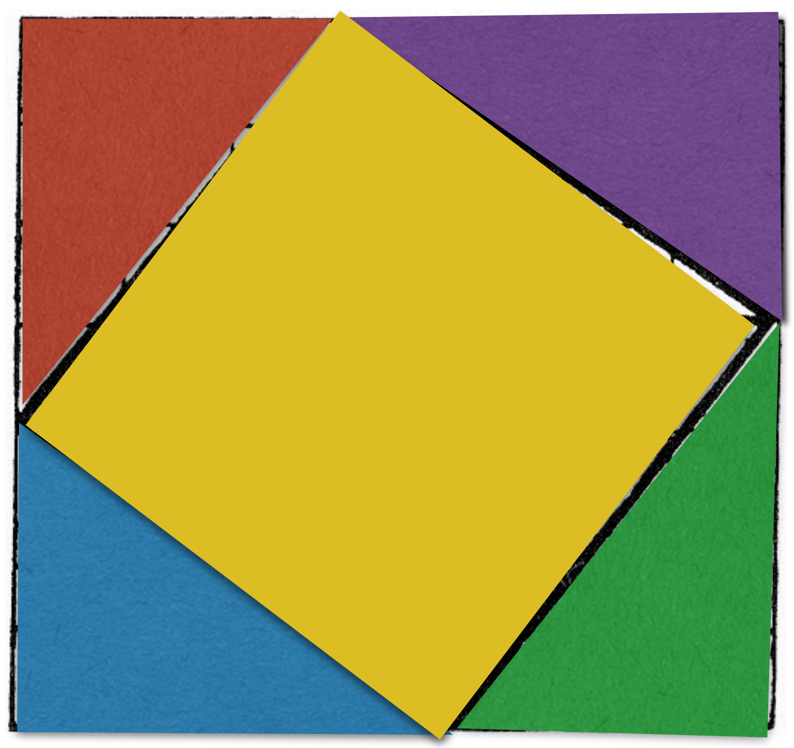

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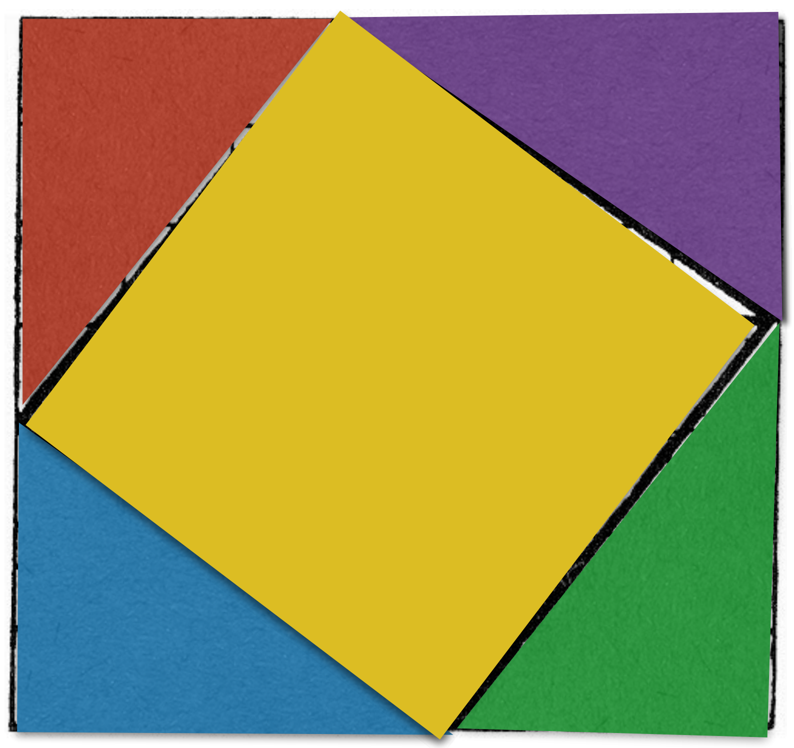

4 triangles + middle square = Whole square

Second *Possible* Proof

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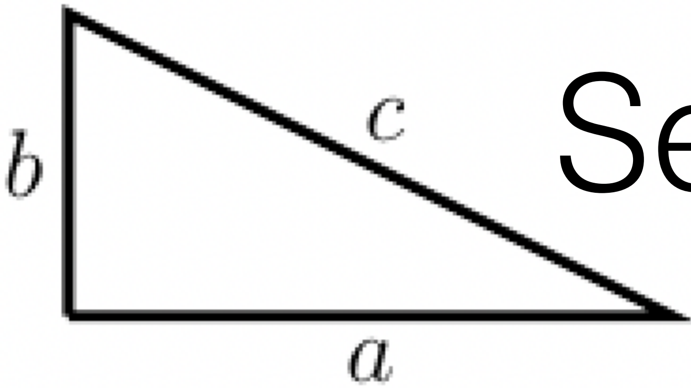

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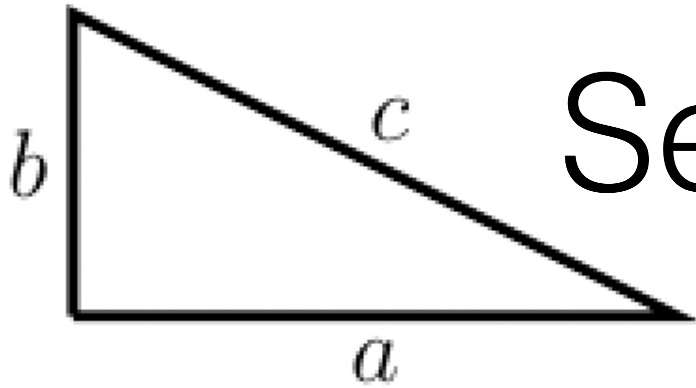

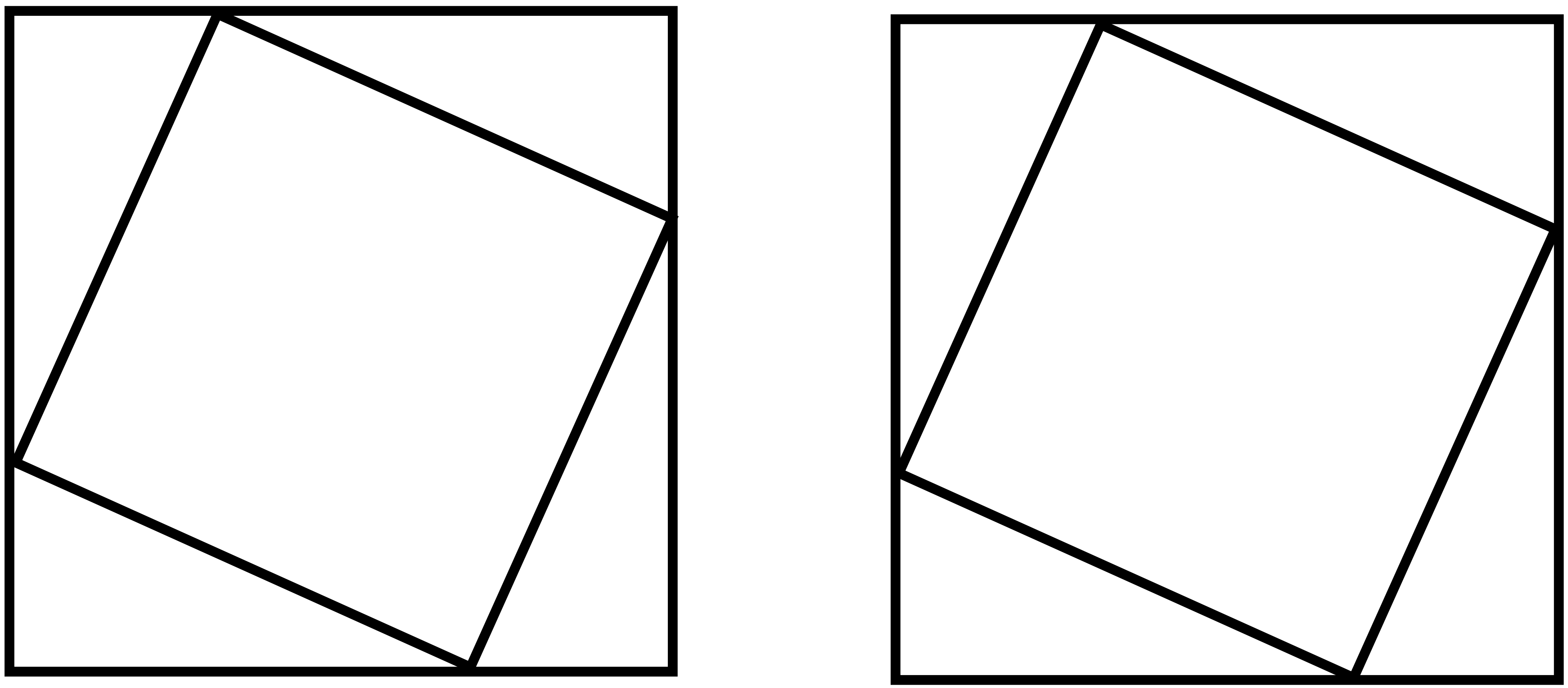

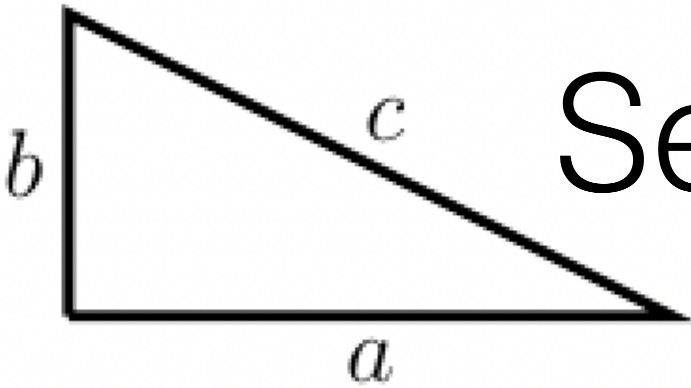

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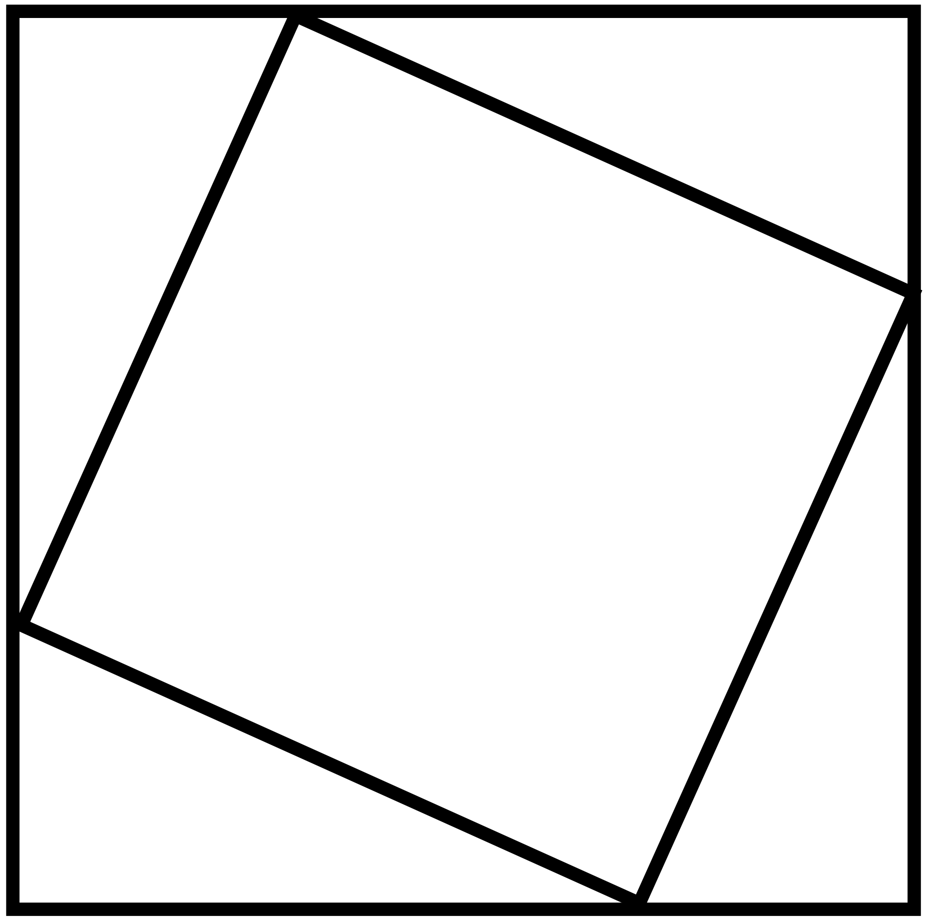

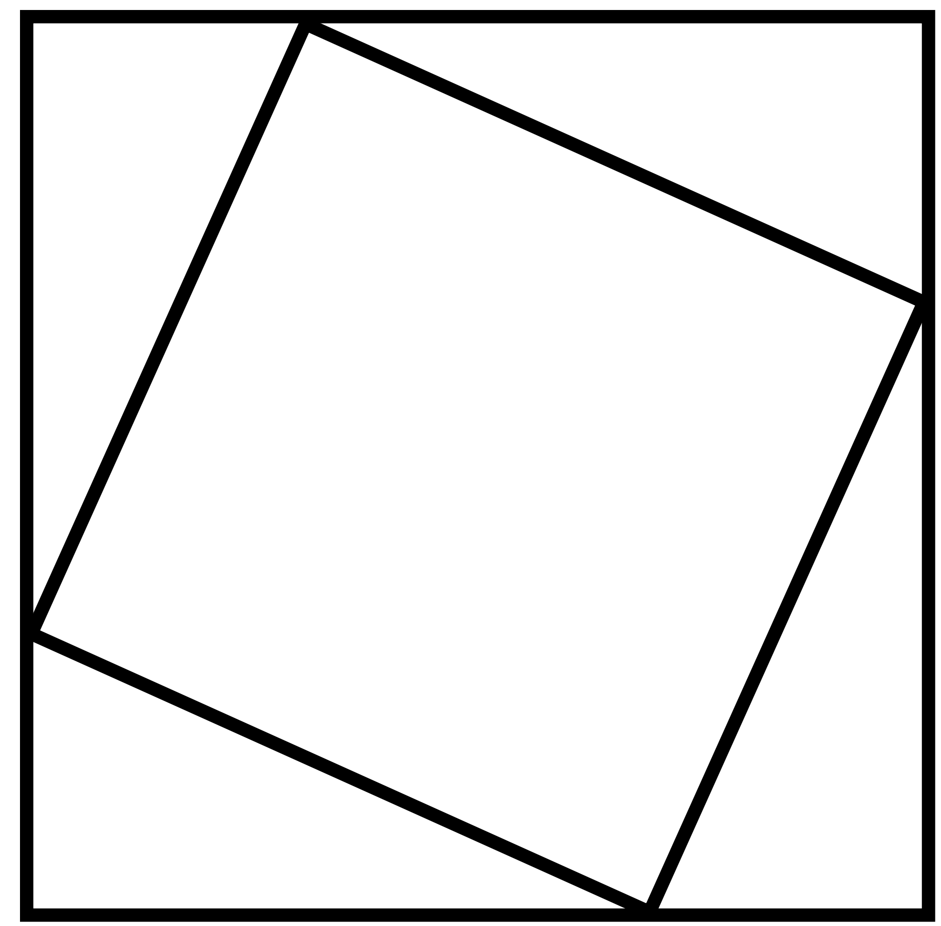

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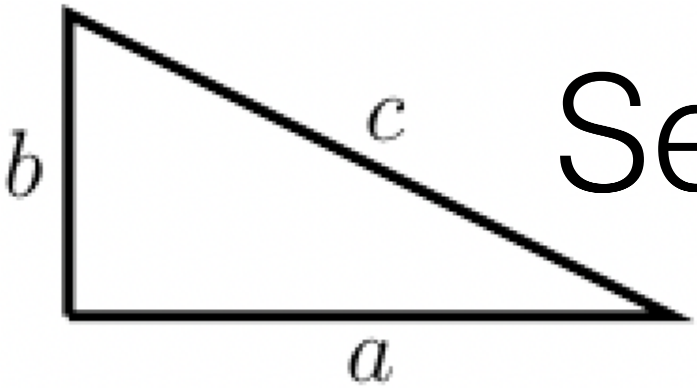

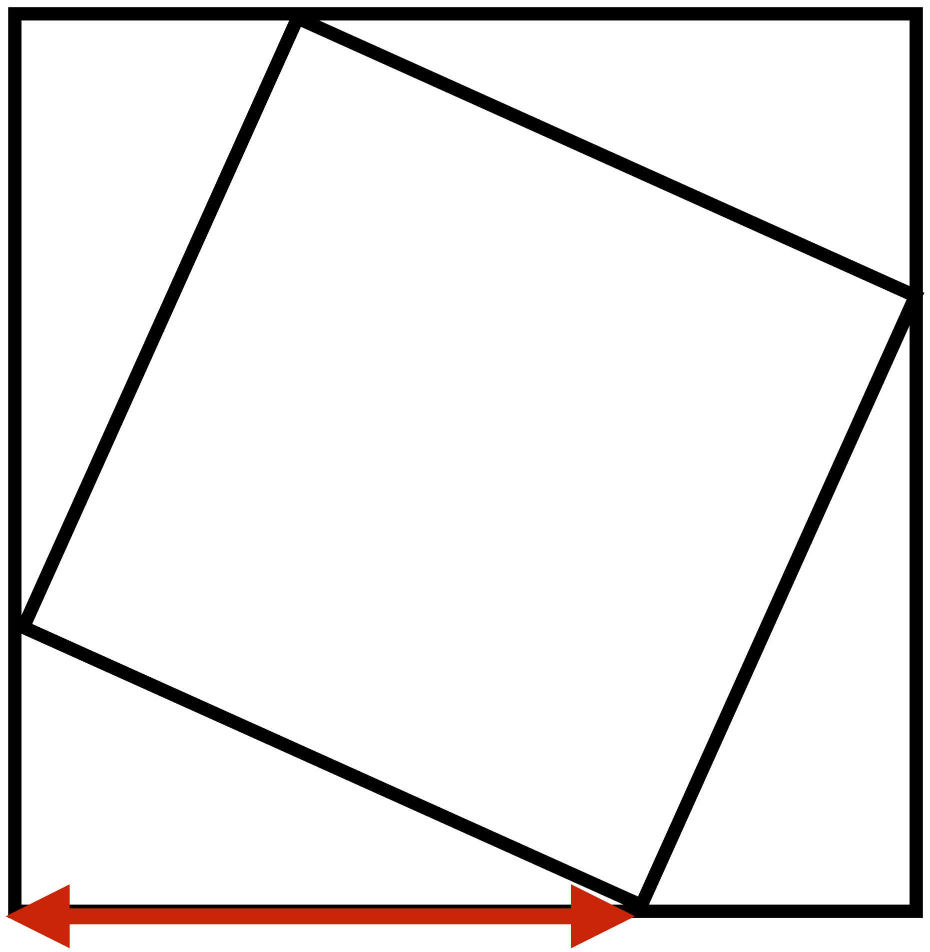

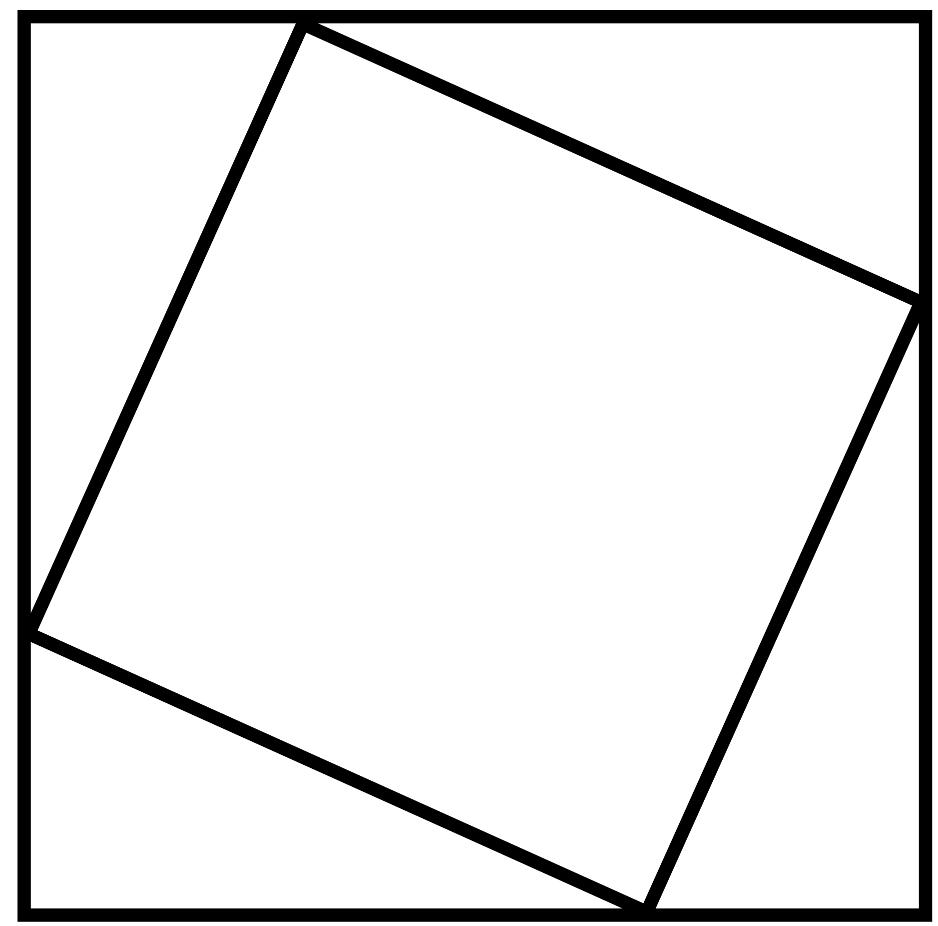

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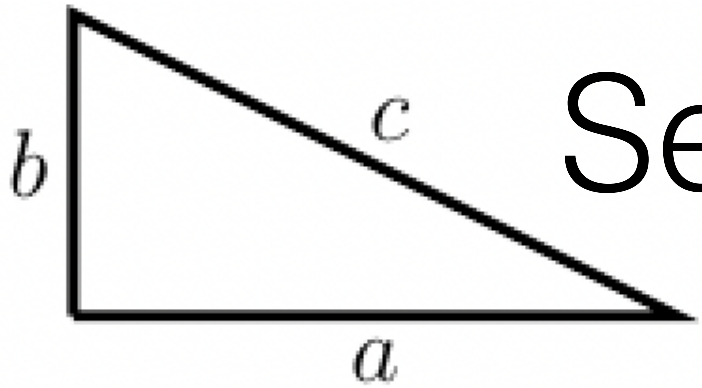

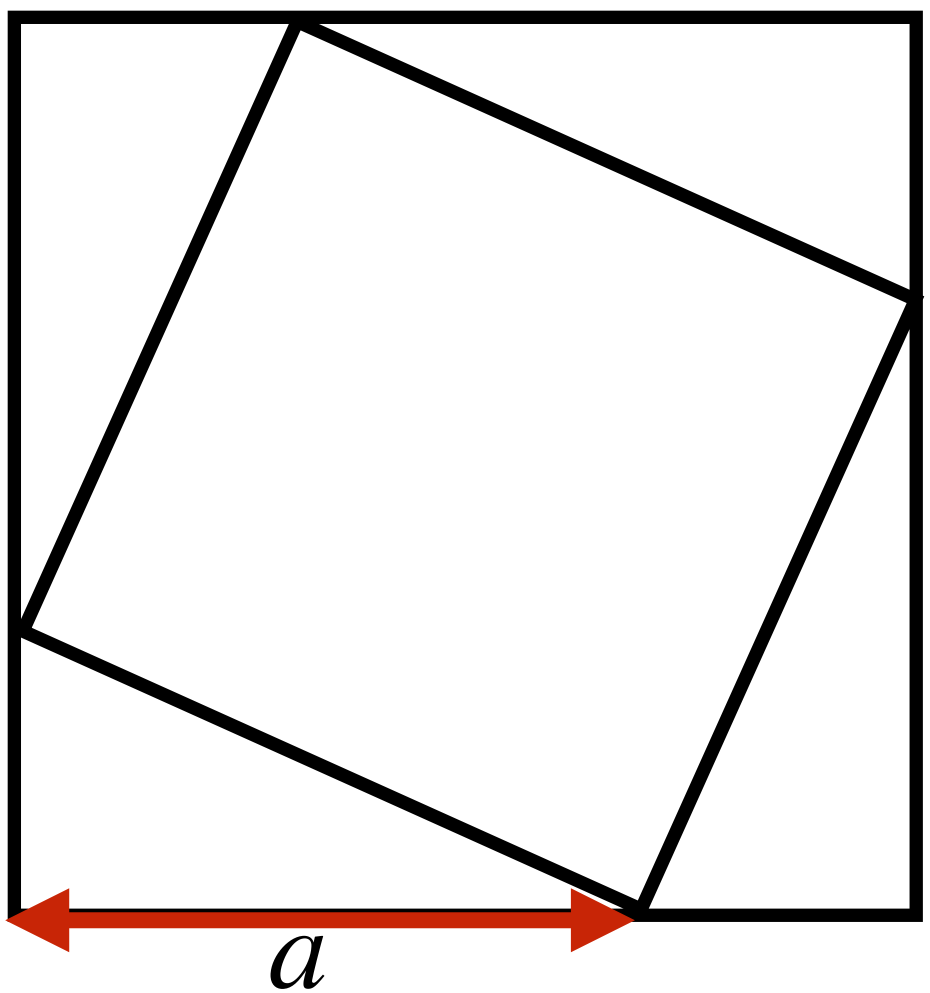

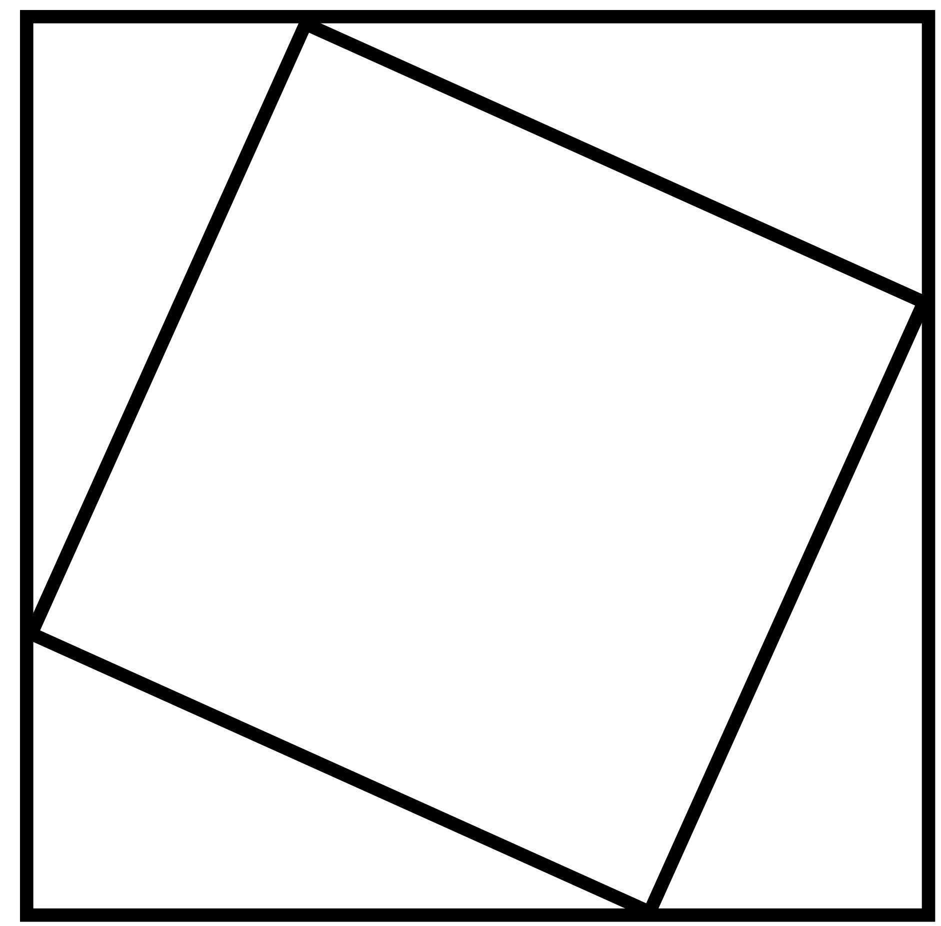

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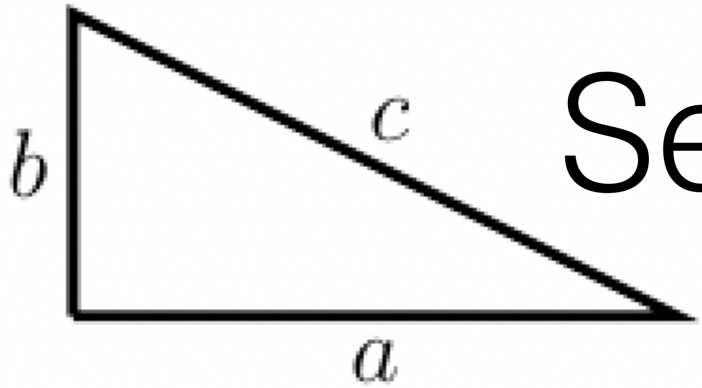

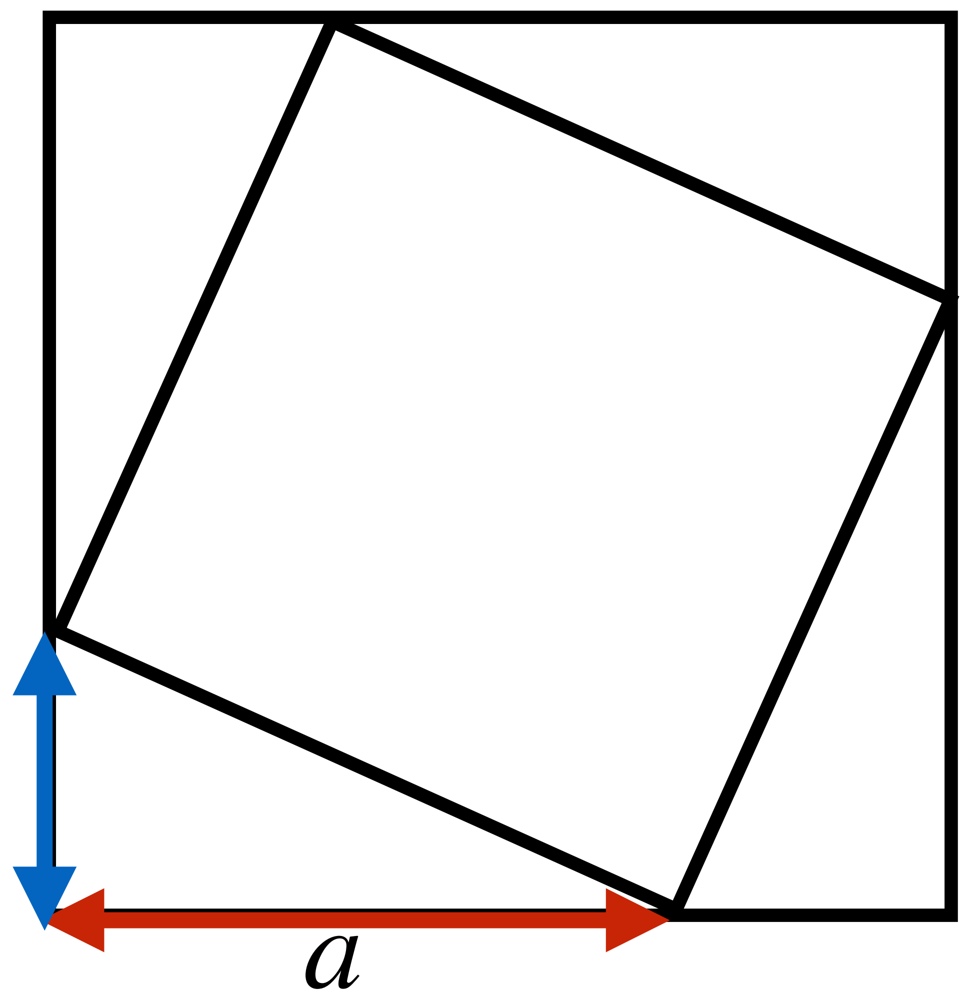

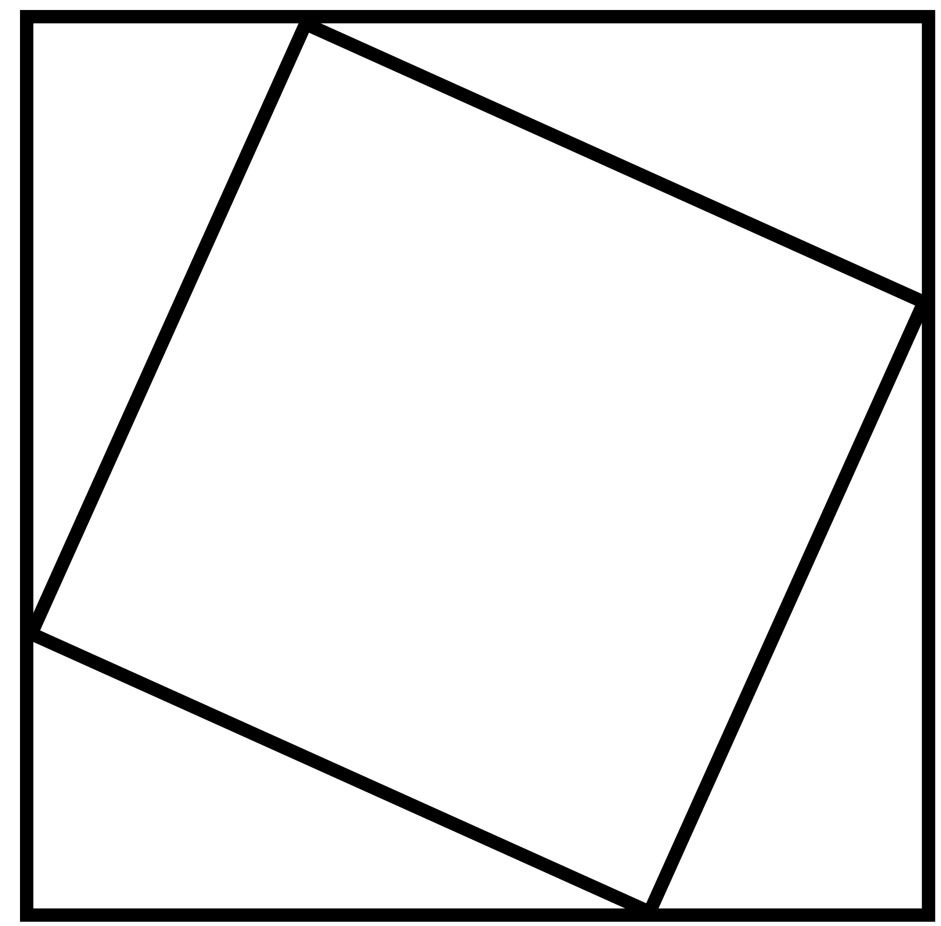

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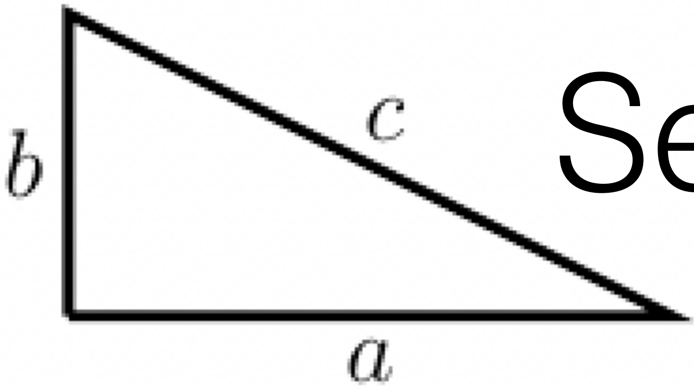

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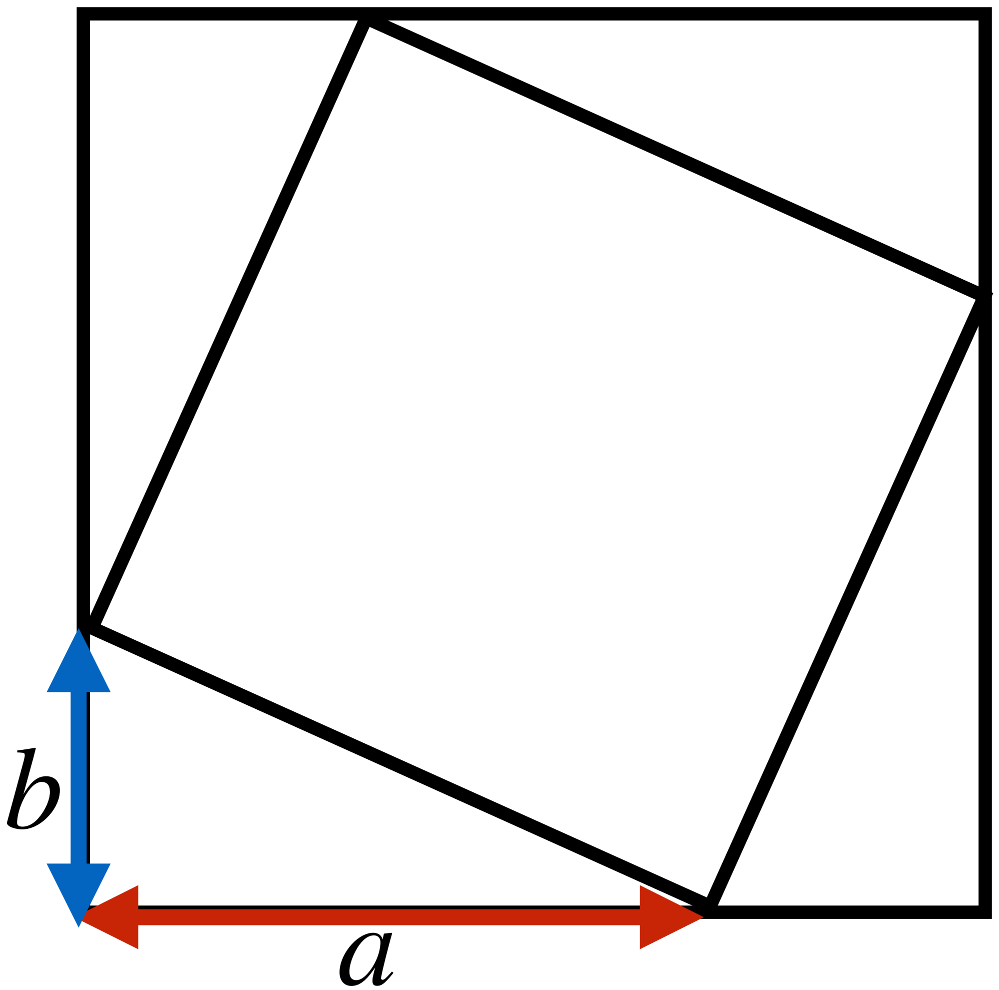

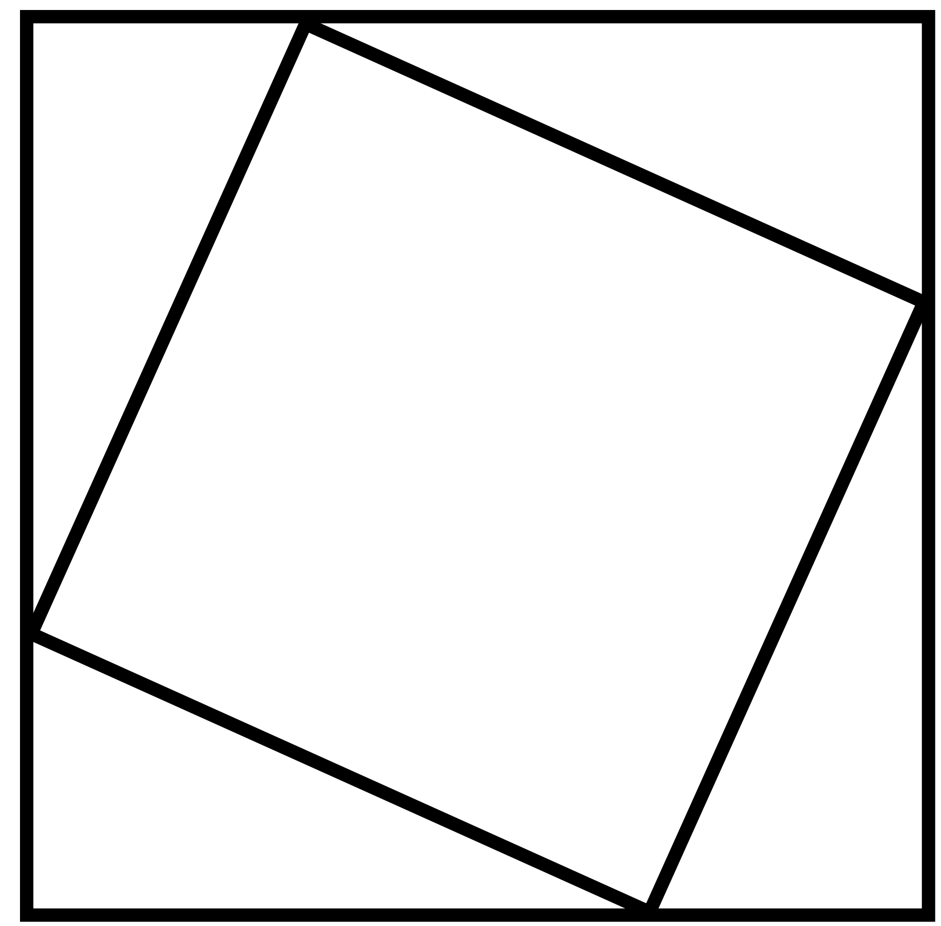

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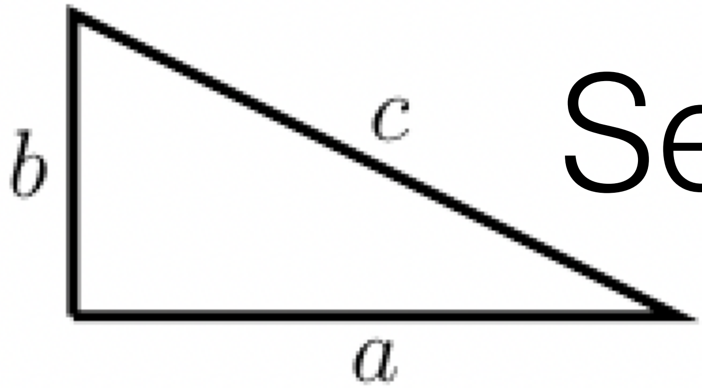
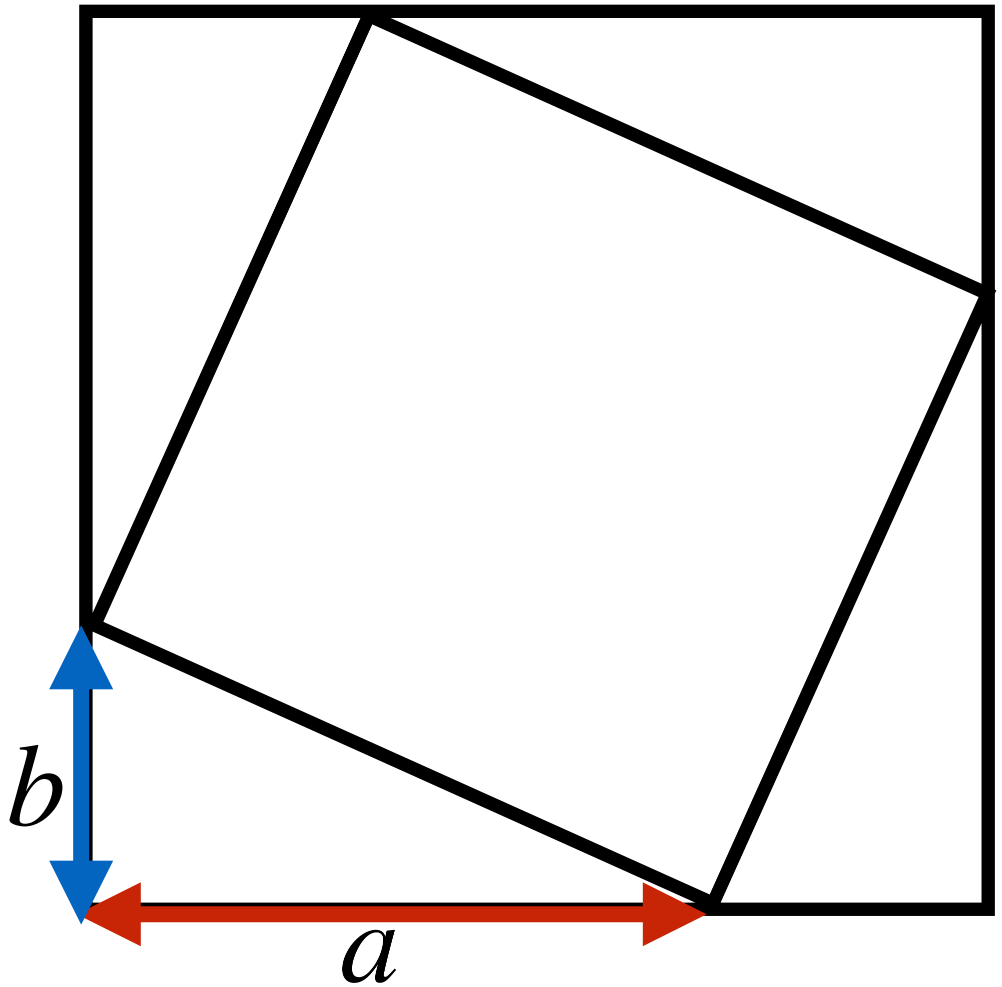
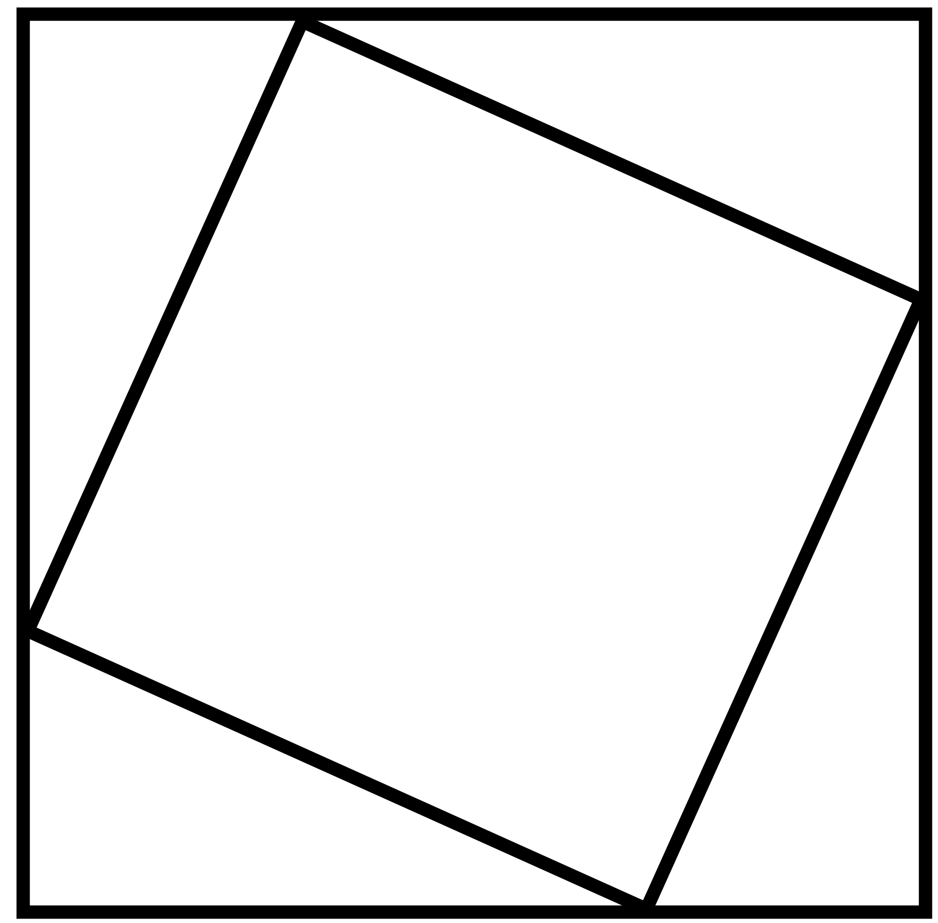


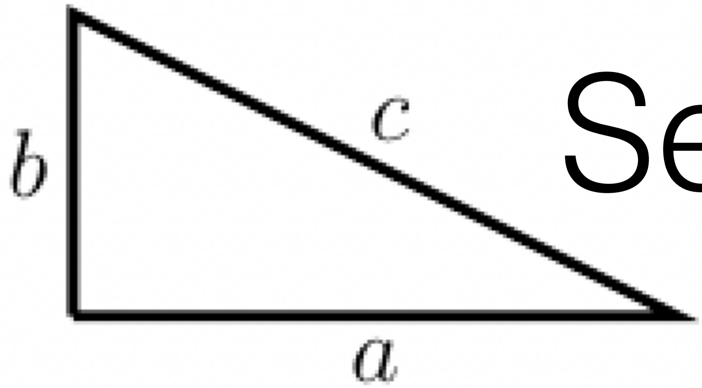


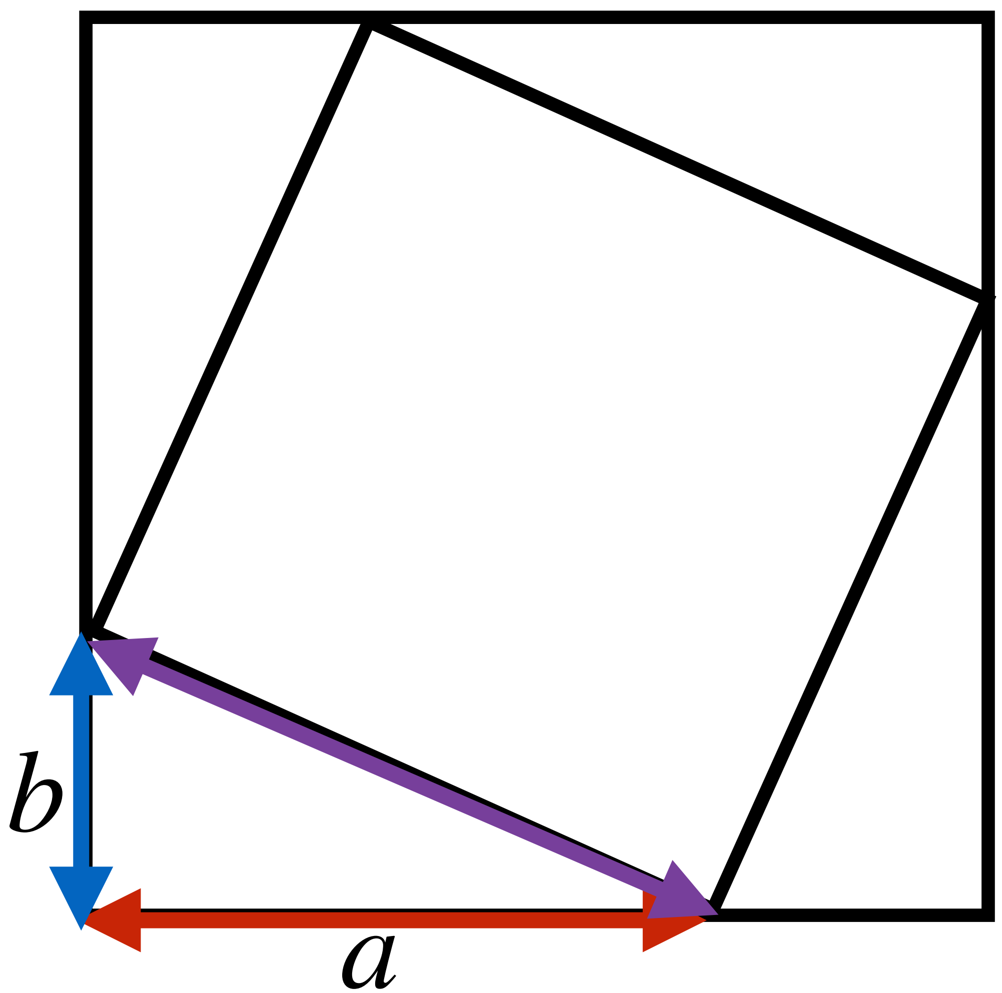


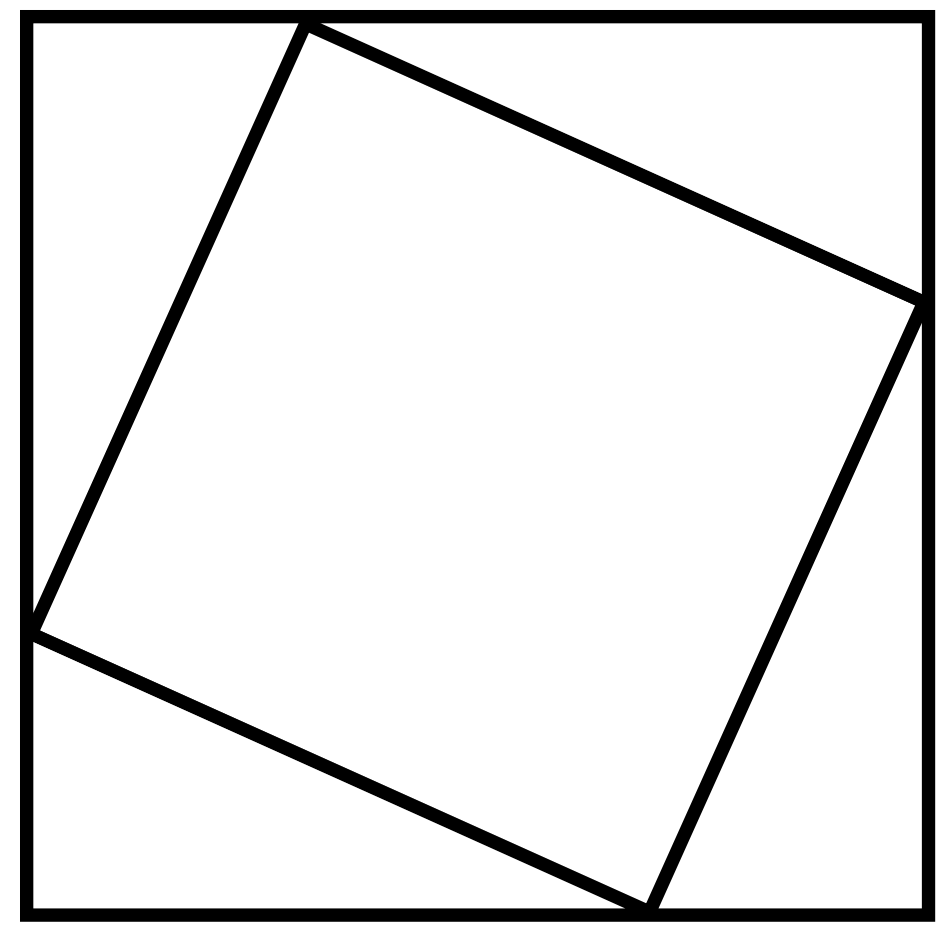
4 triangles + middle square = Whole square

$$4 \cdot \left(\frac{1}{2}a \cdot b\right) \quad +$$

Second *Possible* Proof







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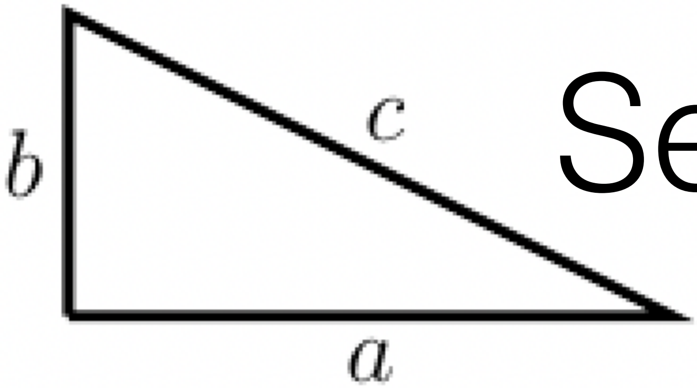

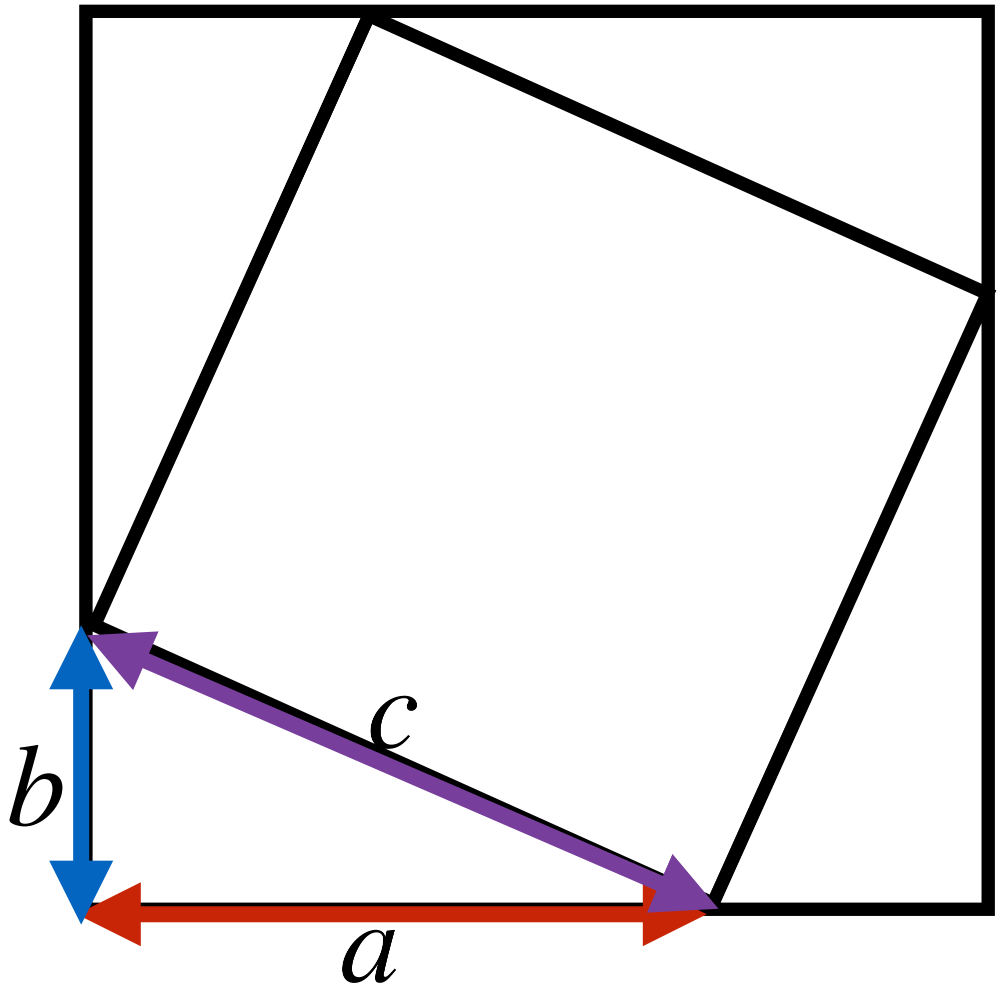

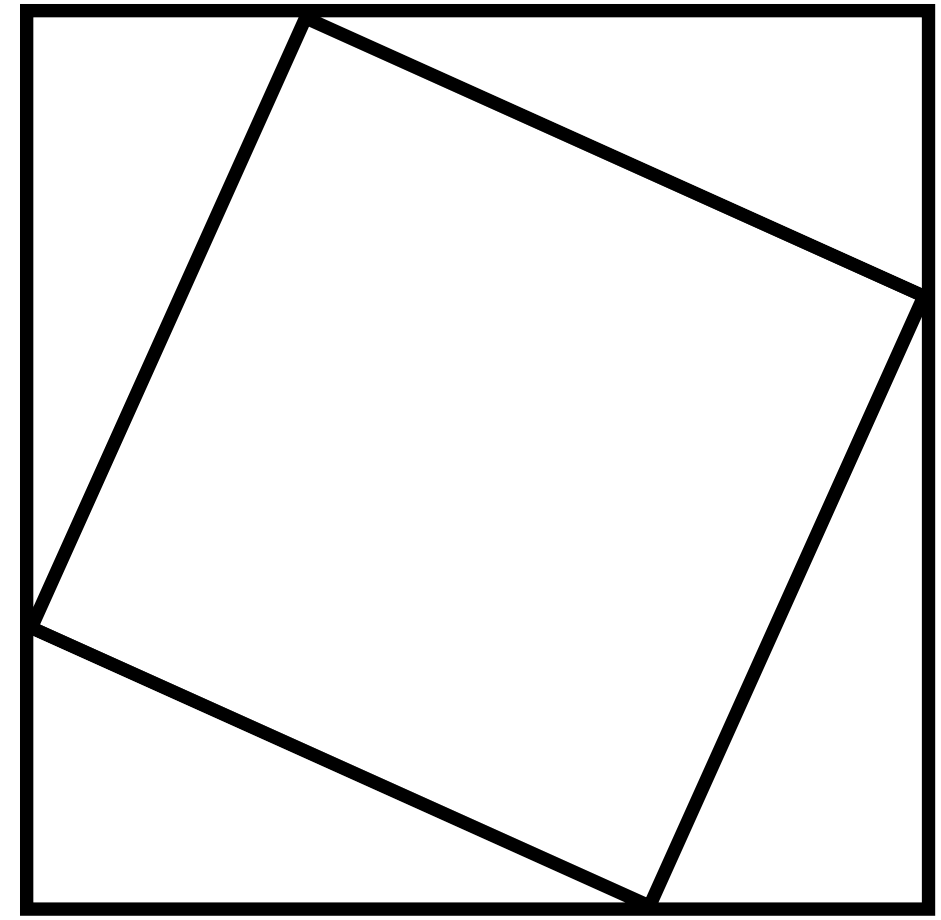

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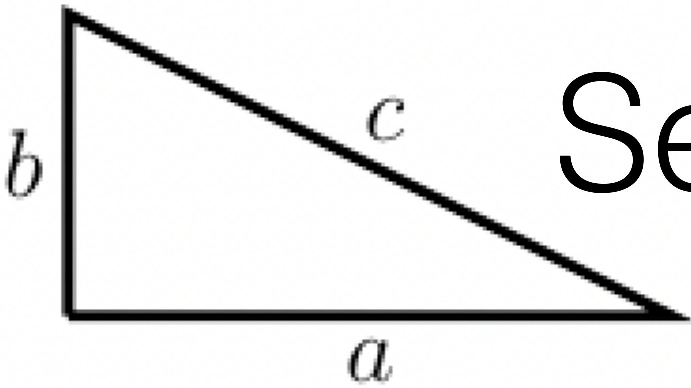

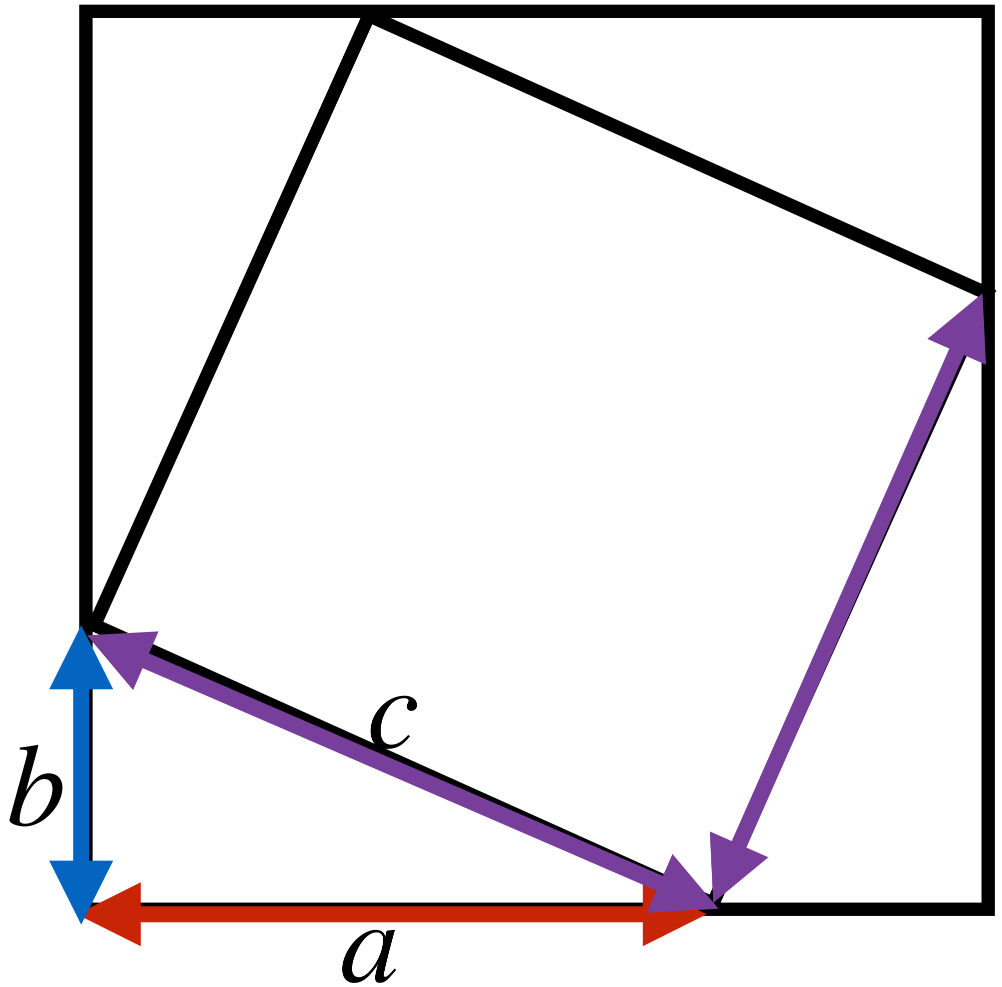

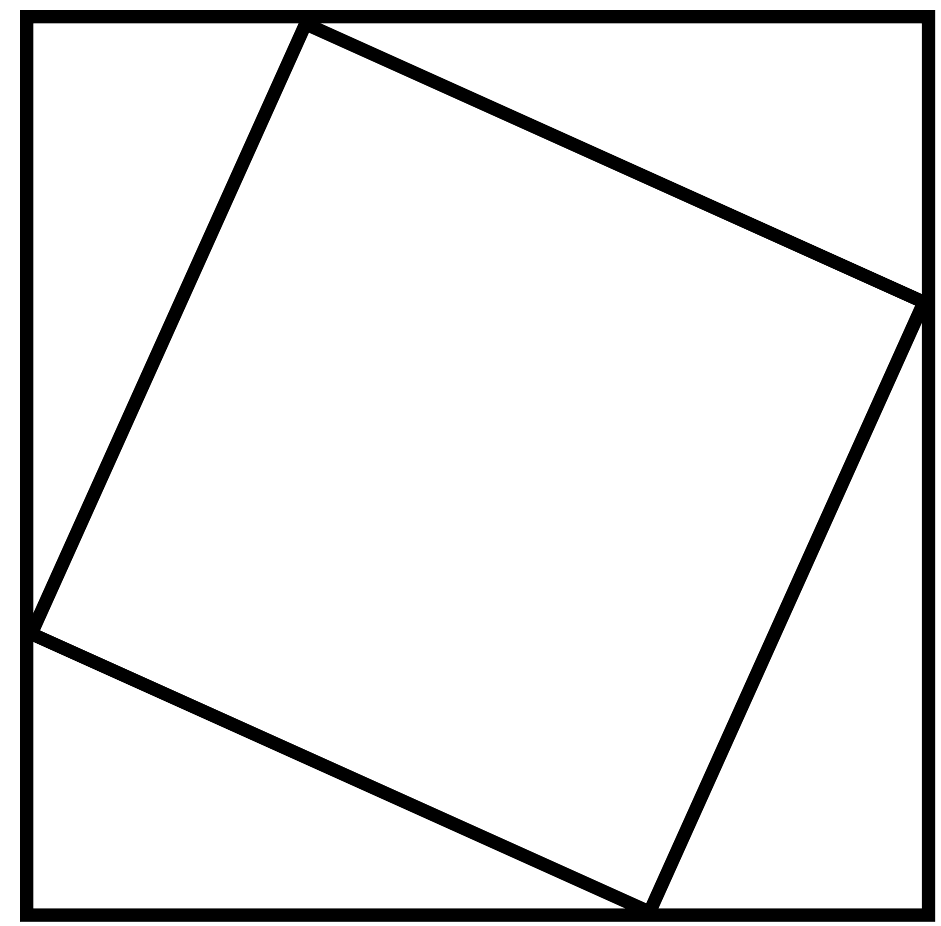

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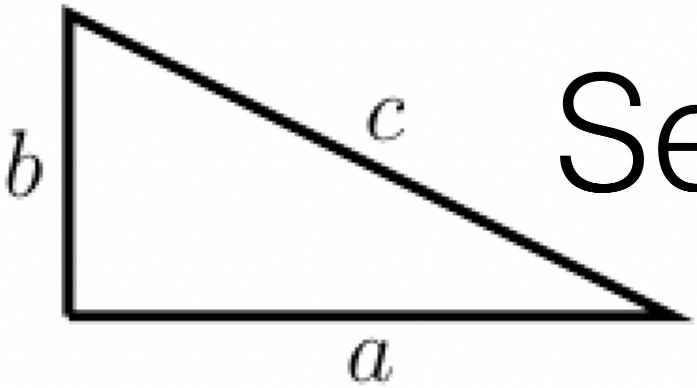

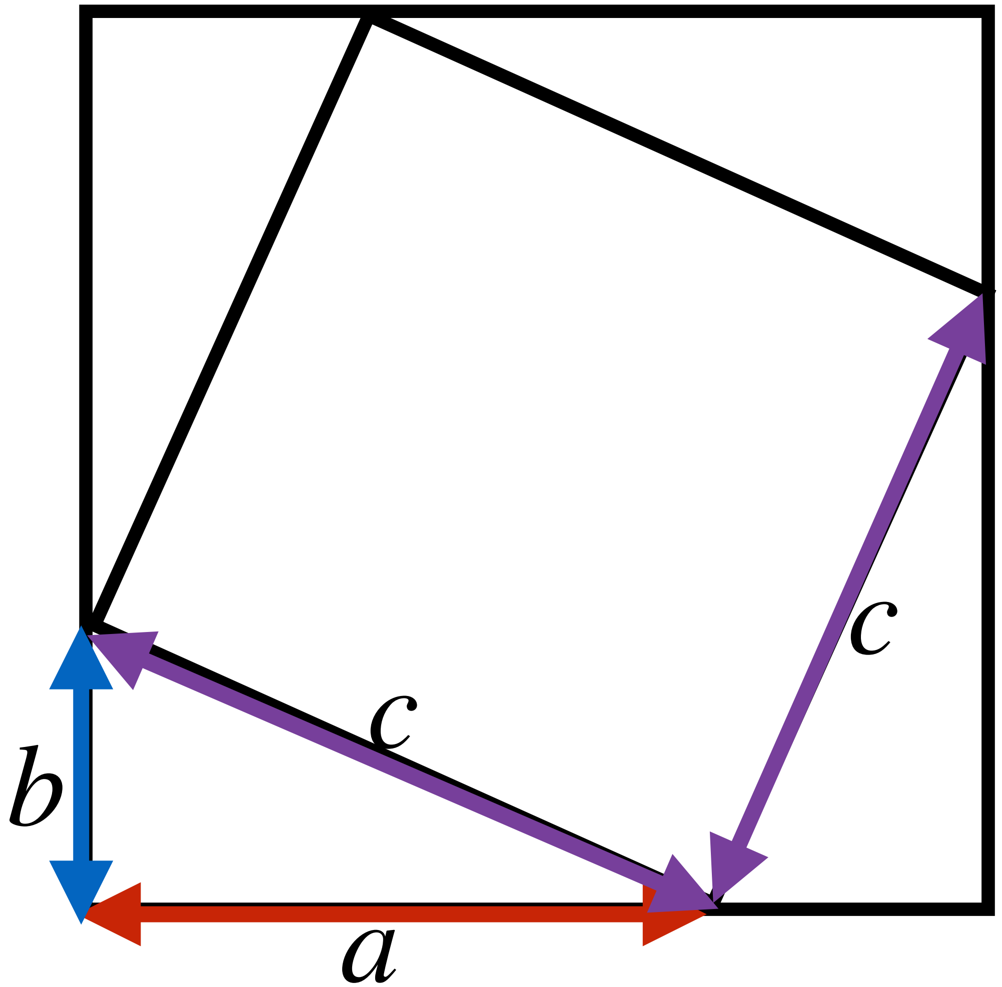

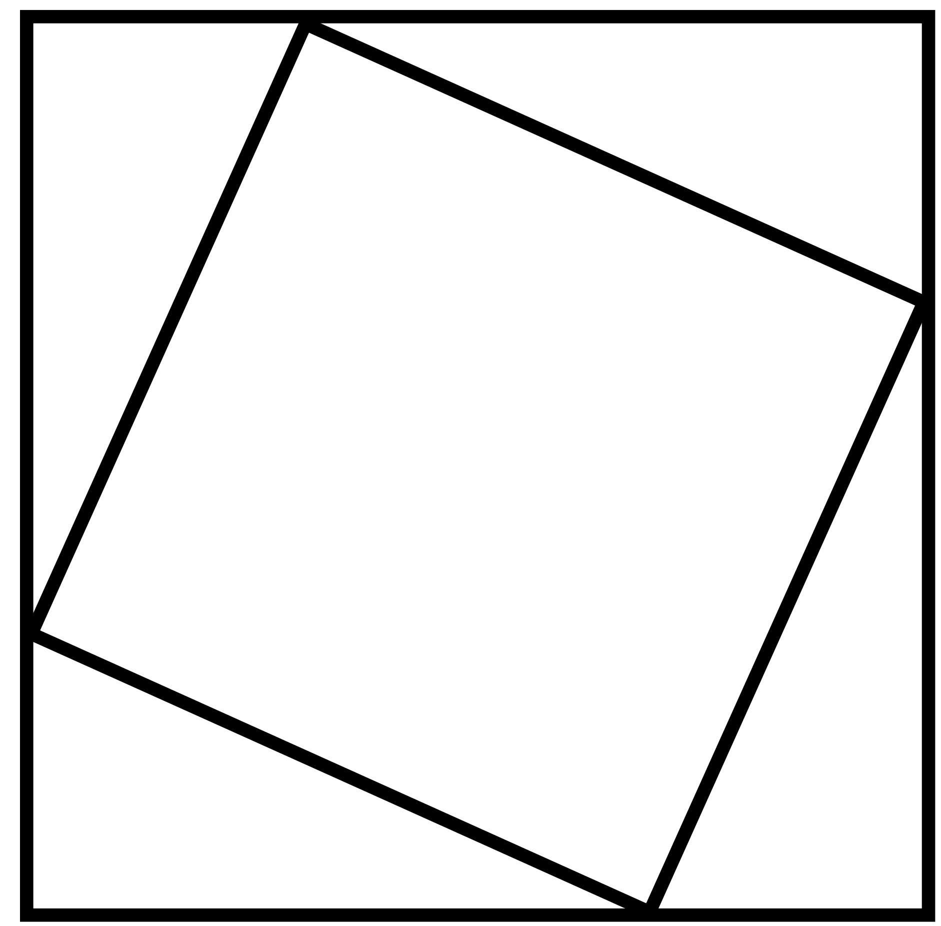

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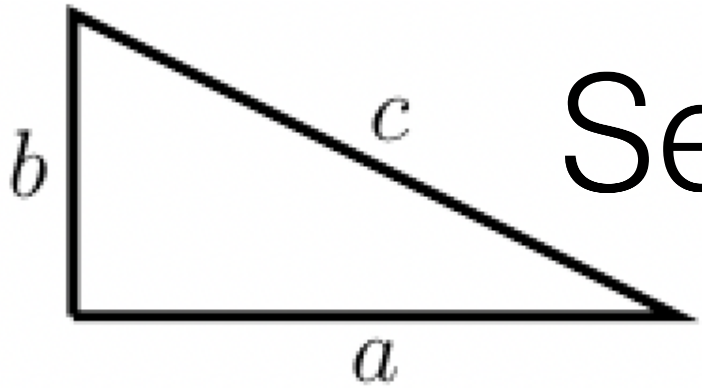

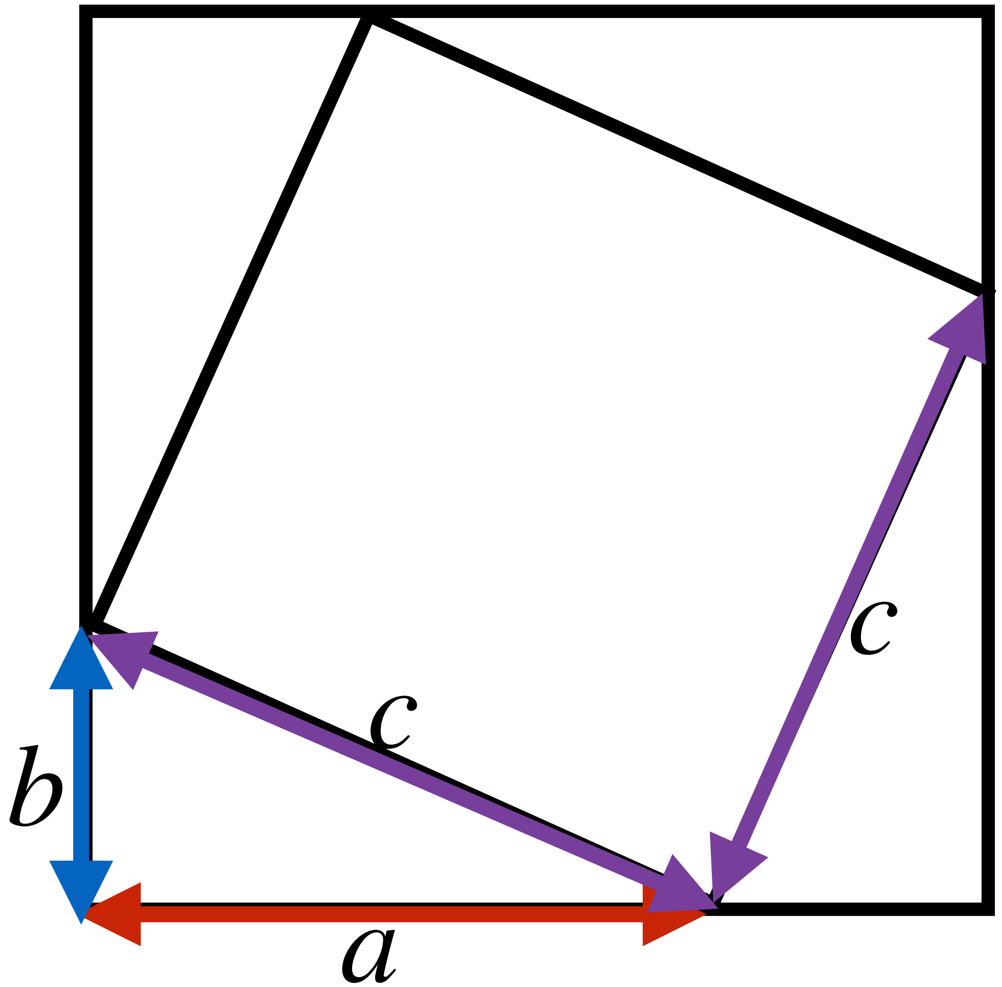

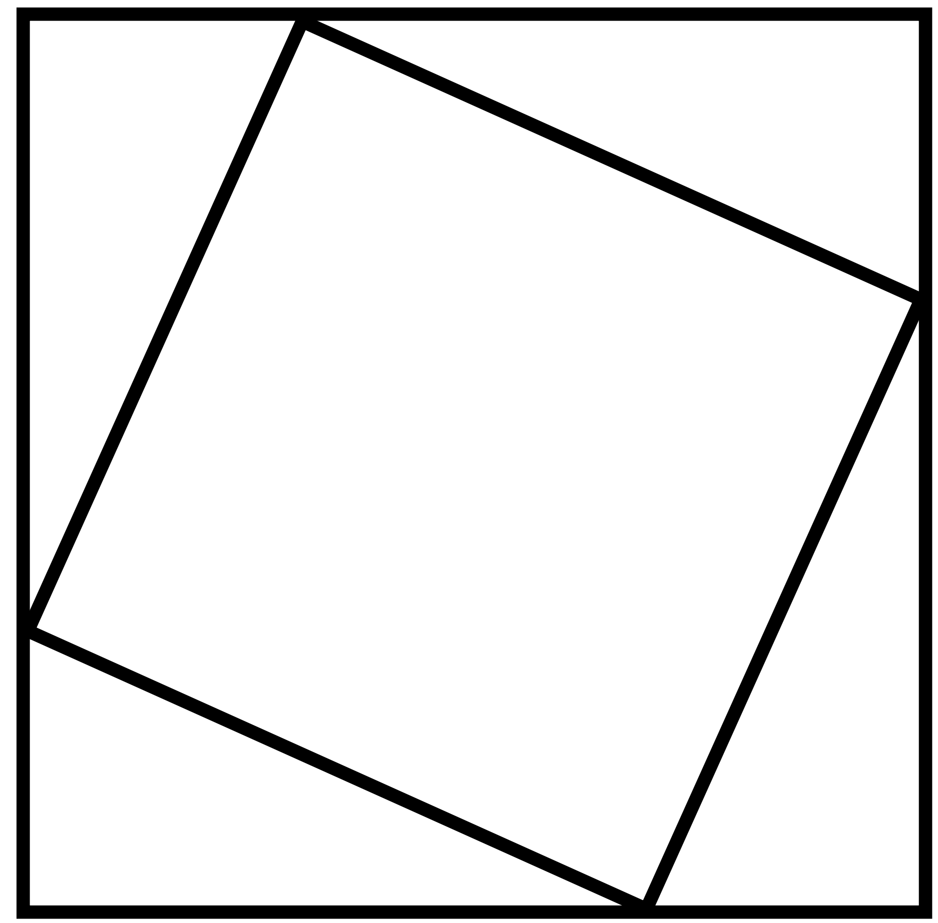

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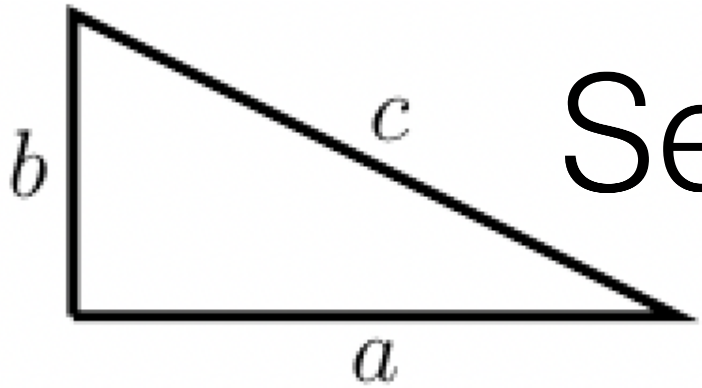
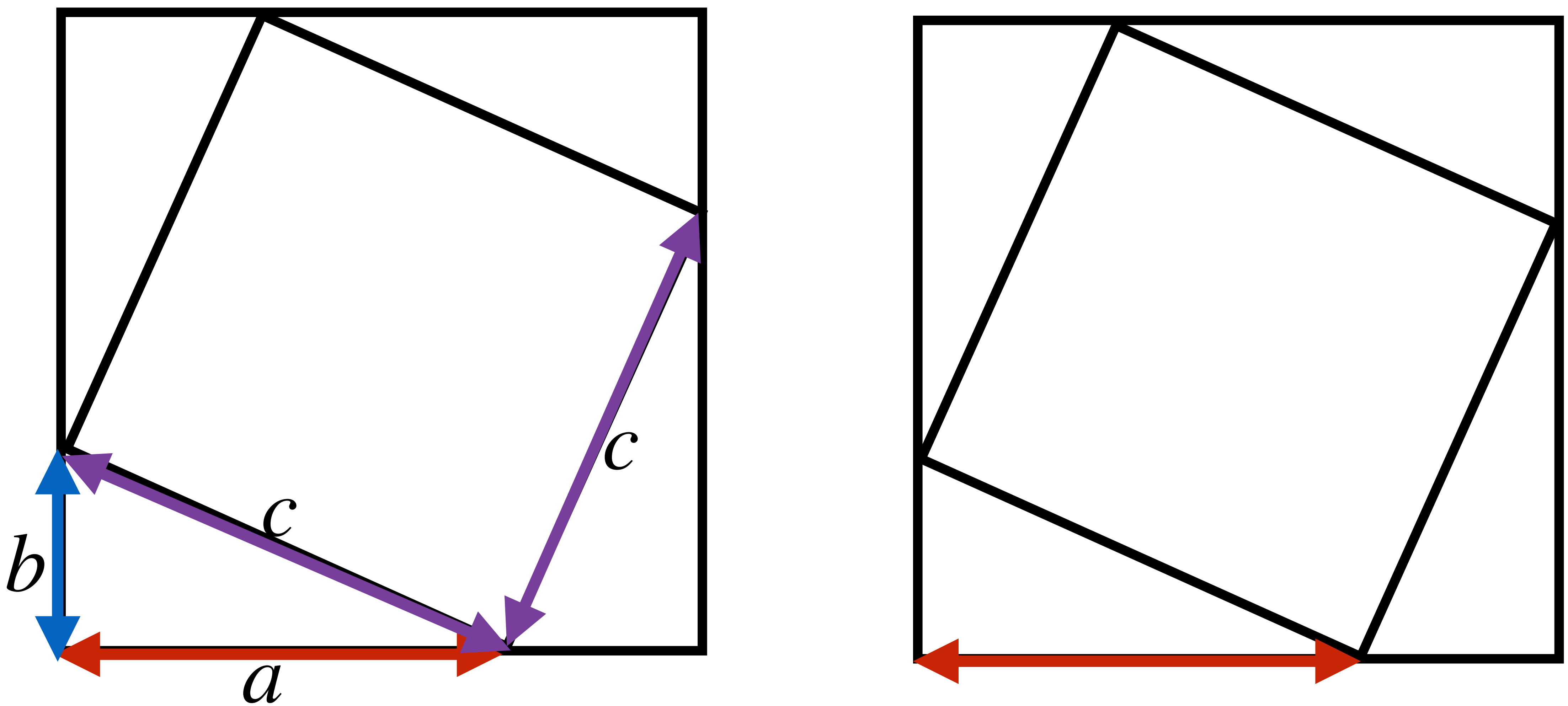
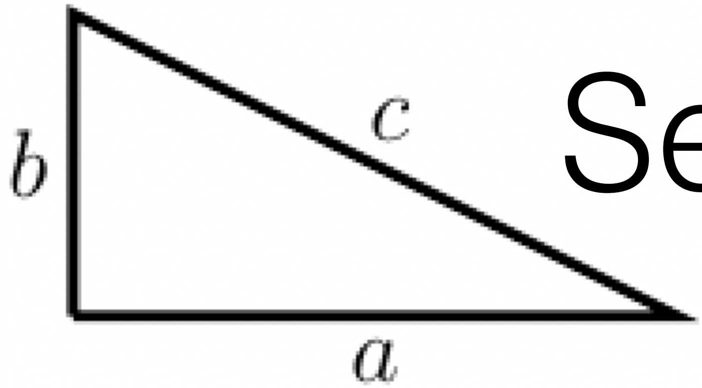


Second *Possible* Proof



4 triangles + middle square = Whole square

$$4 \cdot \left(\frac{1}{2} a \cdot b \right) \quad + \quad c^2$$



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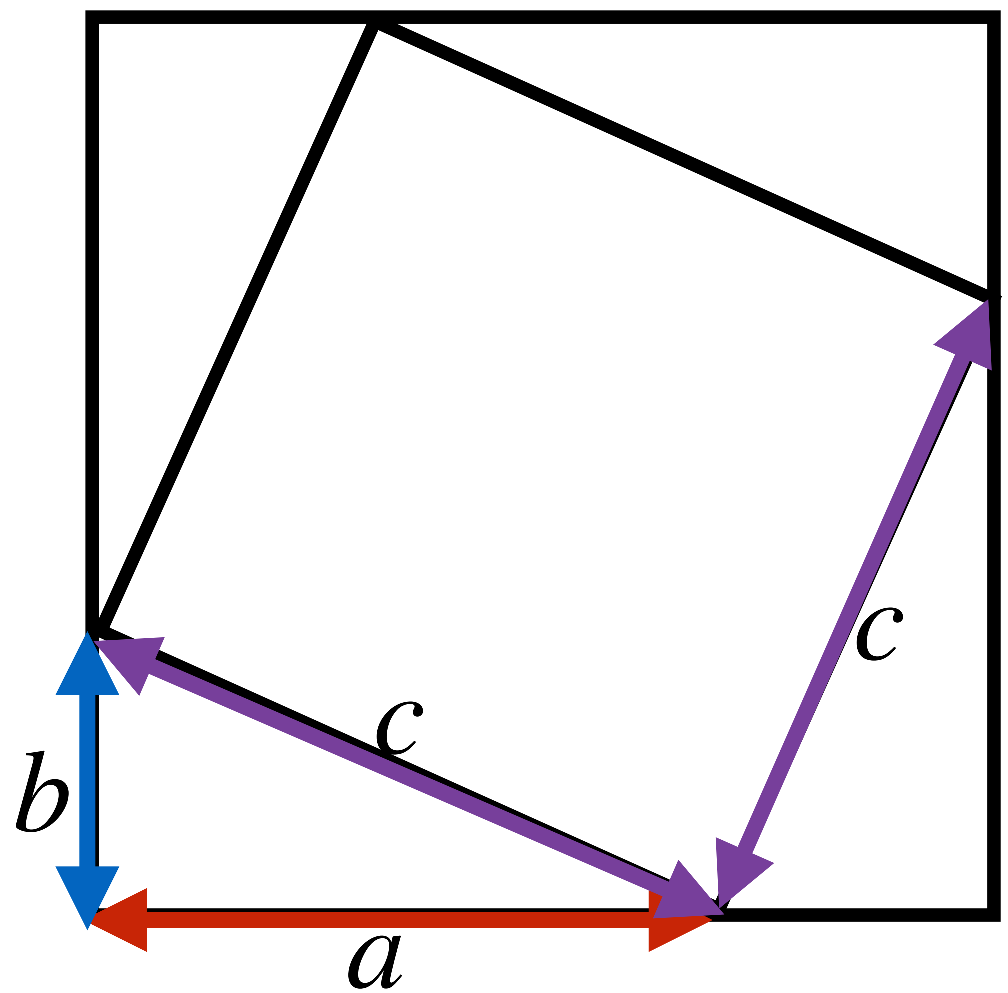

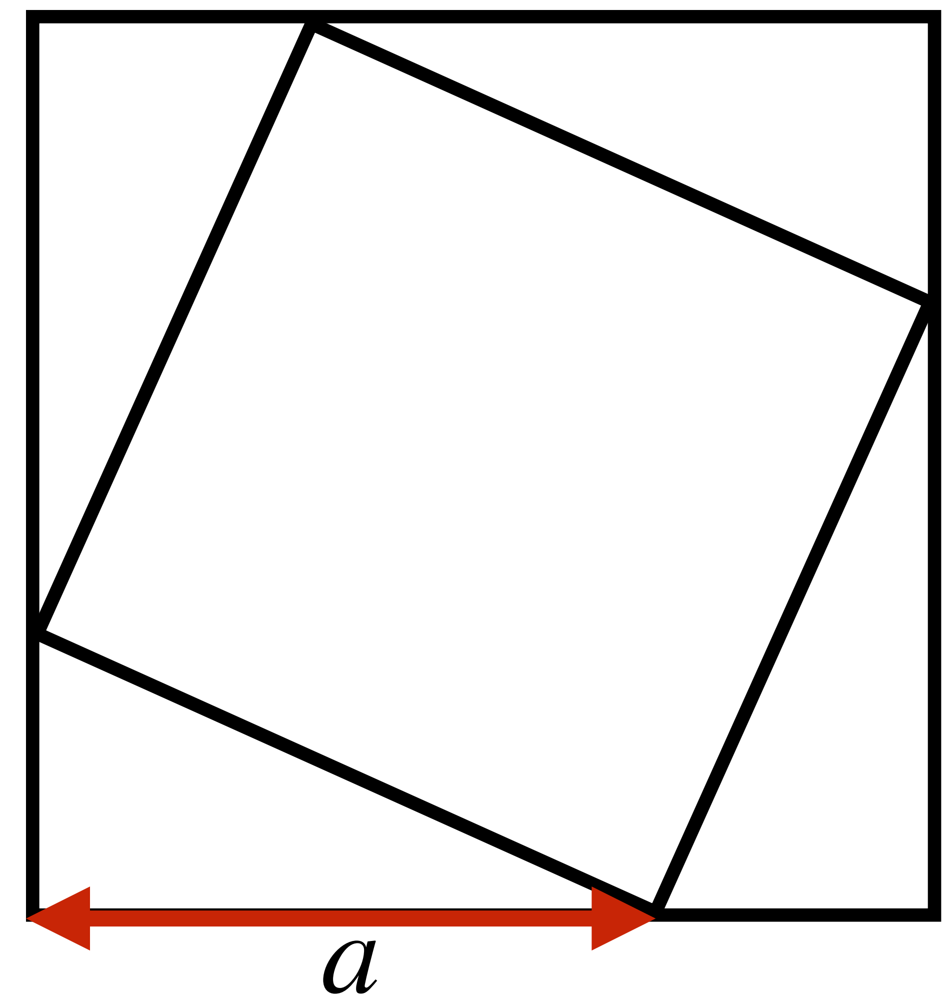

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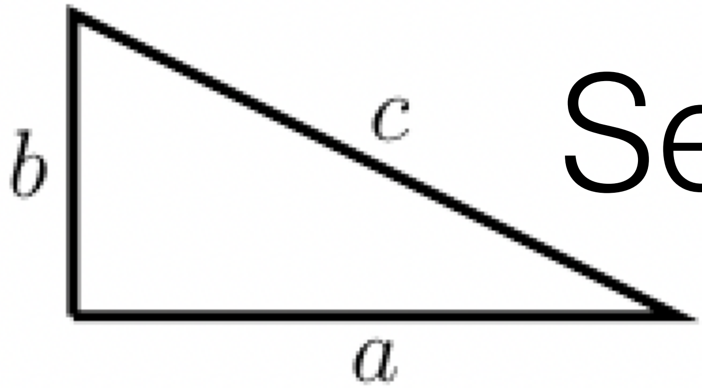

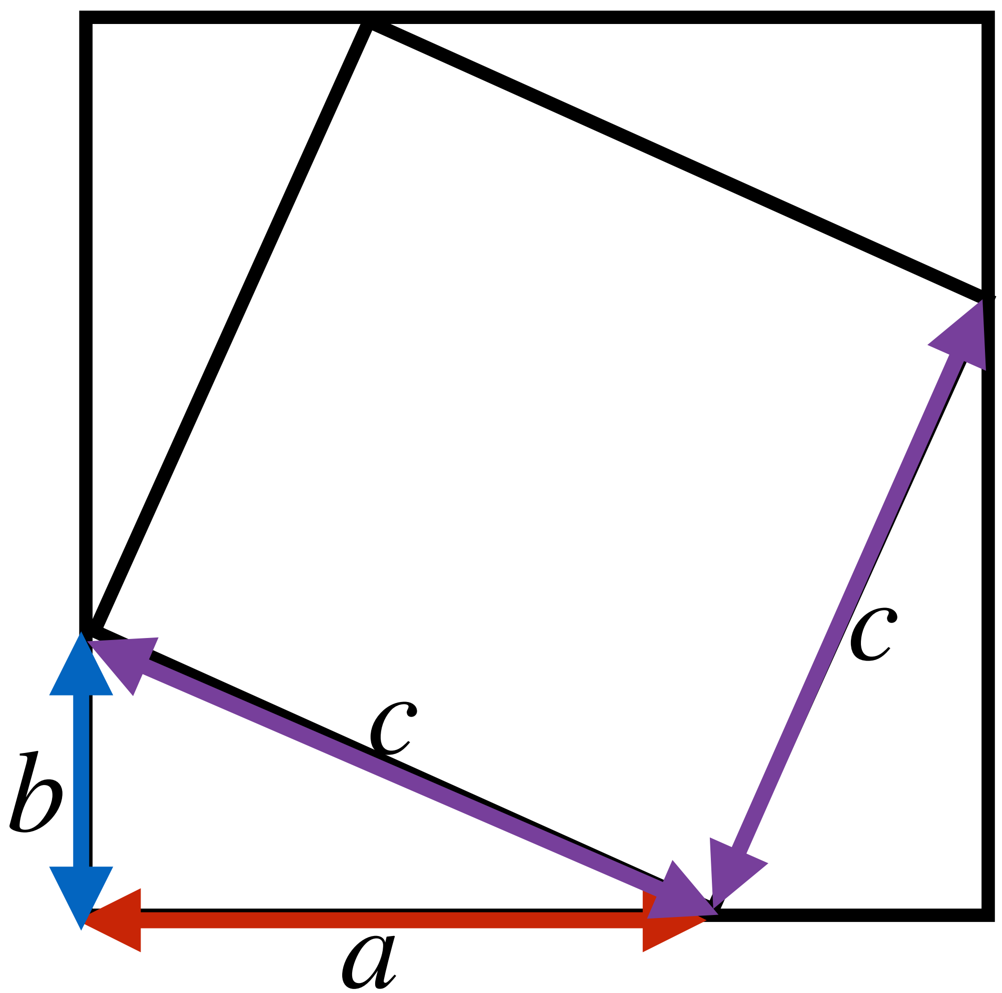

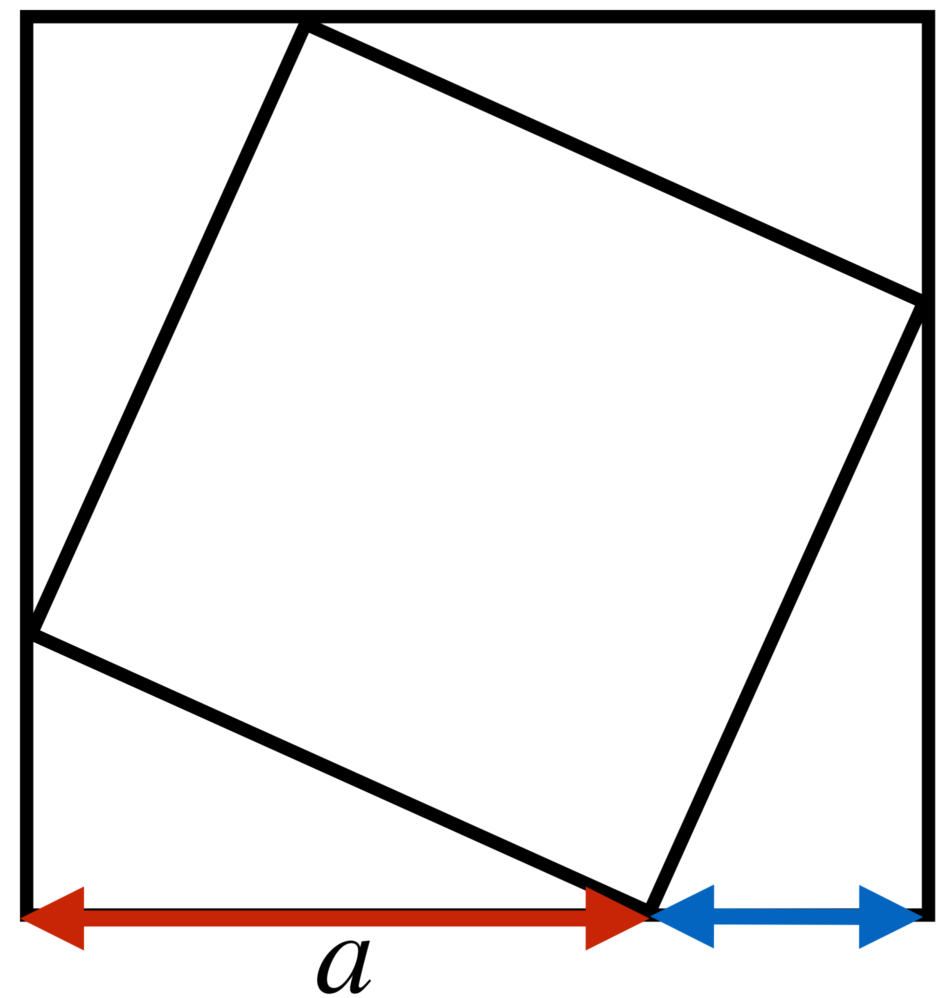

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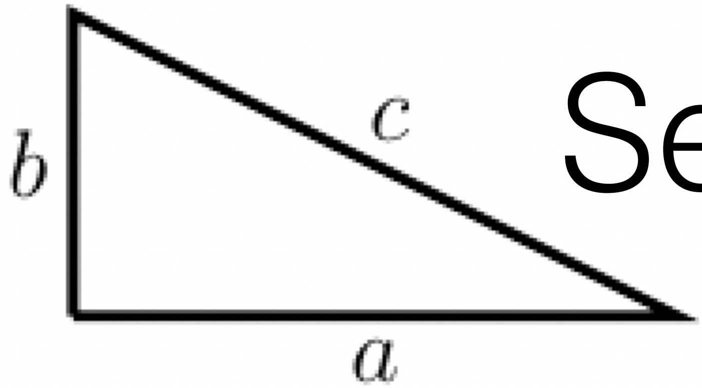

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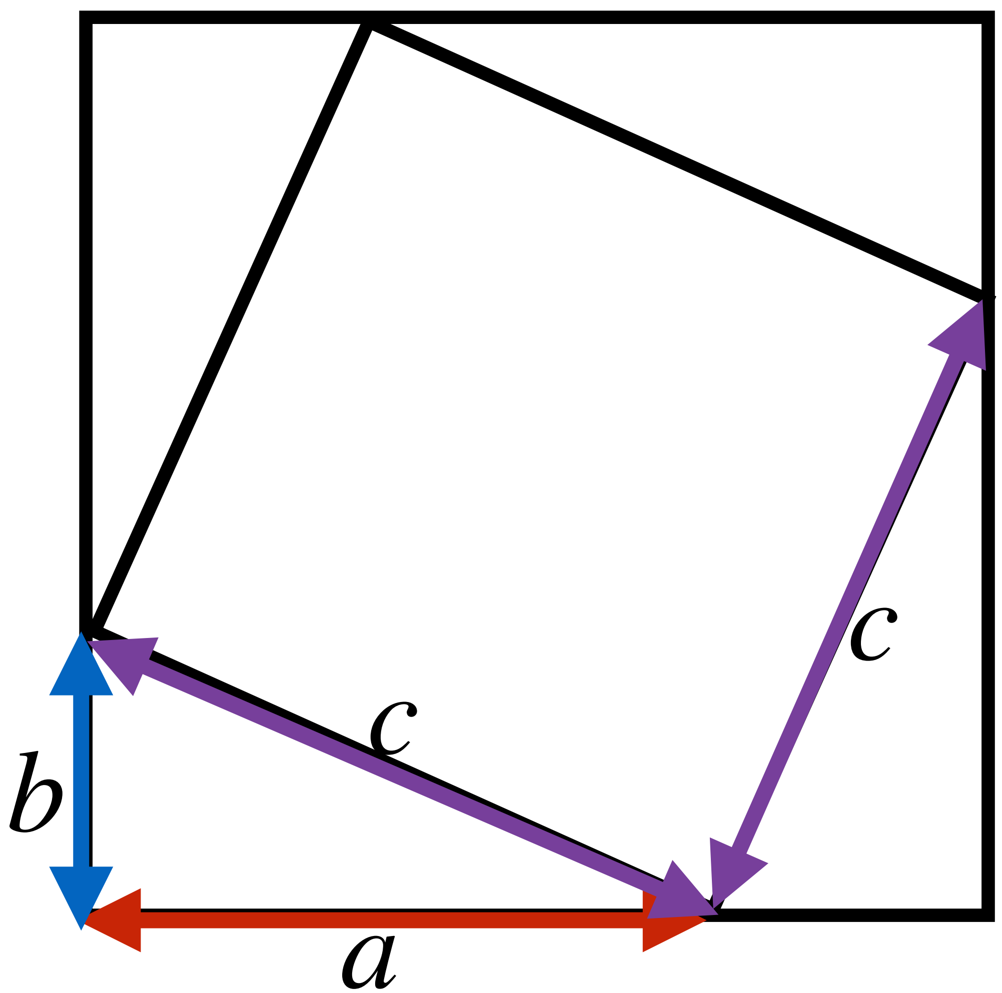

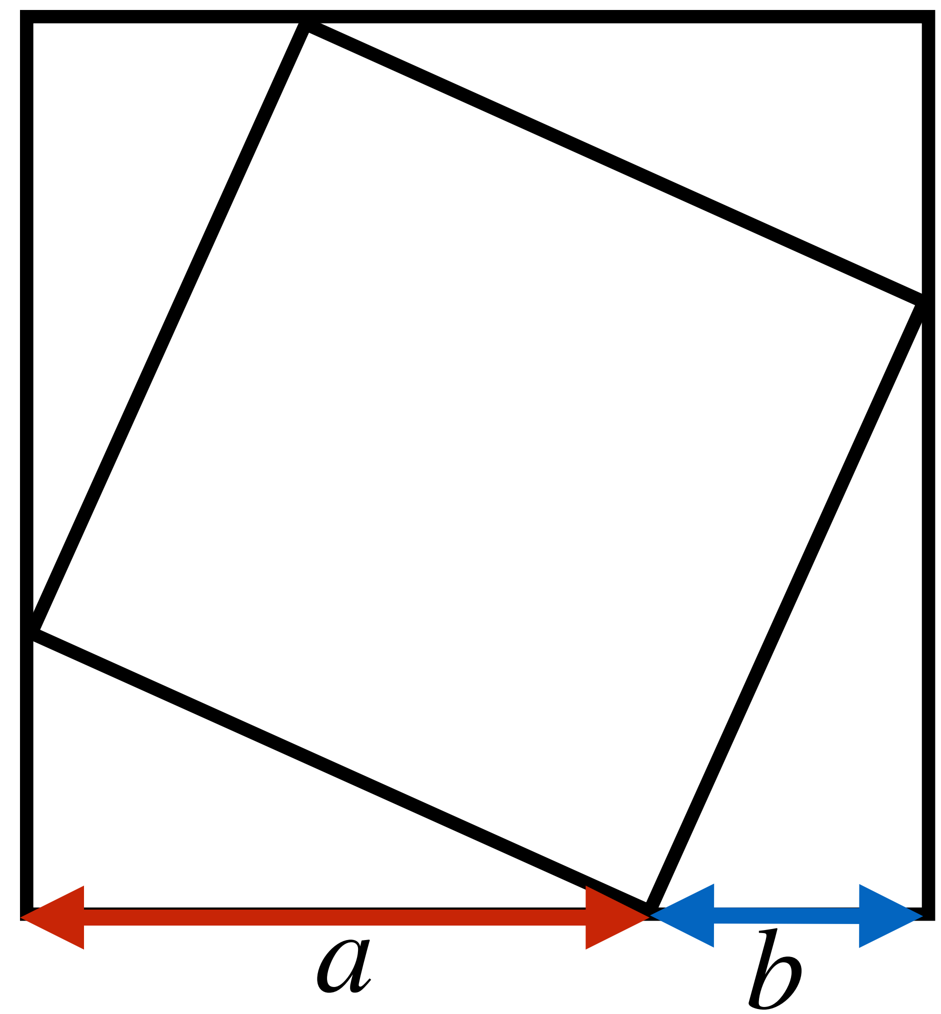

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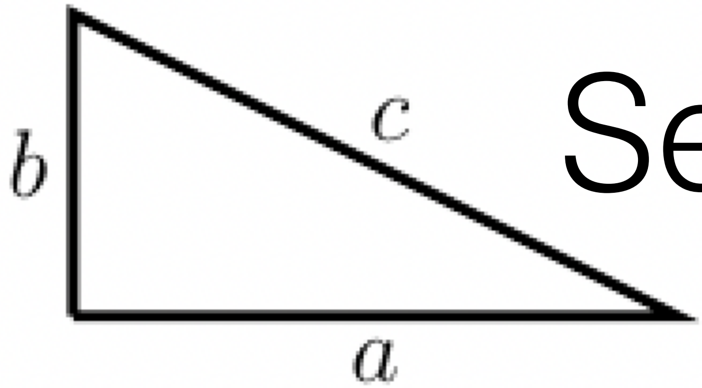

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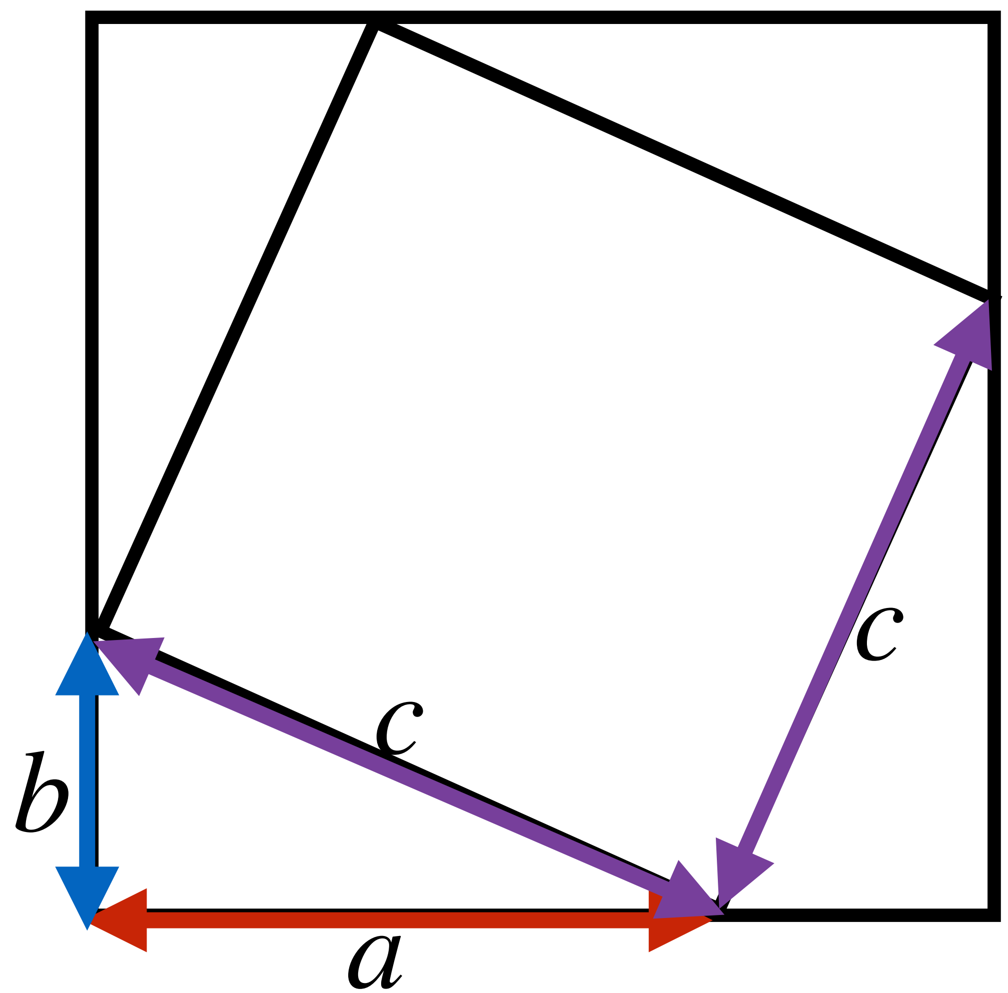

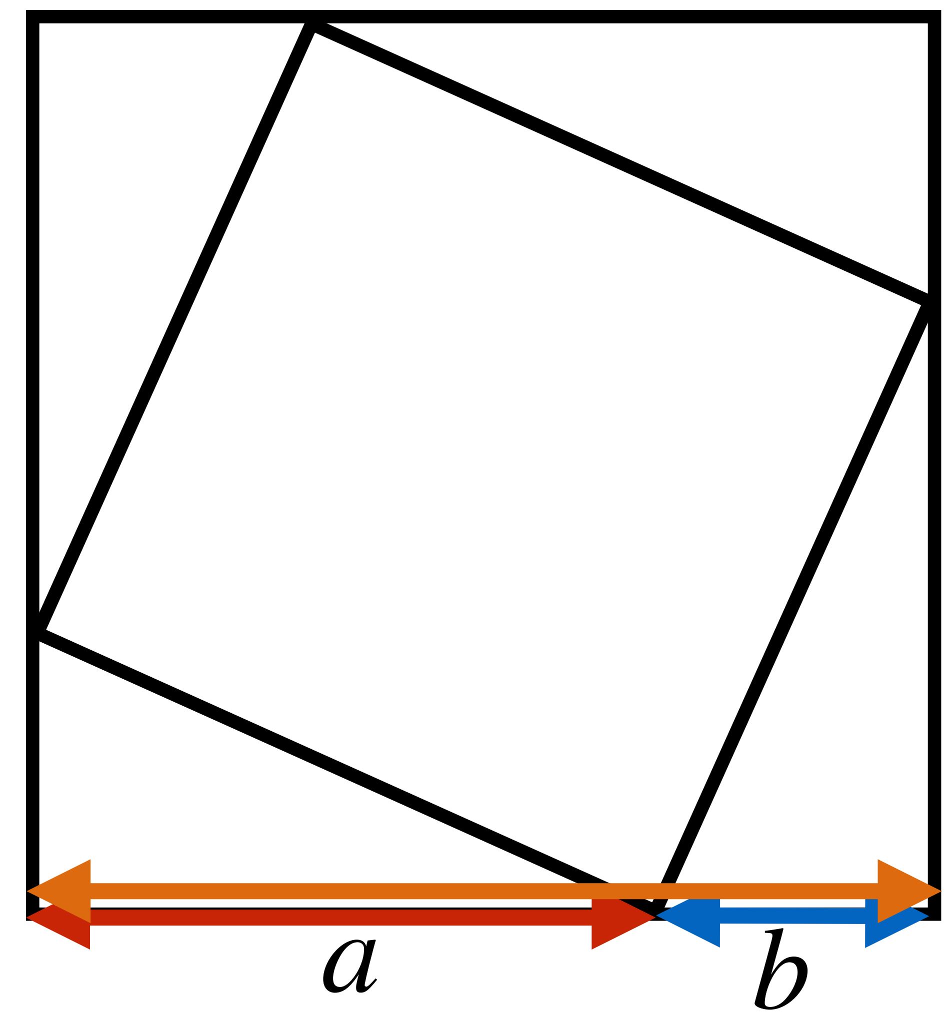

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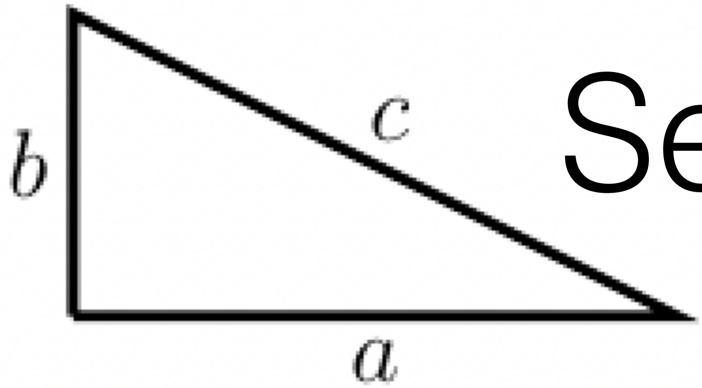

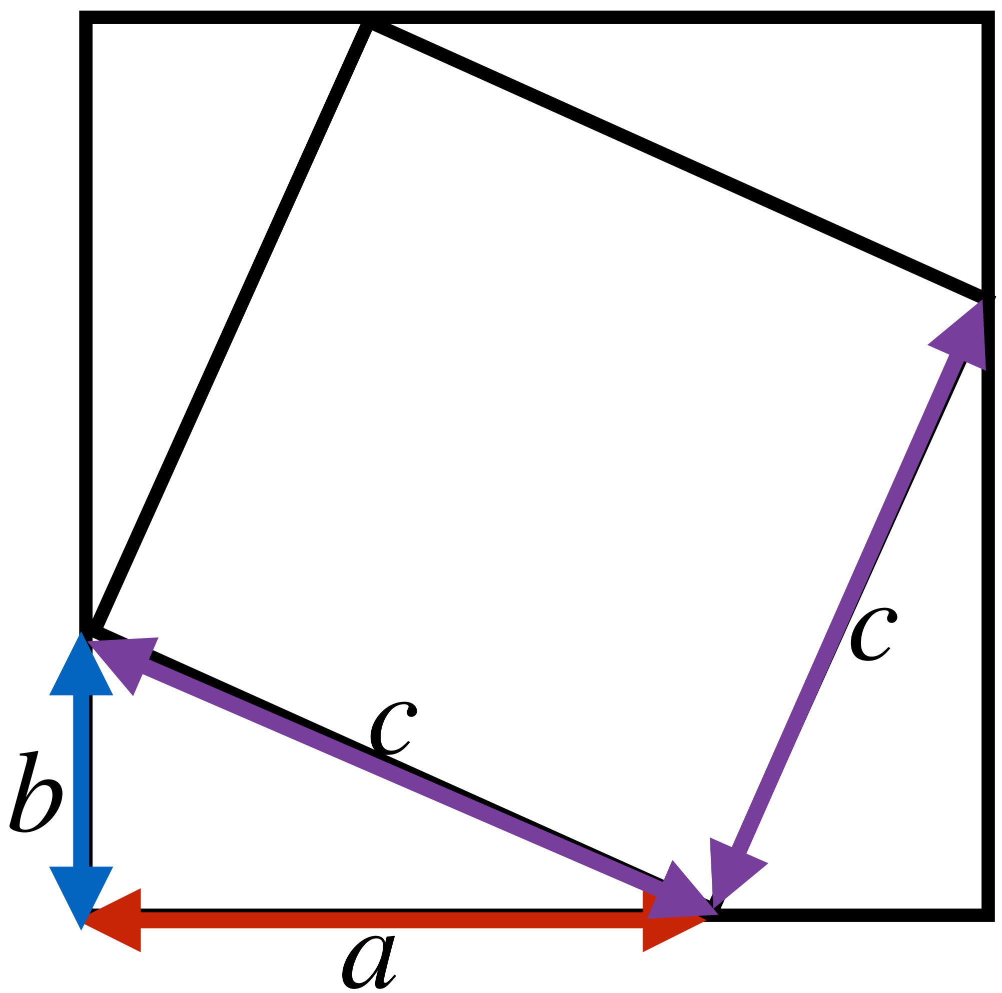

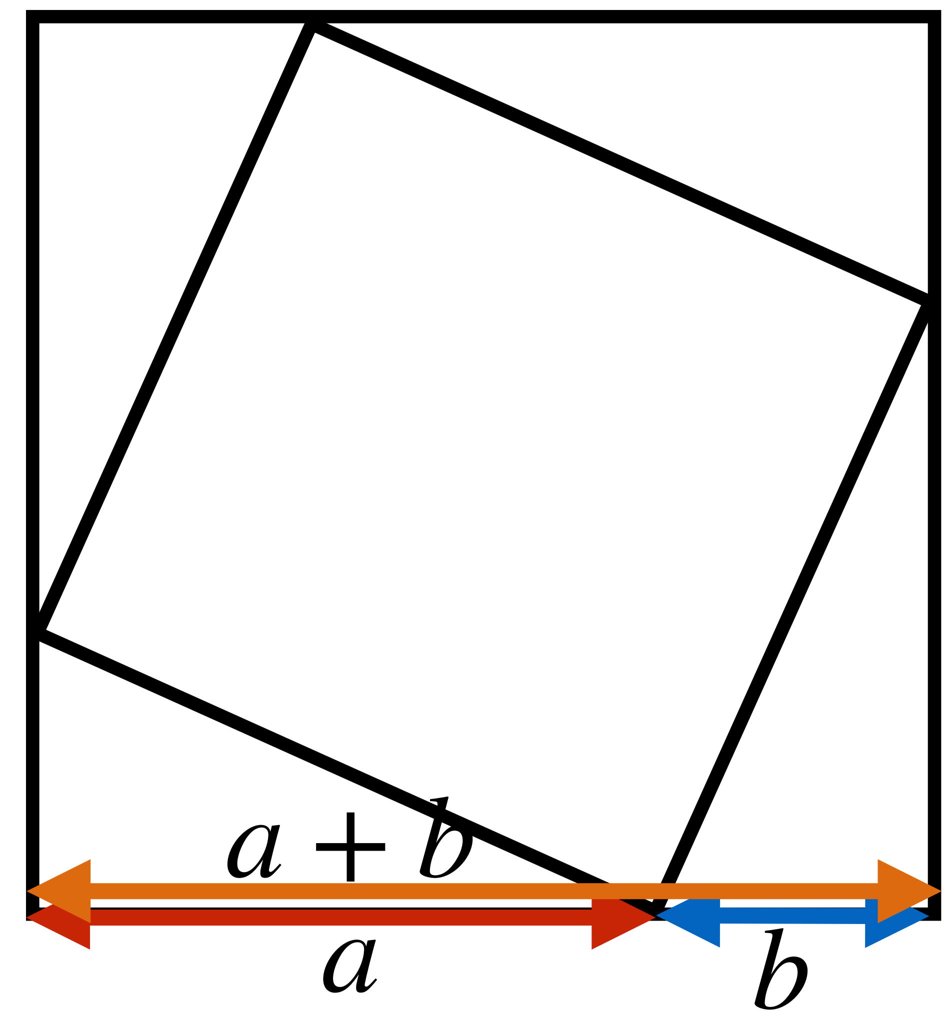

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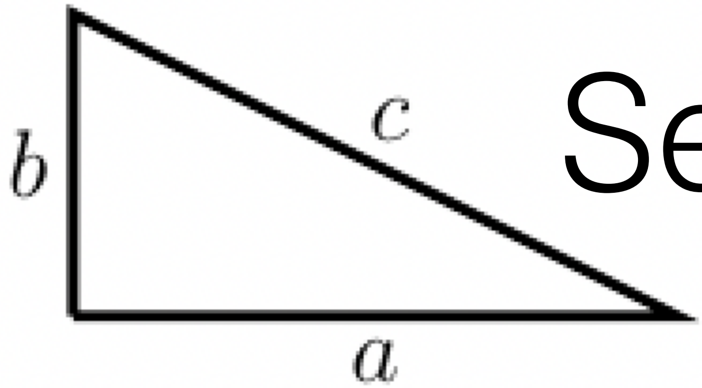

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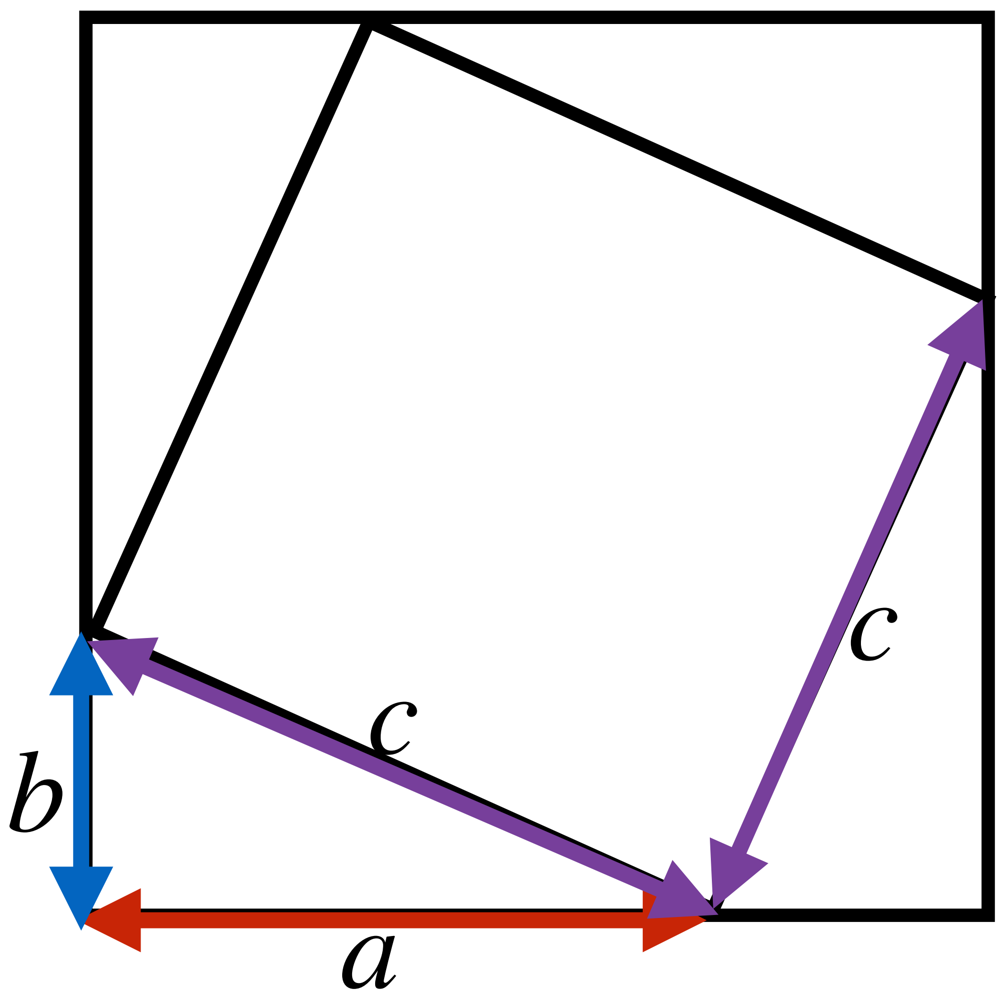

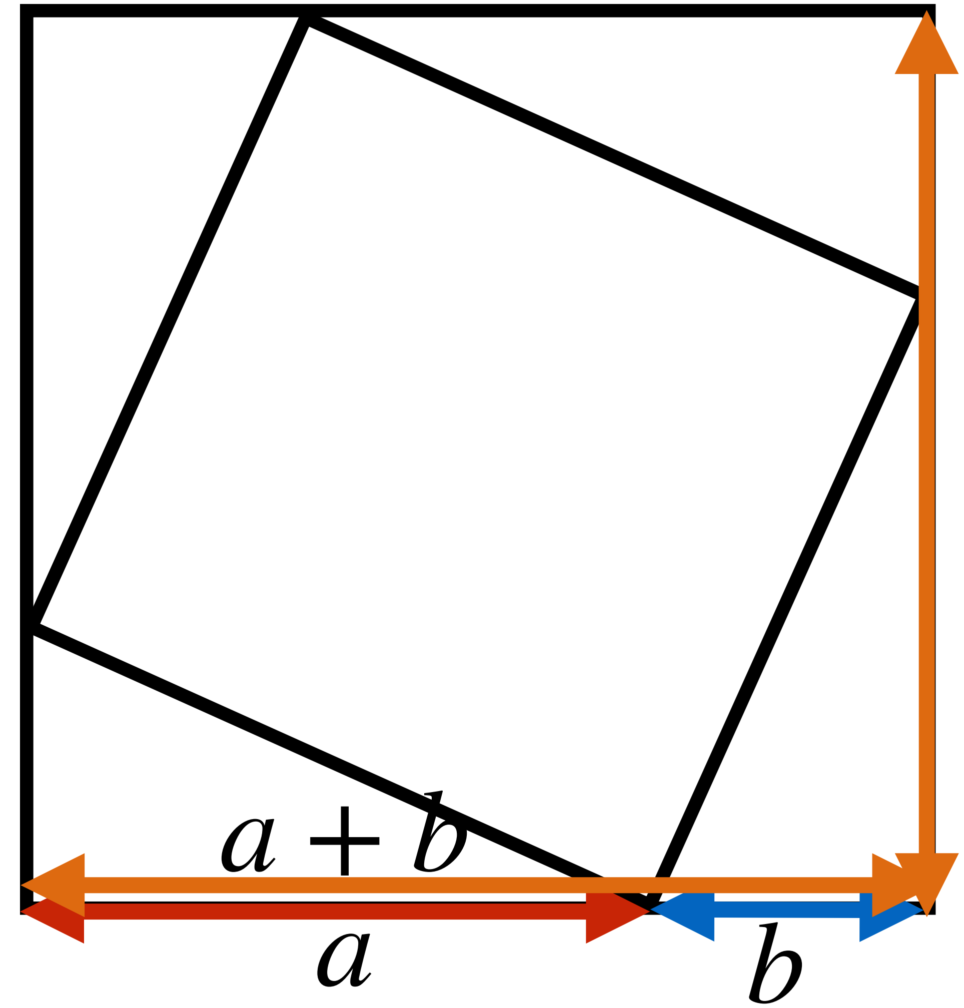

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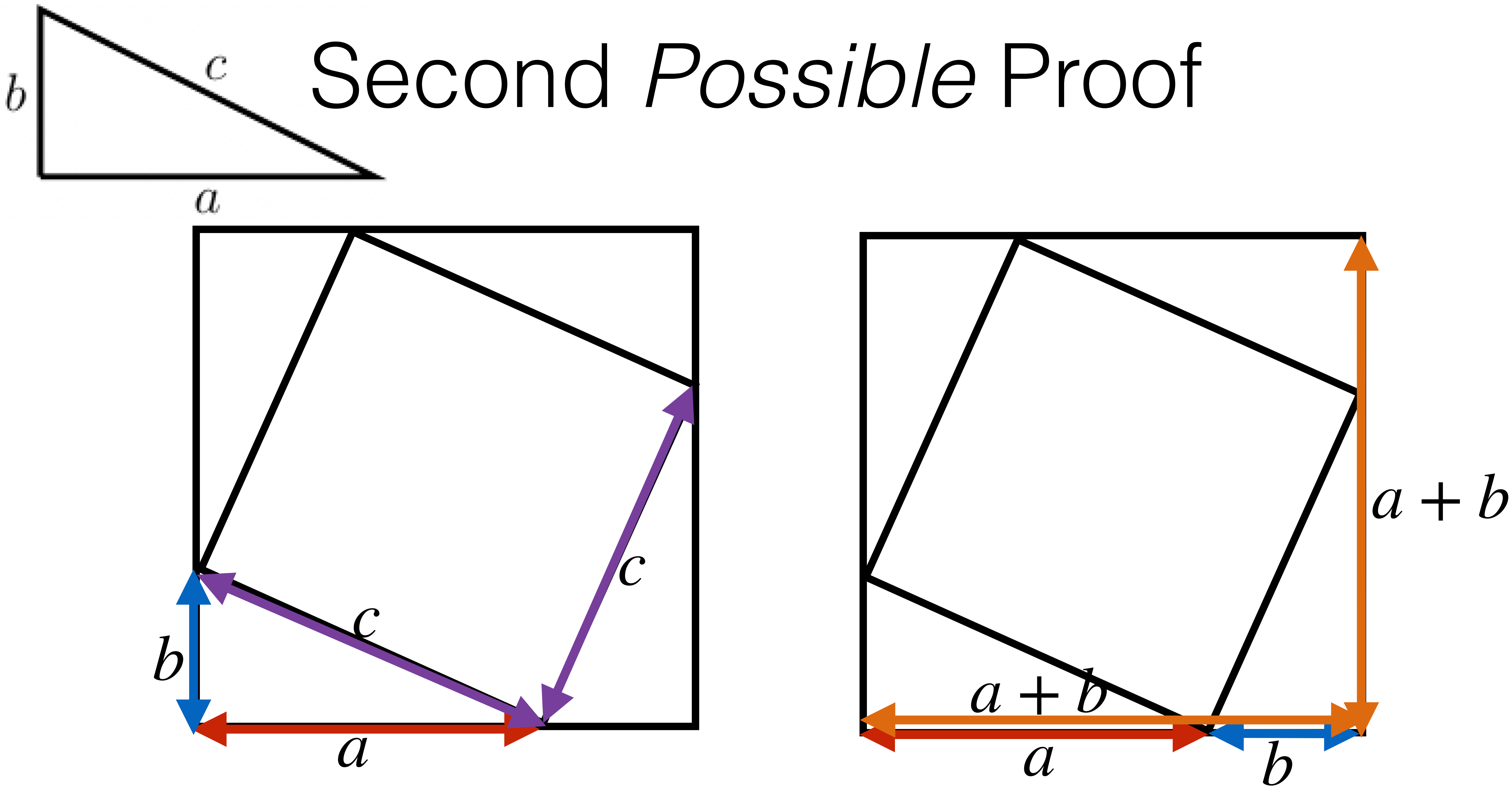

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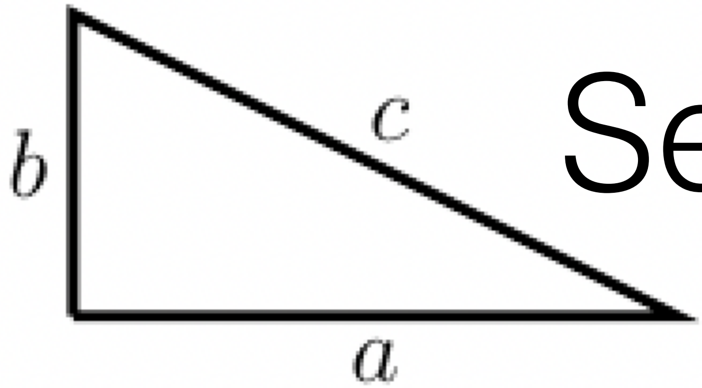

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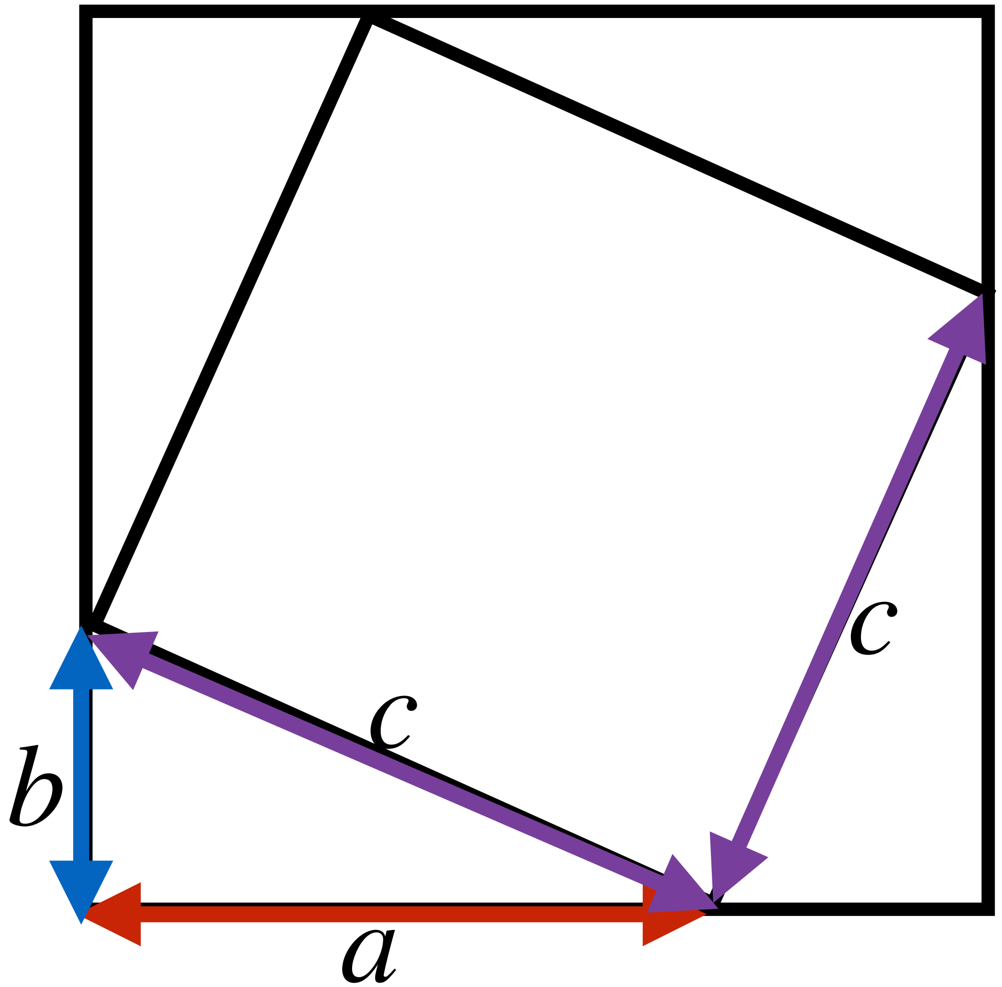
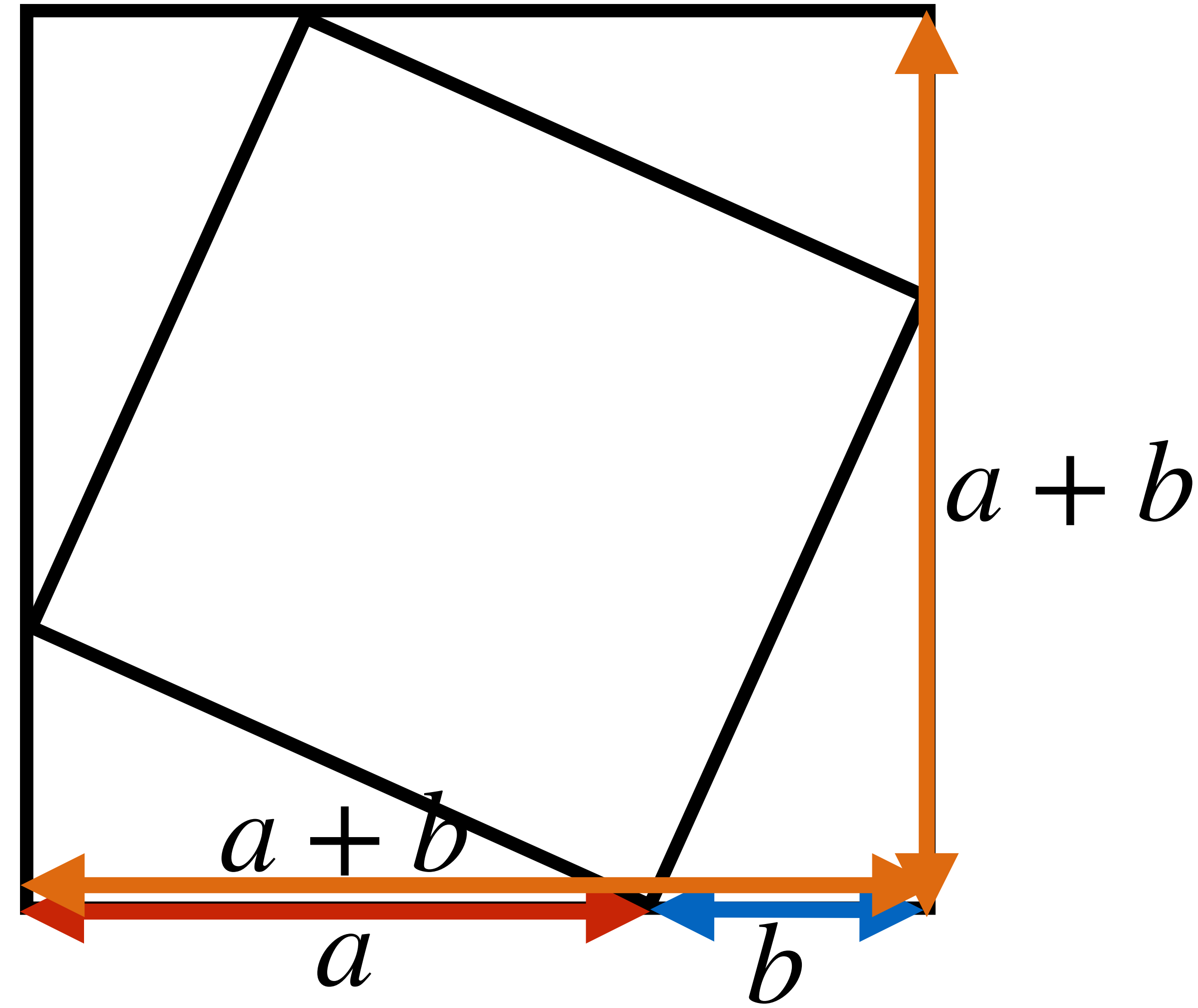
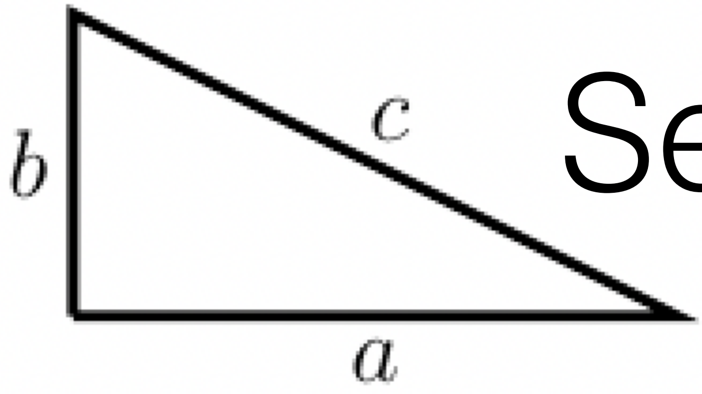




4 triangles + middle square = Whole square

$$4 \cdot \left(\frac{1}{2} a \cdot b\right) + c^2 = (a+b)^2$$



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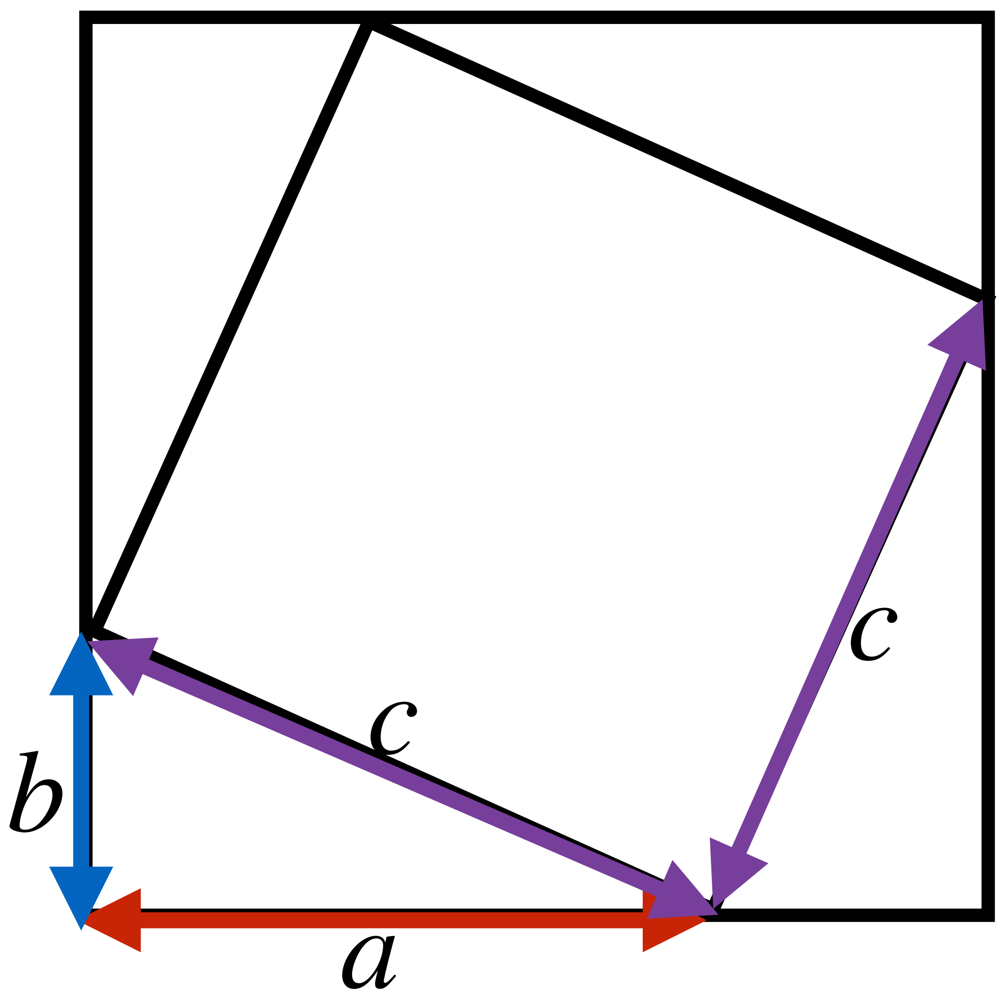

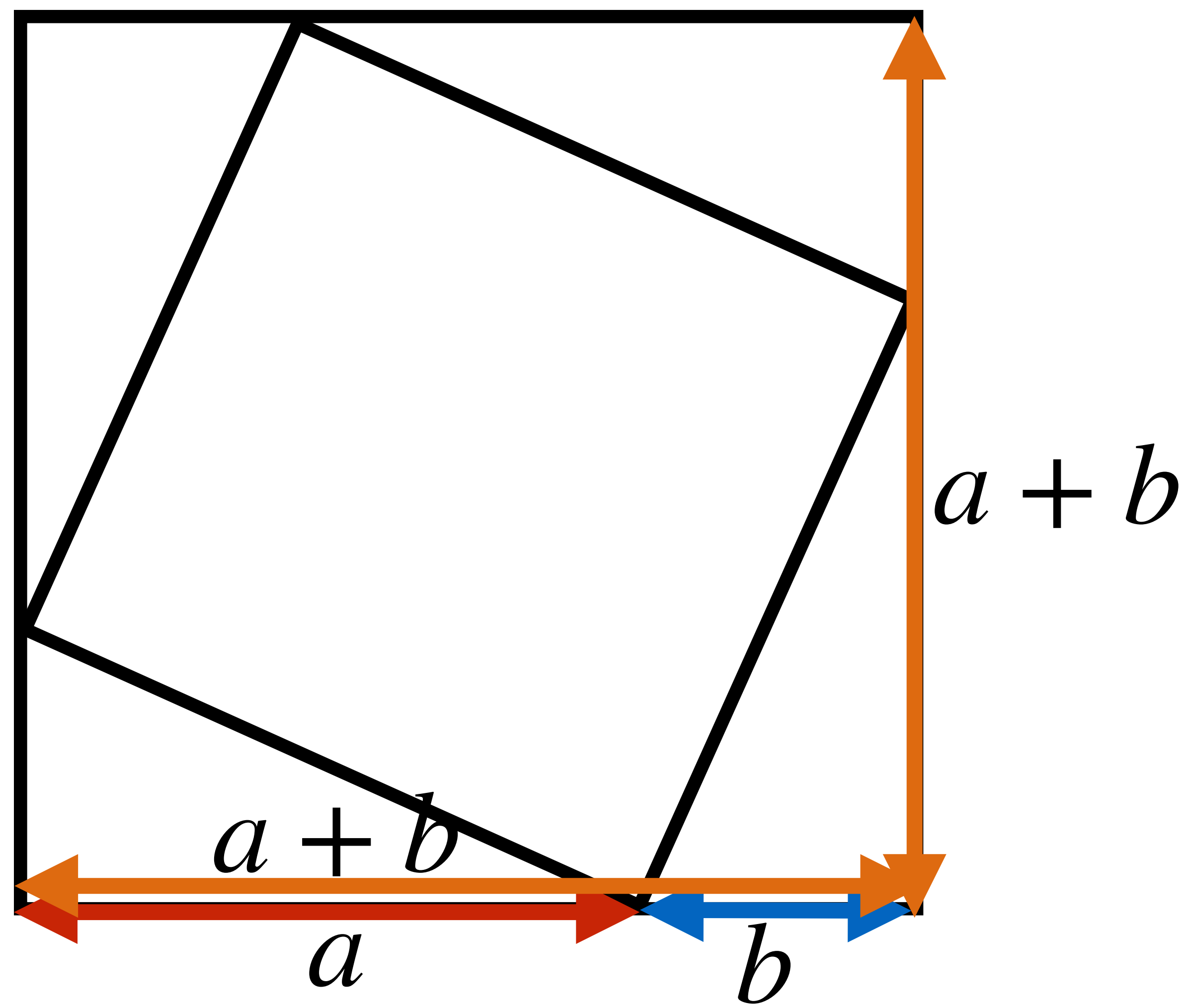

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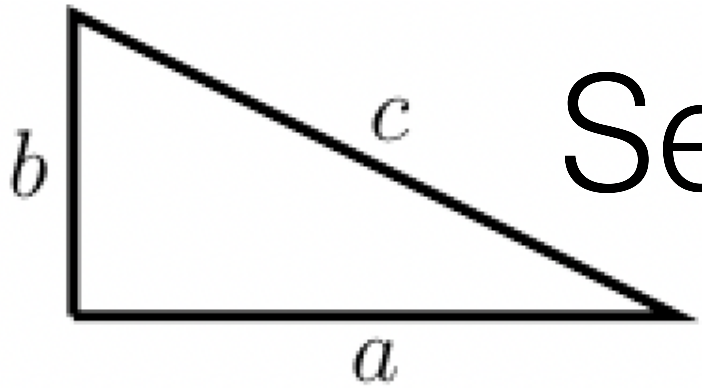
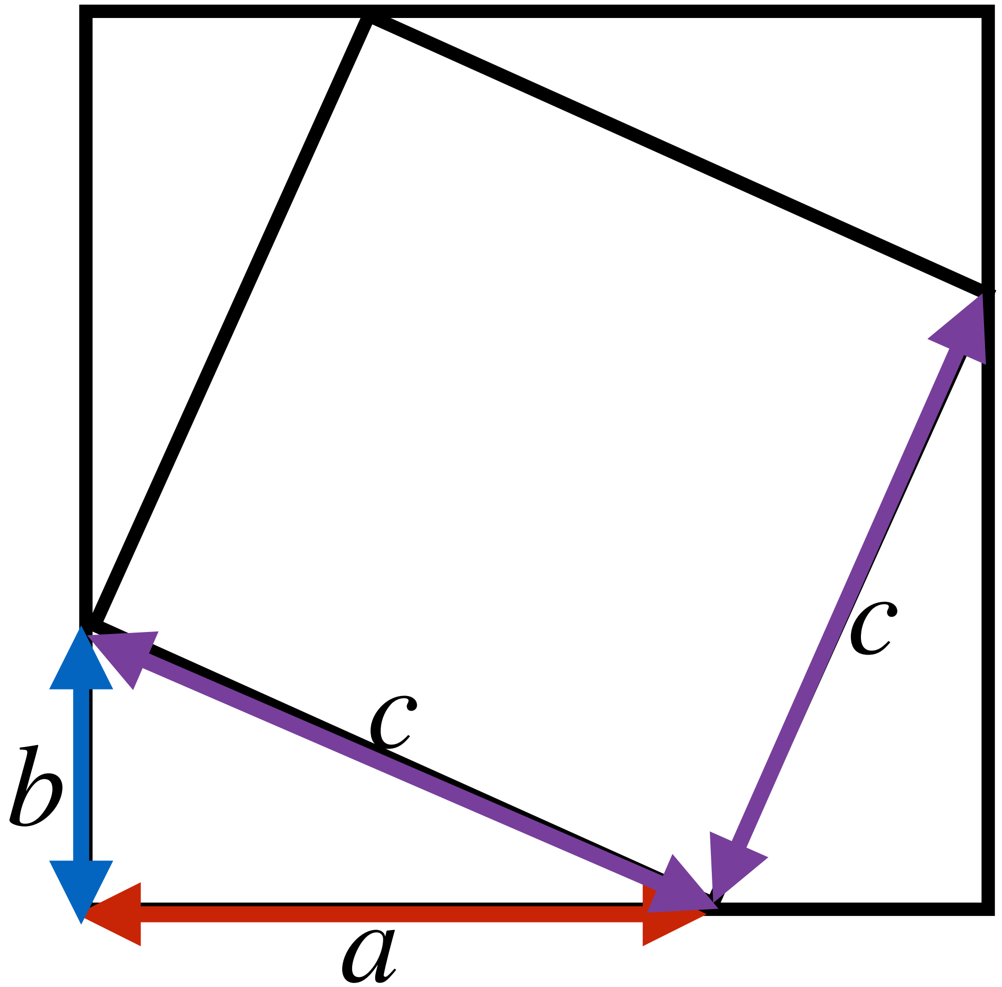
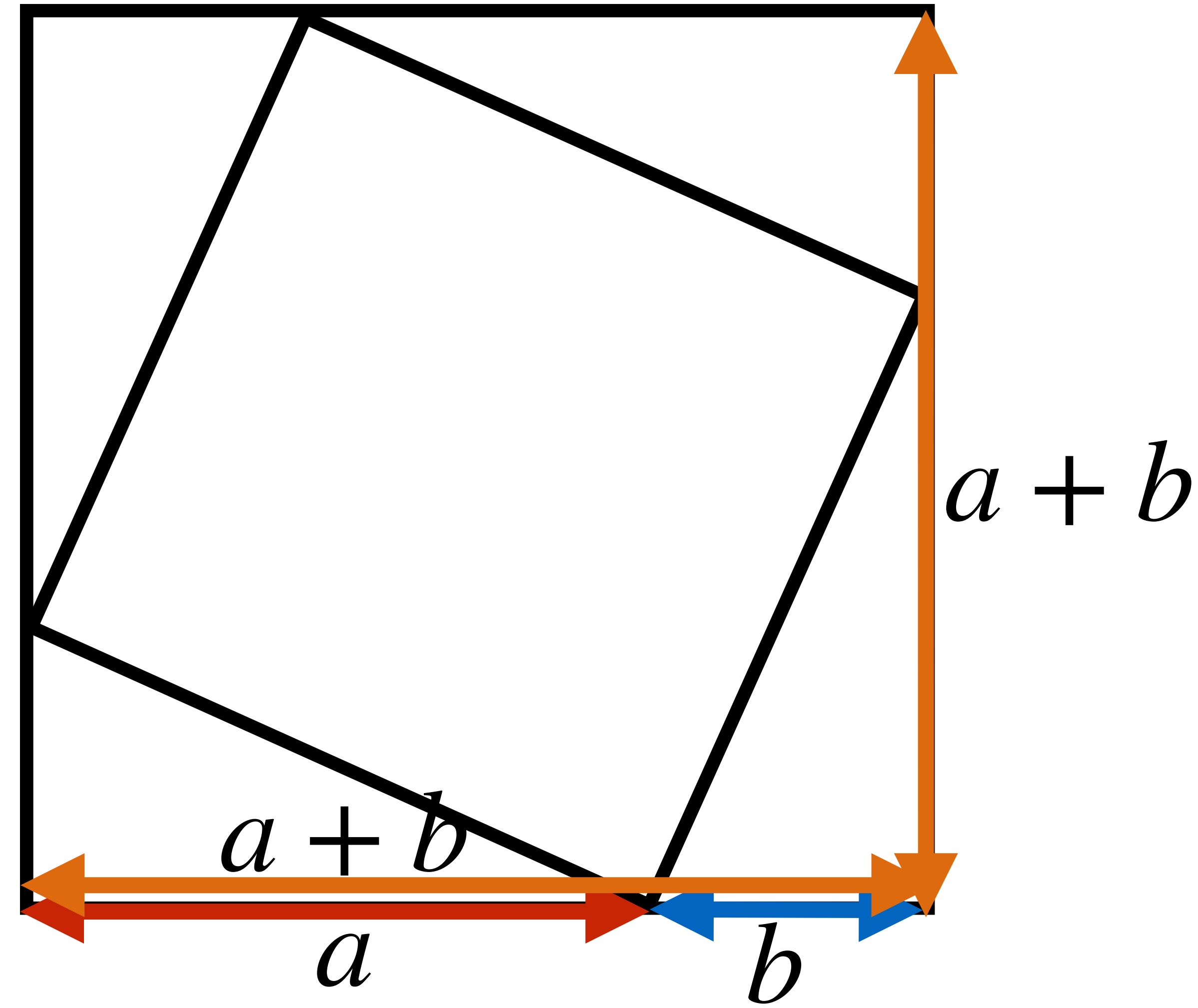
$$4 \cdot \left(\frac{1}{2}a \cdot b\right) \quad + \quad c^2 \quad = \quad (a+b)^2$$

$$2ab \quad + \quad c^2 \quad = \quad a^2 + b^2 + 2ab$$

Second *Possible* Proof







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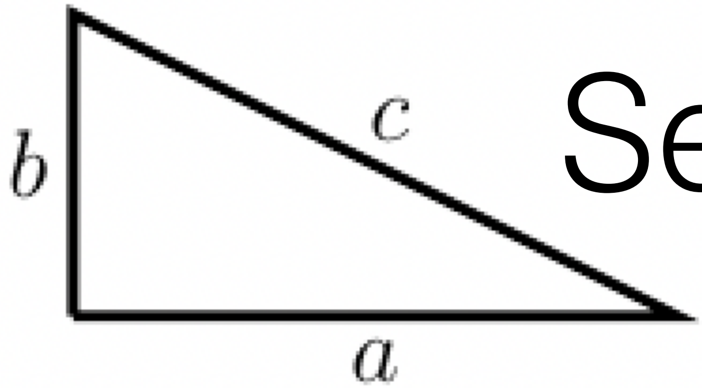

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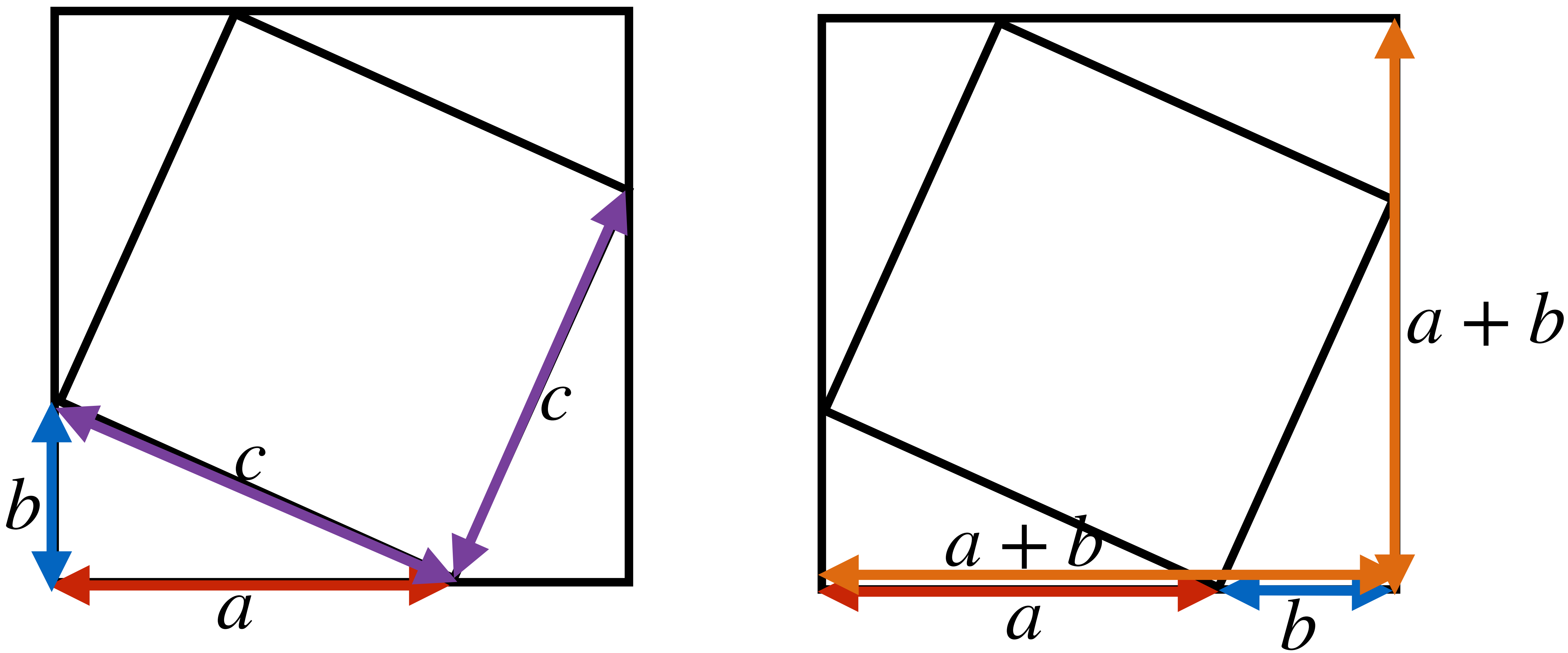

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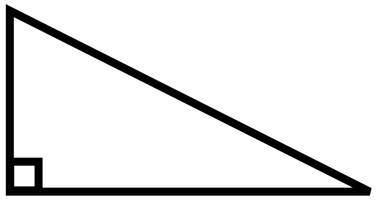
Generalized Pythagorean



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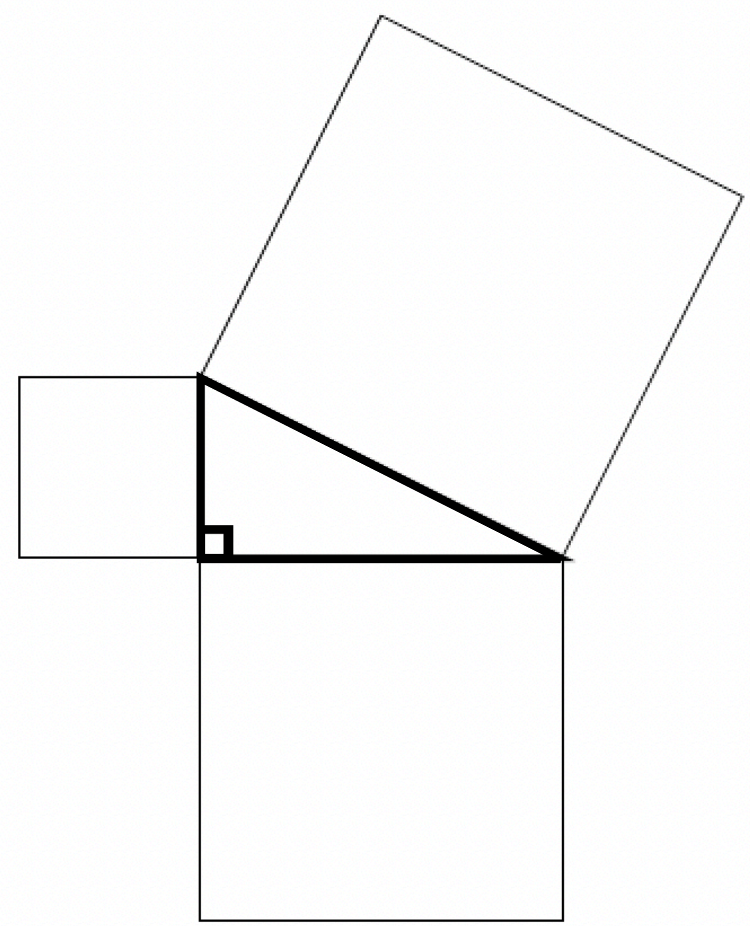
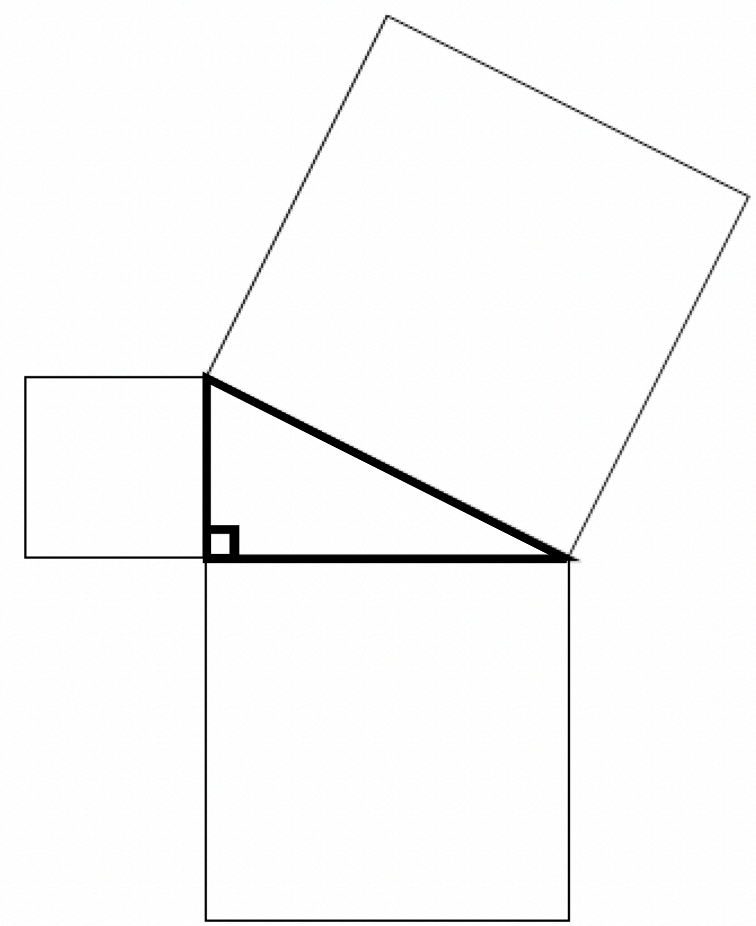


Generalized Pythagorean

If you attach *any* shape to the sides of a right triangle, the area inside the largest one equals the sum of the areas inside the smaller two.



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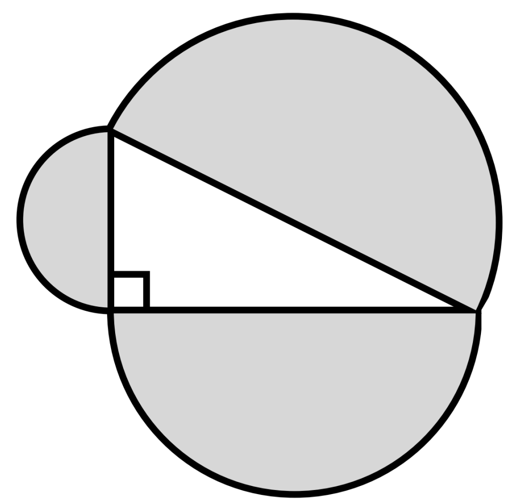

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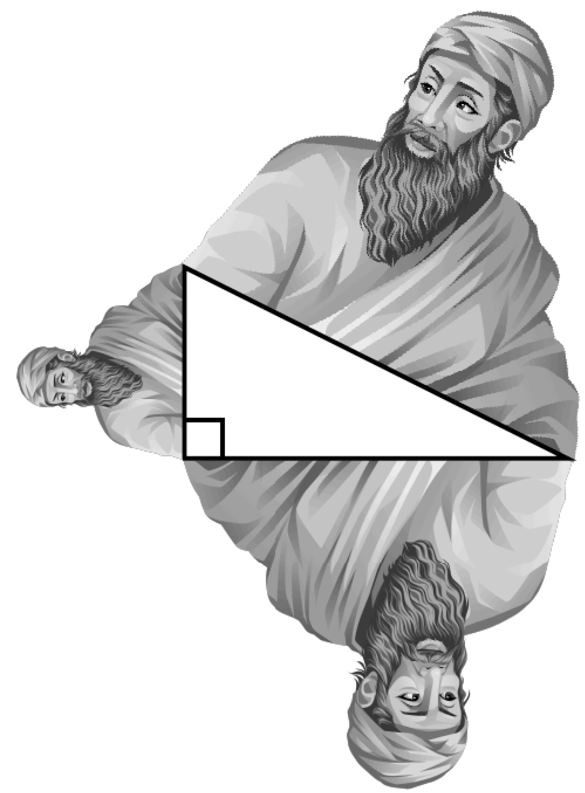

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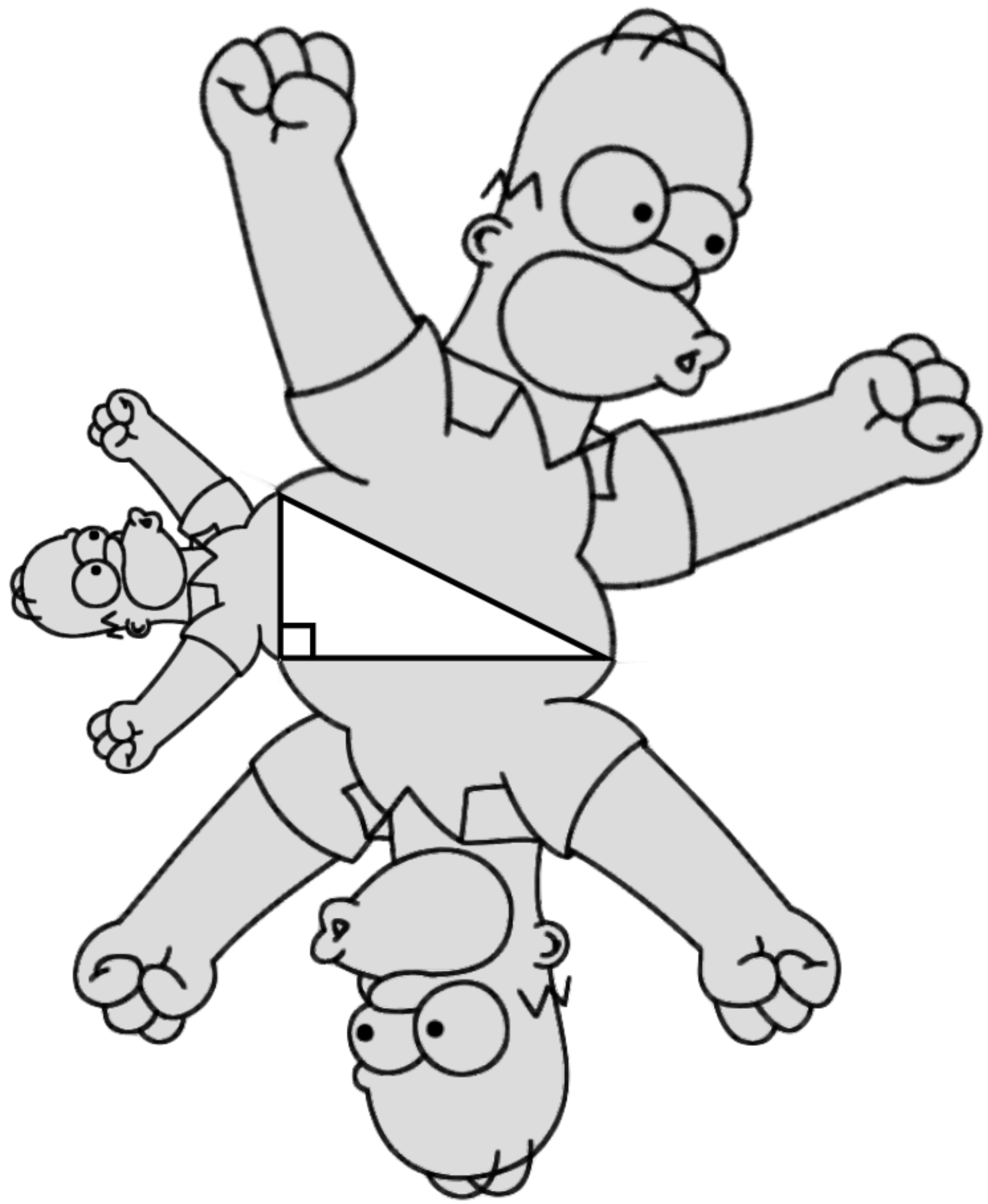
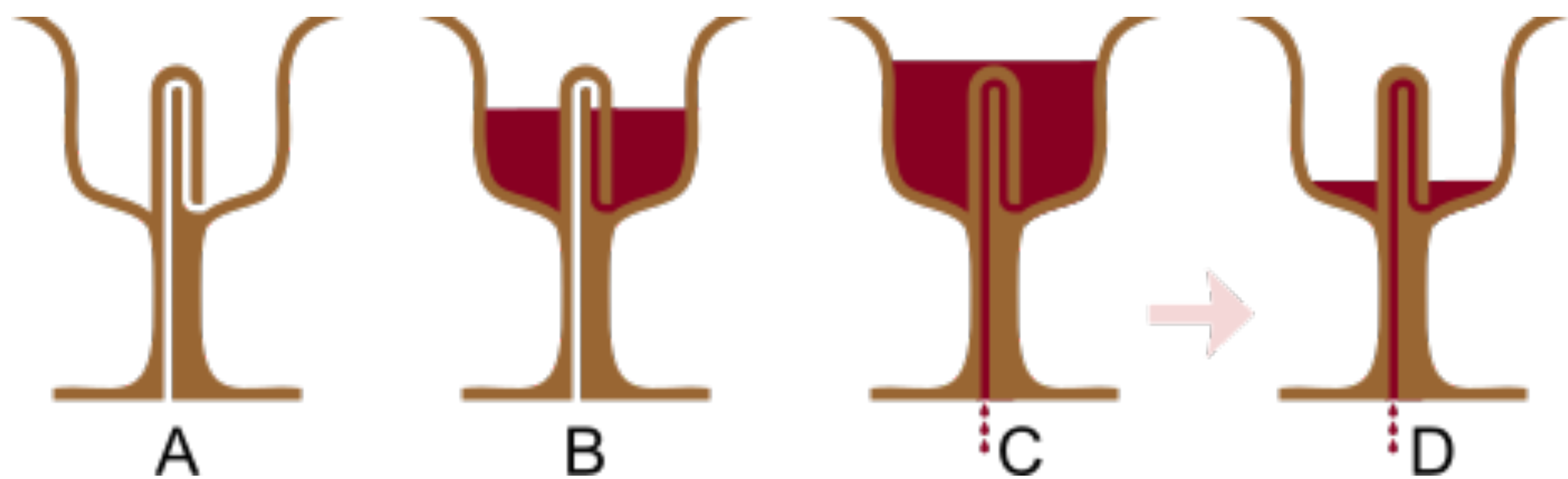
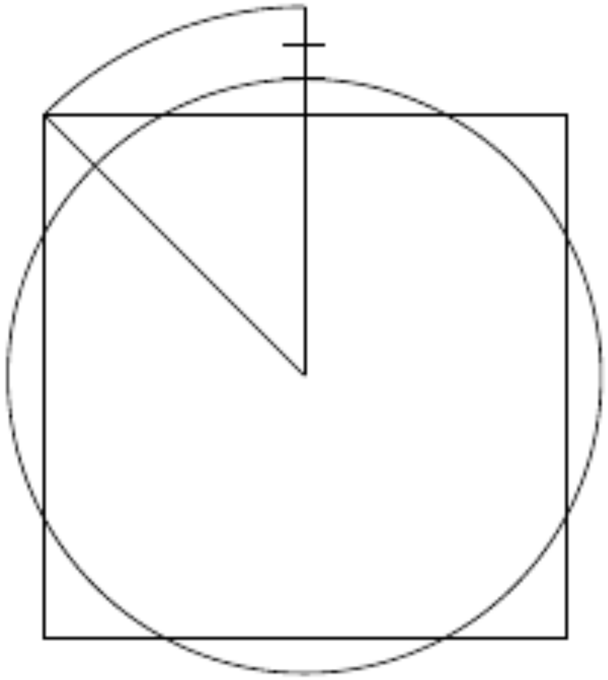
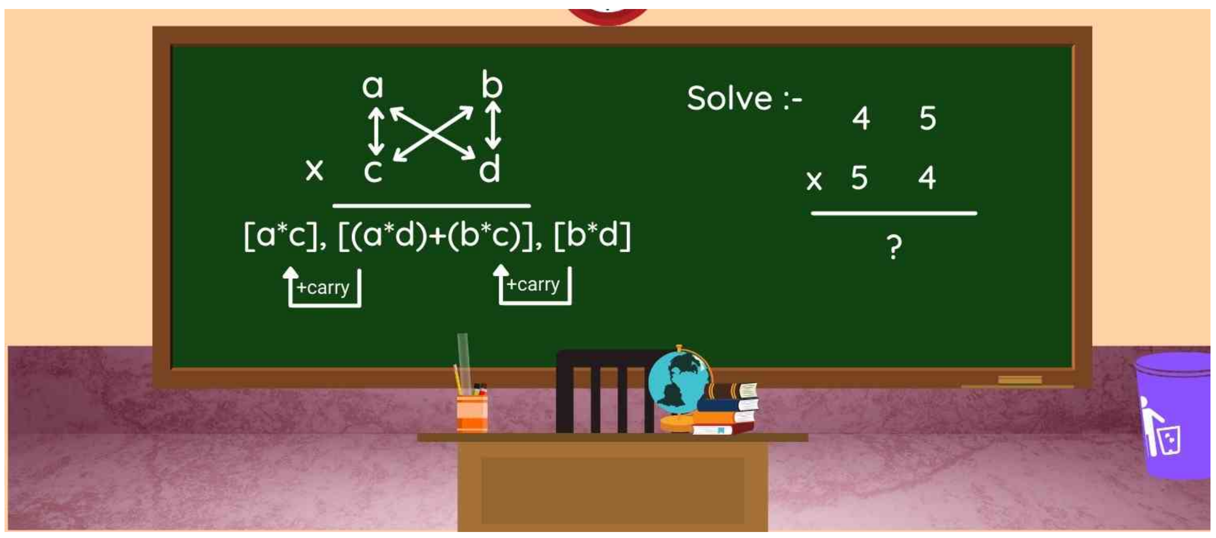


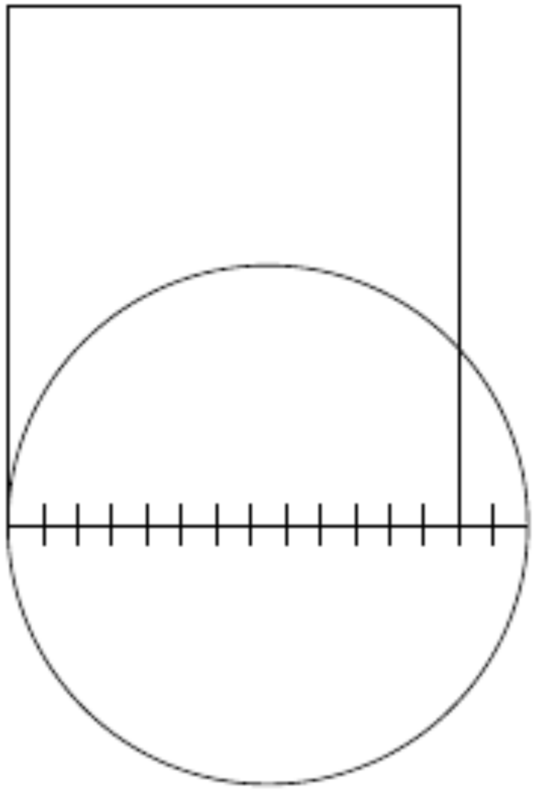
Pythagorean Cup

- A practical joke device whose invention is credited to Pythagoras.
- Also called a “justice cup”

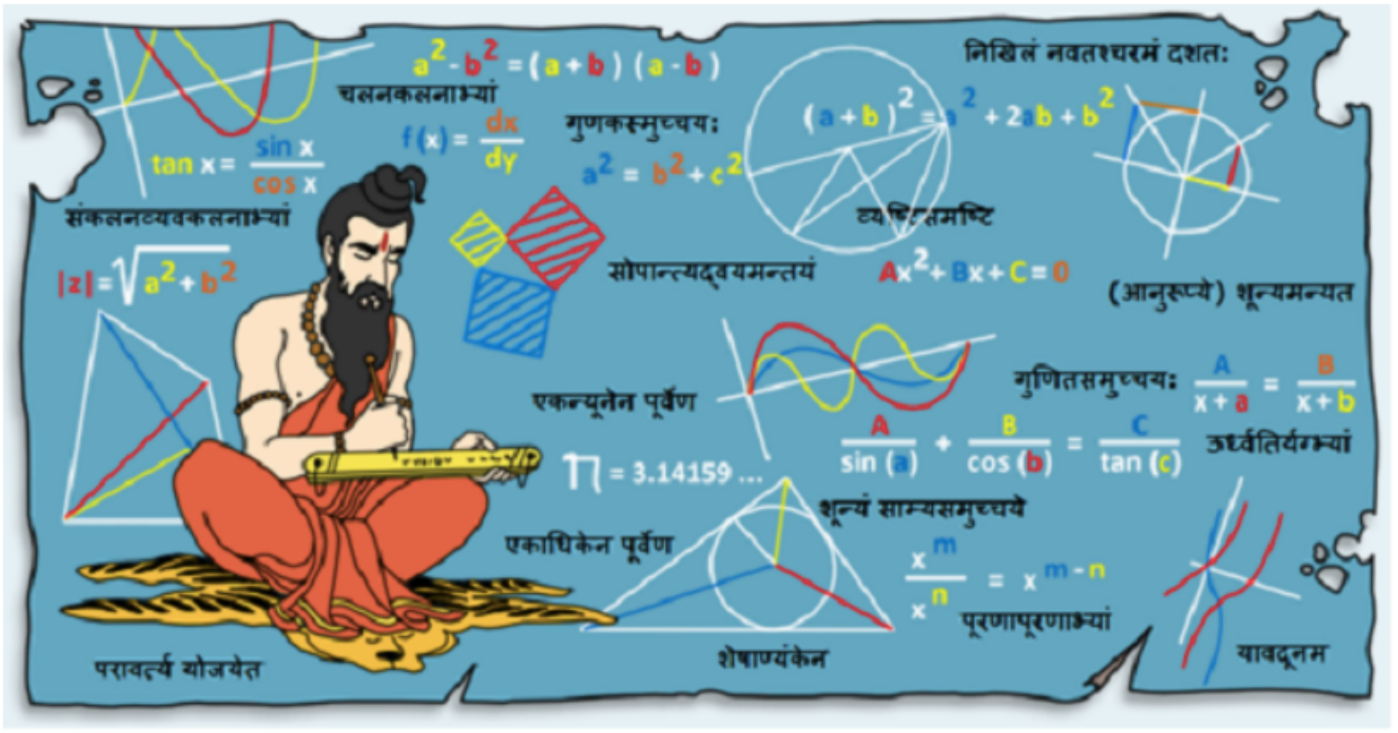


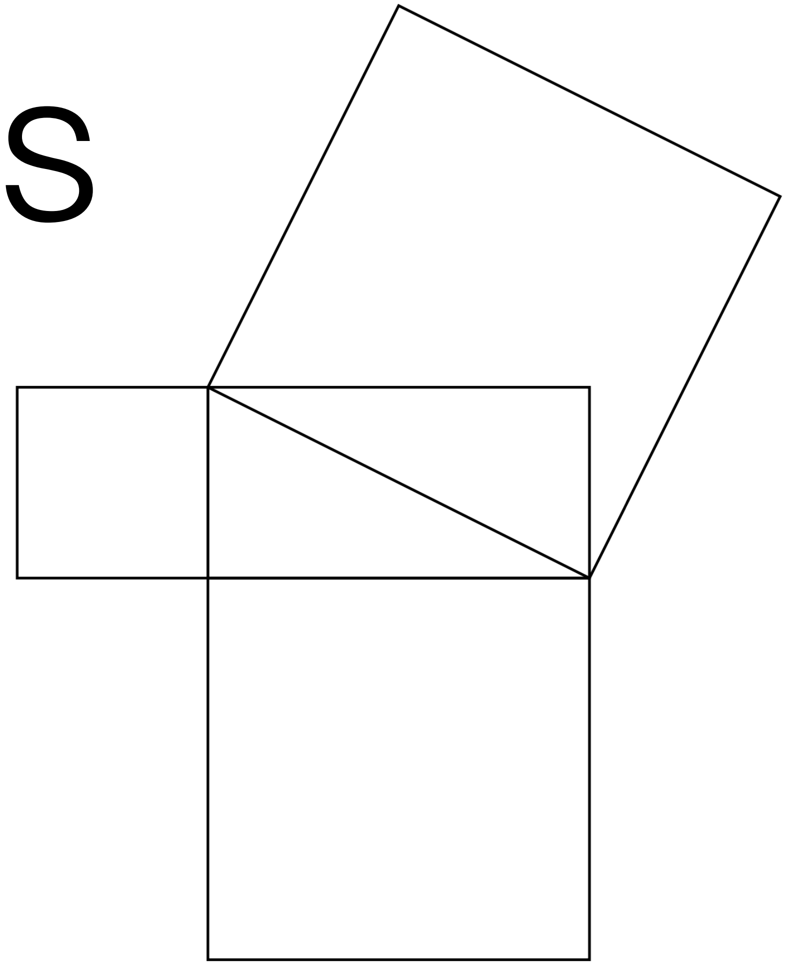






Ancient Indian Mathematics





Ancient India

Ancient India

- The ancient Indians used the Pythagorean theorem well.

Ancient India

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- There is no evidence they proved it, but they used it so well that they may have. They certainly understood it extremely well.

Ancient India

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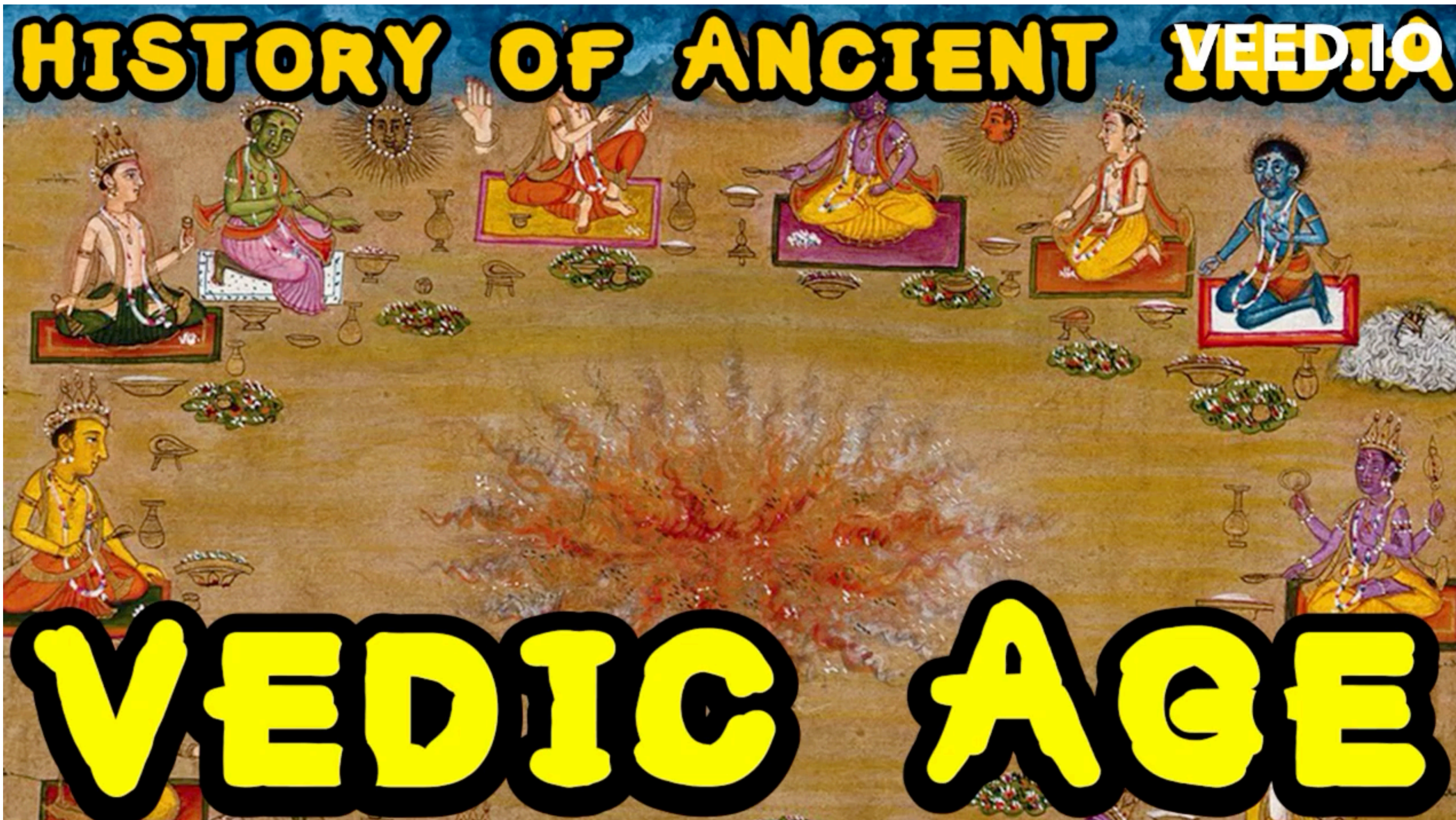

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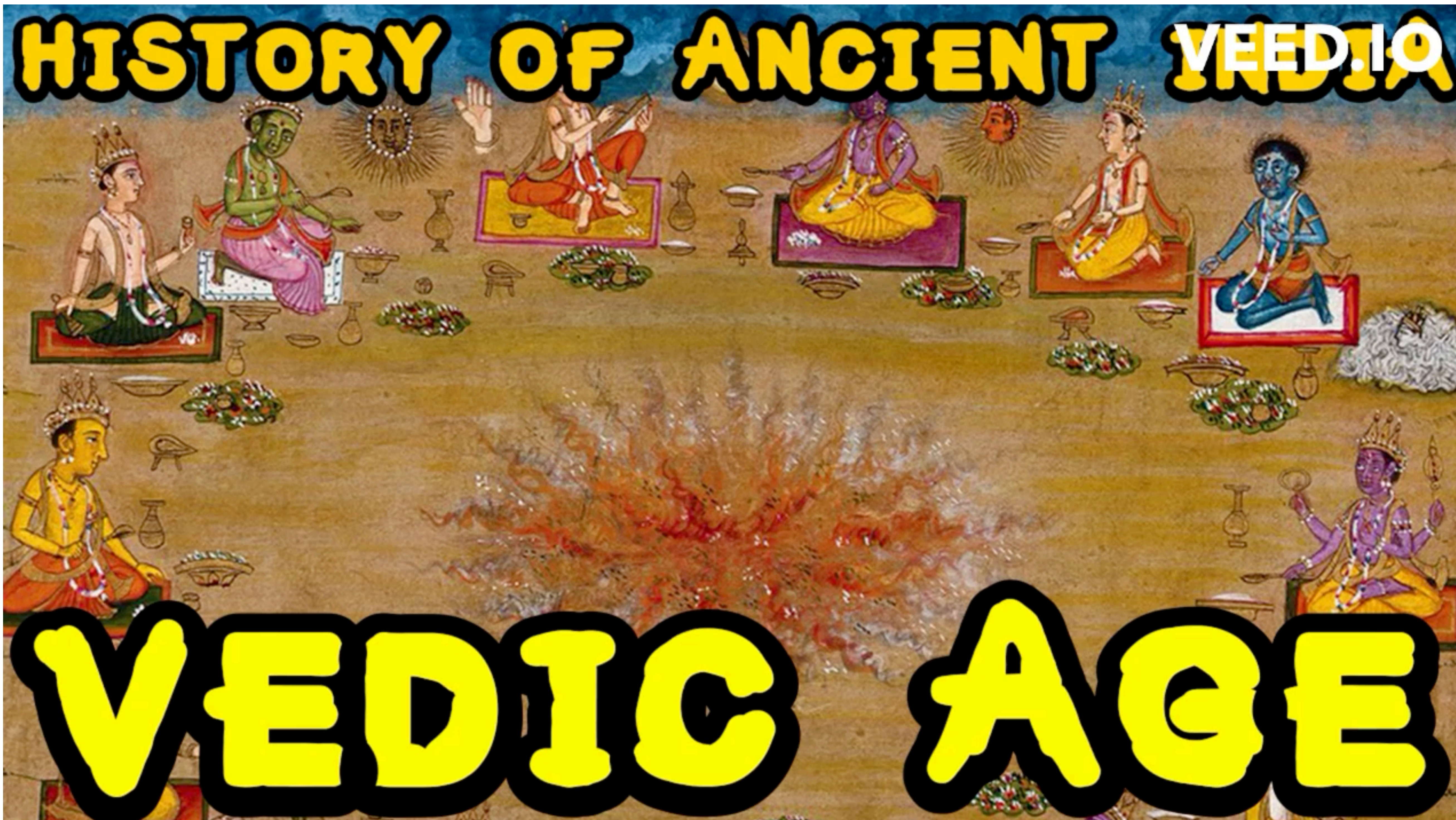

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- Reminder:
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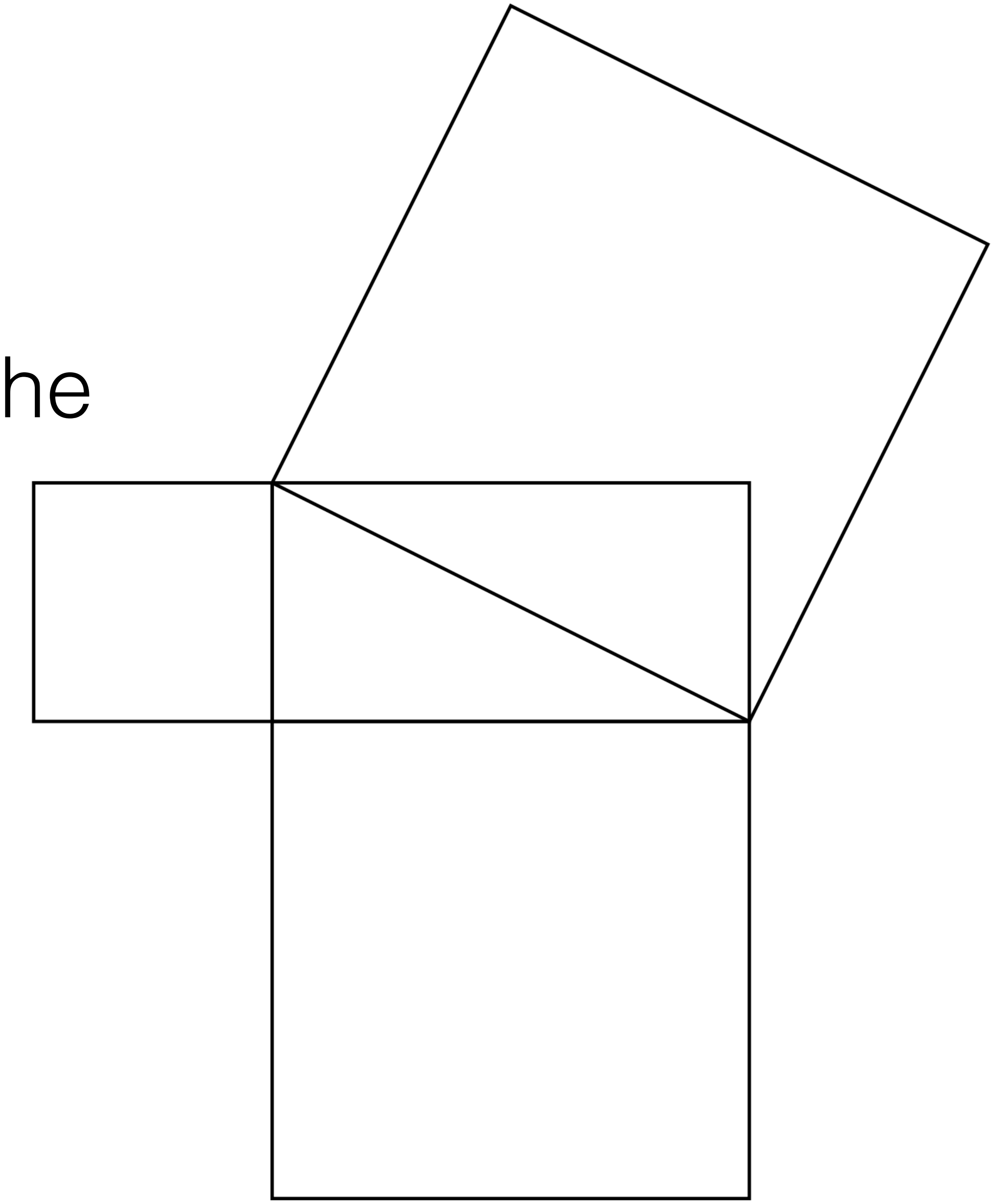

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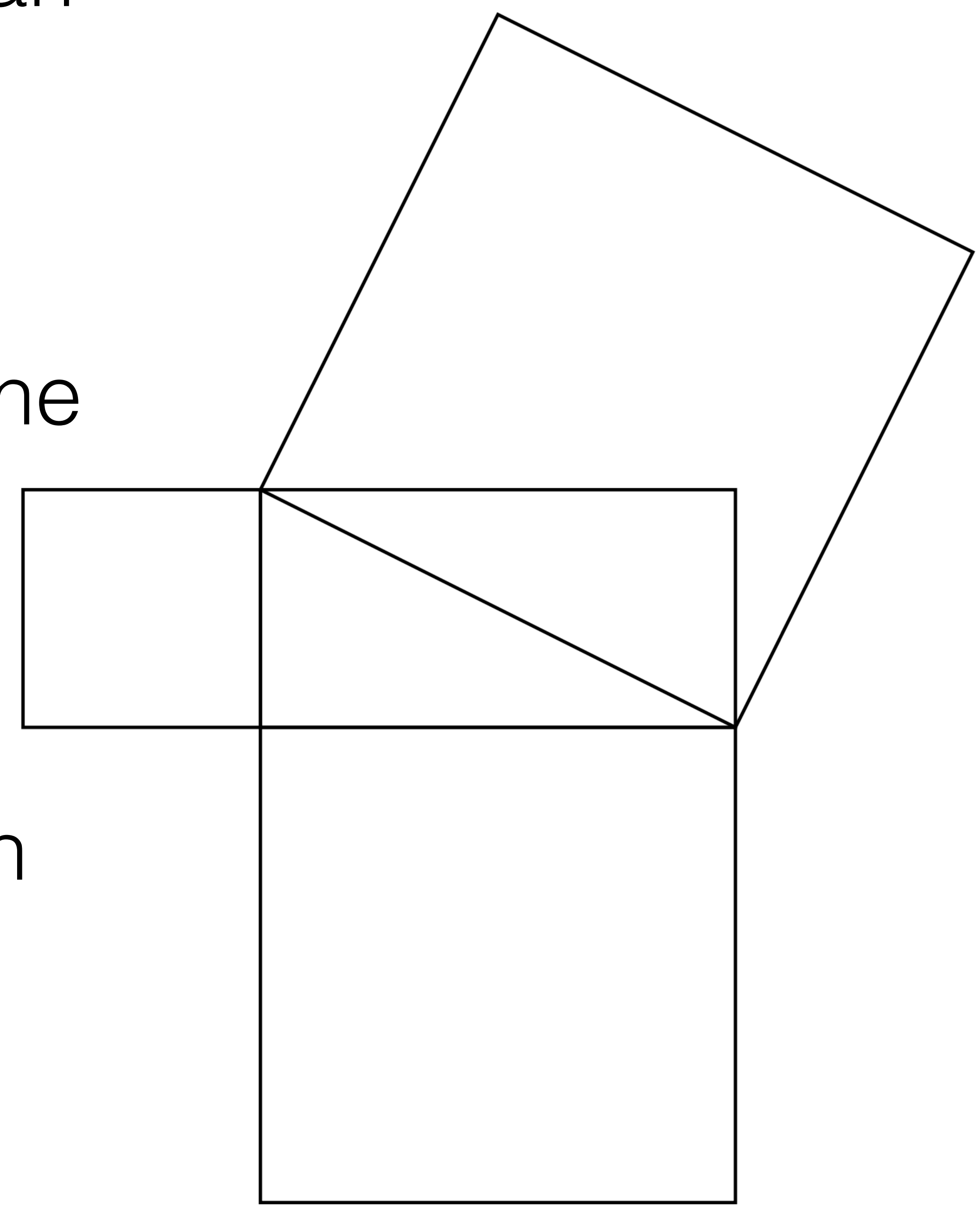

- Three examples of using this in their *sulbasutras*:

Ancient India

- **1. Squaring a pair of squares.** Meaning: Given a pair of squares, create a third square whose area is equal to the sum of the other two.

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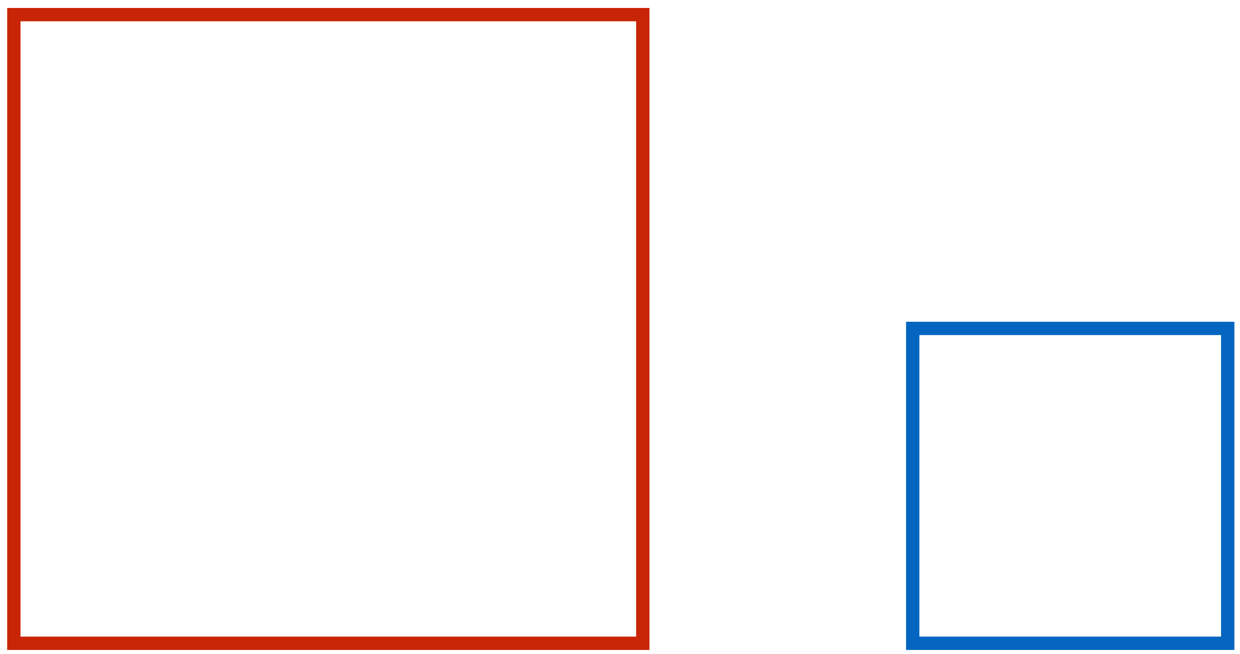

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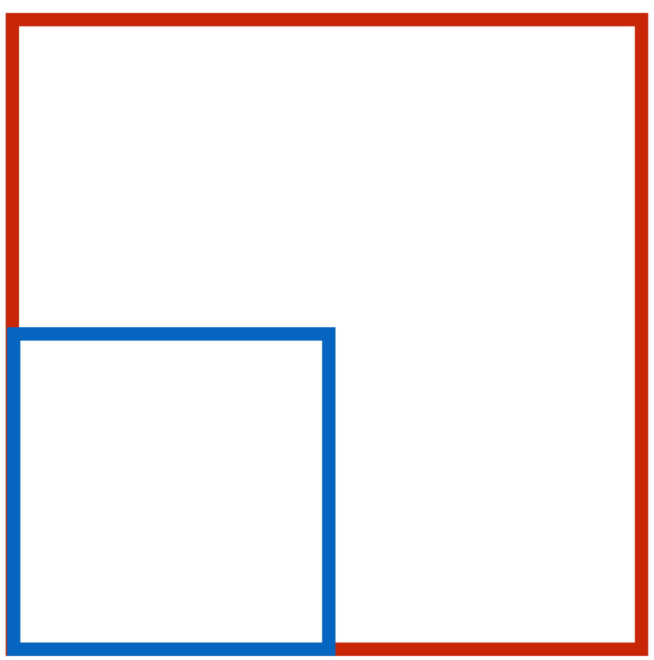

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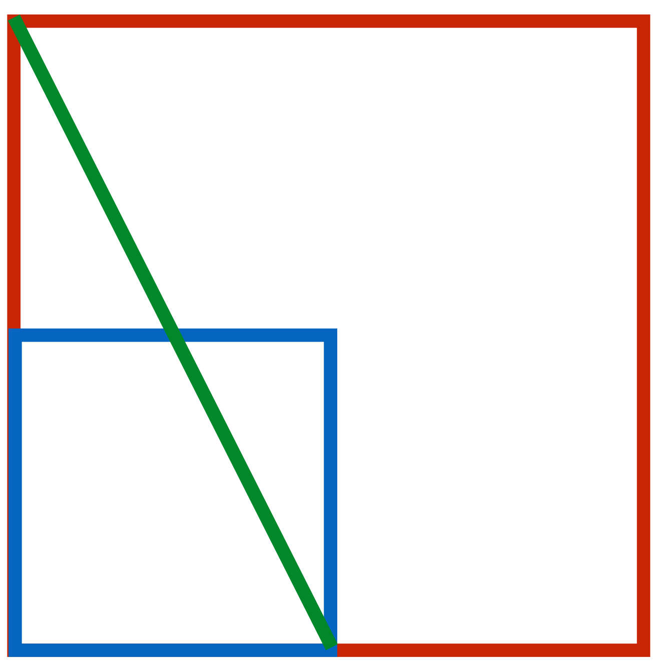

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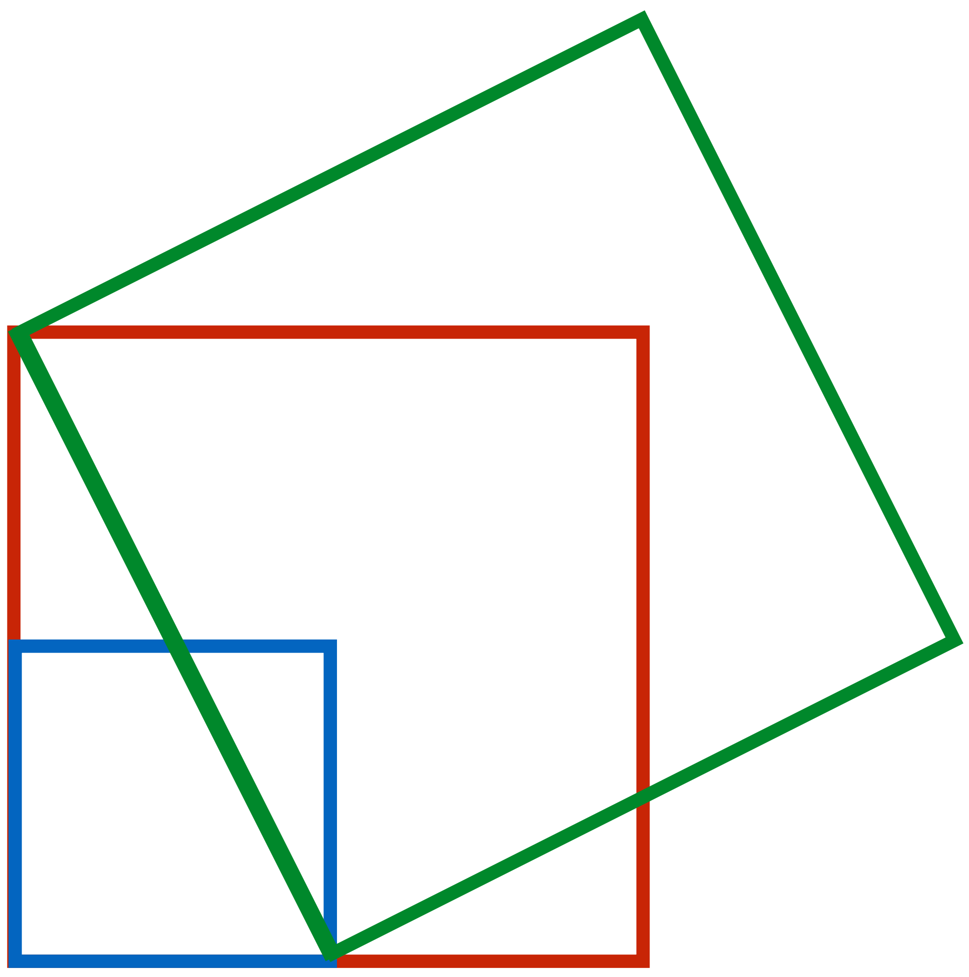

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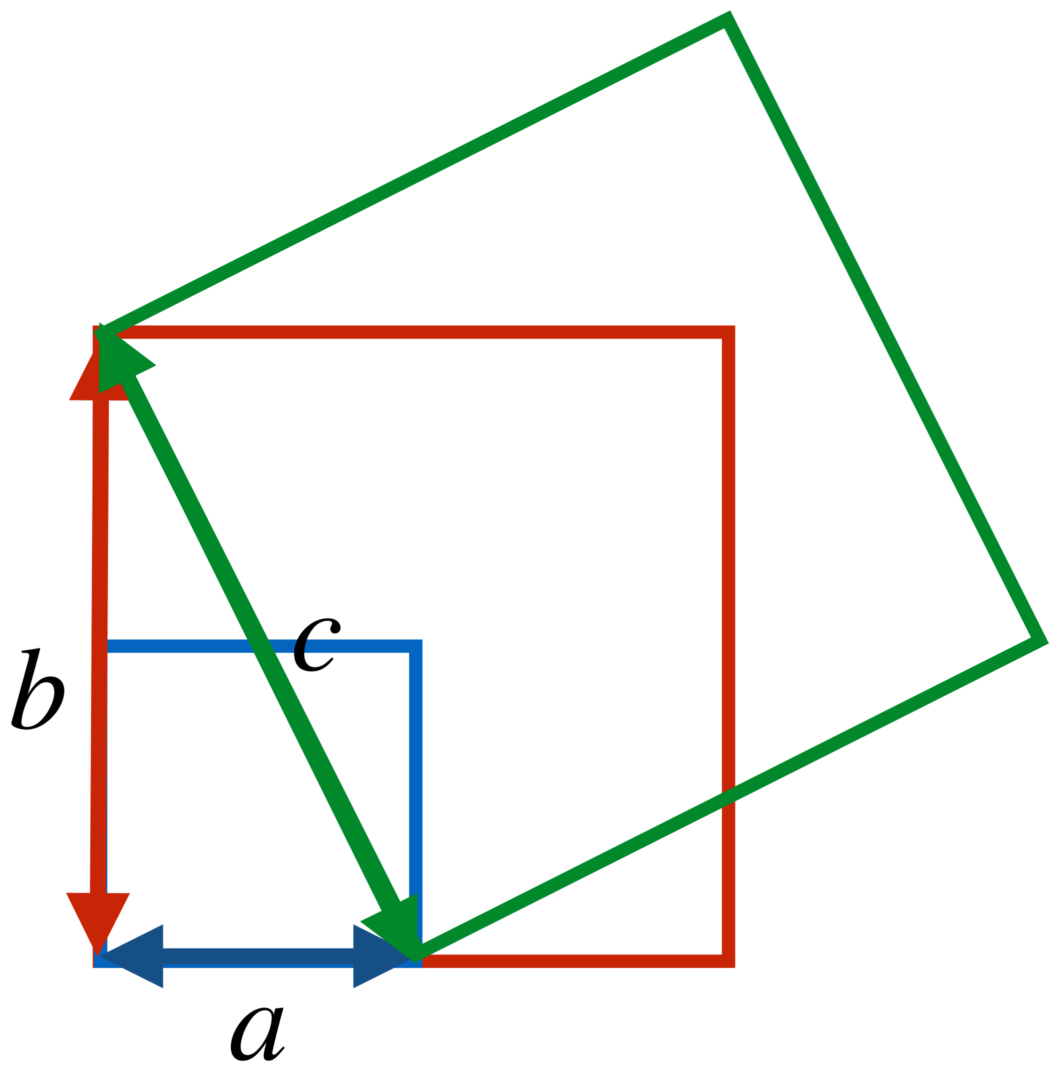

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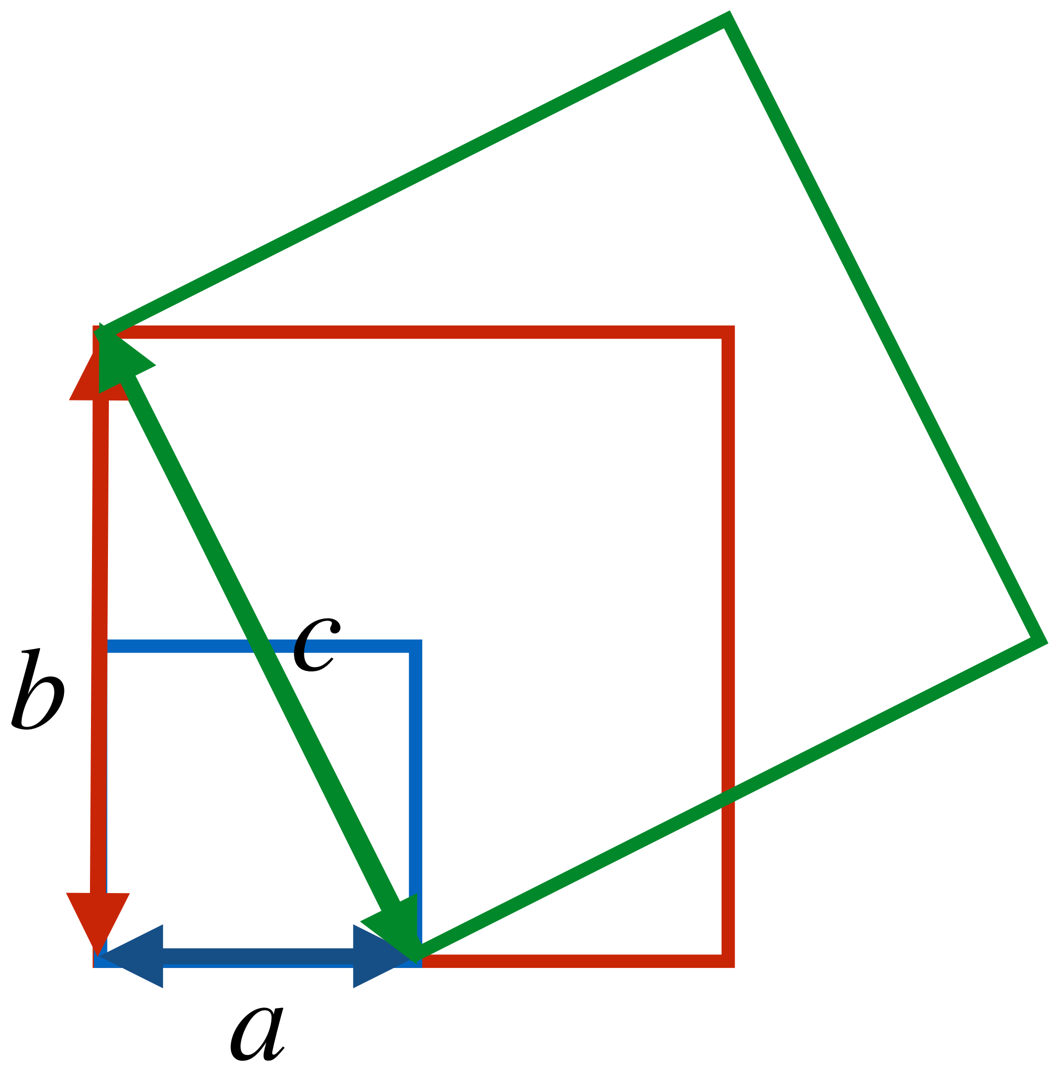

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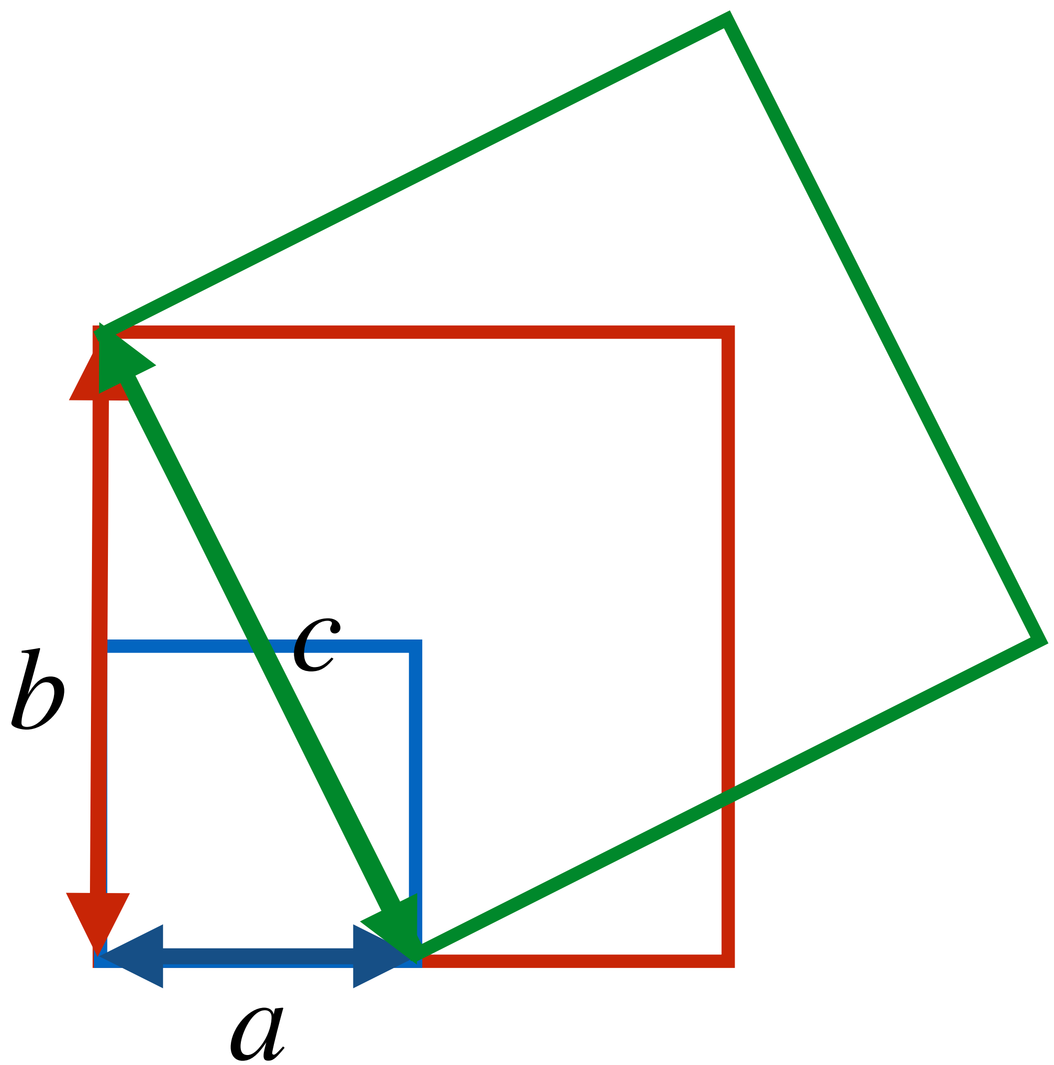

$$\Longrightarrow a^2 + b^2 = c^2$$

$$\Longrightarrow \square + \square = \square$$

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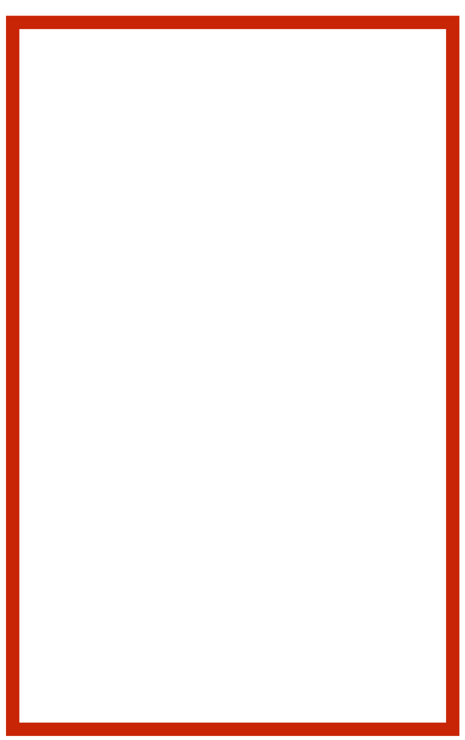

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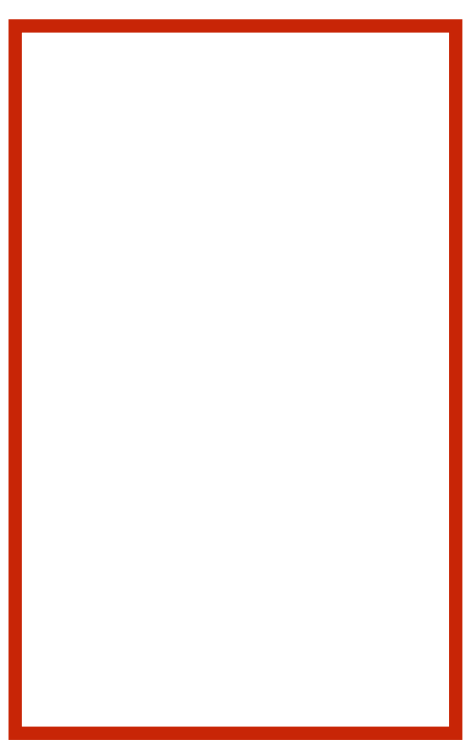

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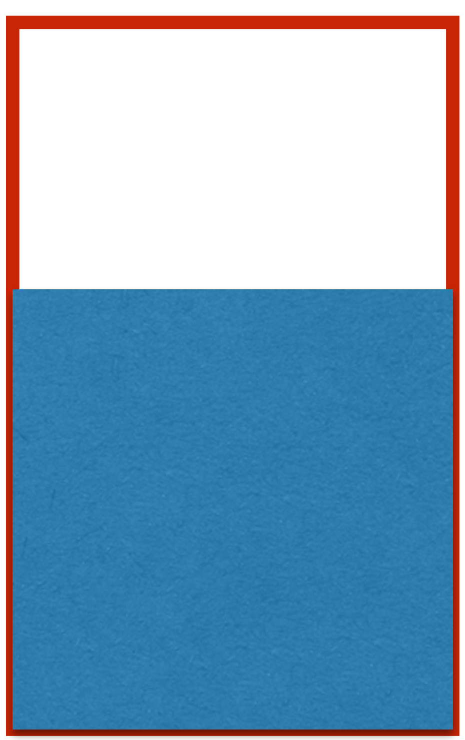

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Ancient India

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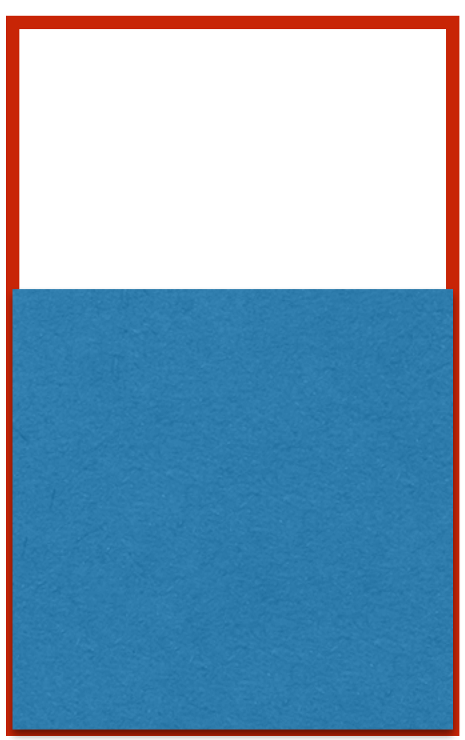

1. Create Square in the bottom of the rectangle.

Total Area =

Ancient India

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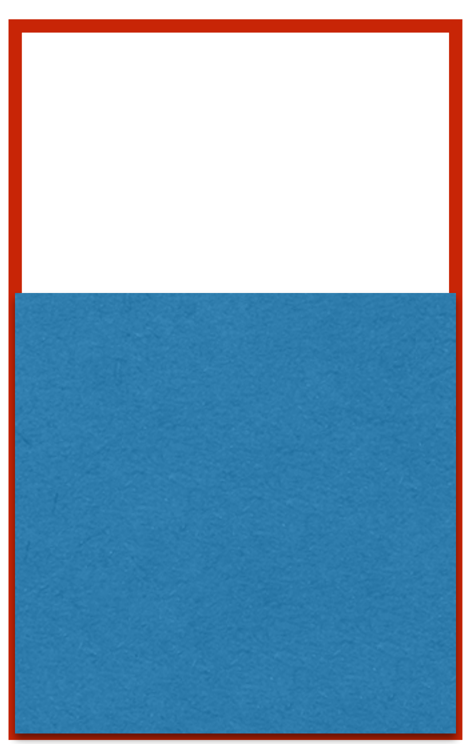
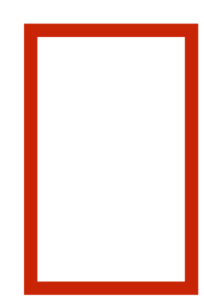


Total Area = 

Ancient India

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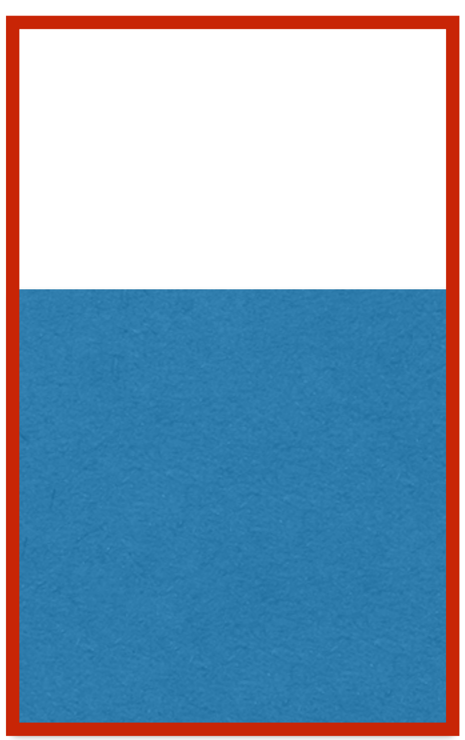
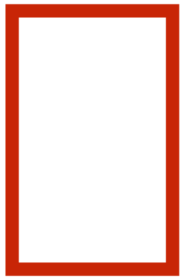


2. Cut the top region in half

Total Area = 

Ancient India

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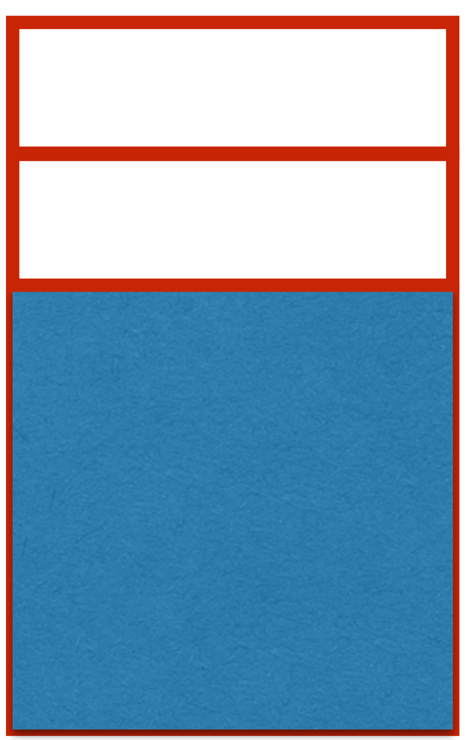
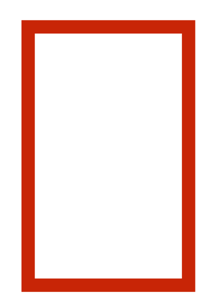
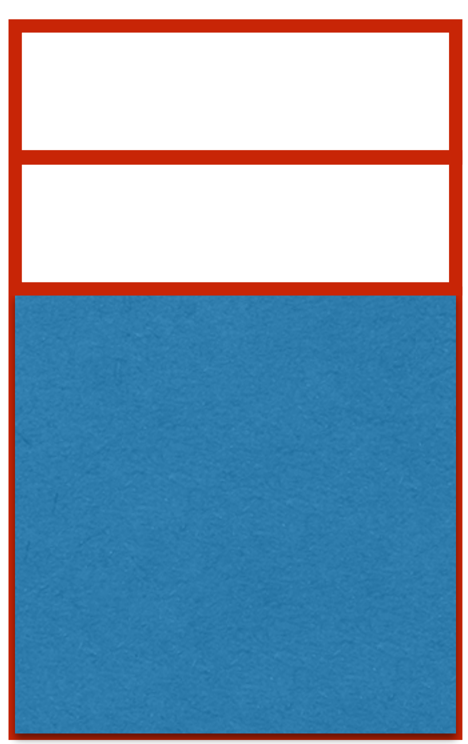


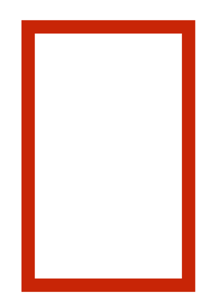
Total Area = 

Ancient India

- **2. Squaring a rectangle.** Meaning: Given a rectangle, create a square whose area is equal to the sum of the other two.

3. Move the top-most rectangle to the right side of the square



Total Area = 

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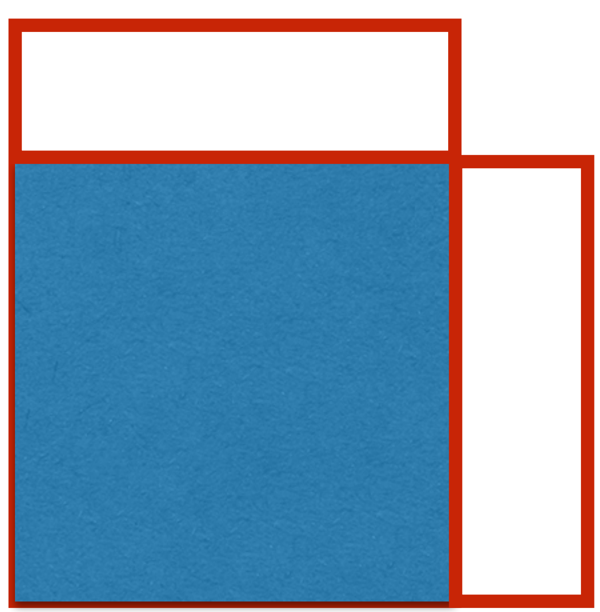
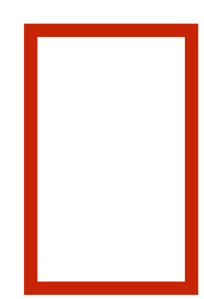
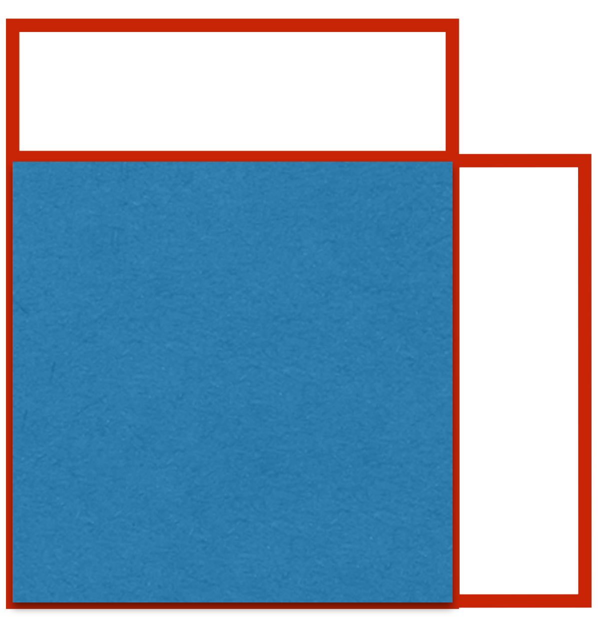


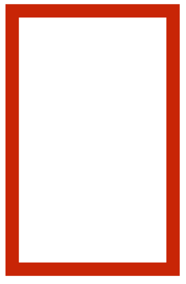
Total Area = 

Ancient India

- **2. Squaring a rectangle.** Meaning: Given a rectangle, create a square whose area is equal to the sum of the other two.

4. Fill in the gap to form a square.



Total Area = 

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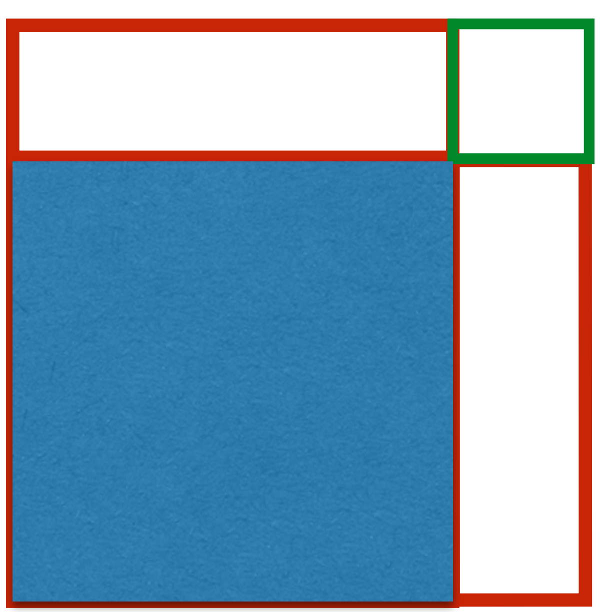
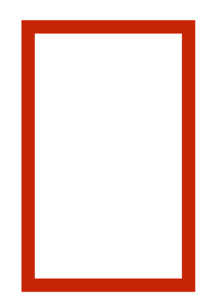


Total Area = 

Ancient India

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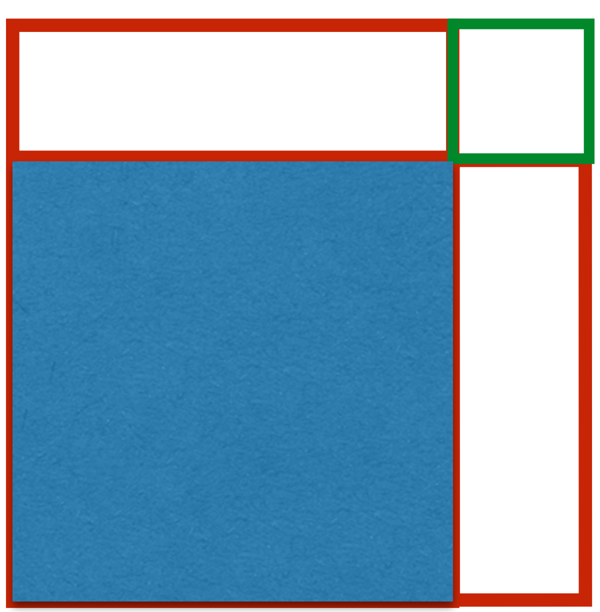
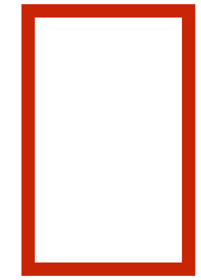
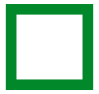


Total Area =  + 

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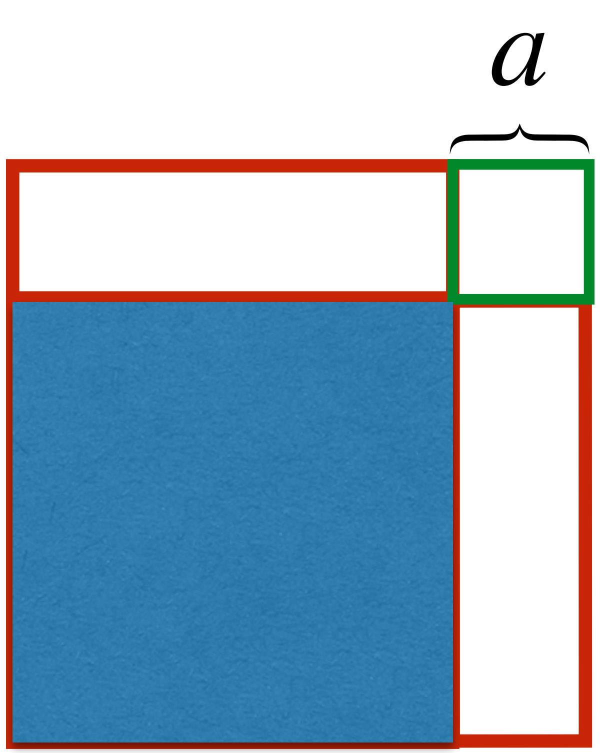

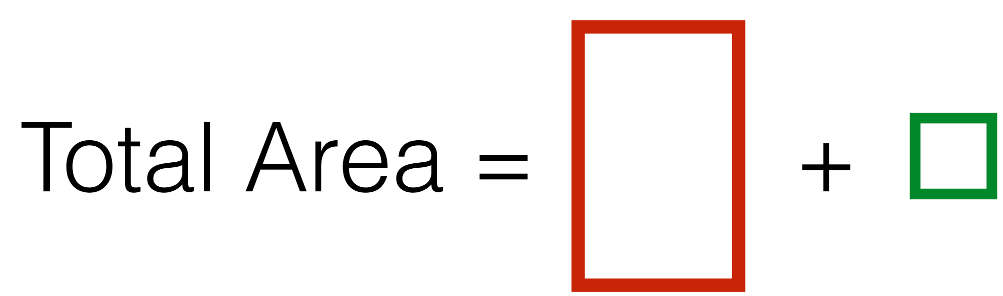

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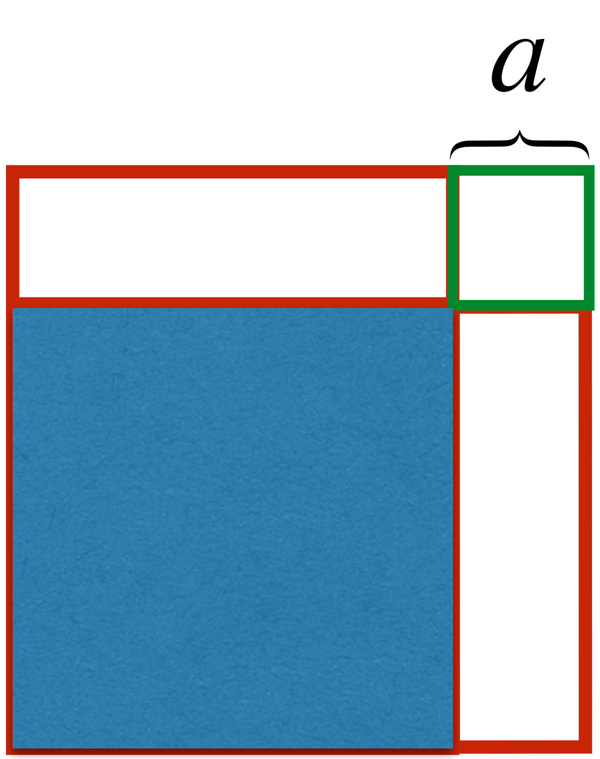
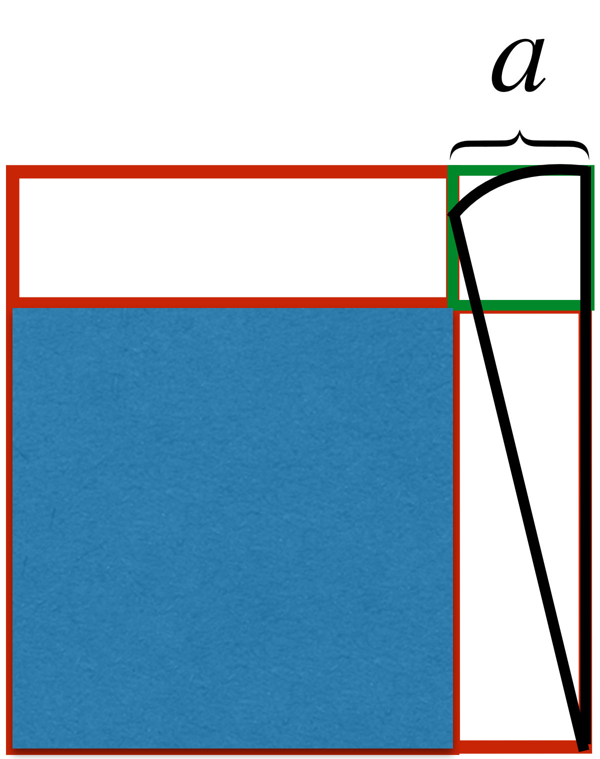


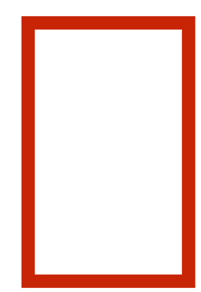
Total Area = $\boxed{}$ + a^2

Ancient India

- **2. Squaring a rectangle.** Meaning: Given a rectangle, create a square whose area is equal to the sum of the other two.

5. Using a compass, make the arc shown.



Total Area =  + a^2

Ancient India

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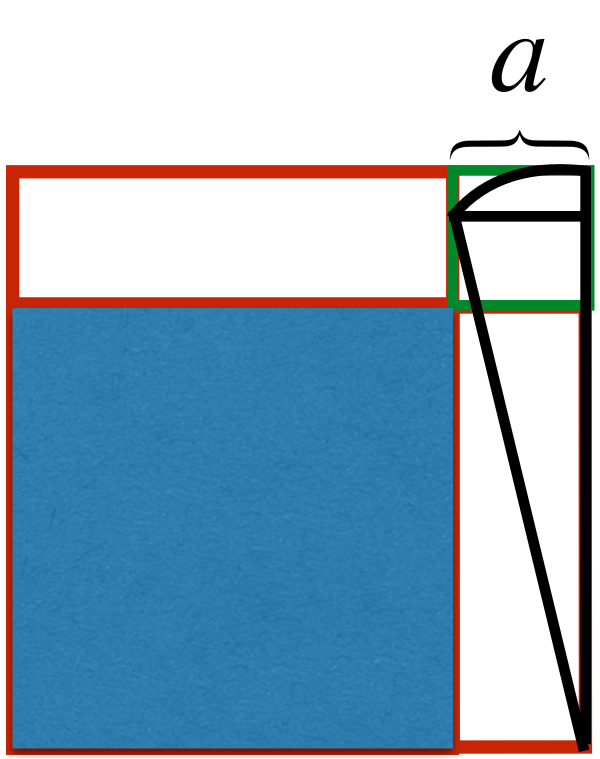
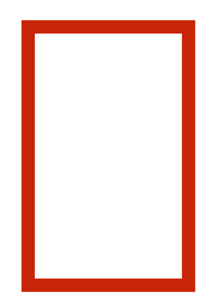


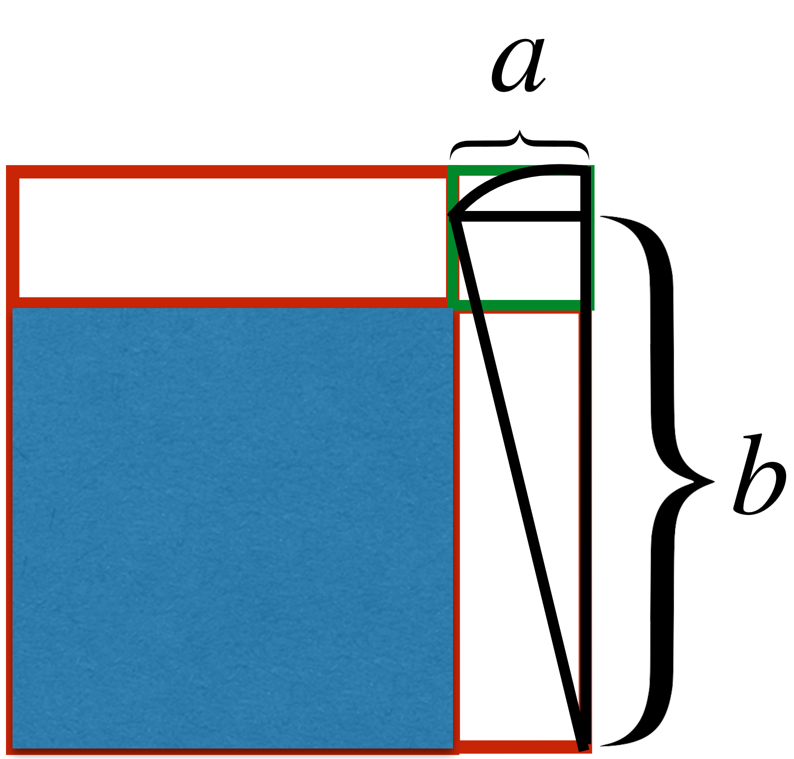
6. Draw the following horizontal line

Total Area =  + a^2

Ancient India

- **2. Squaring a rectangle.** Meaning: Given a rectangle, create a square whose area is equal to the sum of the other two.

7. Label this length b, and let the hypotenuse be c:



Total Area = $\boxed{}$ + a^2

Ancient India

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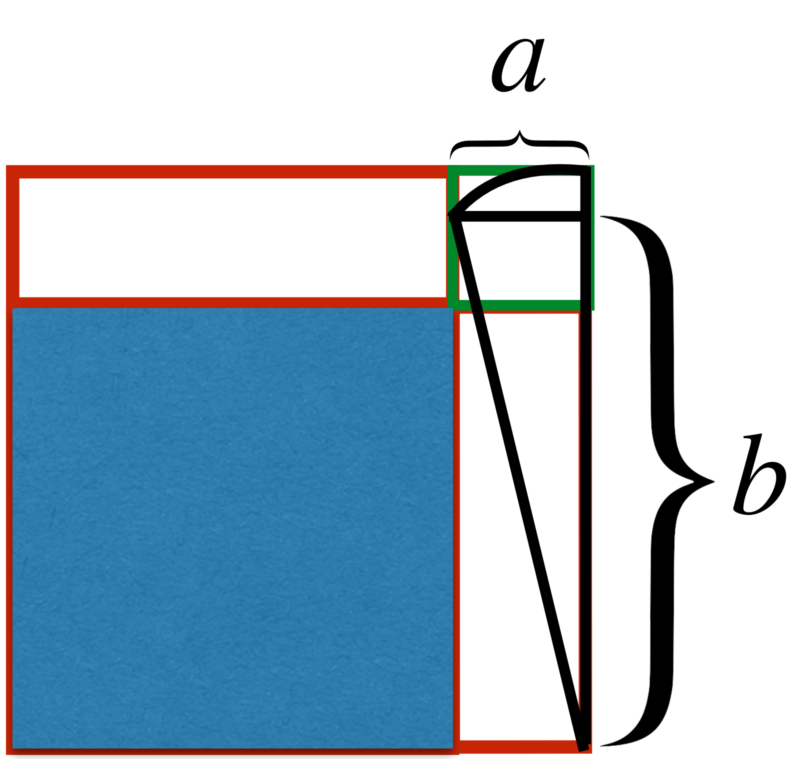

$c^2 = \square + a^2$

Ancient India

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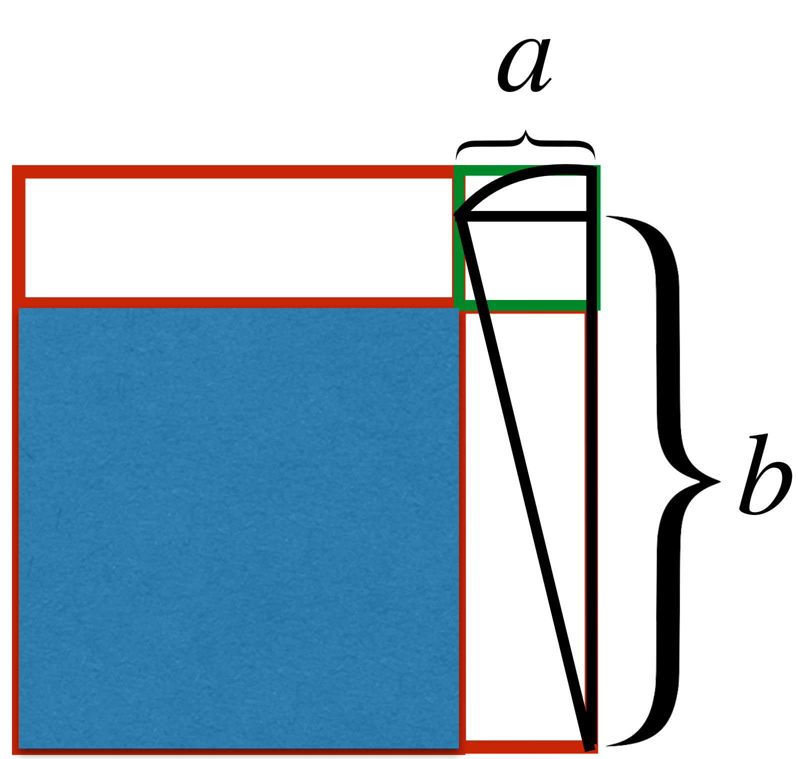

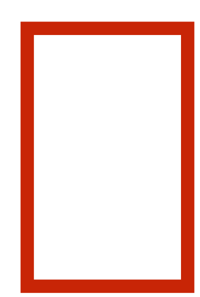

$= c^2 - a^2$

Ancient India

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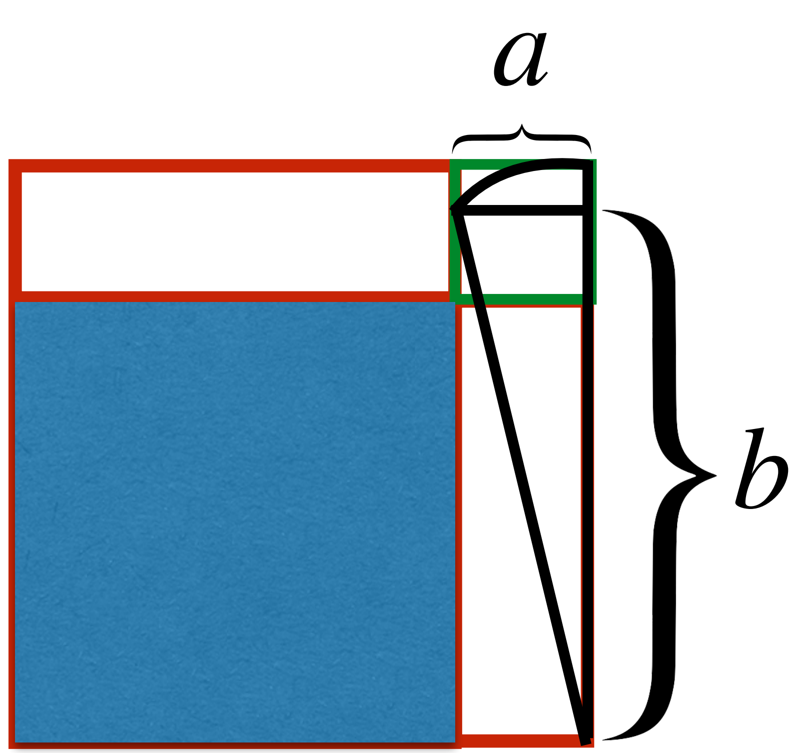
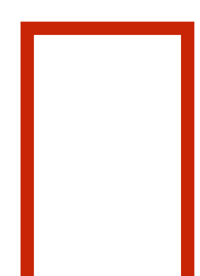


By the Pythagorean theorem,
$$a^2 + b^2 = c^2.$$

 $= c^2 - a^2$

Ancient India

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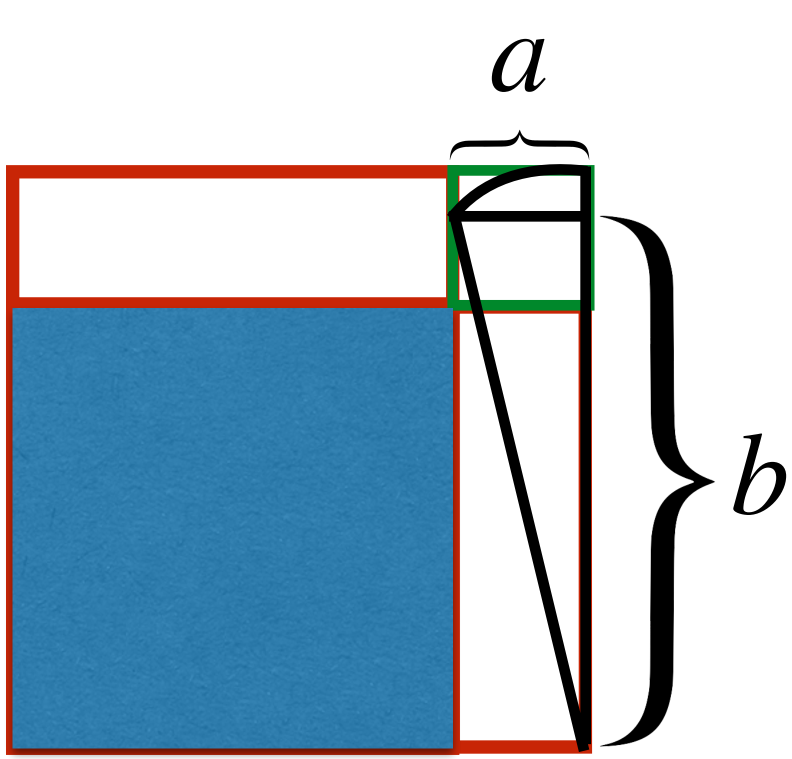

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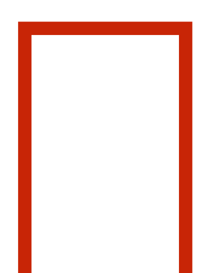
$$\implies b^2 = c^2 - a^2.$$



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Ancient India

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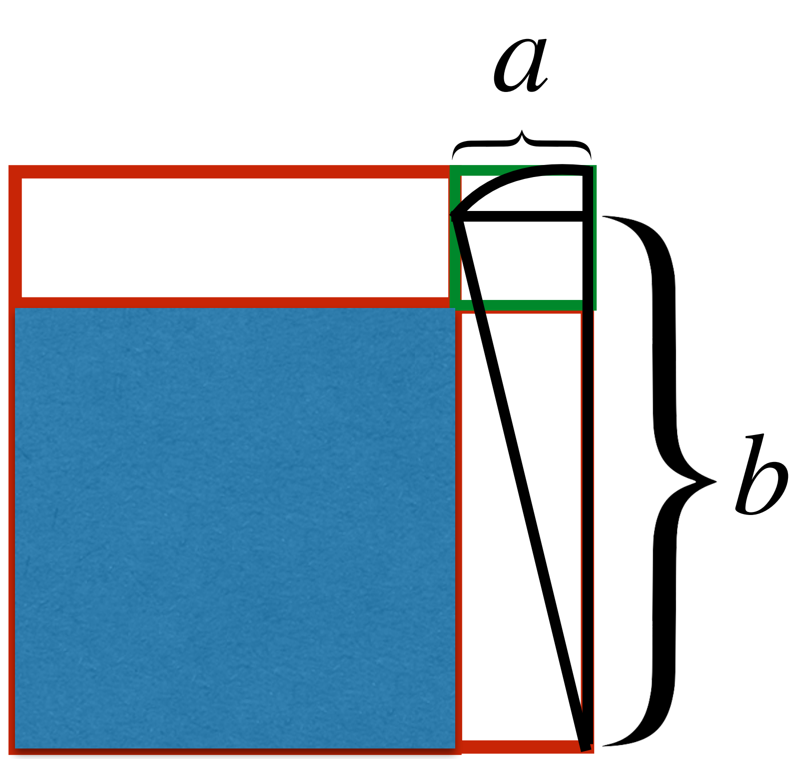

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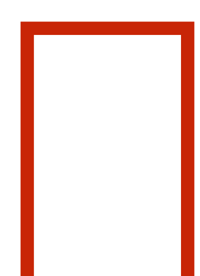
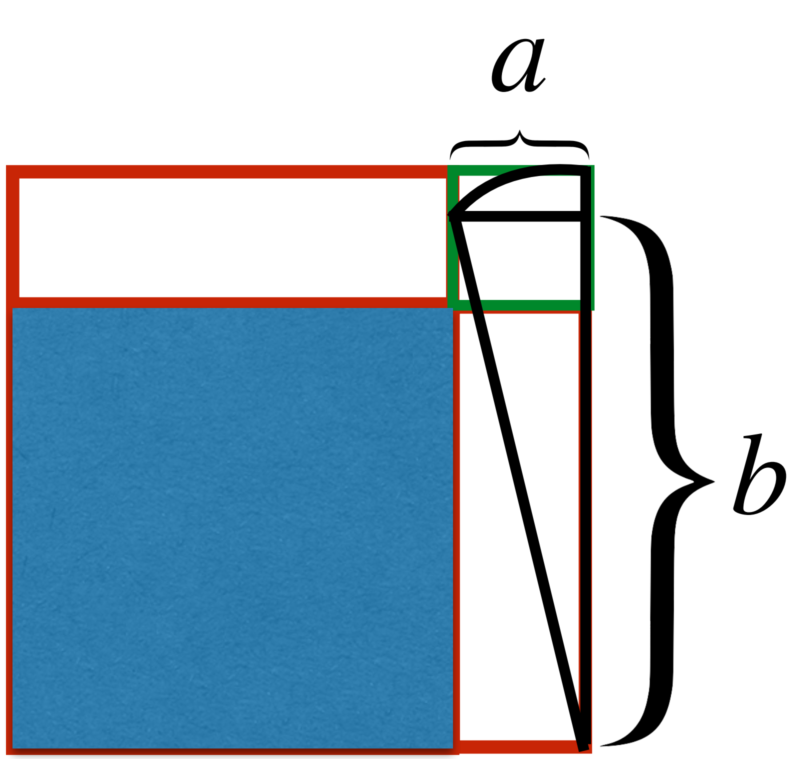


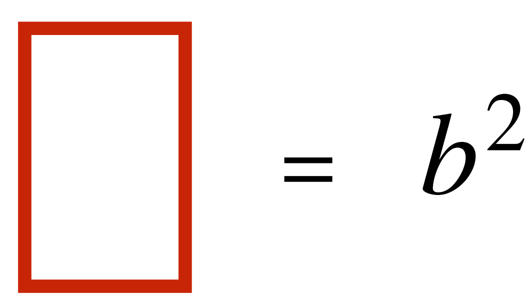
$= \quad b^2$

Ancient India

- **2. Squaring a rectangle.** Meaning: Given a rectangle, create a square whose area is equal to the sum of the other two.

8. Build a square on this line segment.





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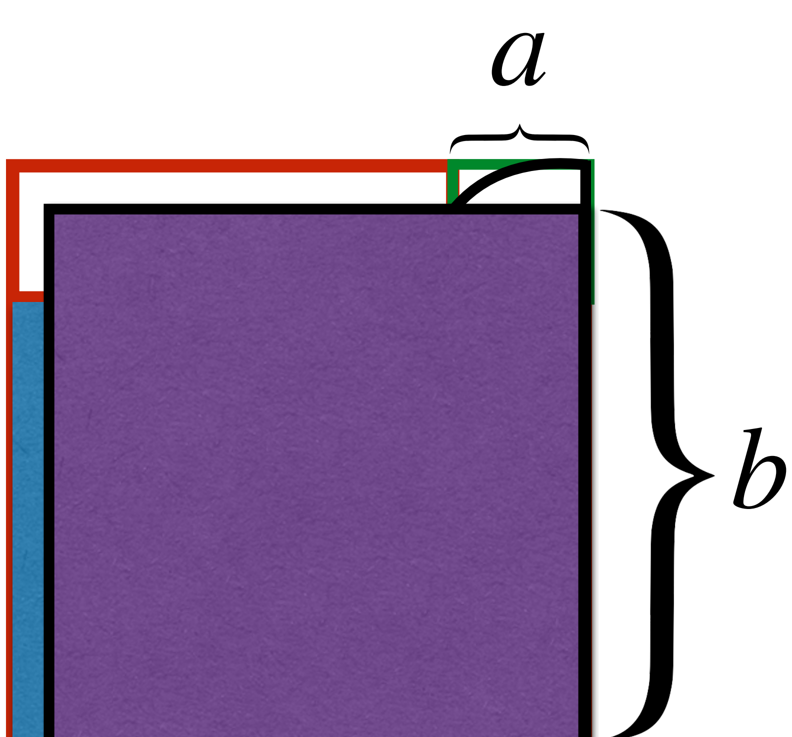
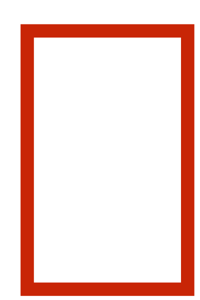
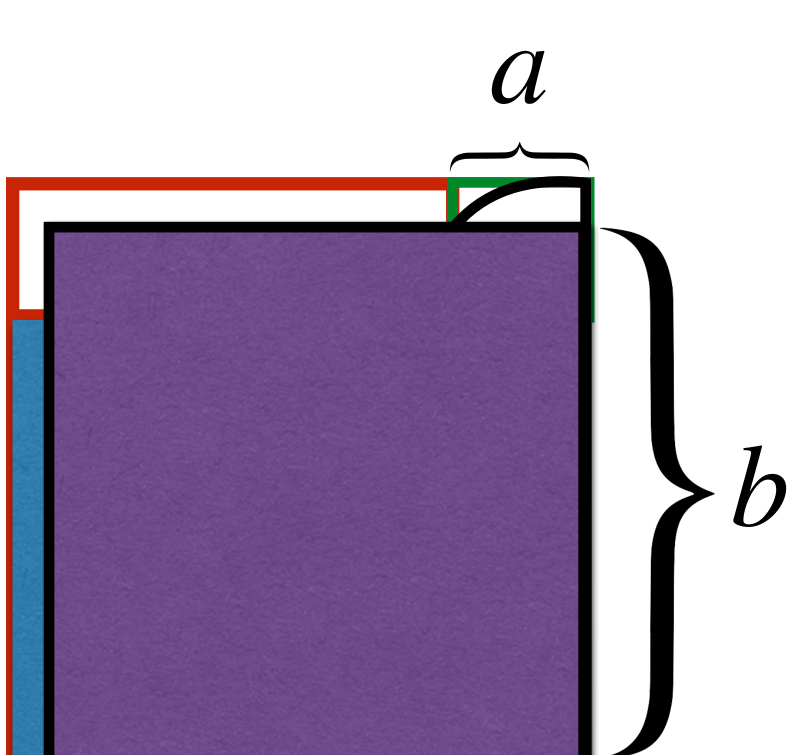


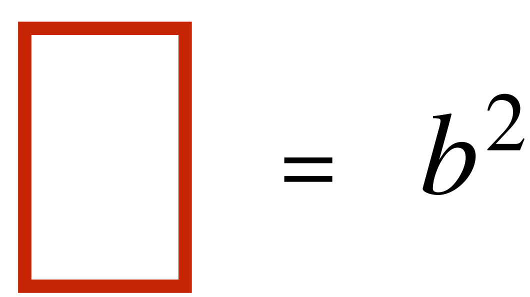
 $= b^2$

Ancient India

- **2. Squaring a rectangle.** Meaning: Given a rectangle, create a square whose area is equal to the sum of the other two.

8. Build a square on this line segment. The area of this square equals b^2.





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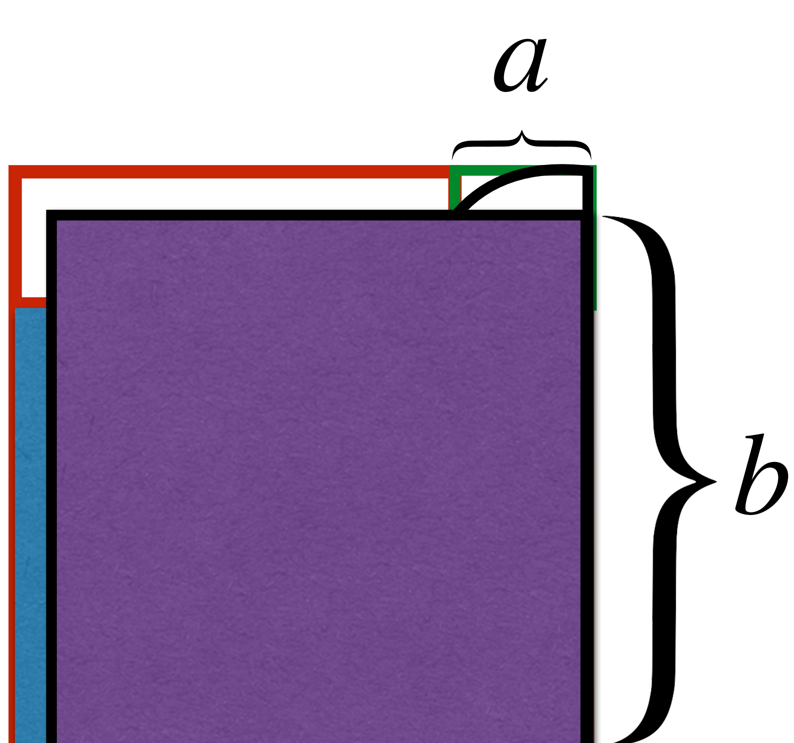
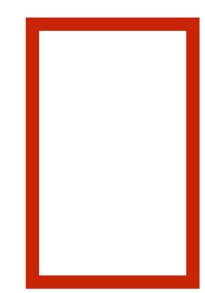
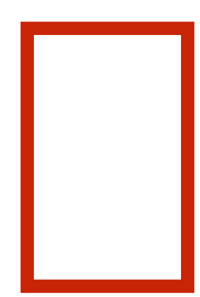


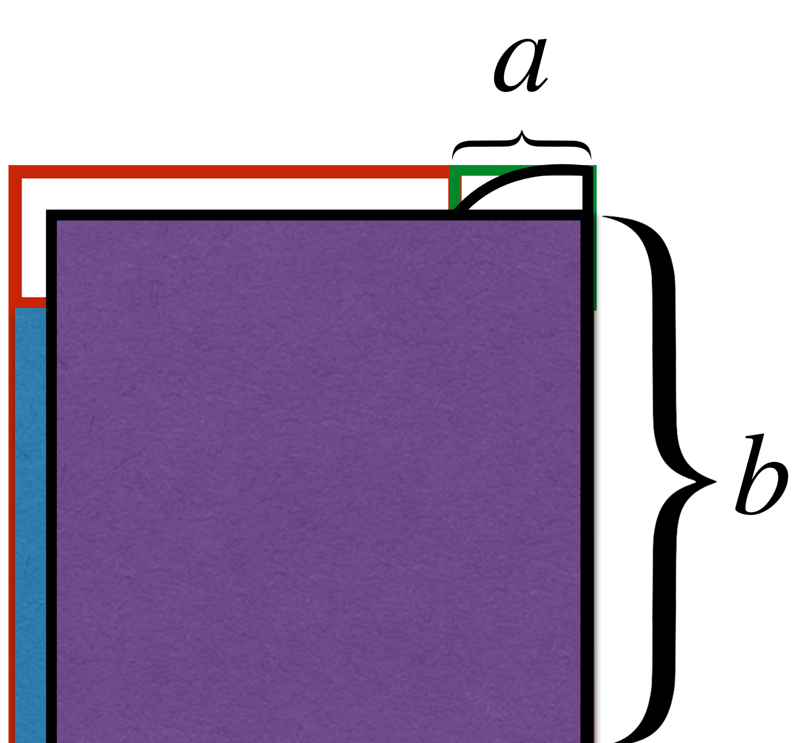
So,  = 

 = b^2

Ancient India

- **2. Squaring a rectangle.** Meaning: Given a rectangle, create a square whose area is equal to the sum of the other two.

9. So, this square has the same area as the original rectangle.



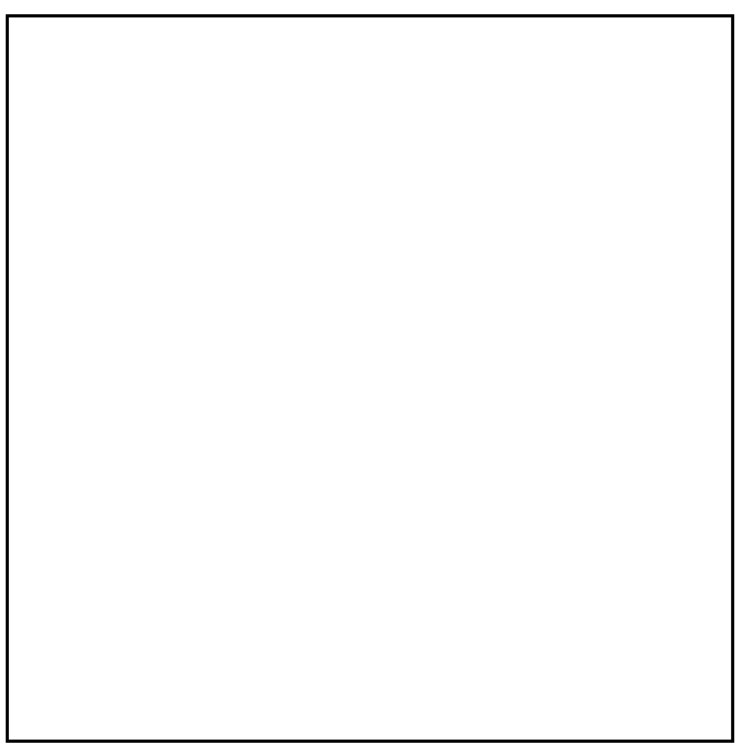
Ancient India

- **3. Circling a square or squaring a circle.** Meaning: Given a square, create a circle whose area is equal to that of the square. Or, given a circle, create a square of the same area.

- Note: While the last two gave perfect constructions, these ones do not. Also, they did not note in their work that these two were imperfect.

Ancient India

- **3. Circling a square or squaring a circle.**



Ancient India

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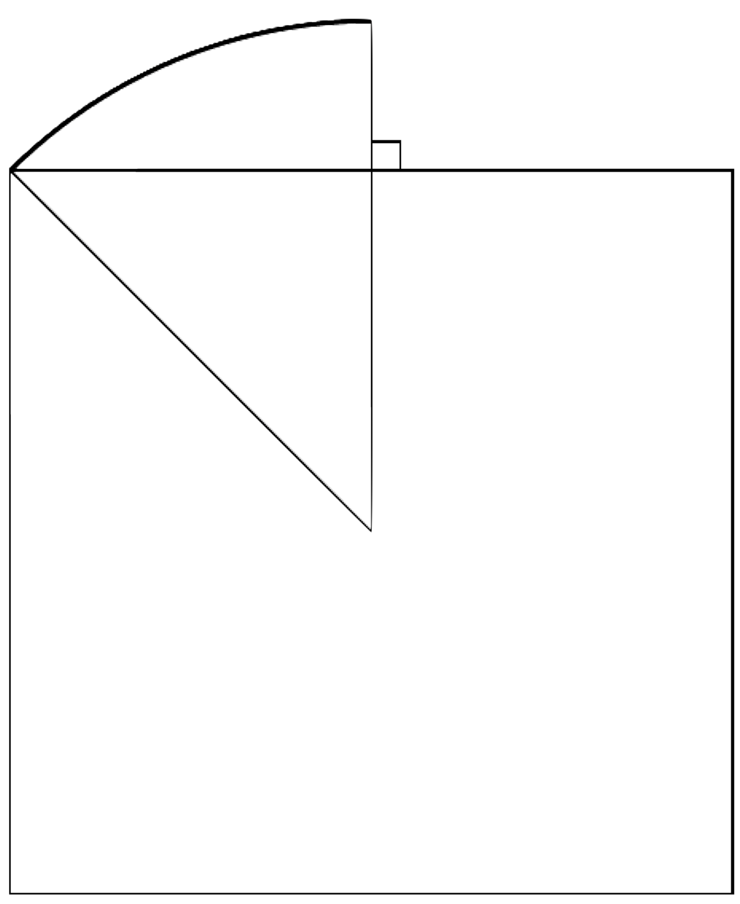

Ancient India

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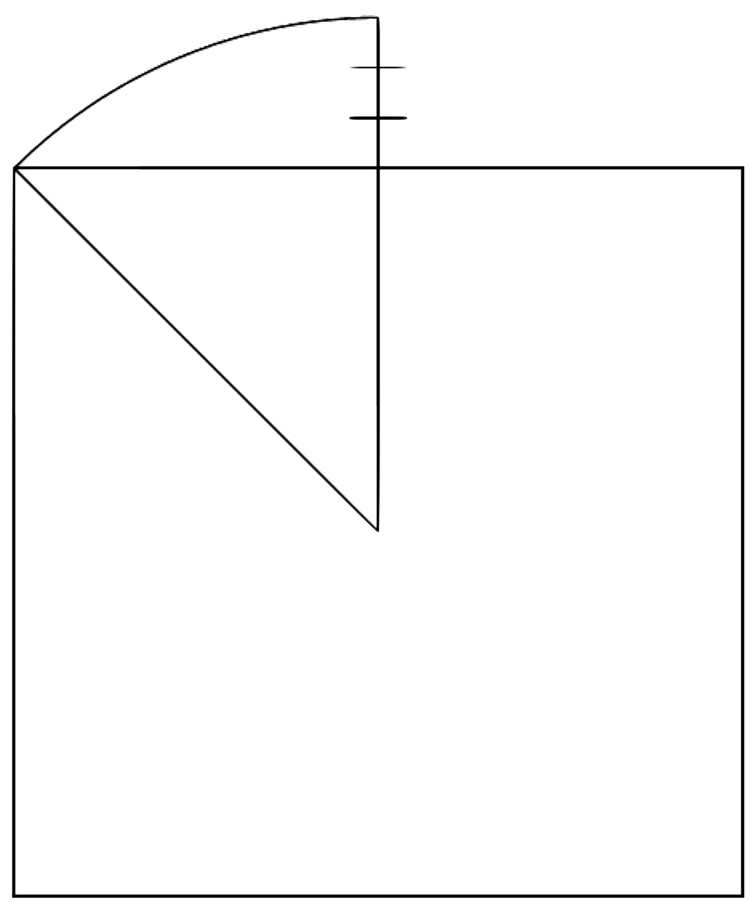

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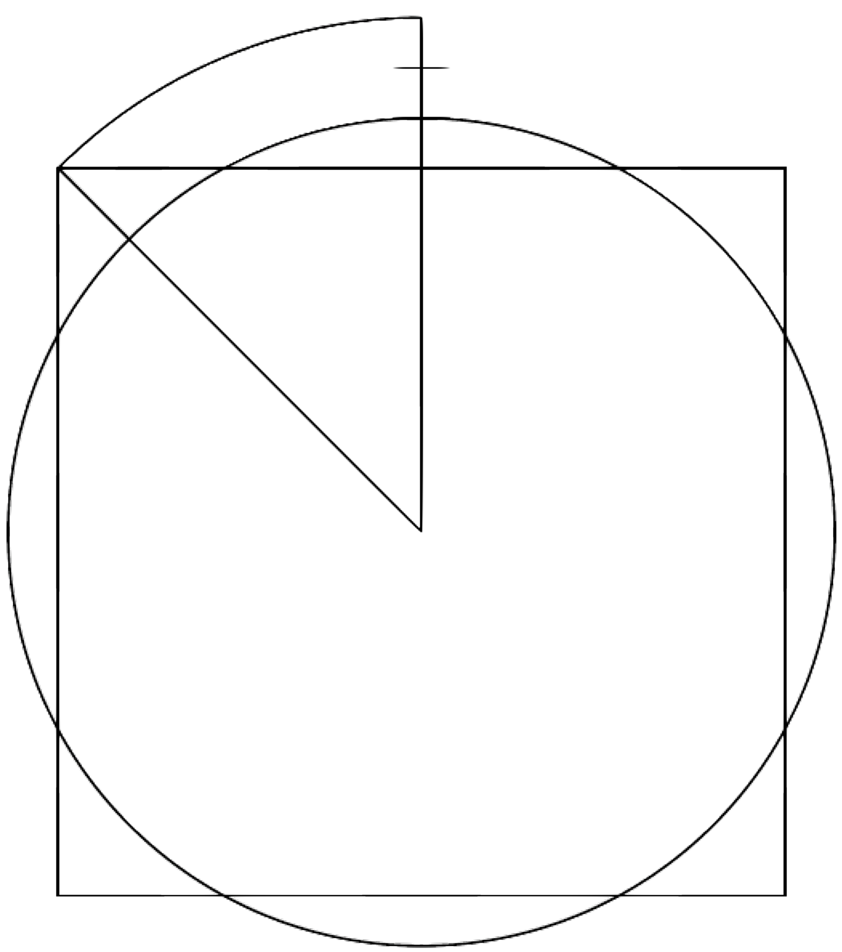

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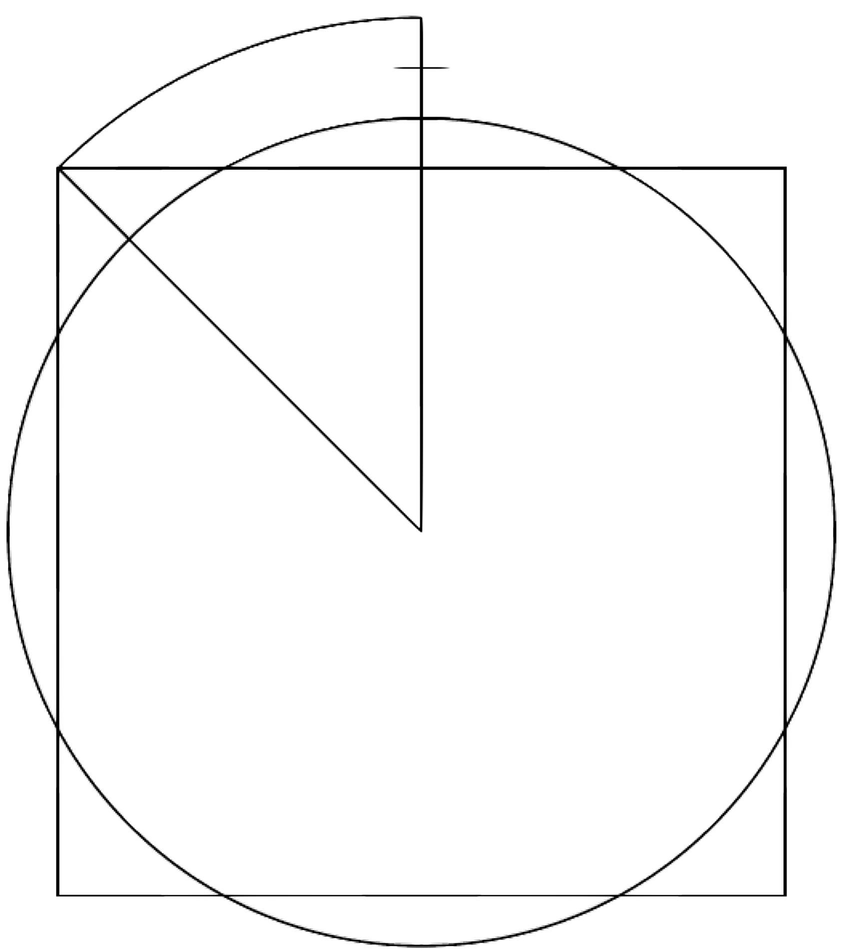

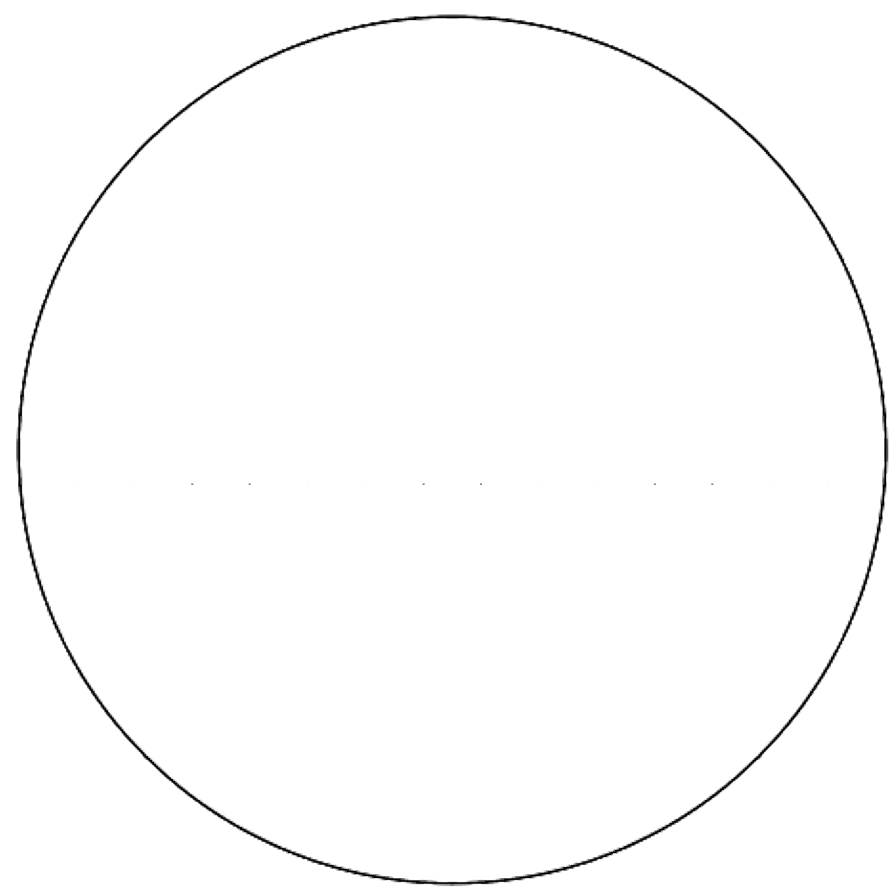

Ancient India

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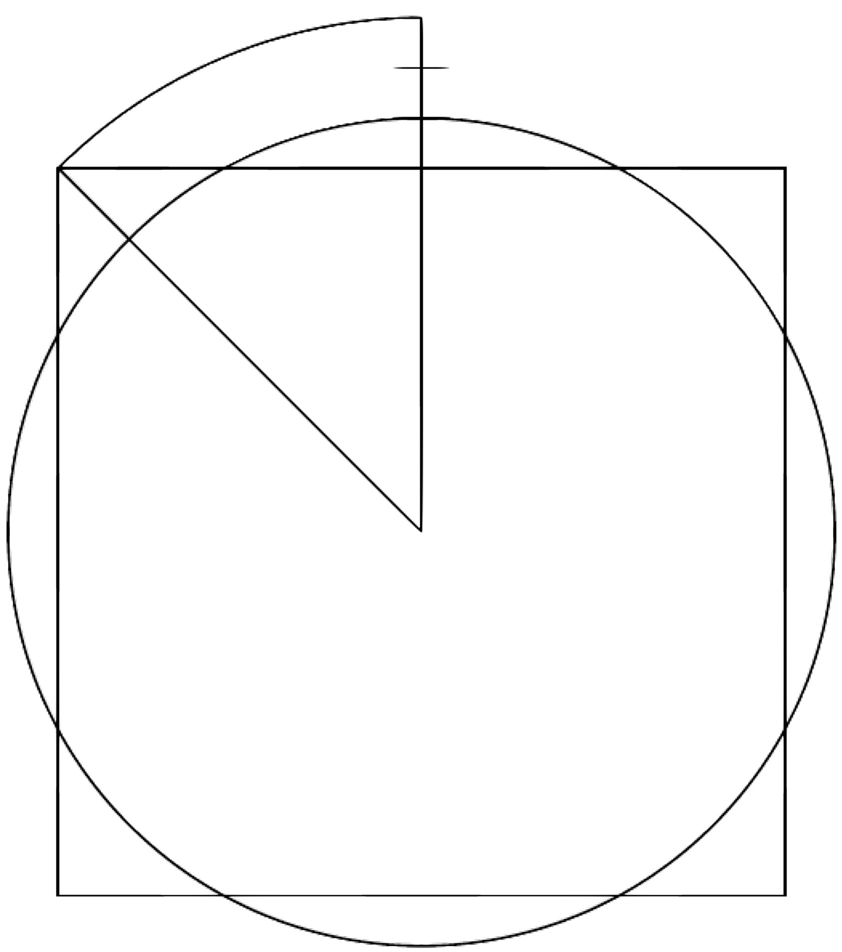

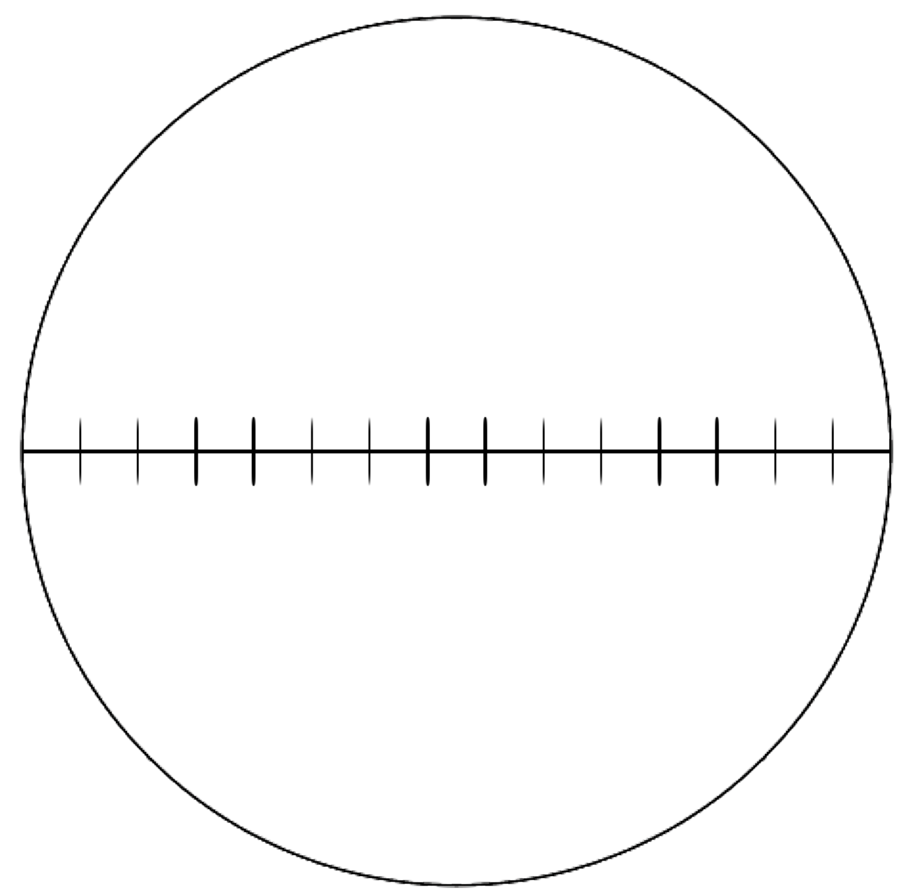

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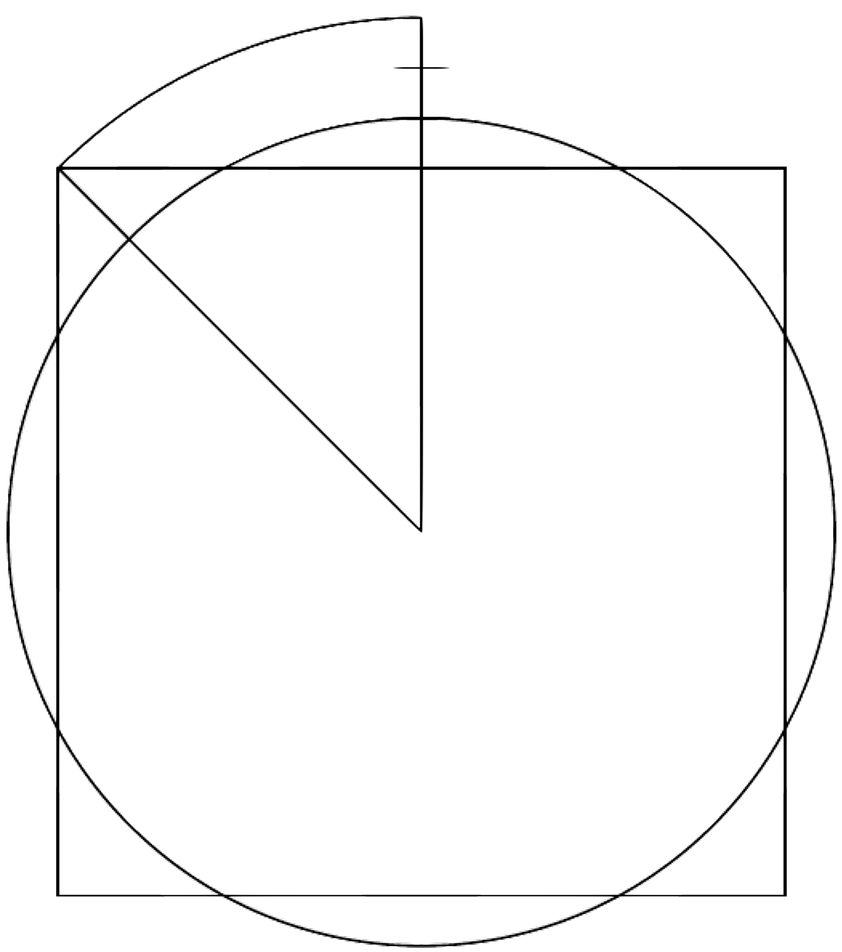

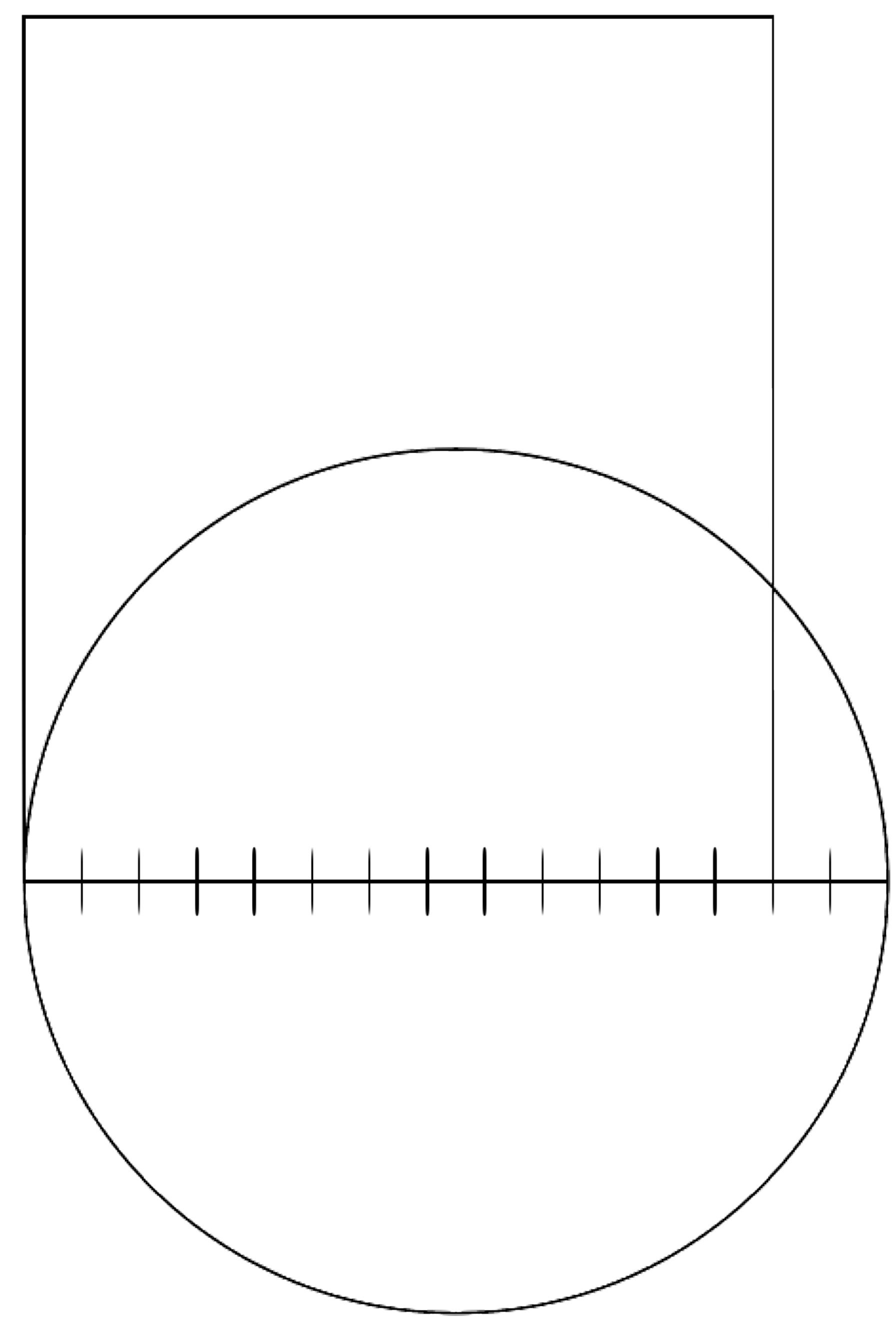

The Aftermath

- The problem of “squaring a circle” is this: Given a circle, use a straightedge and compass to create a square of the same area.

- This was proved to be impossible in 1882. To do so, Ferdinand von Lindemann proved that π is transcendental (i.e., not the root of a polynomial with rational coefficients).

- Proof sketch given in the notes. It’s a neat argument —but is subtle and better read than heard.

Shout-outs!

Shout-outs!

Shout-outs!

Shout-outs!

- Euclid characterized all *primitive Pythagorean triples.* They are all of this form:
  $$a = m^2 - n^2, \quad b = 2mn, \quad c = m^2 + n^2$$

Shout-outs!

- Euclid characterized all *primitive Pythagorean triples.* They are all of this form:
$$a = m^2 - n^2, \quad b = 2mn, \quad c = m^2 + n^2$$

- Pythagoreans were also said to be pioneers in the math of music.

Shout-outs!

Shout-outs!

- A more modern figure who prominently weaved together math and mysticism is Englishman John Dee. In fact, during his life in England, math was considered by many to be pseudo-magical, and in 1555 Dee was arrested for the crime of "calculating."

Shout-outs!

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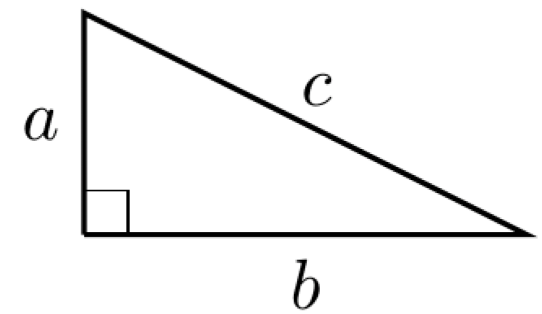
- Lastly: My favorite proof of the Pythagorean theorem.

Shout-outs!

Shout-outs!

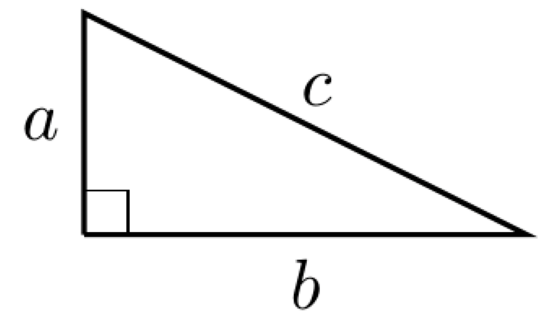
Take any right triangle:



Goal: $a^2 + b^2 = c^2$

Shout-outs!

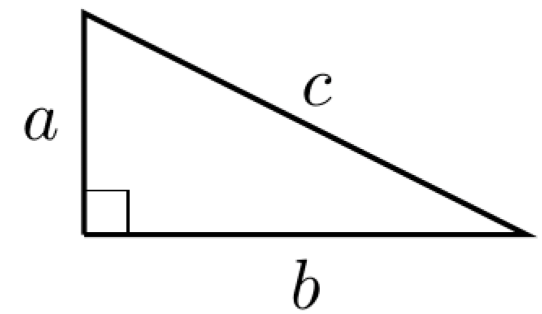
Take any right triangle:



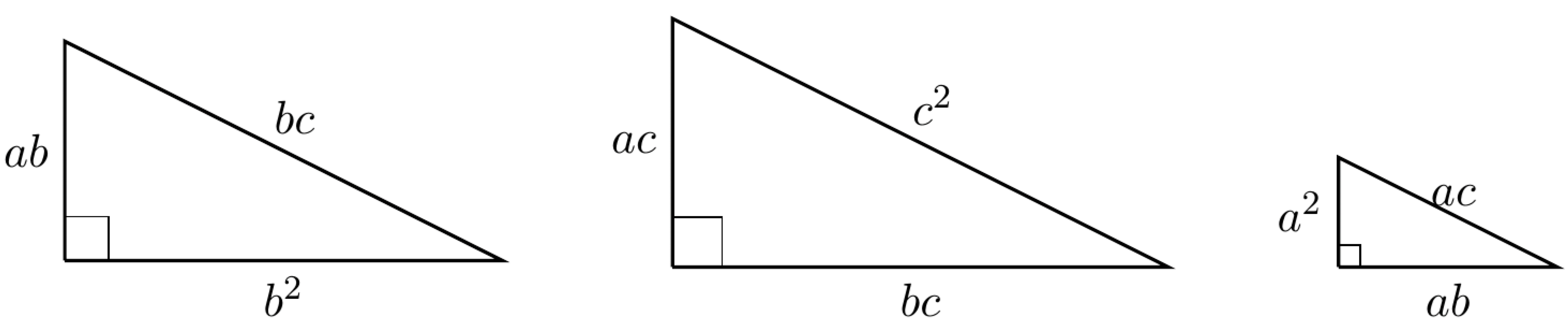
Goal: $a^2 + b^2 = c^2$

Shout-outs!

Take any right triangle:



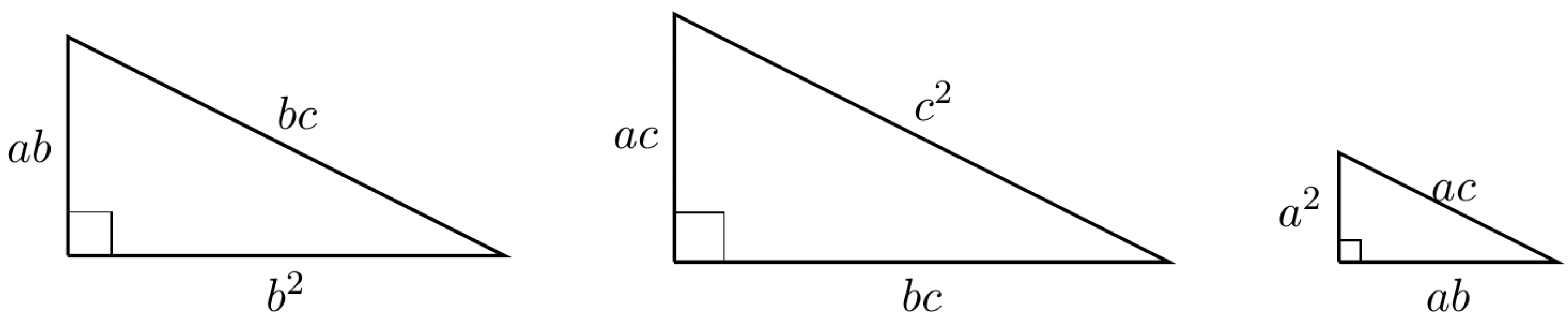
Now, scale up this triangle three times, first by a factor of b, next by a factor of c, and last by a factor of a. This produces these three similar triangles:



Goal: $a^2 + b^2 = c^2$

Shout-outs!

Now, scale up this triangle three times, first by a factor of b, next by a factor of c, and last by a factor of a. This produces these three similar triangles:



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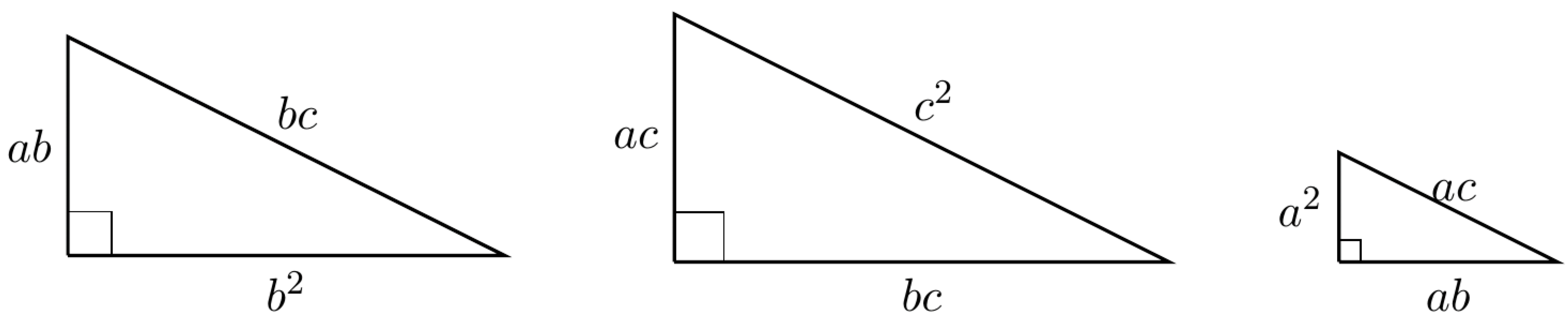
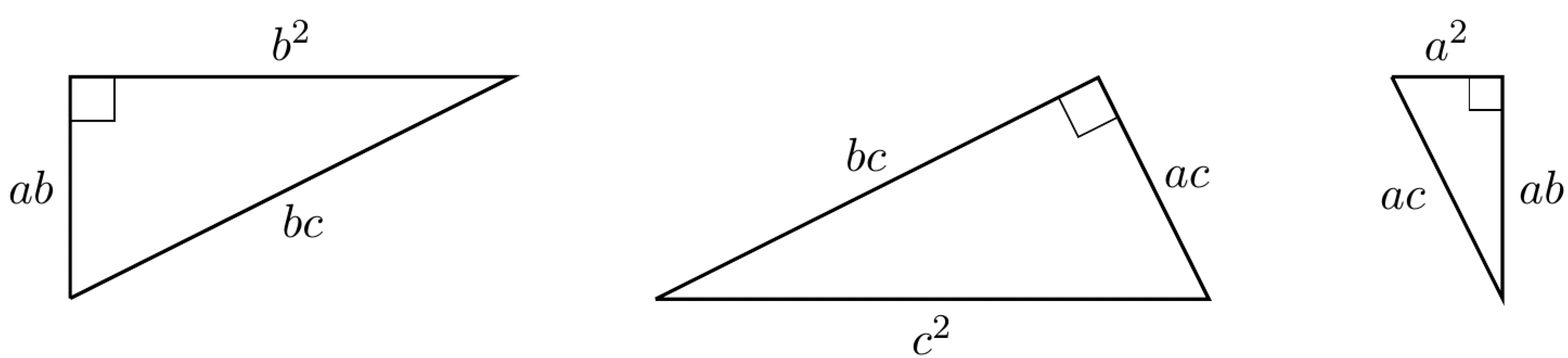


Next, by simply rotating or mirroring them, we arrive here:



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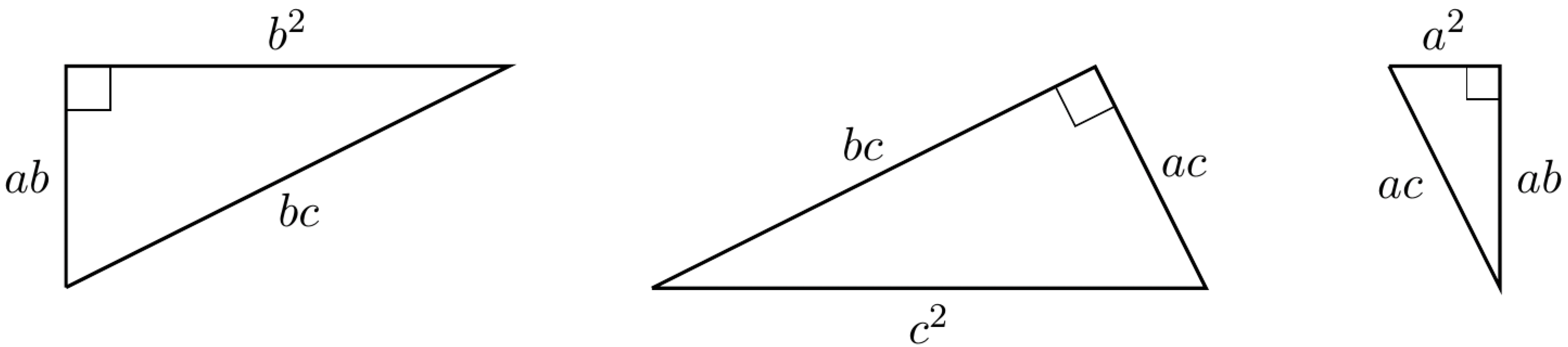

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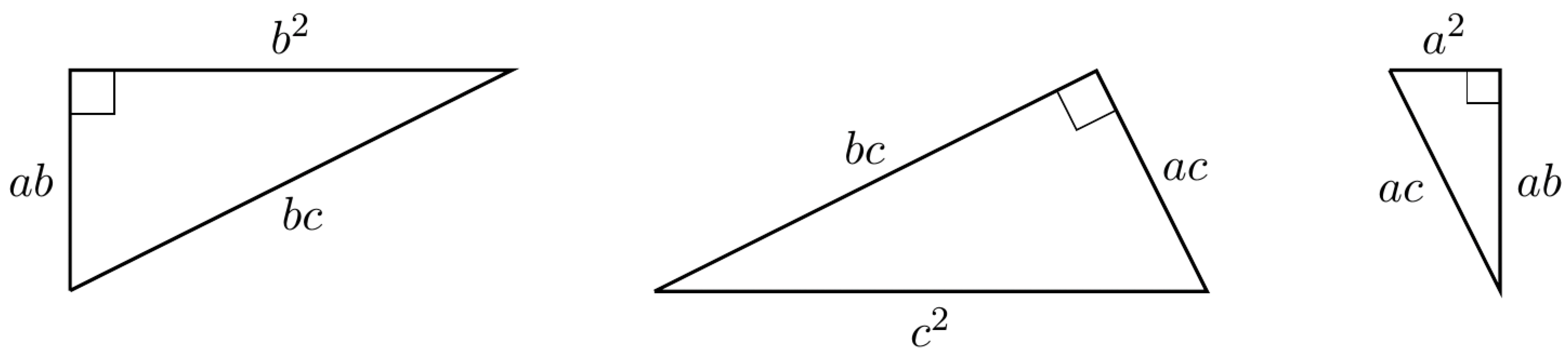
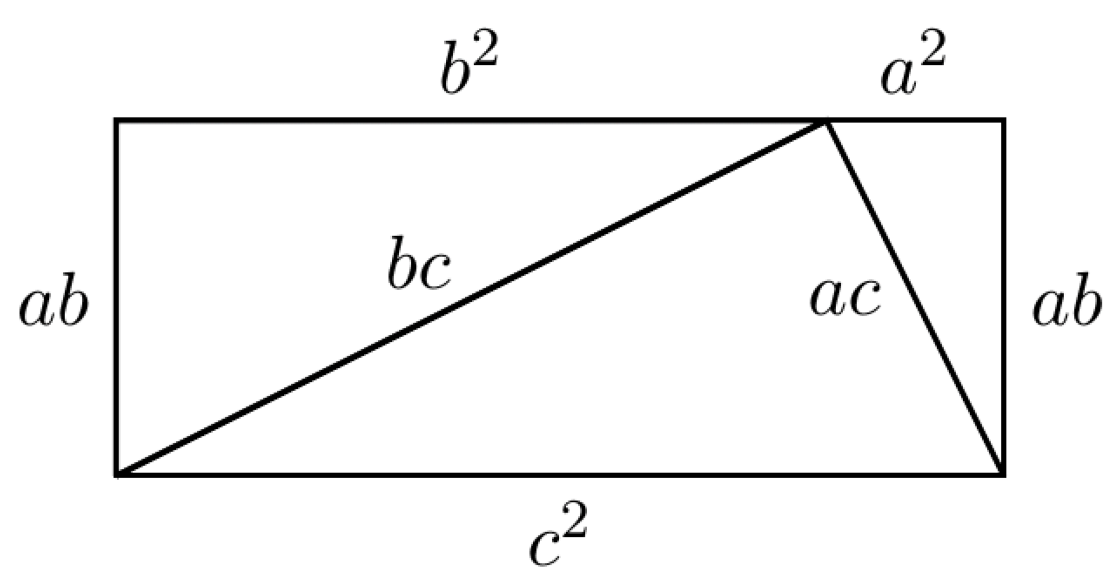


And, finally, piece them together:



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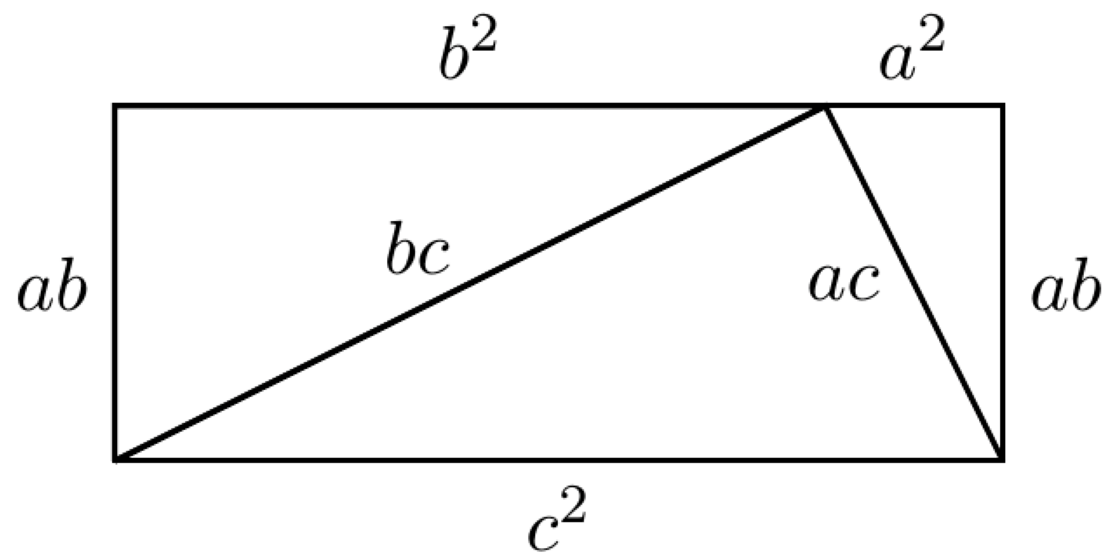

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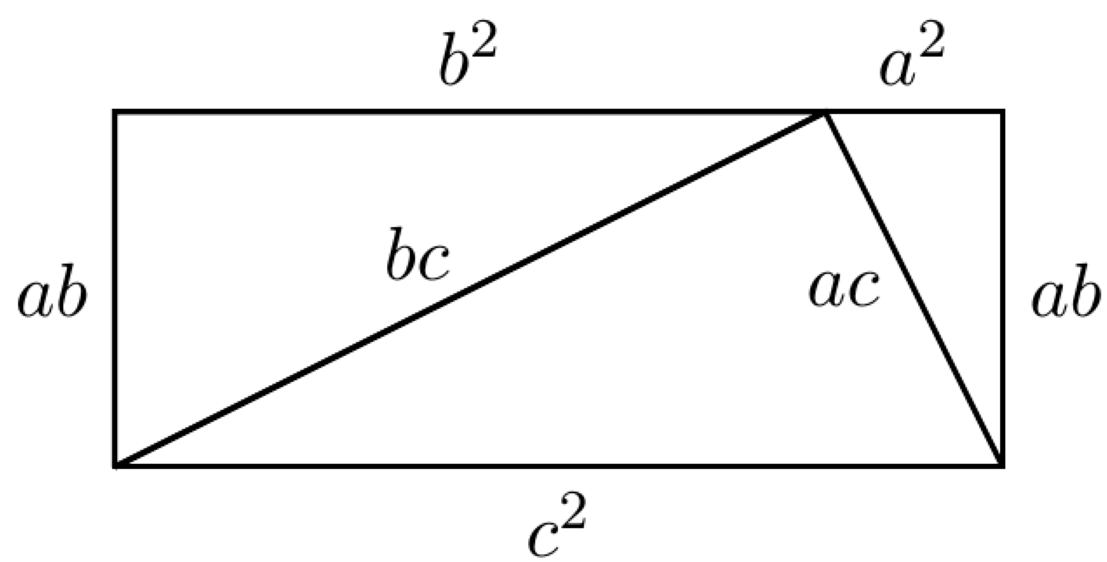

This produces a rectangle.

Goal: $a^2 + b^2 = c^2$

Shout-outs!

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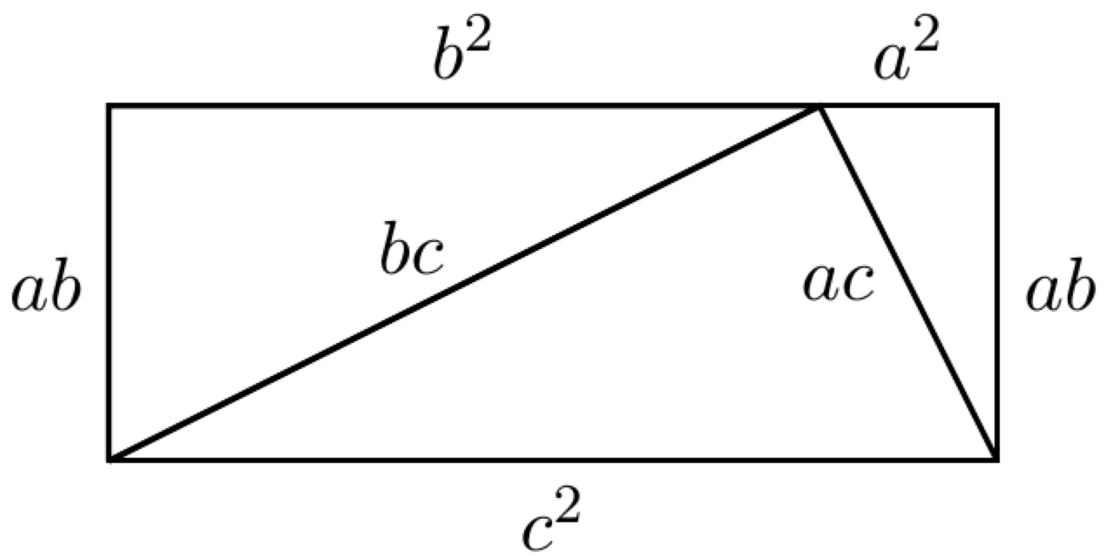

This produces a rectangle.

Utterly trivial fact: The top and bottom of a rectangle have the same length.

Shout-outs!

Goal: $a^2 + b^2 = c^2$

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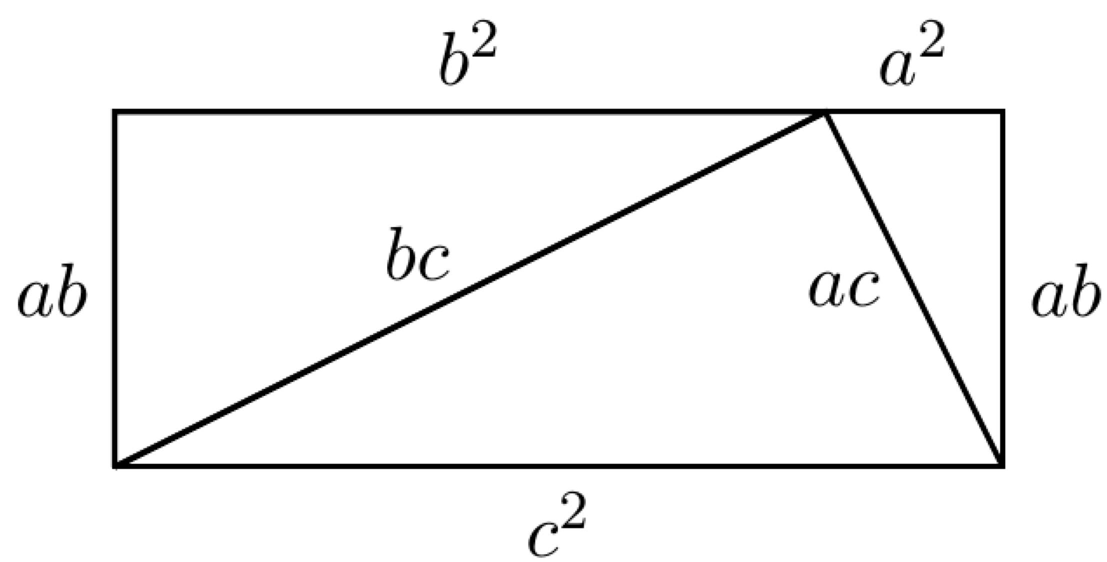

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Thus, $a^2 + b^2 = c^2$.

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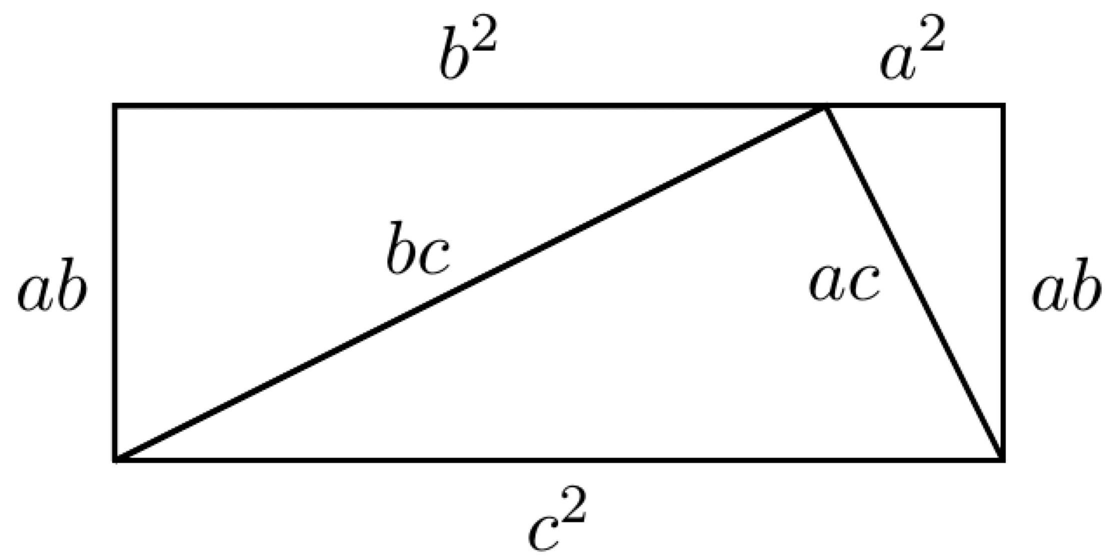

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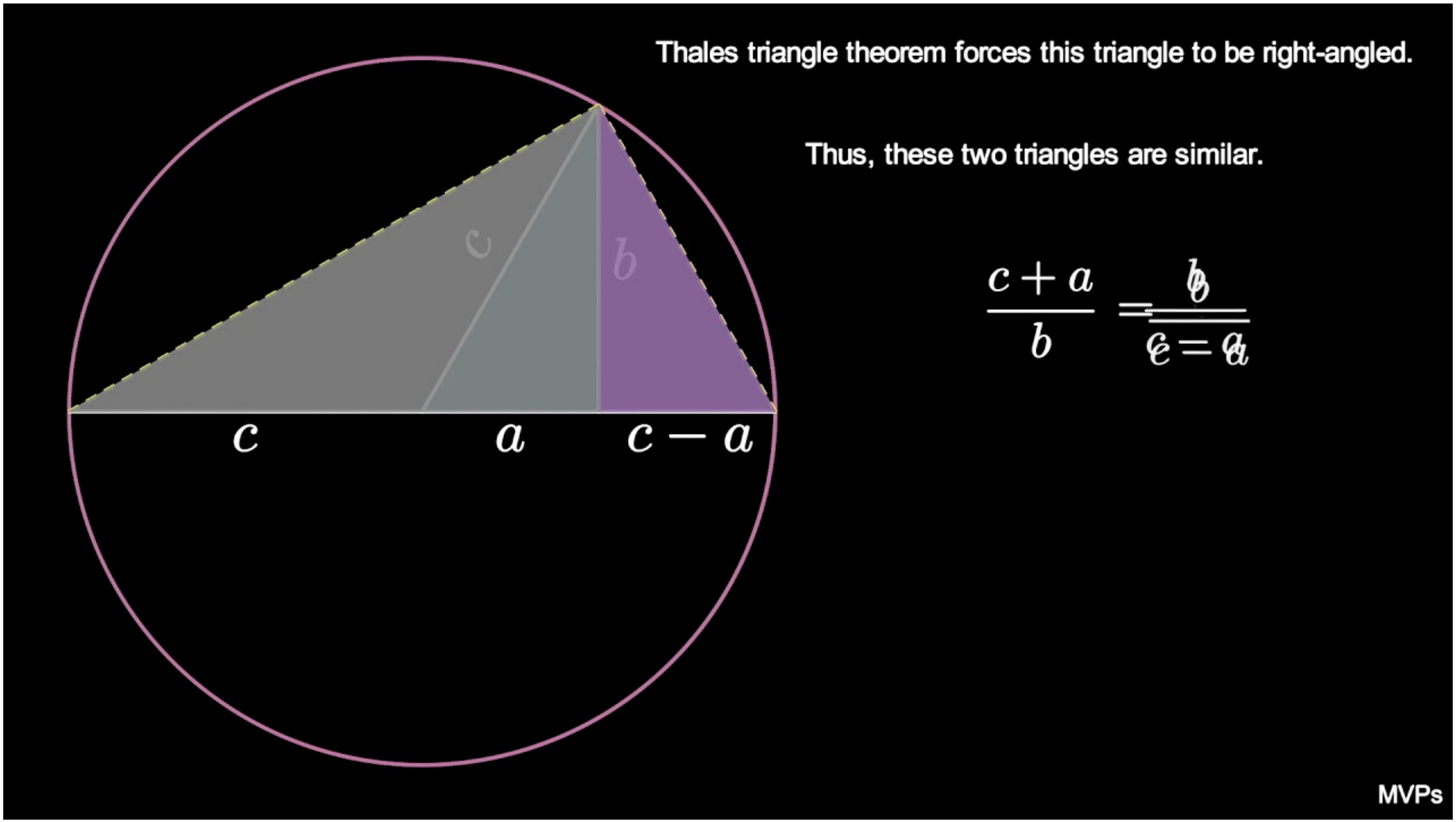
More Proofs of the Pythagorean Theorem

More Pythagorean Proofs



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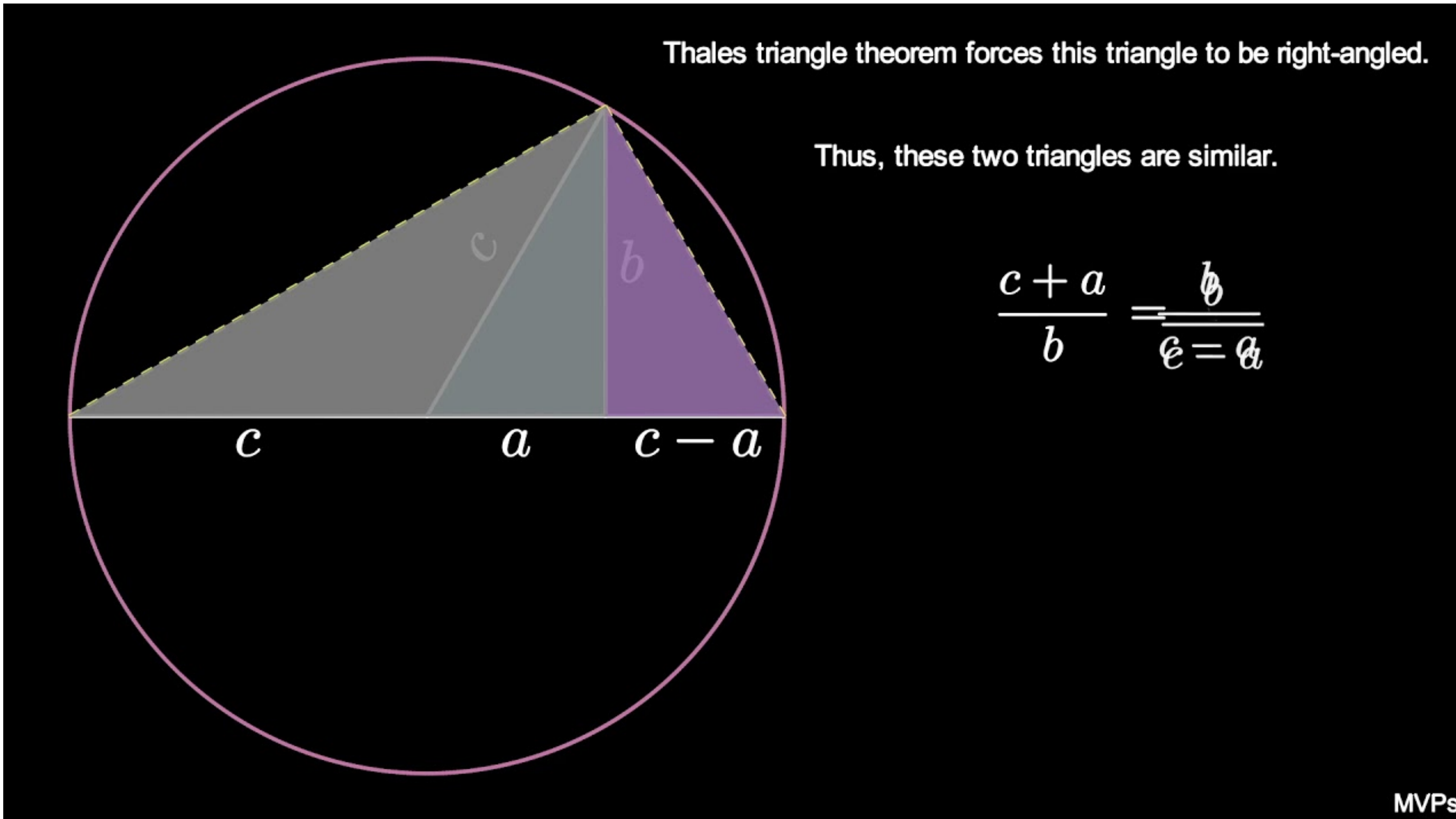
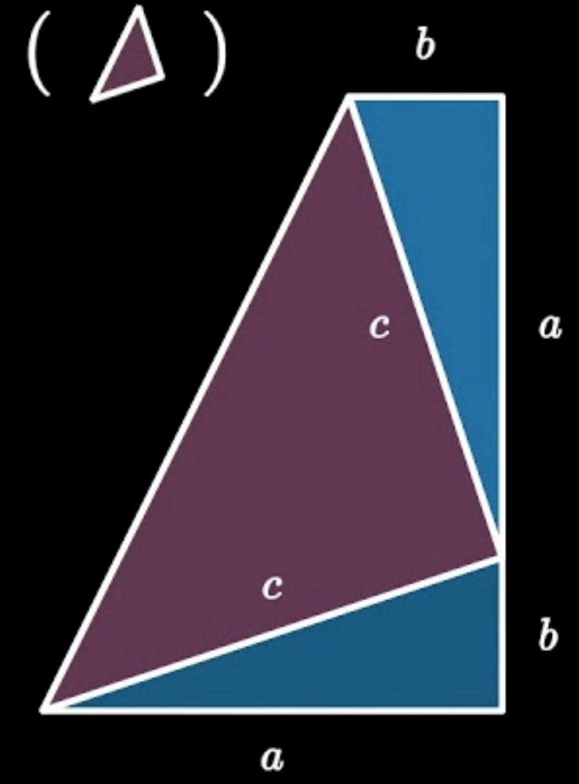


More Pythagorean Proofs

$$\text{Area}\left(\ \right) = \frac{a+b}{2} \cdot (a+b) = \frac{a^2+2ab+b^2}{2}$$

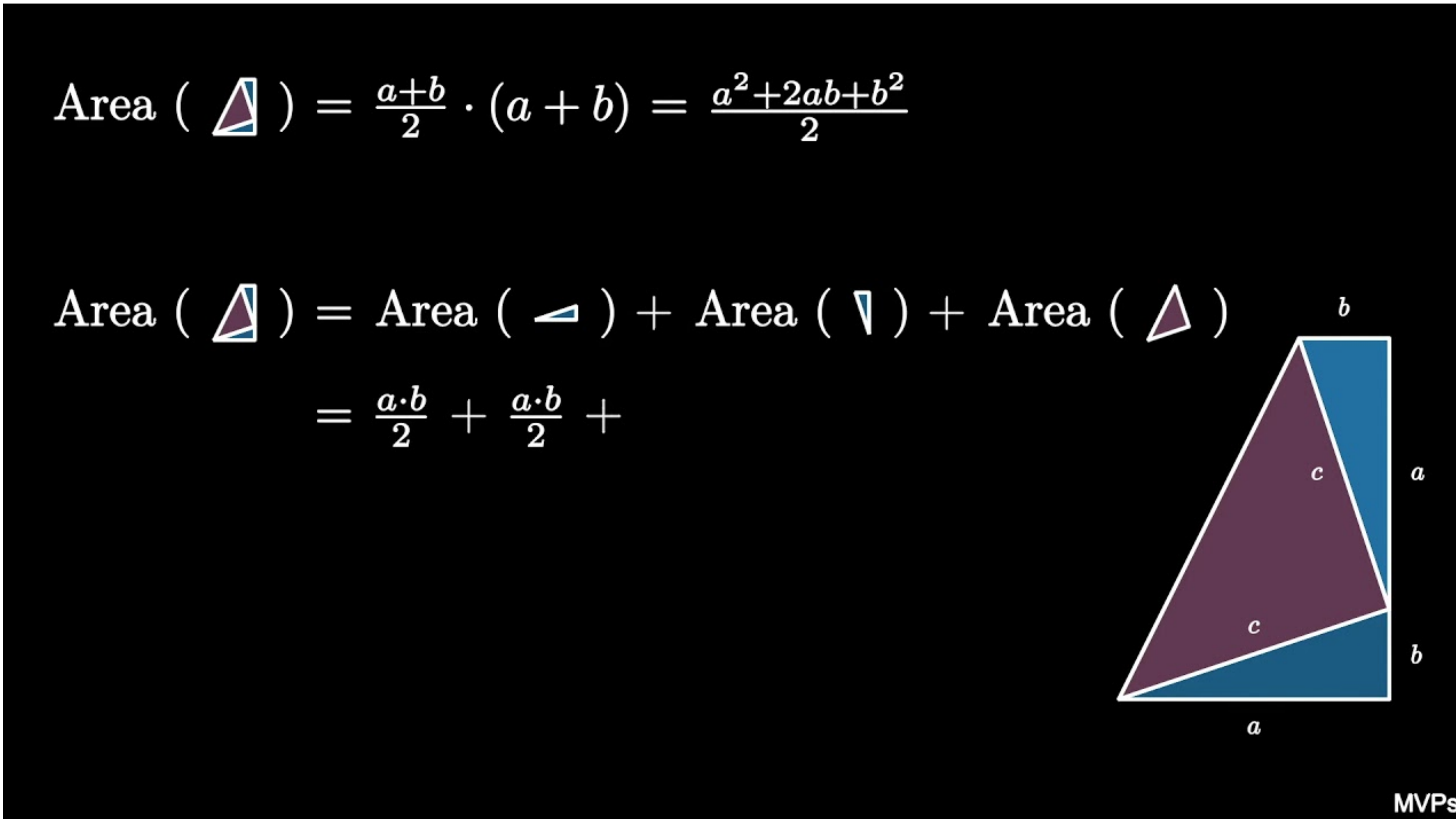
$$\text{Area}\left(\ \right) = \text{Area}\left(\ \right) + \text{Area}\left(\ \right) + \text{Area}\left(\ \right)$$

$$= \frac{a \cdot b}{2} + \frac{a \cdot b}{2} +$$



MVPs

More Pythagorean Proofs



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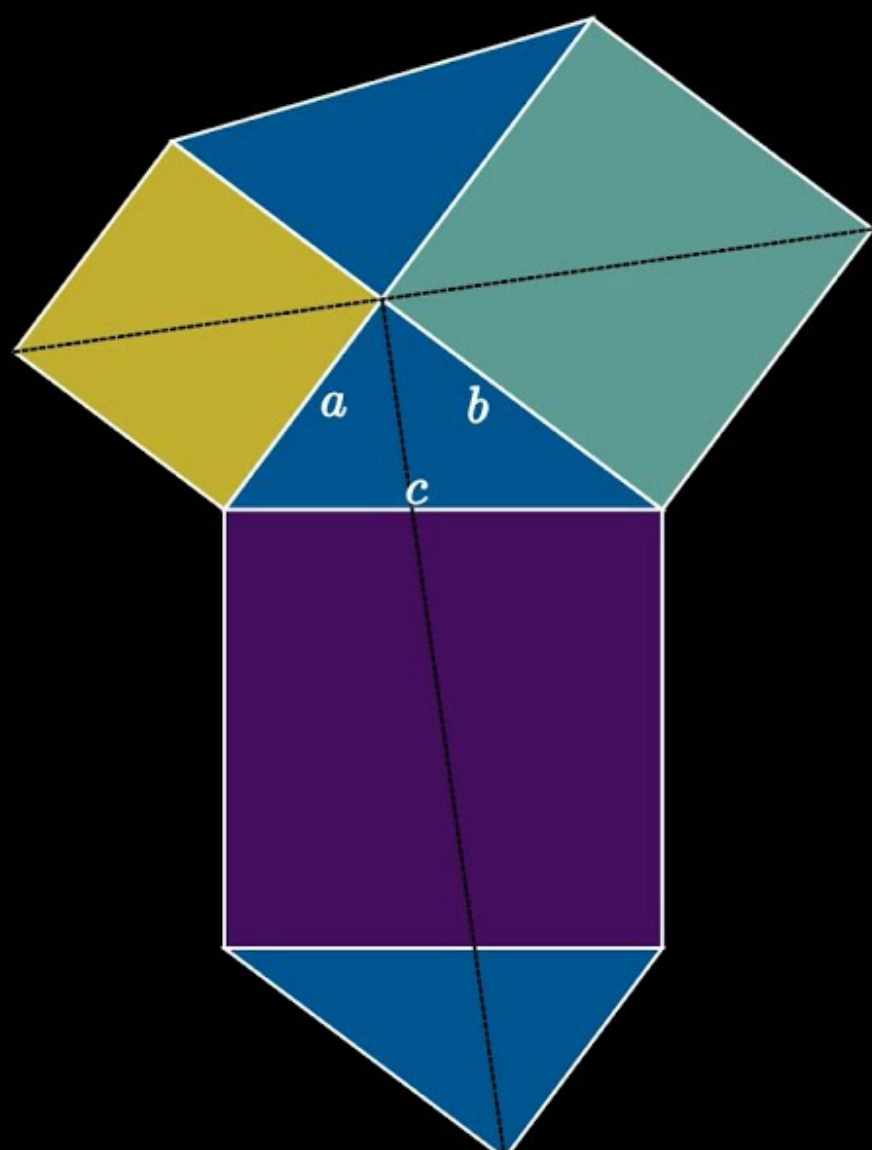

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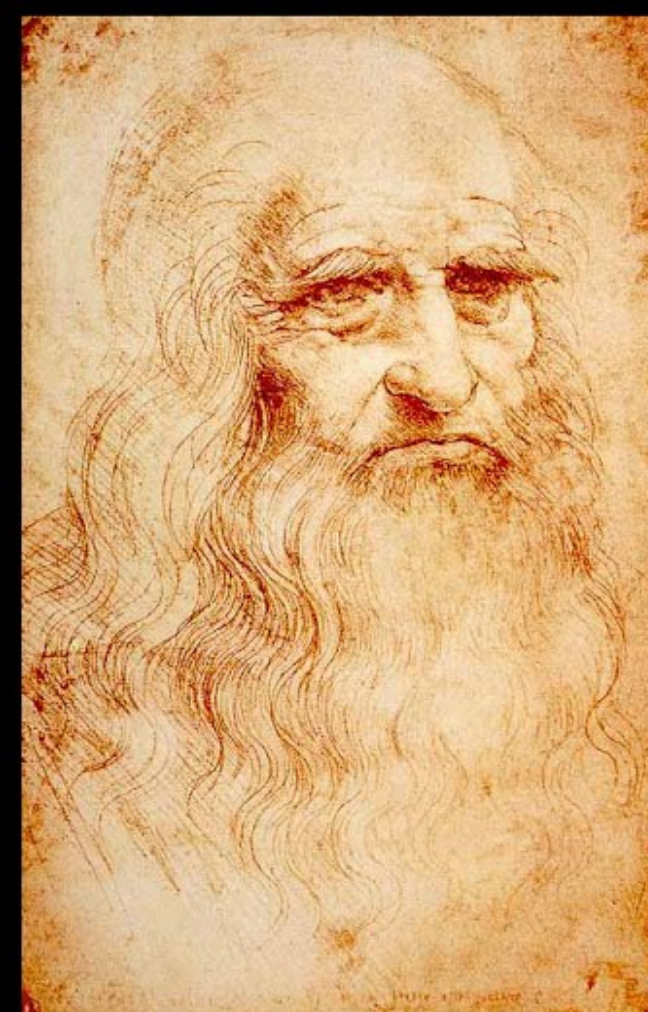

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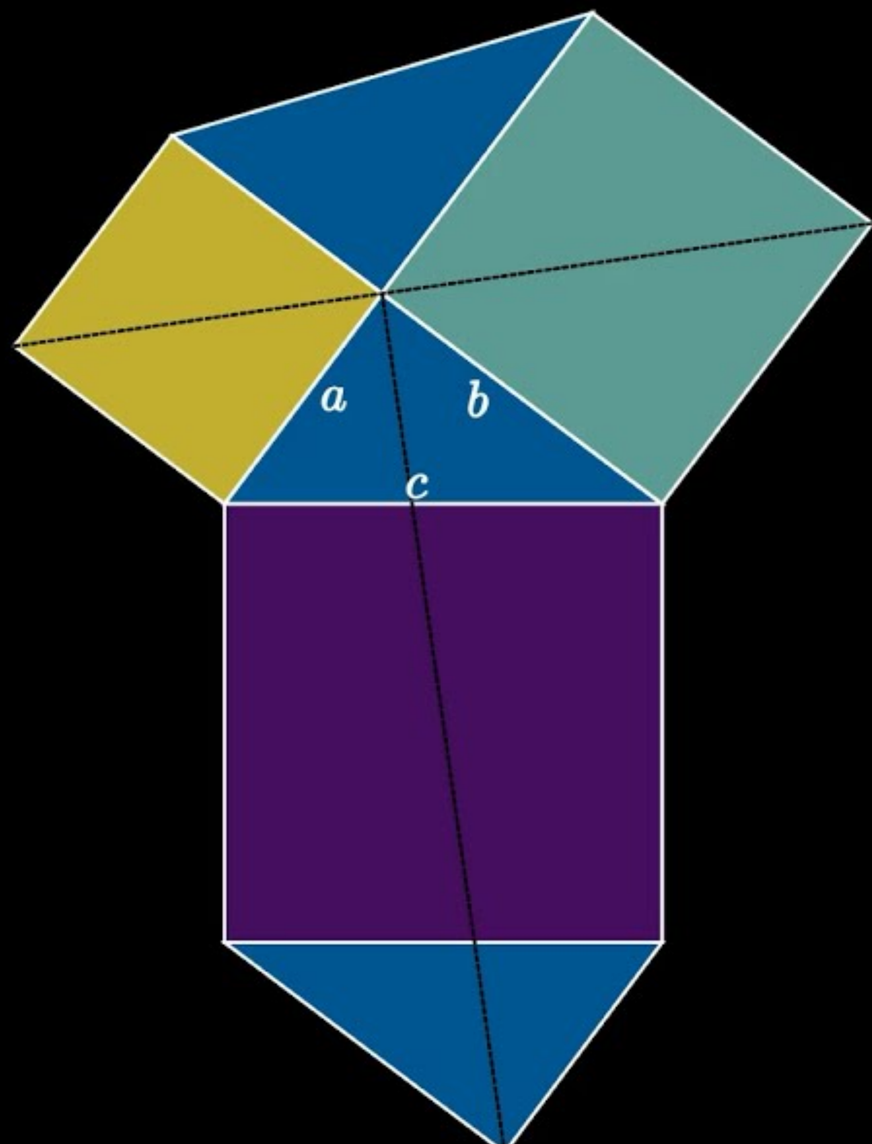

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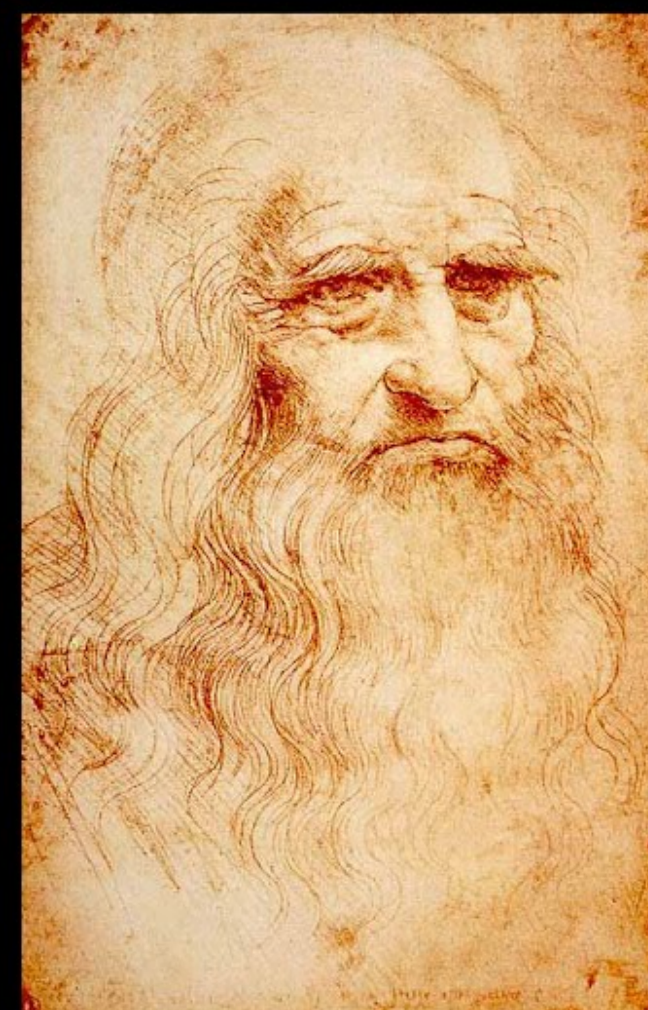

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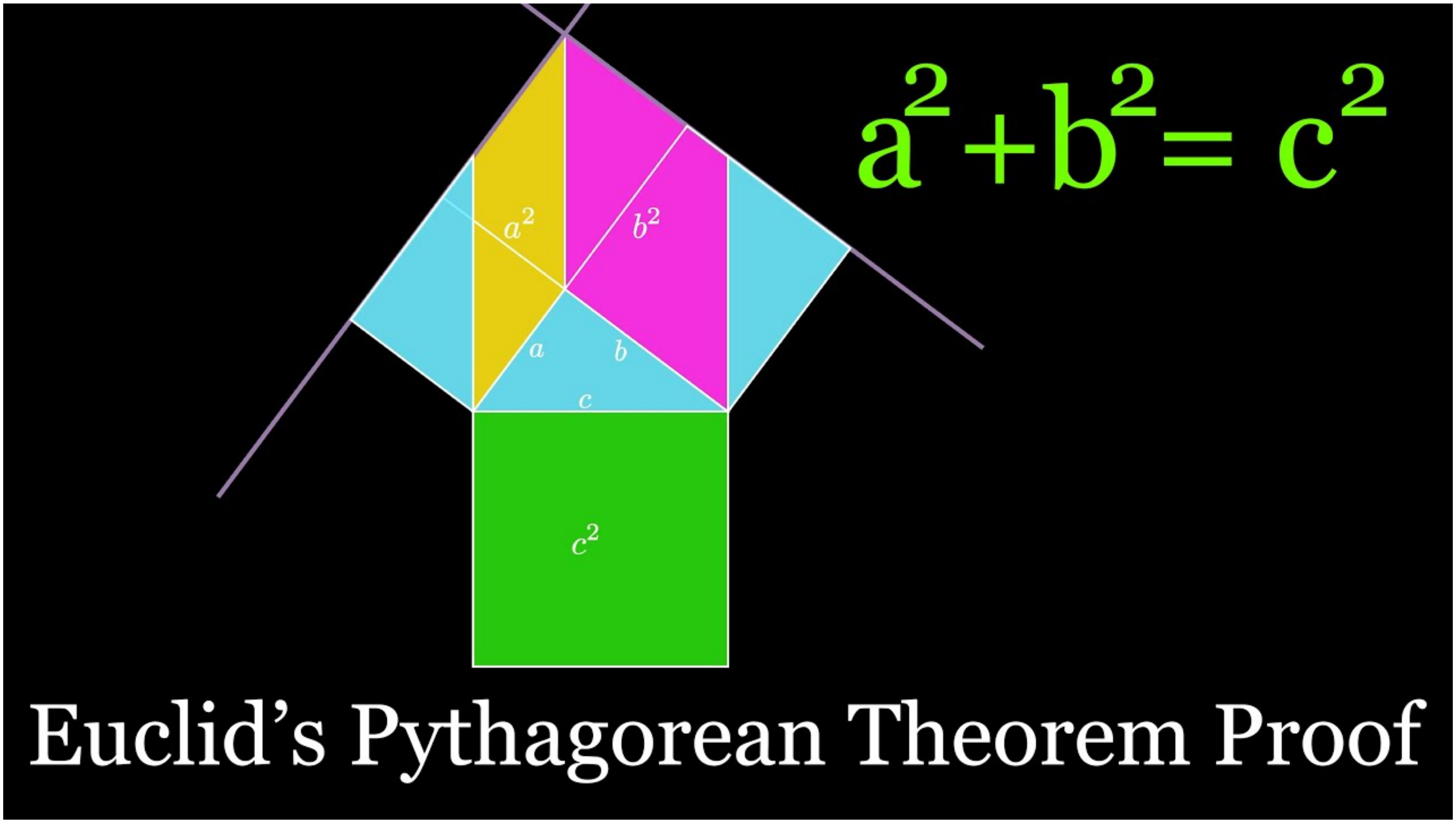

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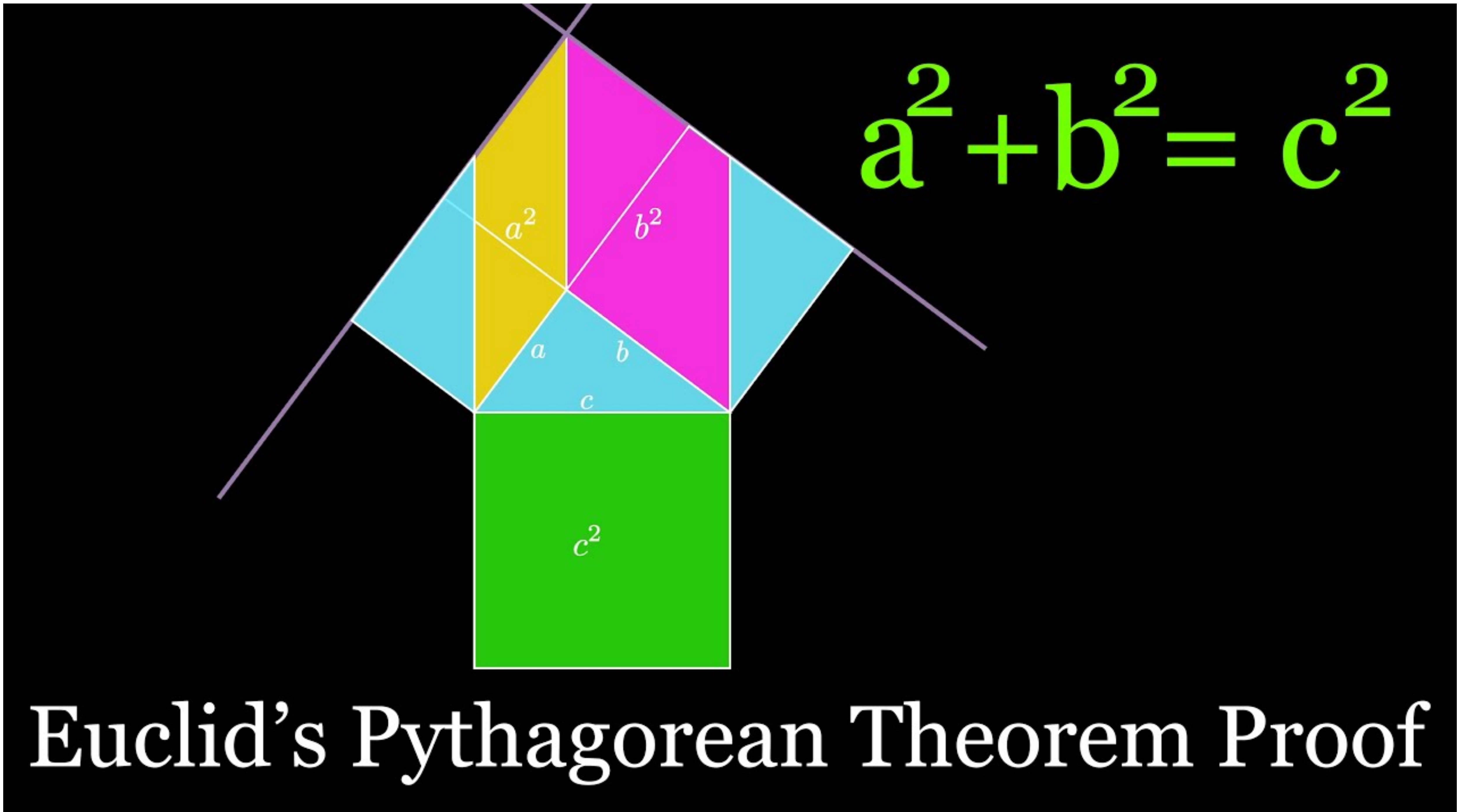

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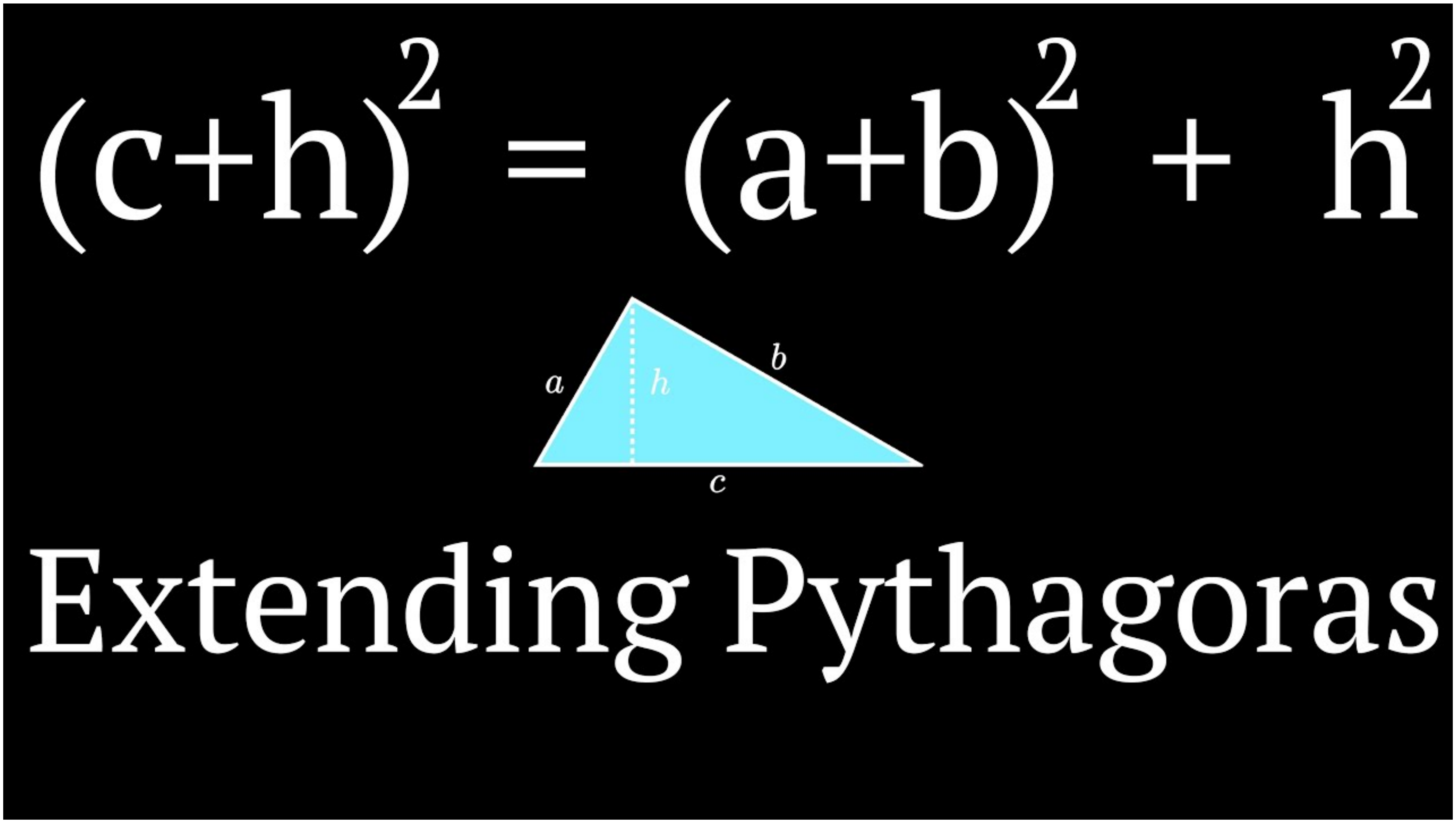

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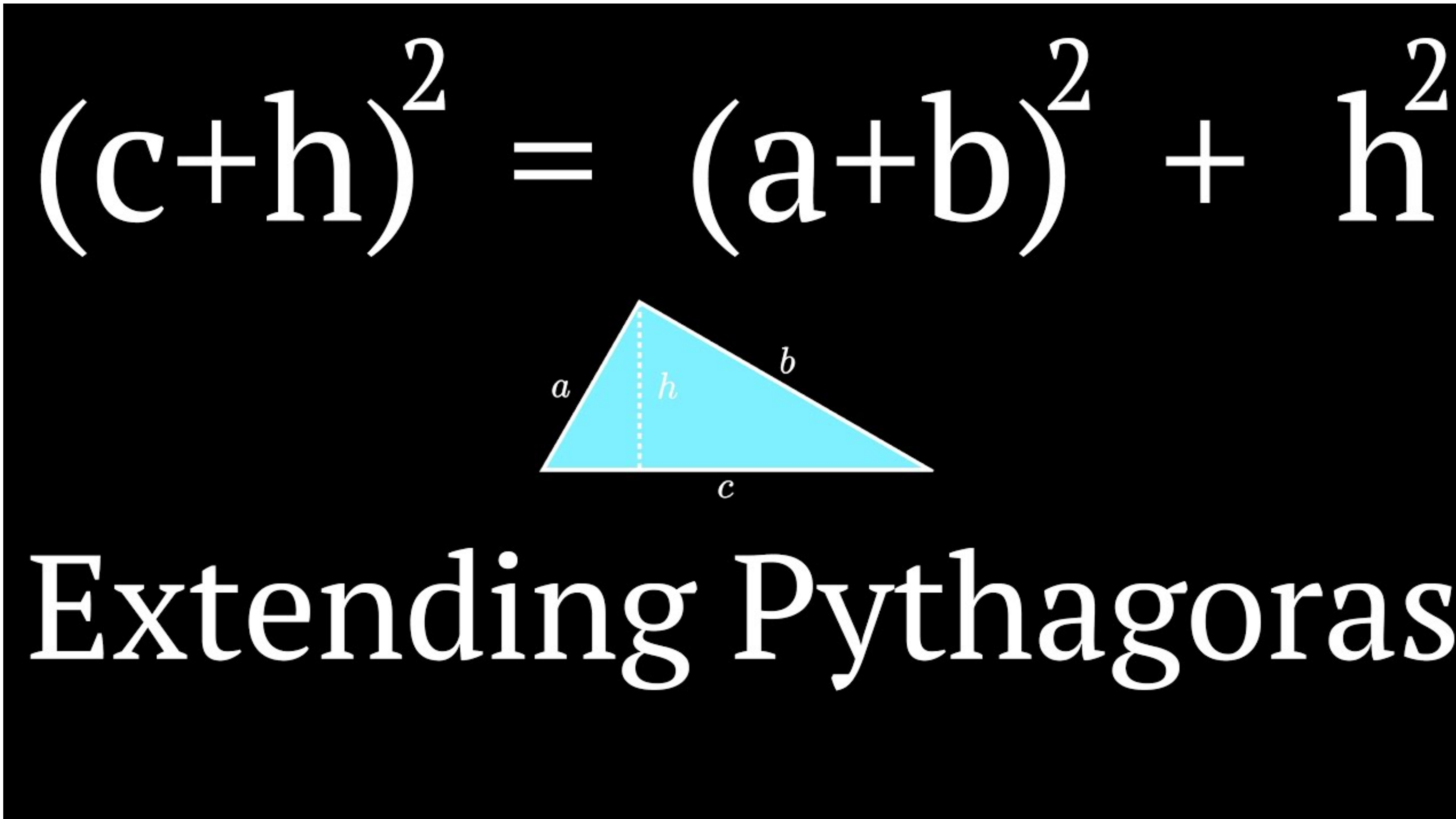

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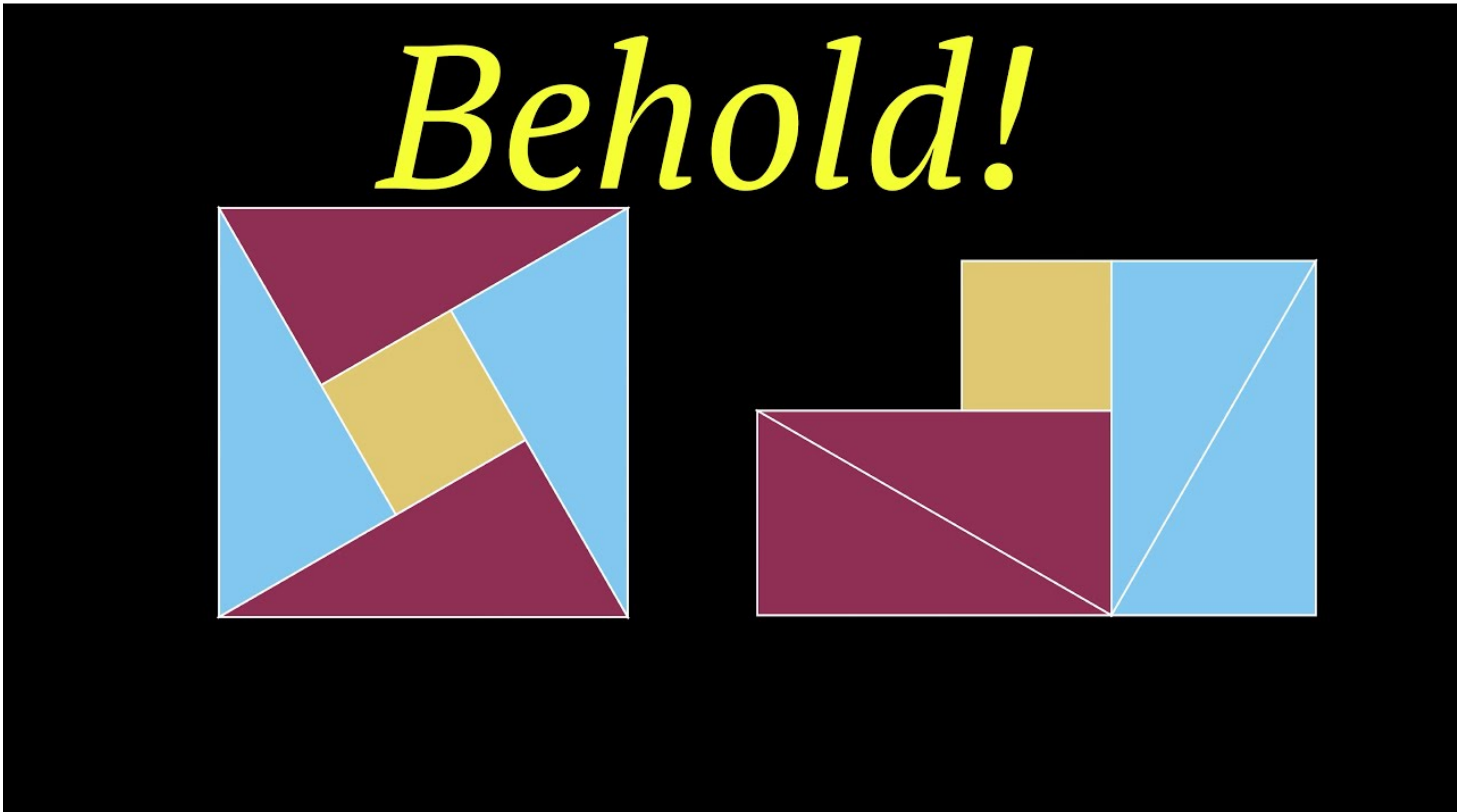

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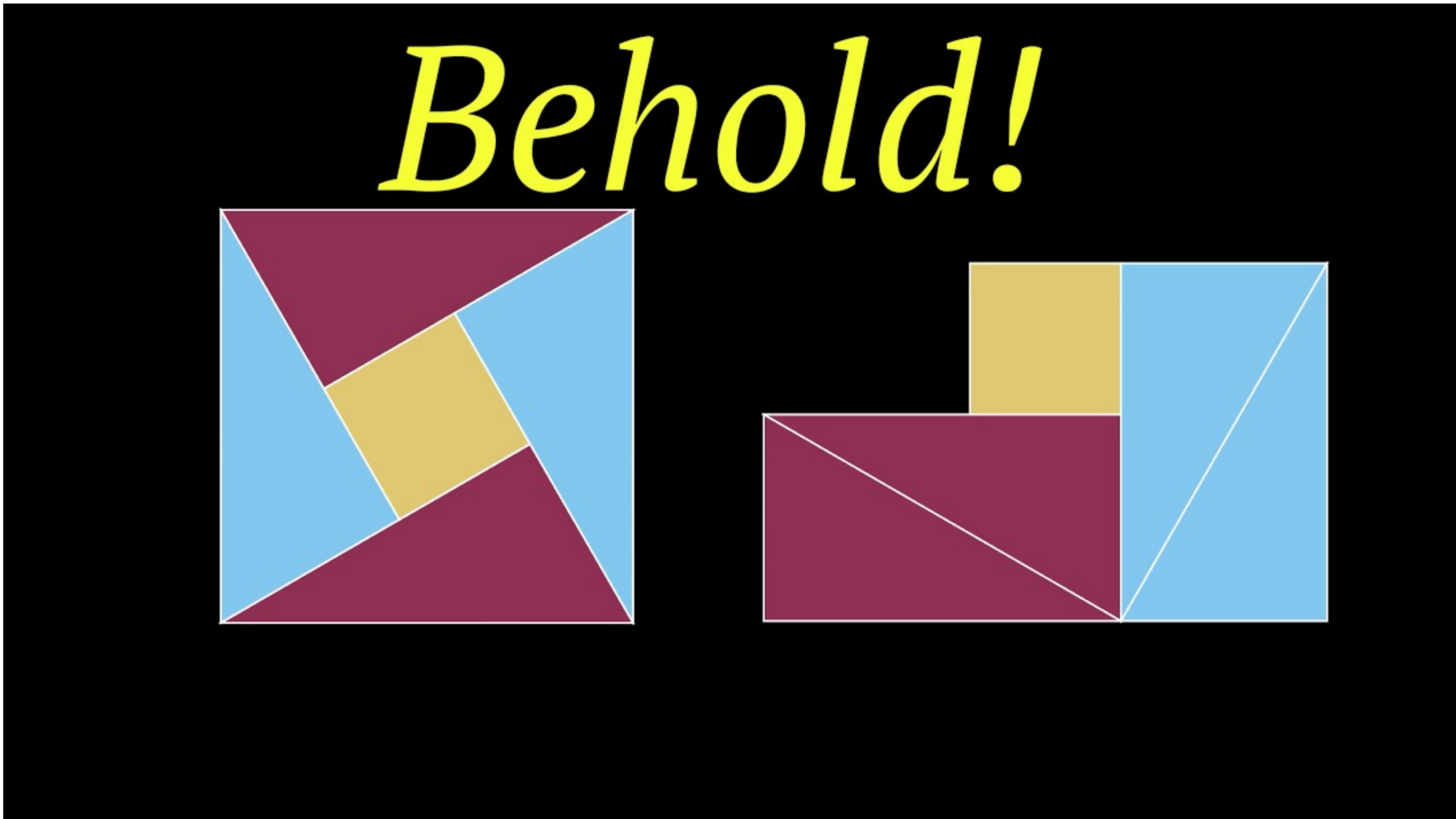

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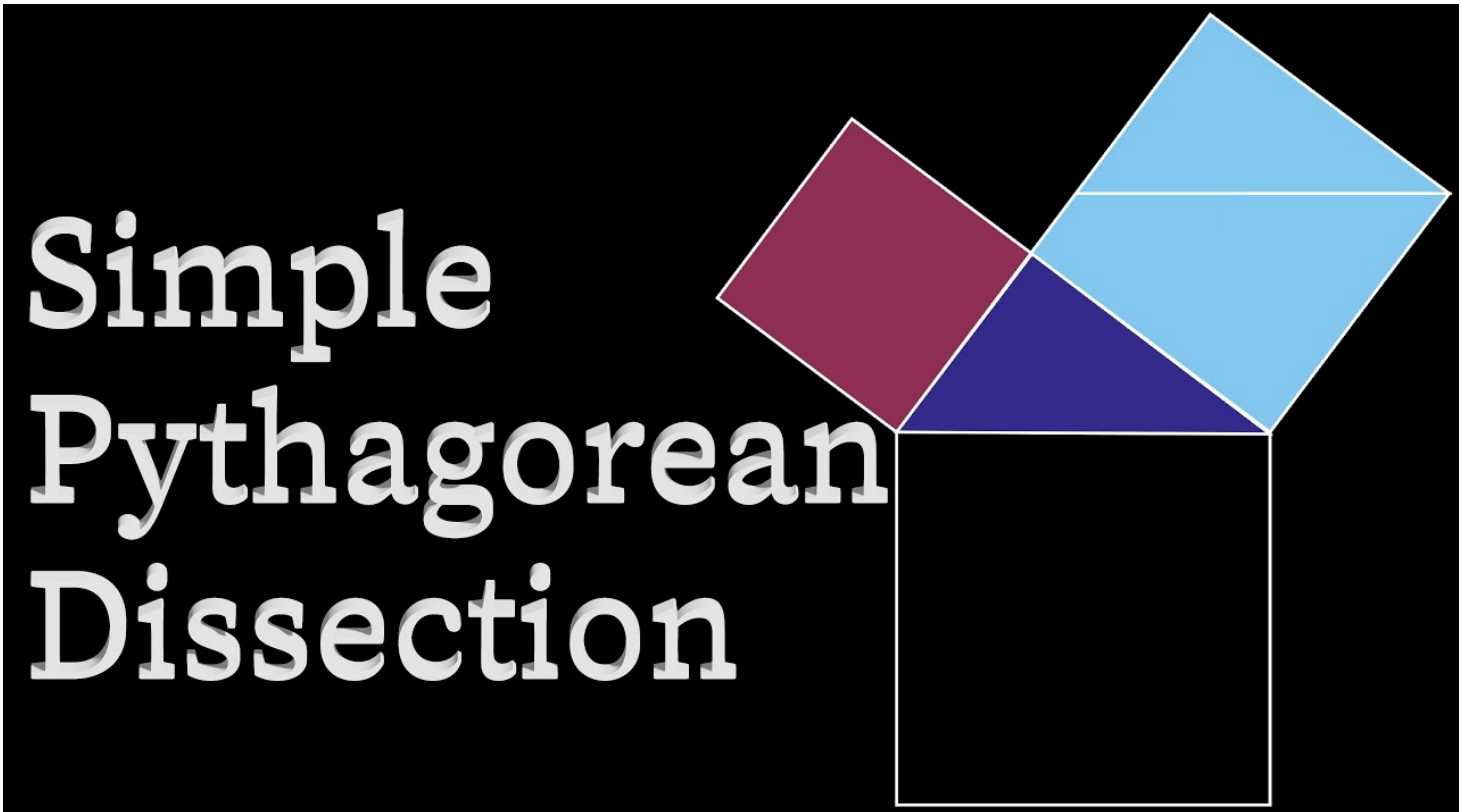

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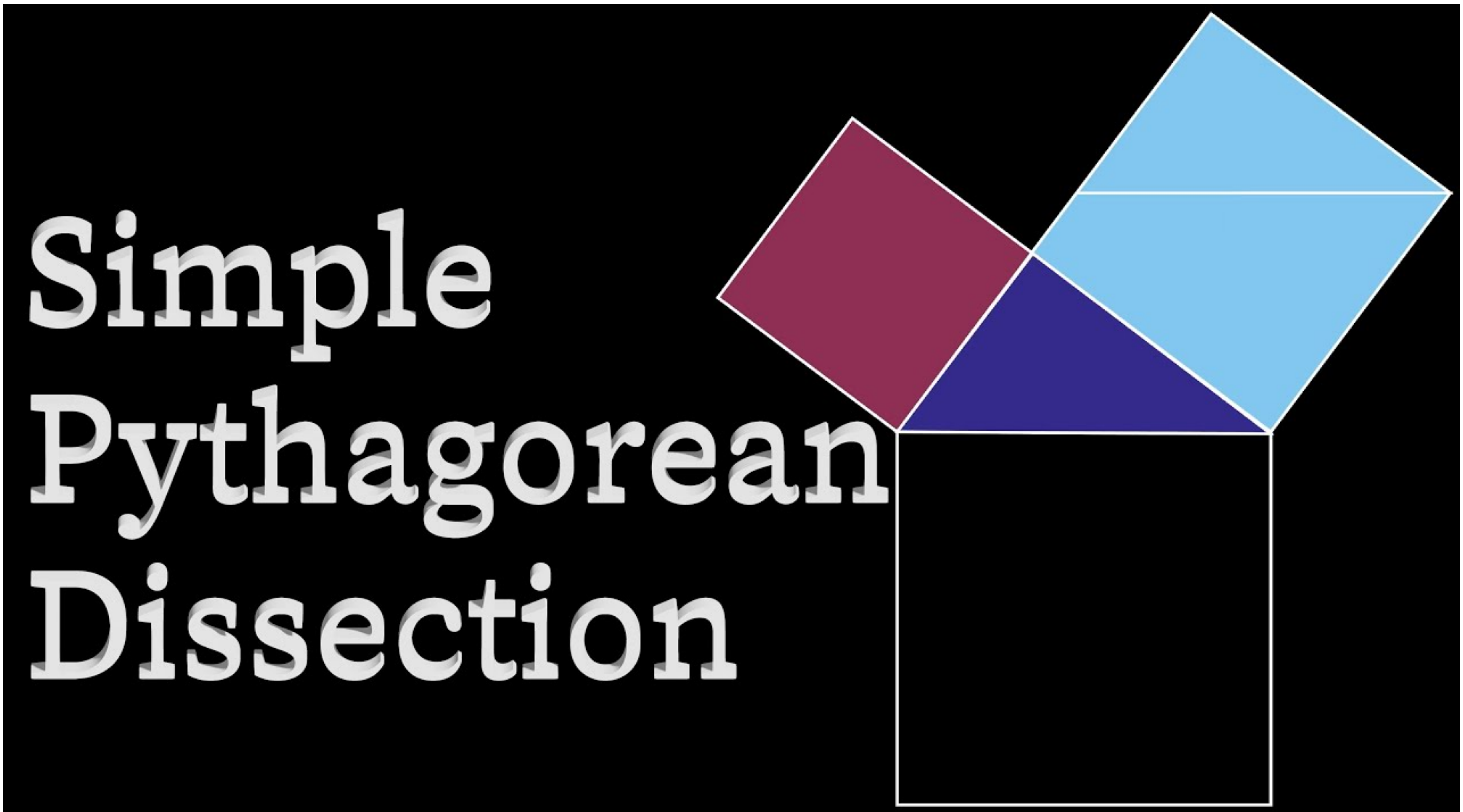

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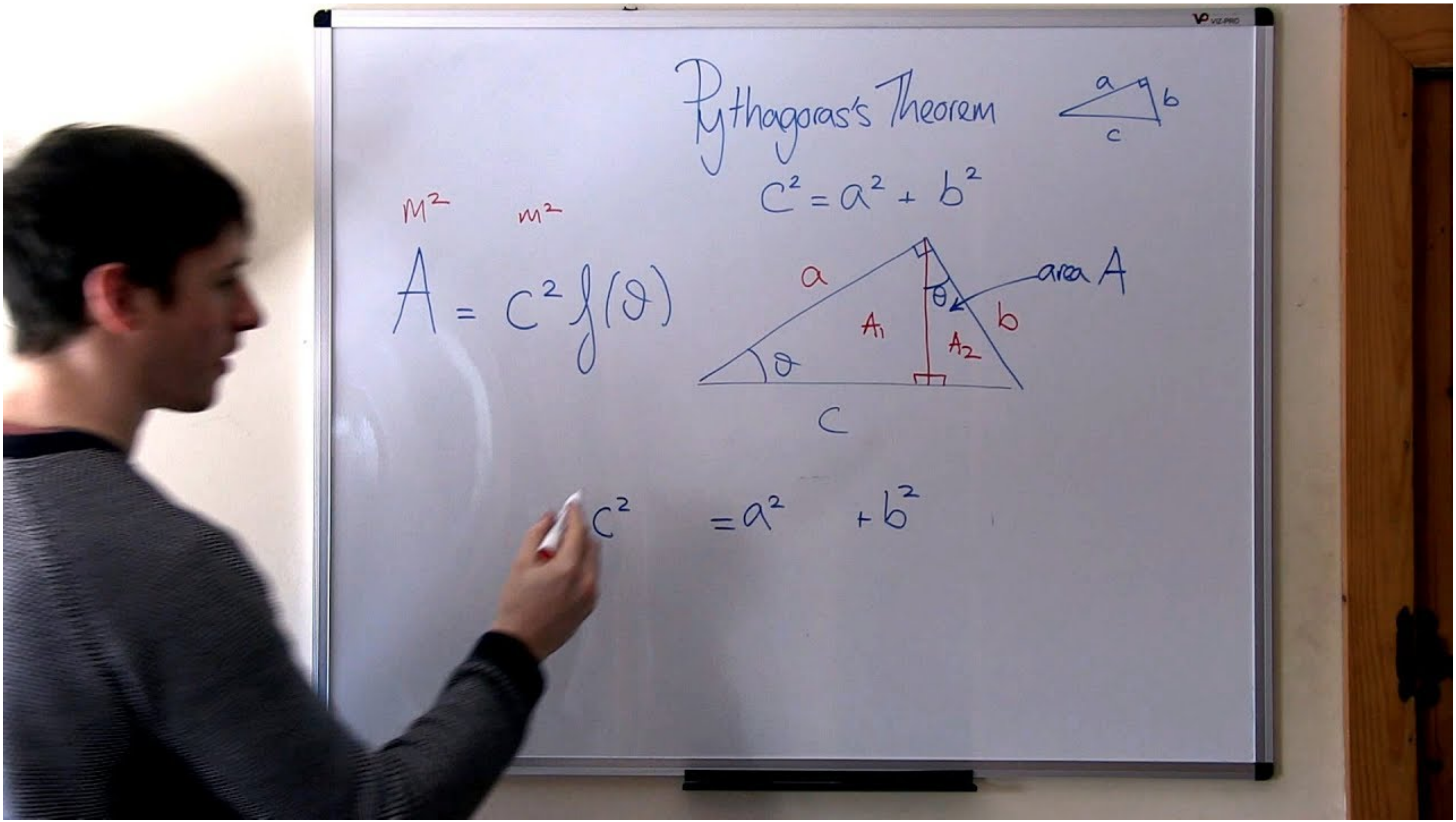

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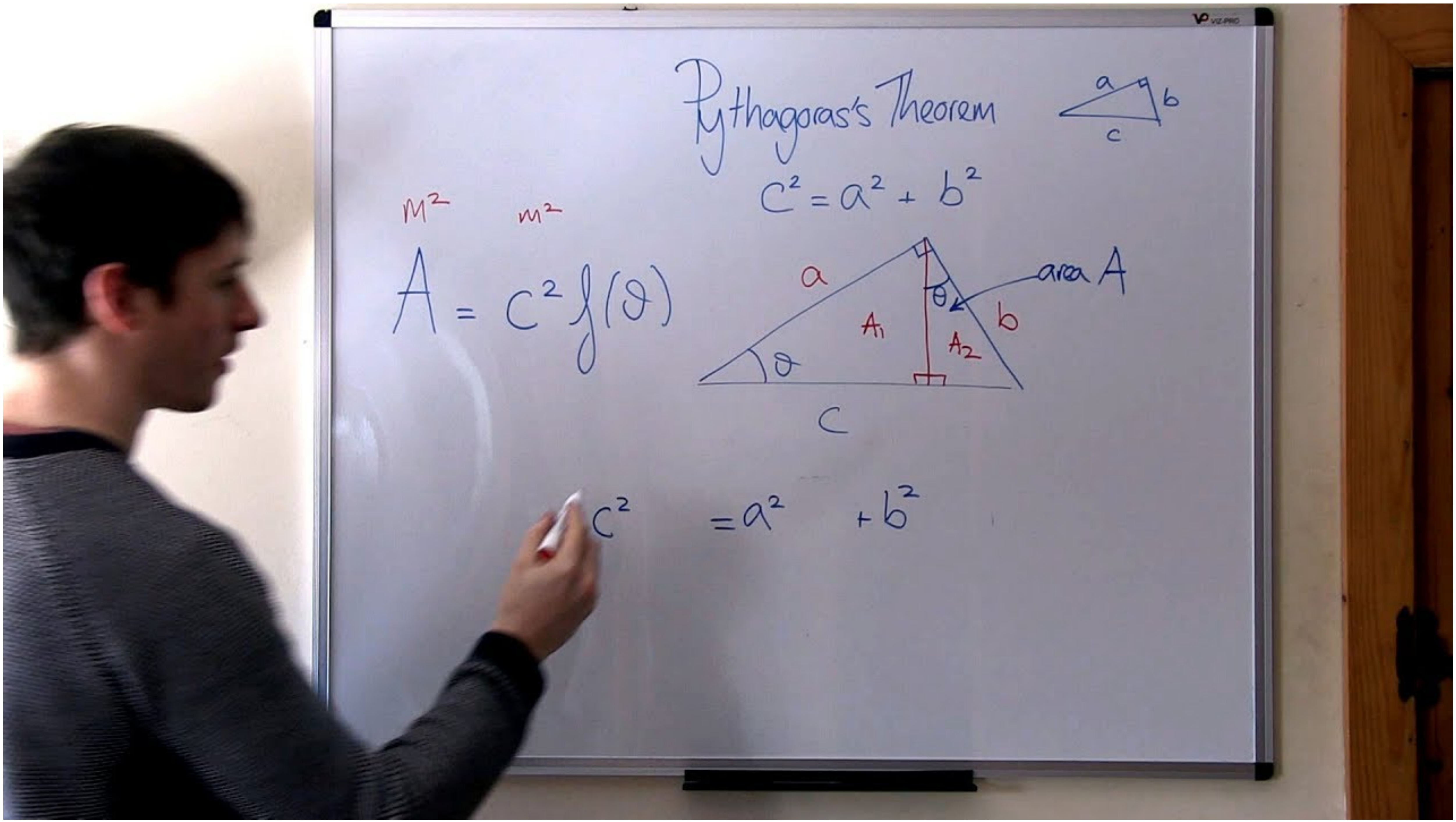

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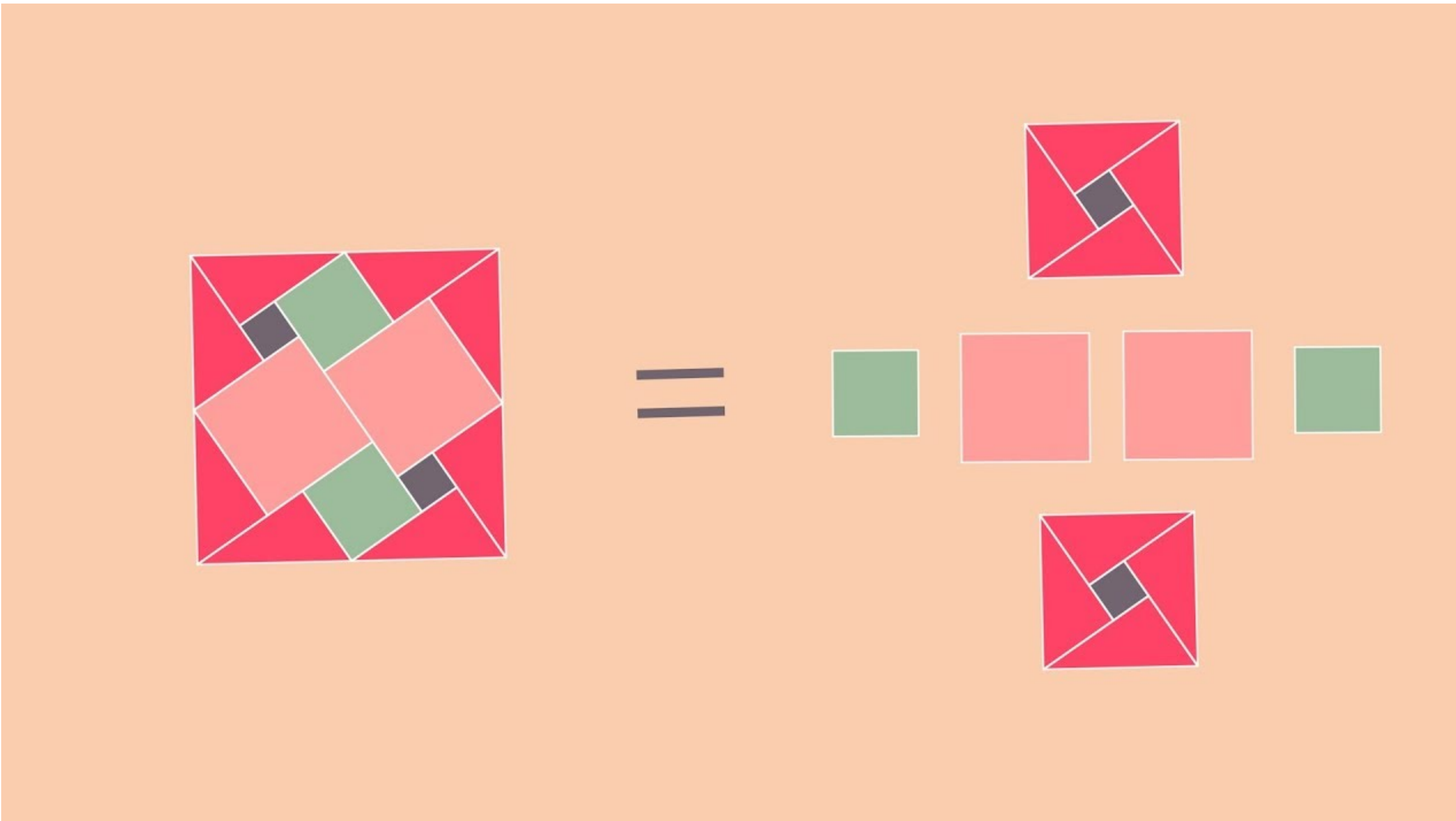

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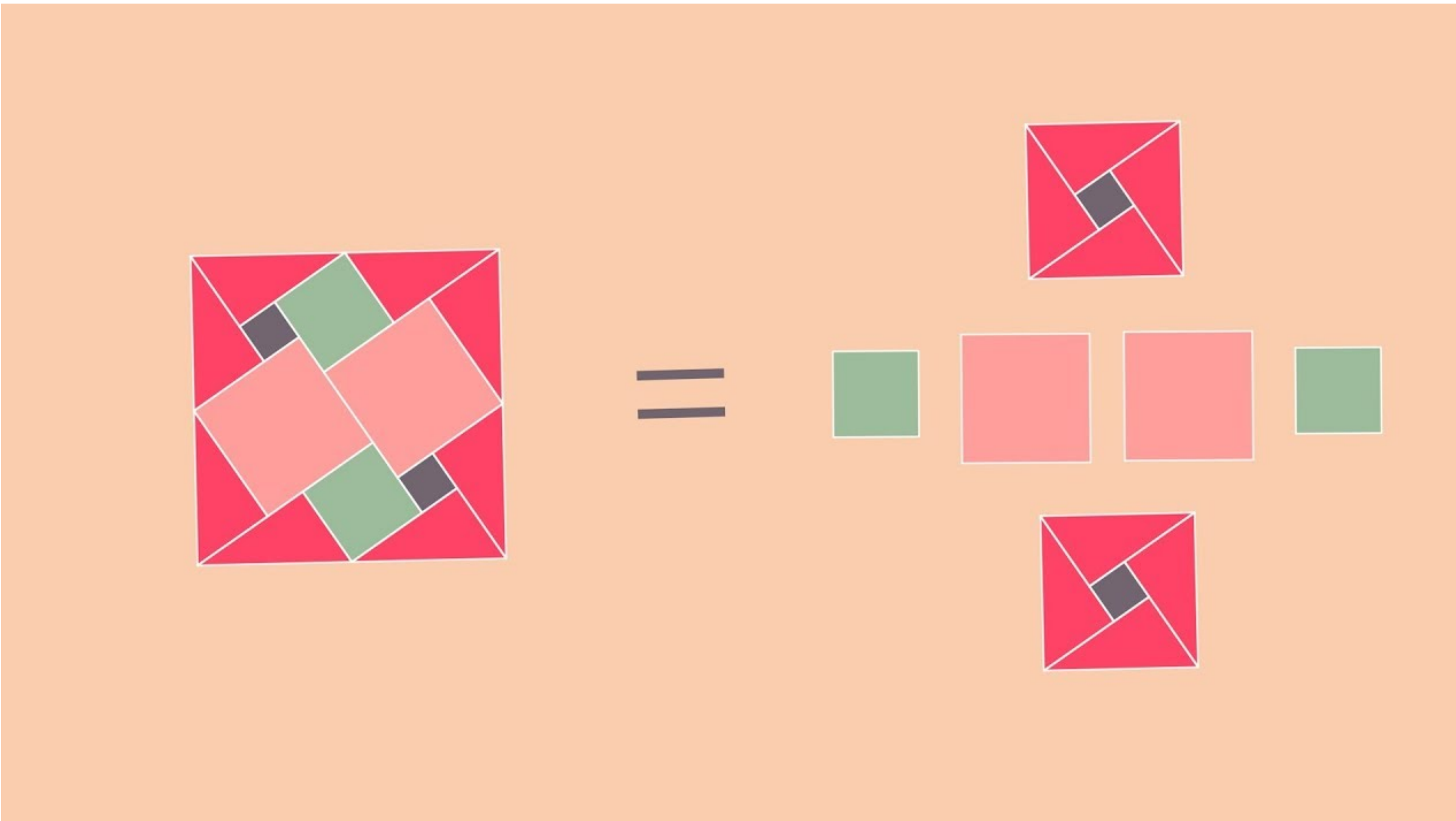

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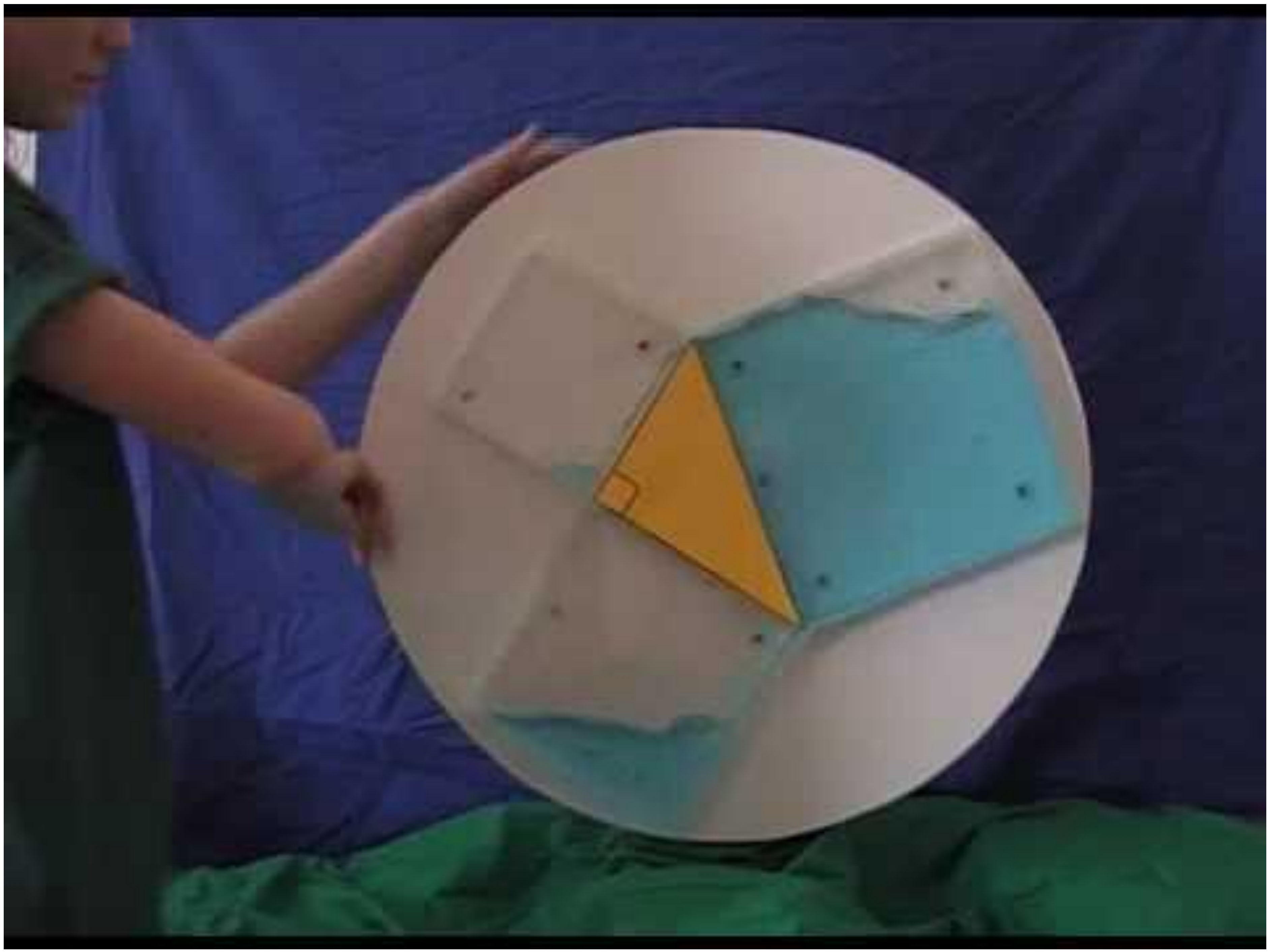

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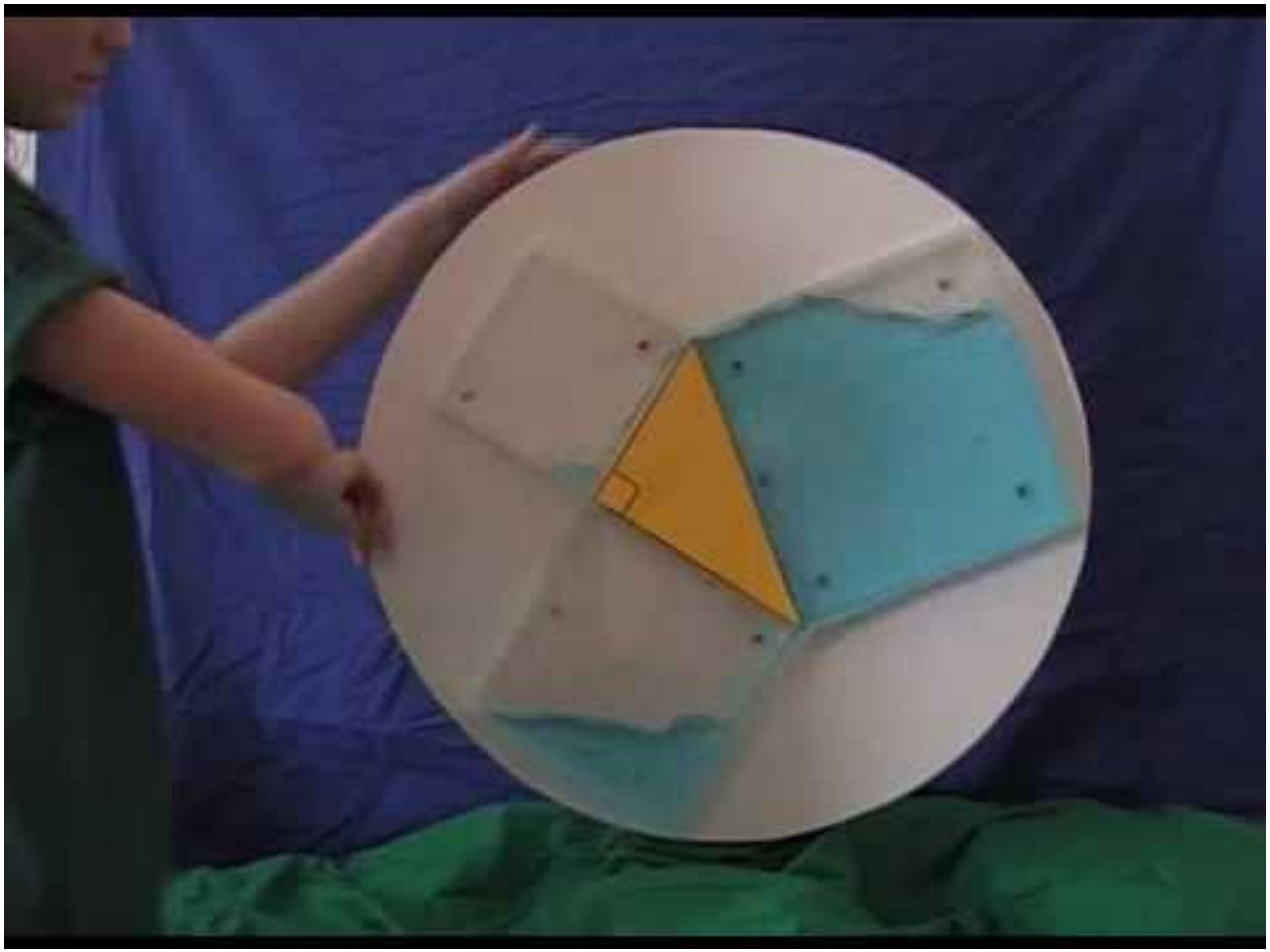

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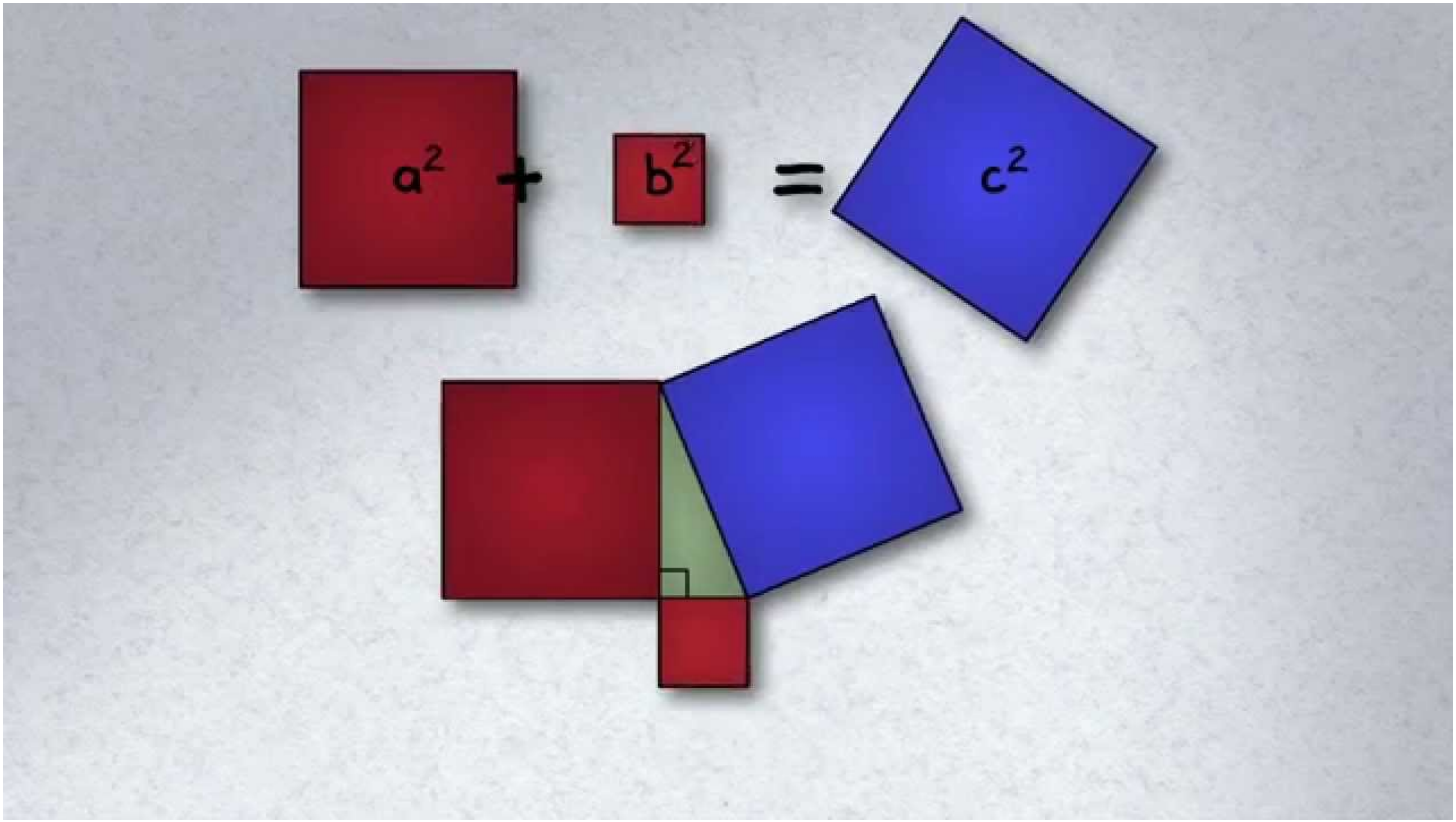

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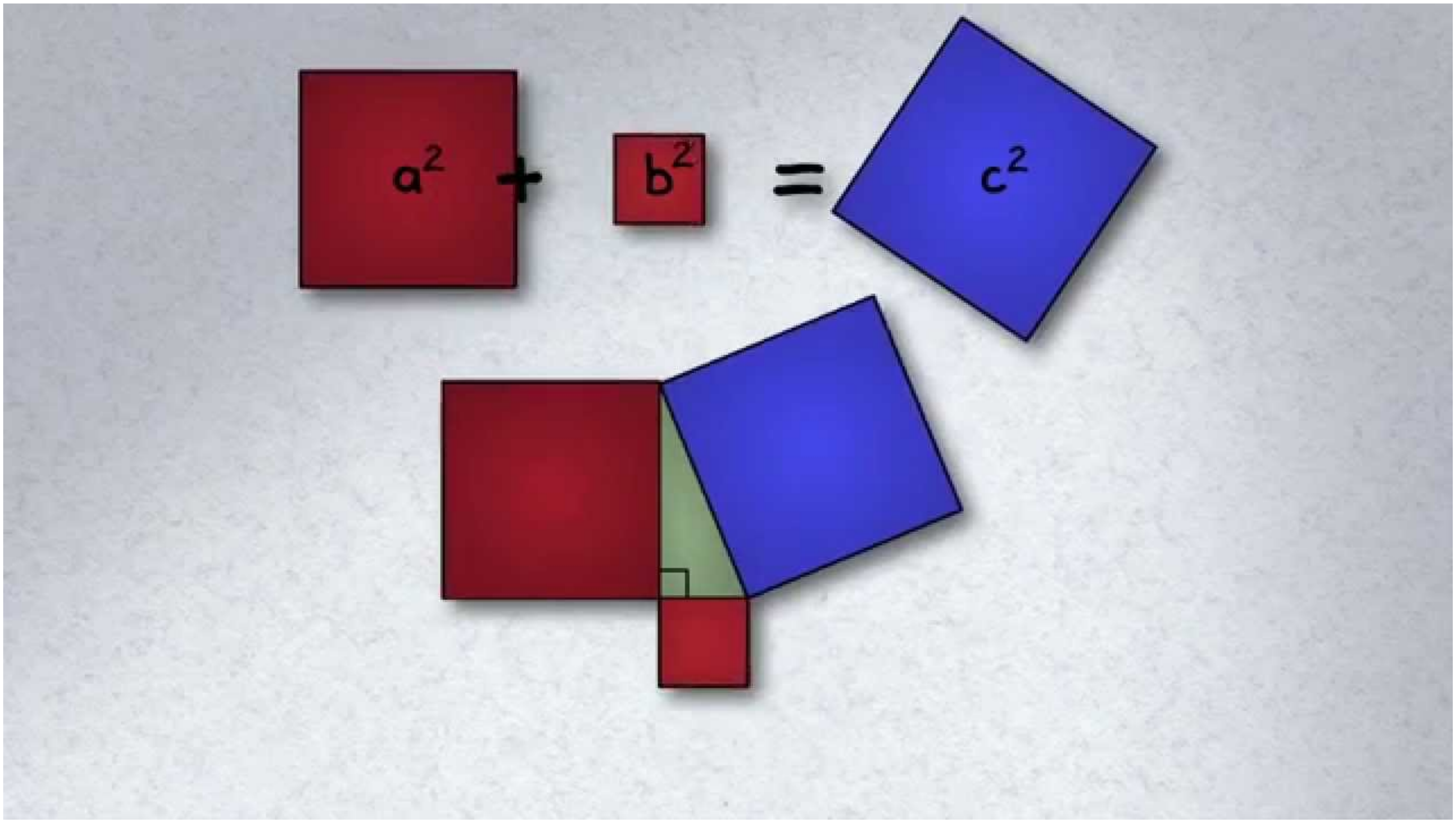

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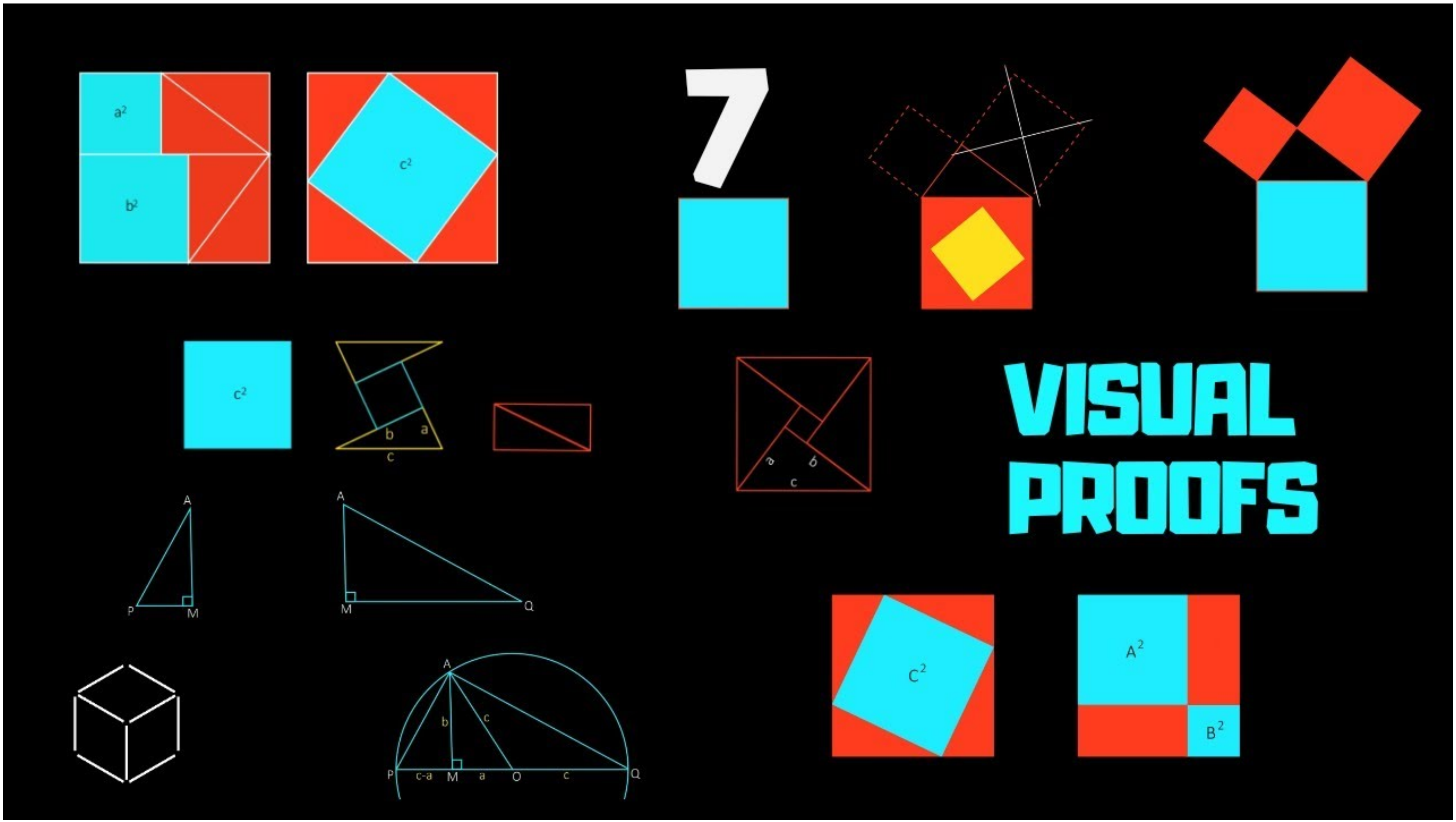

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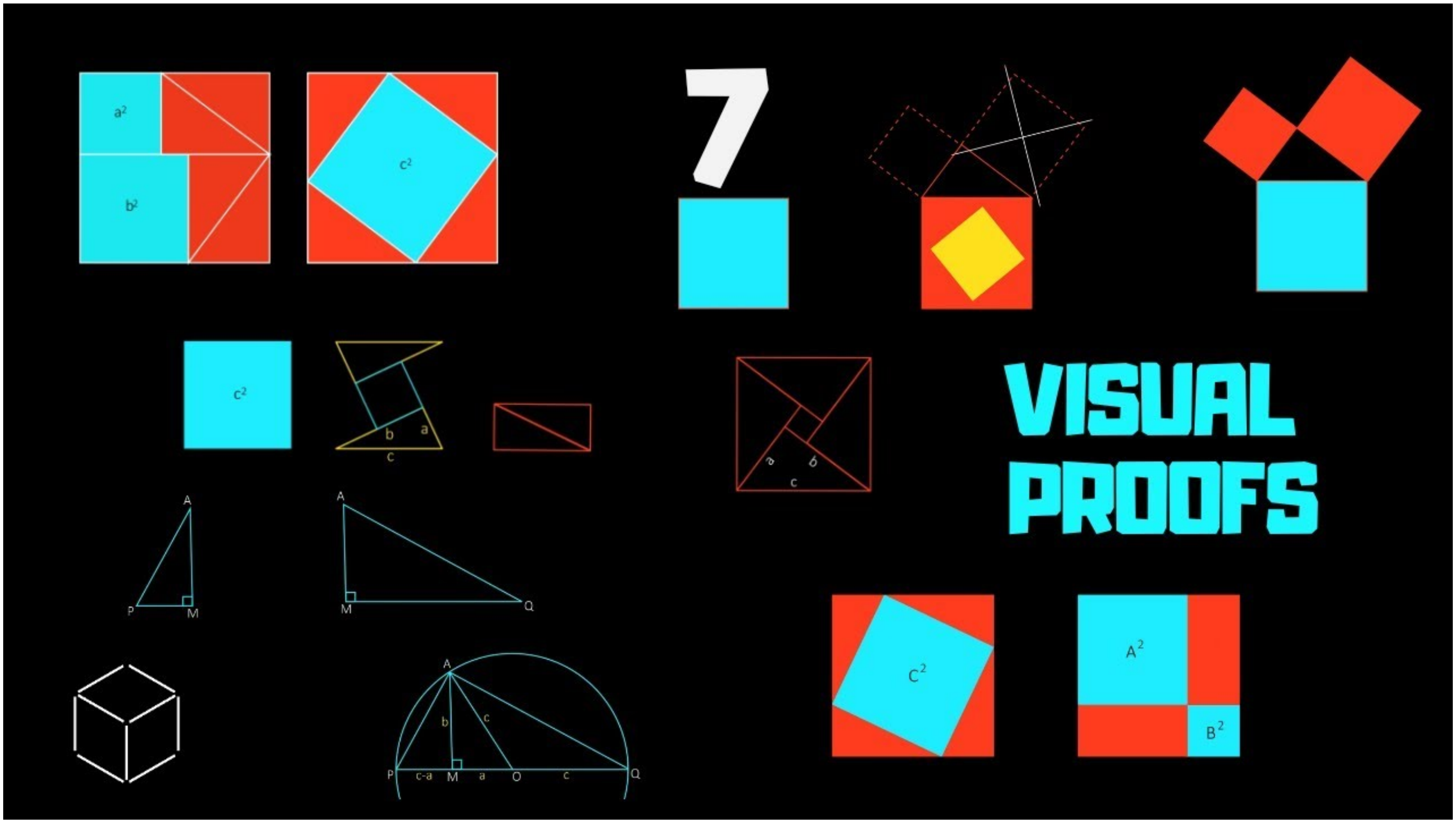

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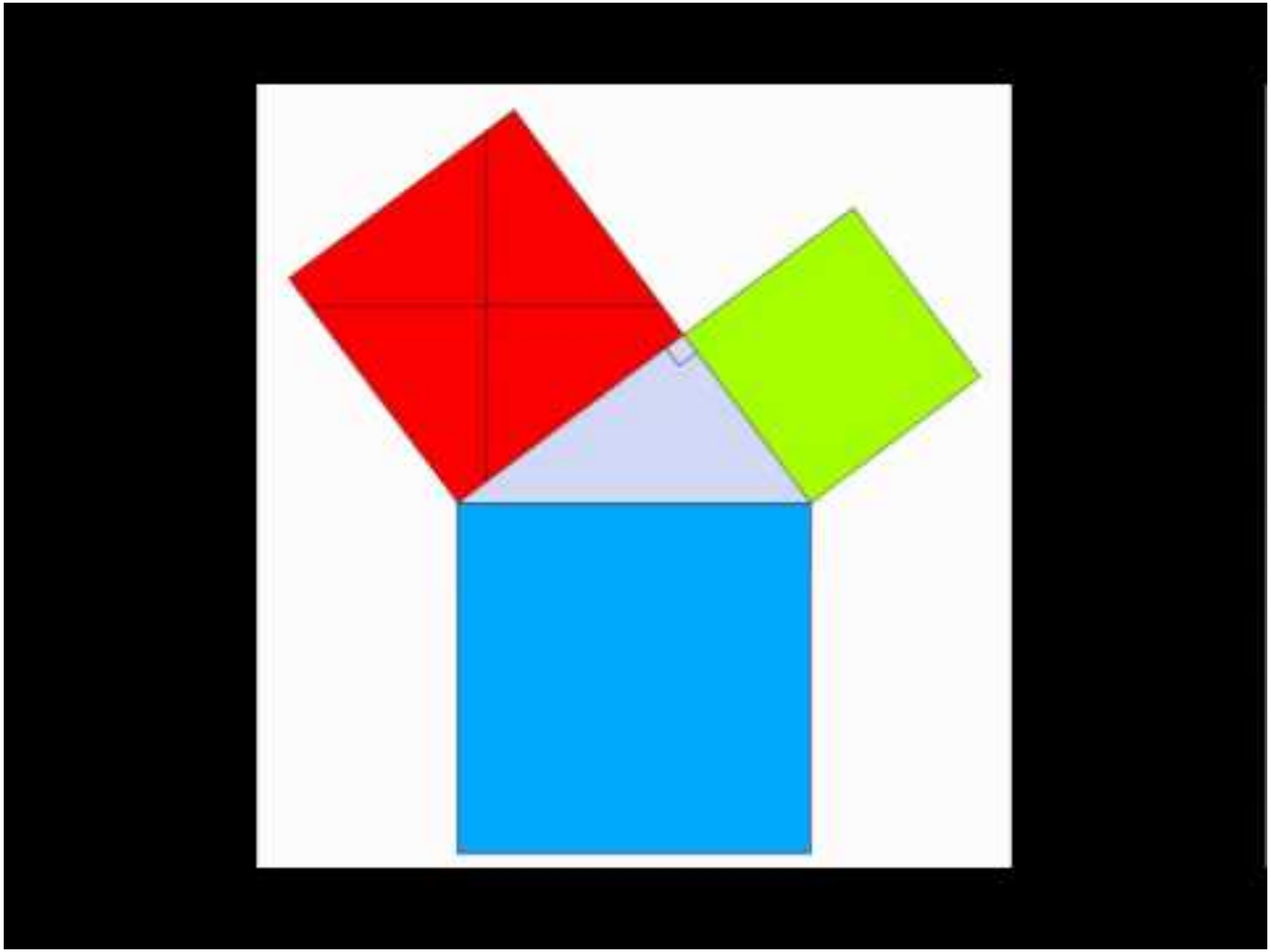

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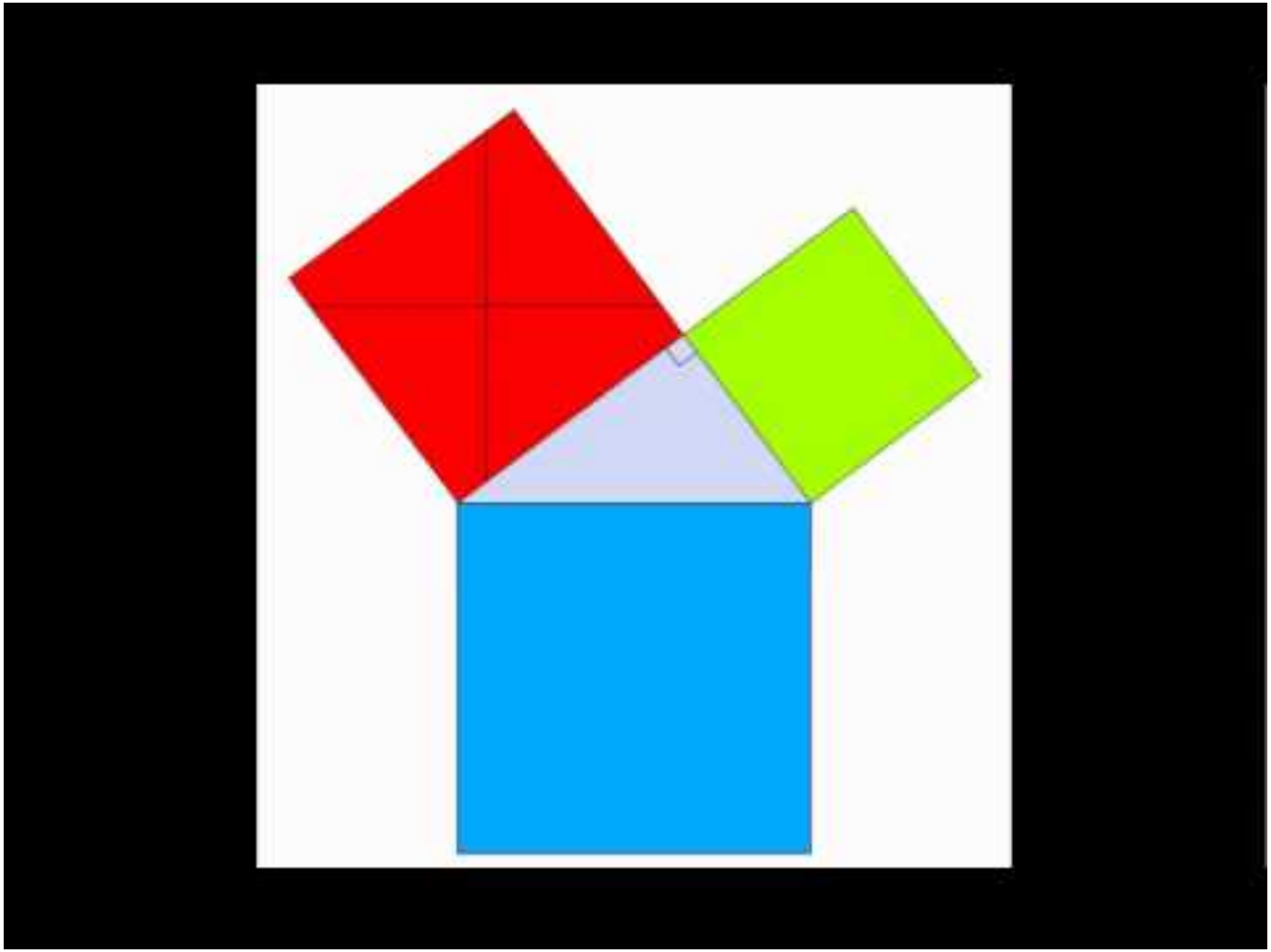
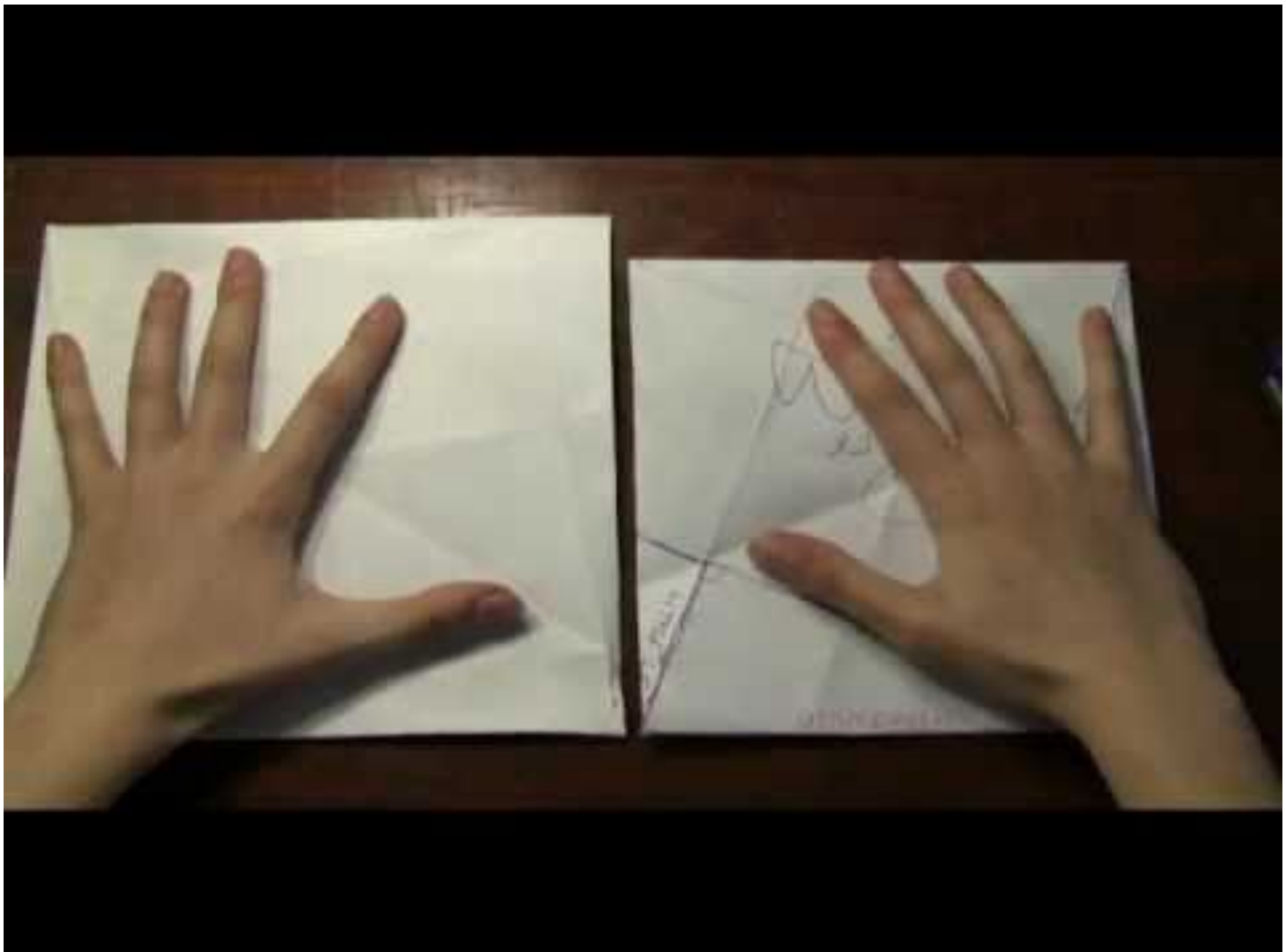


Pythagorean Theorem



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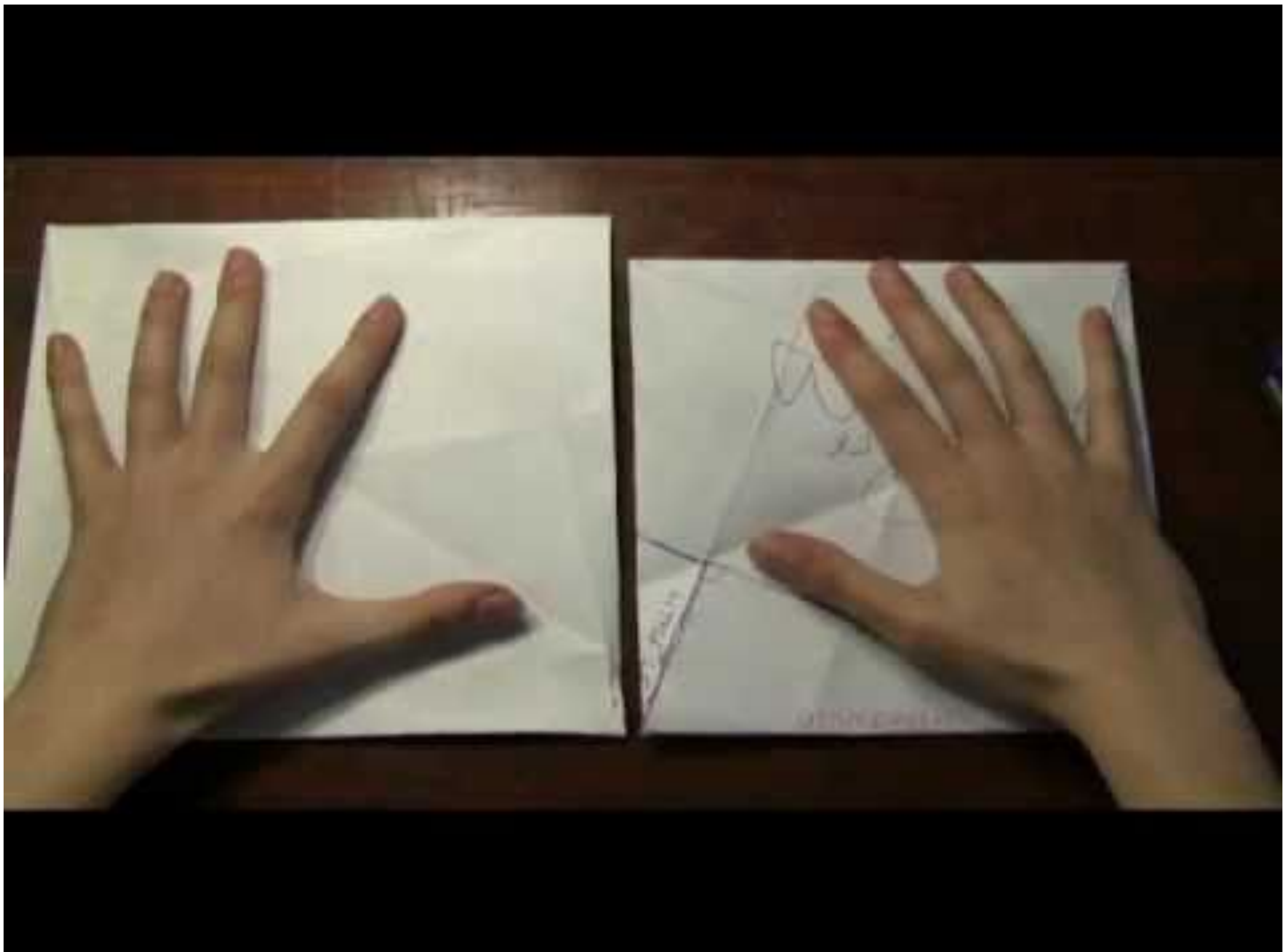


4WWL


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People’s History

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> My brother ...taught me what he had been learning in those 3 months; that is, the *Practical* part of *Common Arithmetick* ...This was my first entry into *Mathematicks*, and all the *Teaching* I had. ...I did thenceforth prosecute it, (at School and in the University) not as a formal Study, but as a pleasing Diversion, at spare hours; as books of *Arithmetick*, or others *Mathematical* fel occasionally in my way. ...For Mathematicks, (at that time, with us) was scarce looked upon as *Academical* Studies, but rather *Mechanical*; as the business of *Traders*, *Merchants*, *Seamen*, *Carpenters*, *Suveyors of Lands*, or the like; and perhaps *Almanack-makers in London*. And amongst more than Two hundred Students (at that time) in our College, I do not know of any Two (perhaps not any) who had more of *Mathematicks* than I, (if so much) which was then but little; And but very few, in that whole University. For the Study of *Mathematicks* was at that time more cultivated in London than in the Universities.

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- Instead, an example from Roman land surveyor Marcus Junius Nipsius.

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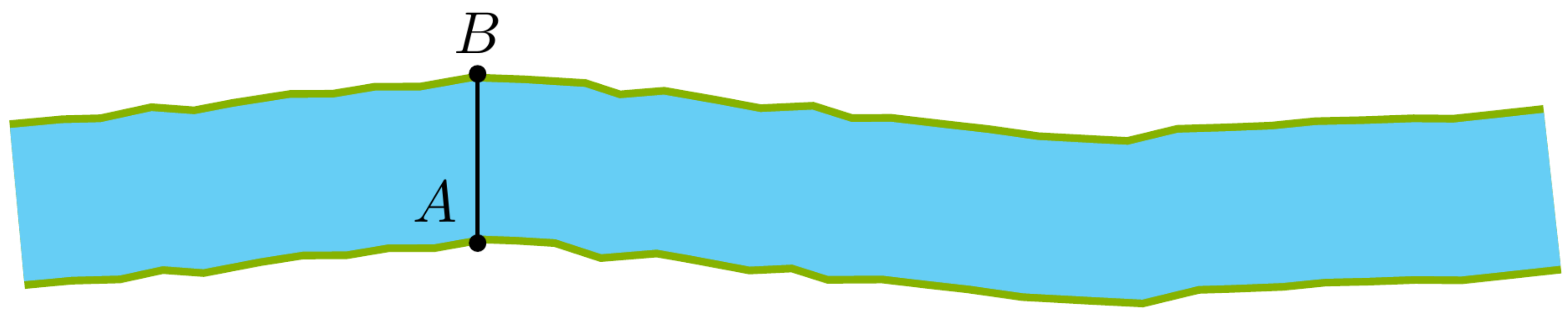

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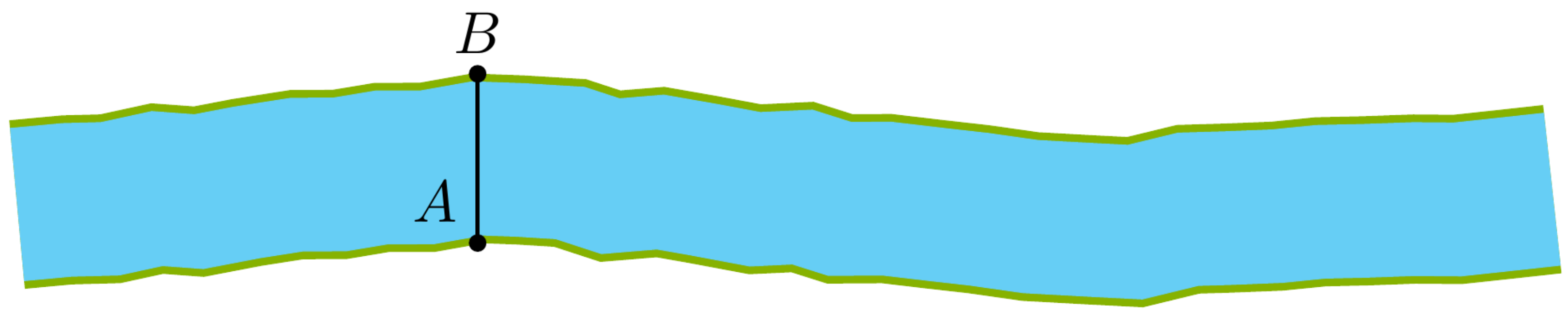

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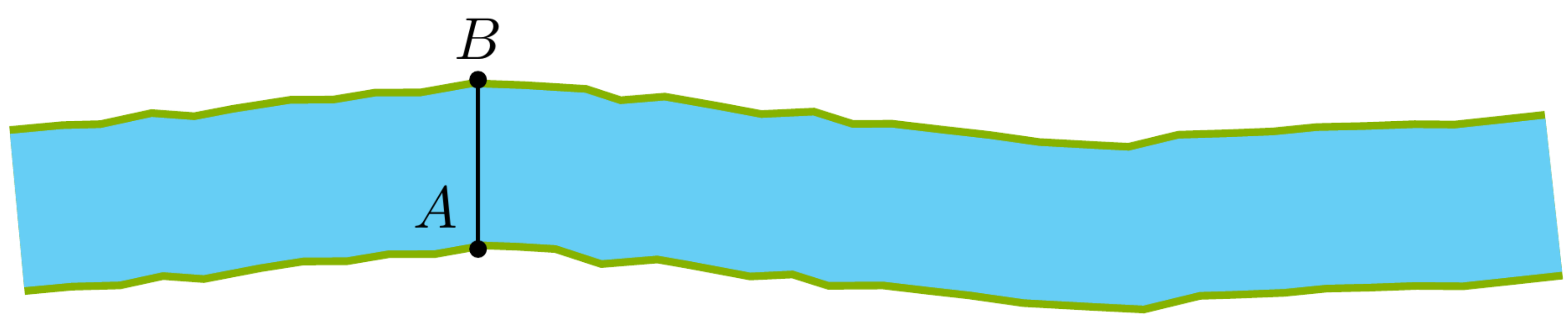
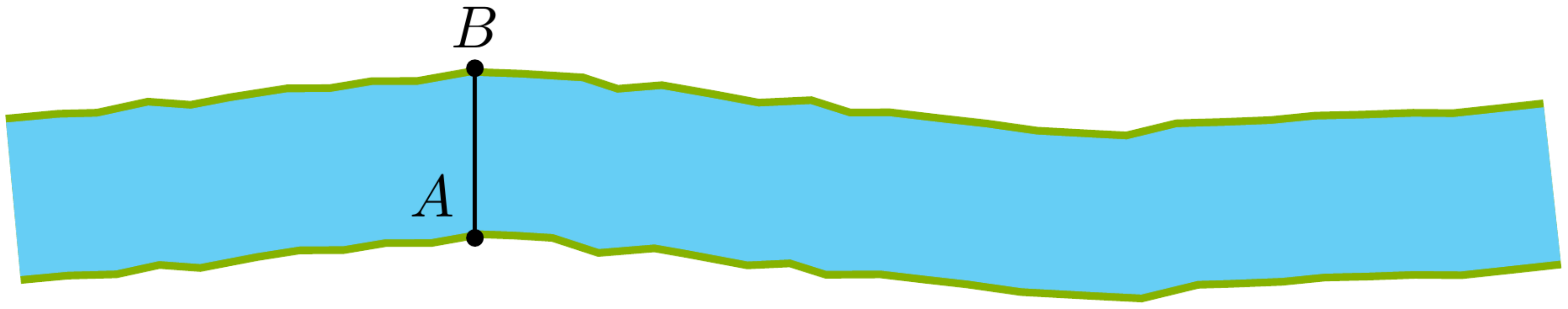


- Maybe, $\boldsymbol{B}$ is some tree, or something else in sight.

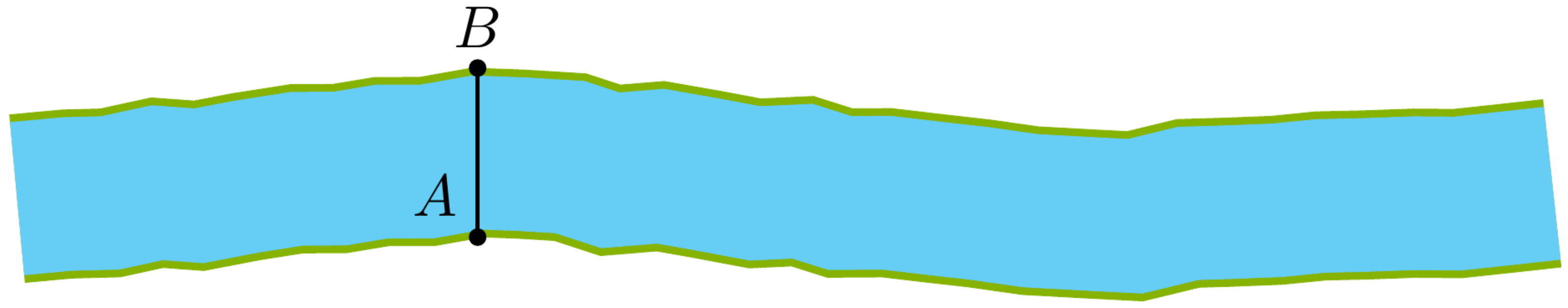
- Rules: You must stay on your side of the river.

People's History of Surveying



People's History of Surveying

- Idea: Extended $\overline{AB}$ to a point C. And choose a point D so that $\angle BCD$ is a right angle.



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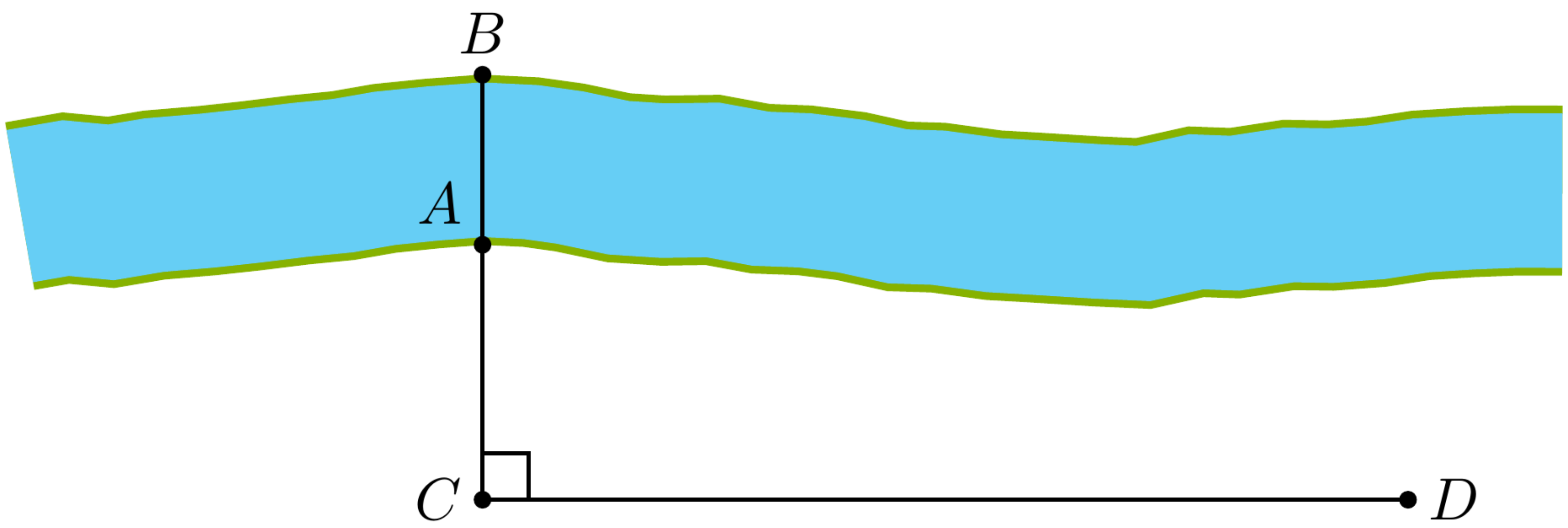

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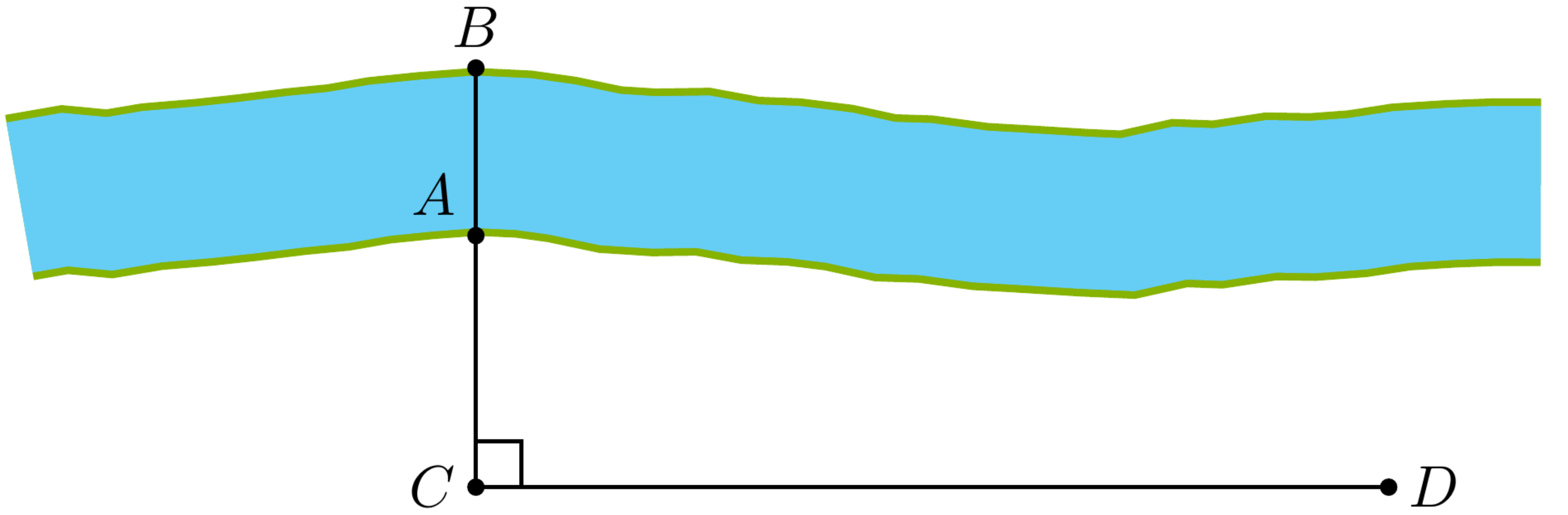
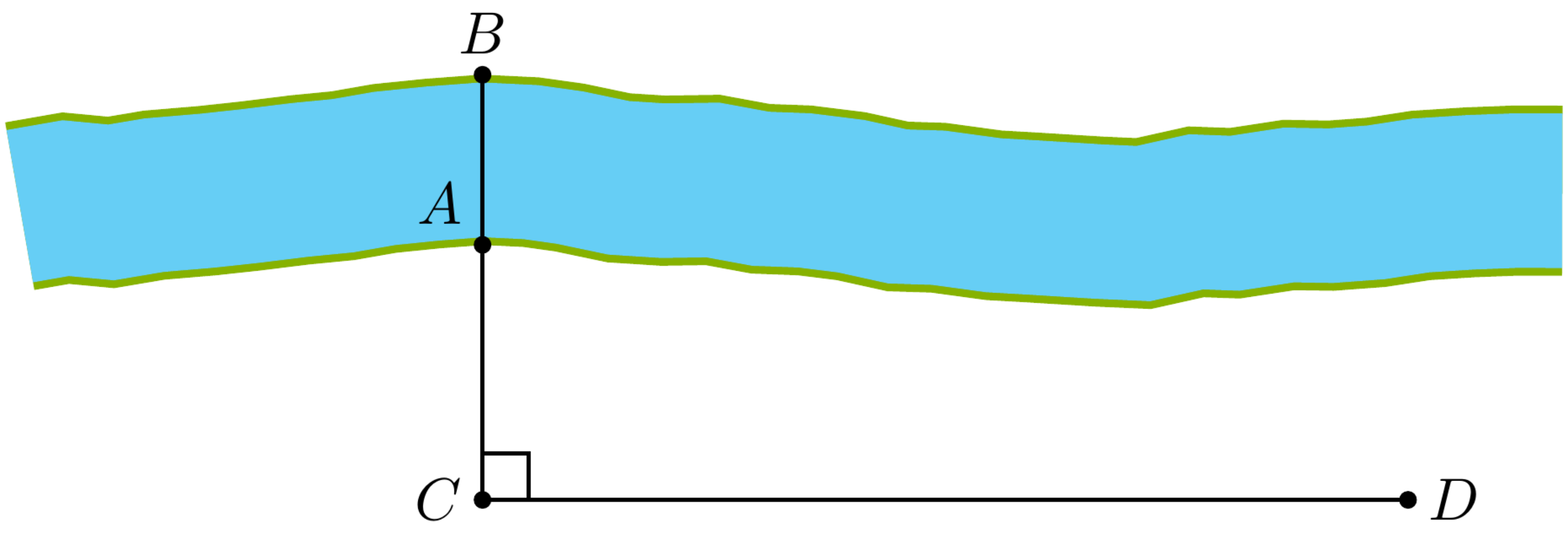


People's History of Surveying

- Make another right angle, this time at D, drawing a ray downward



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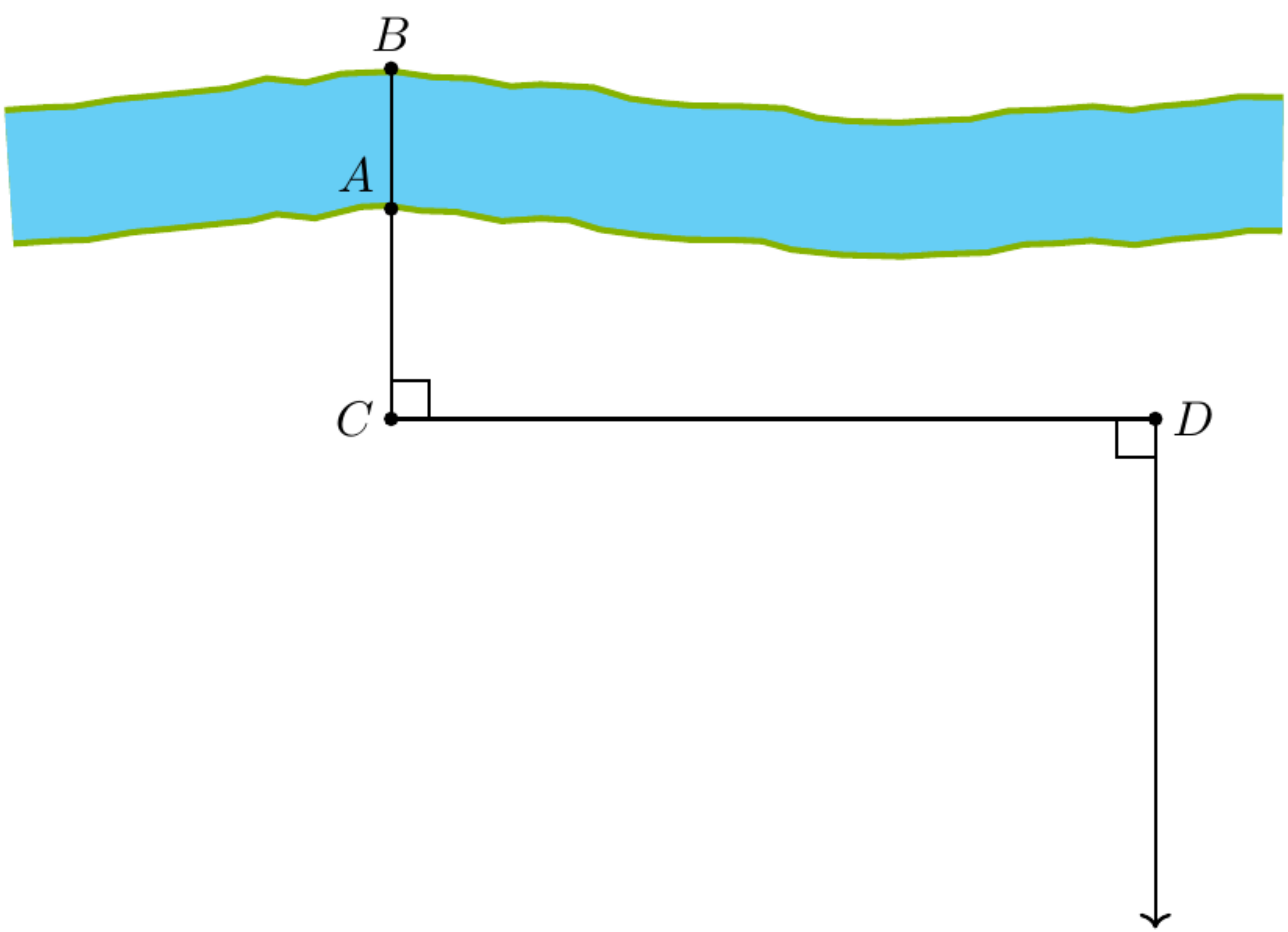

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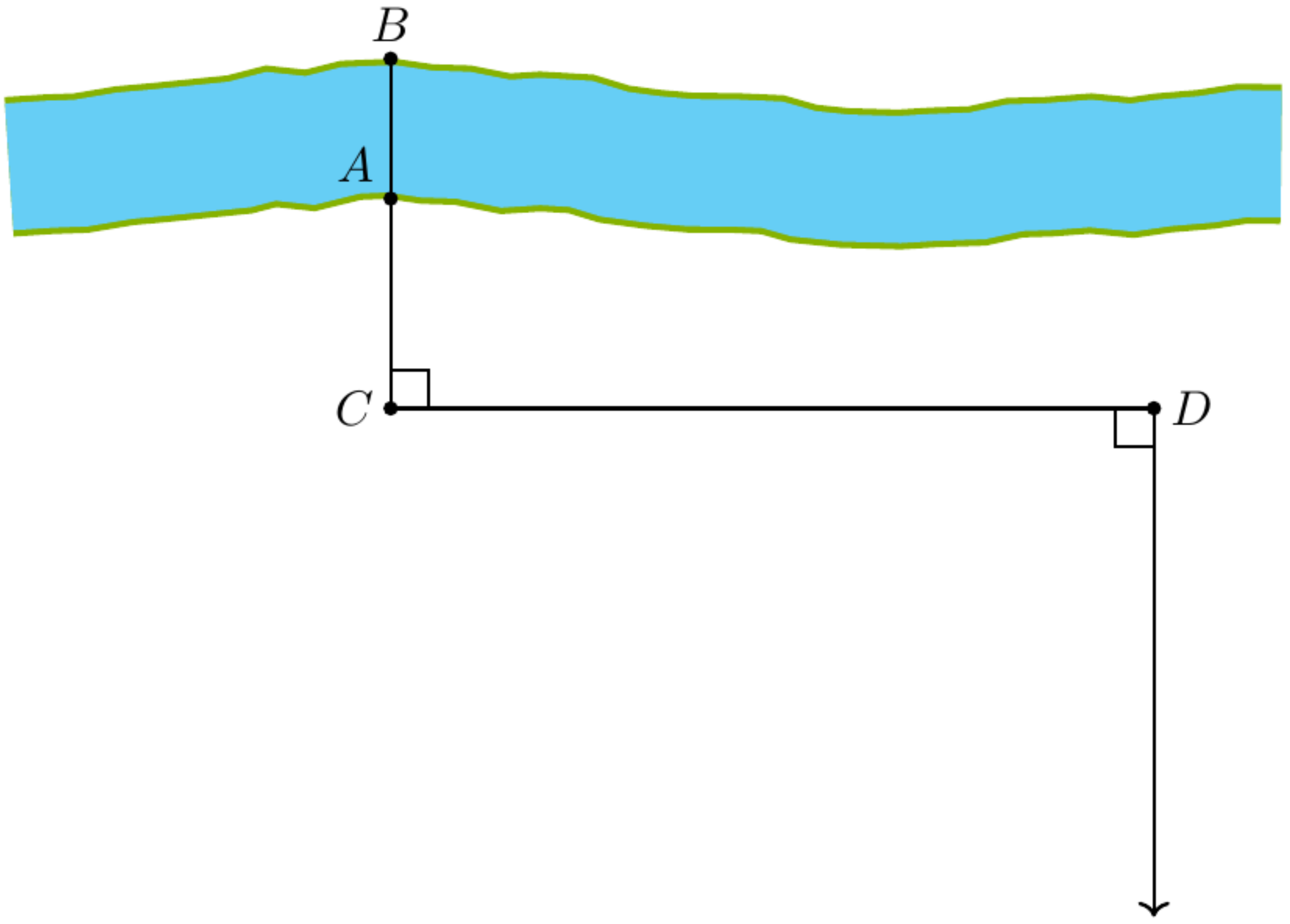
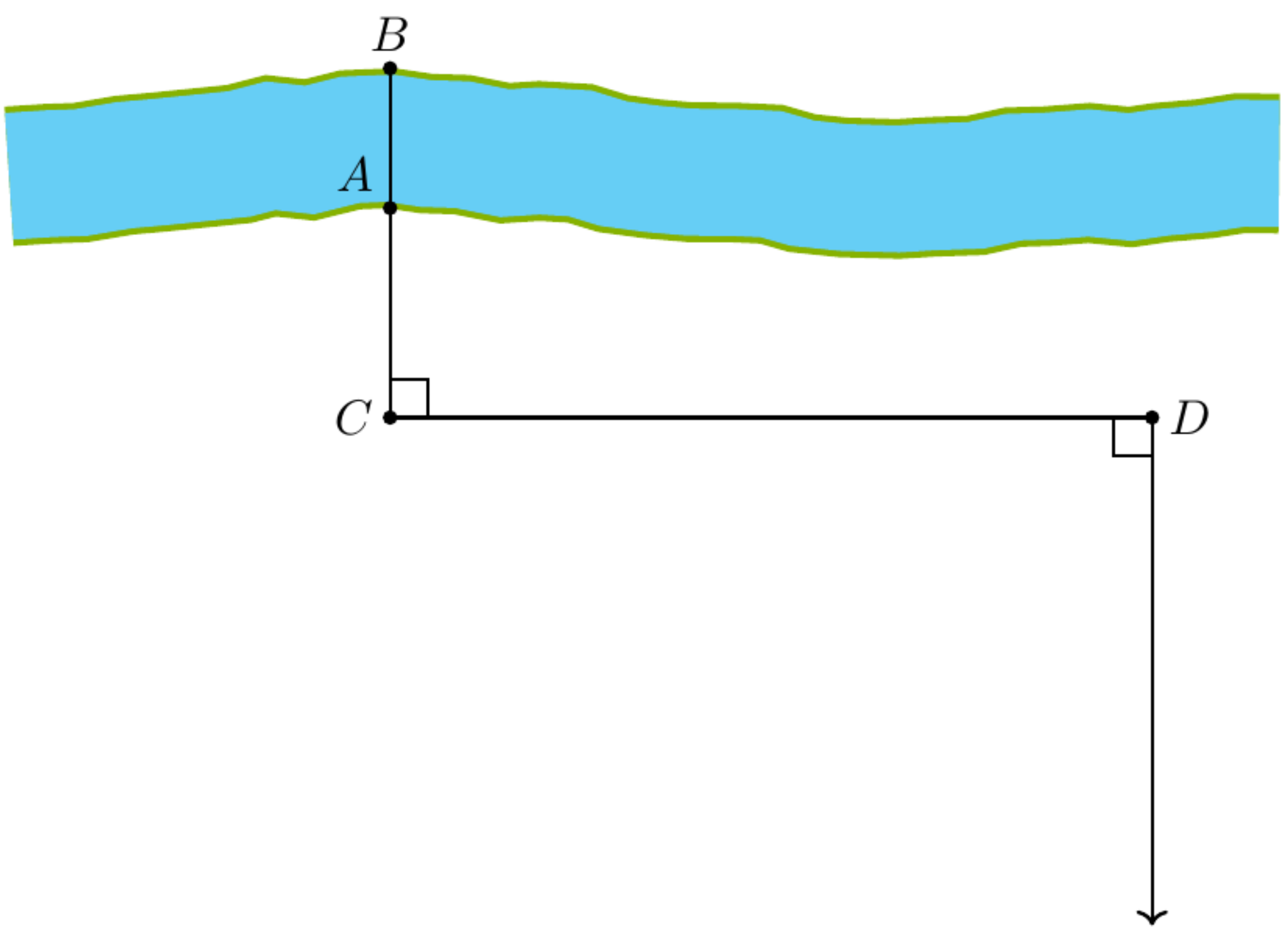


People's History of Surveying

- Bisect $\overline{CD}$, call the midpoint E. Then, draw $\overline{BE}$, and continue this line until it meets the ray.



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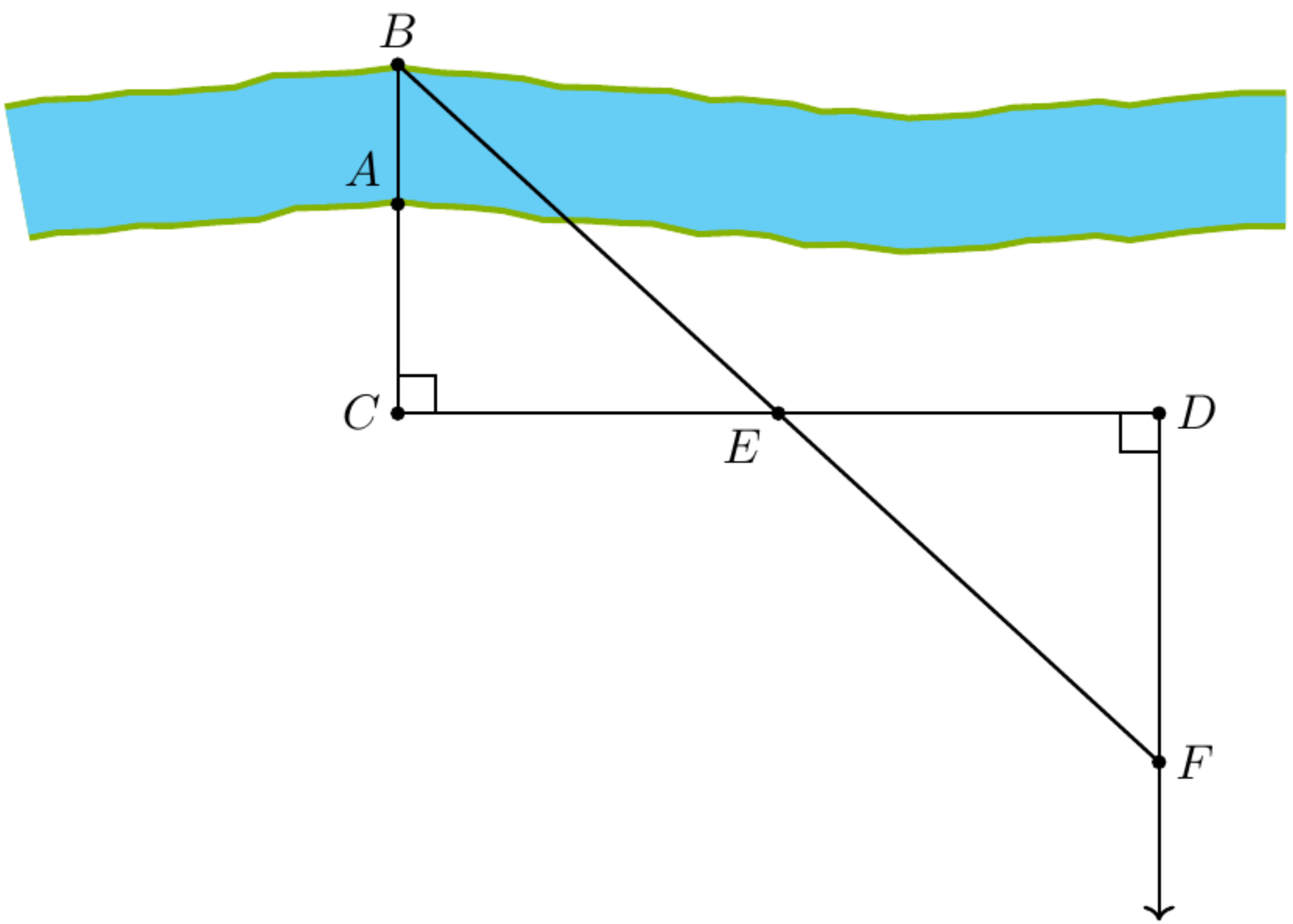

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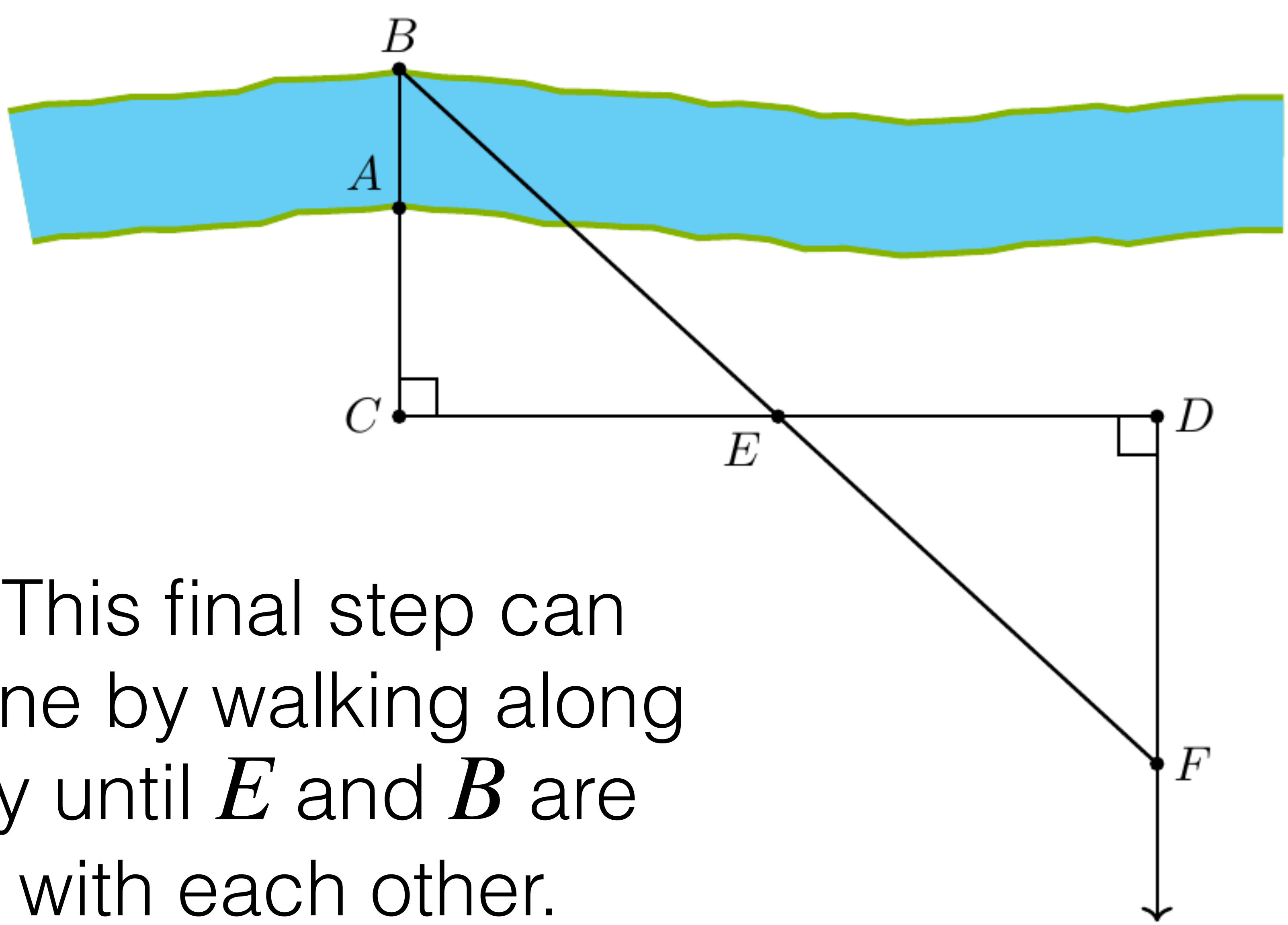
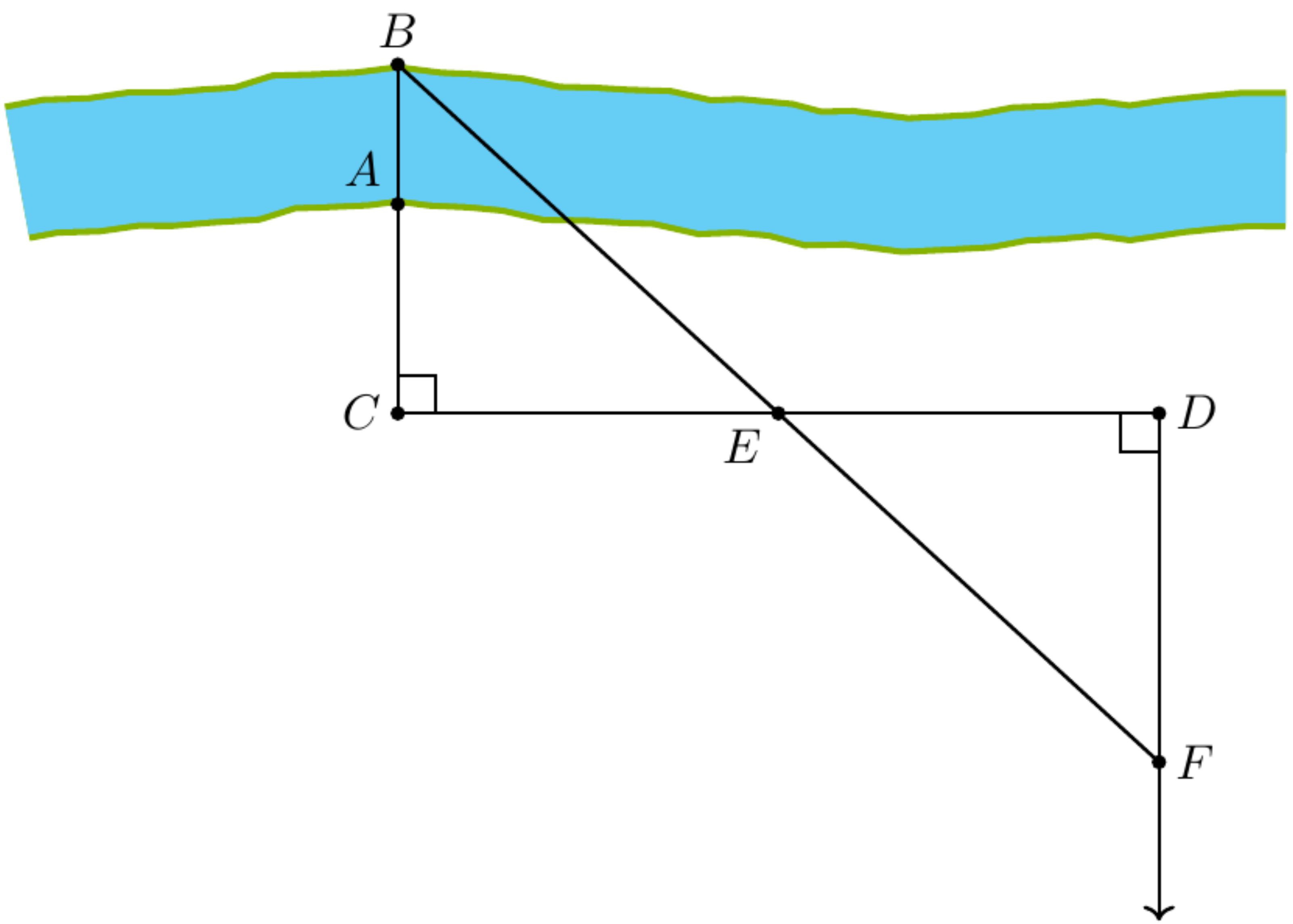


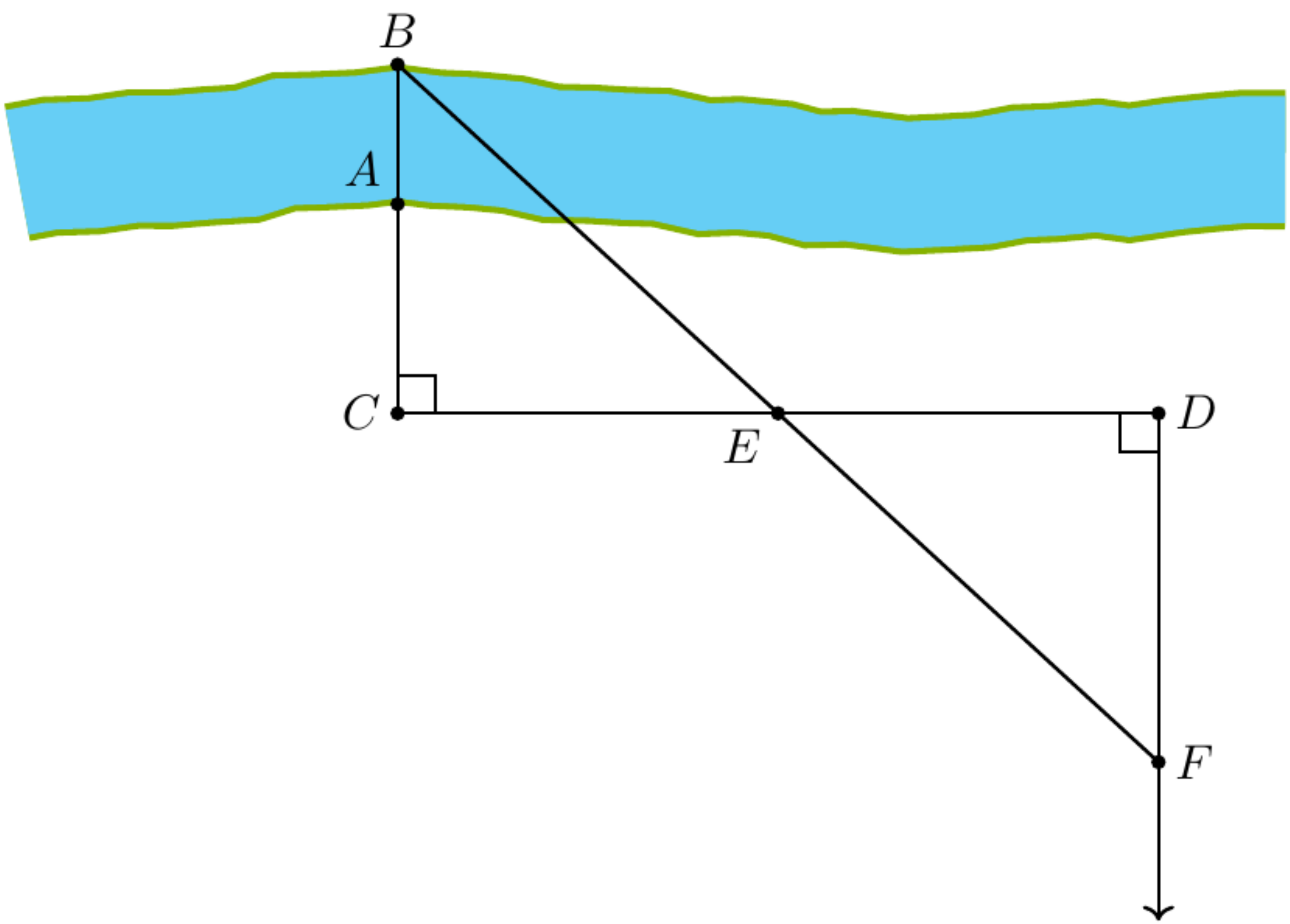
- Note: This final step can be done by walking along the ray until E and B are in line with each other.

People's History of Surveying



People's History of Surveying

- Mathematician: $\triangle ECB$ and $\triangle EDF$ are congruent.



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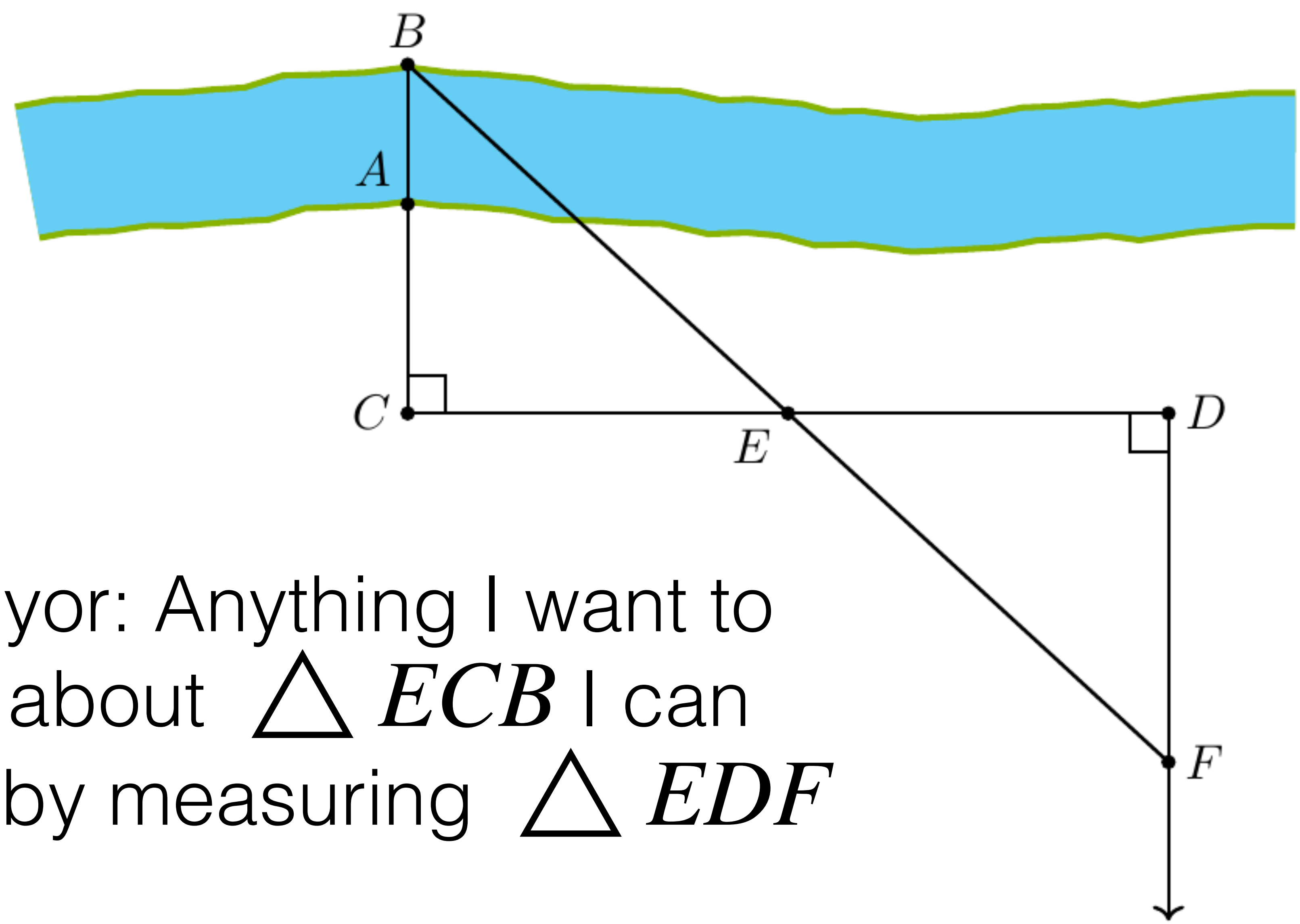
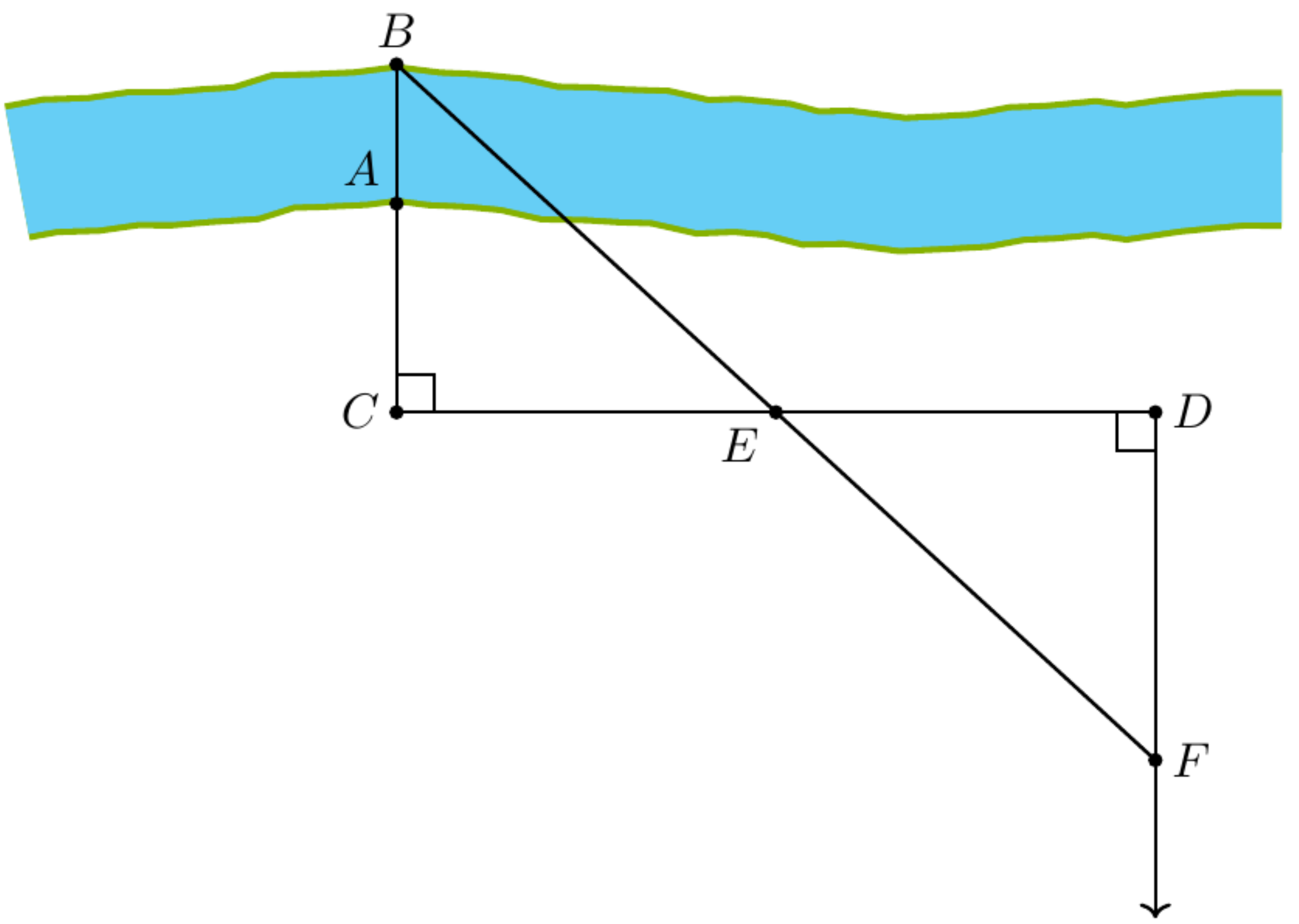


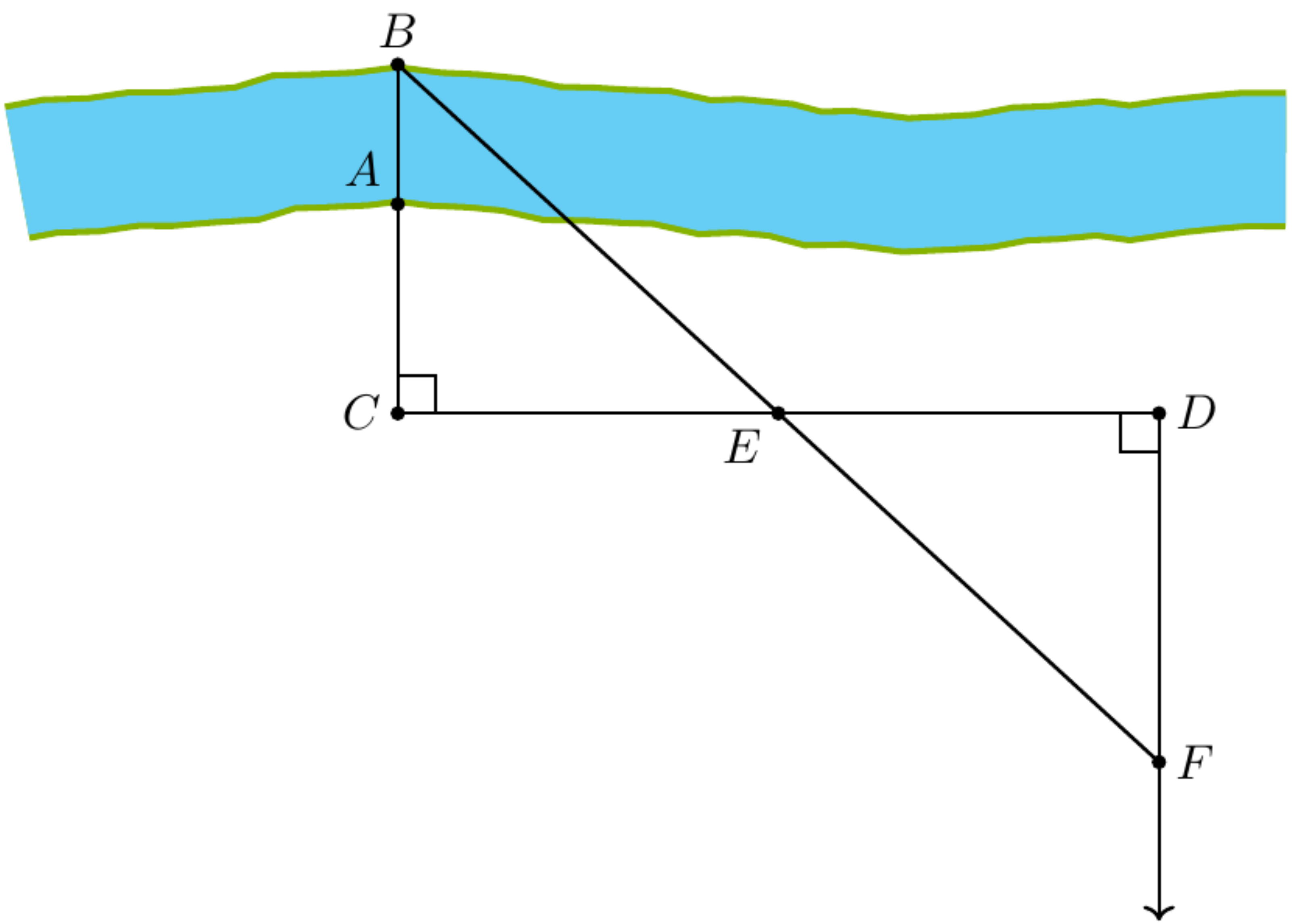
- Surveyor: Anything I want to know about $\triangle ECB$ I can learn by measuring $\triangle EDF$

People's History of Surveying



People's History of Surveying

- To find the distance $\overline{AB}$, add the point G.



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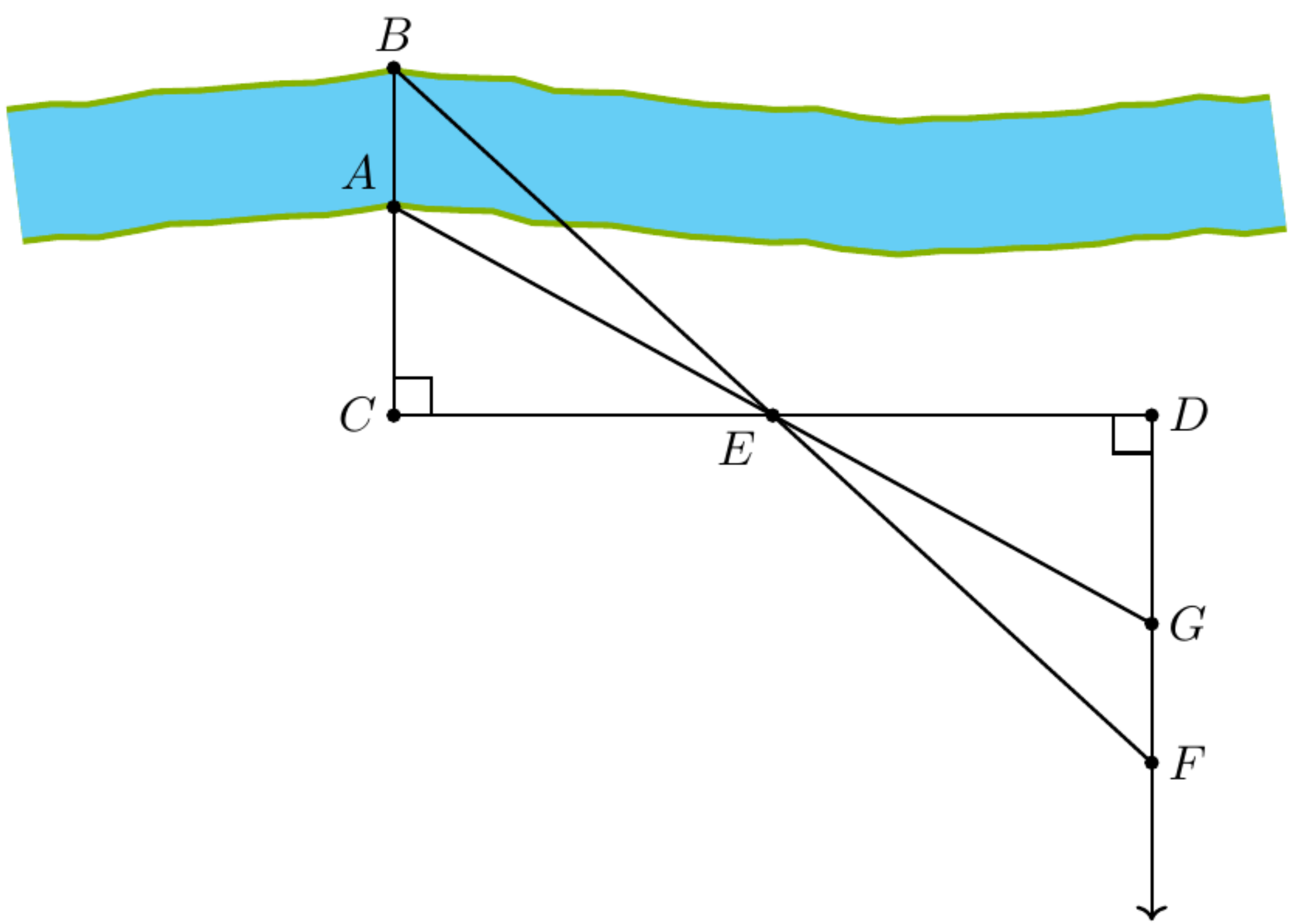

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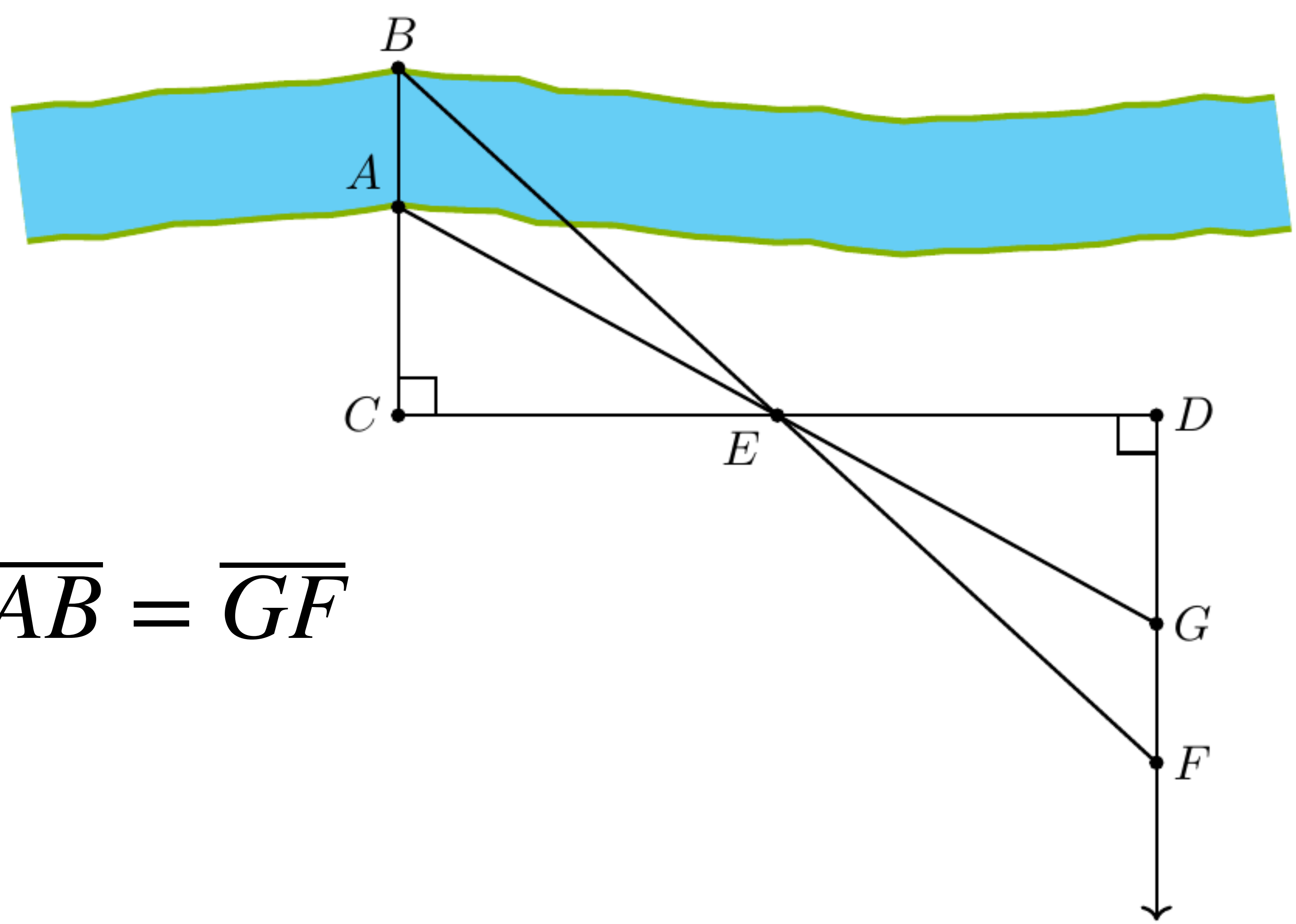

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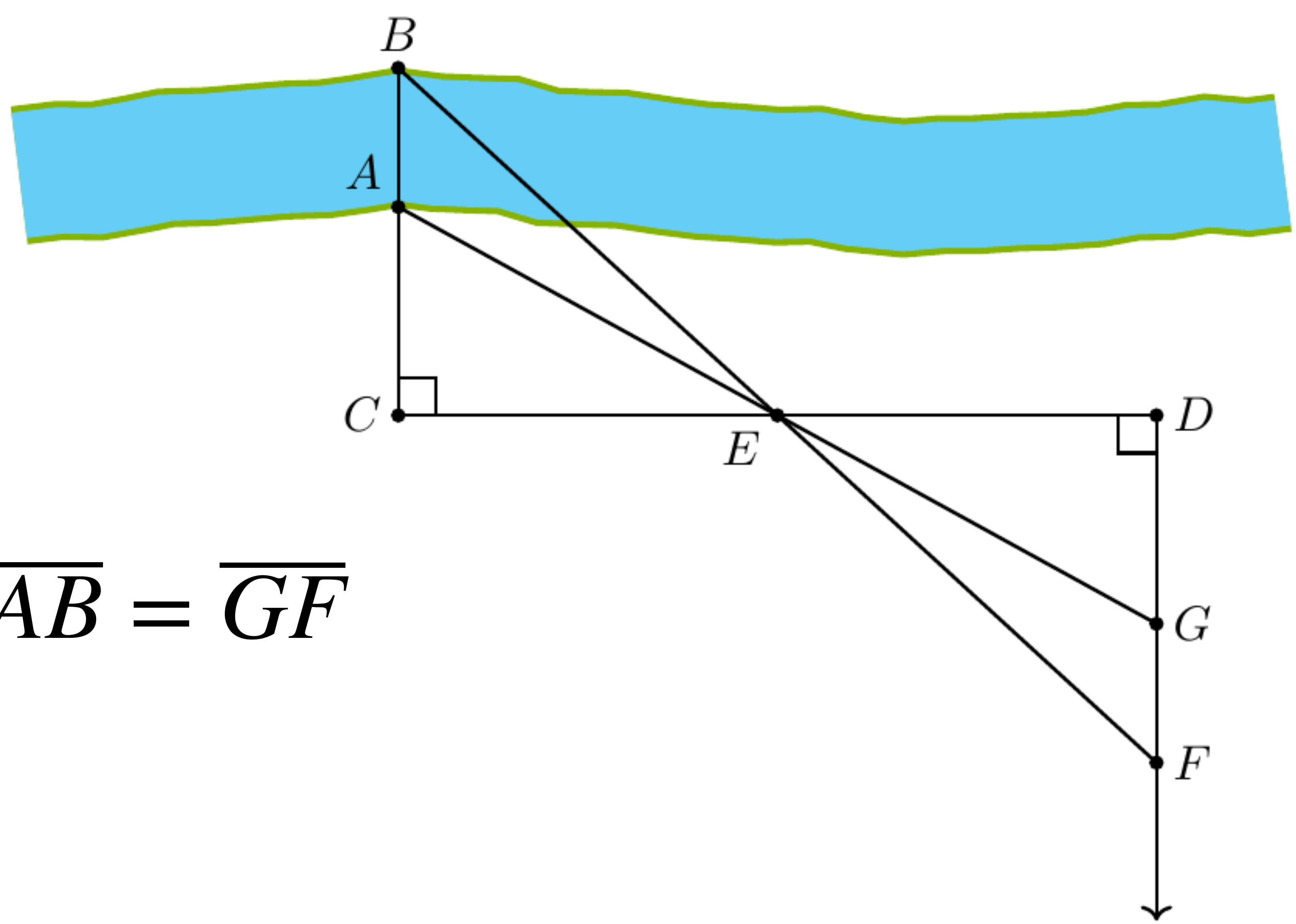
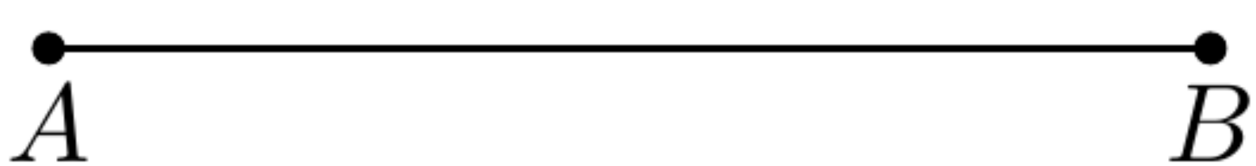


- Now, $\overline{AB} = \overline{GF}$
- Cool!

People's History of Surveying

- A related problem: An object is at an inaccessible point A. You are at B. How do you find this distance?



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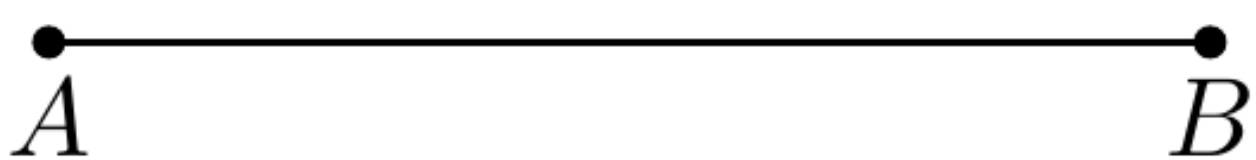
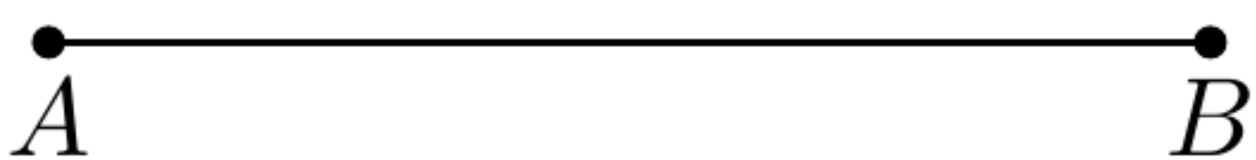


People's History of Surveying

- Extend $\overline{AB}$ to a point C, draw a perpendicular at C to a point D.



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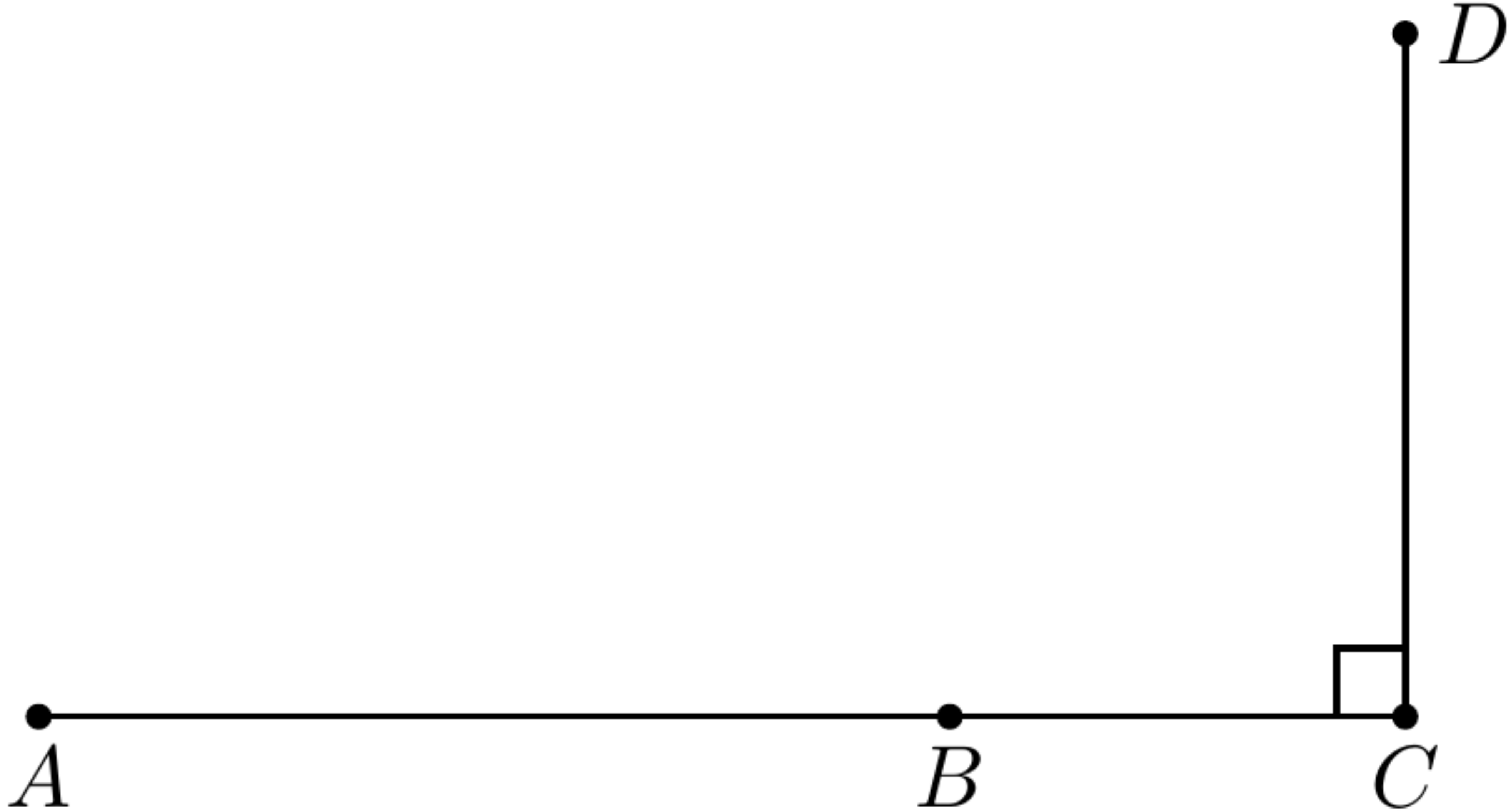

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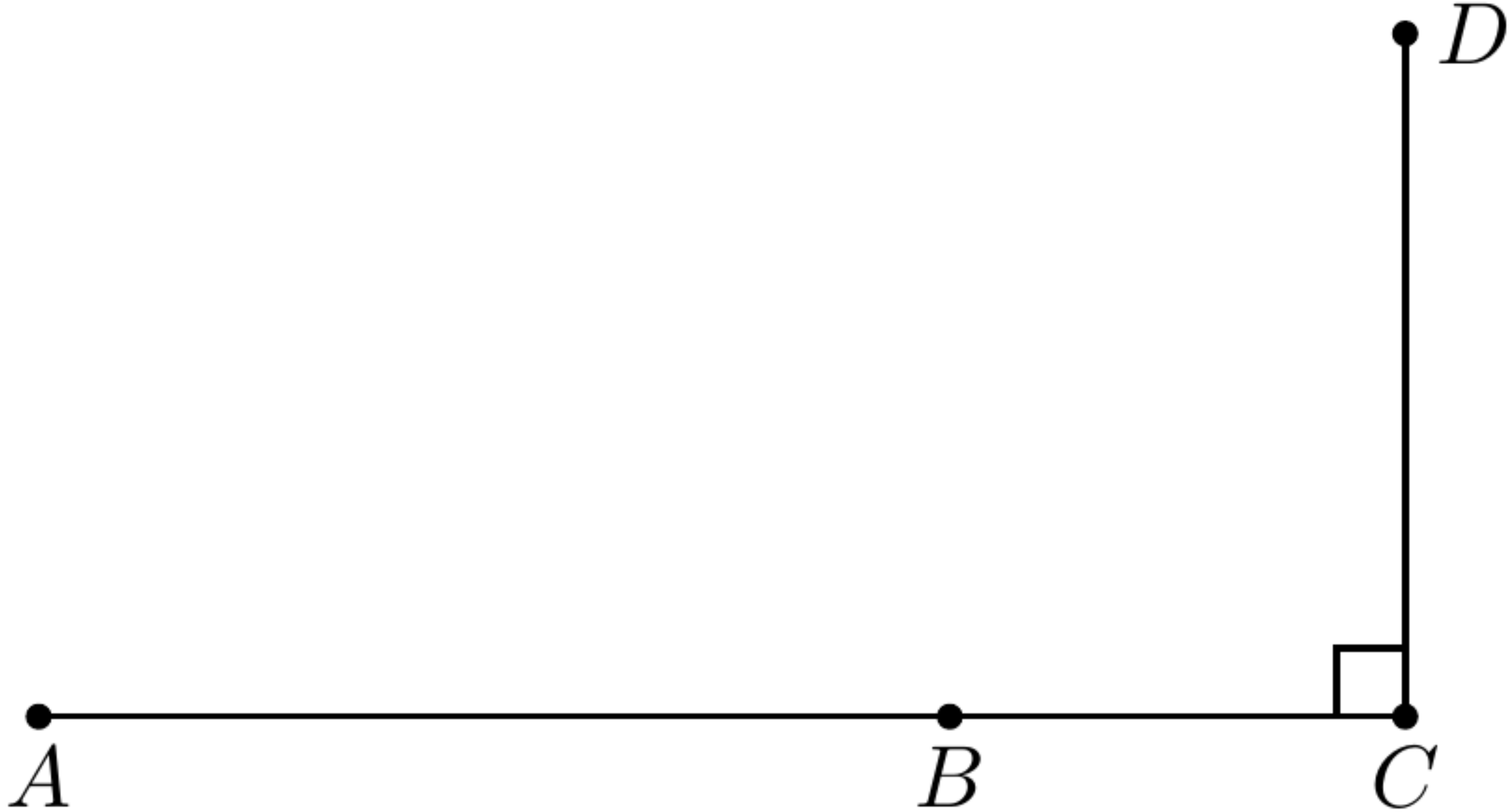
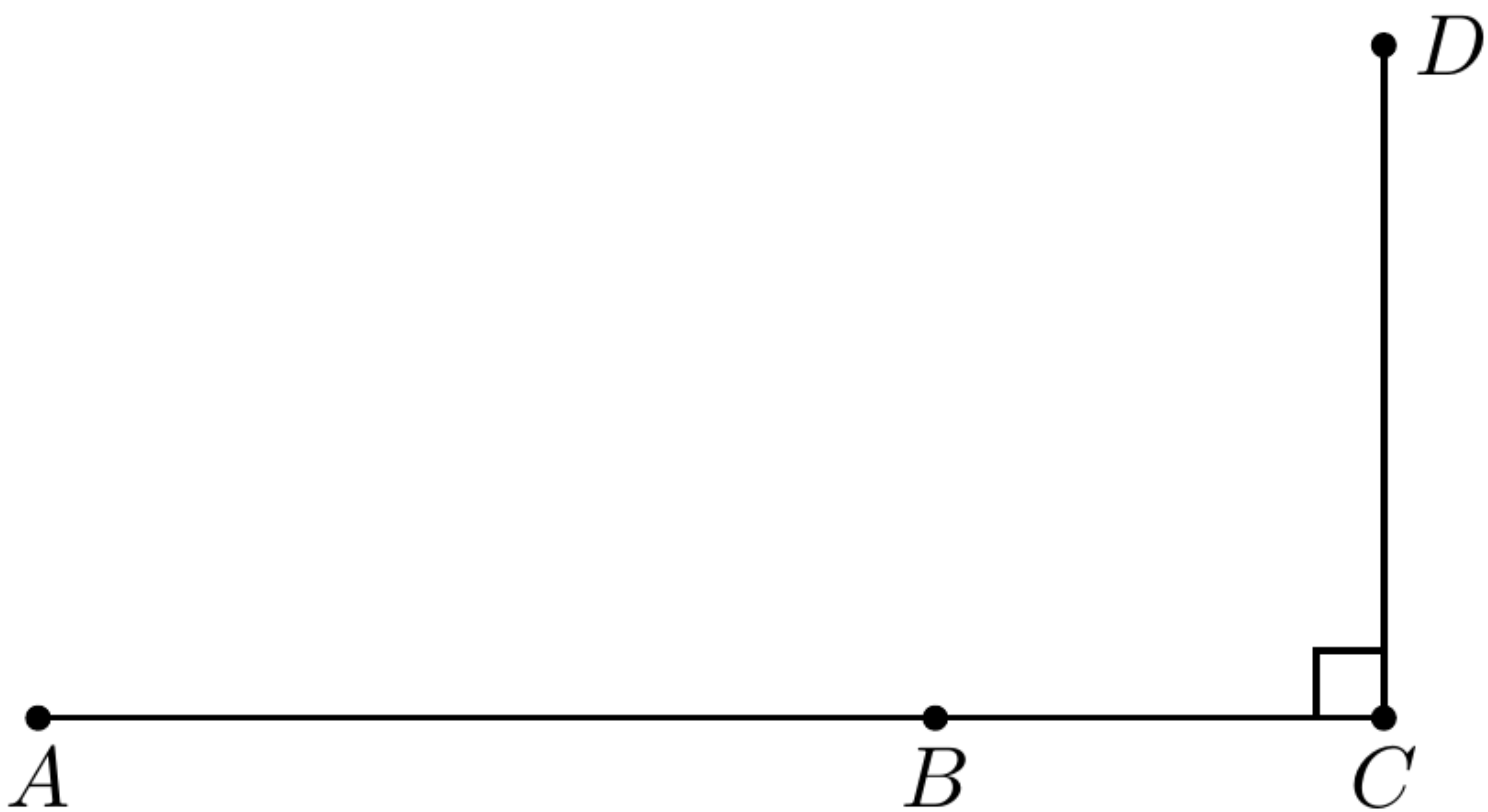


People's History of Surveying

- Walk along the perpendicular to B until you are in line with A and D.



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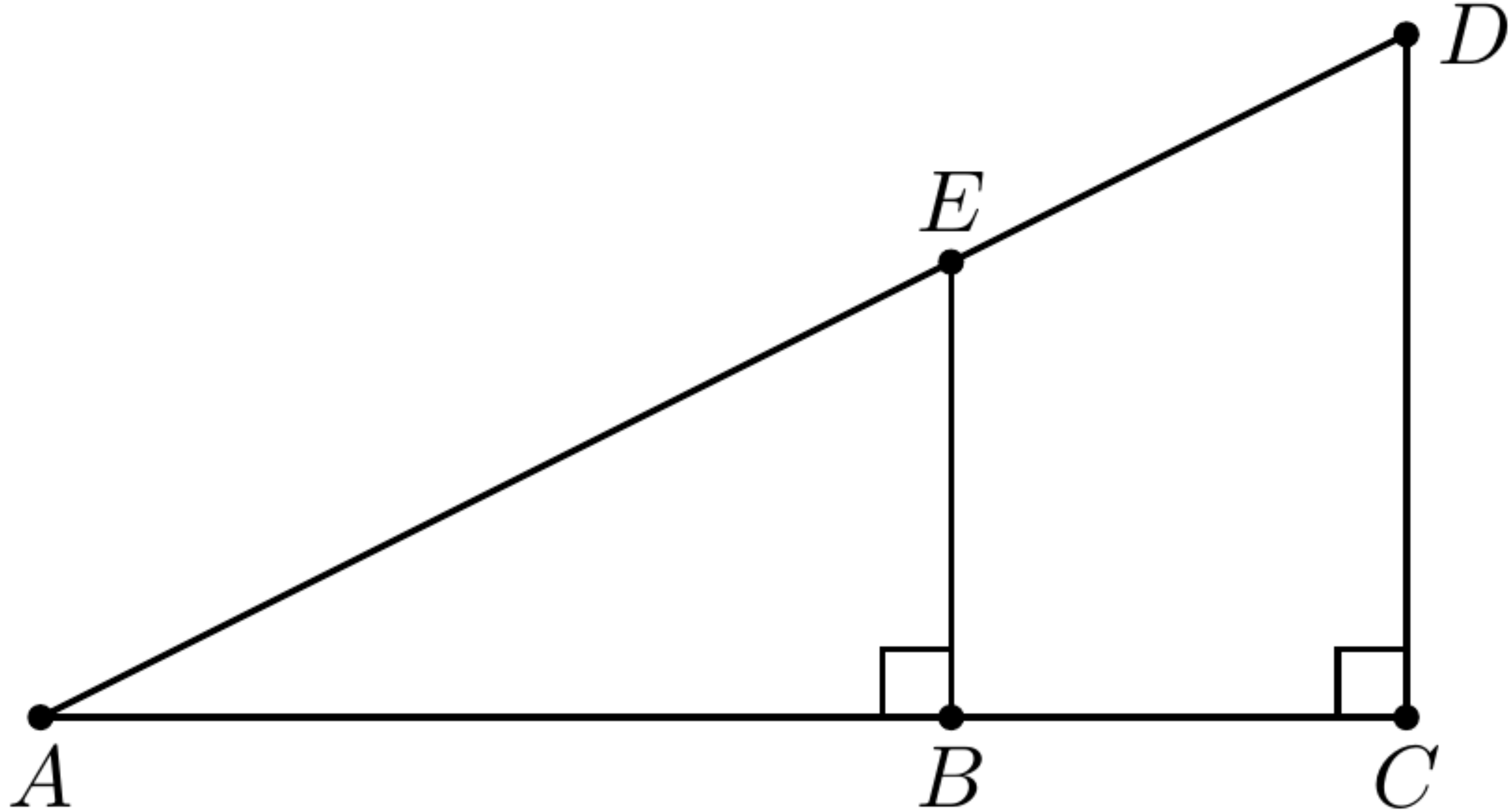

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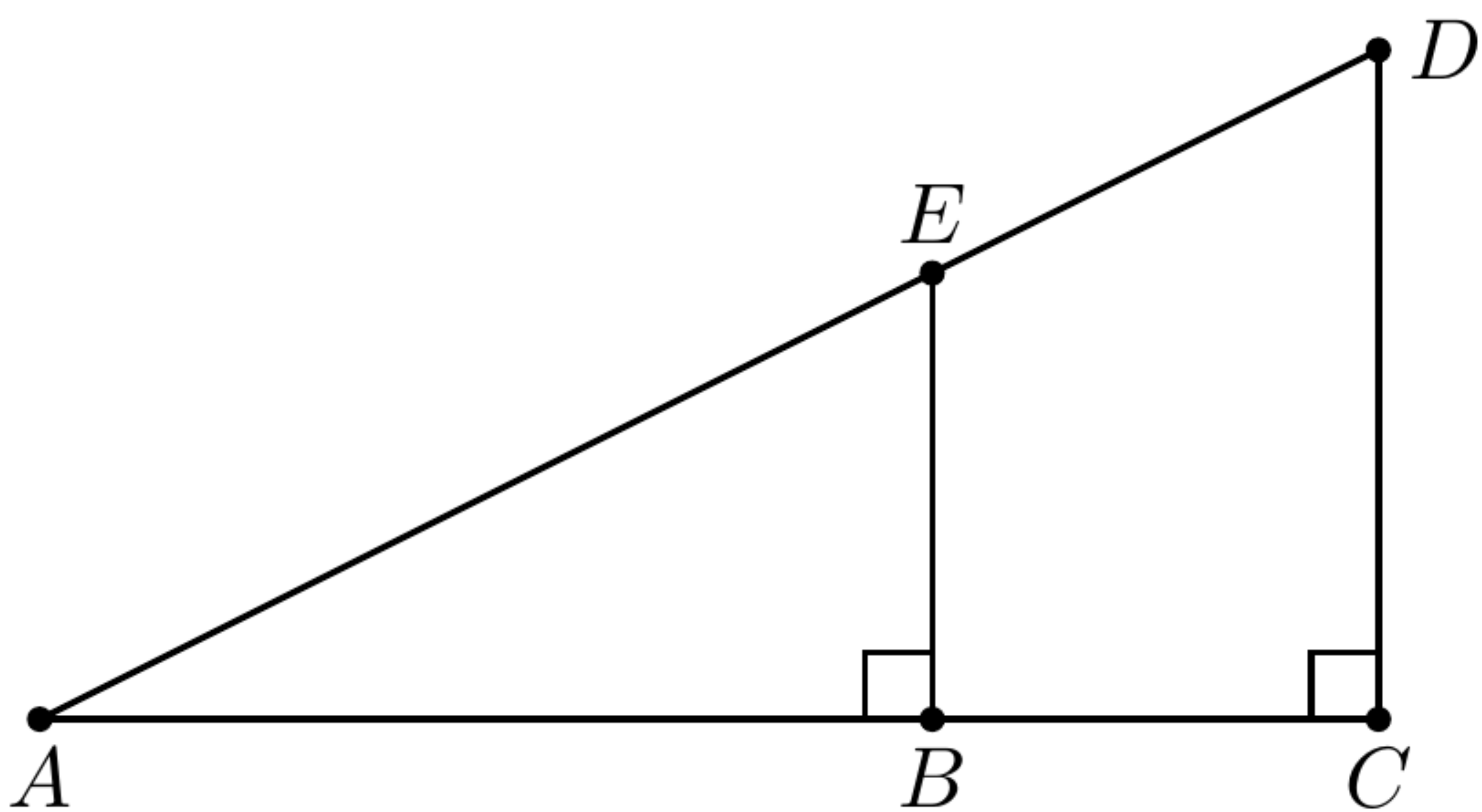
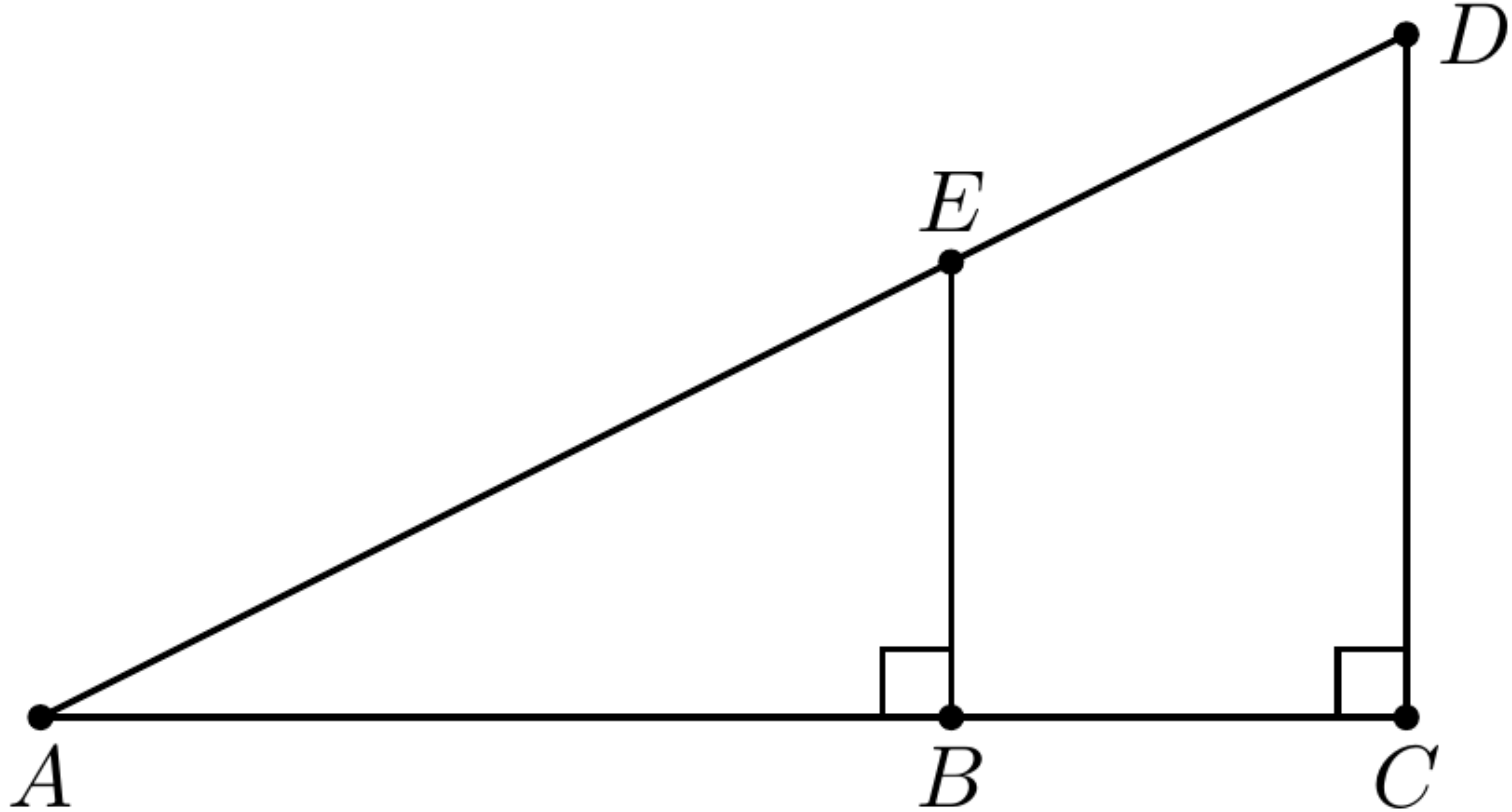


People's History of Surveying

- By similar triangles, $$\frac{\overline{AB}}{\overline{BE}} = \frac{\overline{AC}}{\overline{CD}}$$



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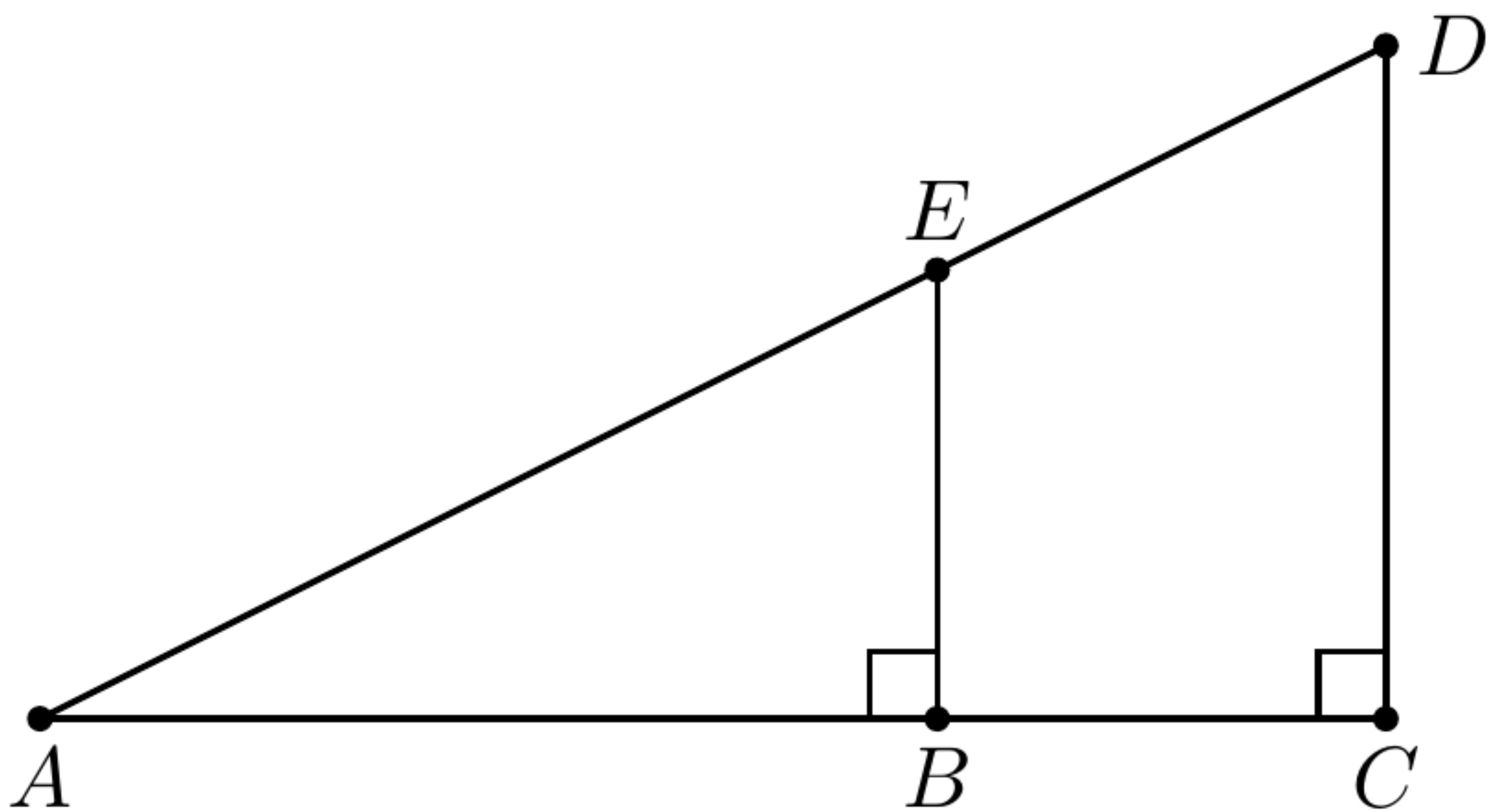

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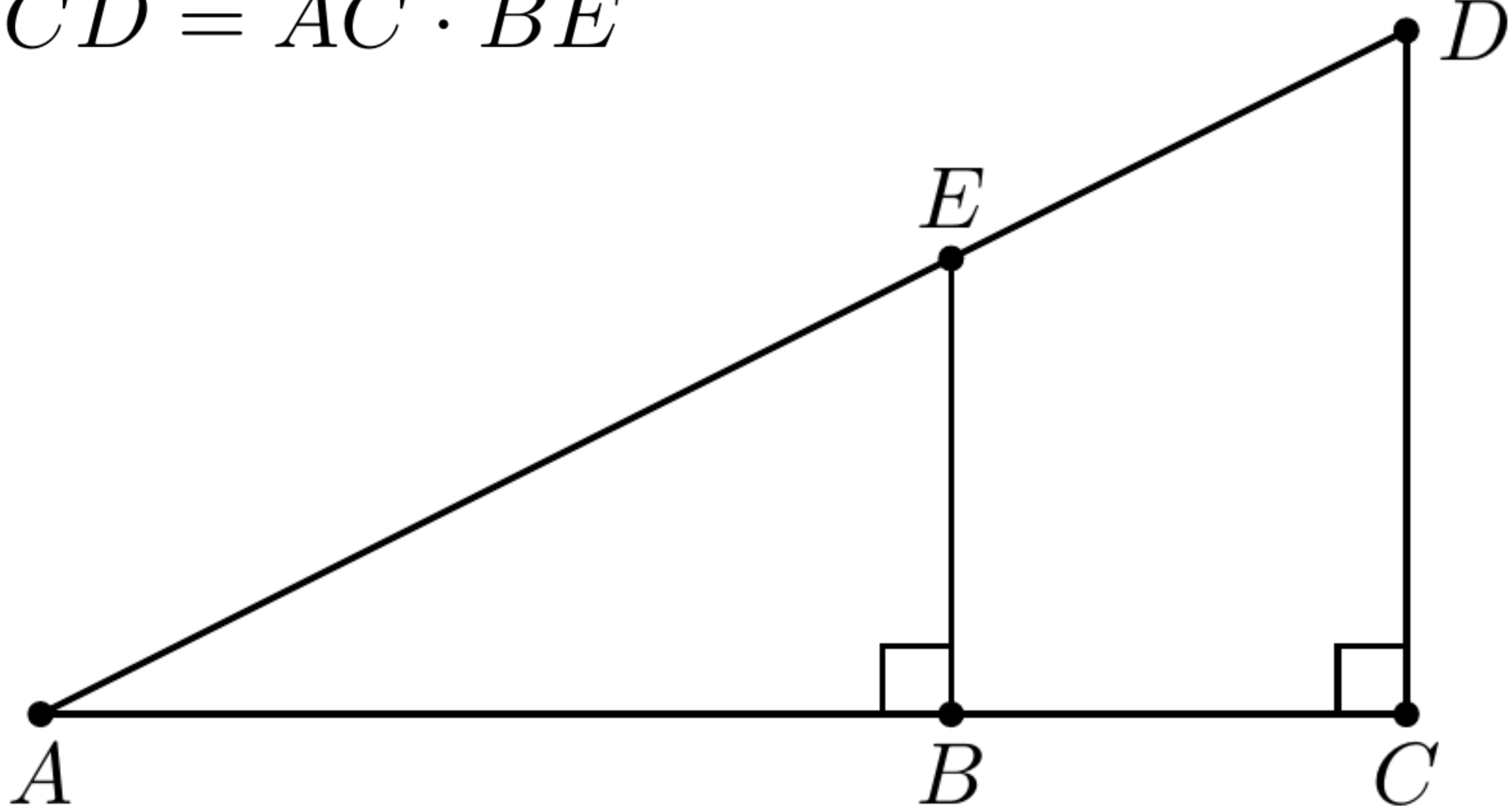

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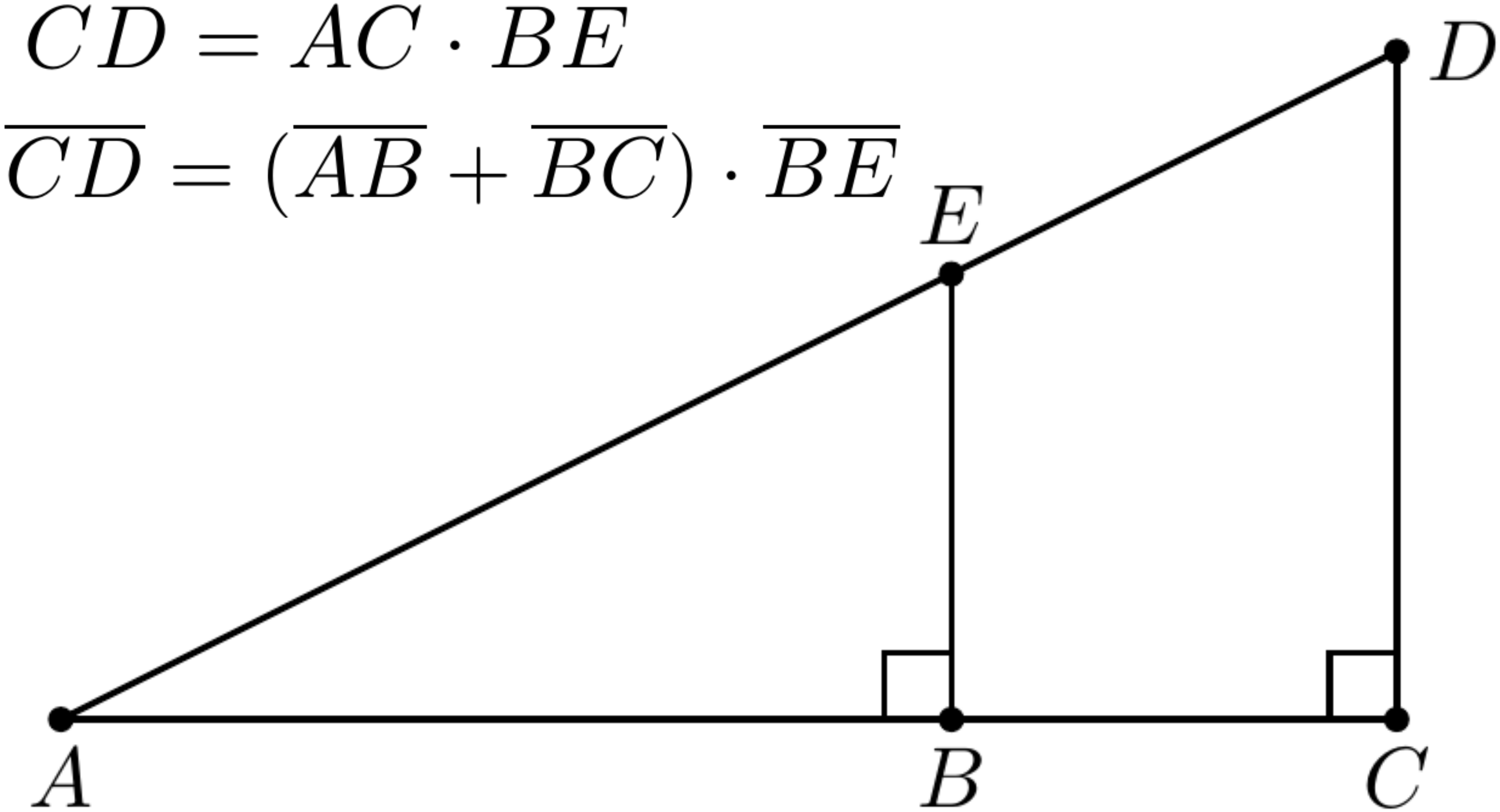

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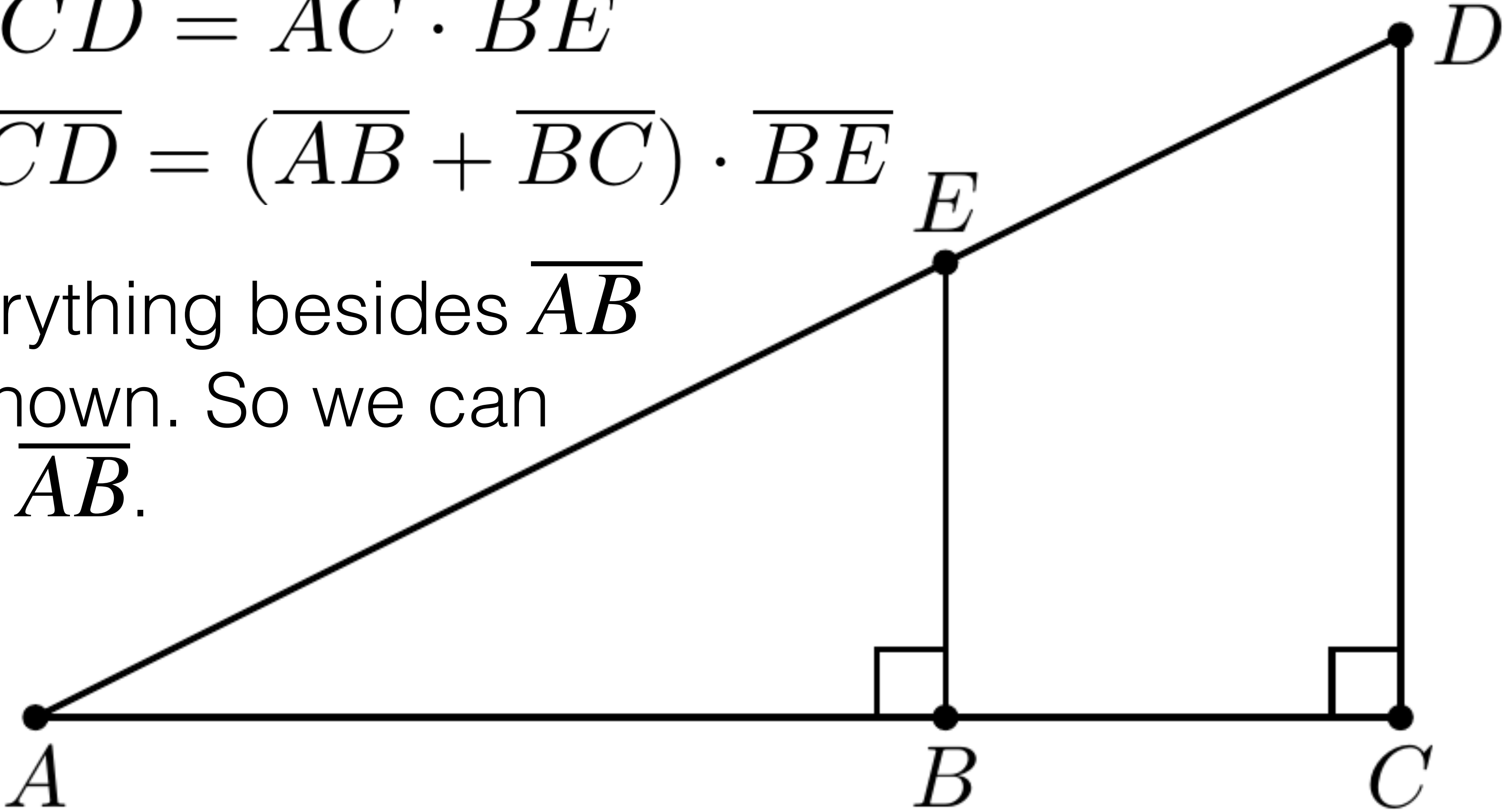
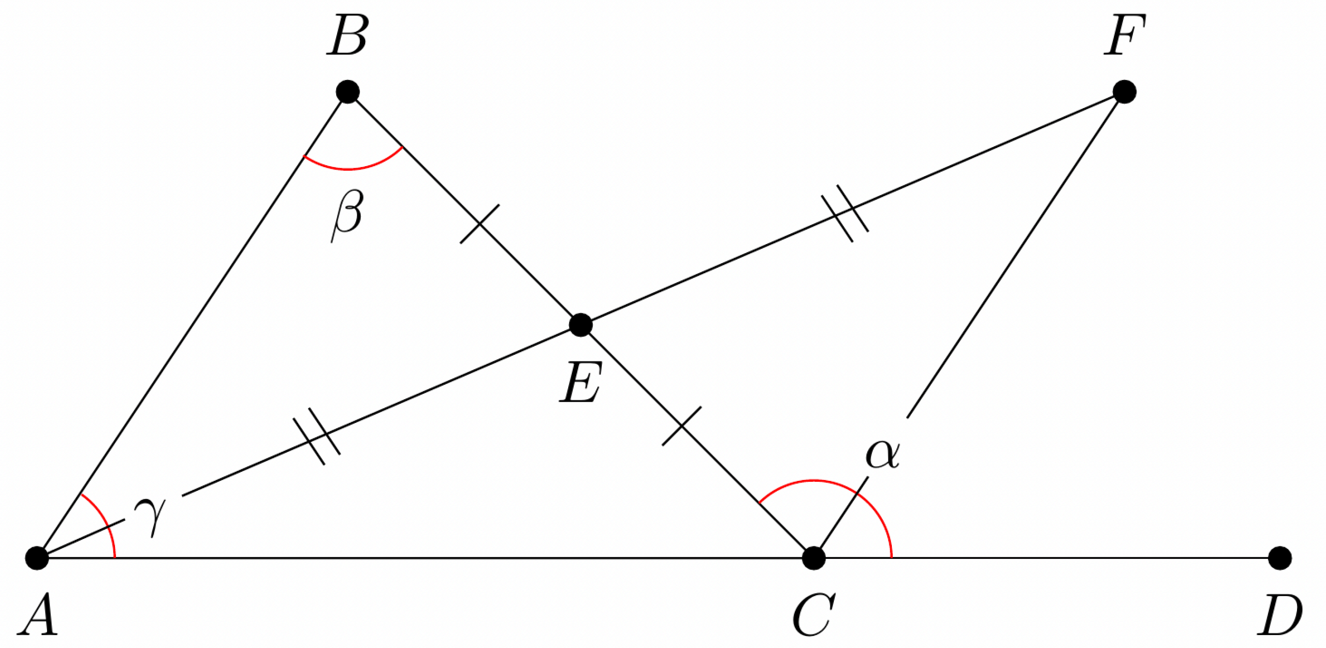
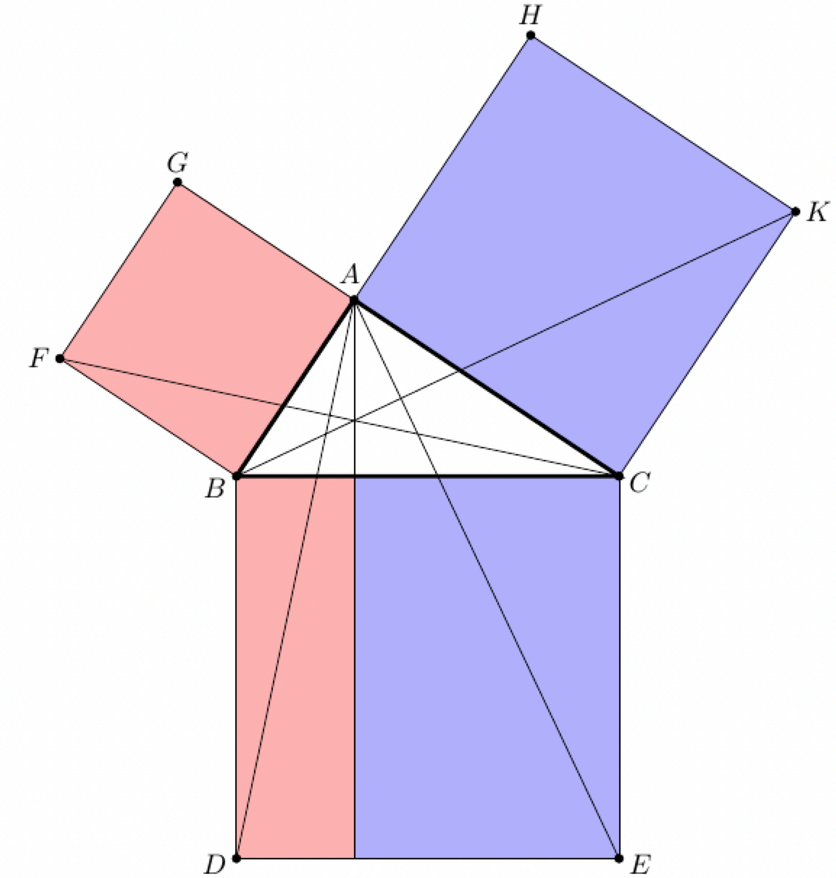
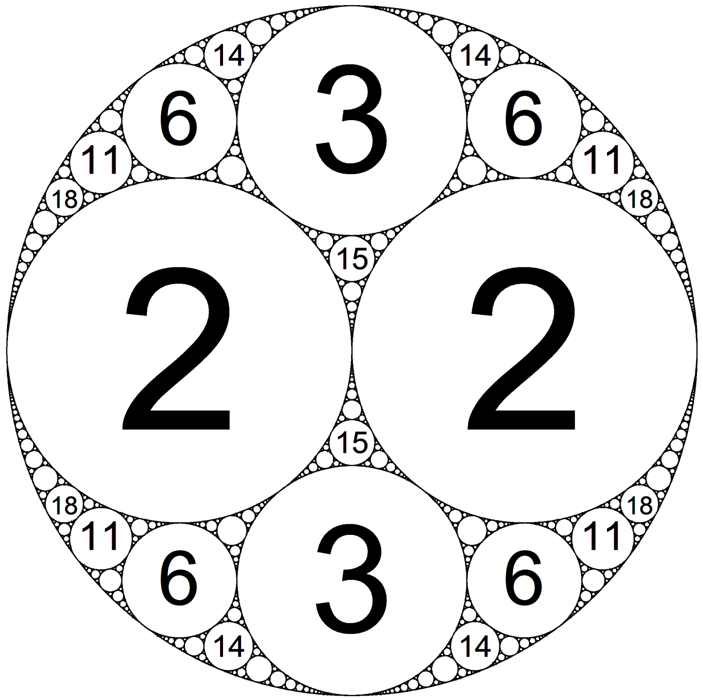
- Everything besides $\overline{AB}$ is known. So we can find $\overline{AB}$.







Chapter 4: Euclidean Geometry



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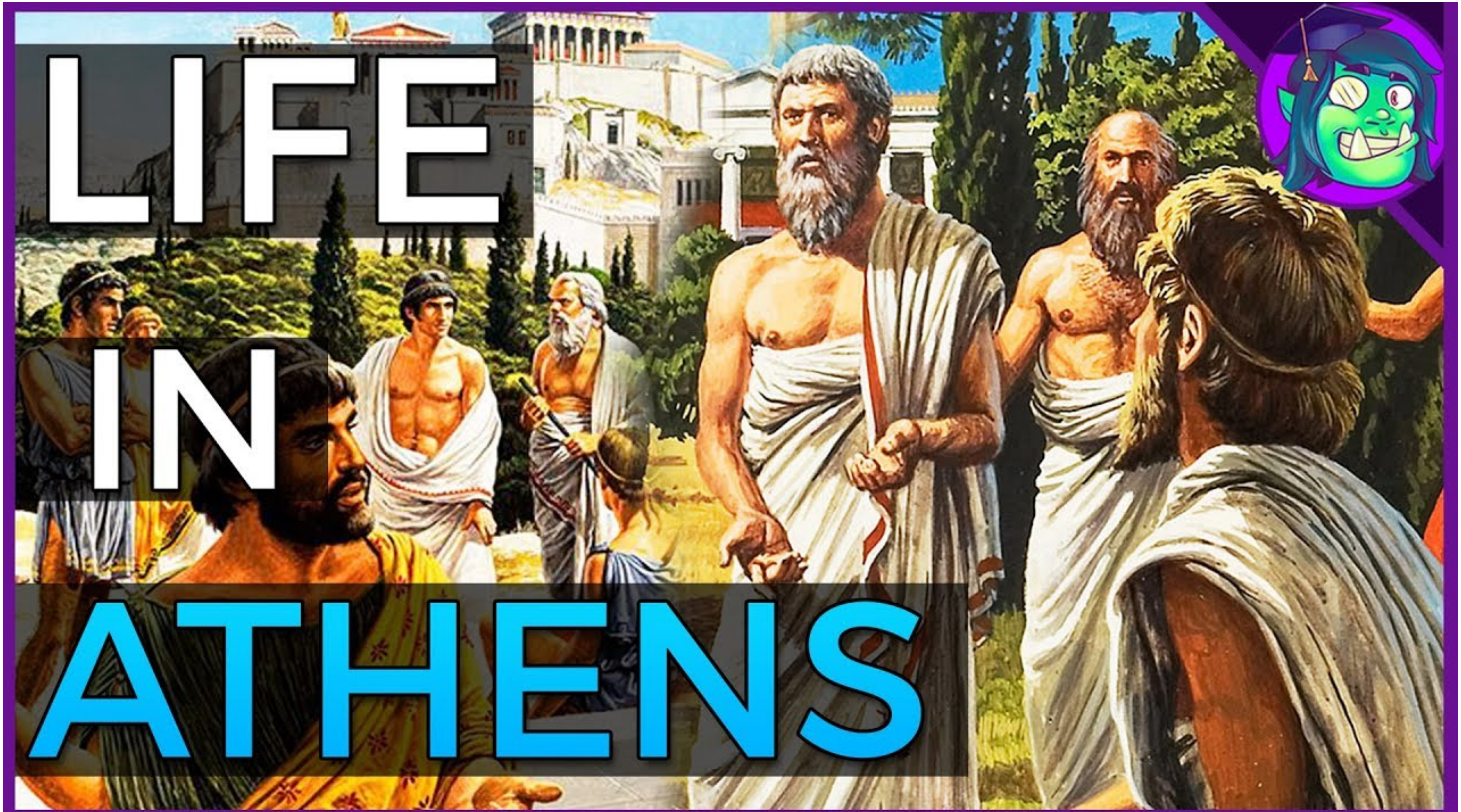
- Many great Greek mathematicians studied at the Academy. Greatest of them was Eudoxus of Cnidus.

Academy in Athens

Life in Ancient Athens



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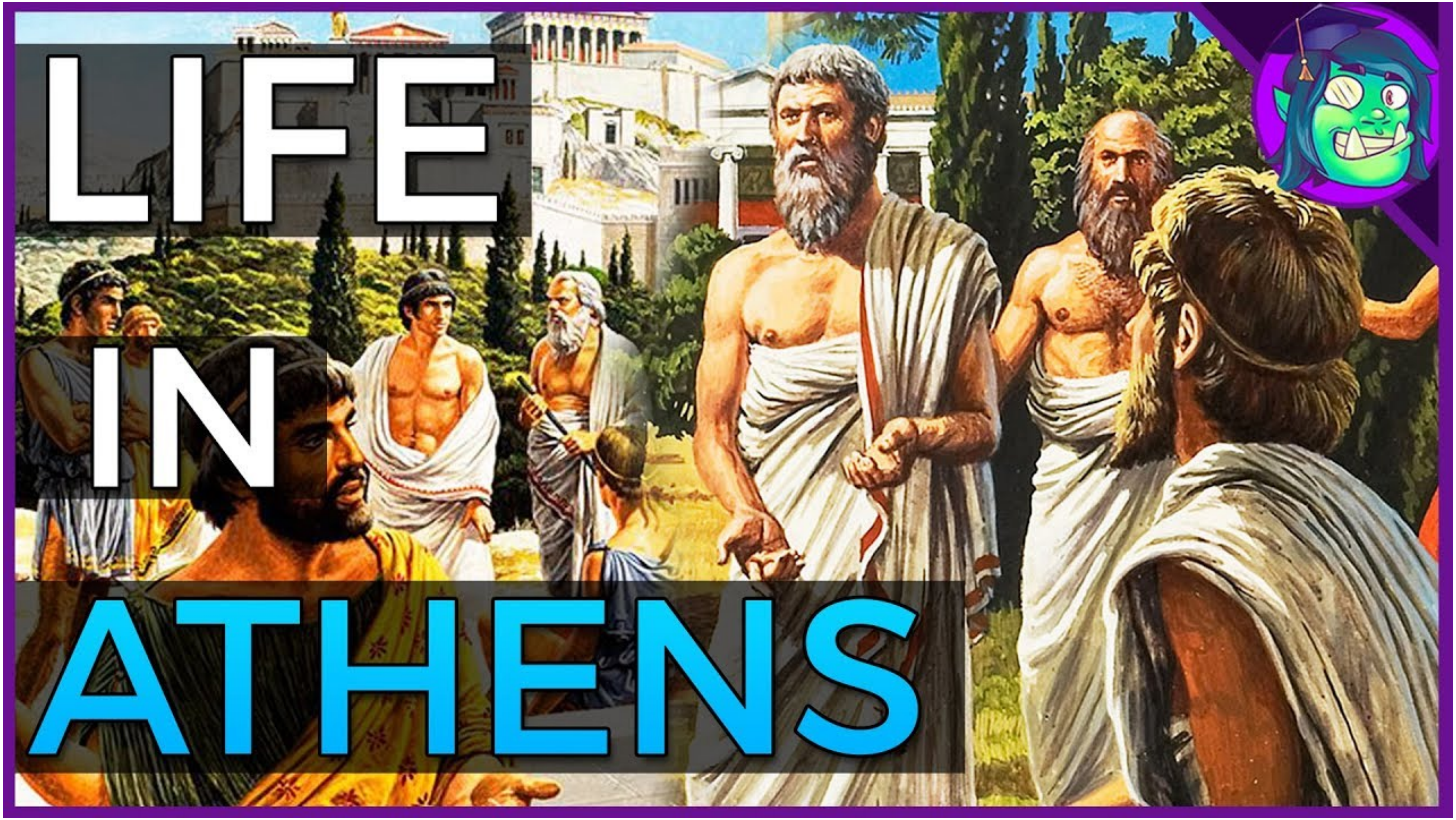

The Elements

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- The major sources were likely the Pythagoreans, Hippocrates and Eudoxus. Many of the proofs are believed to be original to Euclid.

- Its success is due to its logical and axiomatic presentation. This deductive style is the central approach to mathematics today.

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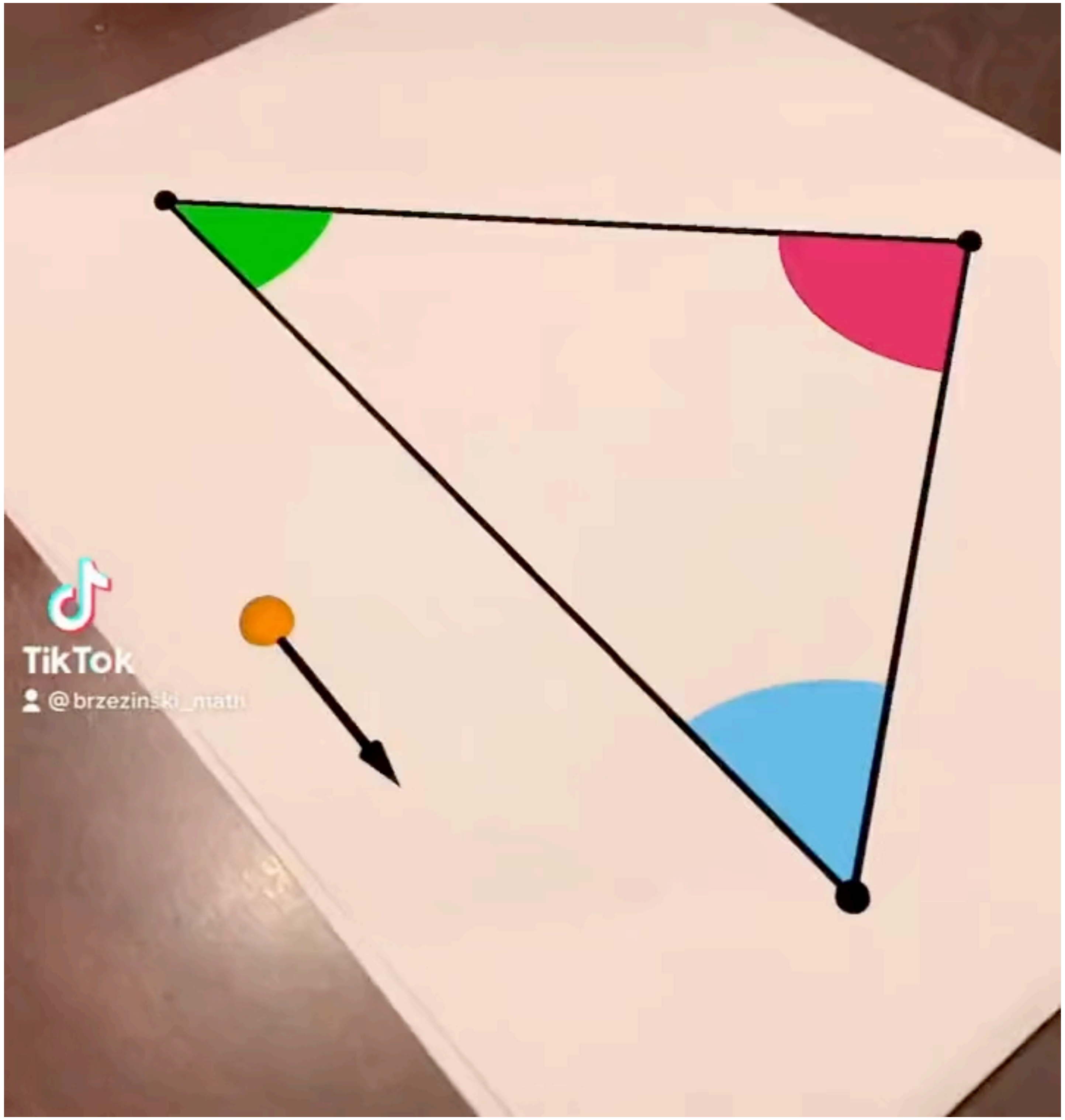
- Euclid’s postulates allow him to use a straightedge and compass, which play a central role.

Definitions, Common Notions and Postulates

Proving Propositions

Triangle Angle Sum



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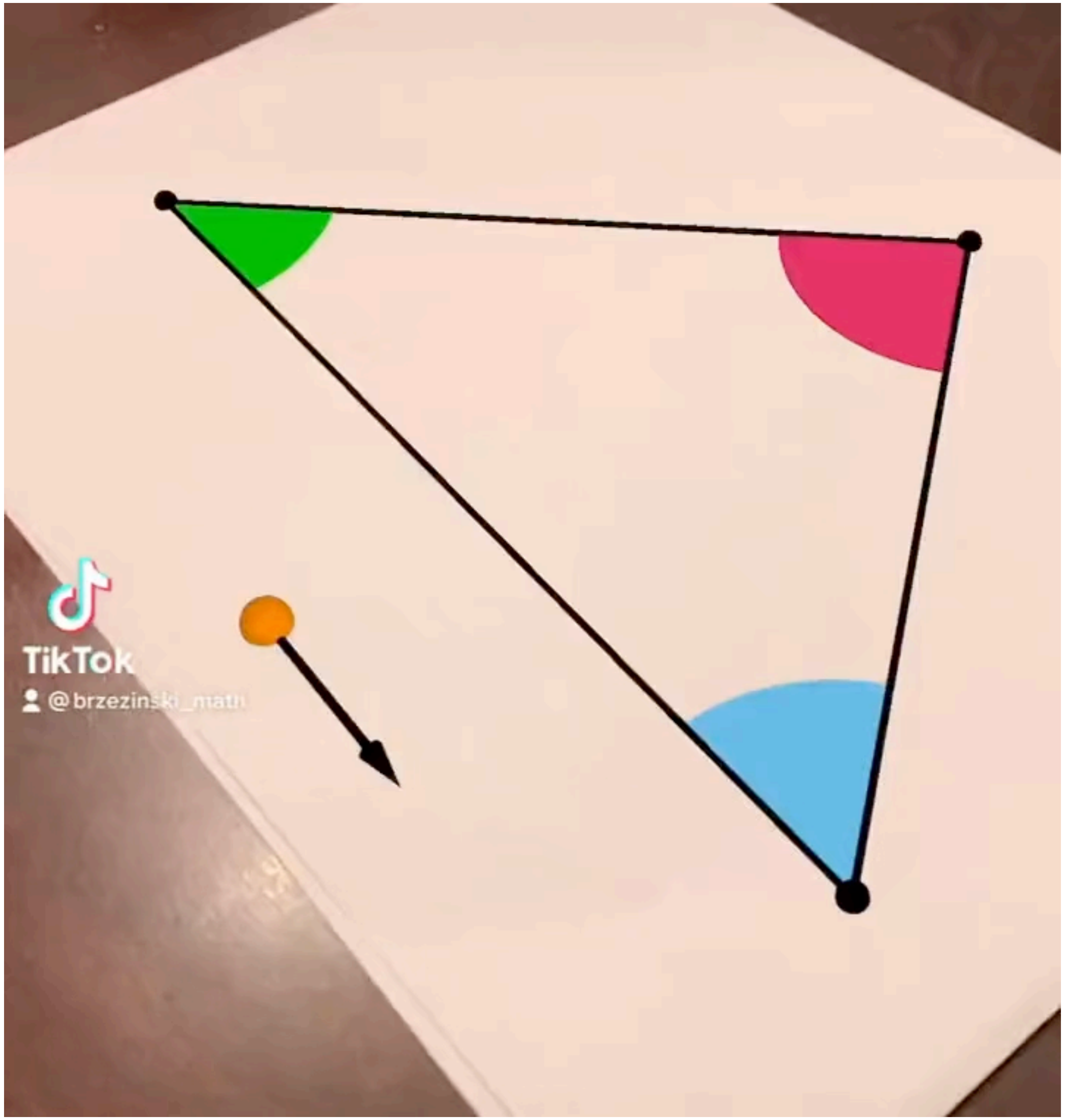
Triangle Angle Sum



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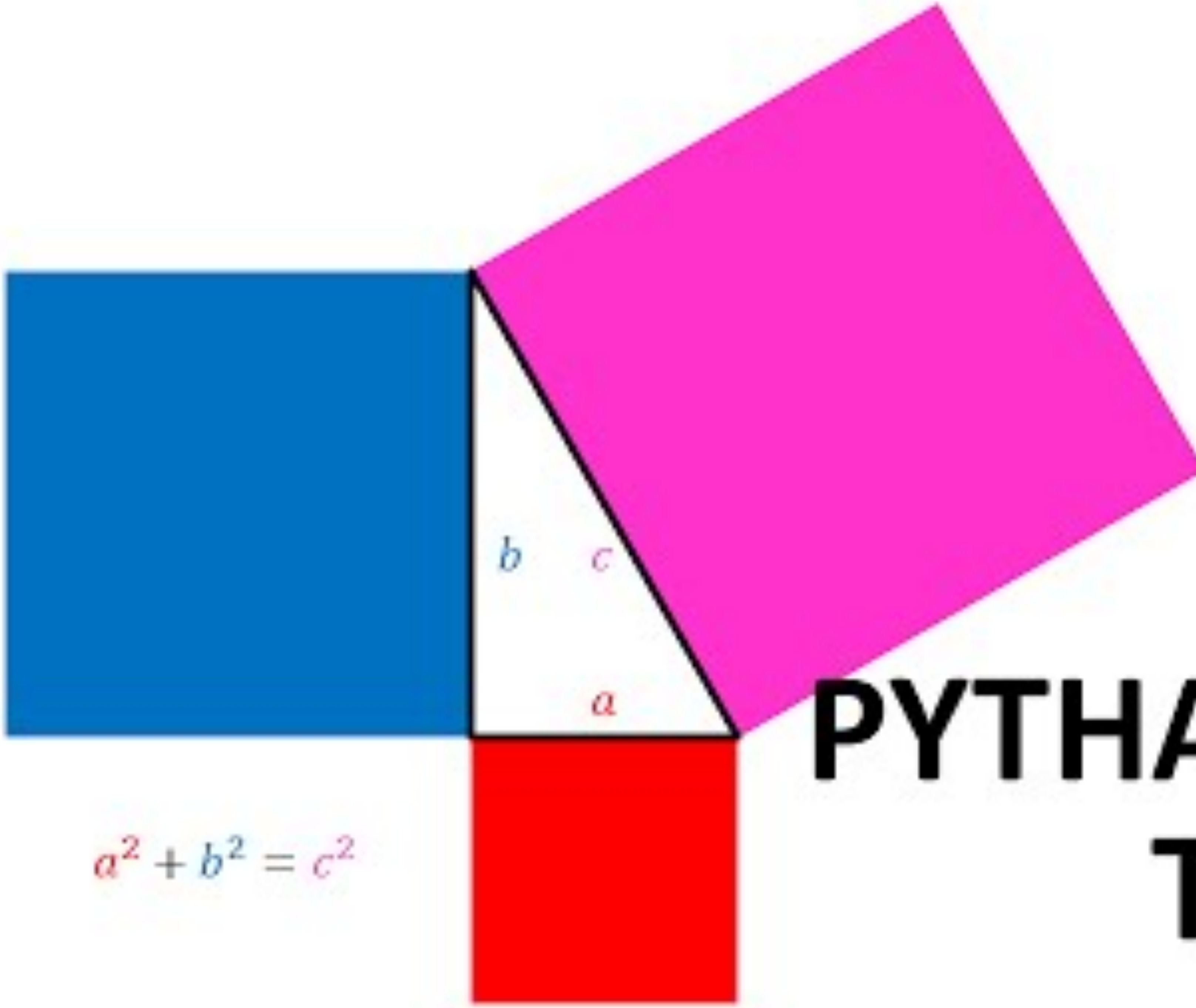
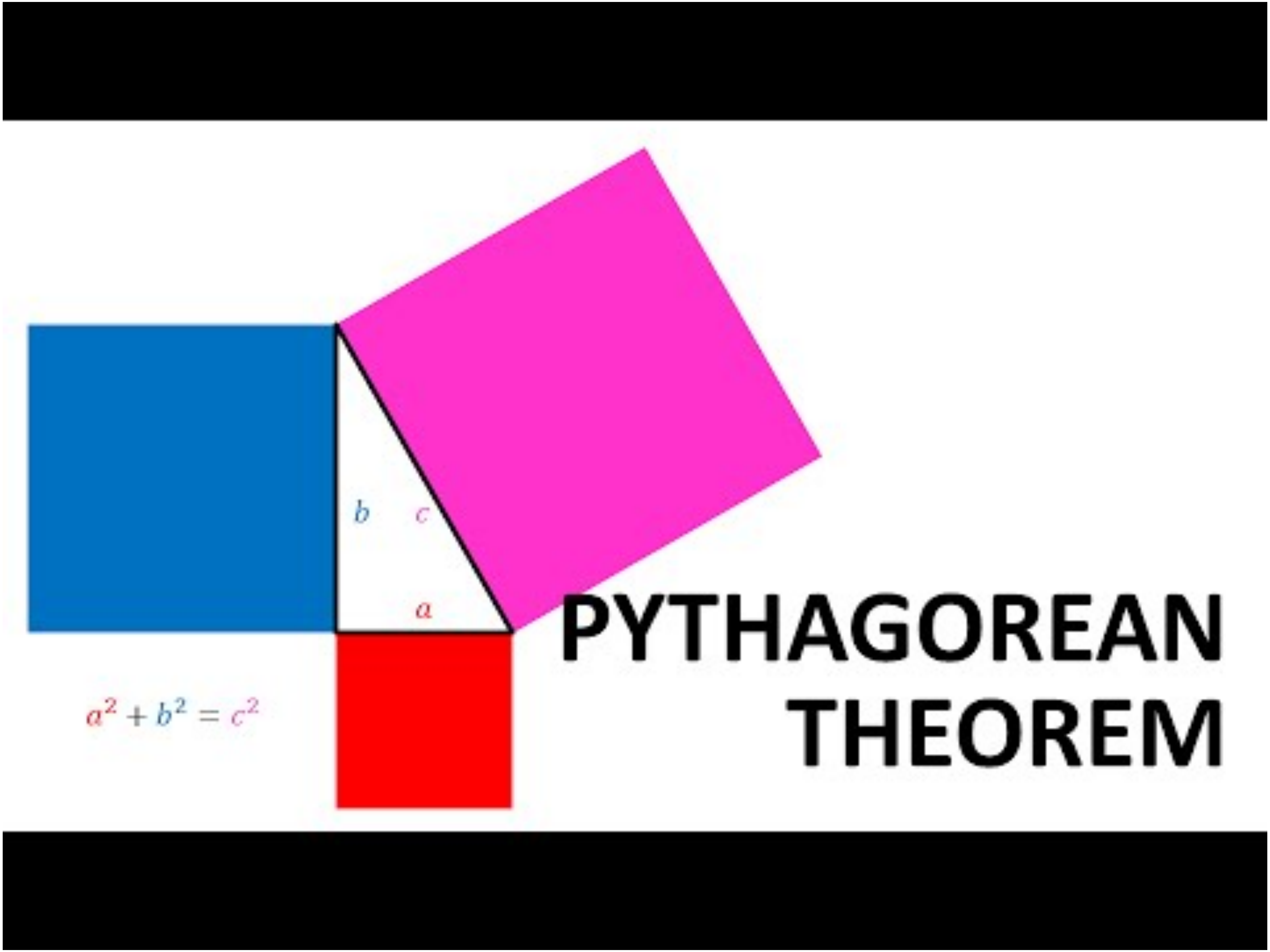


PYTHAGOREAN THEOREM



PYTHAGOREAN
THEOREM
b
c
a
$a^2 + b^2 = c^2$

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- Ancient Greeks also know how to construct a regular pentagon, and given a regular polygon with m sides, they could construct a regular polygon with $2m$ sides.

- Major open question: For which values of n is the regular n-gon constructible with a straightedge and compass?

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- Then, at 24, he entirely classified which regular polygons are constructible.

- He did not prove one direction of his theorem, though. That was done by Pierre Wantzel.

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- A Fermat prime is a prime of the form $2^{2^k} + 1$.

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- Book VI: Proportions between lines and shapes

The Elements

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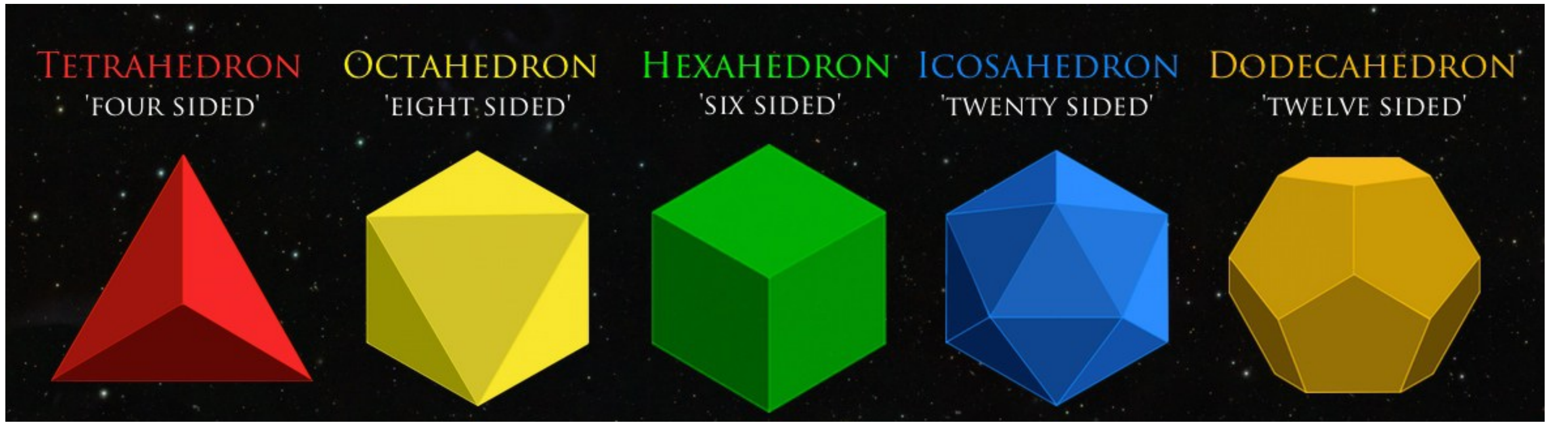
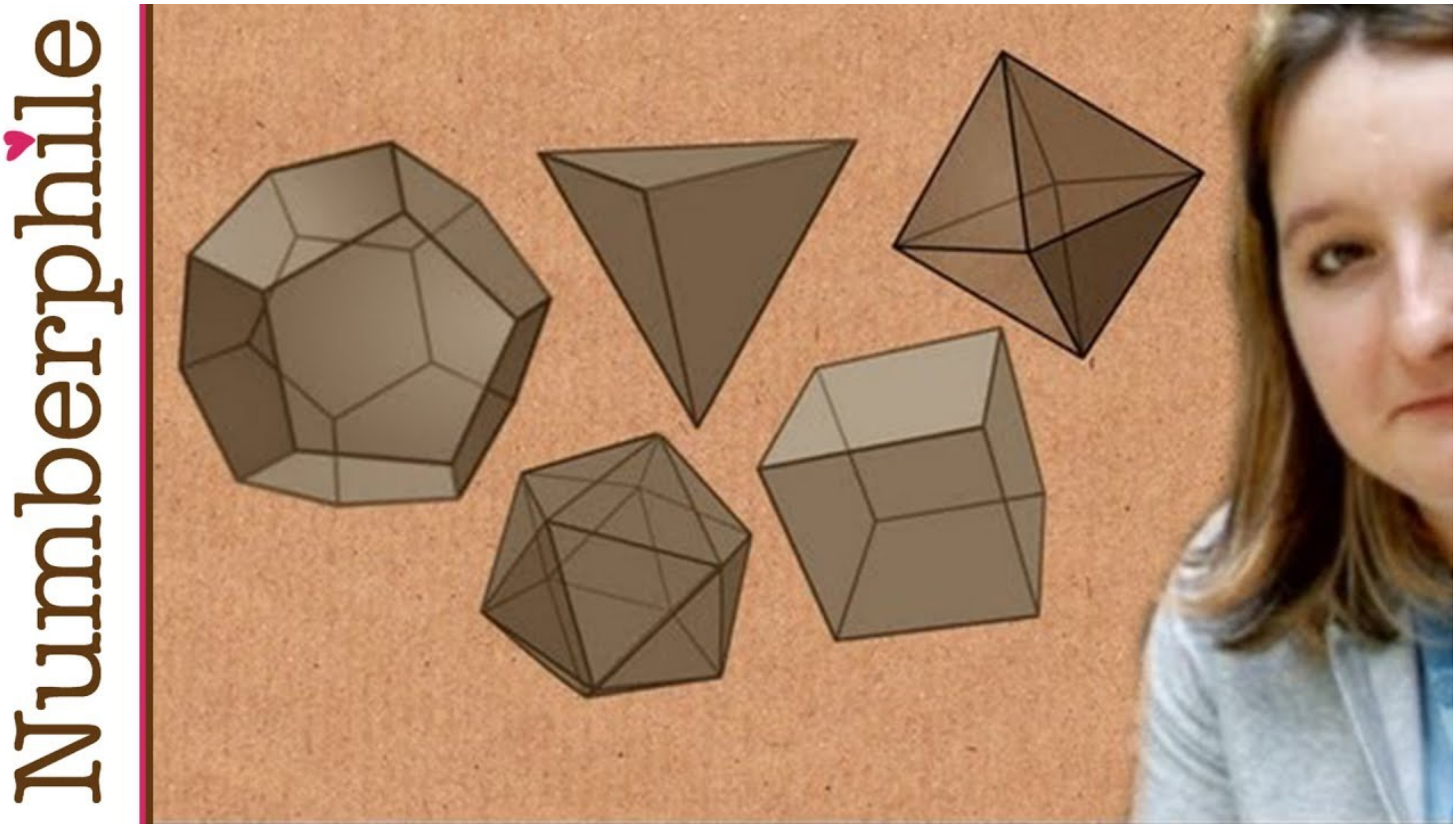
- Book XIII: Platonic solids



Platonic solids



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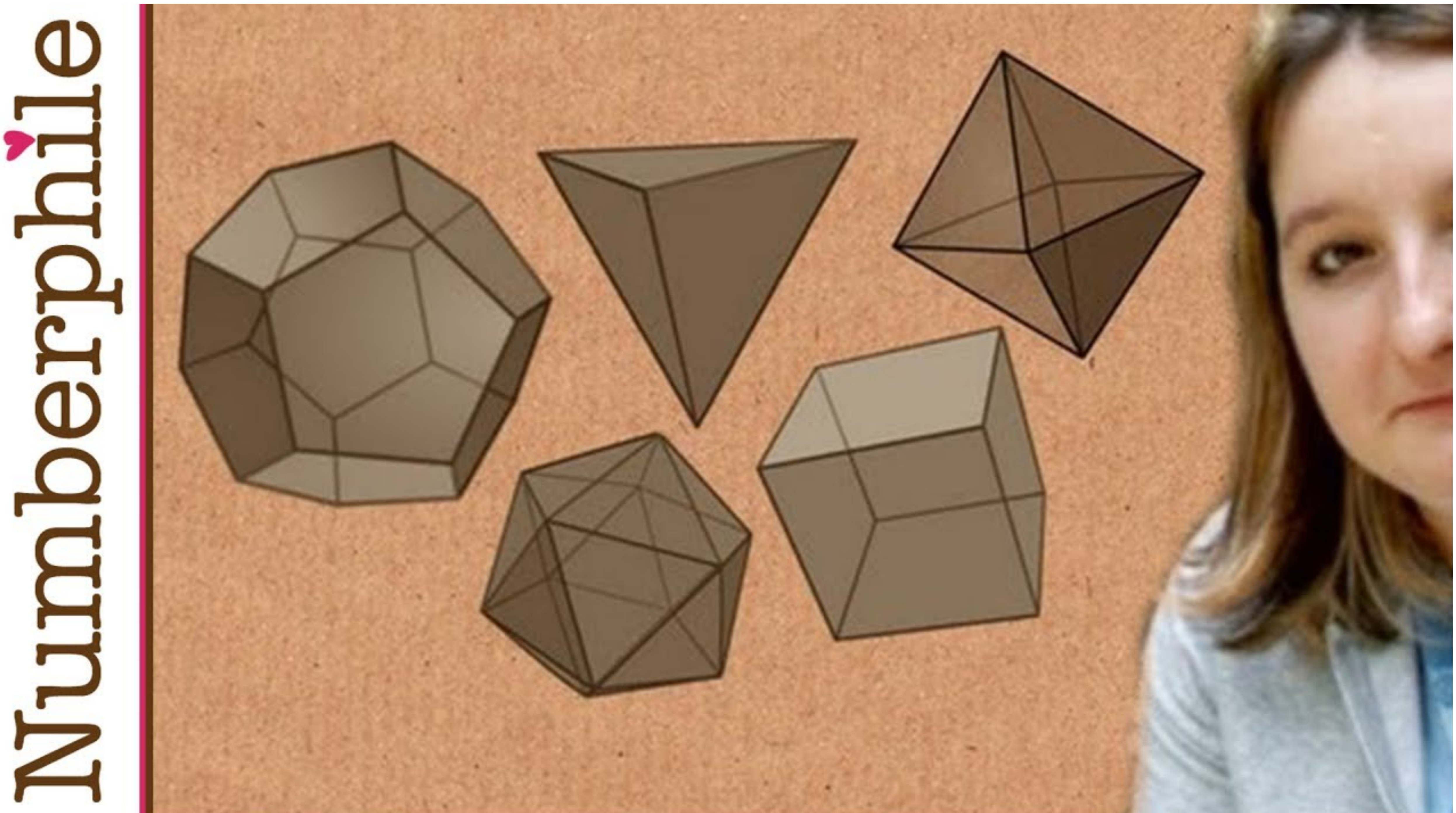
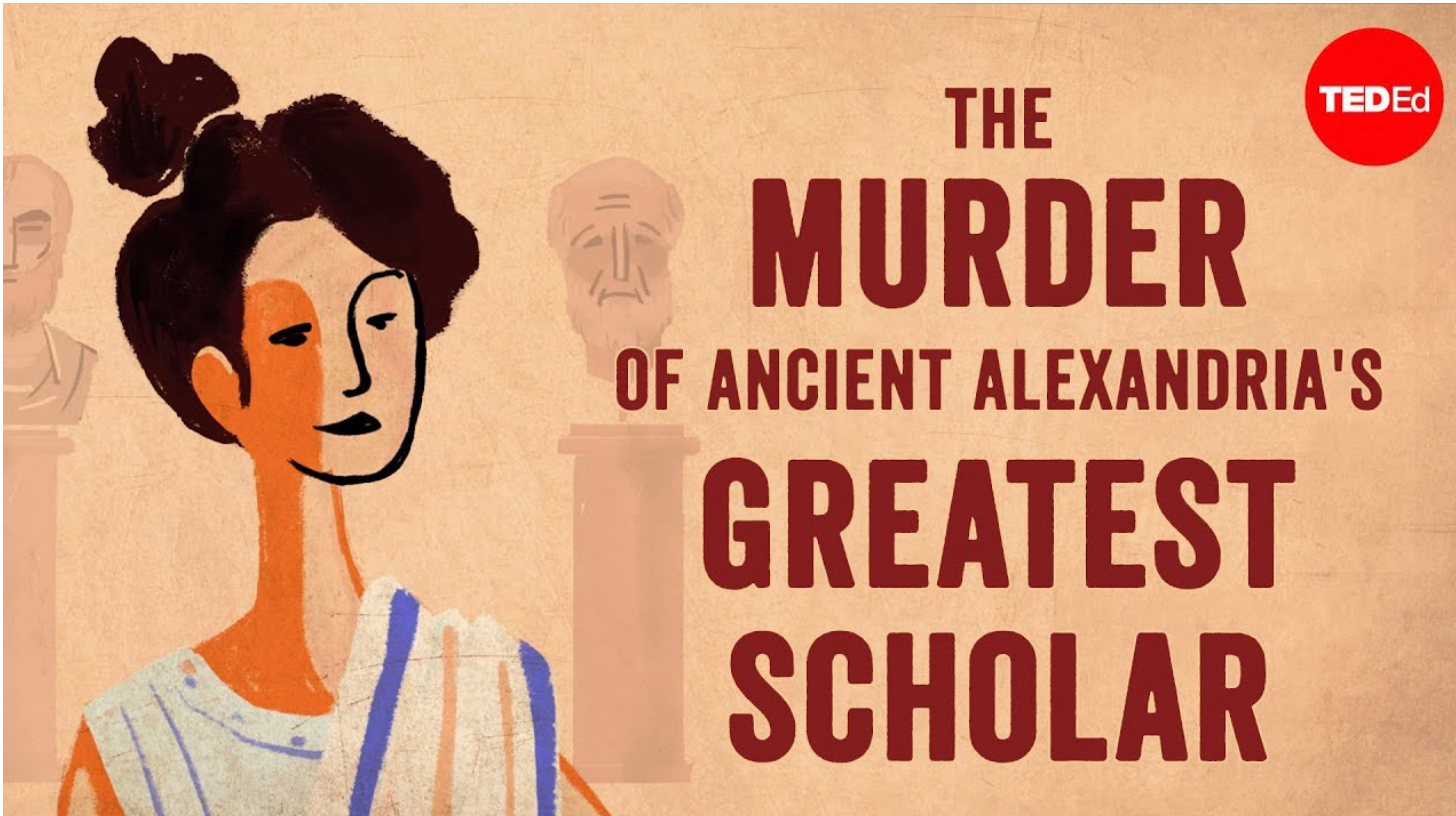


Hypatia



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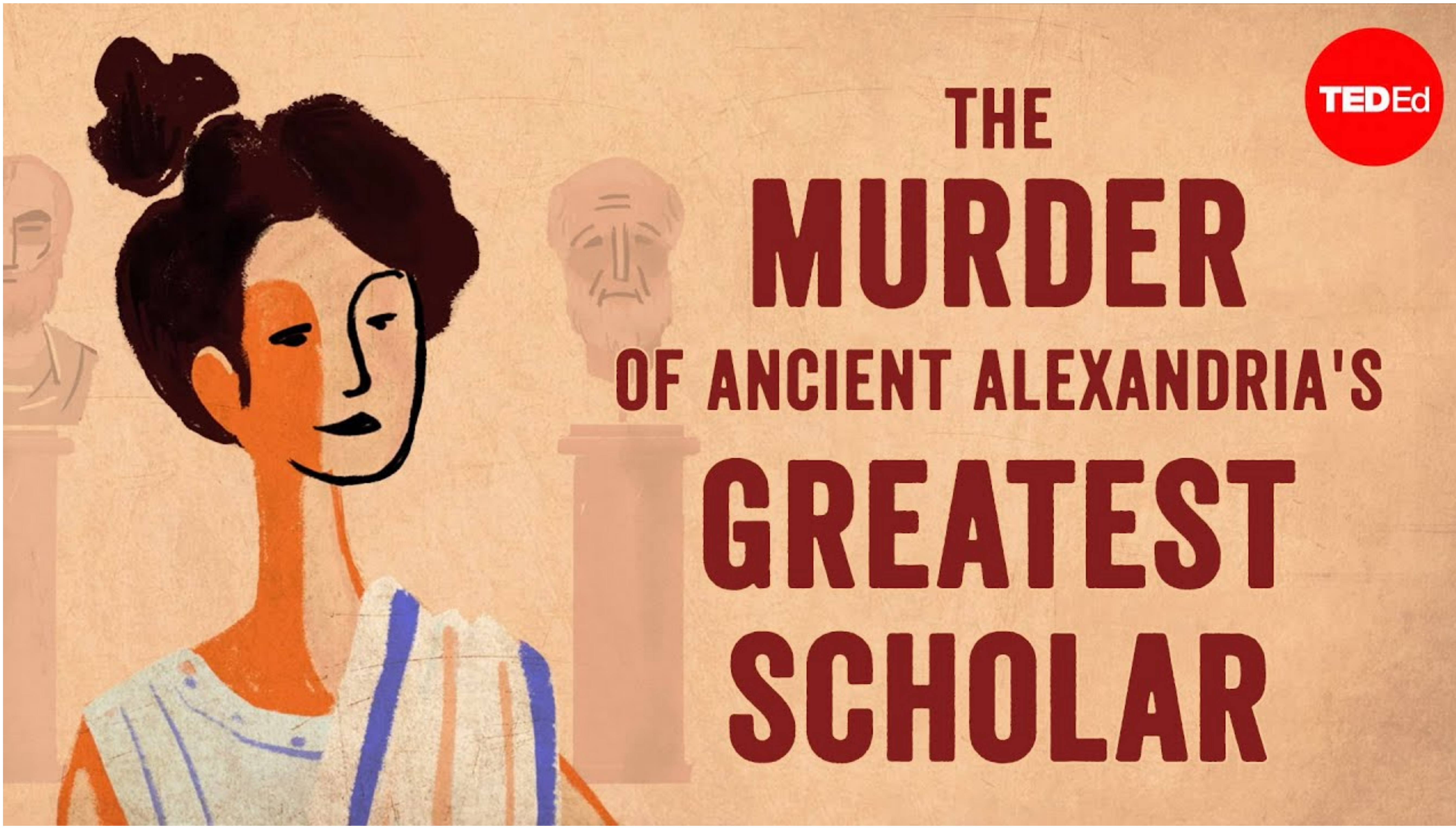

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- She was a first-class thinker and highly respected philosopher, mathematician and teacher.

- Her life ended in tragedy, as she was brutally murdered for defending religious freedom.

The Legacy of Axiomatic Thinking

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- Can that postulate be deduced from the other four? Unfortunately, no.

- In this way, the parallel postulate is *undecidable* if your axioms are only the first four.

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- This implies that there exist non-Euclidean geometries. Instead of the parallel postulate, you can have an axiom saying that if ℓ is a line and P is a point off of ℓ, then there are 2+ lines through P that are parallel to ℓ. Or no lines that are parallel.

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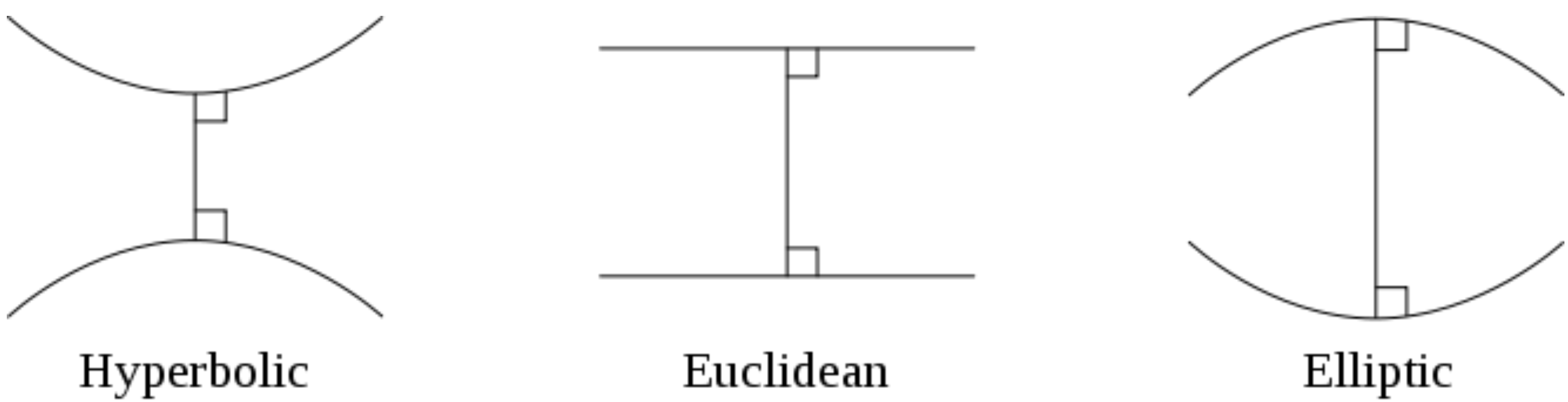

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- From these, not only is set theory built, but number theory, too.

- Kurt Gödel proved that every set of axioms that leads to number theory must have undecidable statements. There must be statements that can neither be proven nor disproven.

Approximating $\sqrt{2}$

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$$\sqrt{2} \approx 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34}.$$

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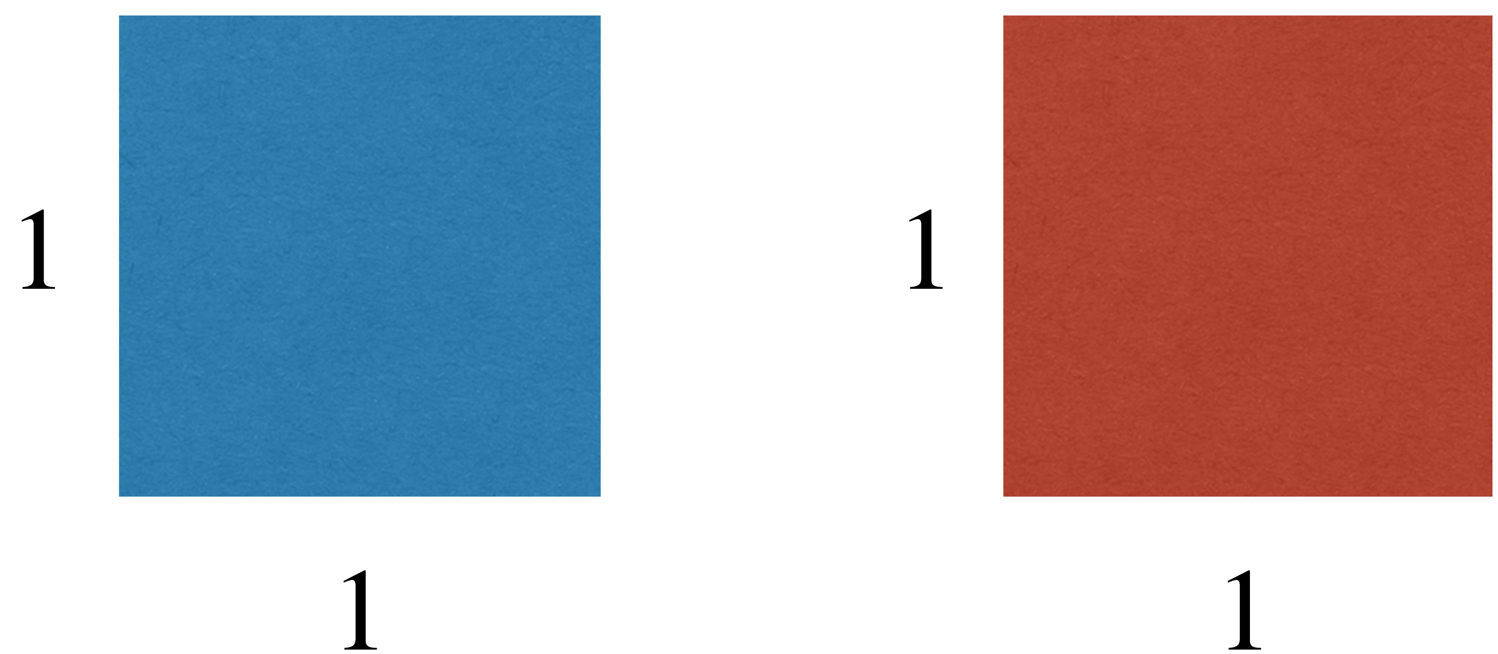
Approximating $\sqrt{2}$

Suppose you have two 1×1 squares



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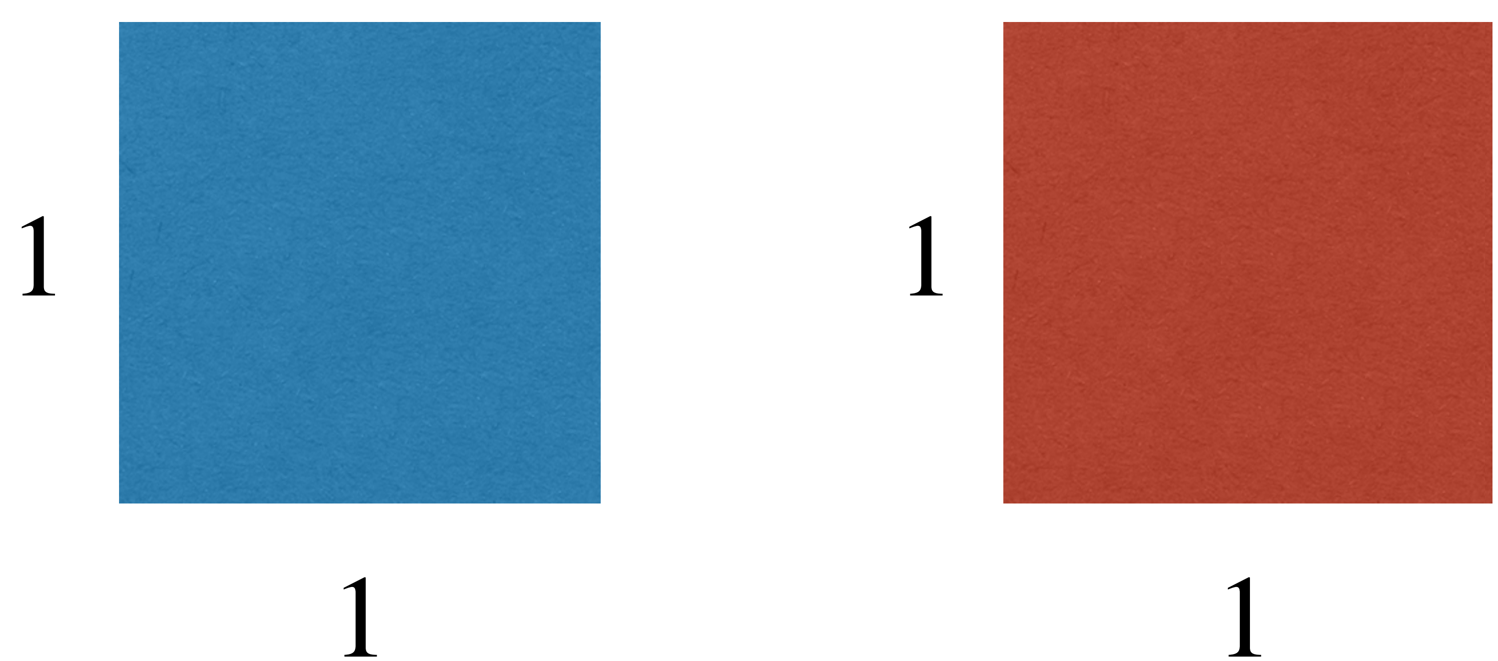

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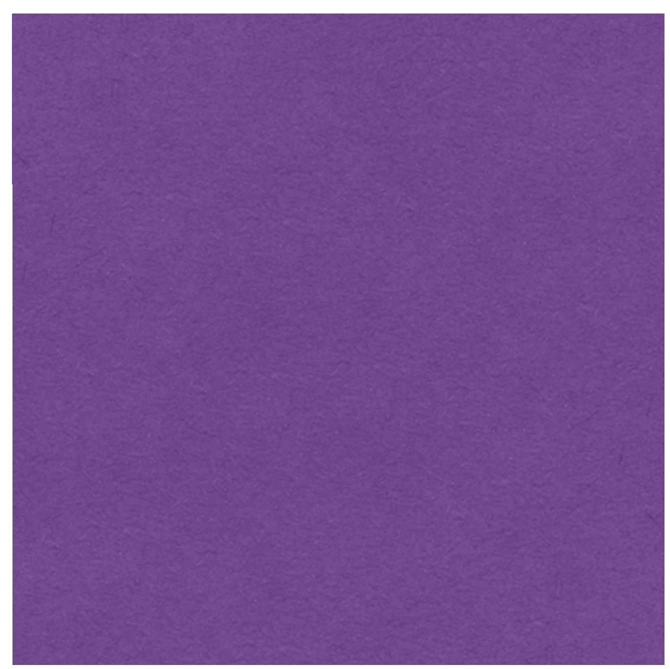
If you could combine them into a single square, it would be a $\sqrt{2} \times \sqrt{2}$ square



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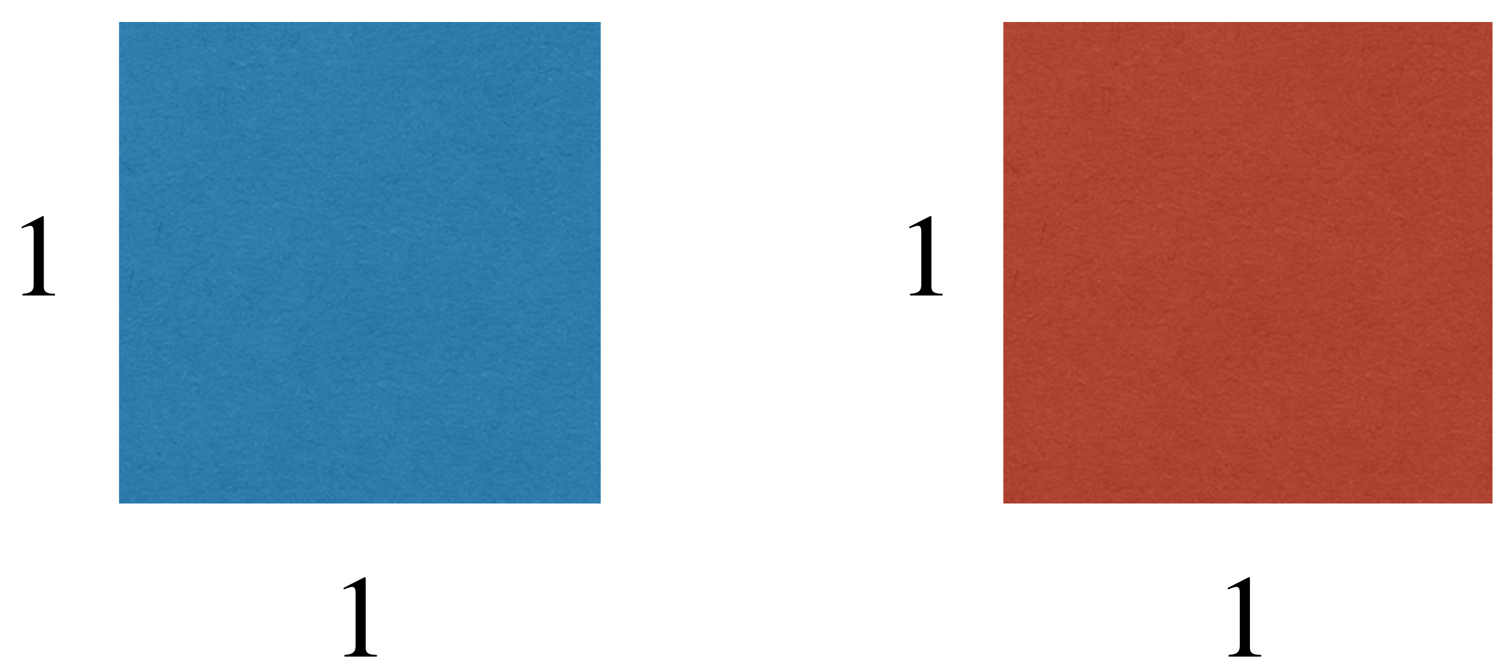

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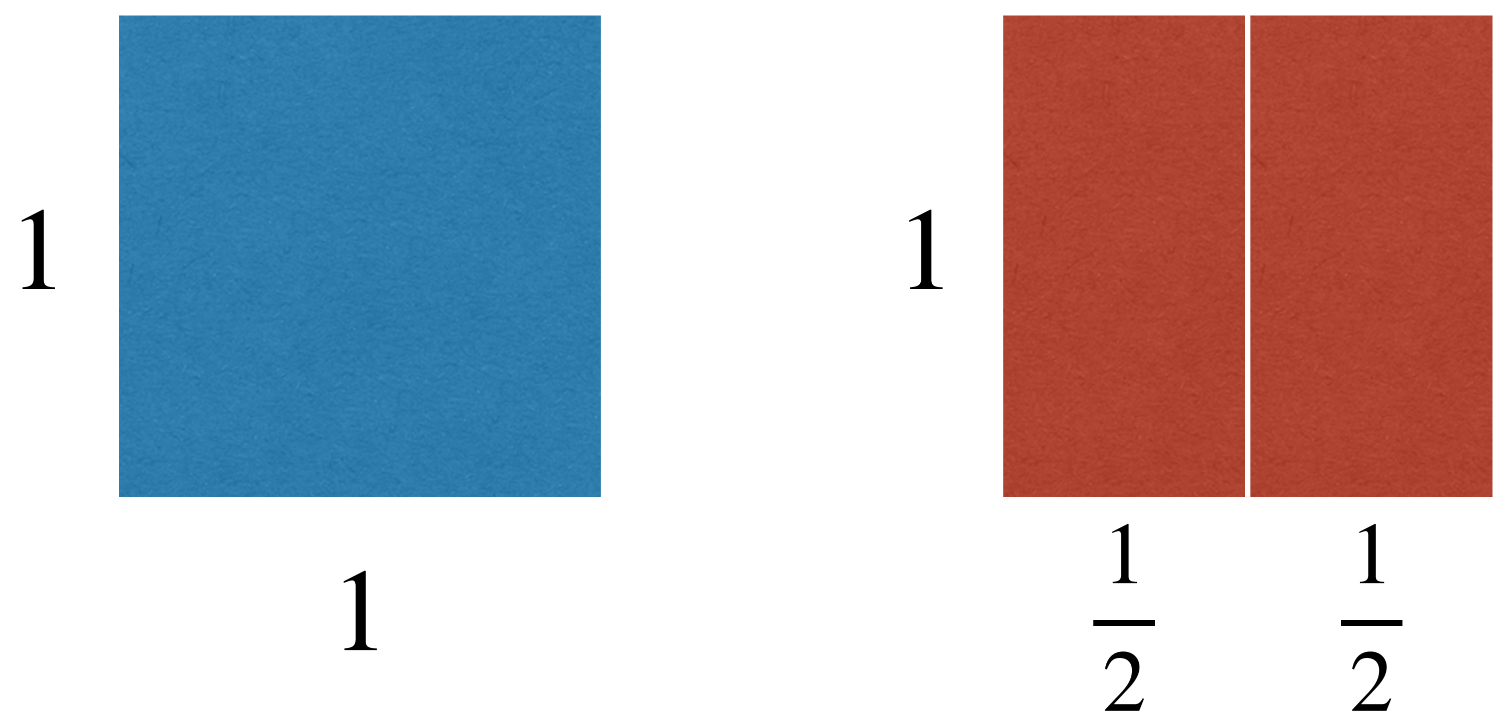

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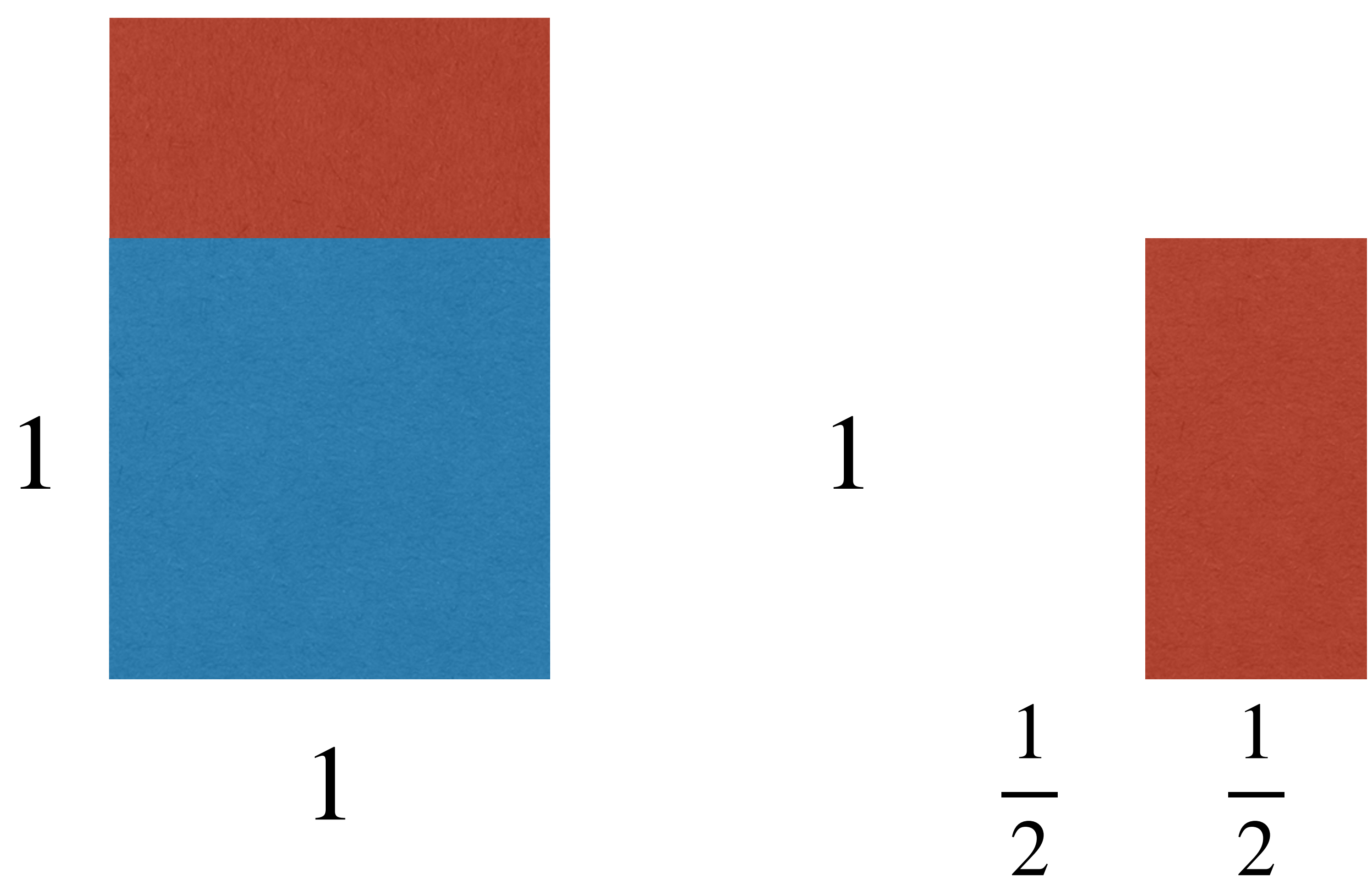

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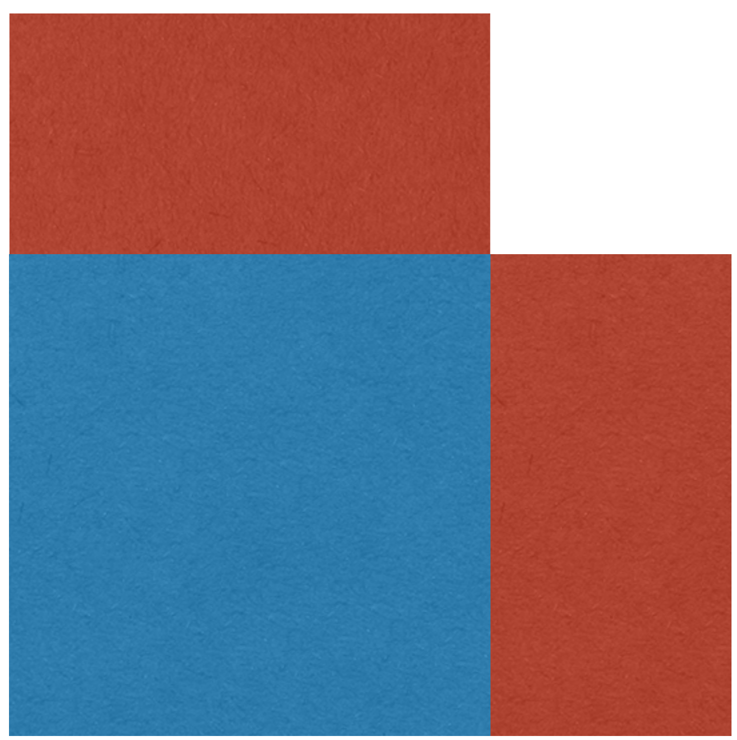

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$\frac{1}{2}$ $\frac{1}{2}$

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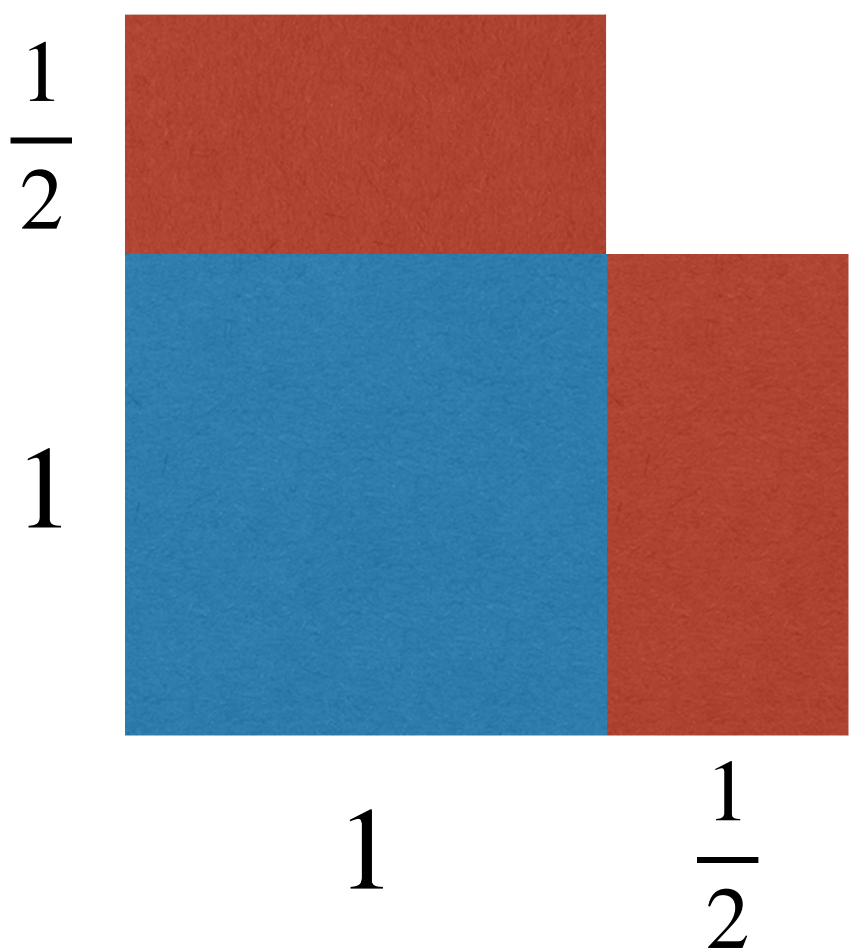

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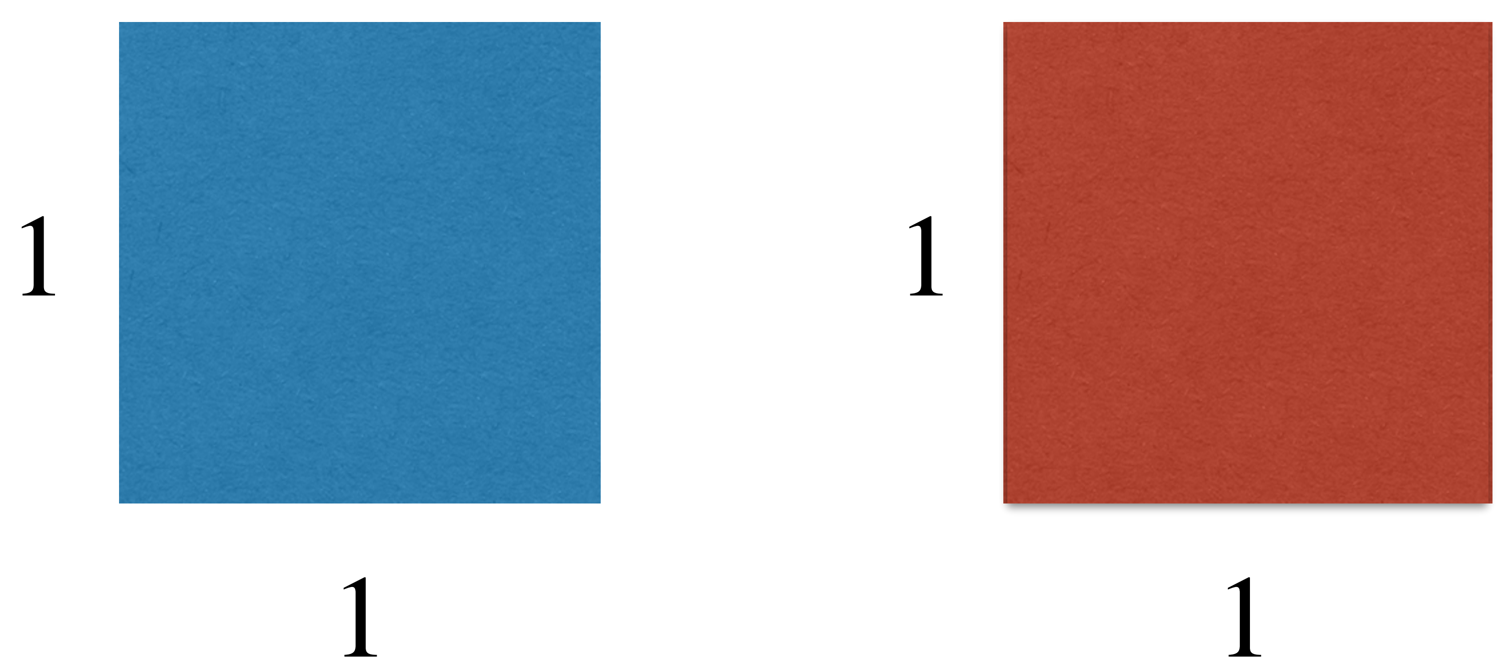

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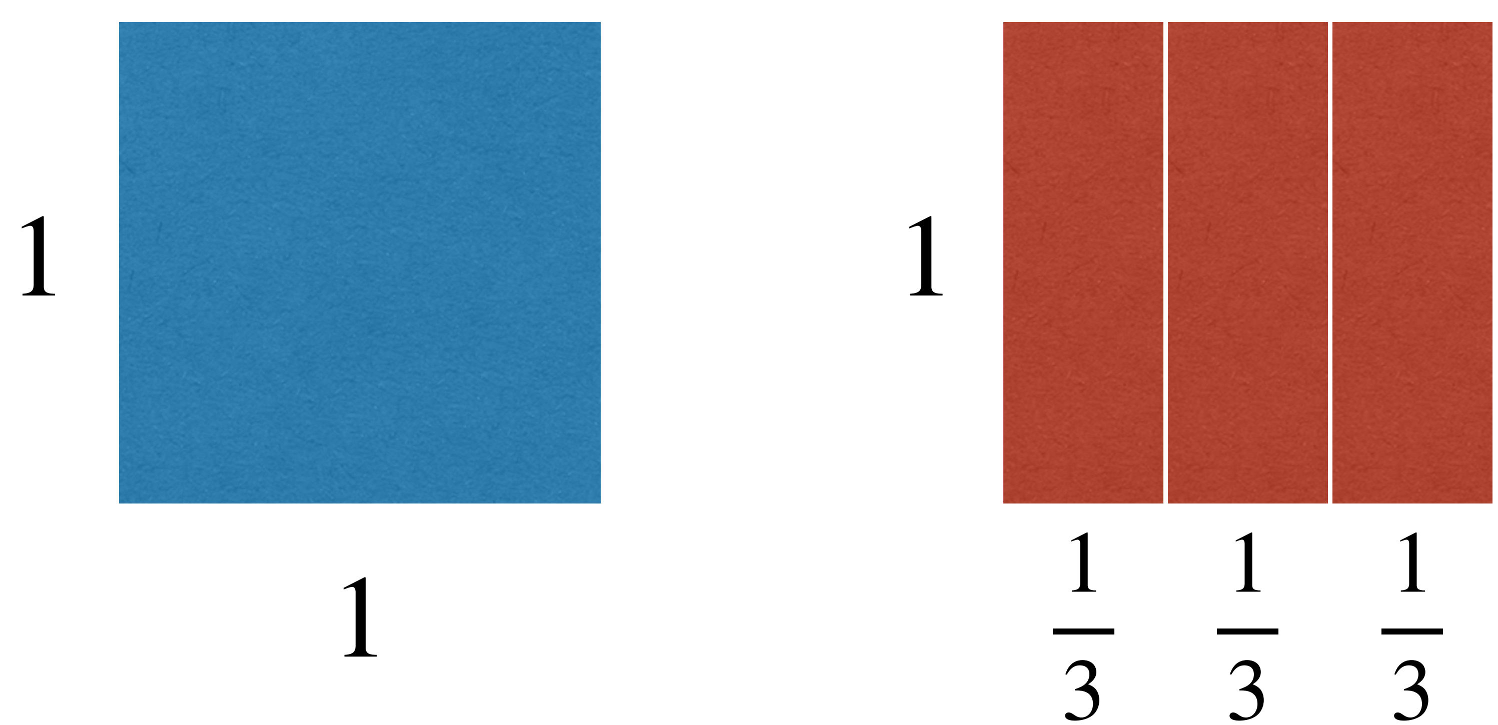

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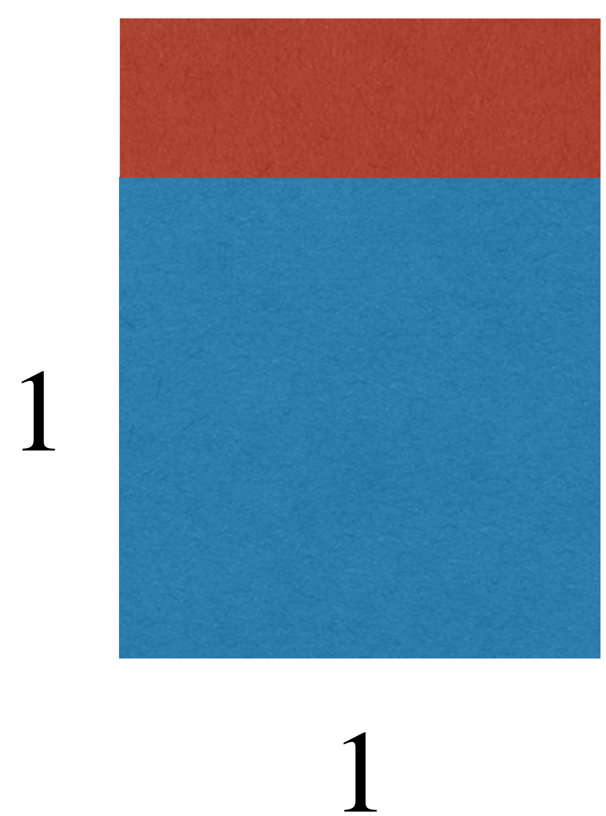

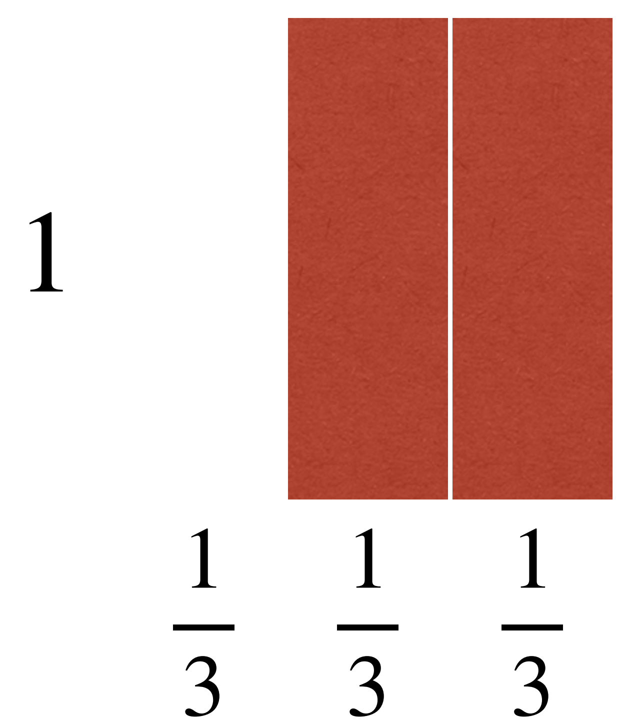

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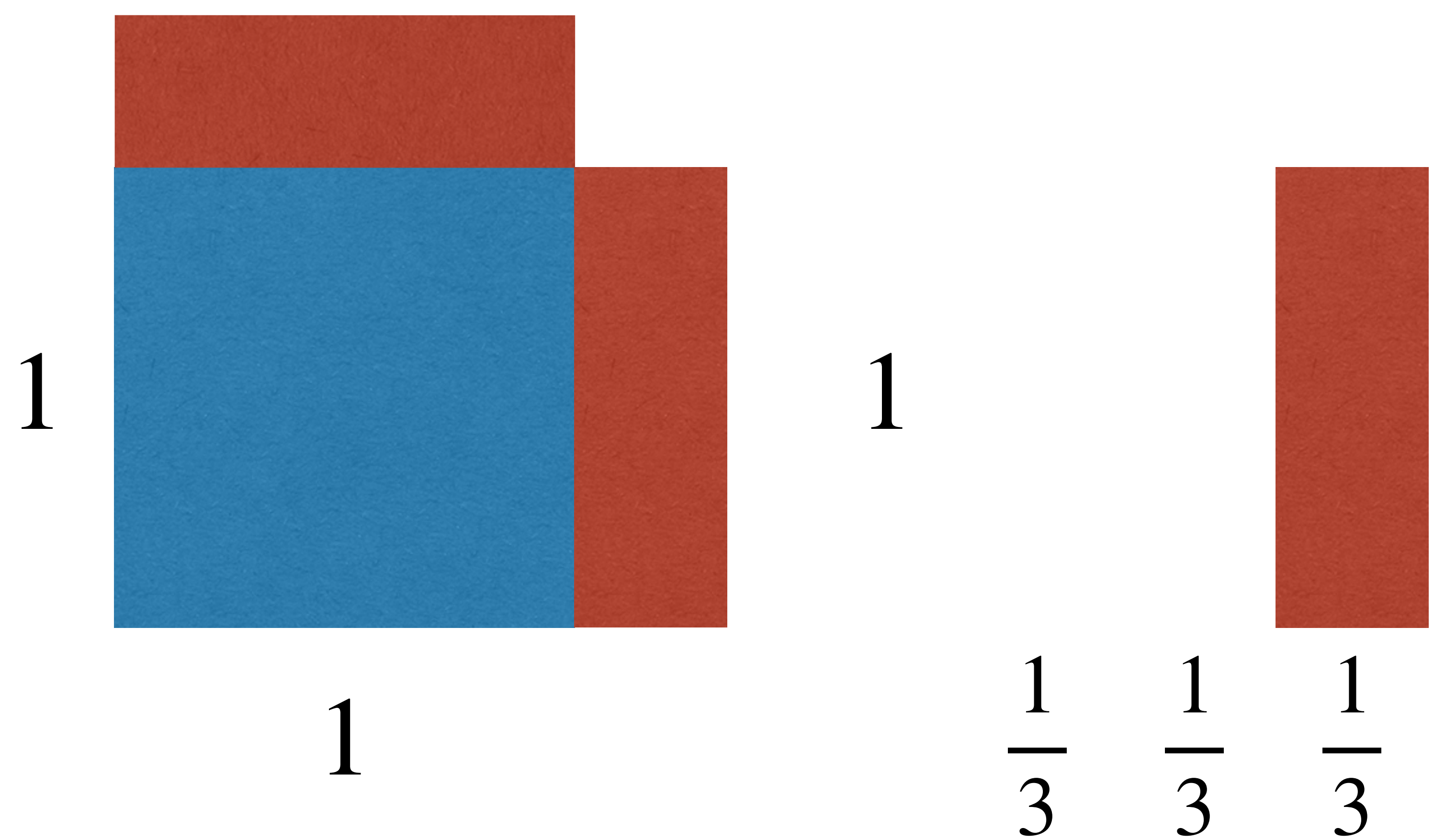

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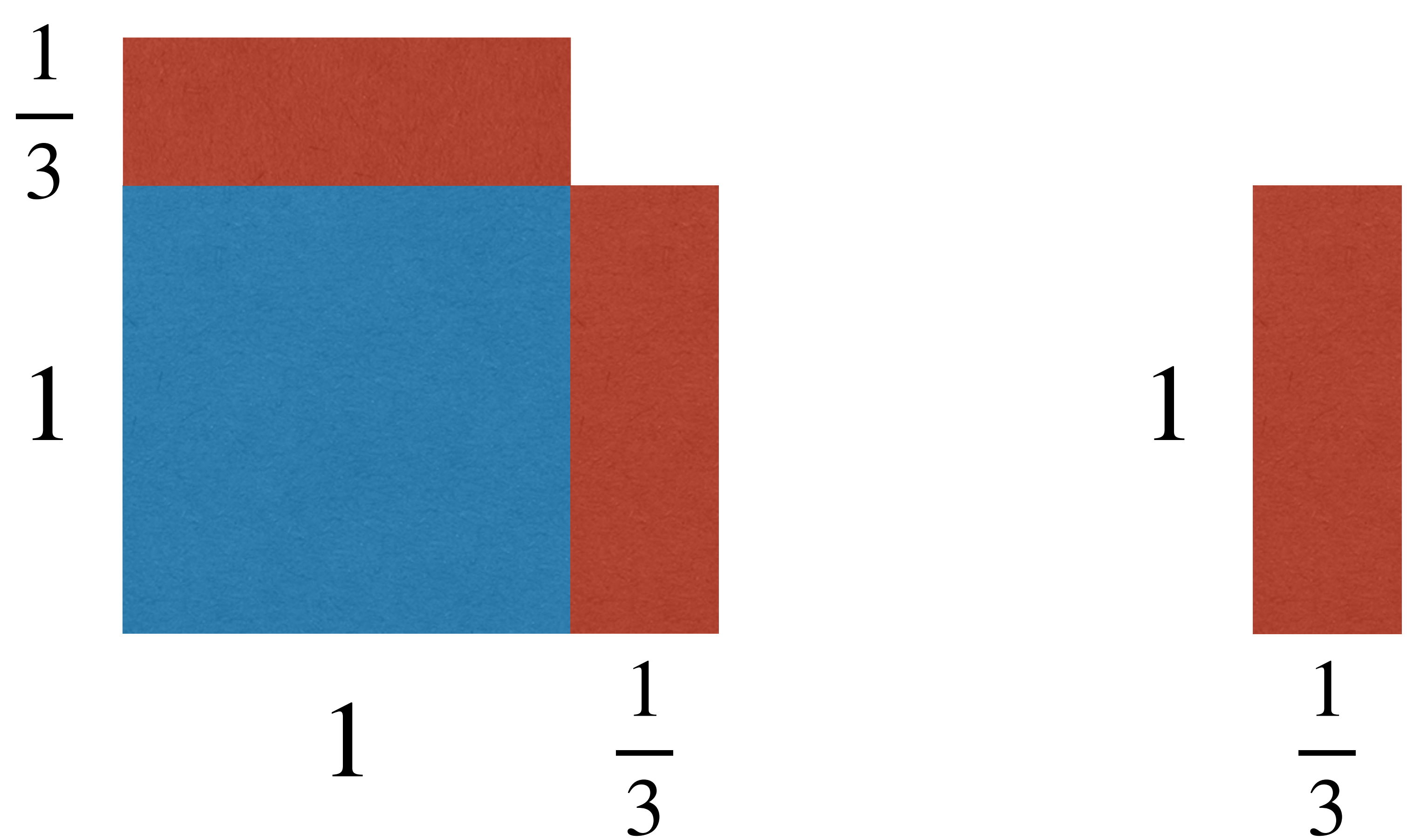

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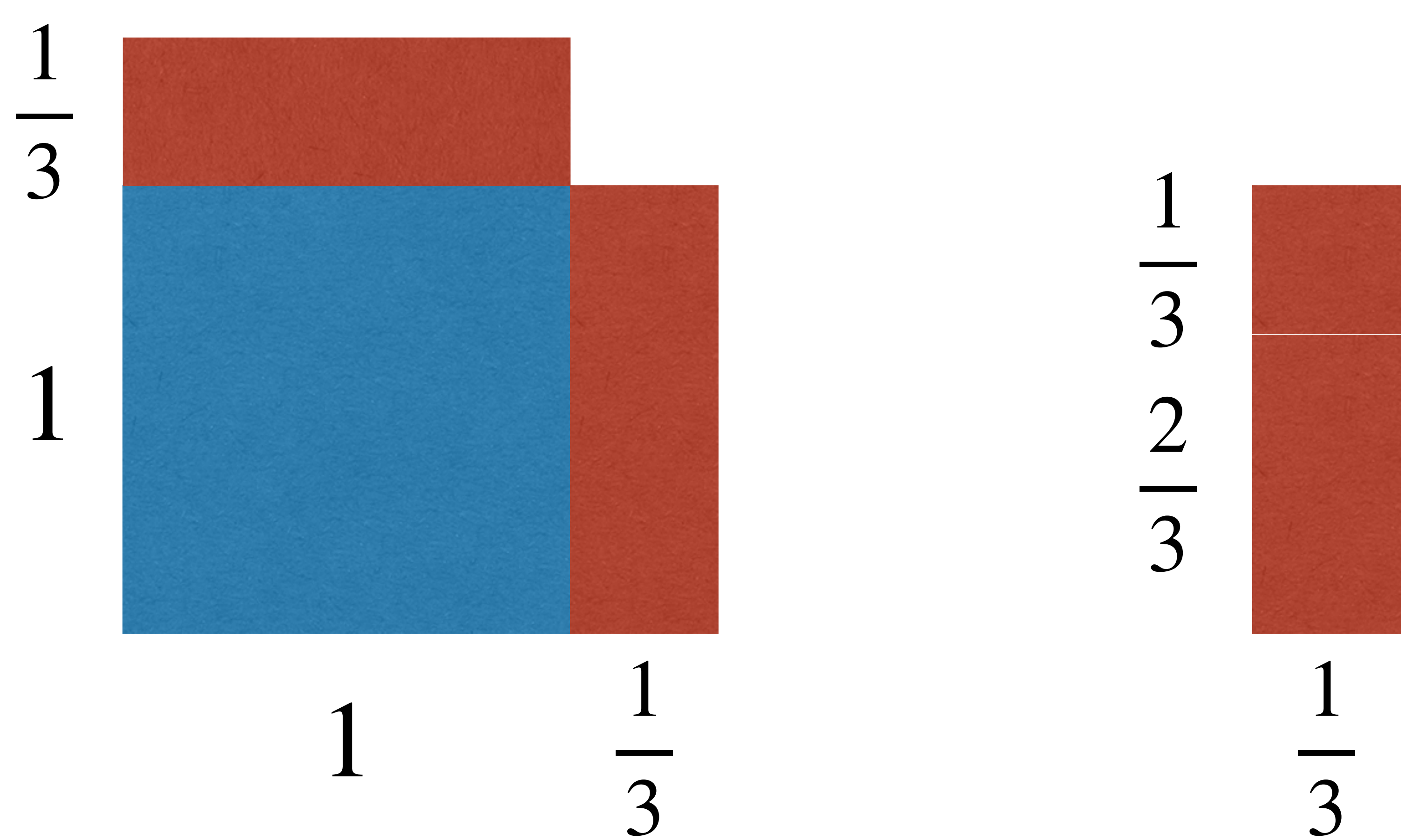

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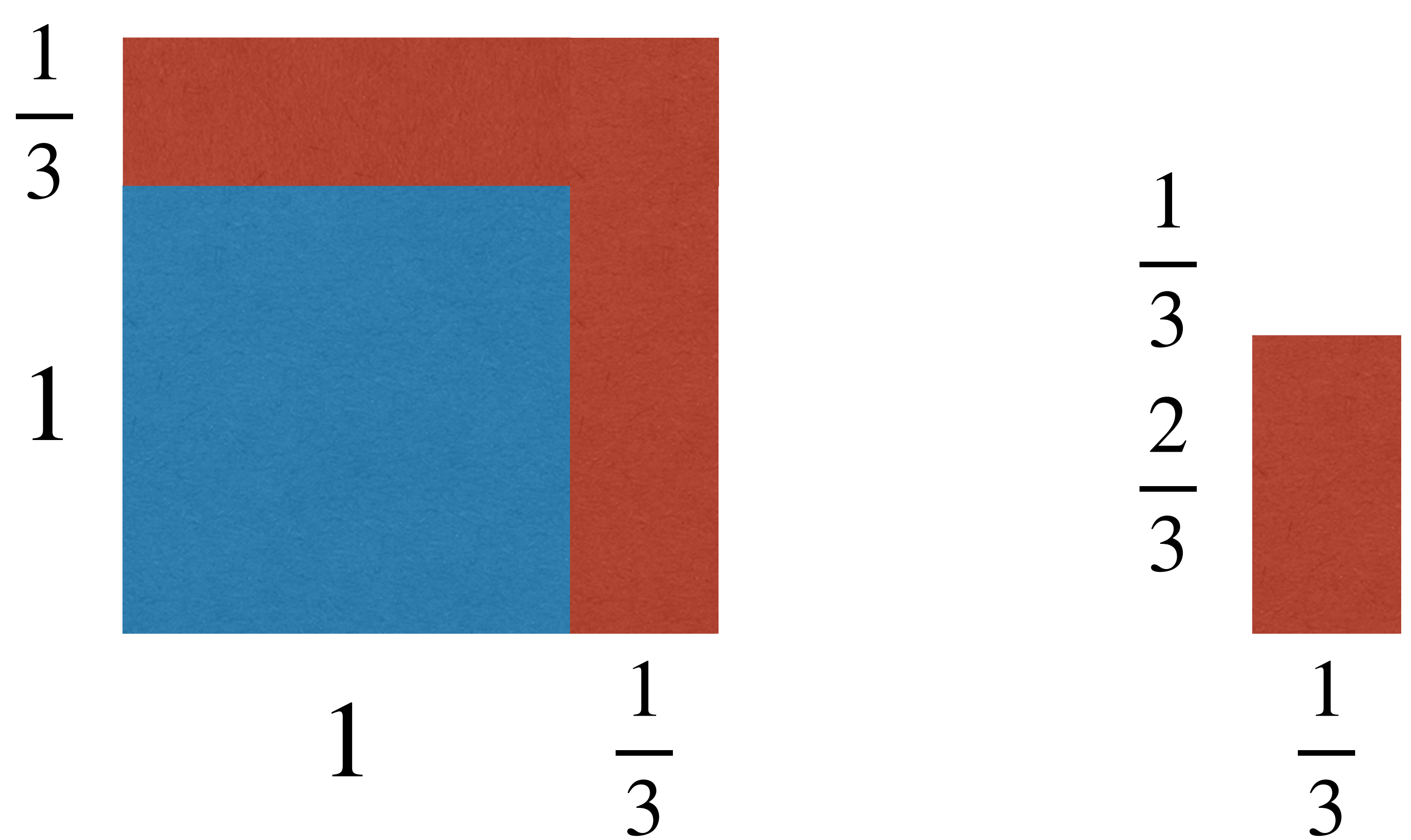

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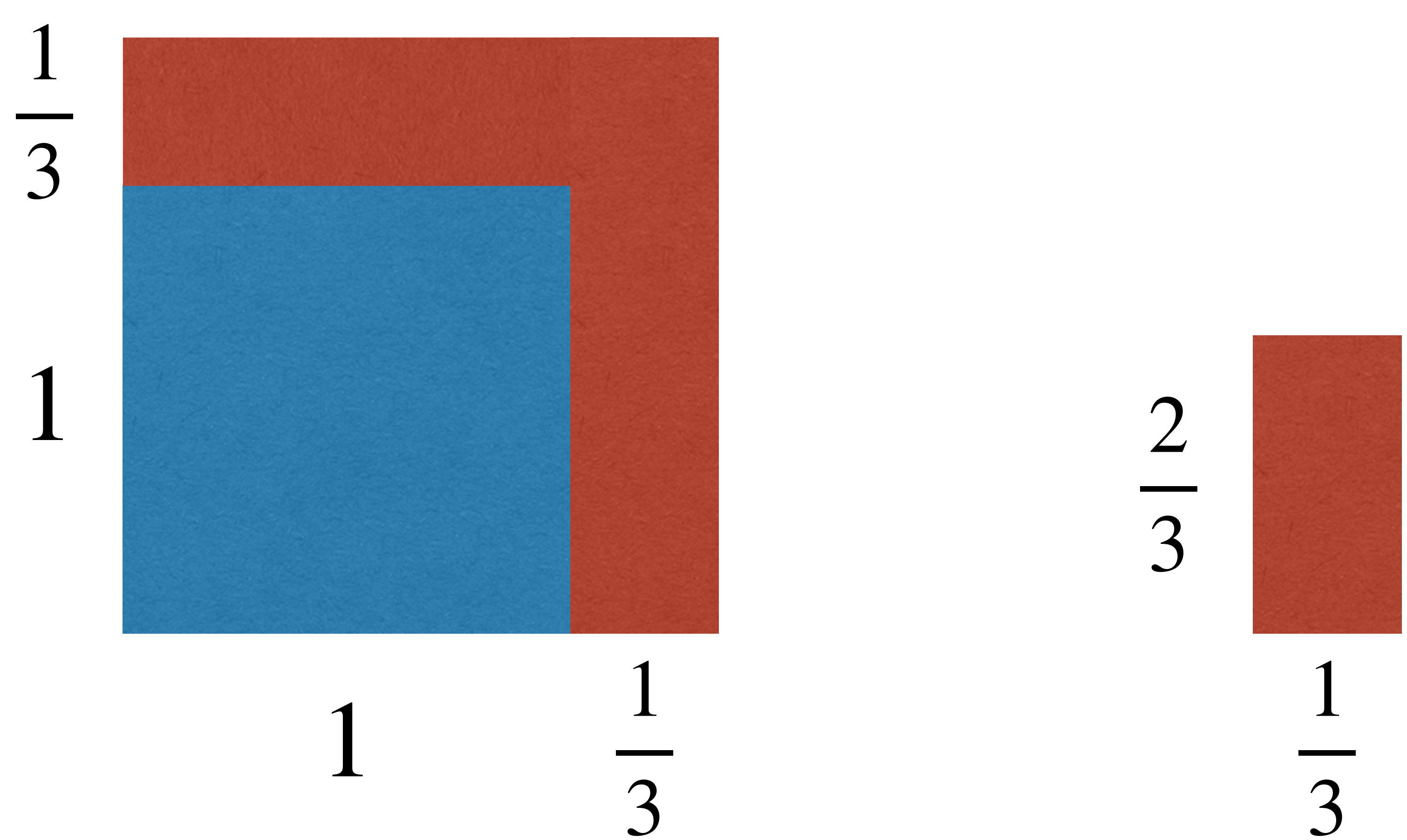

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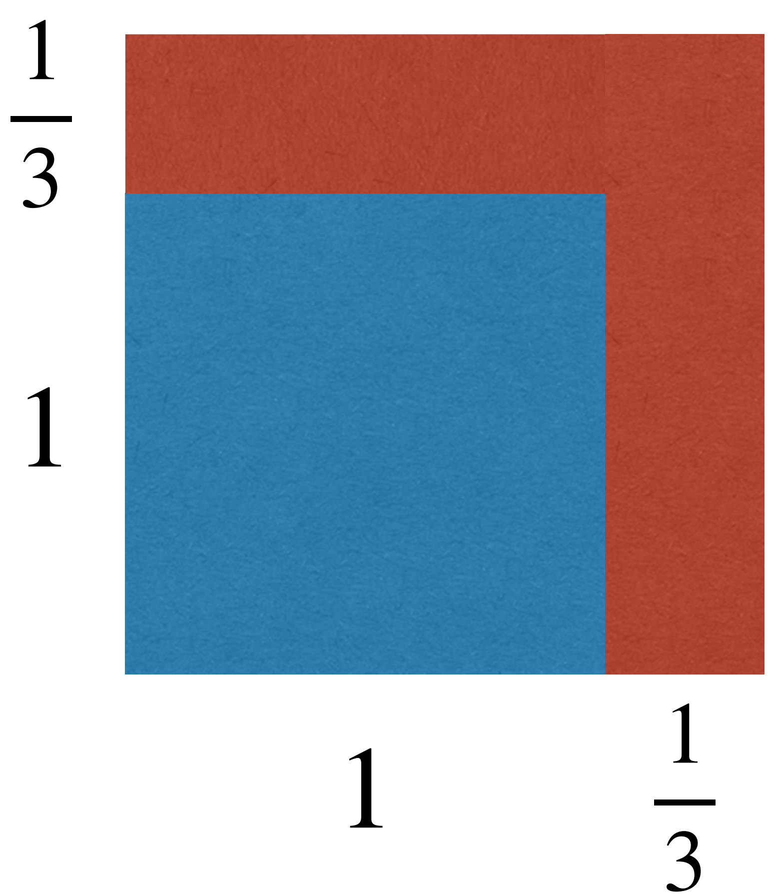
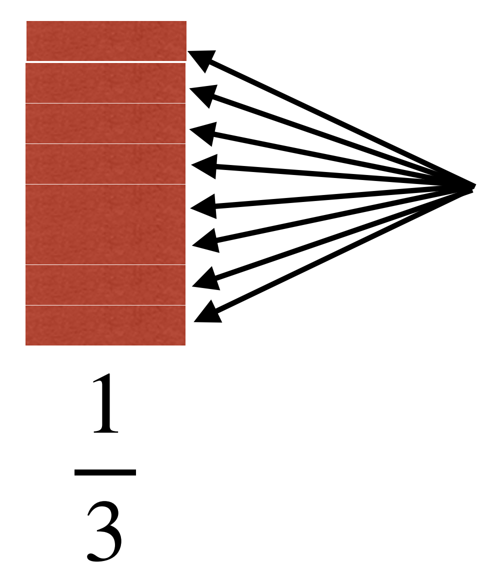
Divide into 8 pieces





$$\frac{1}{8} \cdot \frac{2}{3} = \frac{1}{3 \cdot 4}$$

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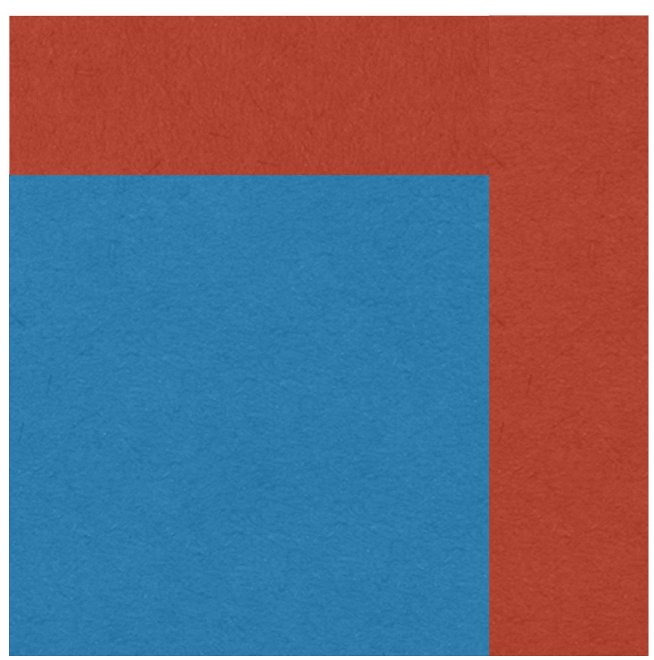
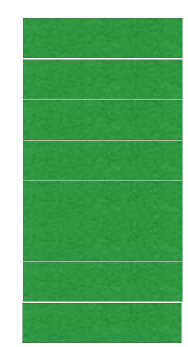
Place them on the diagram





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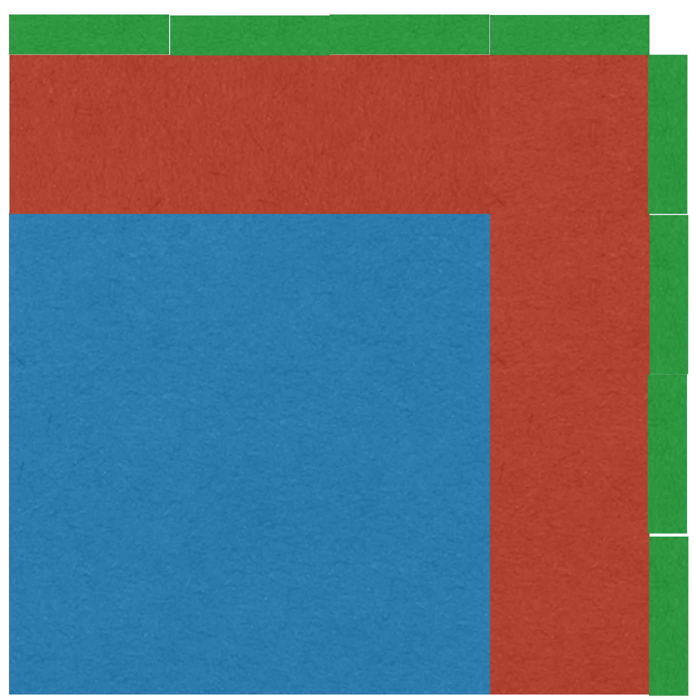
1



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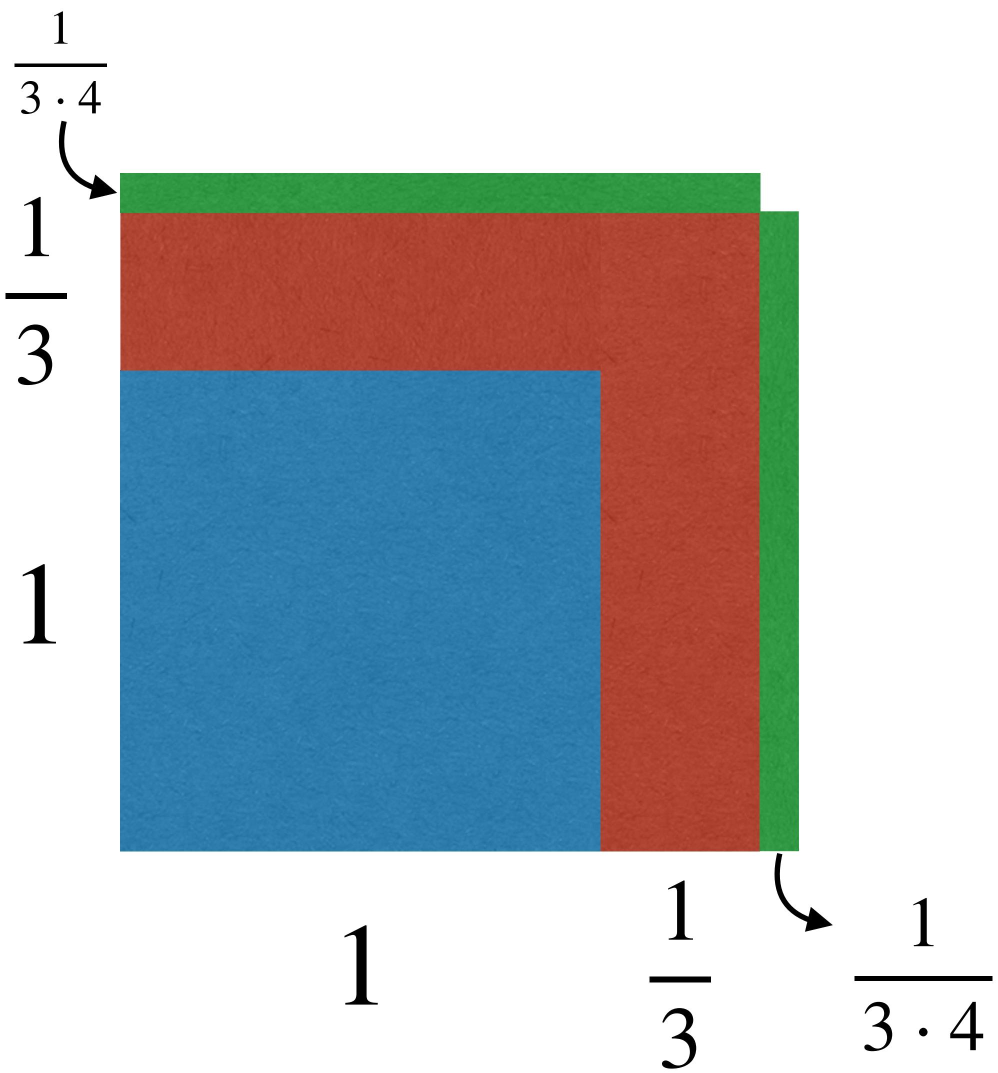

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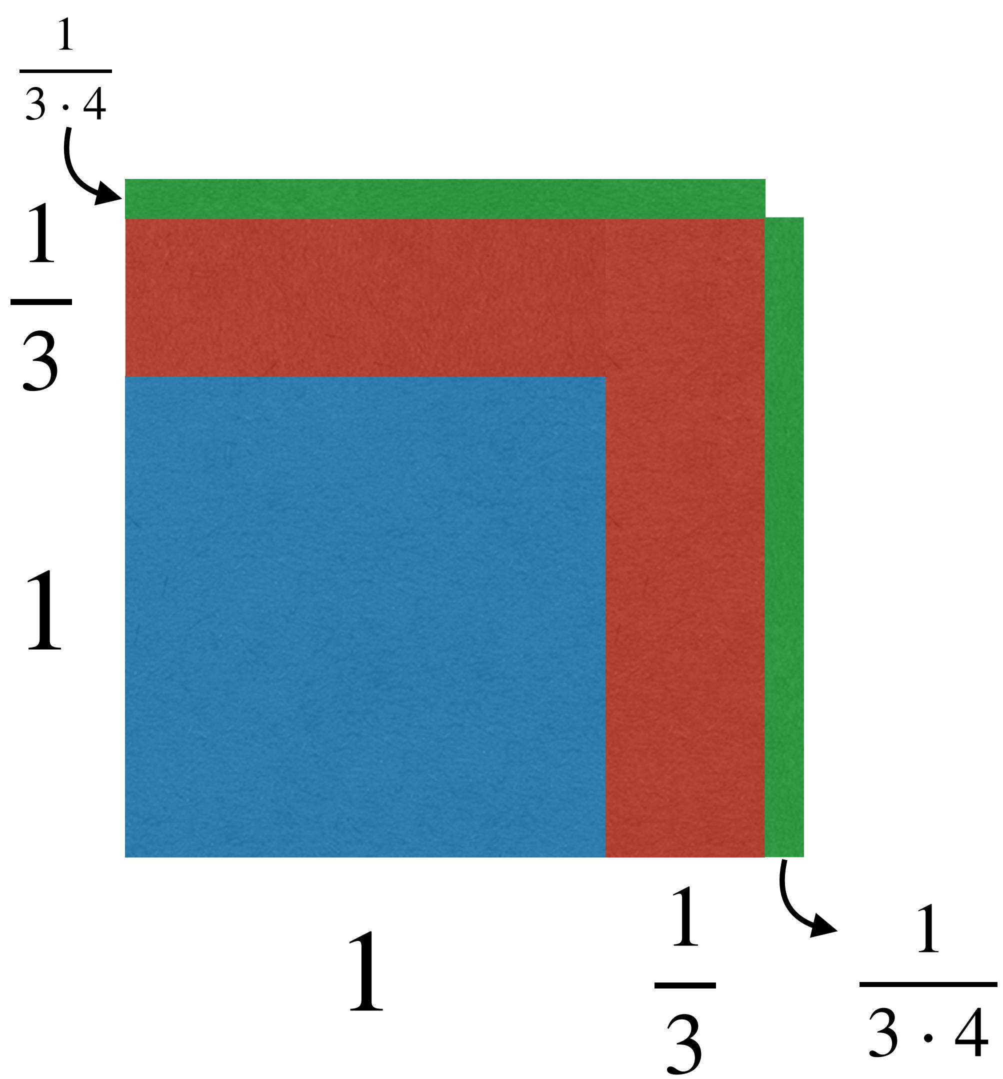

This is not a square, but it's close. And it does use all of the original two squares.

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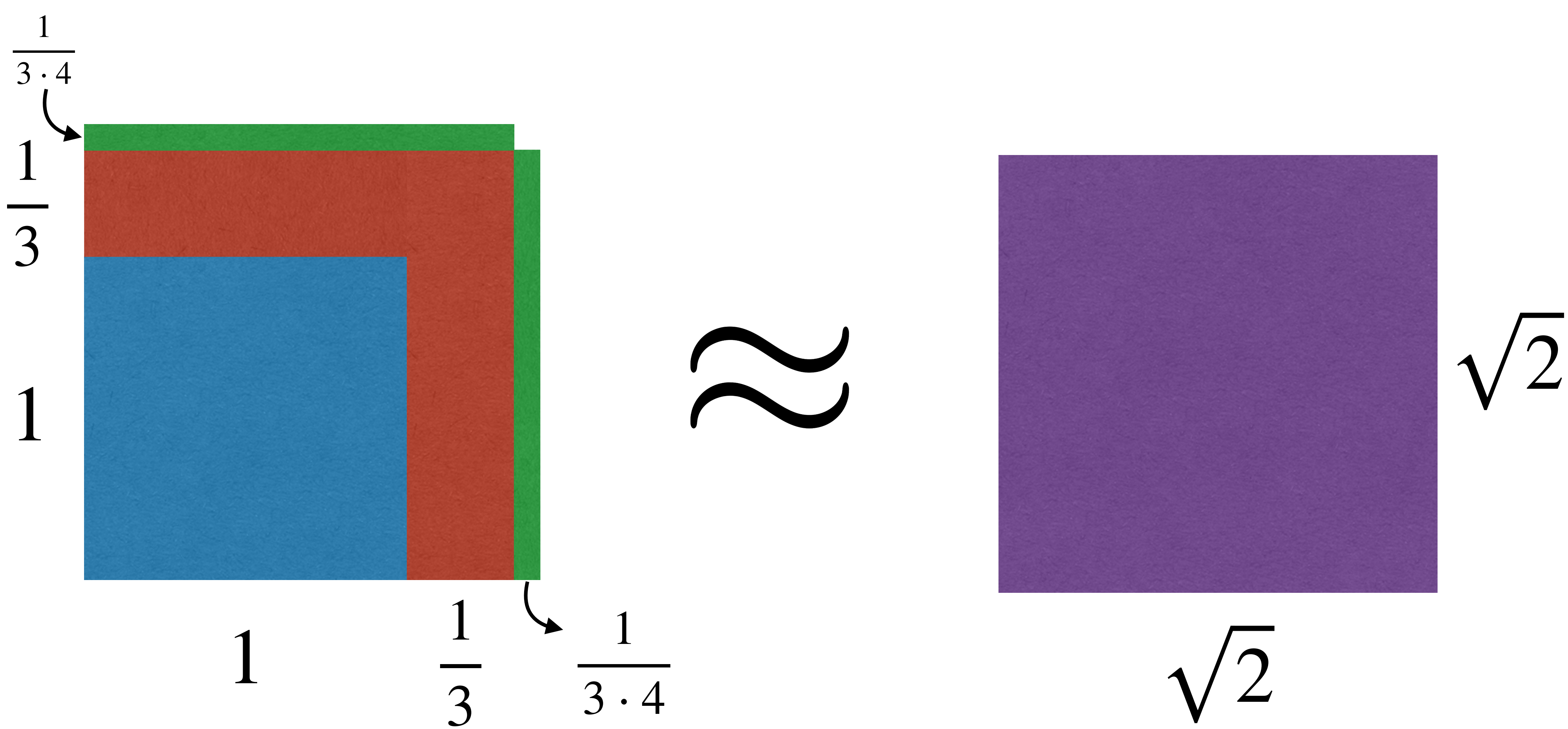

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$$1 + \frac{1}{3} + \frac{1}{3 \cdot 4} \approx \sqrt{2}$$

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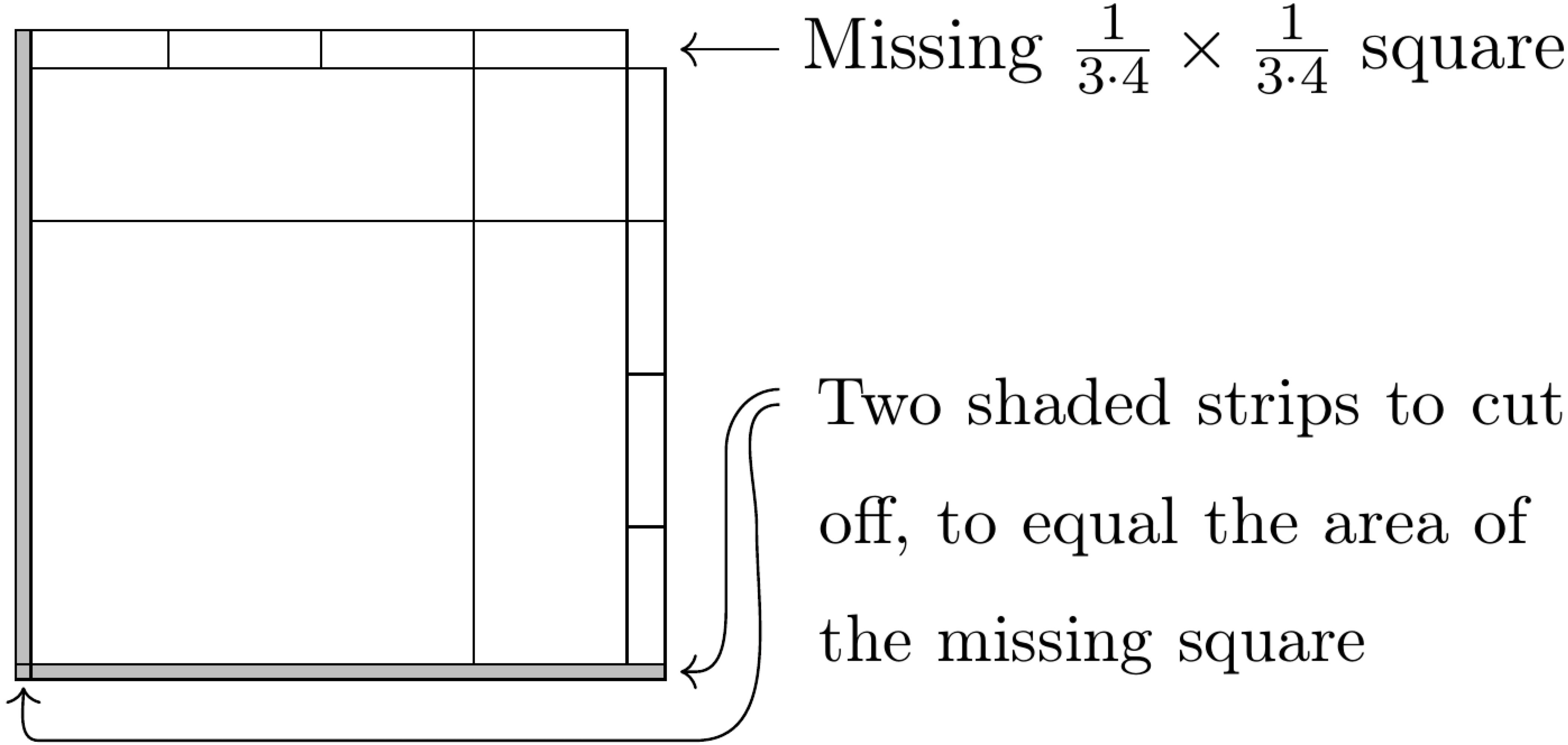

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$$W = \frac{1}{3 \cdot 4 \cdot 34}. \quad \checkmark$$

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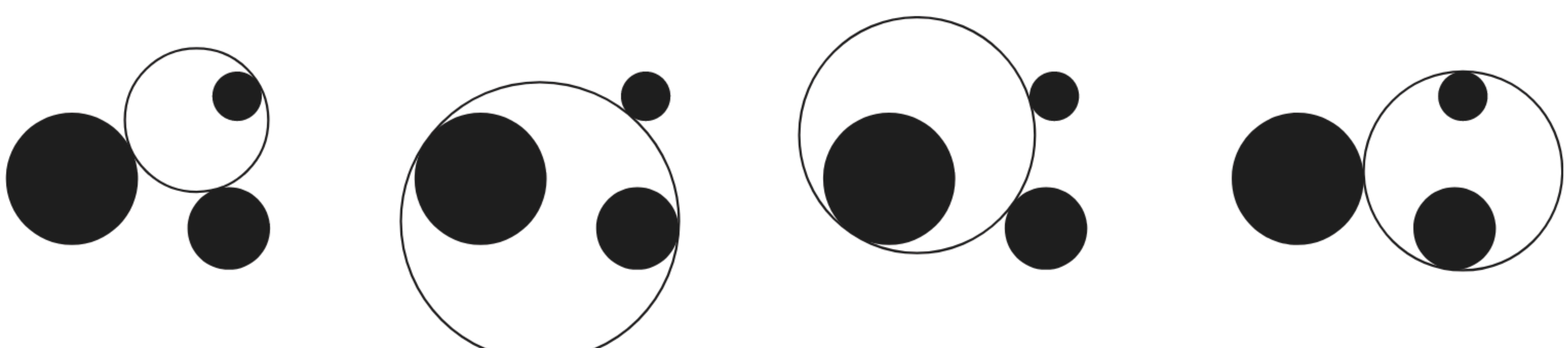

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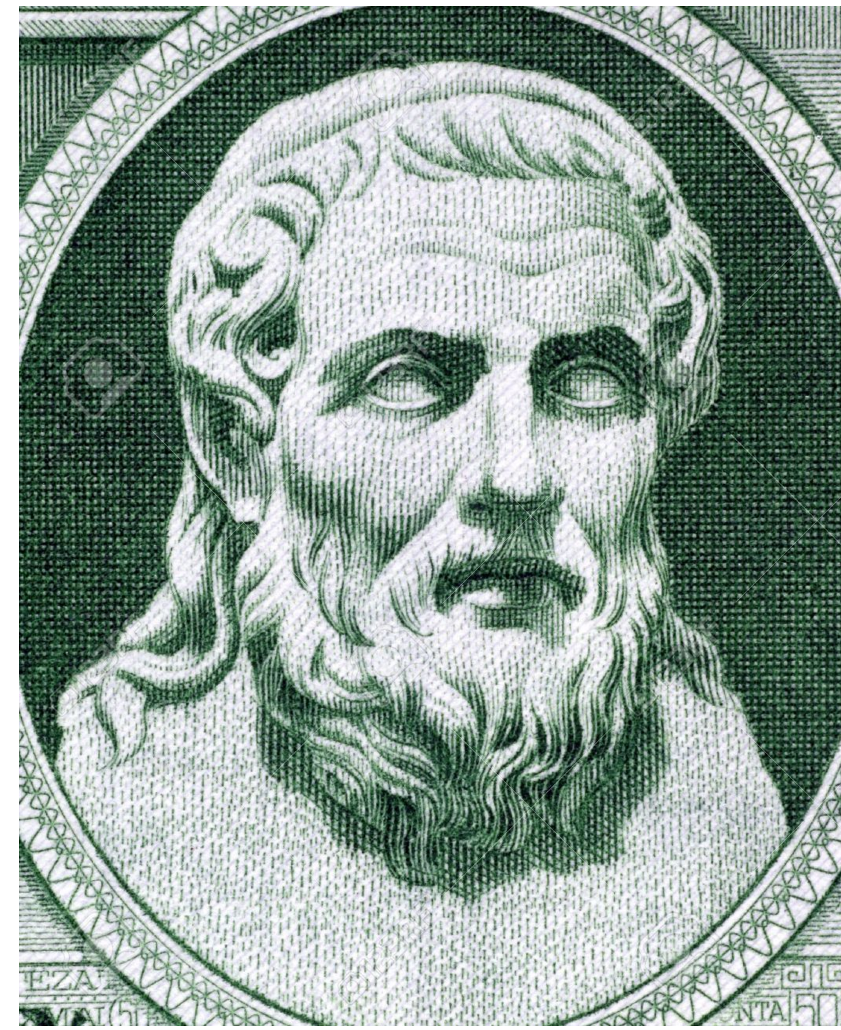

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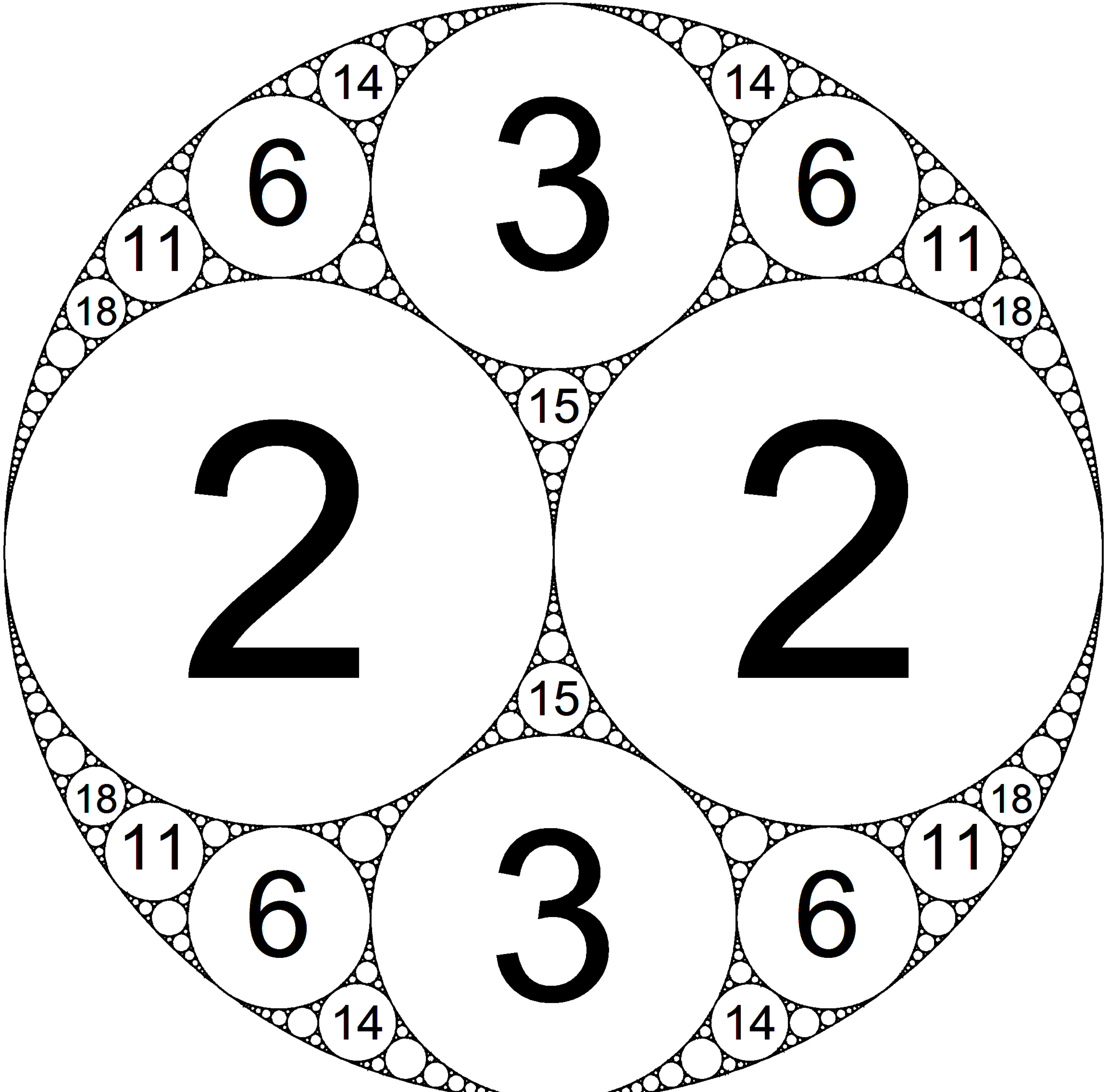

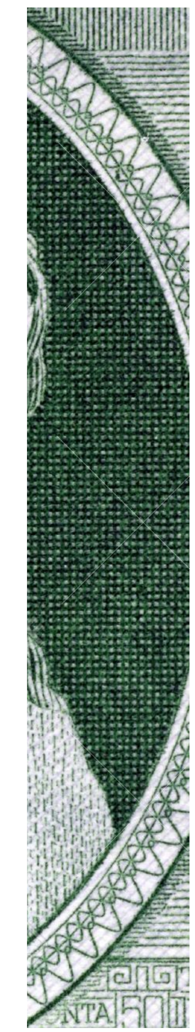

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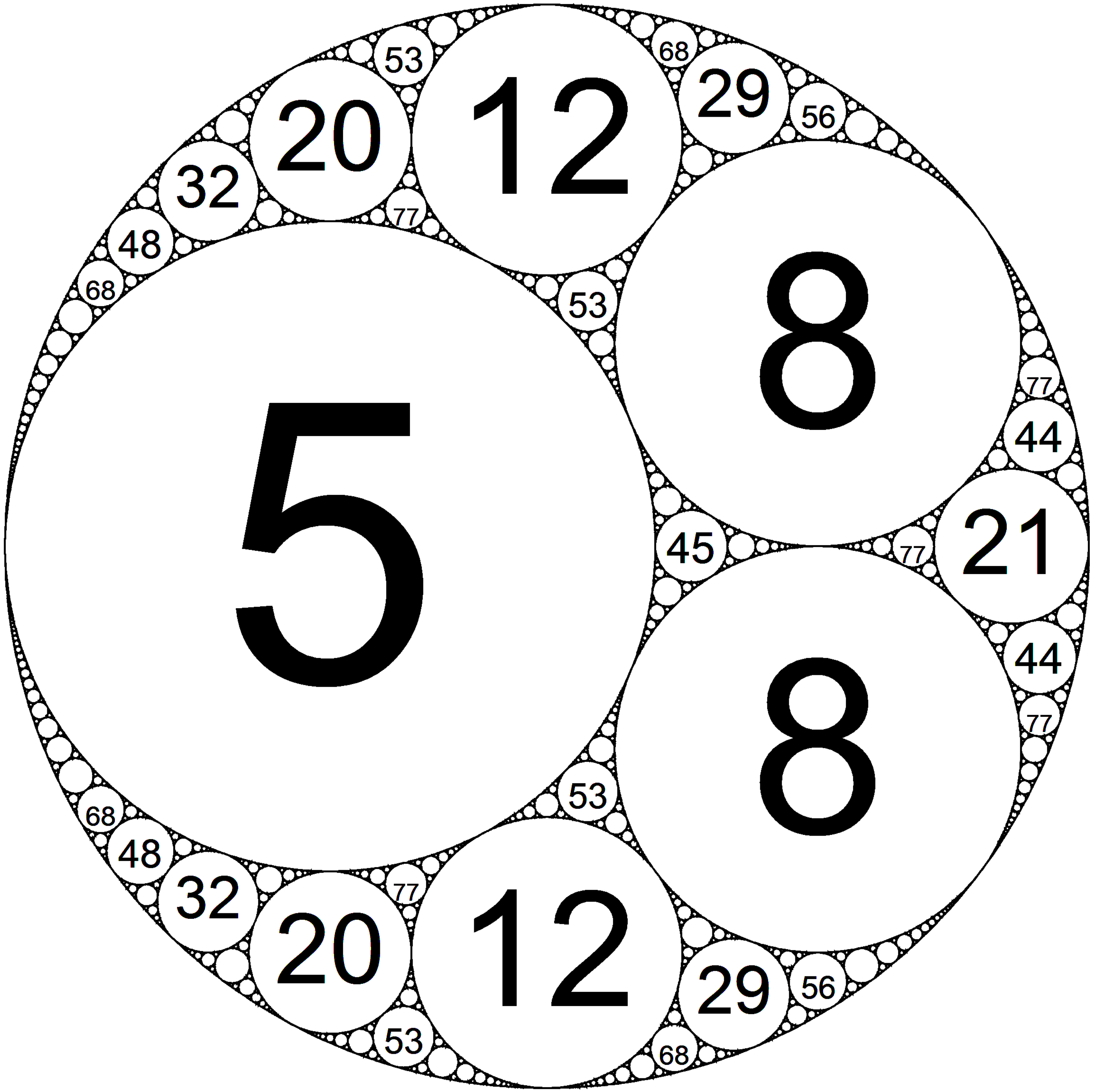
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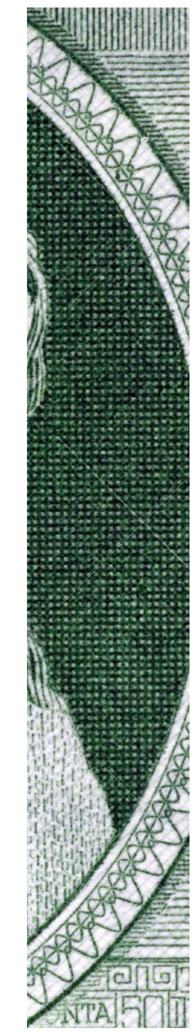

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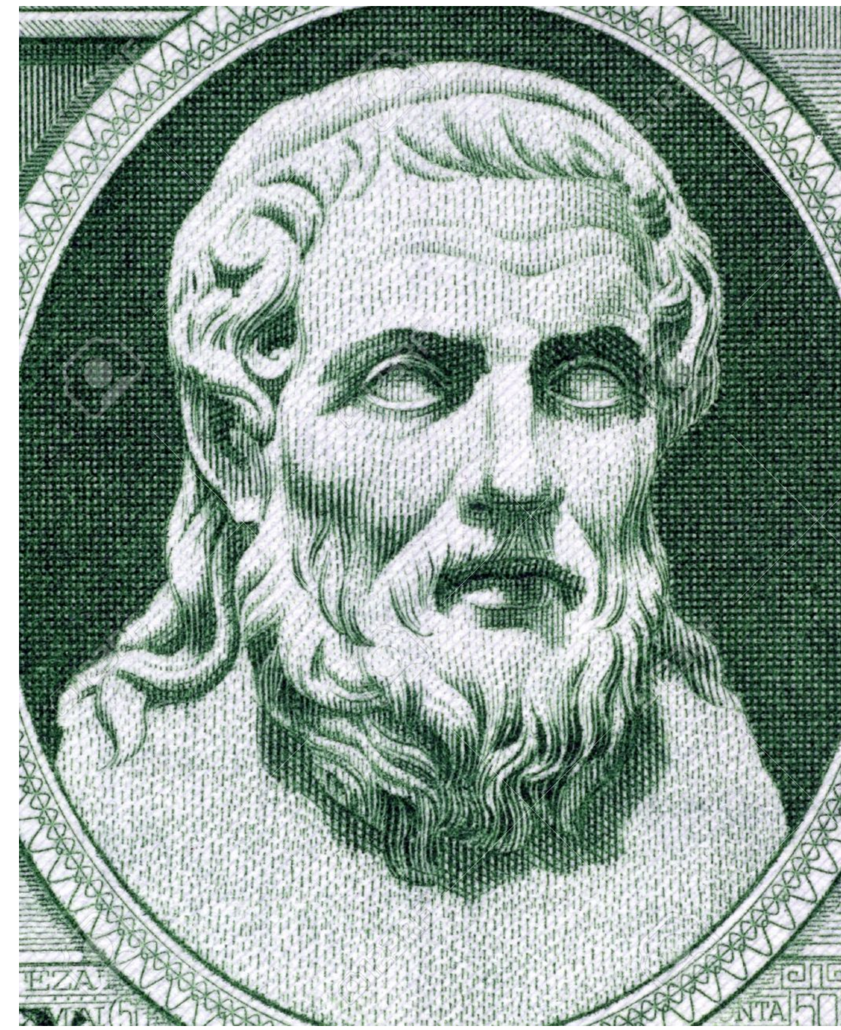

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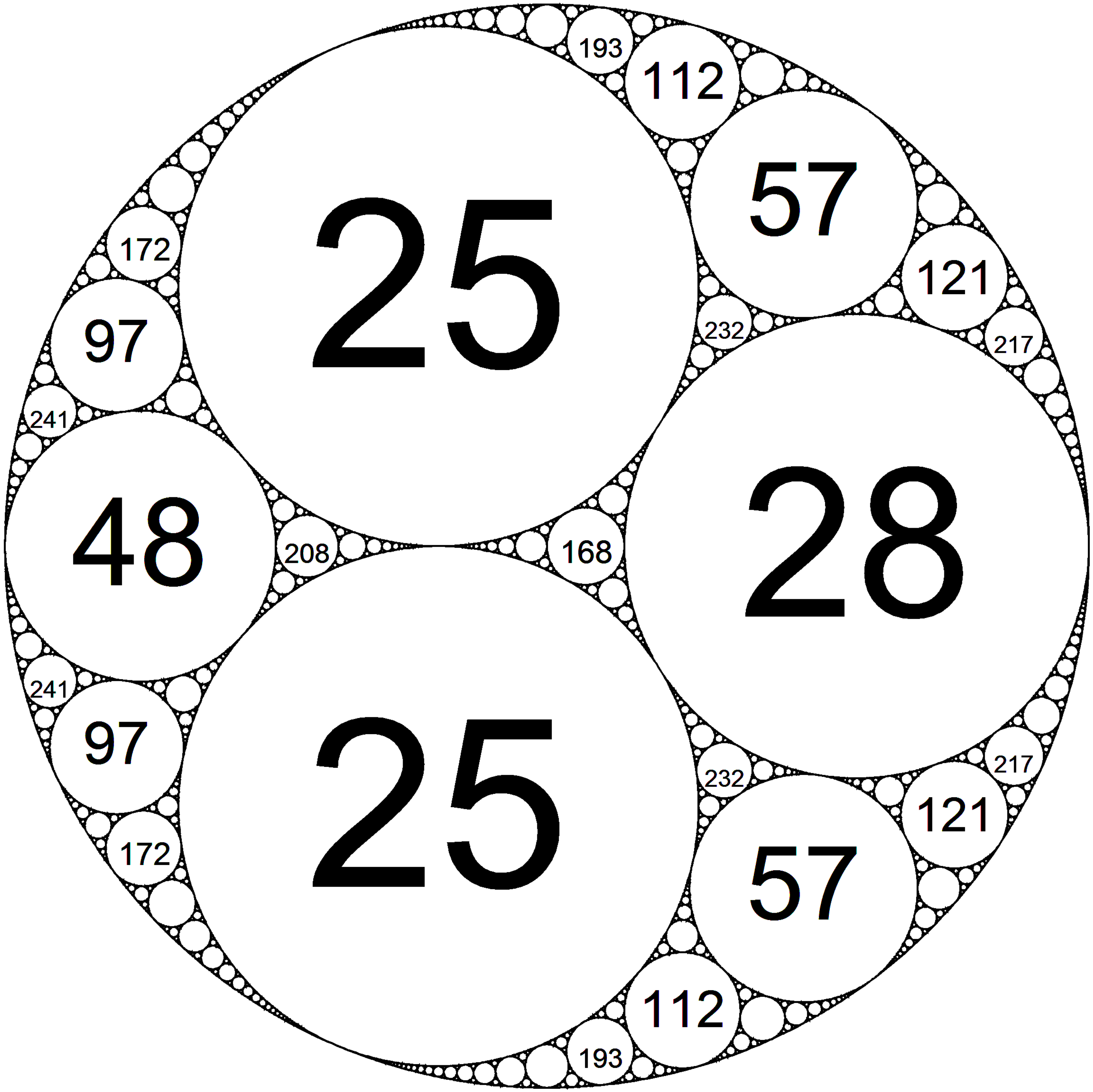
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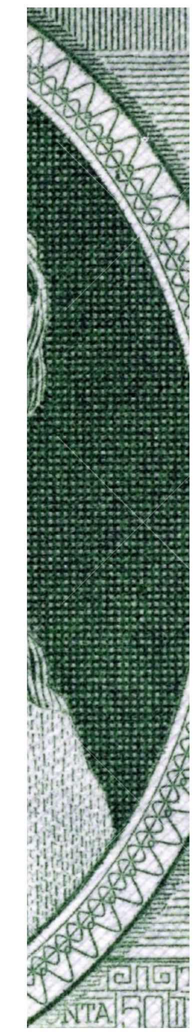
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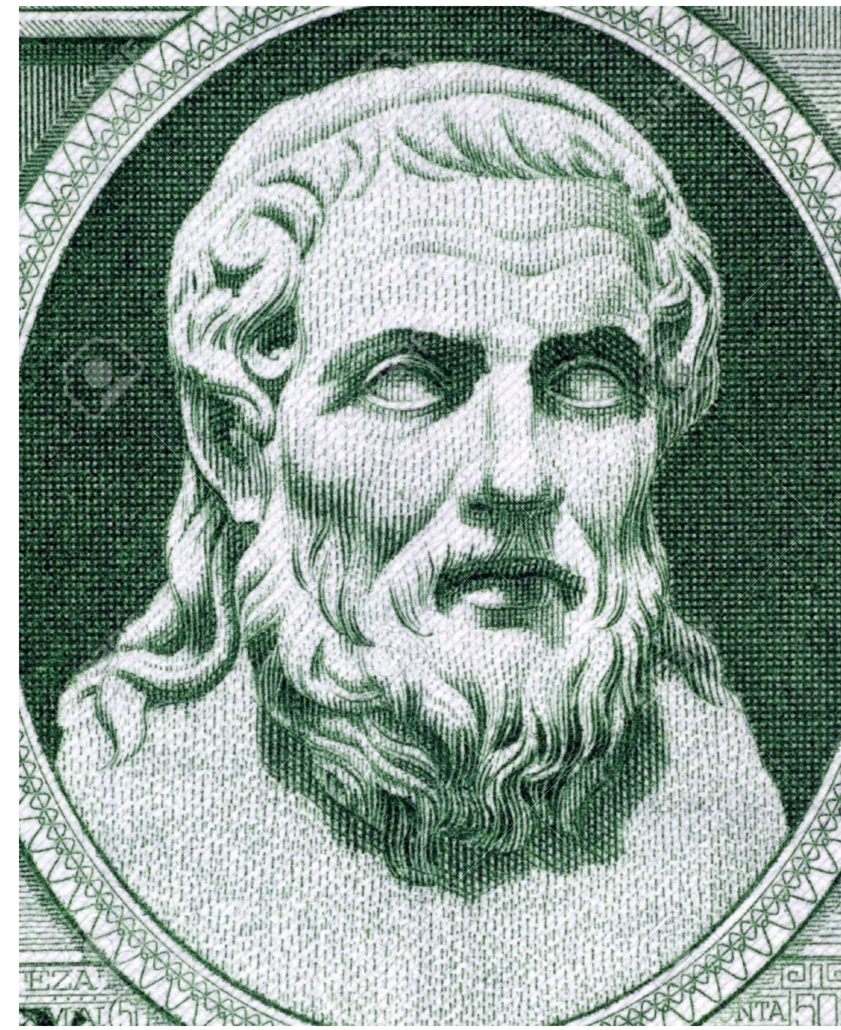

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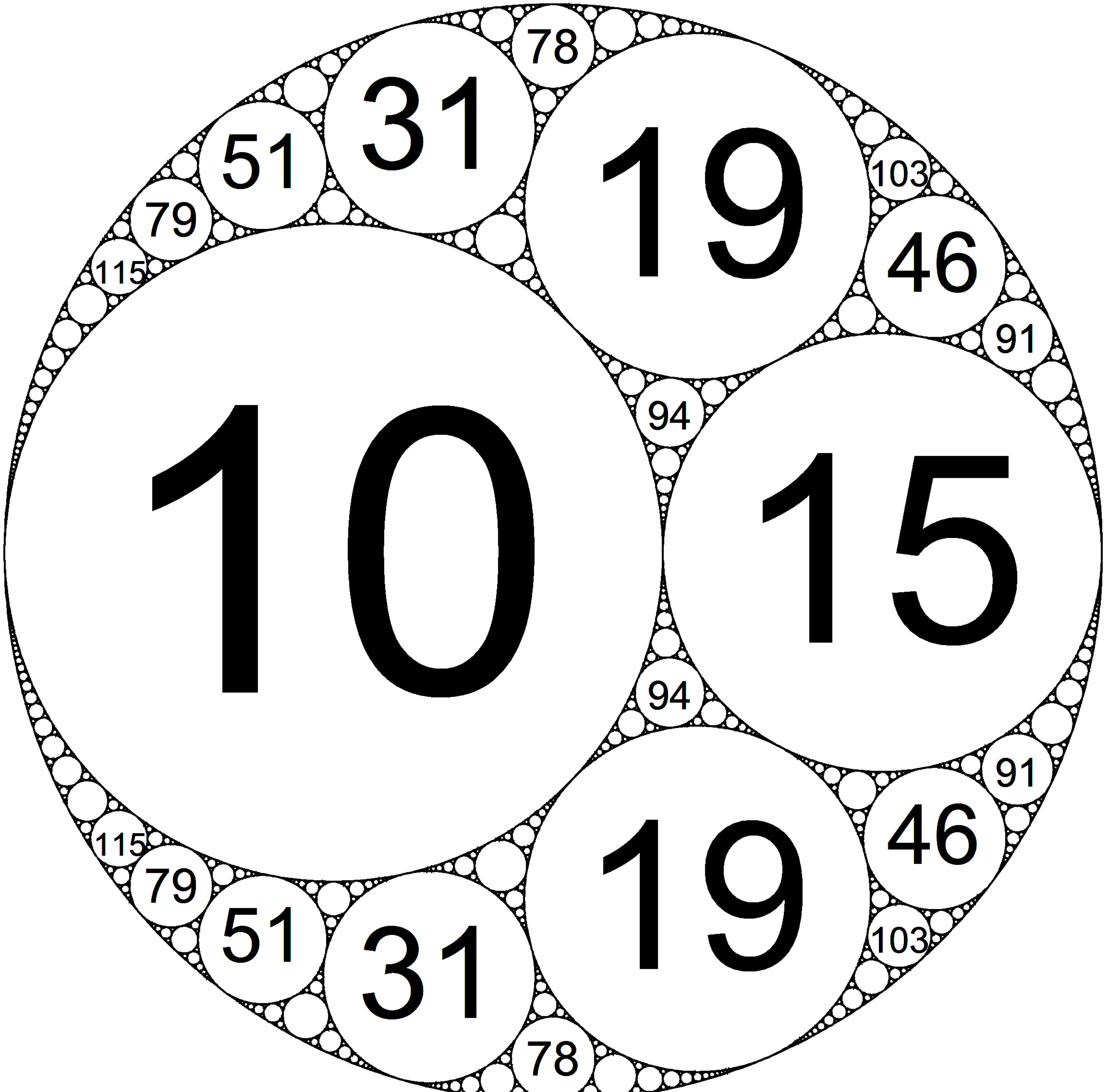

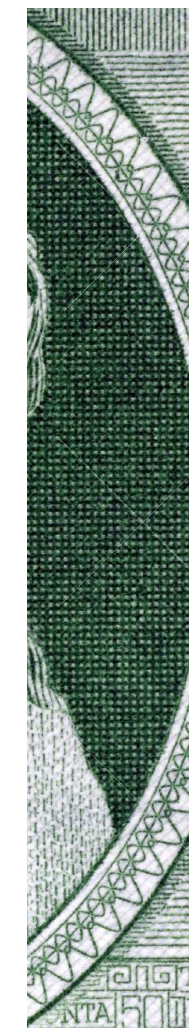

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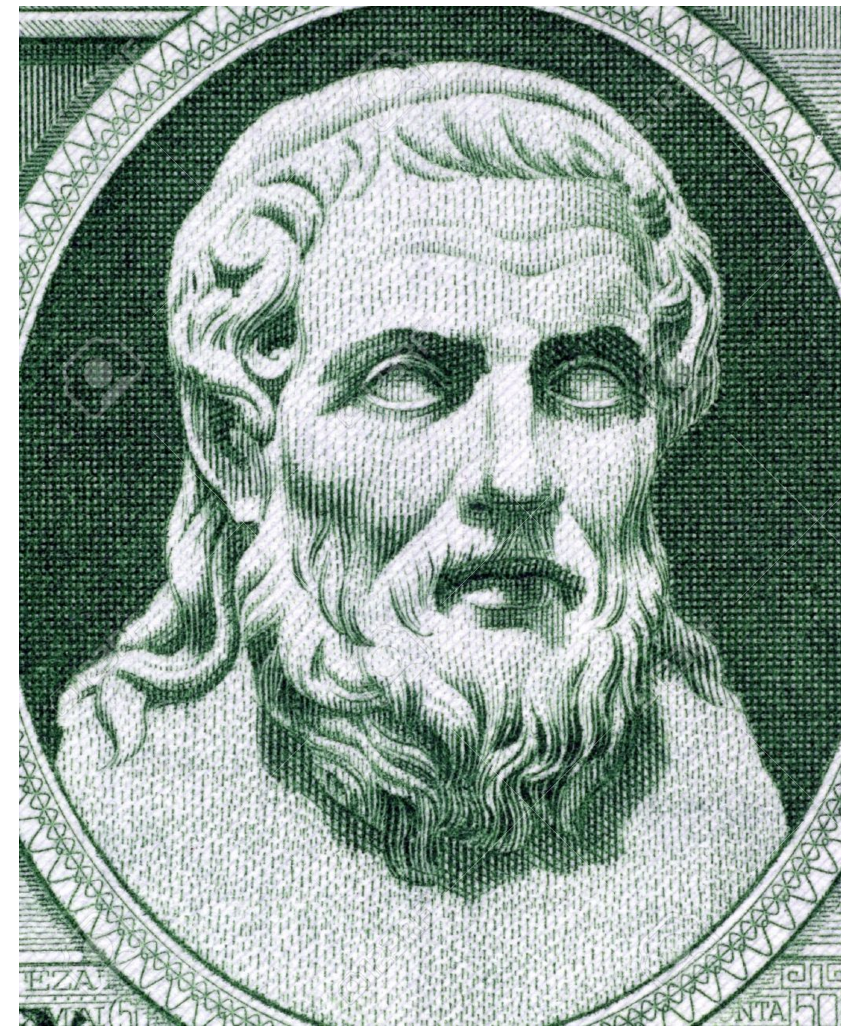

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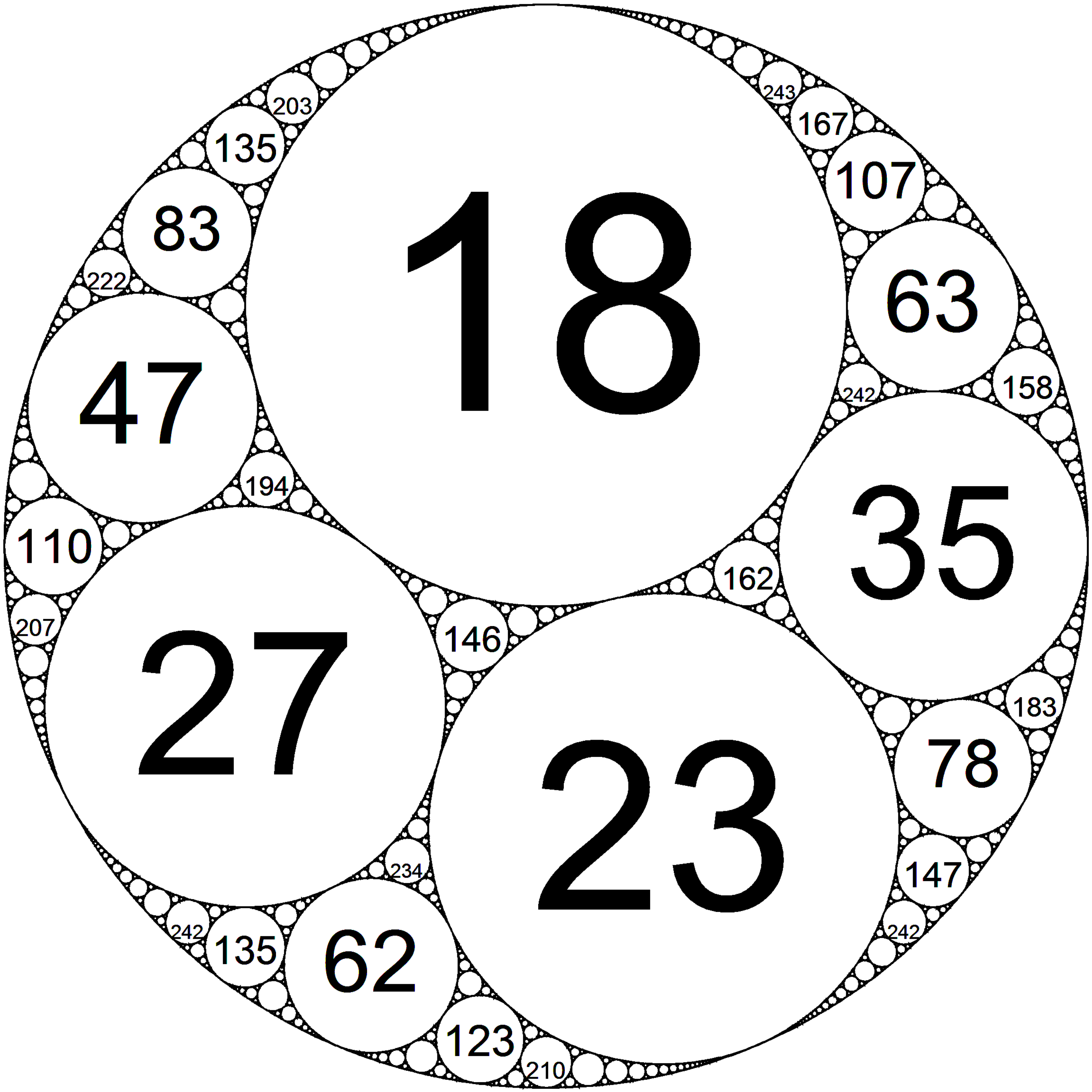

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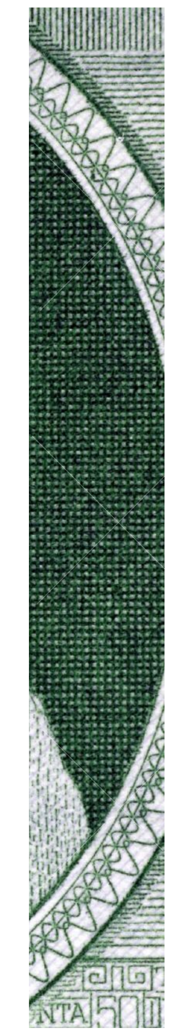

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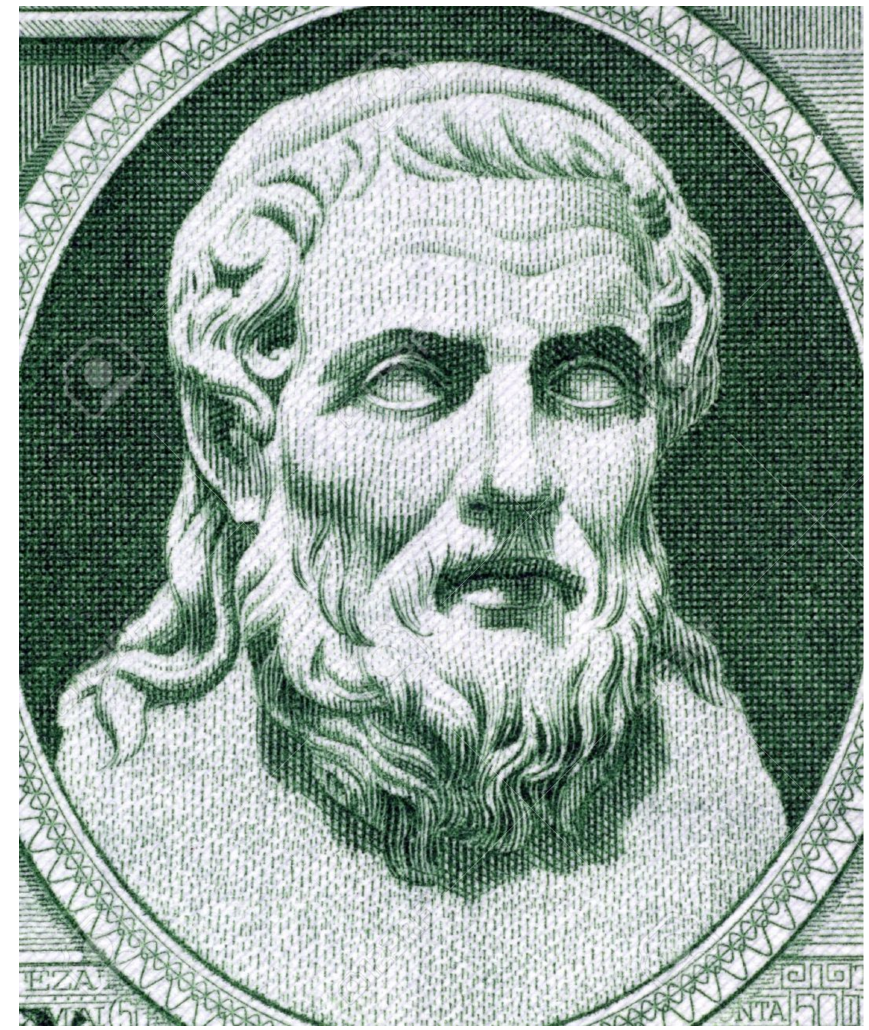

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- This was proved in 1998 by Thomas Hales.

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- For her work, Viazovska won the 2022 Fields medal. She was the second woman in history to win it.

School of Athens

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People’s History

People's History of Geometries