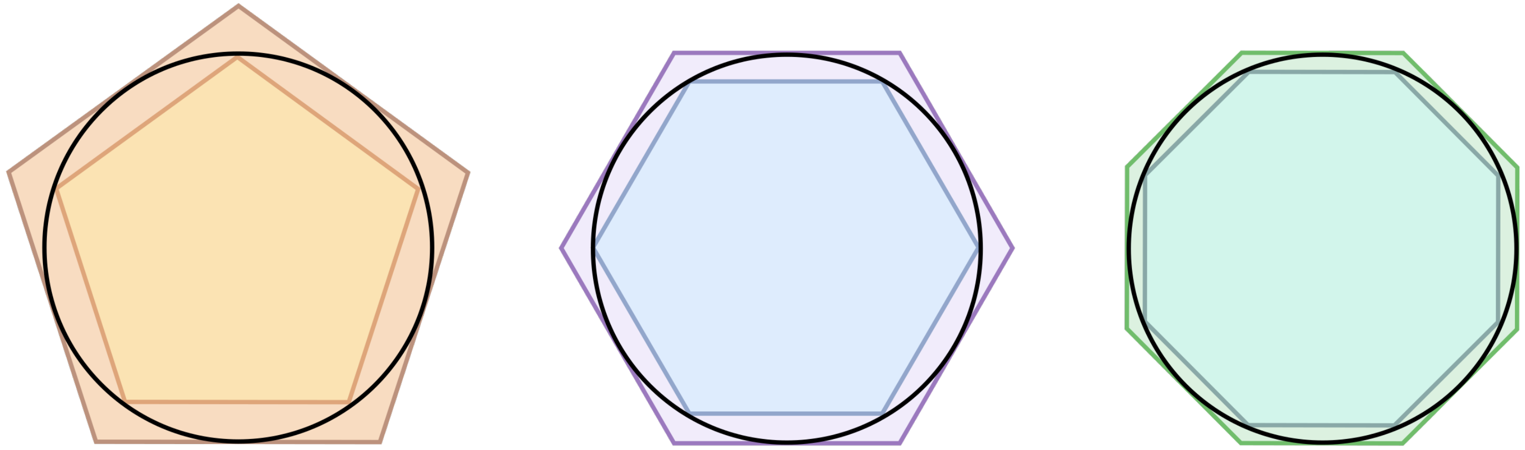
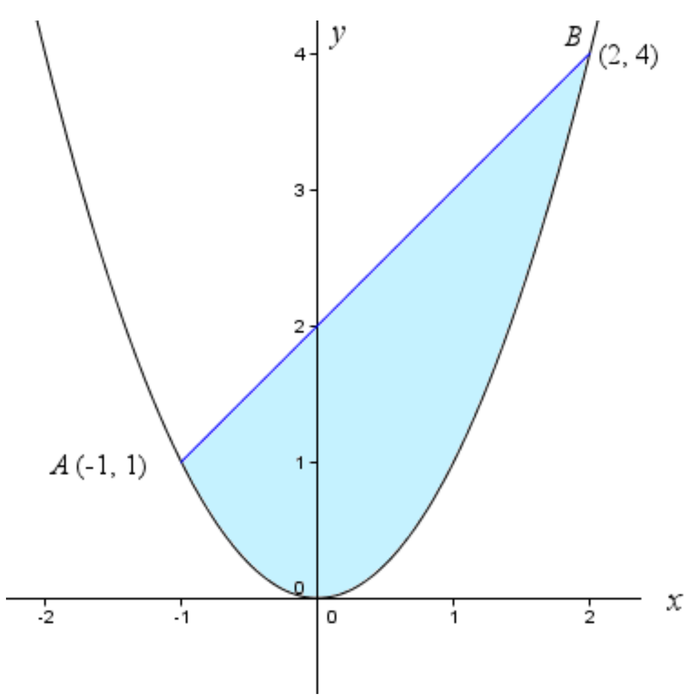
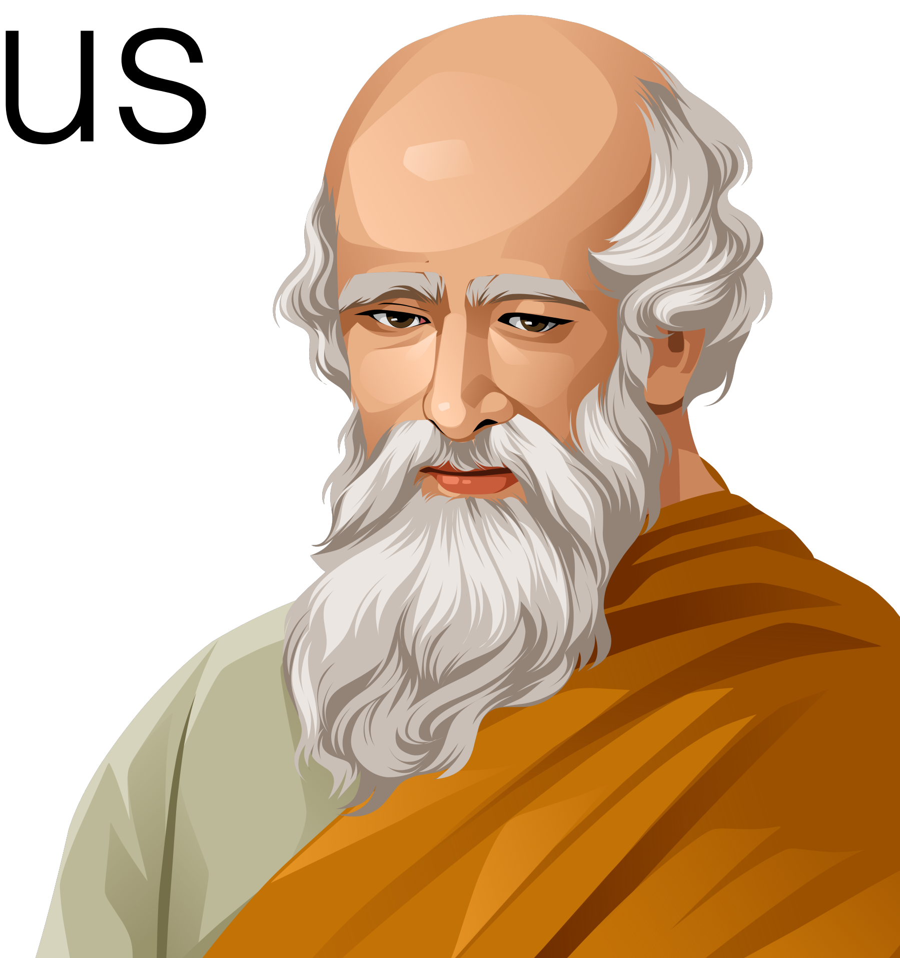


Chapter 7: Early Calculus





Zeno’s Paradoxes

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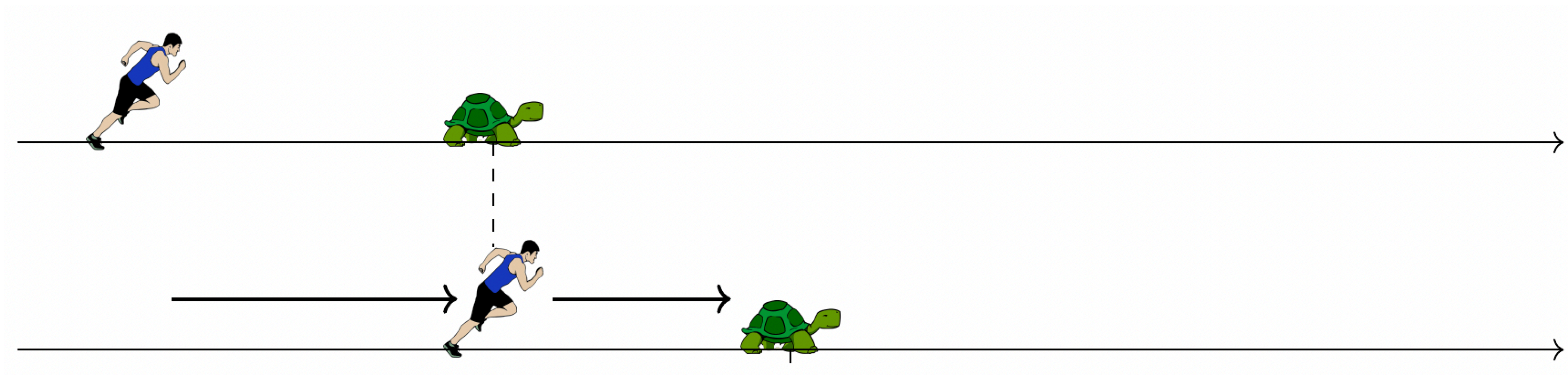
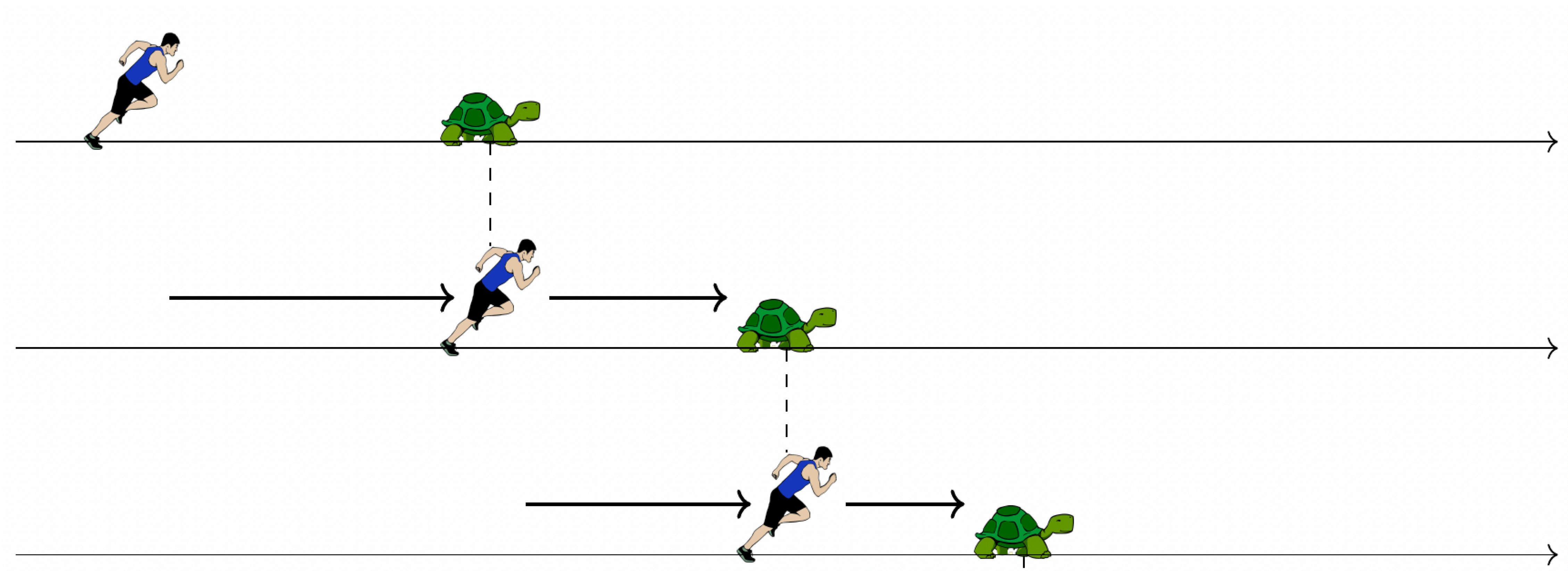


Zeno's Paradoxes



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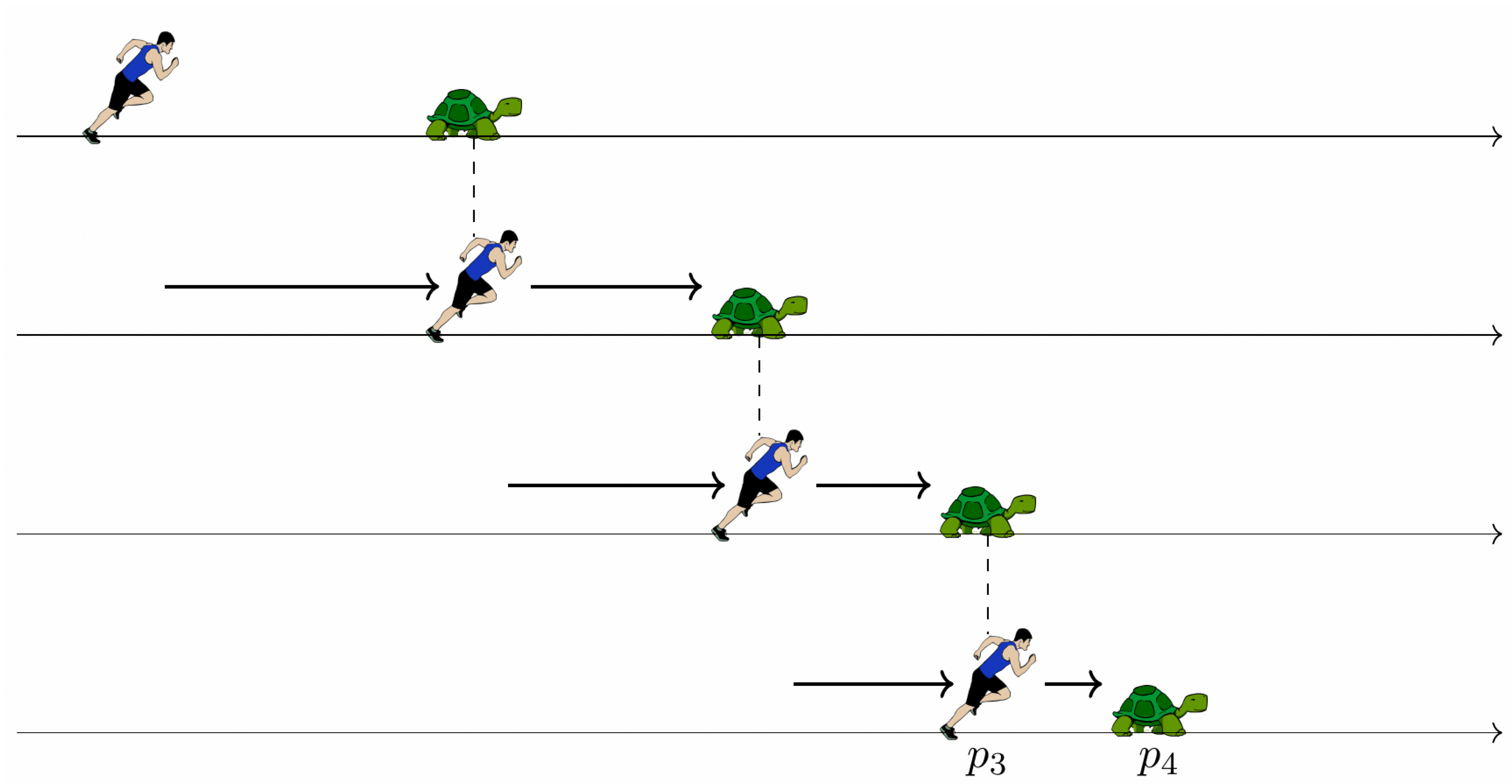
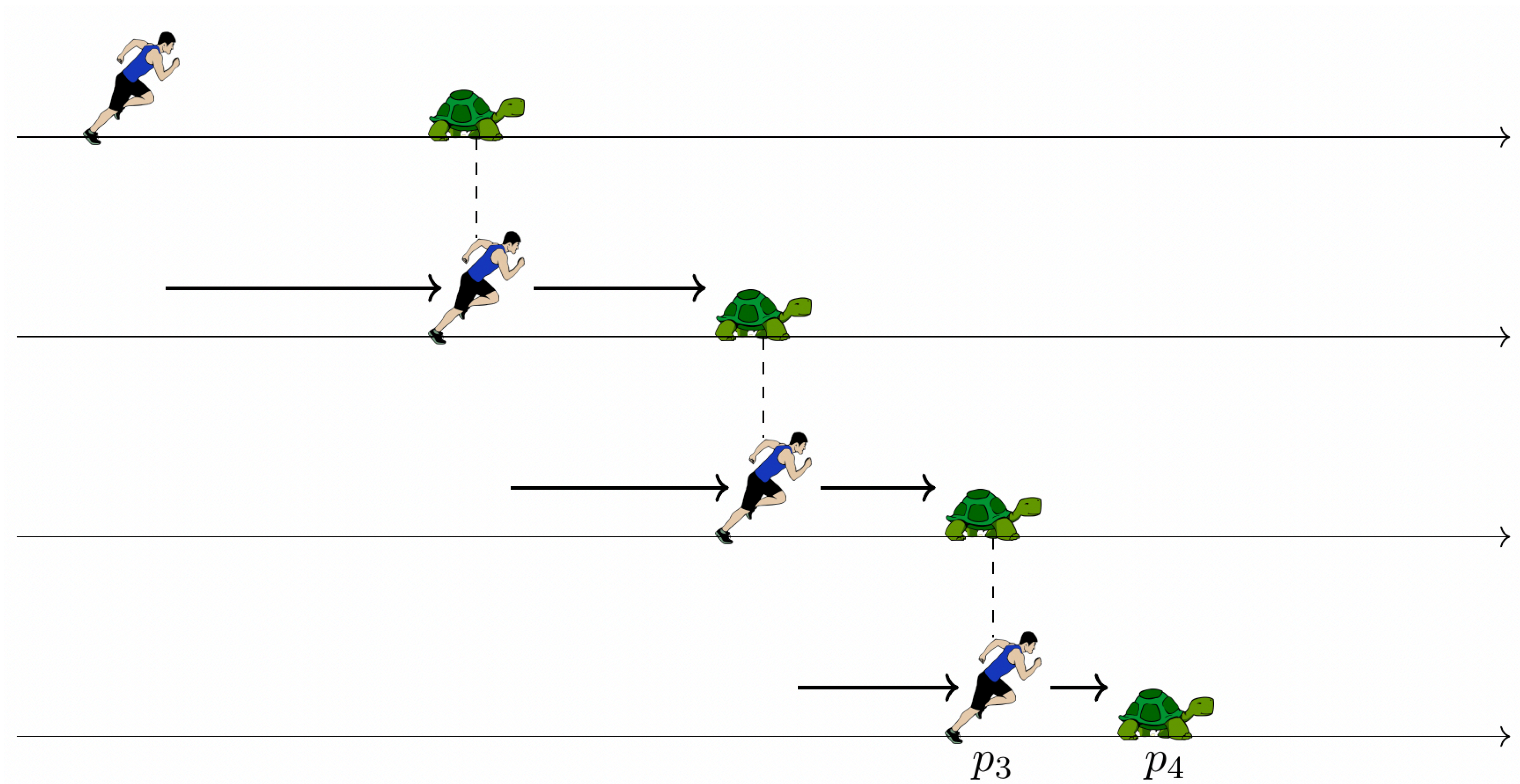
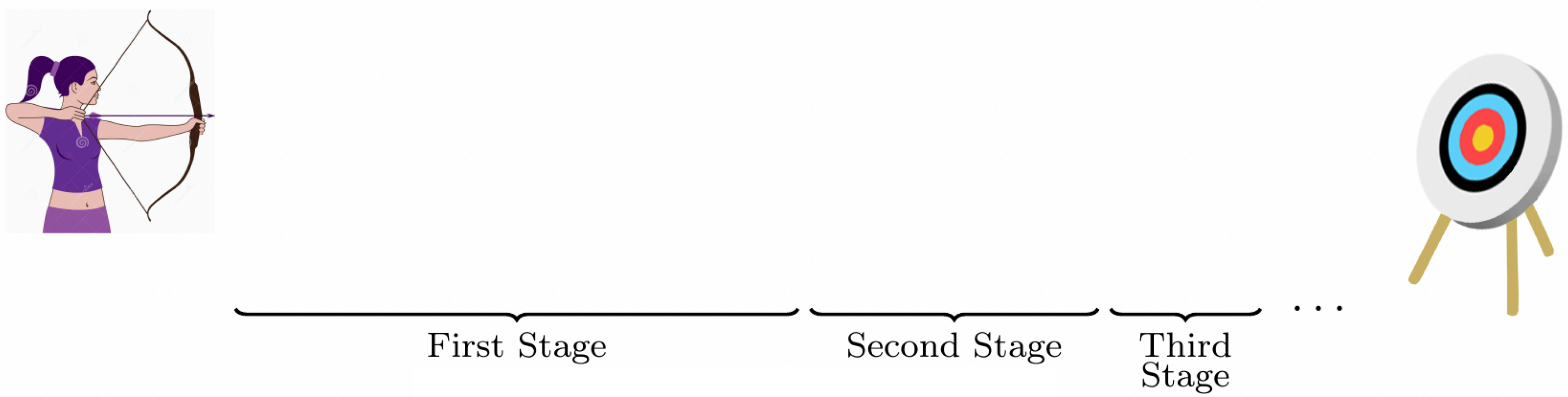


Zeno's Paradoxes

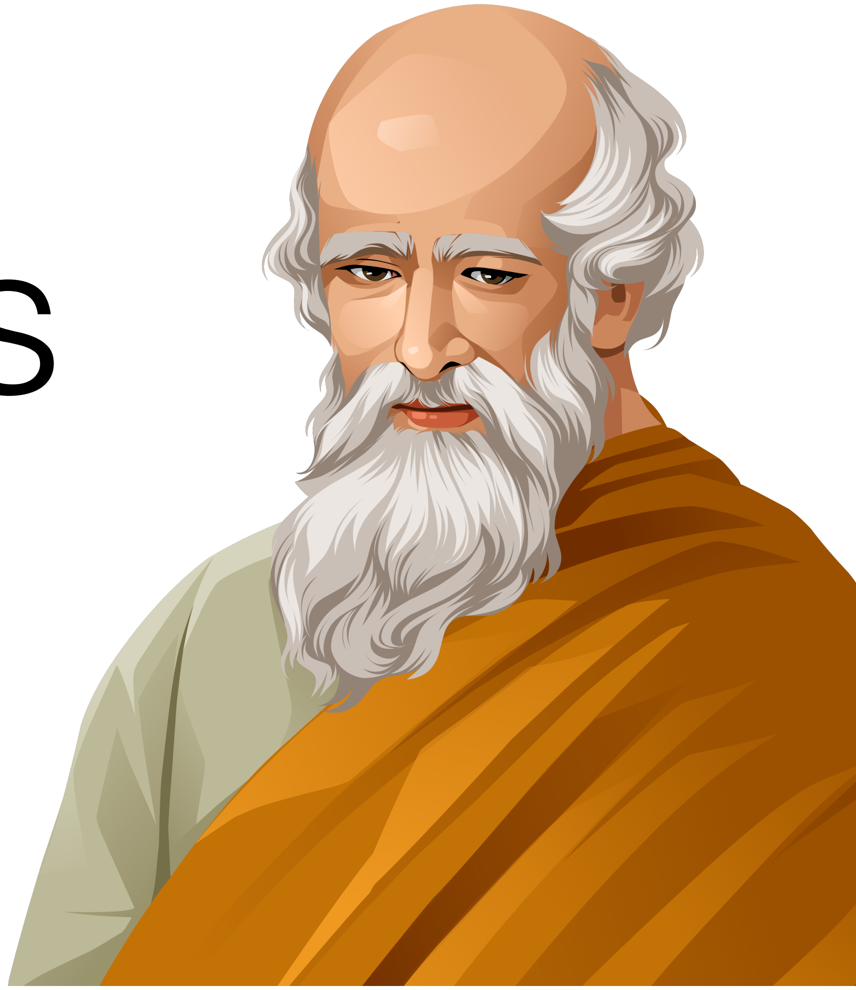


Zeno: If Achilles is always behind, he can never surpass the tortoise, so he is guaranteed to lose.

Zeno's Paradoxes



Archimedes



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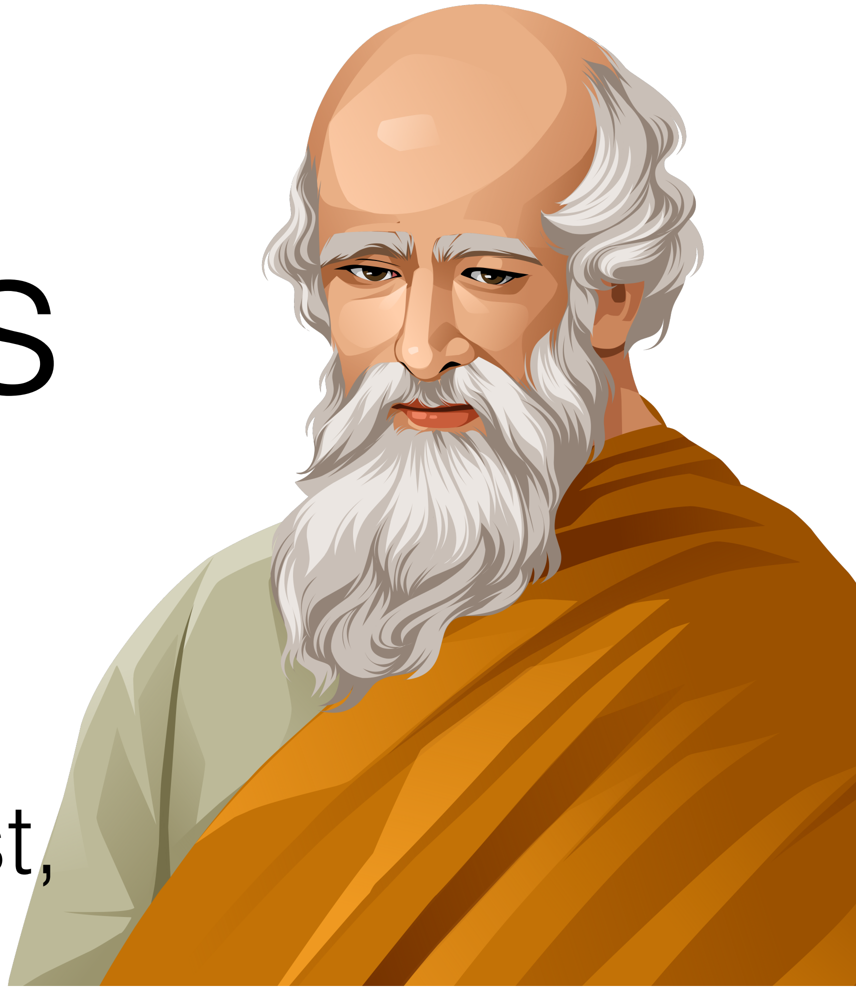

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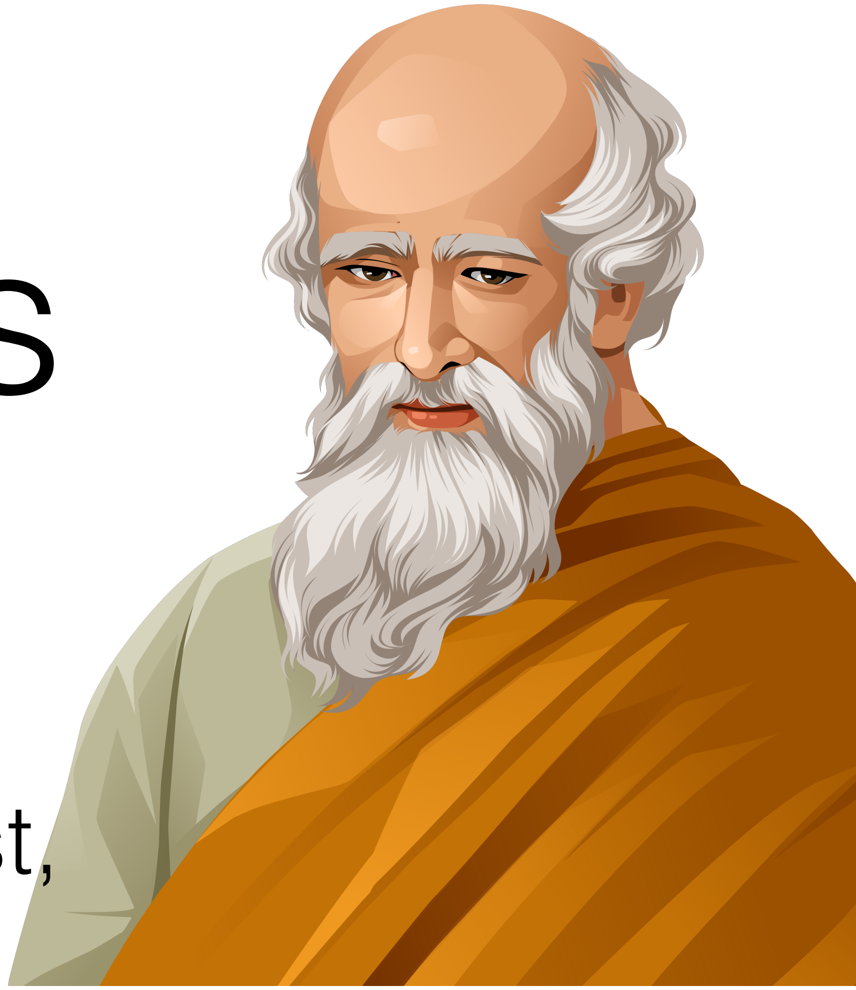
Archimedes



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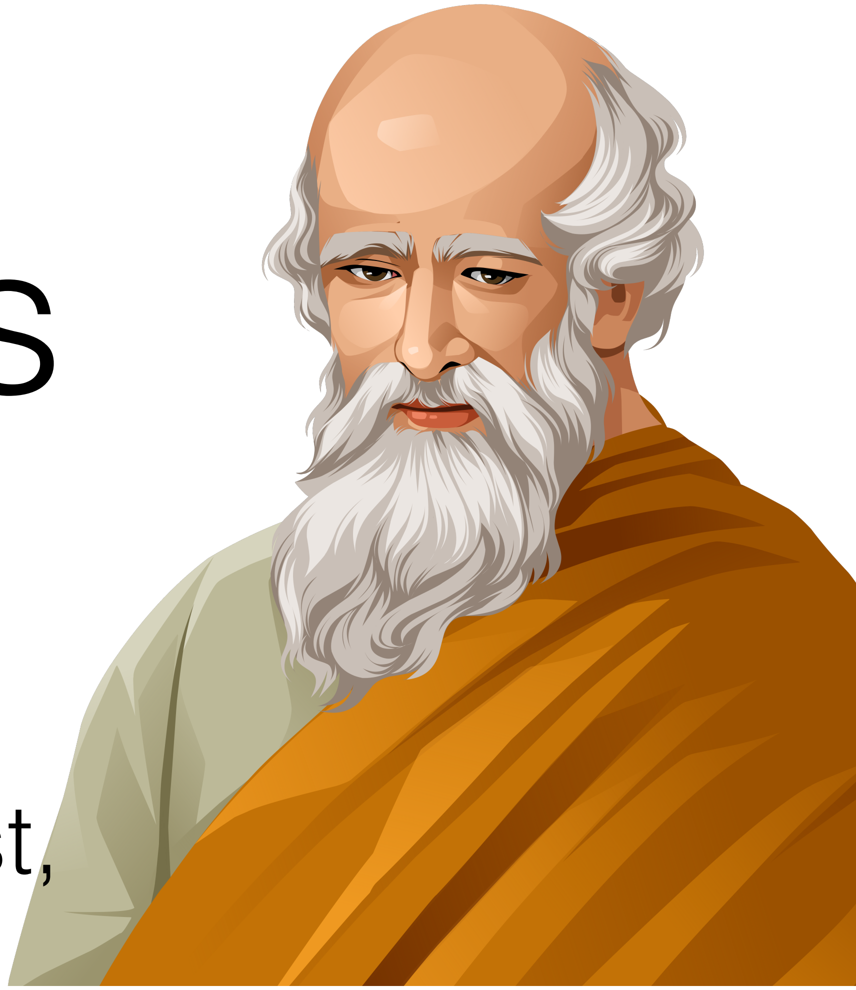

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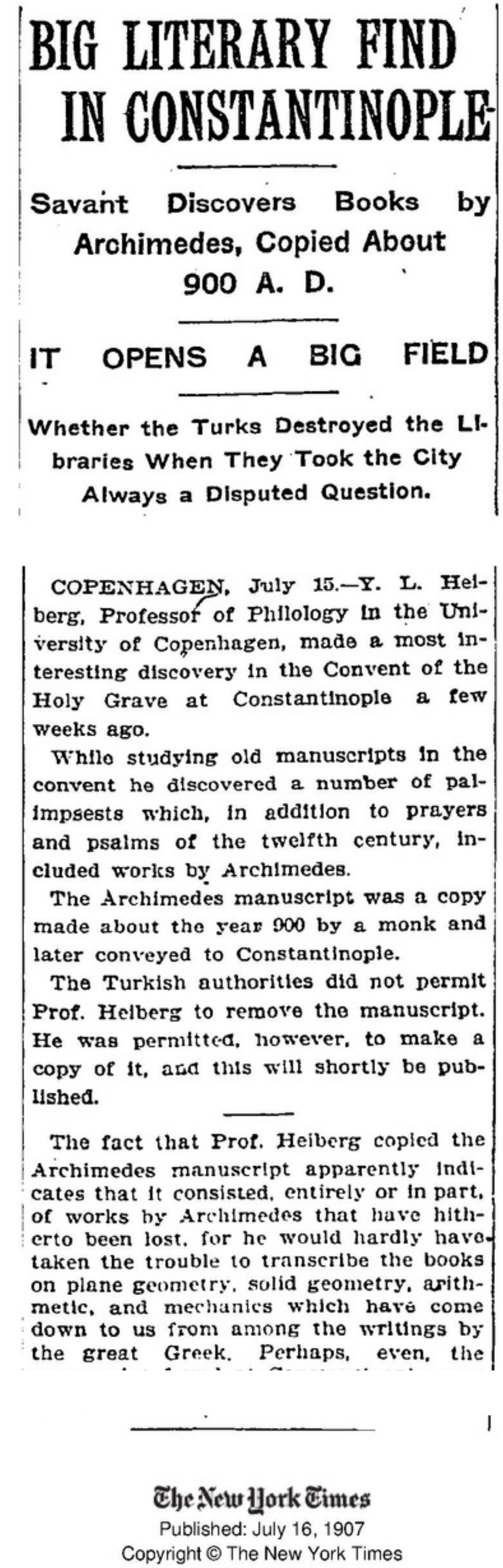


BIG LITERARY FIND IN CONSTANTINOPLE

Savant Discovers Books by Archimedes, Copied About 900 A. D.

IT OPENS A BIG FIELD

Whether the Turks Destroyed the Libraries When They Took the City Always a Disputed Question.

COPENHAGEN, July 15.—Y. L. Heiberg, Professor of Philology in the University of Copenhagen, made a most interesting discovery in the Convent of the Holy Grave at Constantinople a few weeks ago.

While studying old manuscripts in the convent he discovered a number of palimpsests which, in addition to prayers and psalms of the twelfth century, included works by Archimedes.

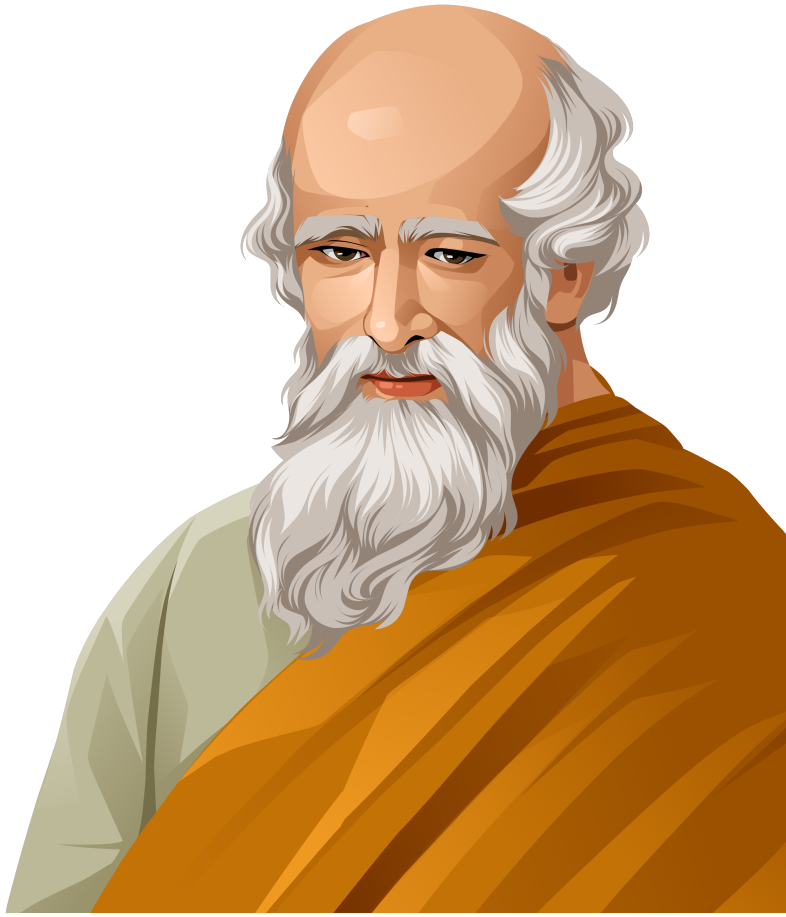
The Archimedes manuscript was a copy made about the year 900 by a monk and later conveyed to Constantinople.

The Turkish authorities did not permit Prof. Heiberg to remove the manuscript. He was permitted, however, to make a copy of it, and this will shortly be published.

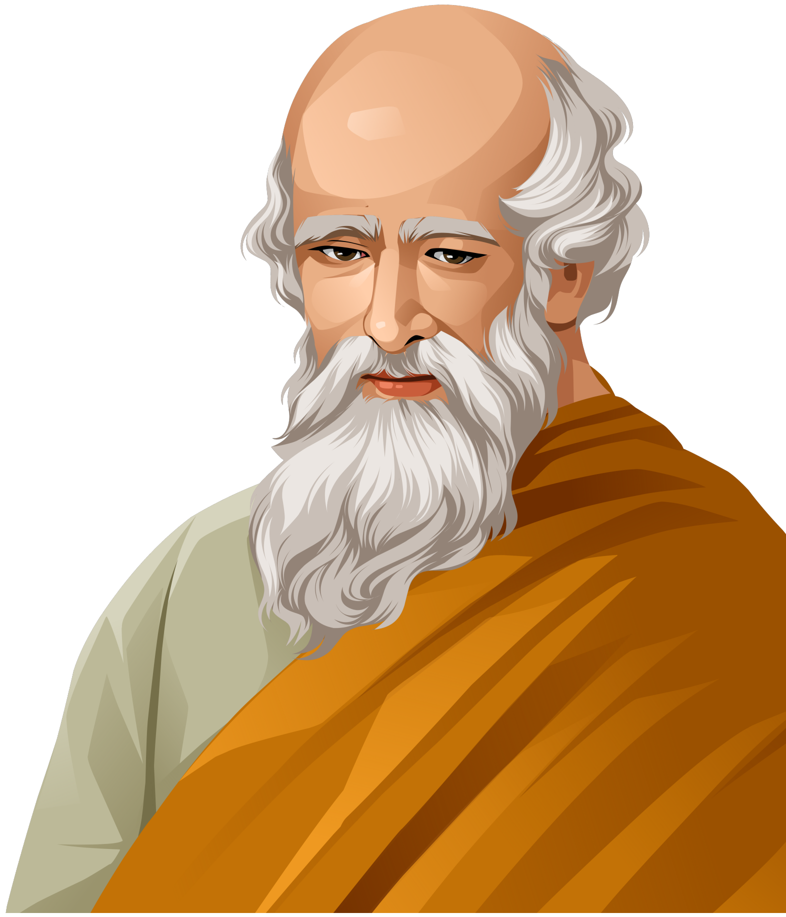
The fact that Prof. Heiberg copied the Archimedes manuscript apparently indicates that it consisted, entirely or in part, of works by Archimedes that have hitherto been lost, for he would hardly have taken the trouble to transcribe the books on plane geometry, solid geometry, arithmetic, and mechanics which have come down to us from among the writings by the great Greek. Perhaps, even, the

The New York Times
Published: July 16, 1907

Archimedes



Archimedes



- Defended his hometown of Syracuse against the Romans using his "war machines."

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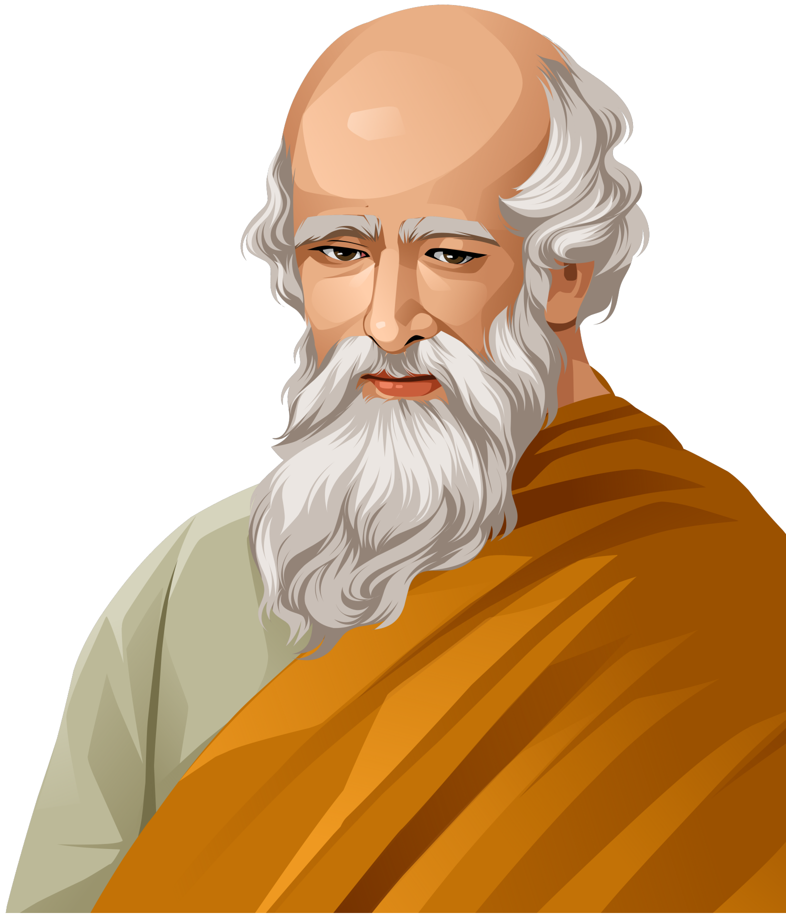

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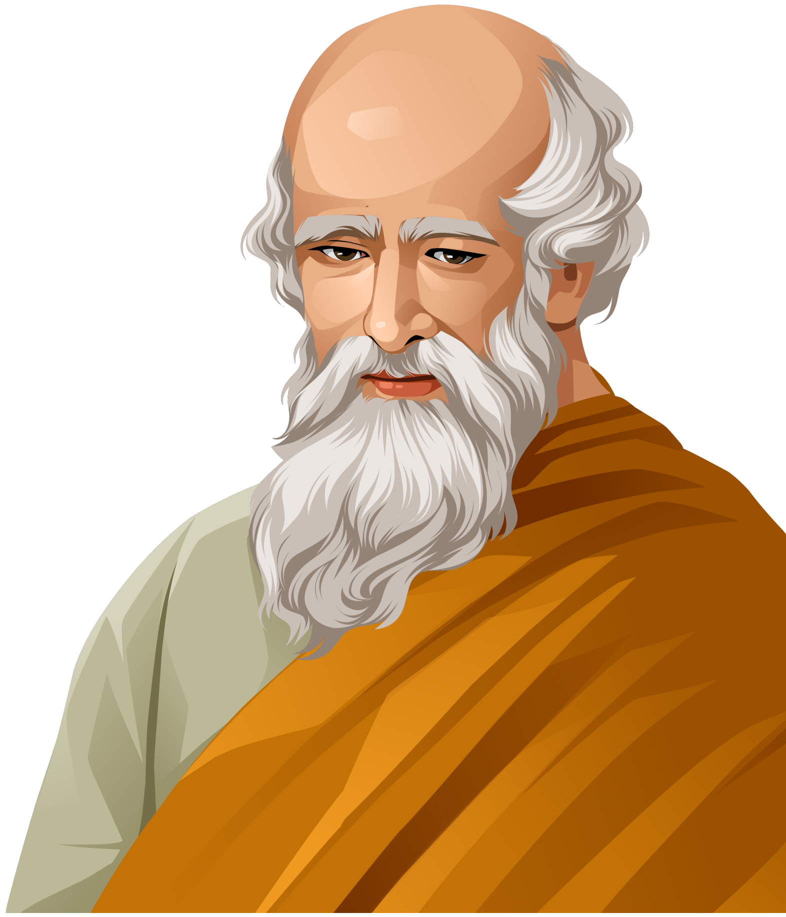

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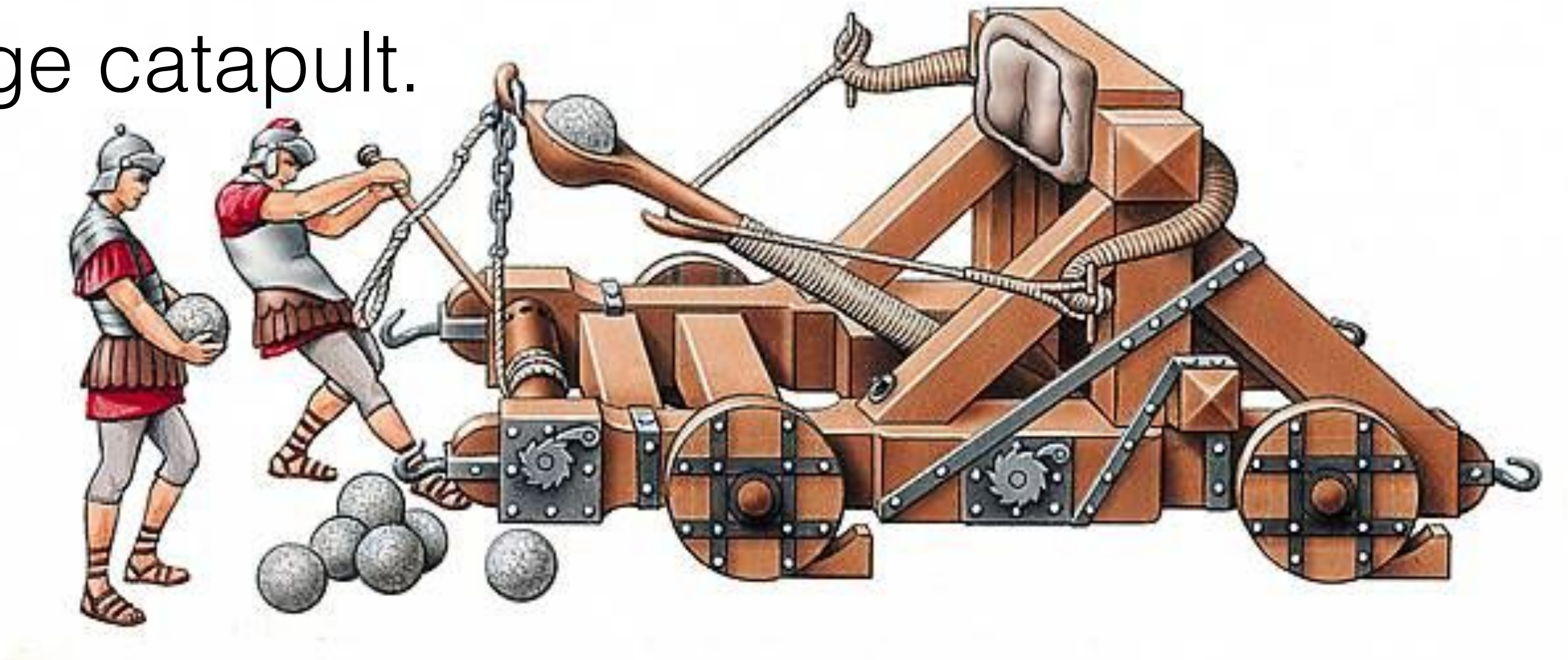

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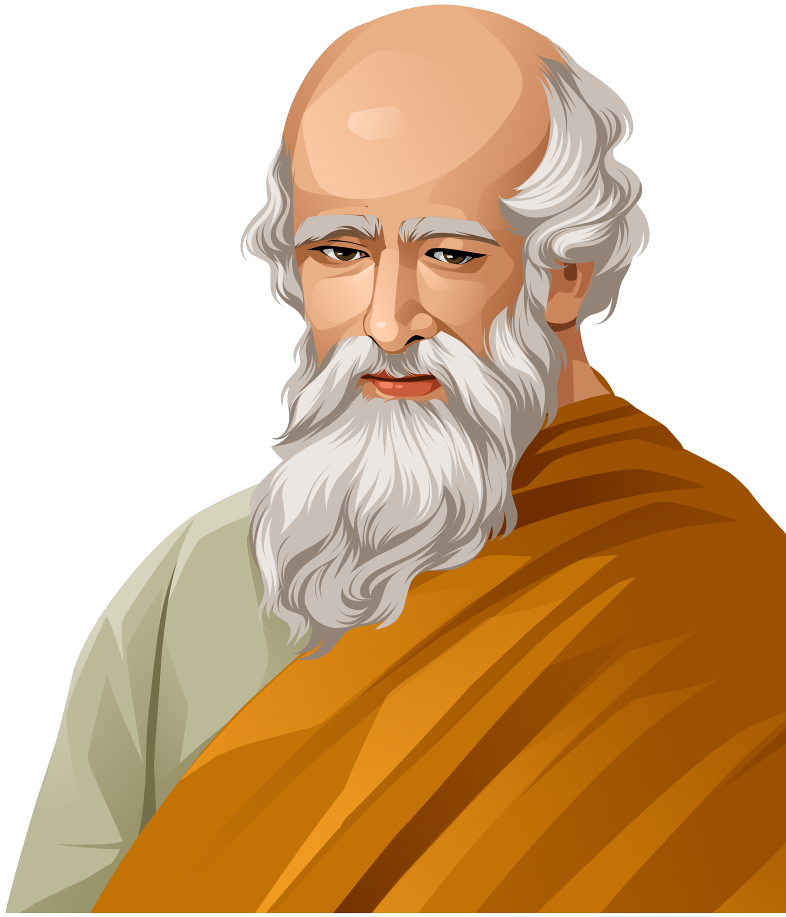

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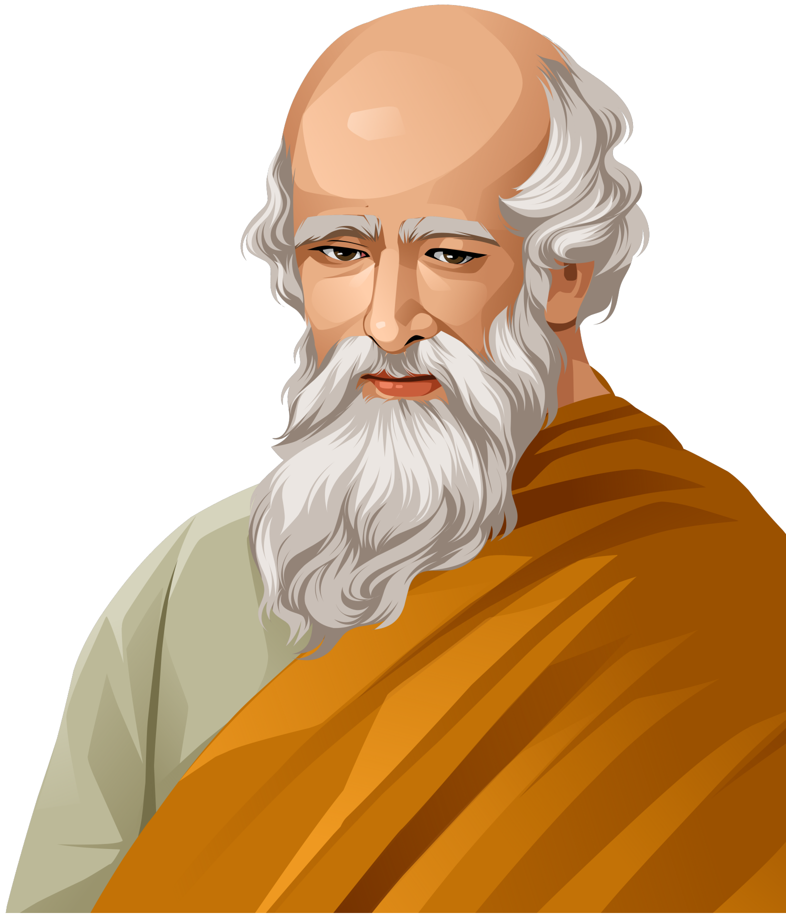

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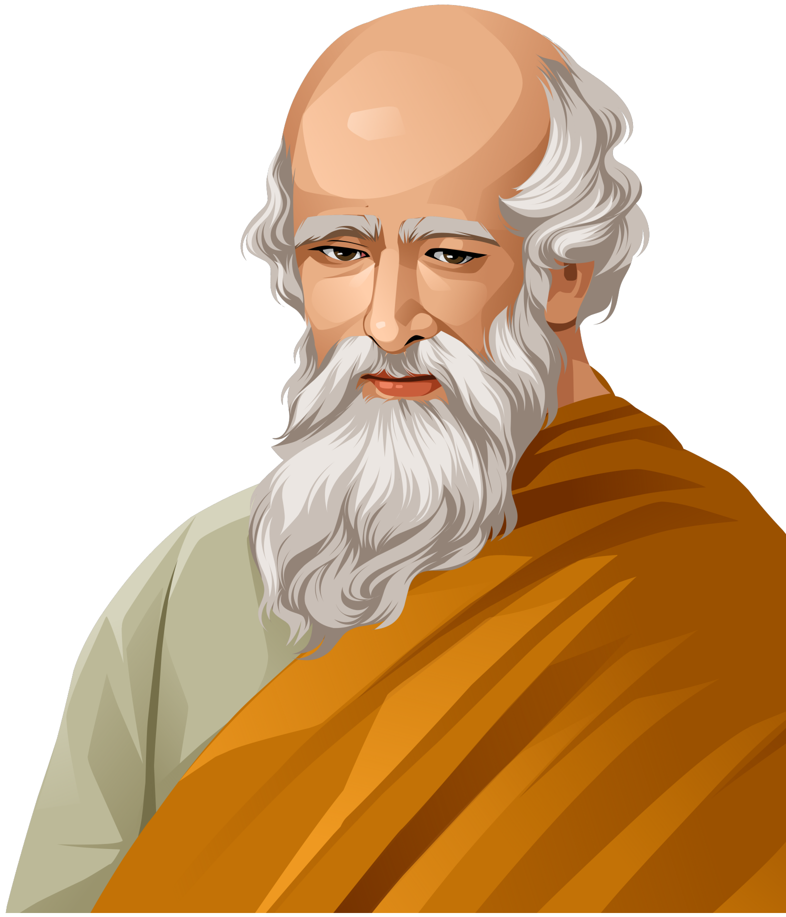

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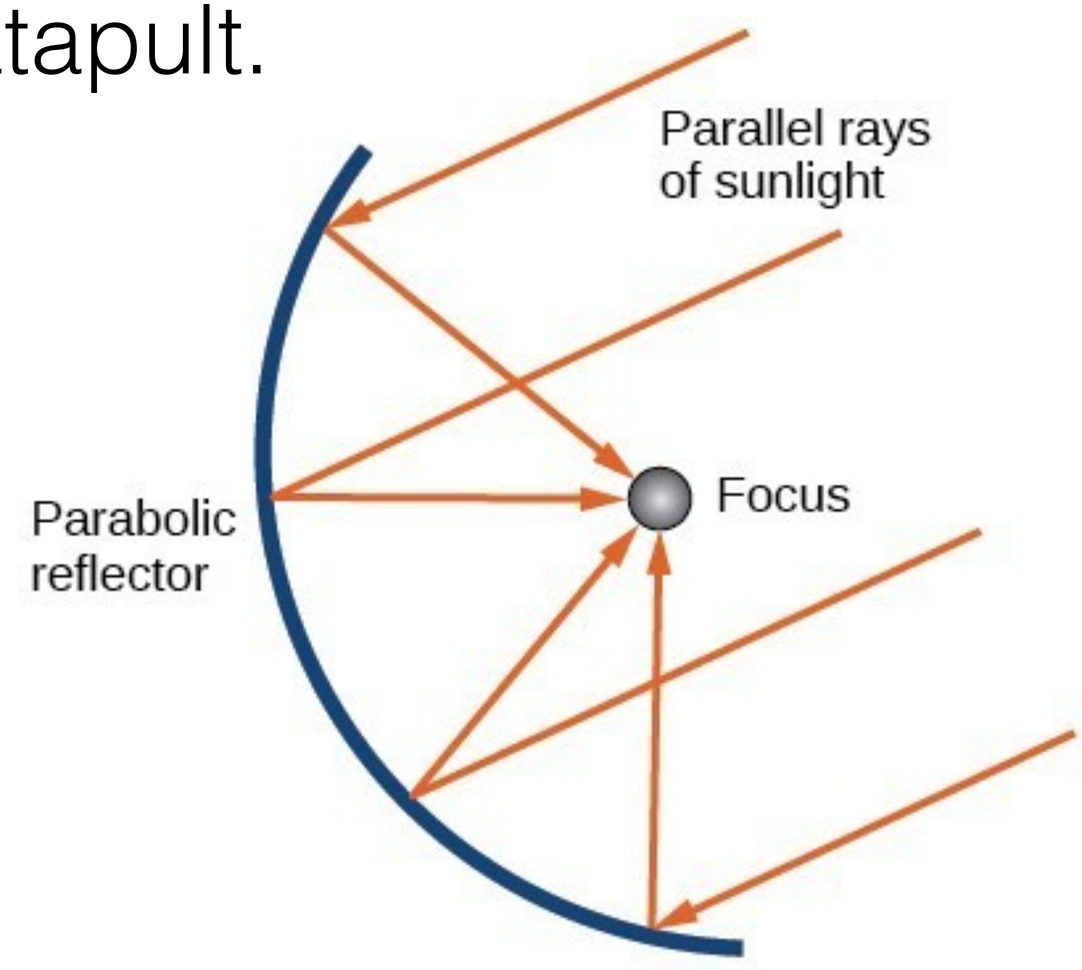

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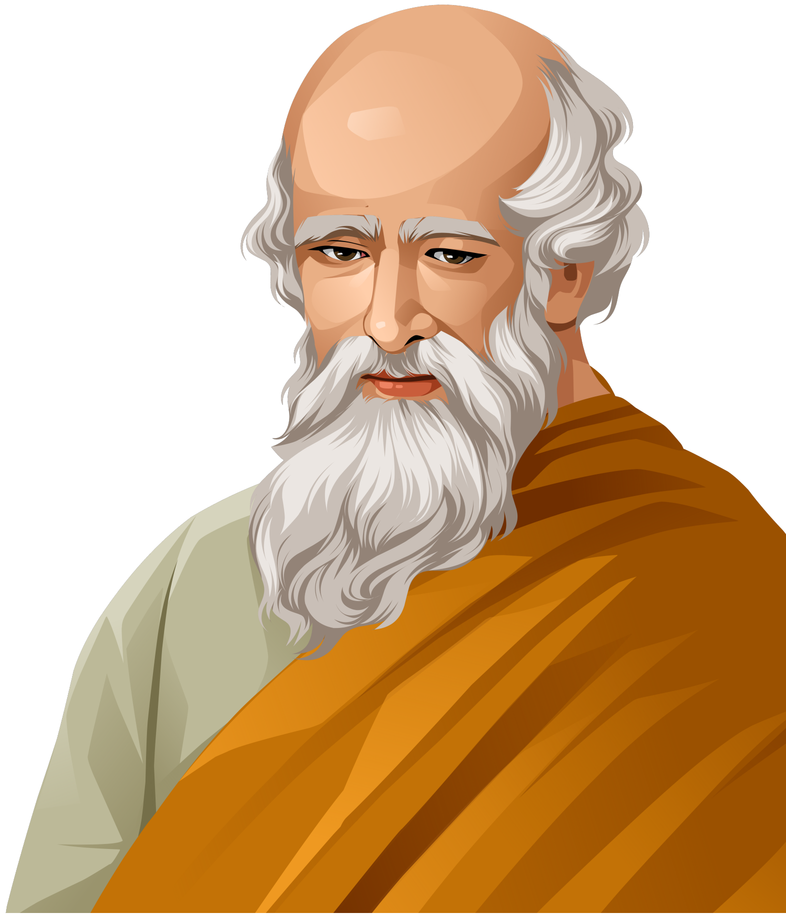

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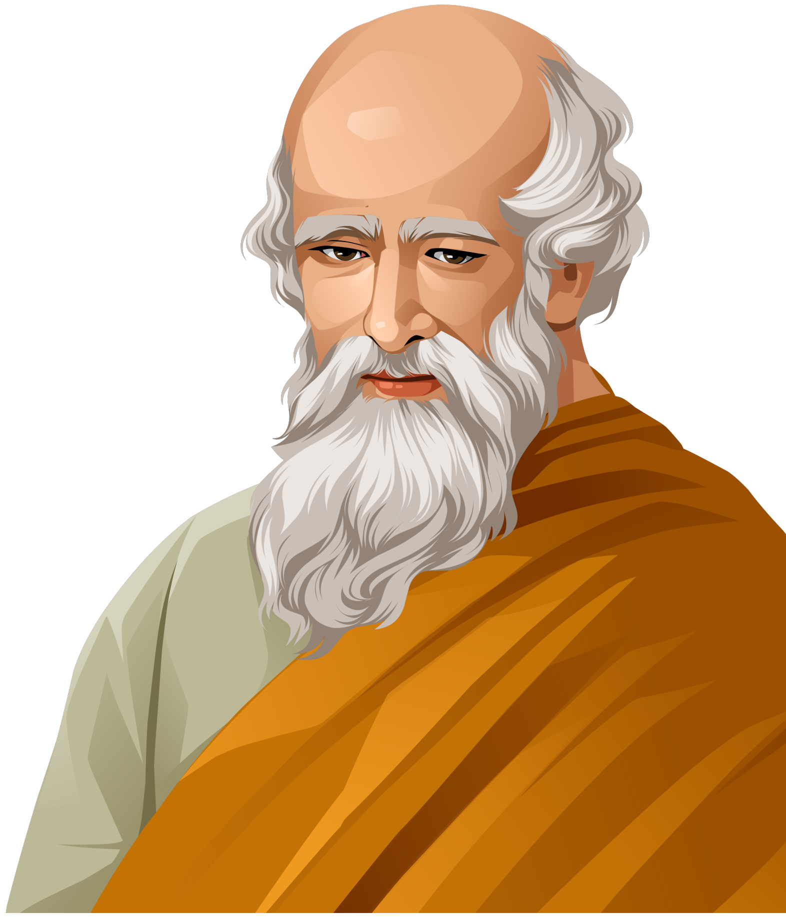

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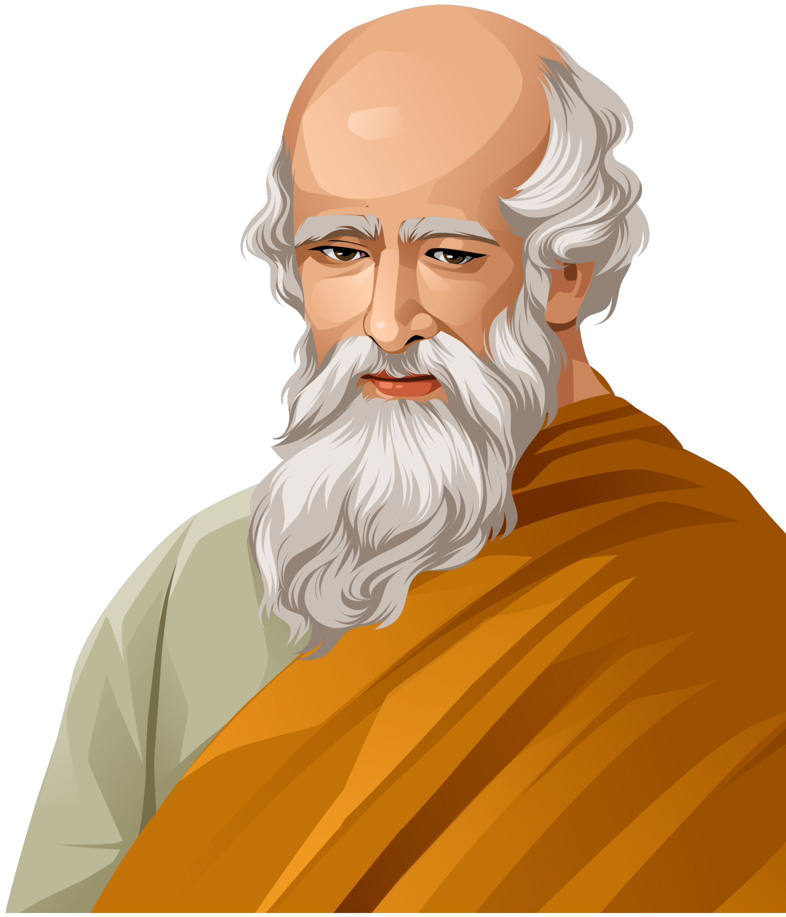

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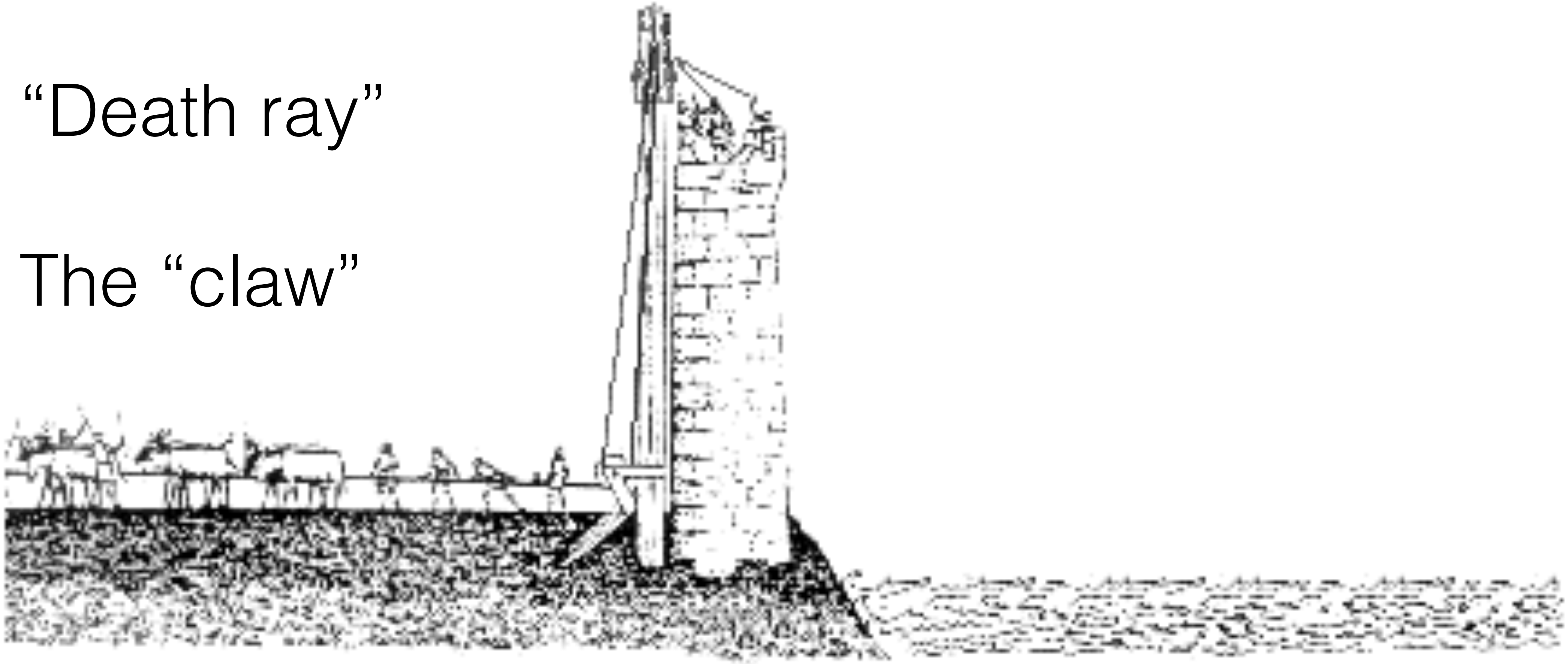

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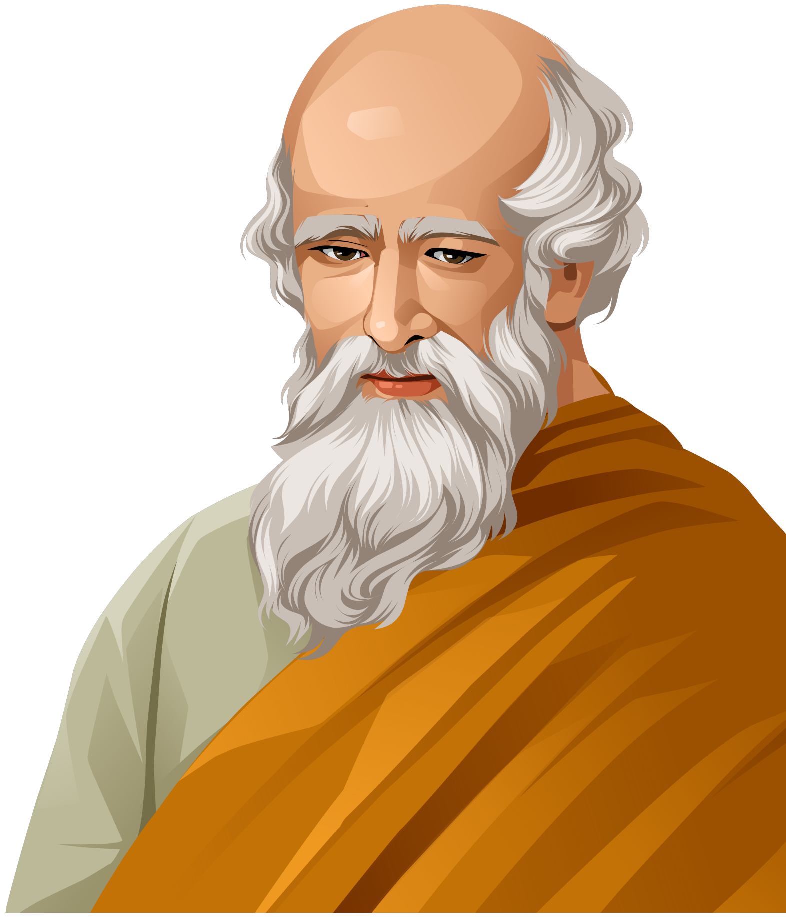

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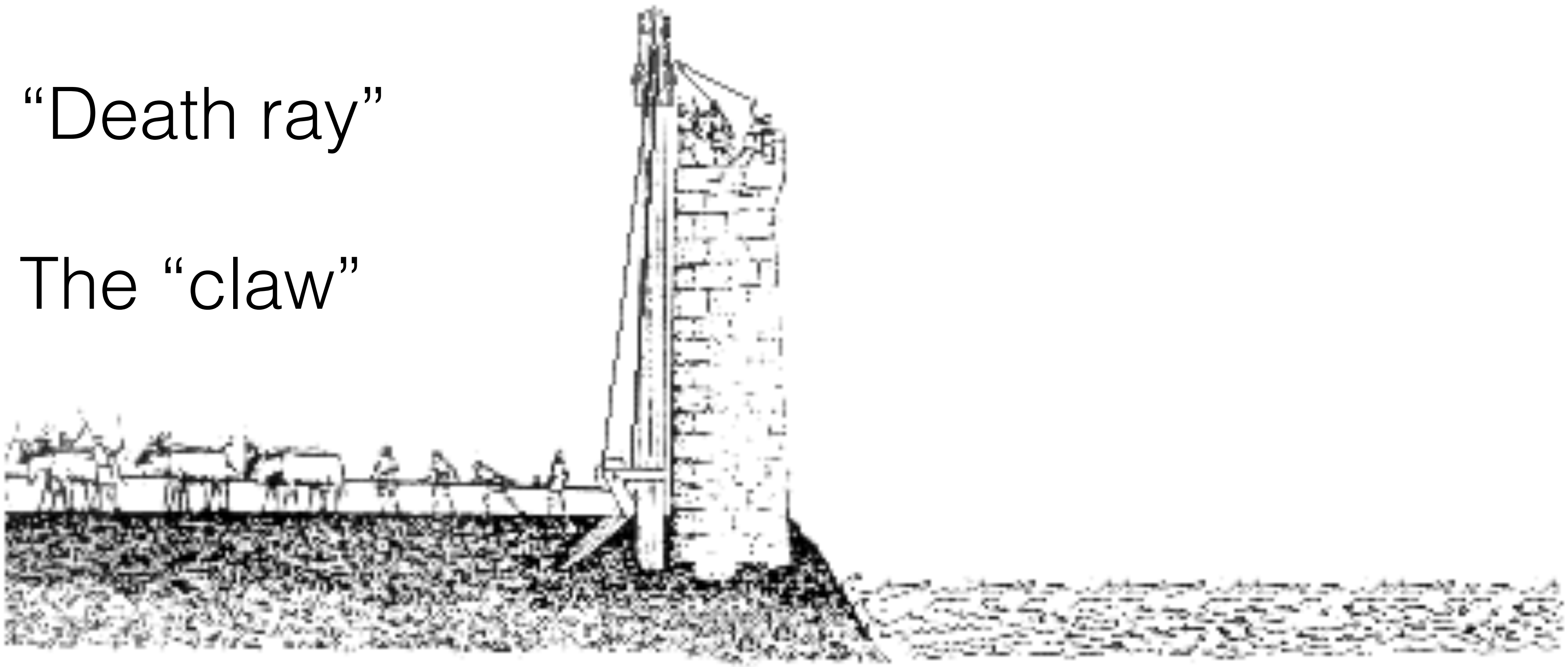

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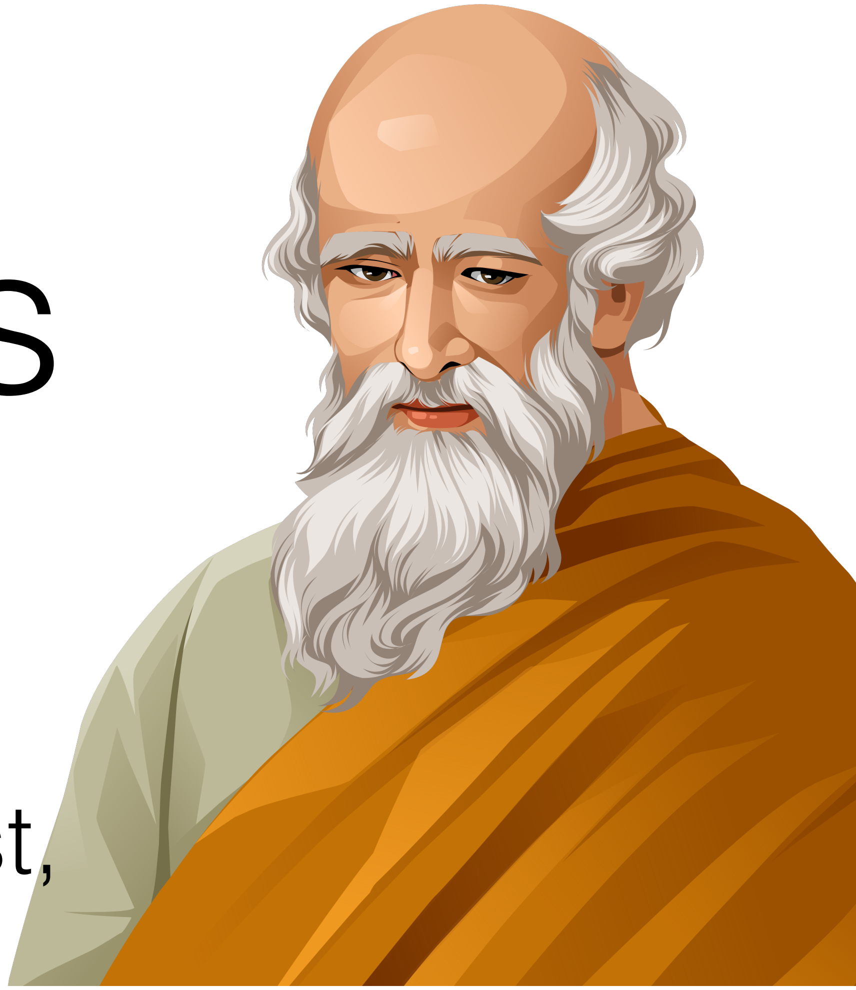

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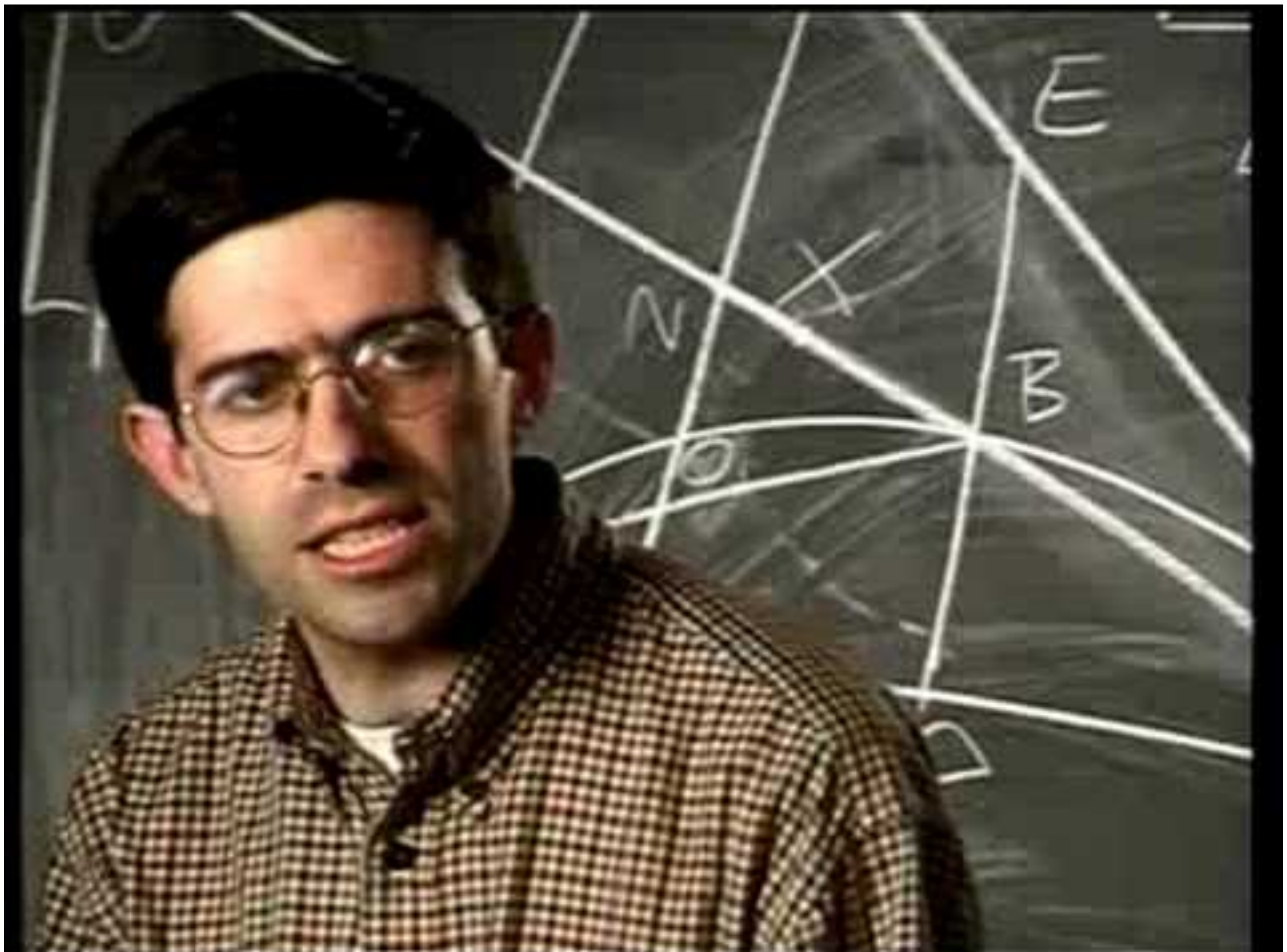

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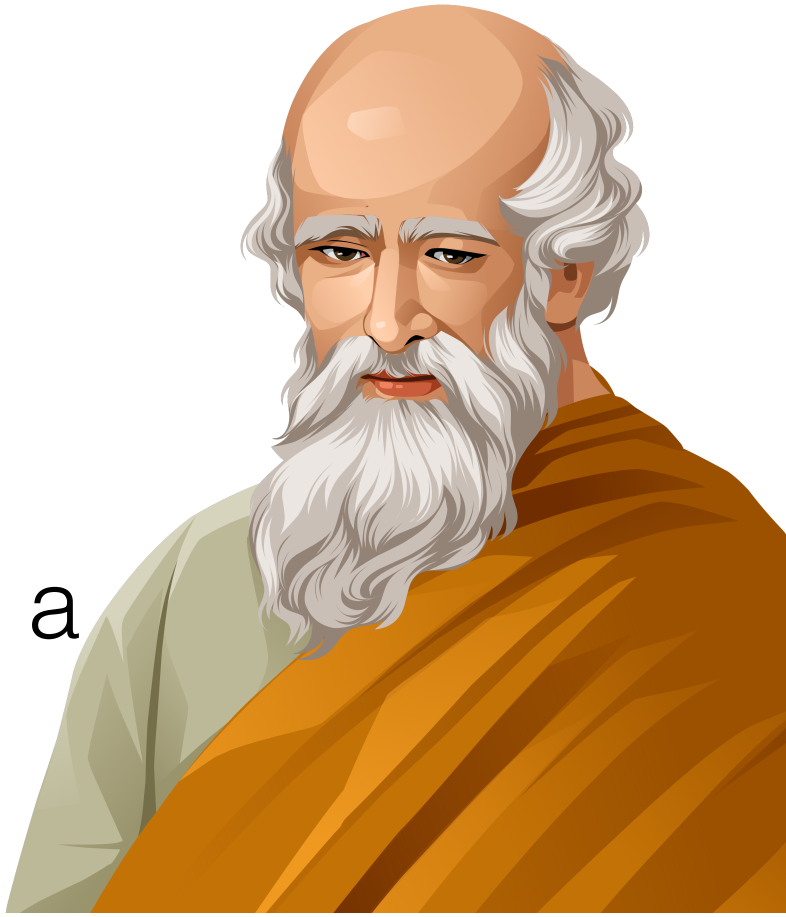
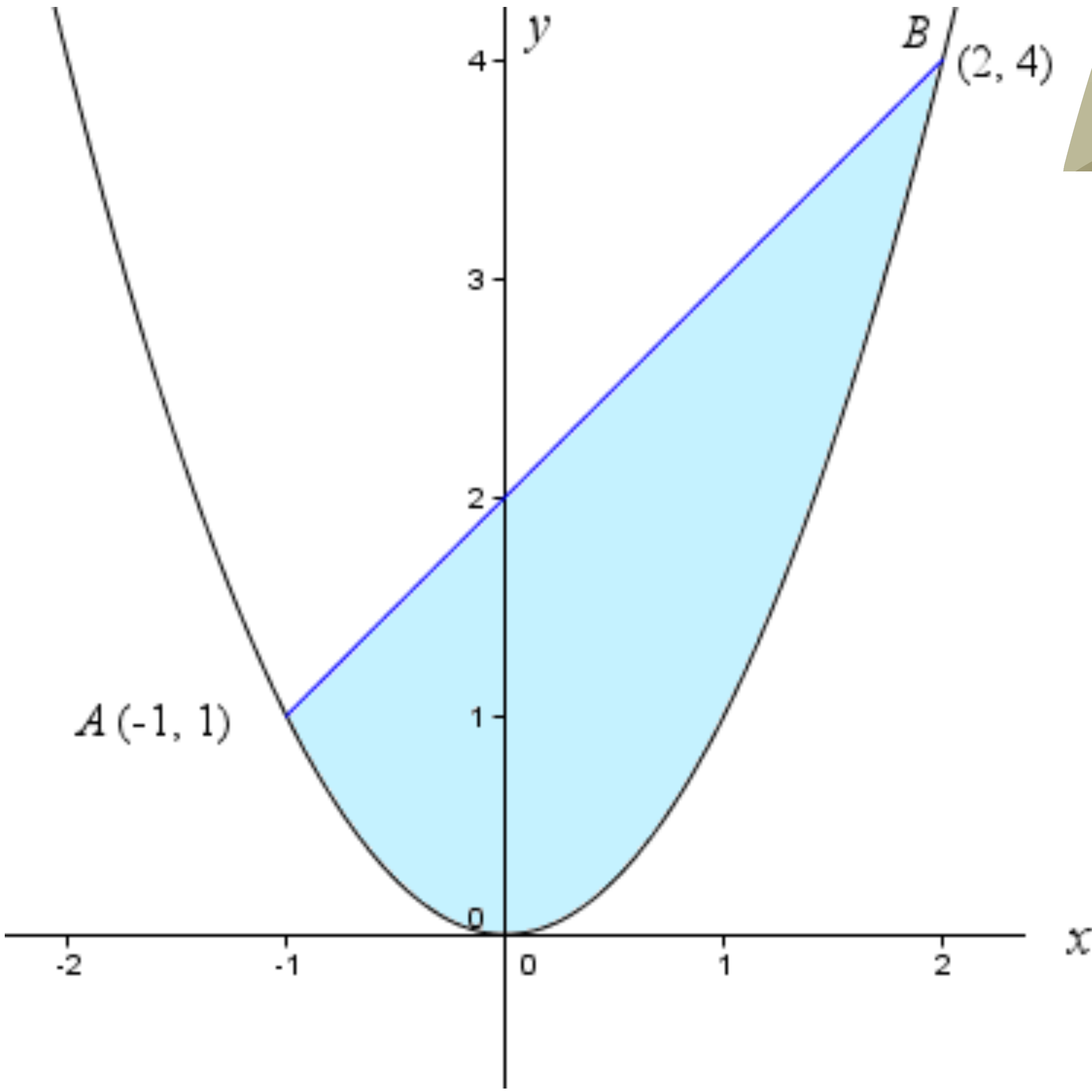
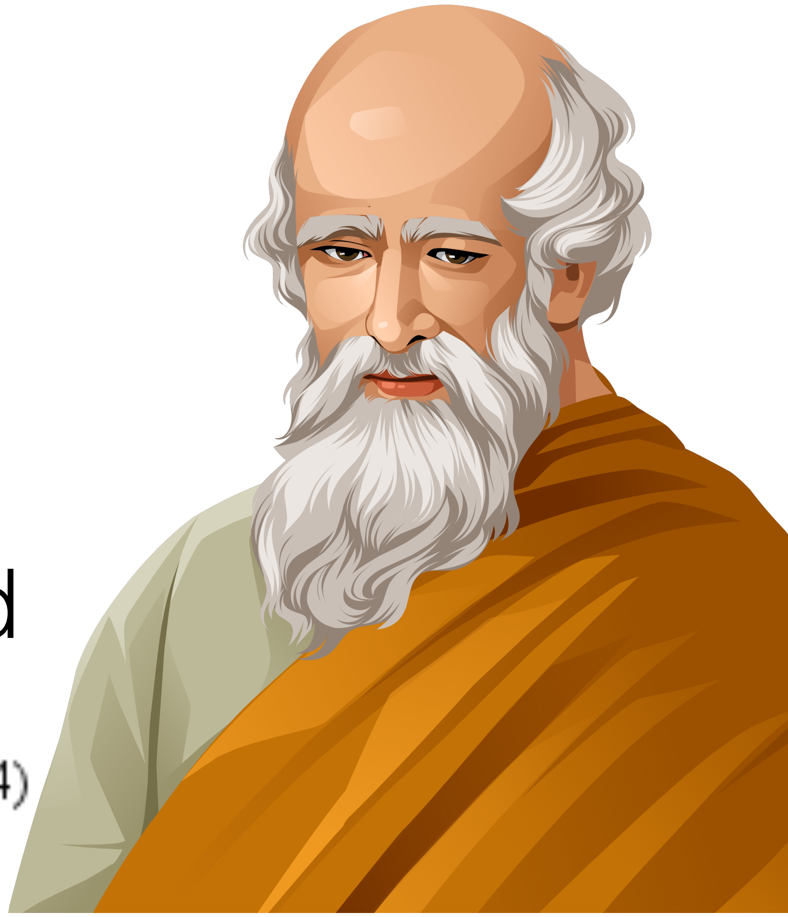


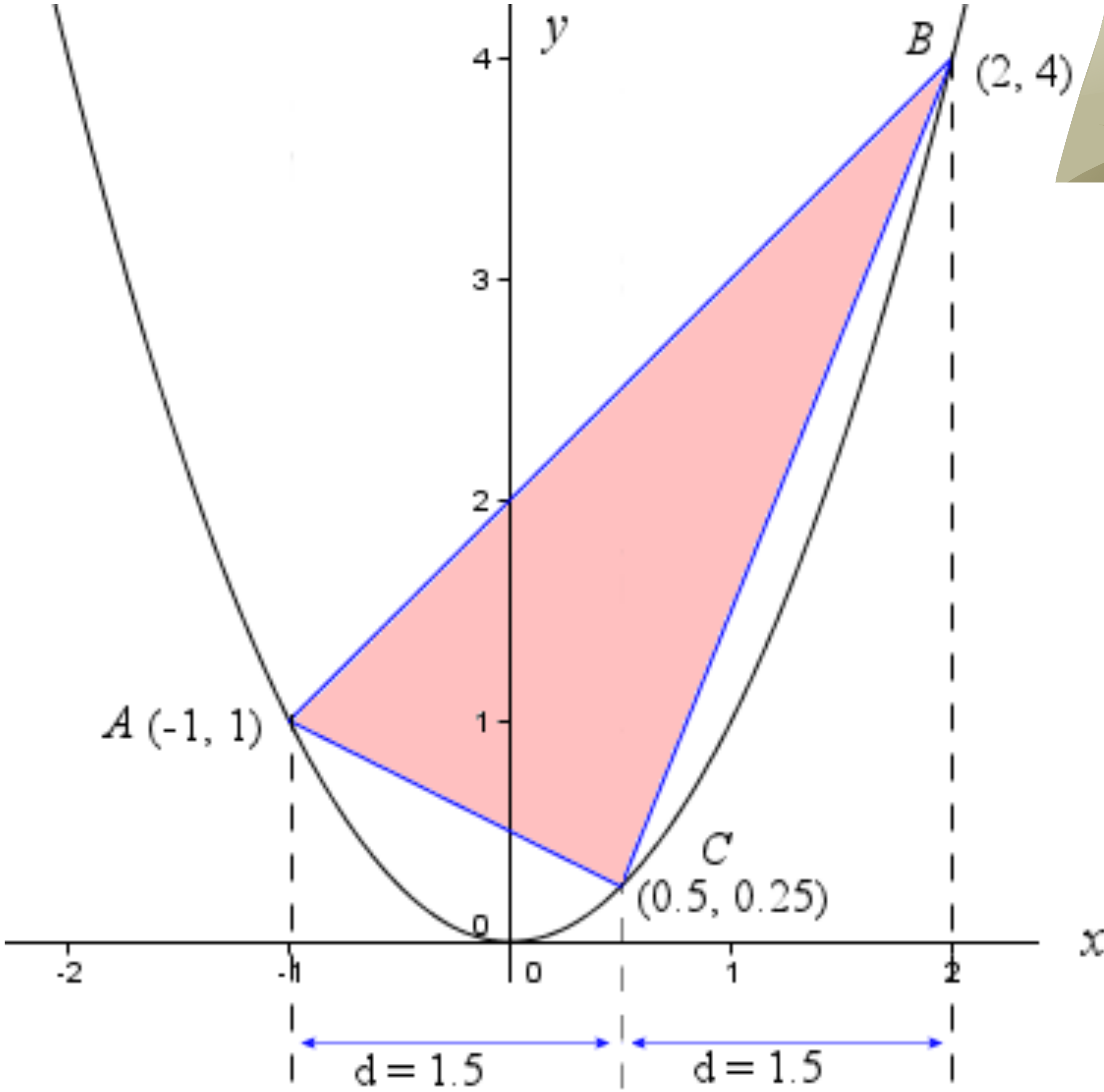
- Find the area between a parabola and a line:



Archimedes

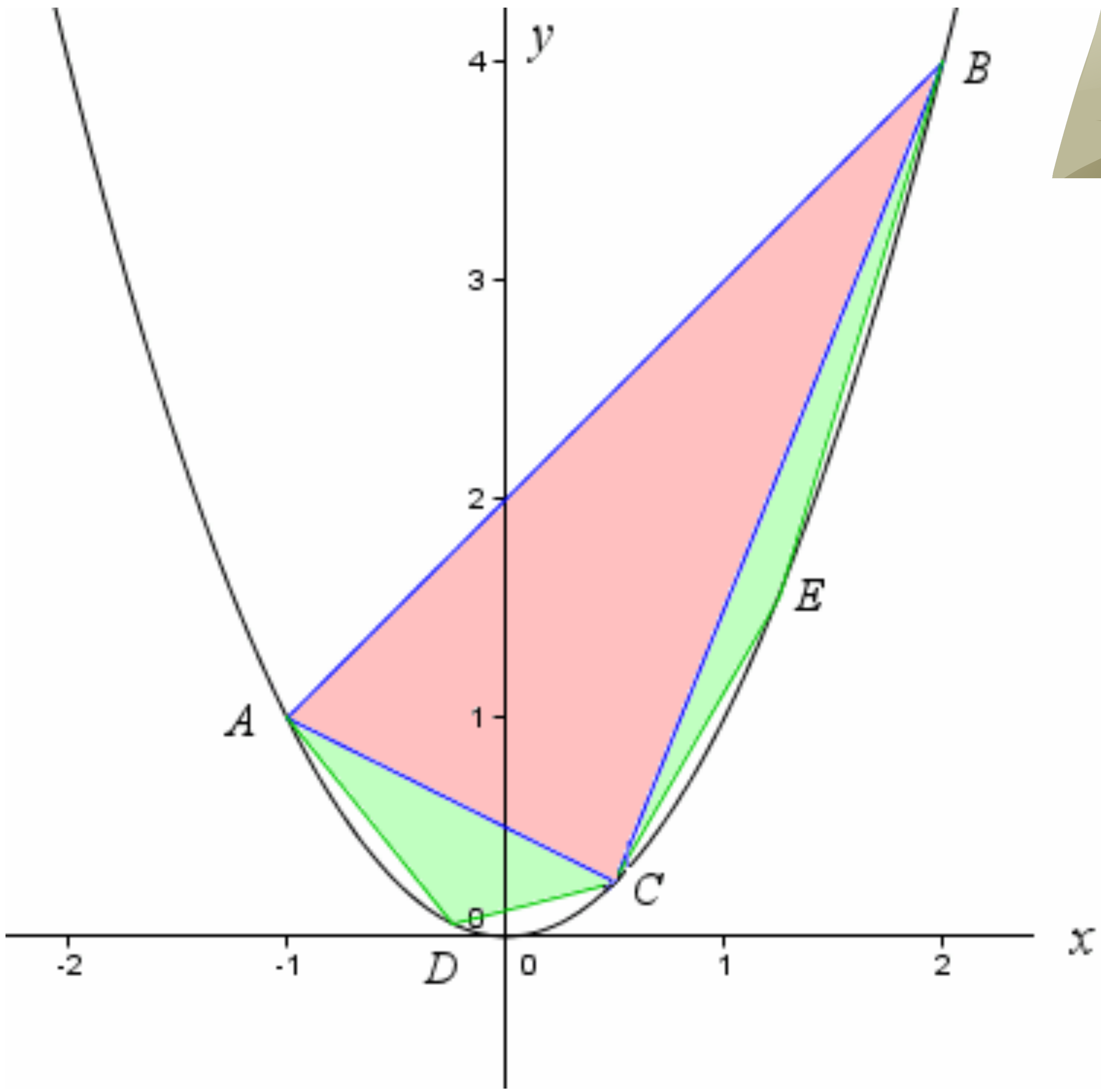


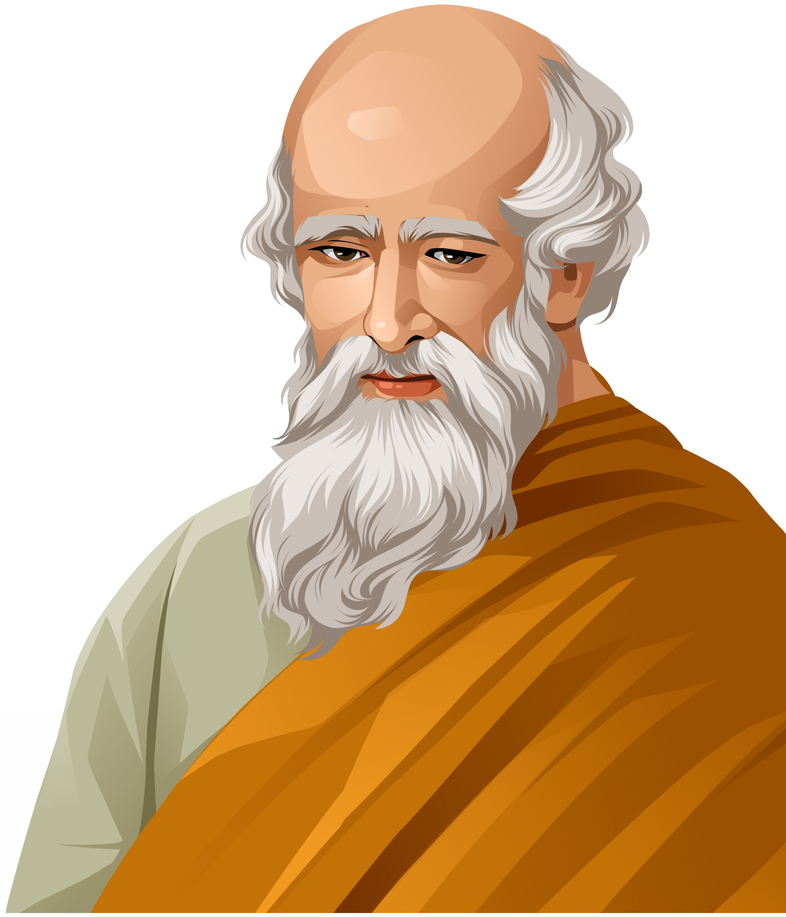
- Add a triangle with the same base and height.



Archimedes

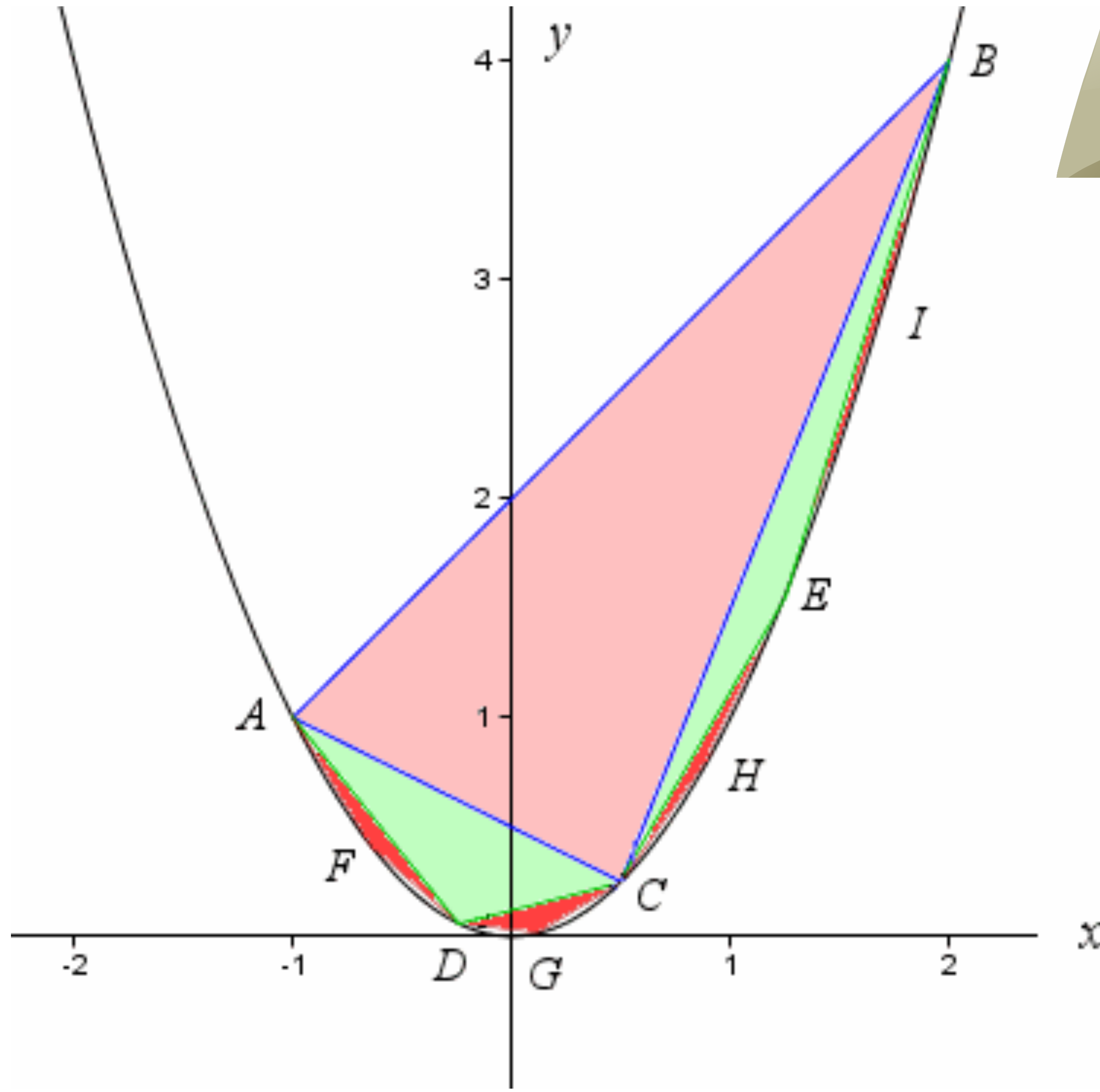
- To the empty places, place similar triangles.

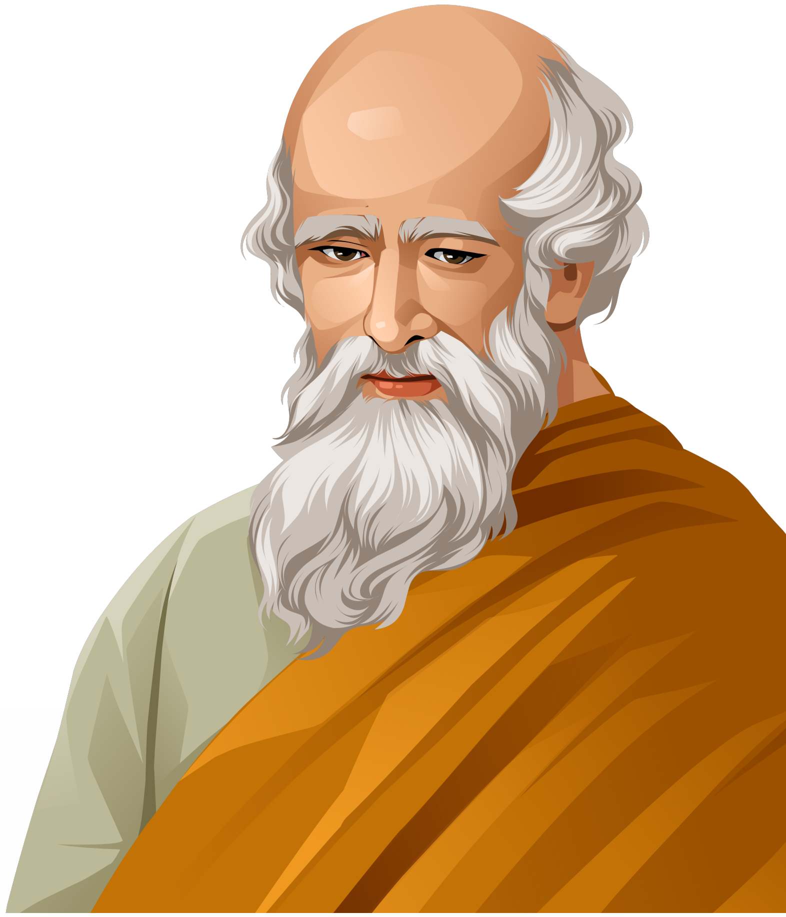




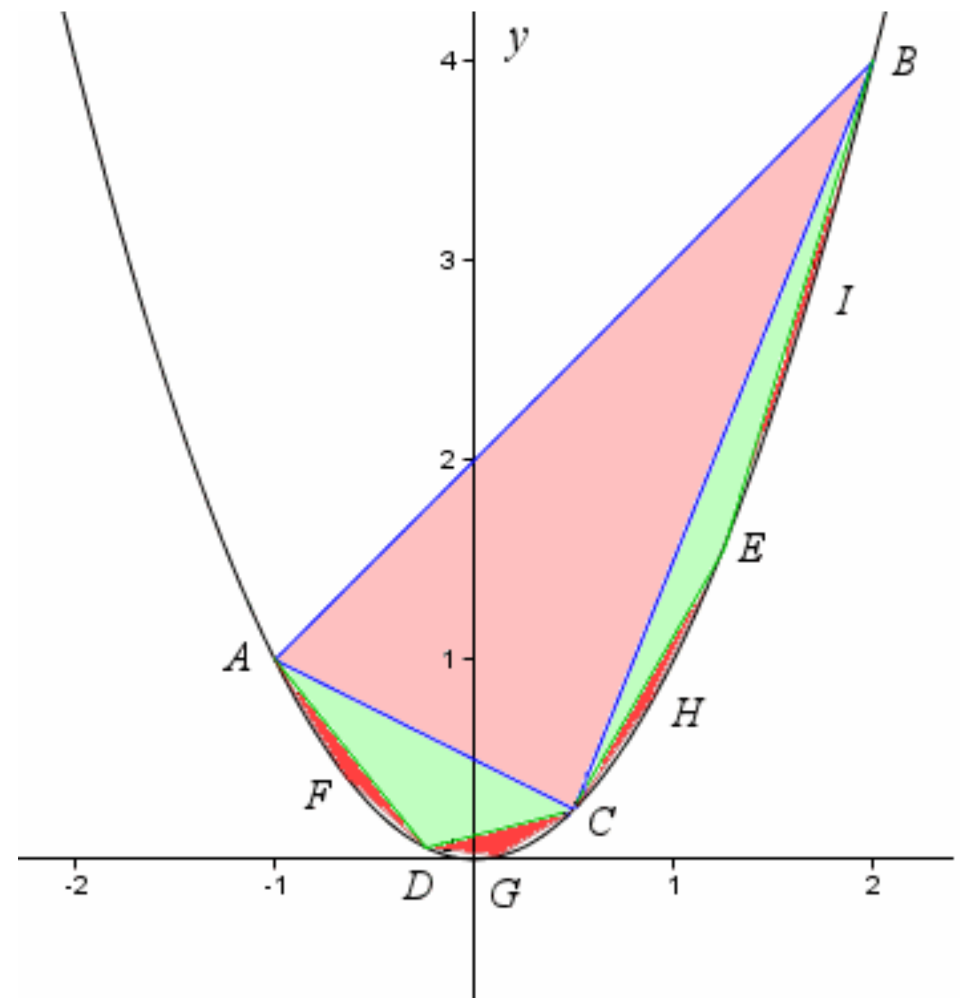
Archimedes

- Continue this forever.



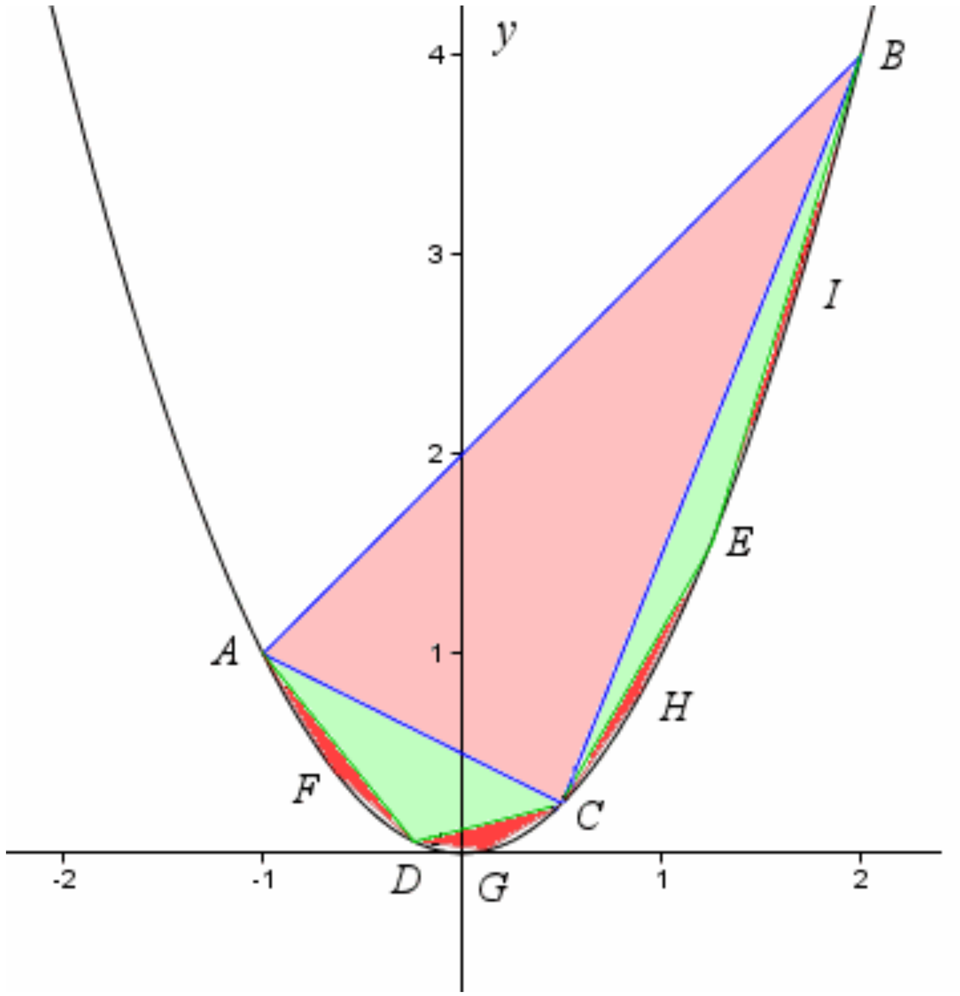


Archimedes



Archimedes

- Archimedes showed that each triangle has 1/8 the area as in the previous step.



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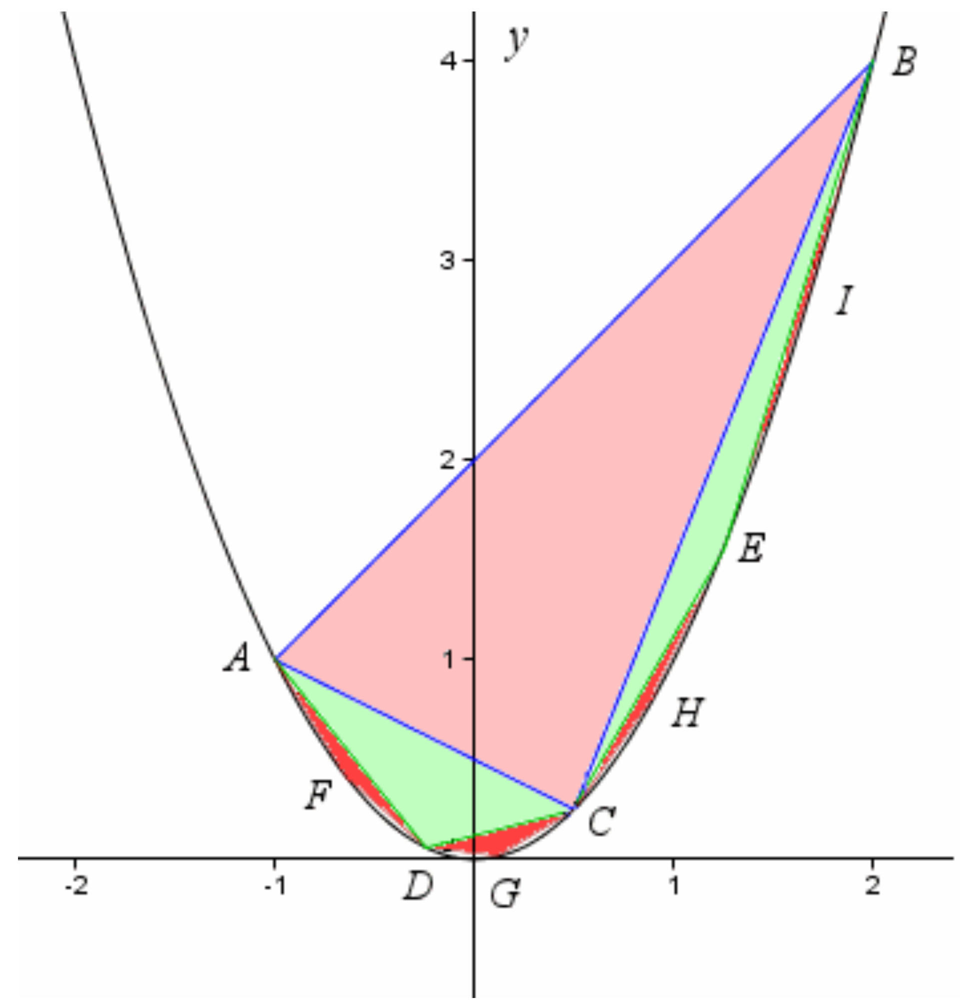
- Archimedes showed that each triangle has 1/8 the area as in the previous step.

- Since there are twice as many triangles each step, the total area each step is 1/4 as much as in the previous step.



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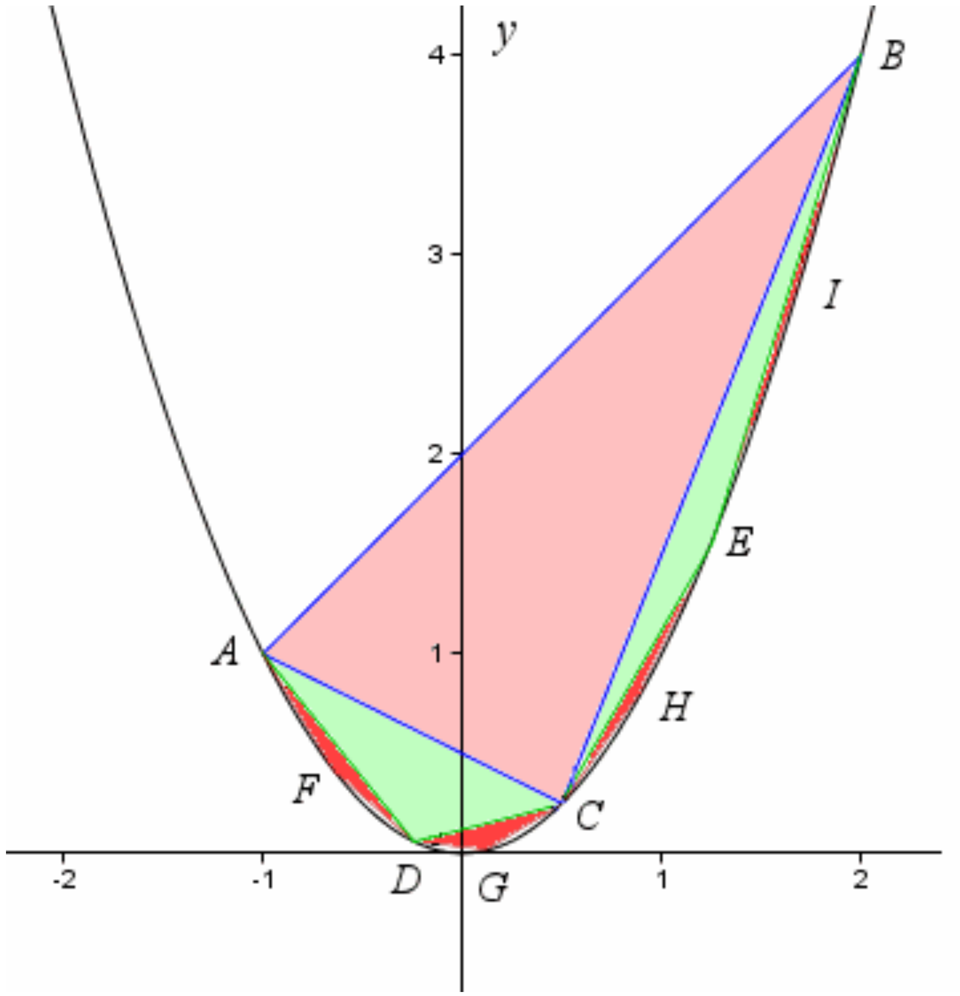
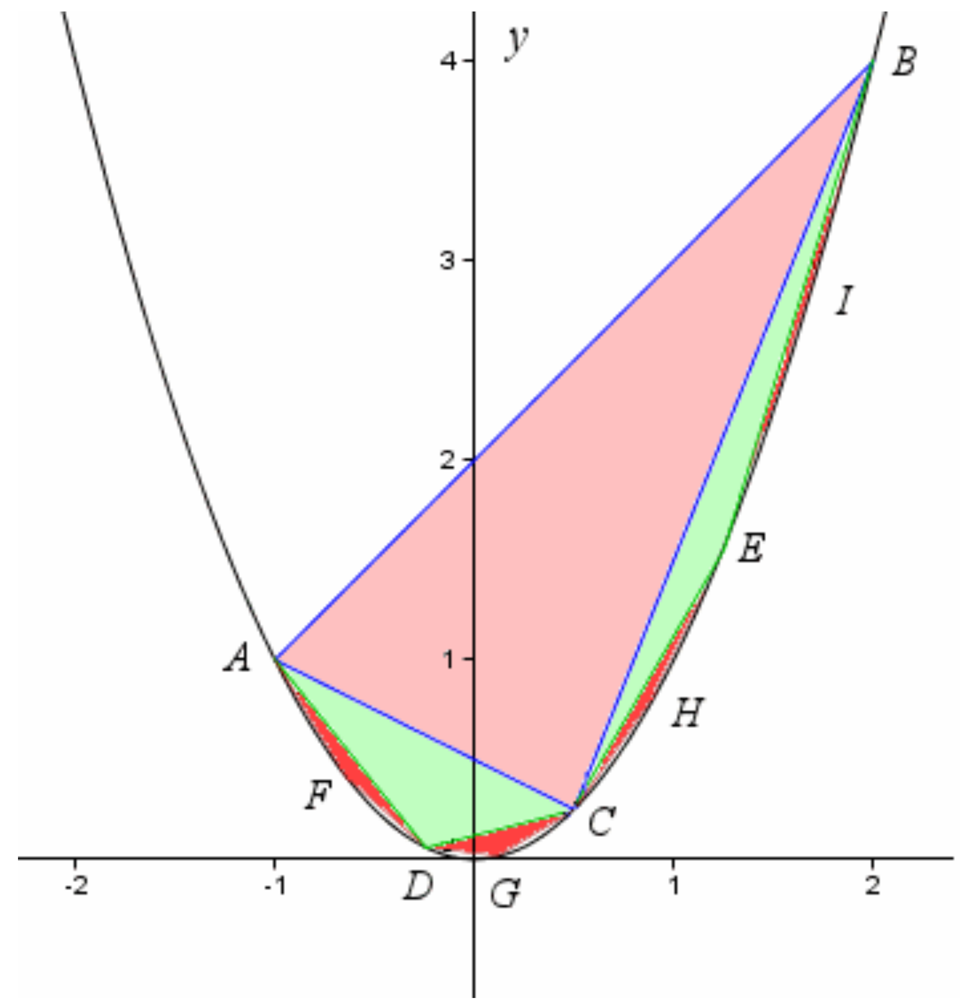


- Thus, if A is the area of the triangle, the area between the line and parabola is

$$A + \frac{1}{4}A + \frac{1}{4^2}A + \frac{1}{4^3}A + \frac{1}{4^4}A + \ldots$$

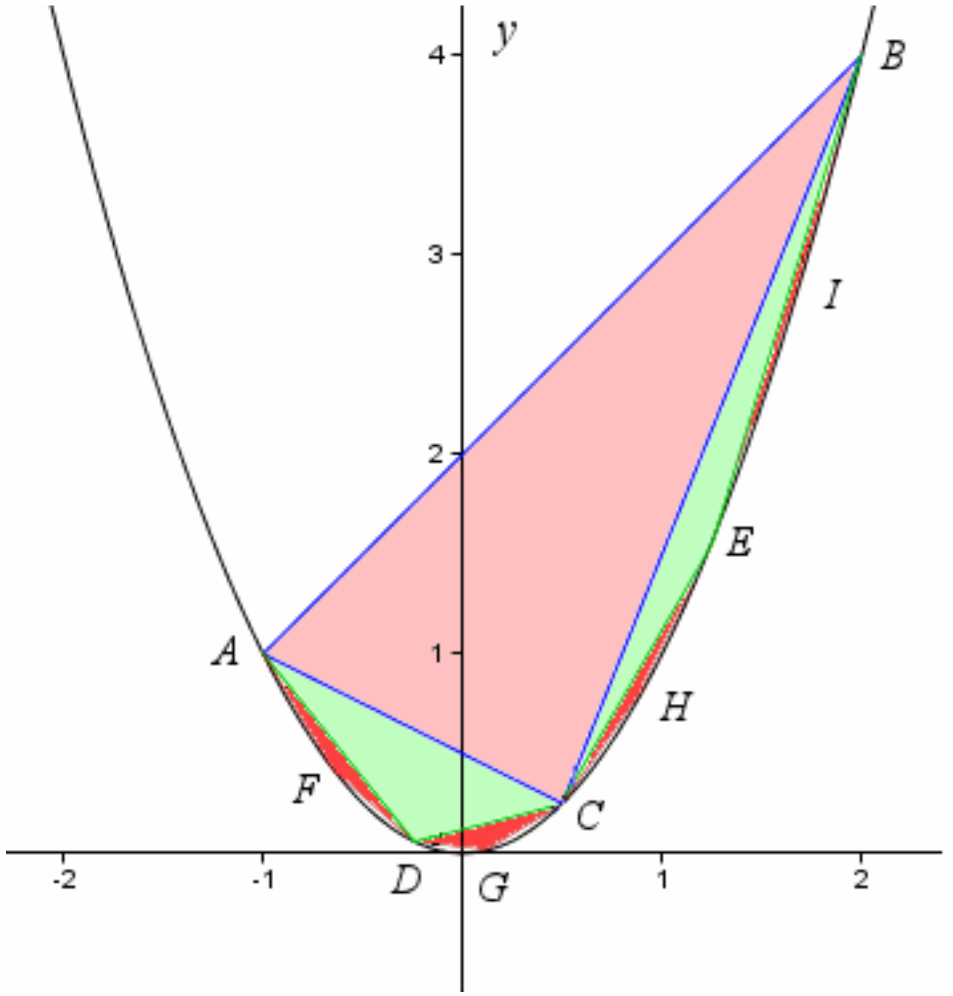
Archimedes



Archimedes

- First, he rewrote this as

$$A + A\left(\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots\right)$$



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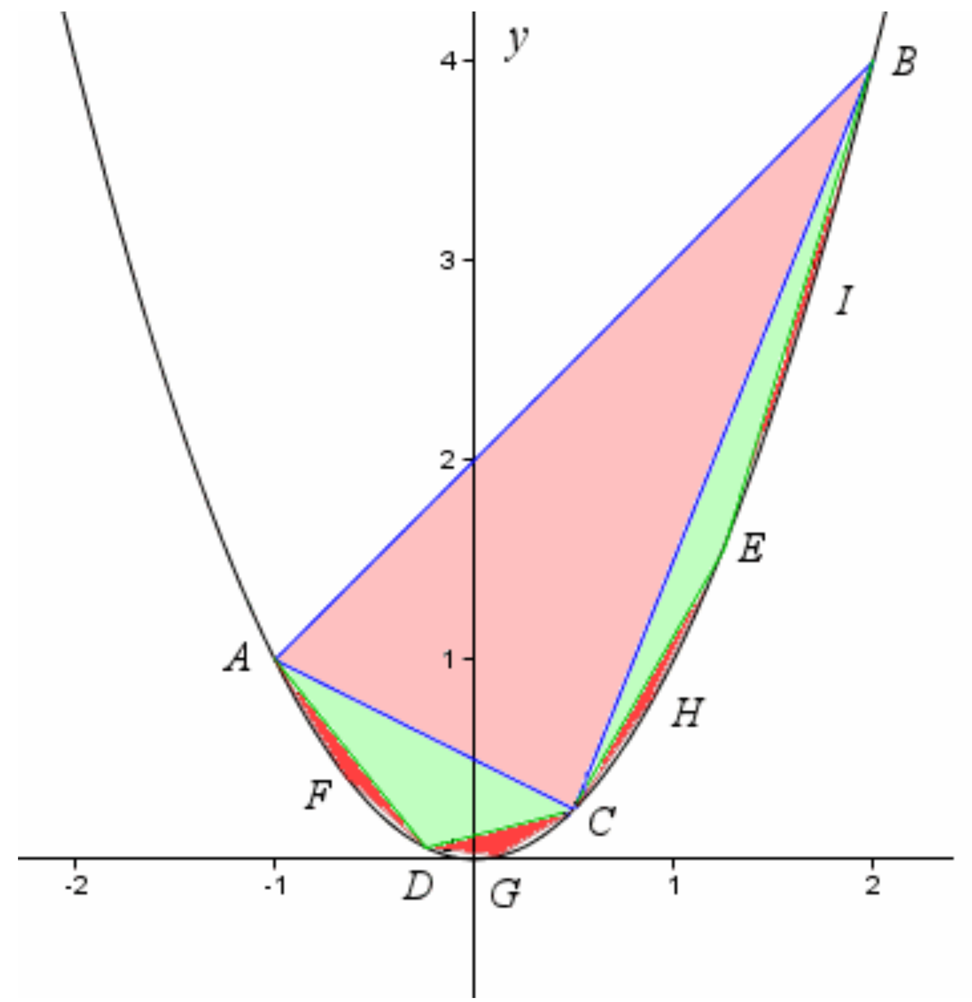
- Finding the area of a triangle is easy, so all Archimedes needed was to find
$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \ldots$$



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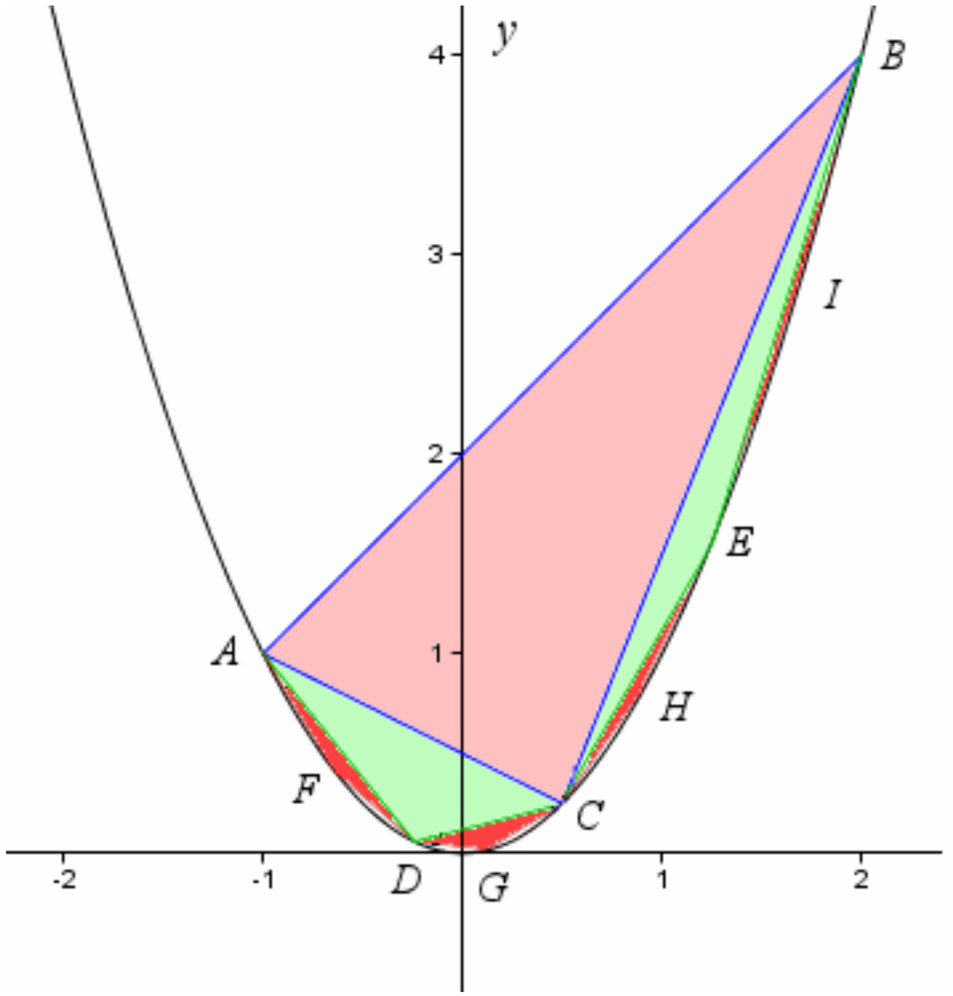

- How do you determine this before knowing a formula?

Archimedes

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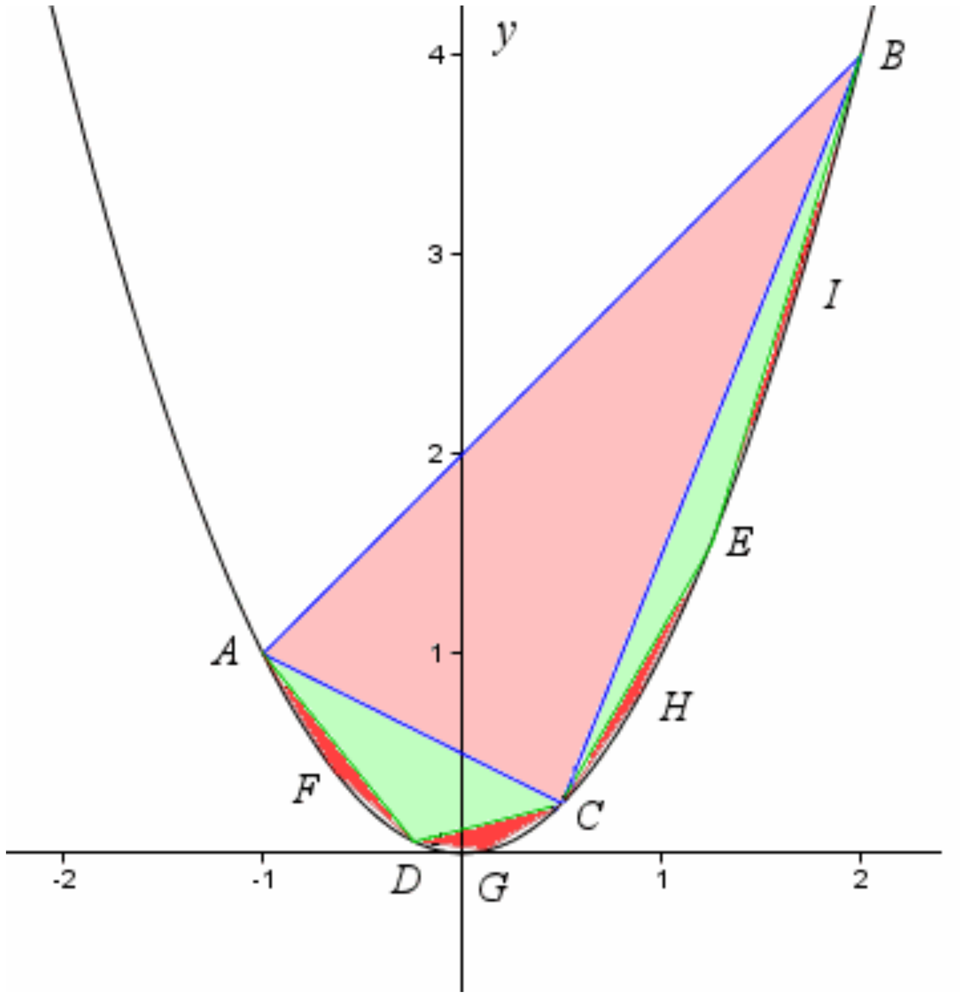

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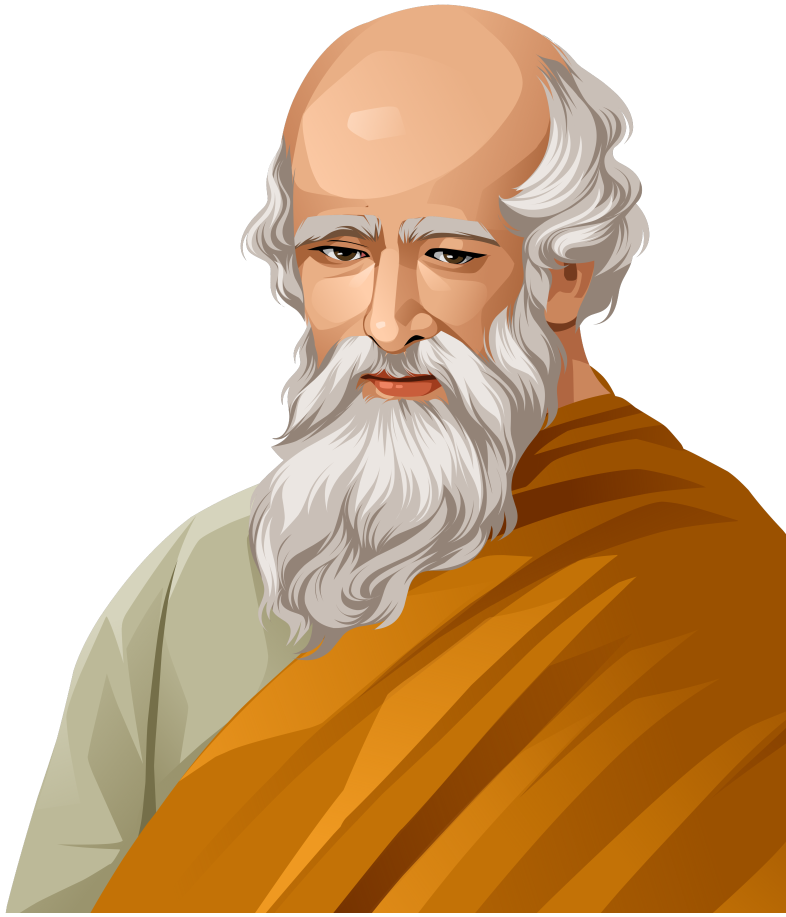
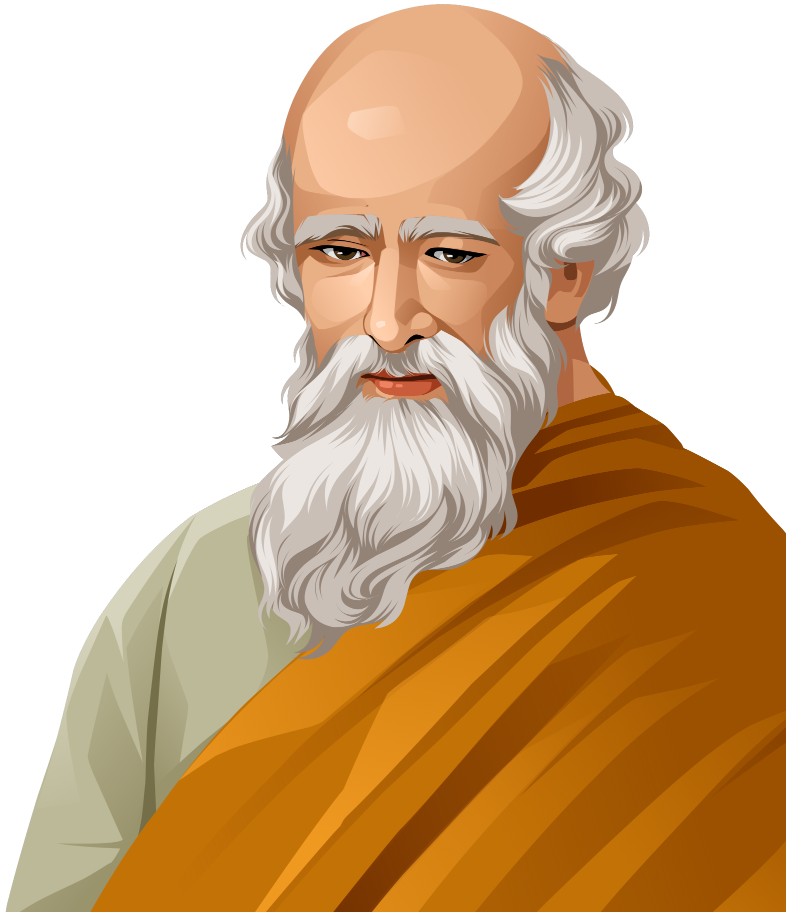
Archimedes

Finding $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \ldots$



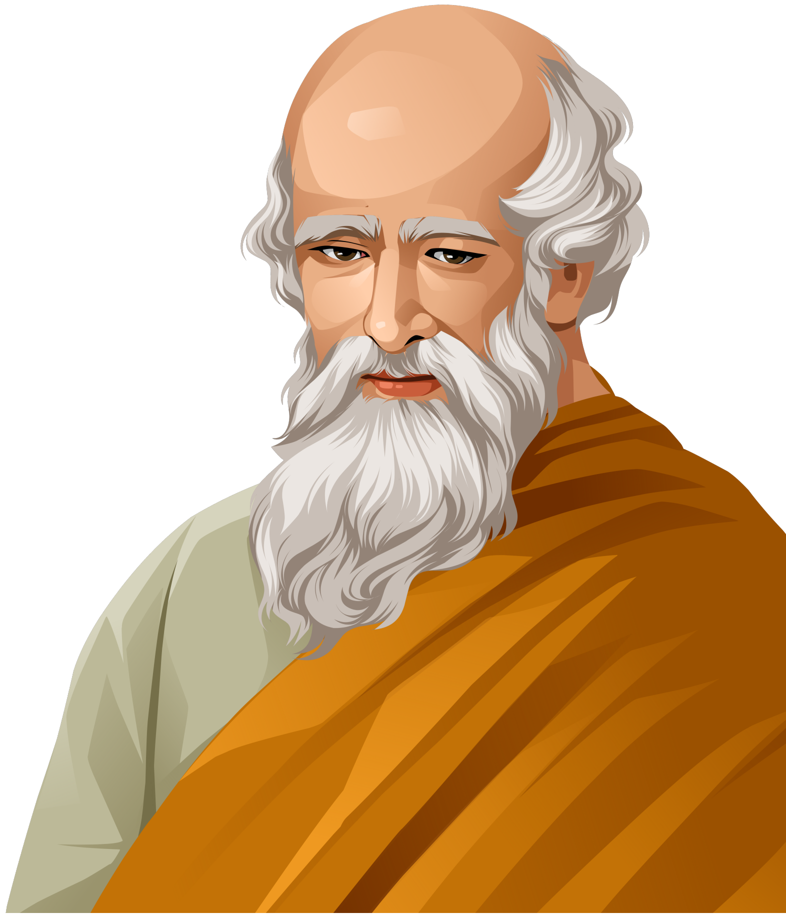
Archimedes



Finding $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \ldots$

Partial sums:

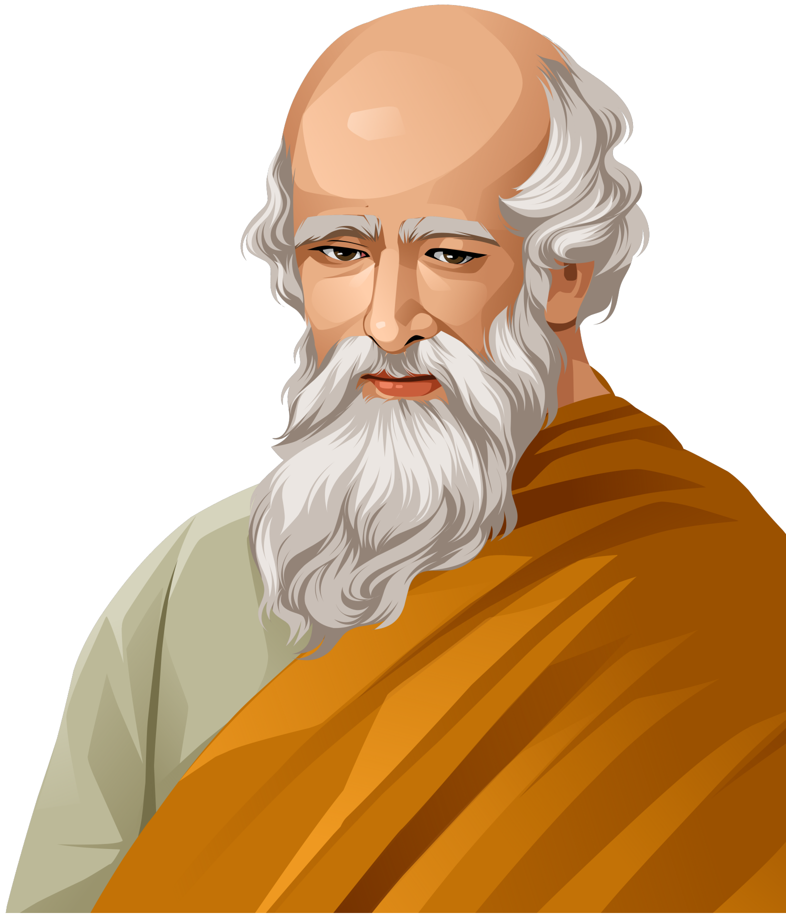
Archimedes



Finding $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\dots$

Partial sums: $\frac{1}{4}=0.25$

Archimedes

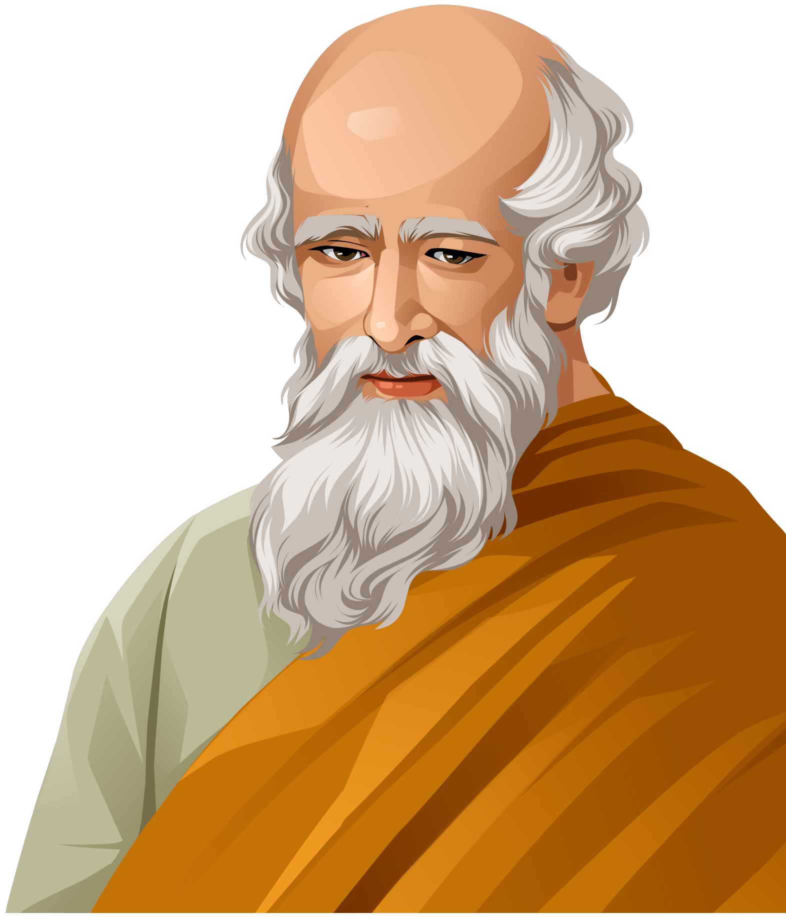


Finding $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\ldots$

Partial sums: $\frac{1}{4}=0.25$

$$\frac{1}{4}+\frac{1}{16}=0.3125$$

Archimedes



Finding $$\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\dots$$

Partial sums:

$$\frac{1}{4}=0.25$$

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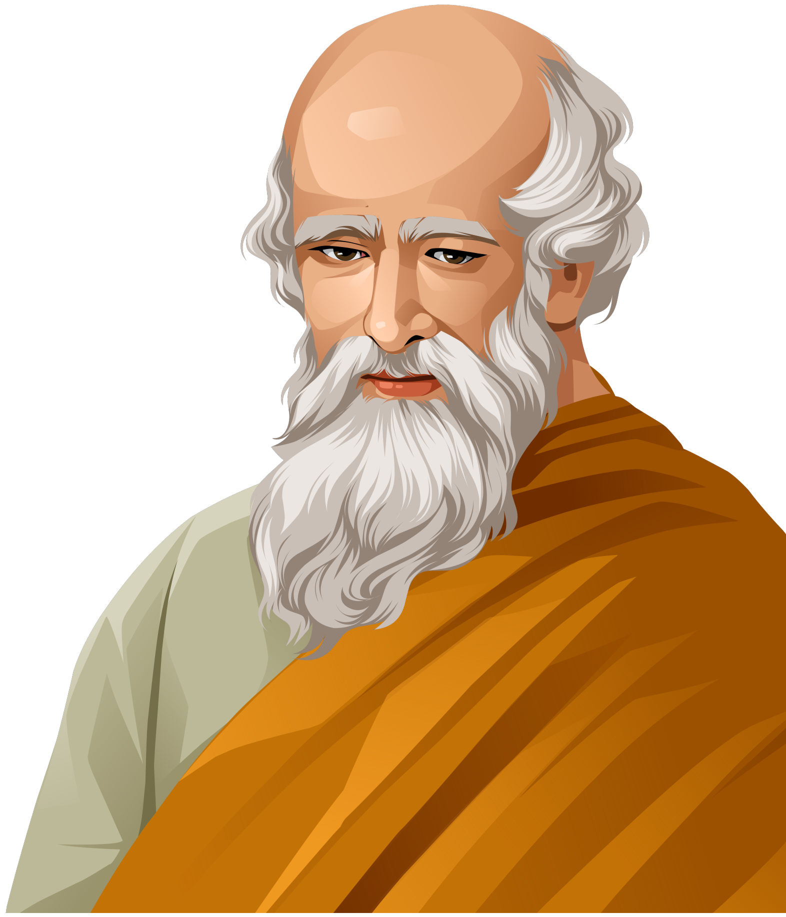

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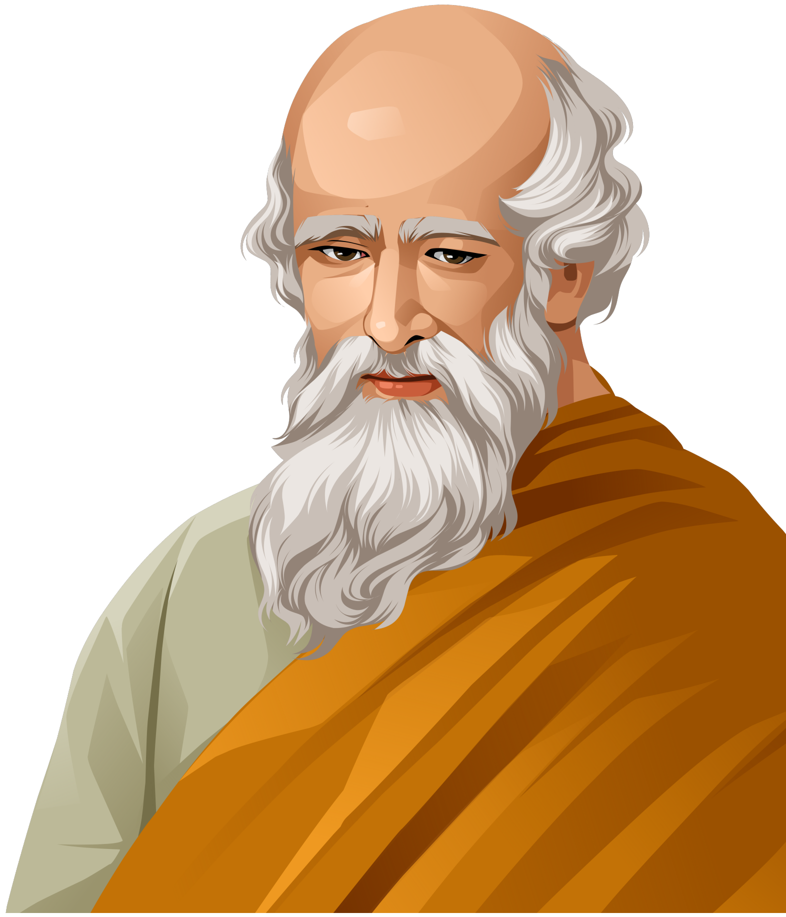
Archimedes



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$$\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}=0.3330078125$$

Archimedes

Showing that $$\frac{1}{3} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \dots$$

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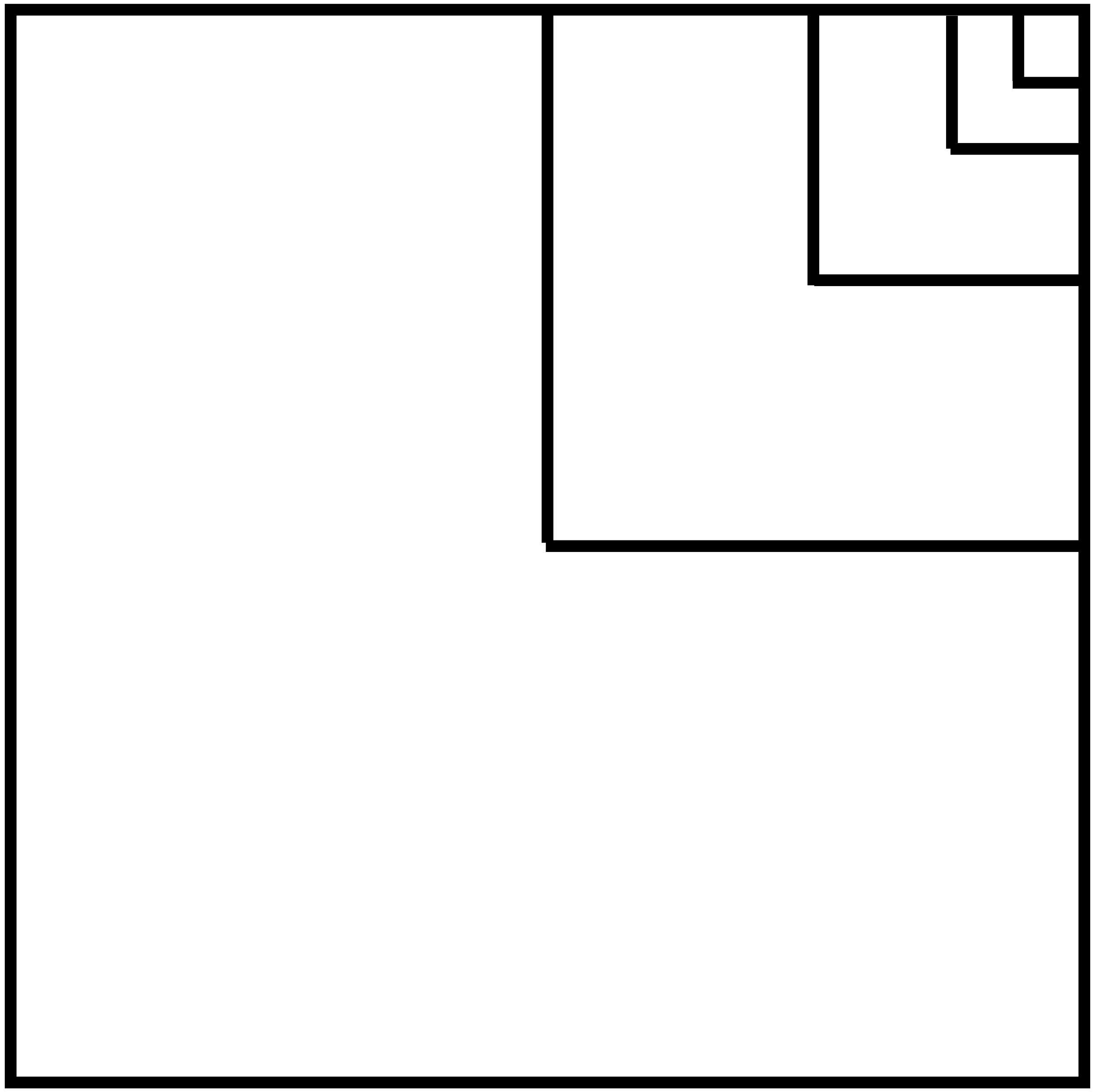
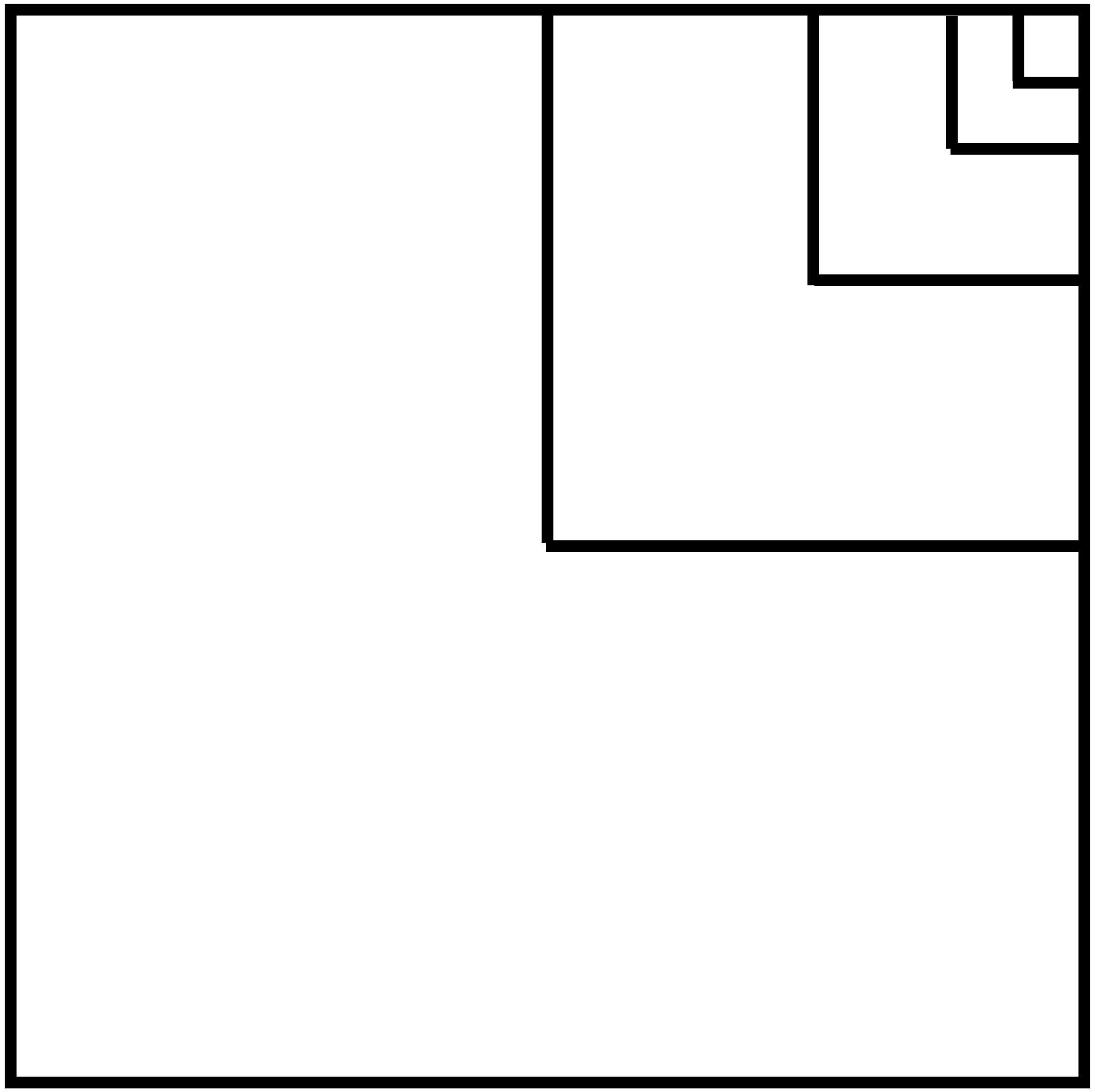
Archimedes' logic relied on this drawing.



Think Like A Math Historian

Archimedes

Showing that $\frac{1}{3} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \ldots$



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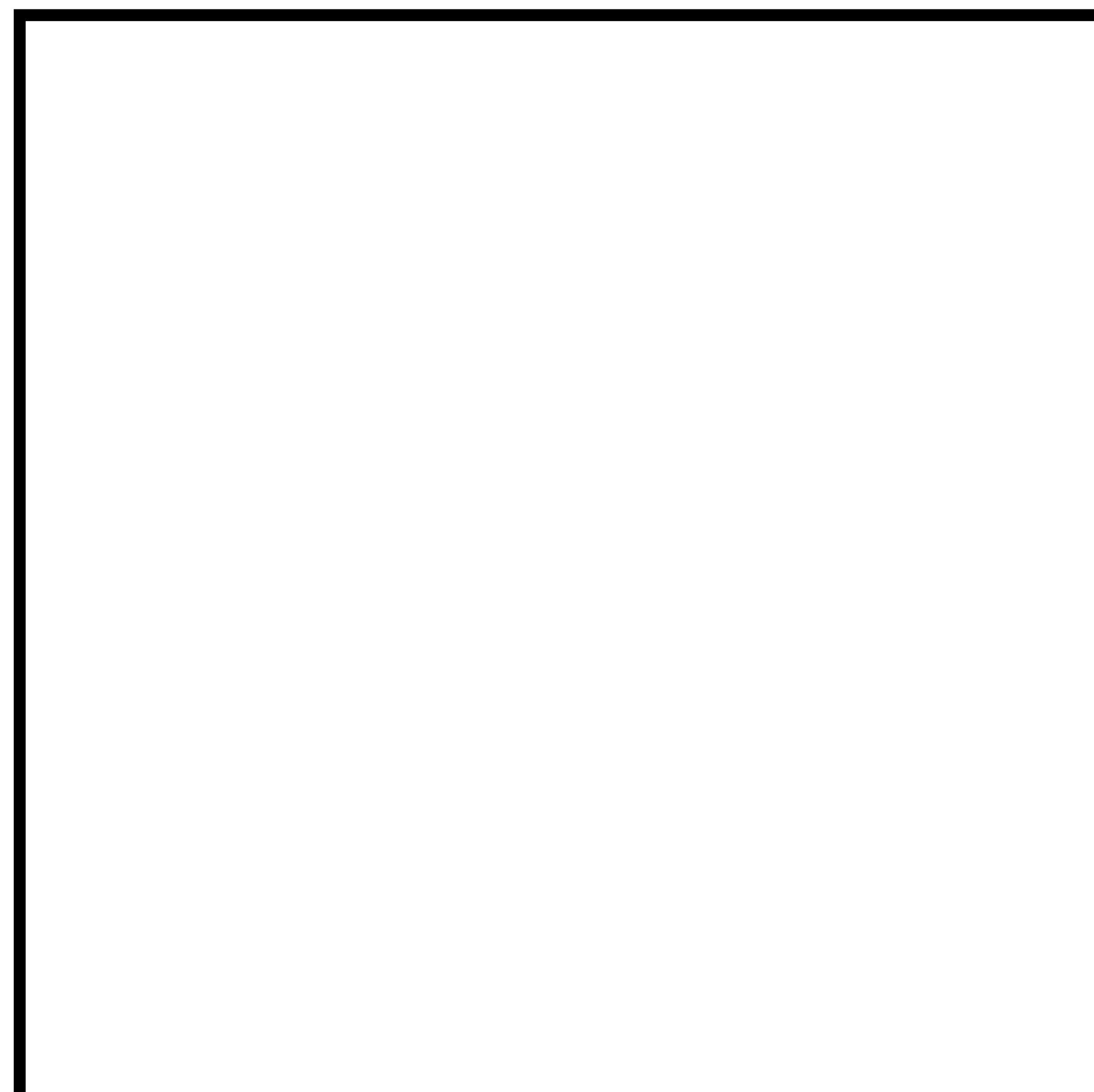

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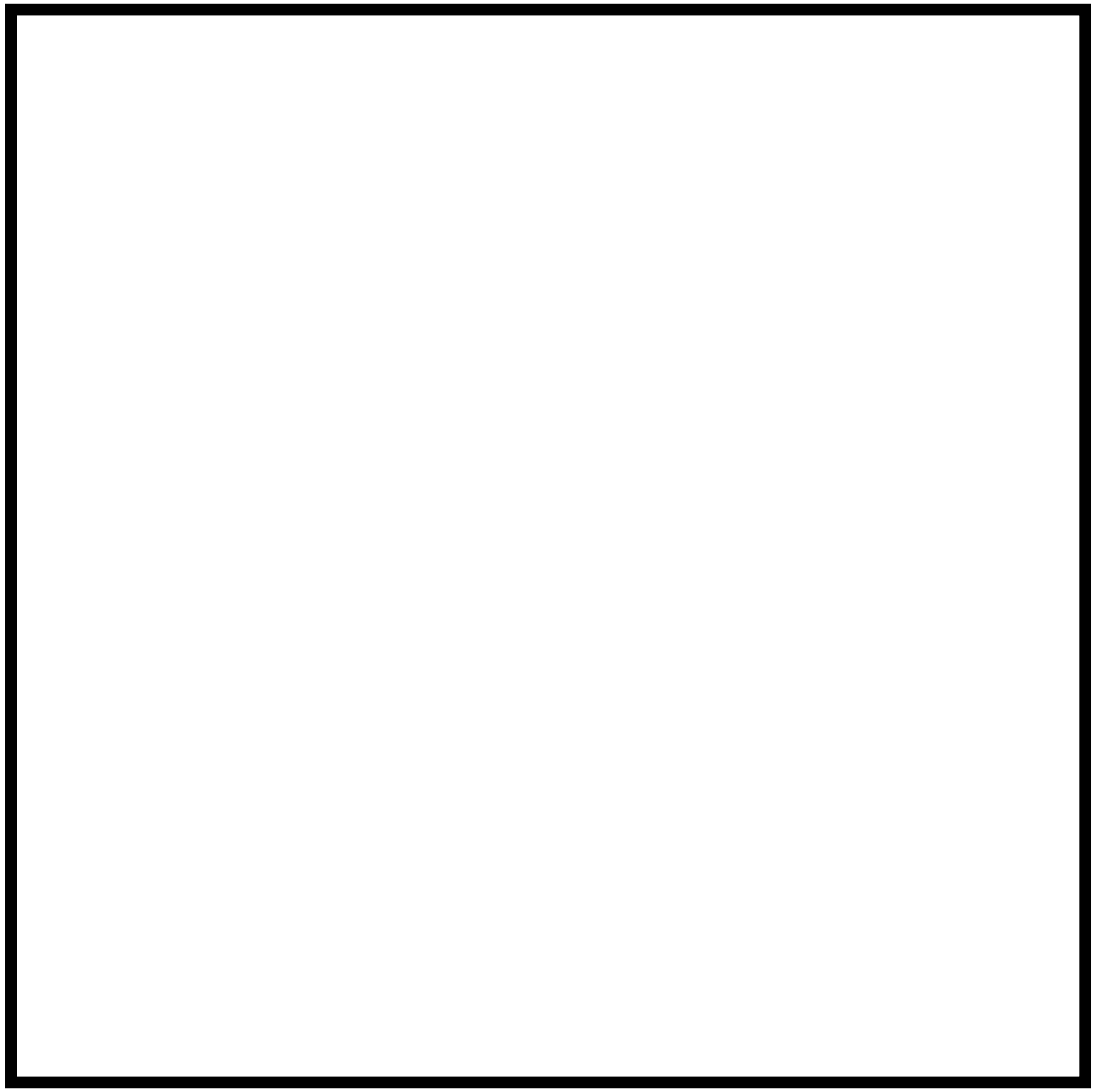
1



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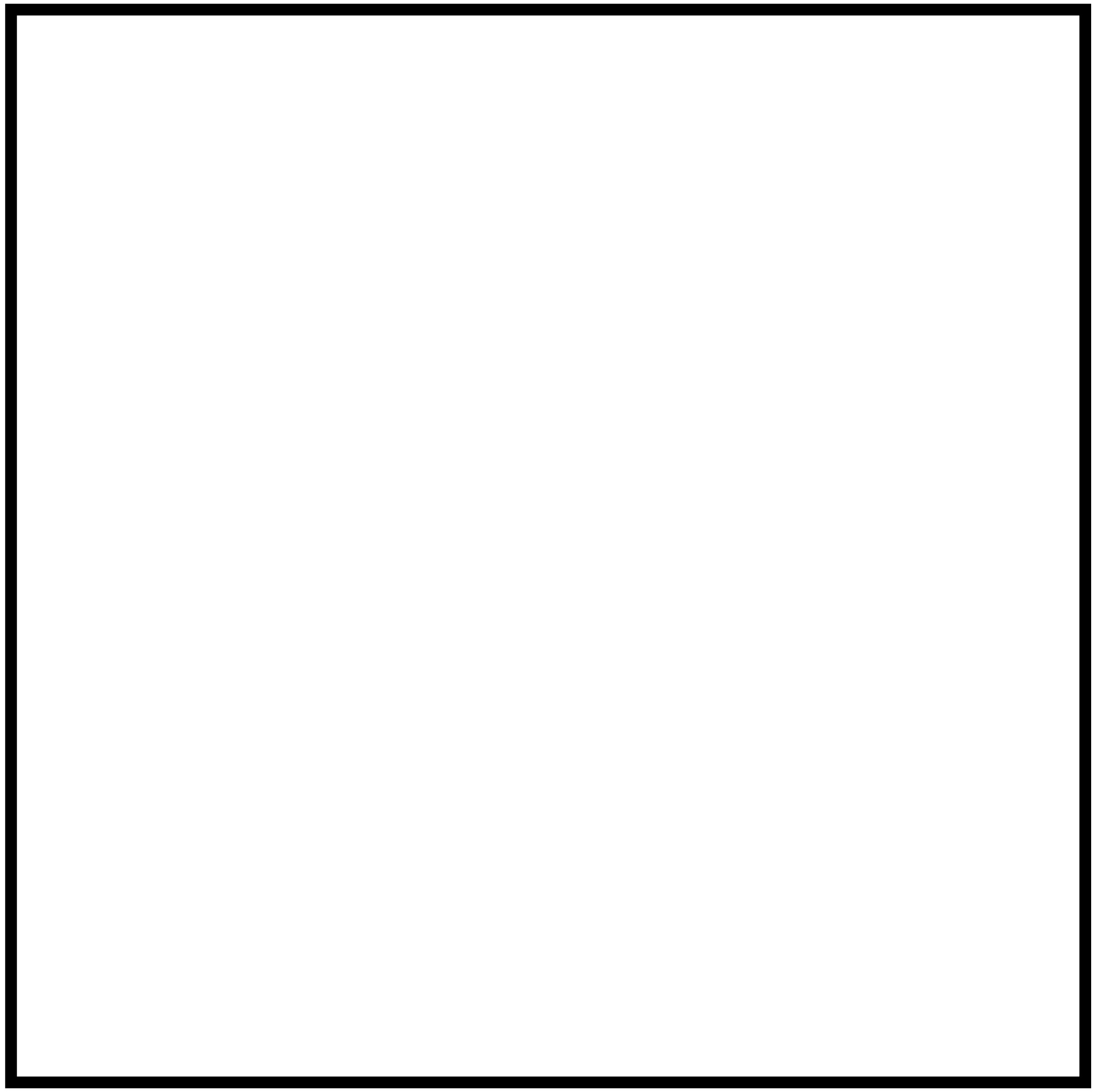
$1 =$



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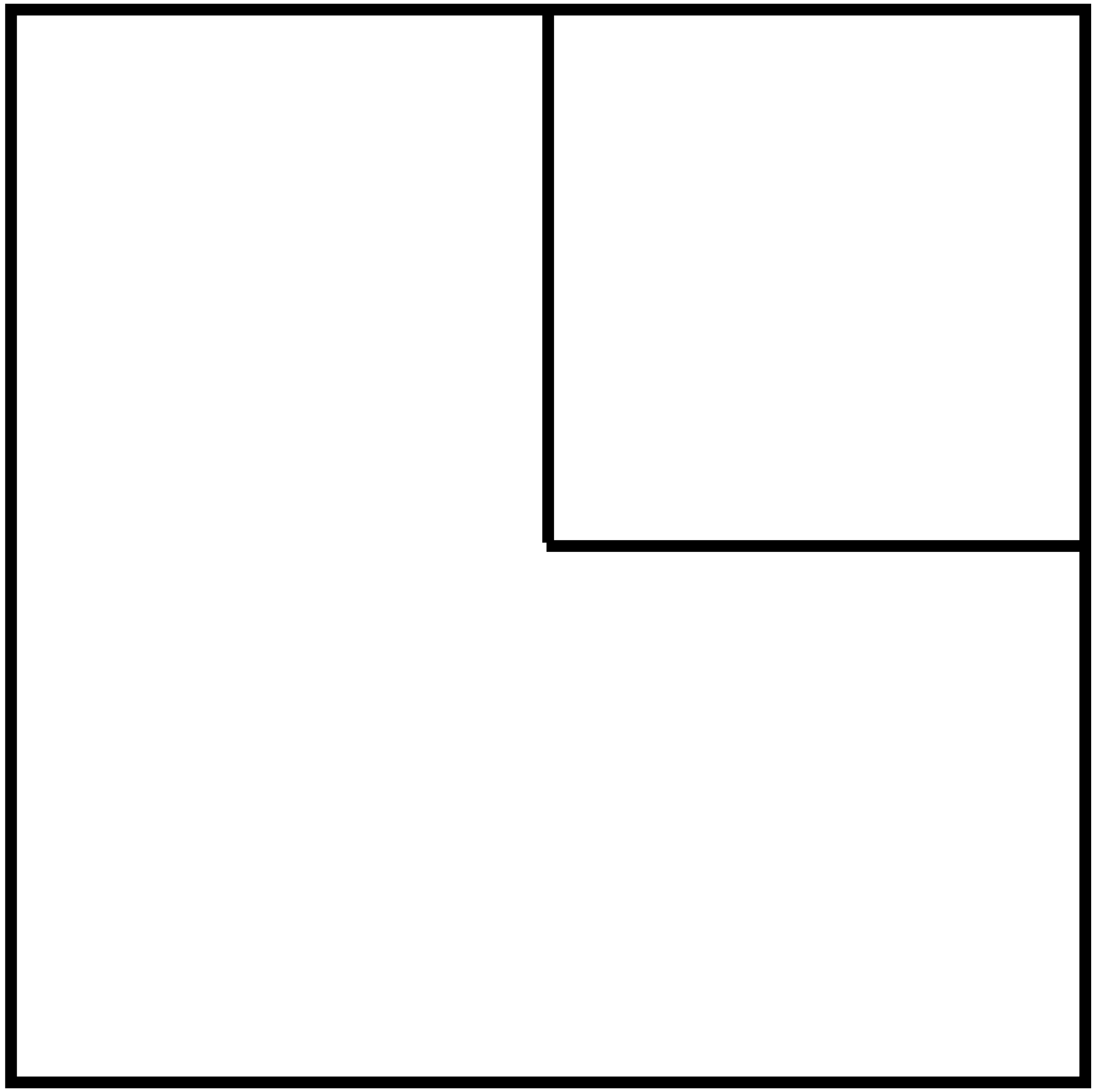

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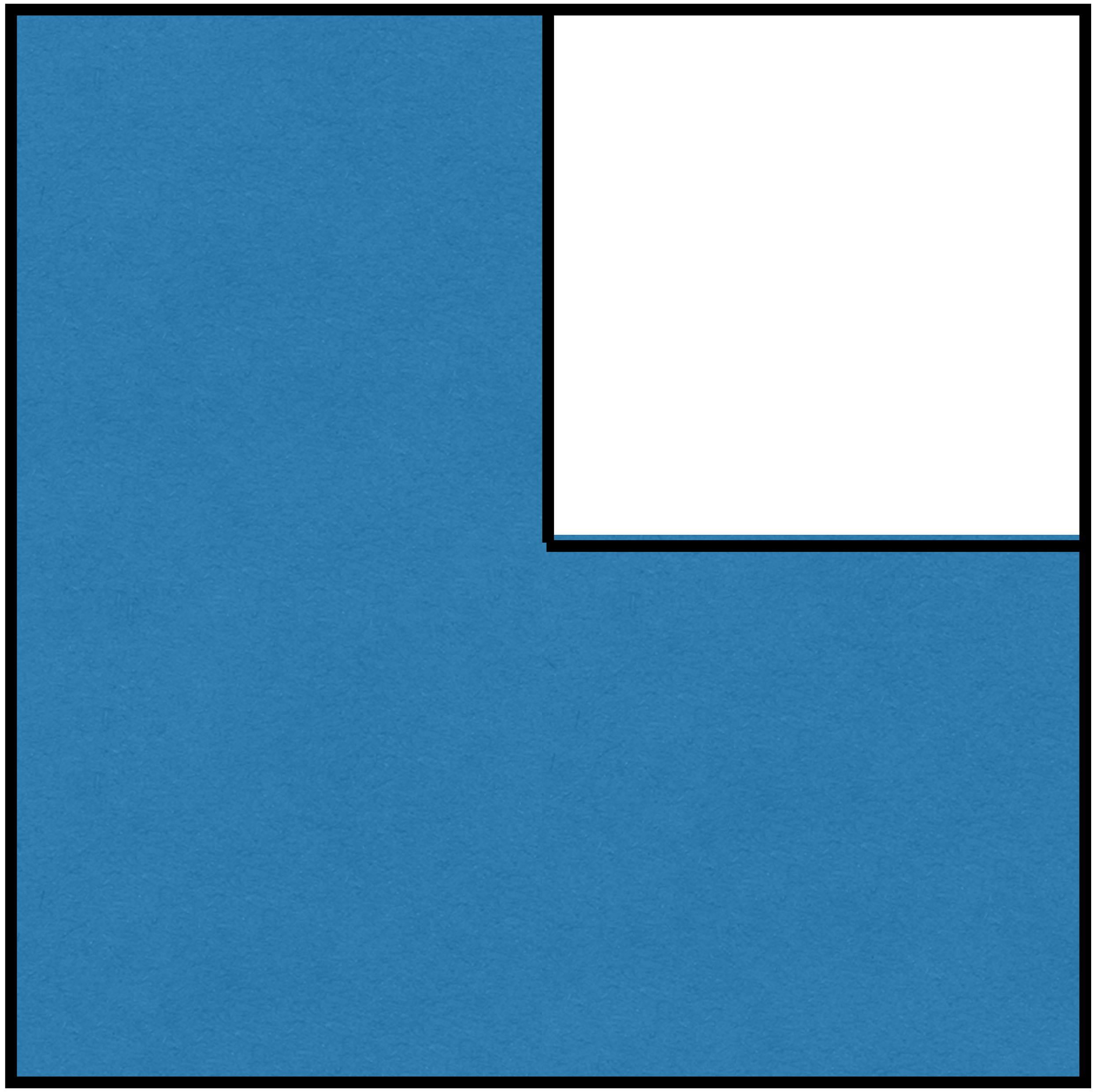

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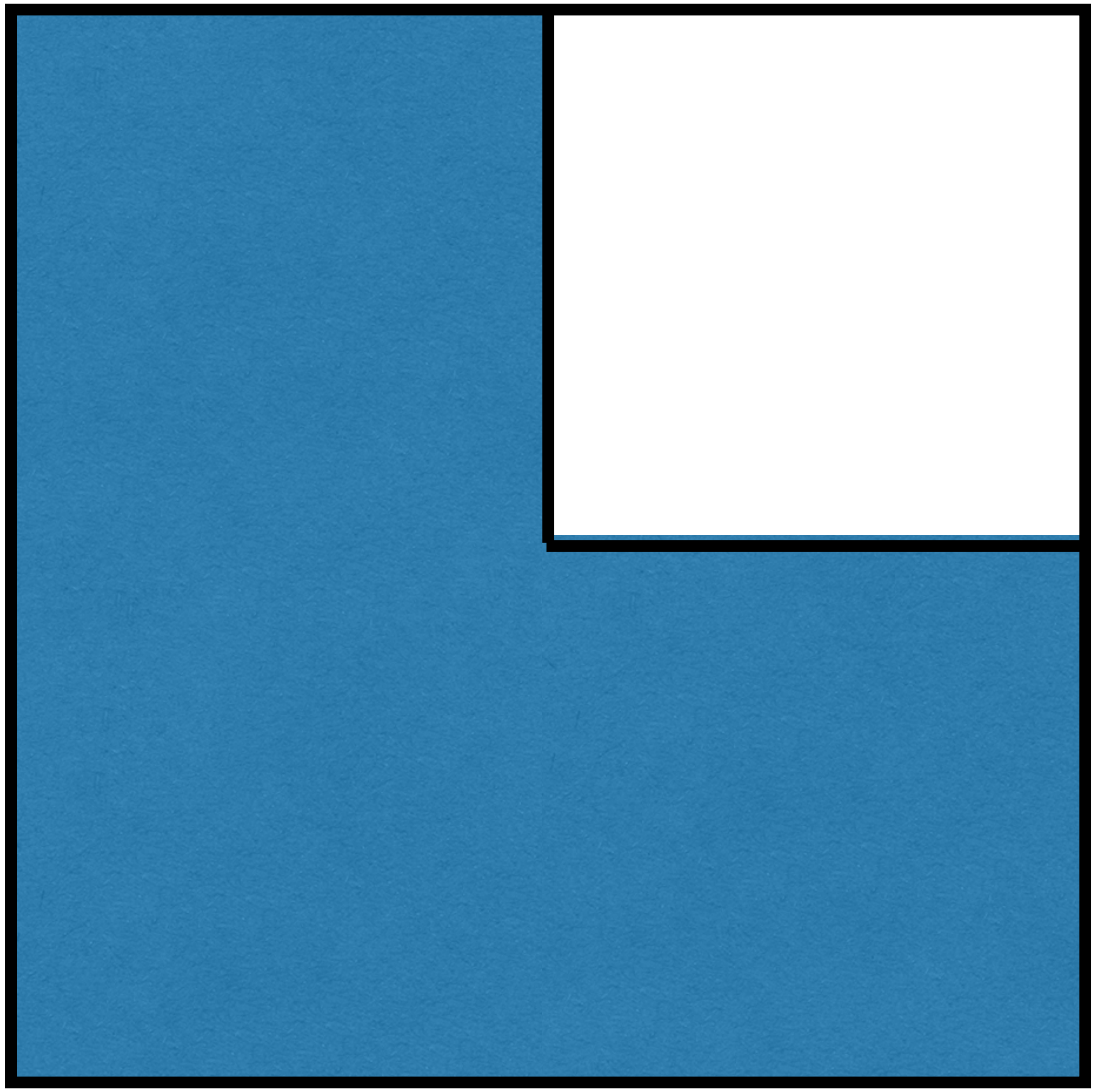
$$1 = \frac{3}{4}$$



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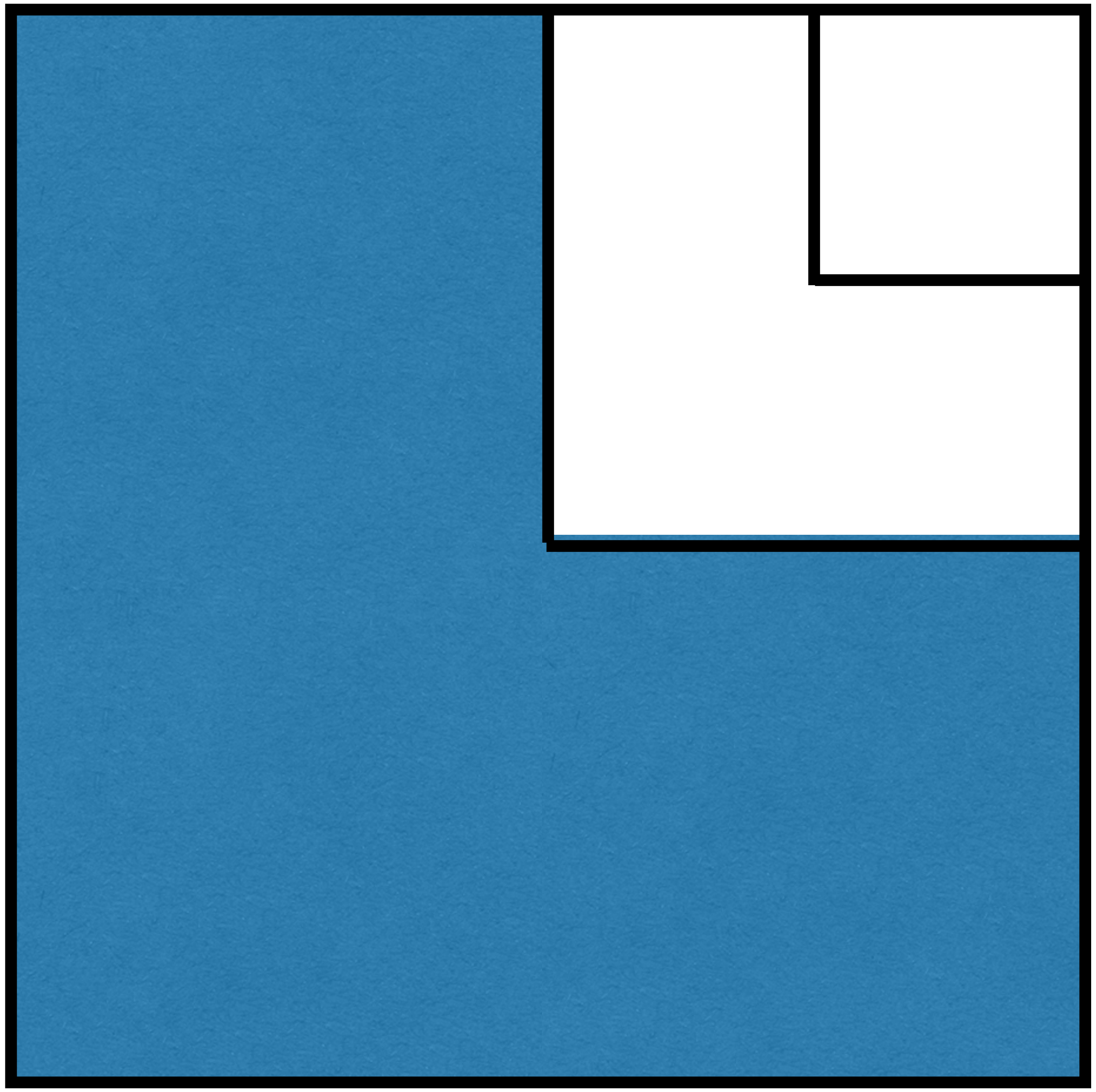

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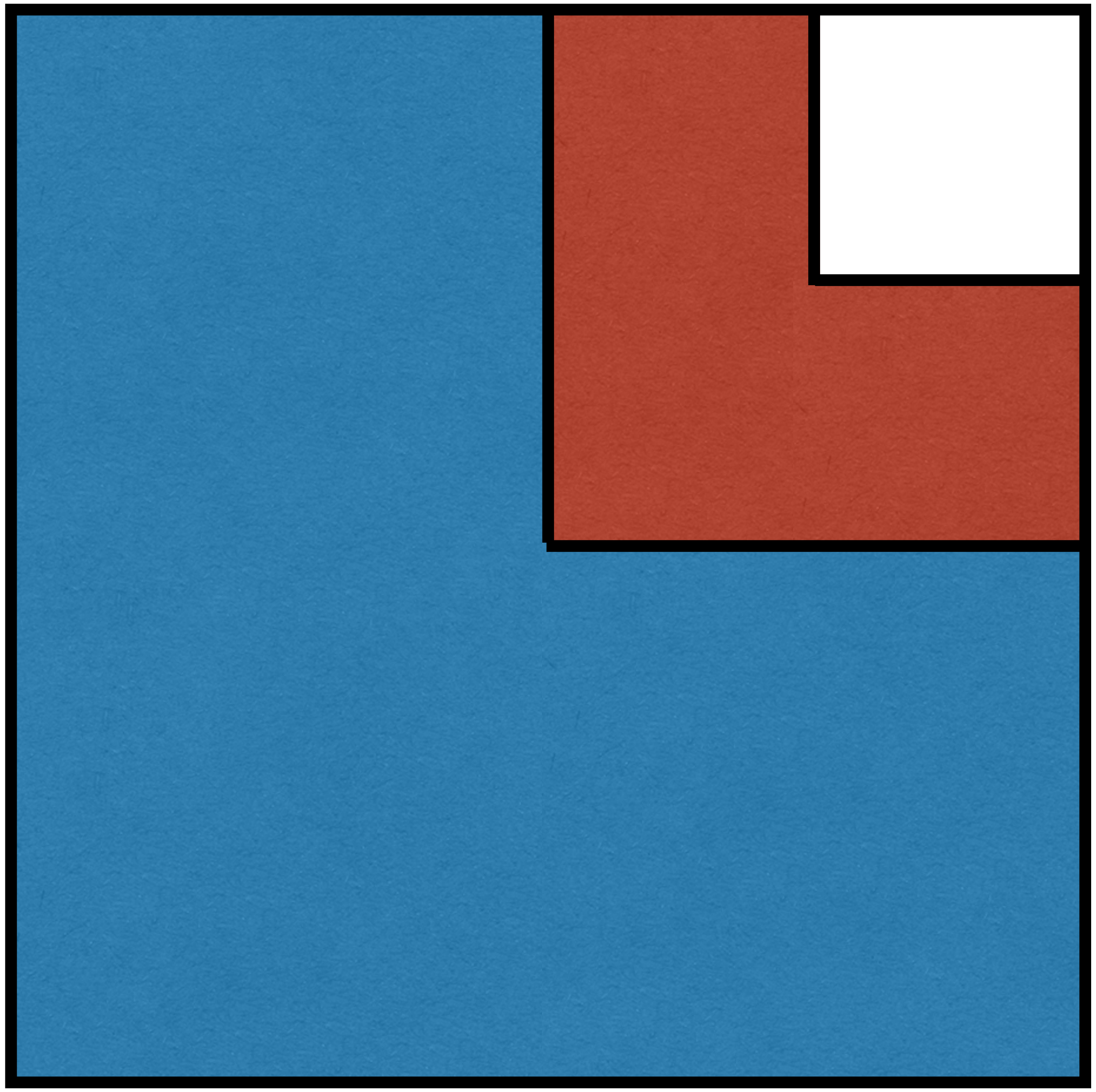

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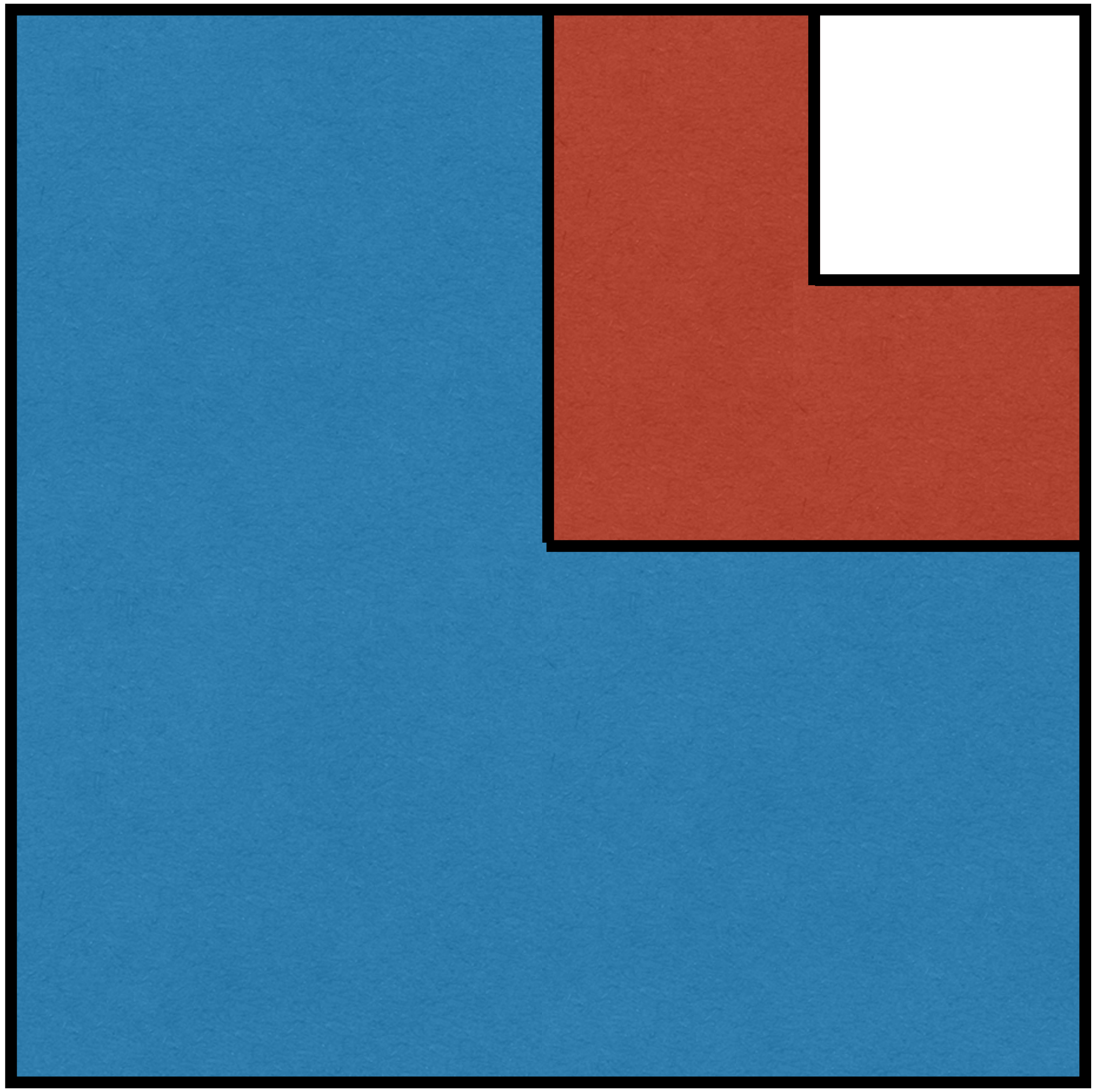

$$1 = \frac{3}{4} + \frac{3}{16}$$

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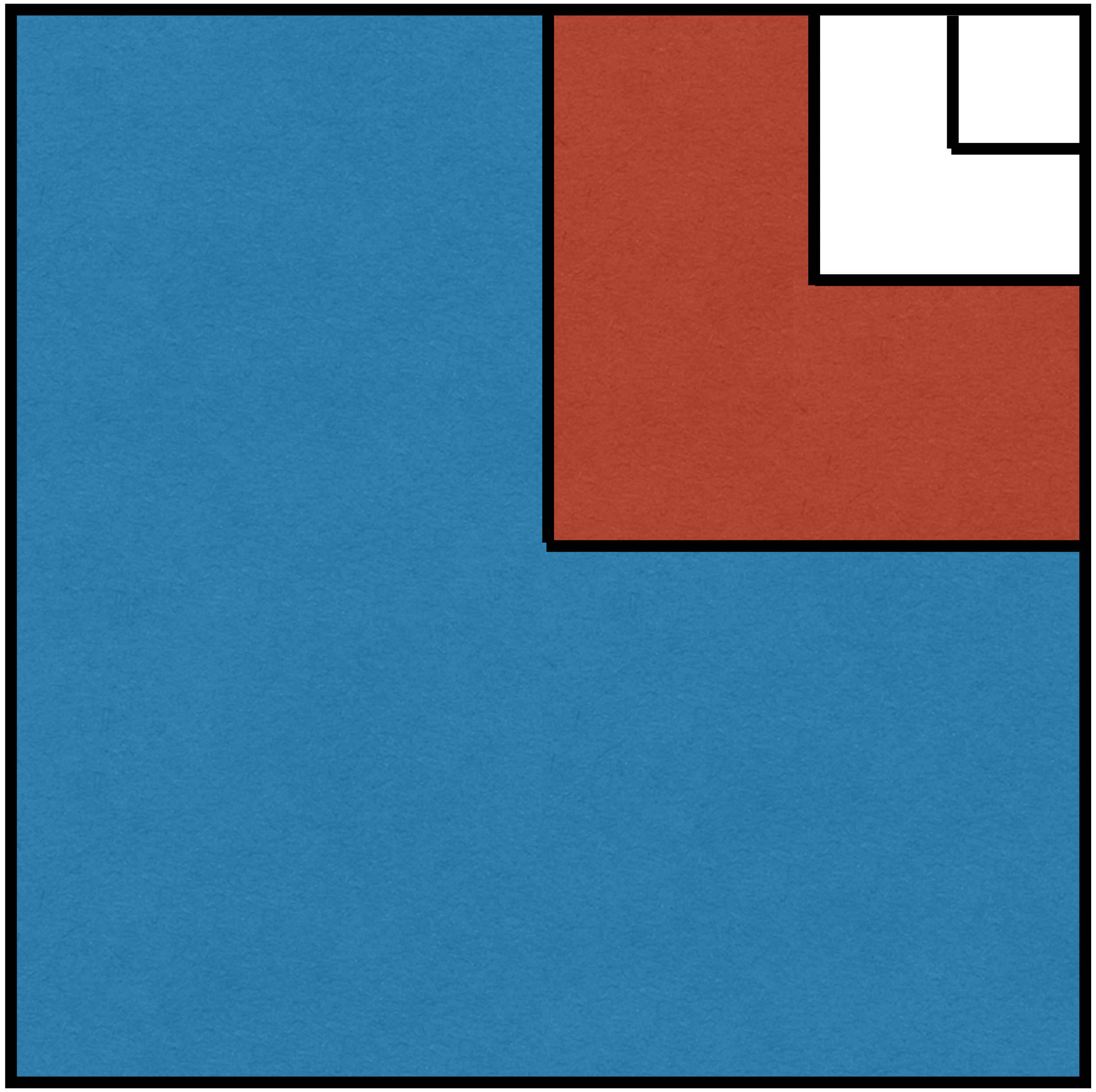

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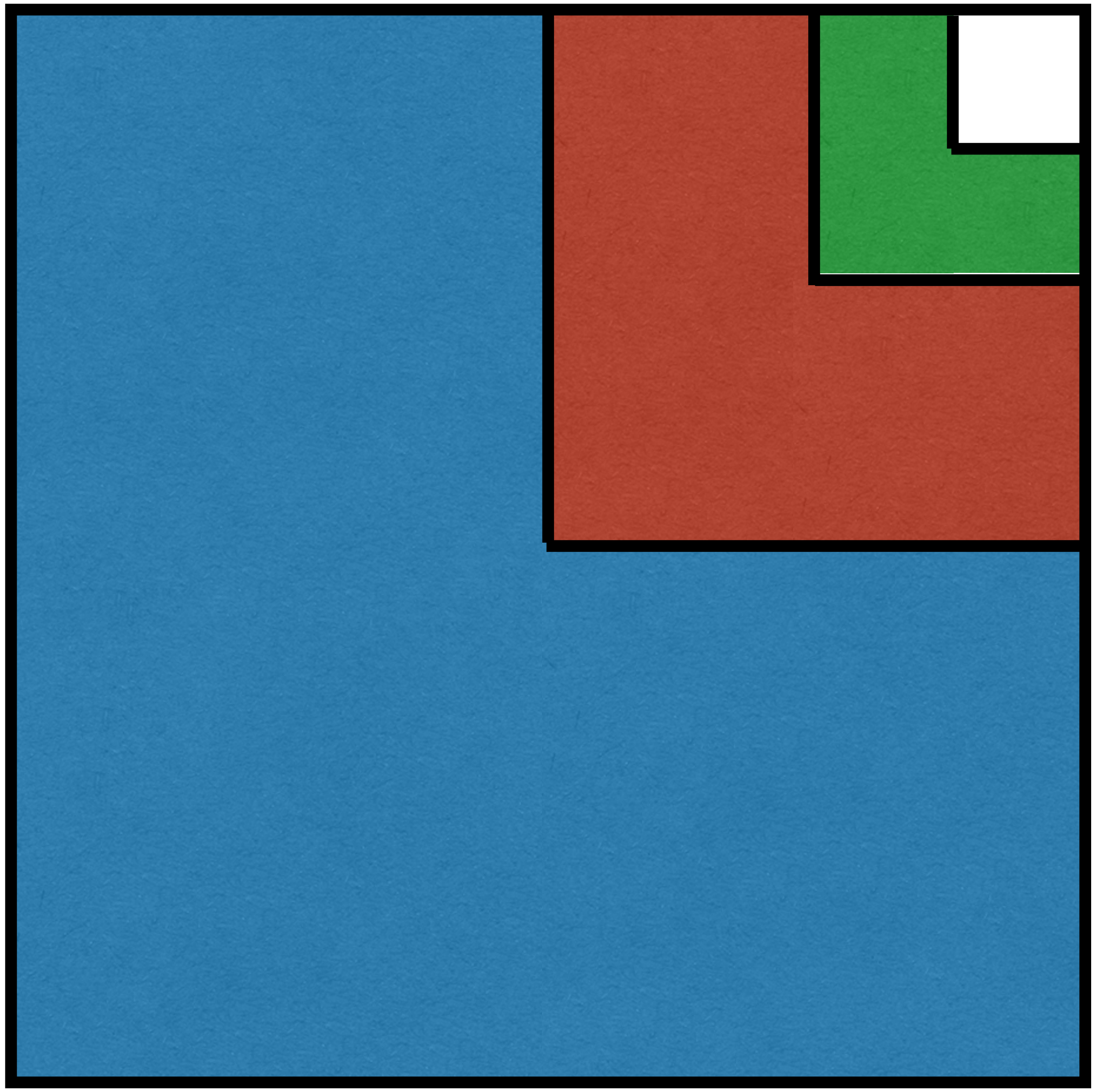

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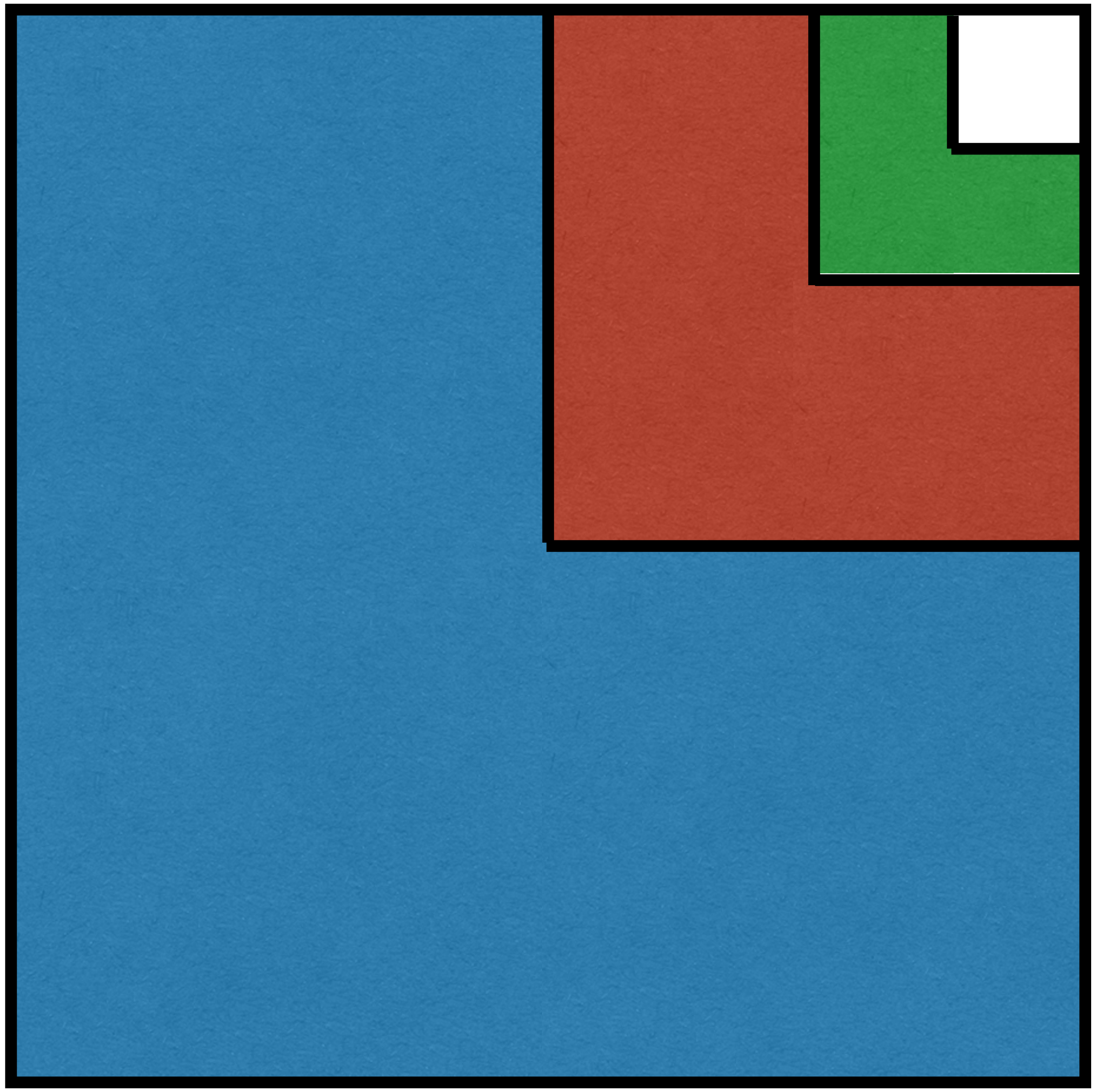

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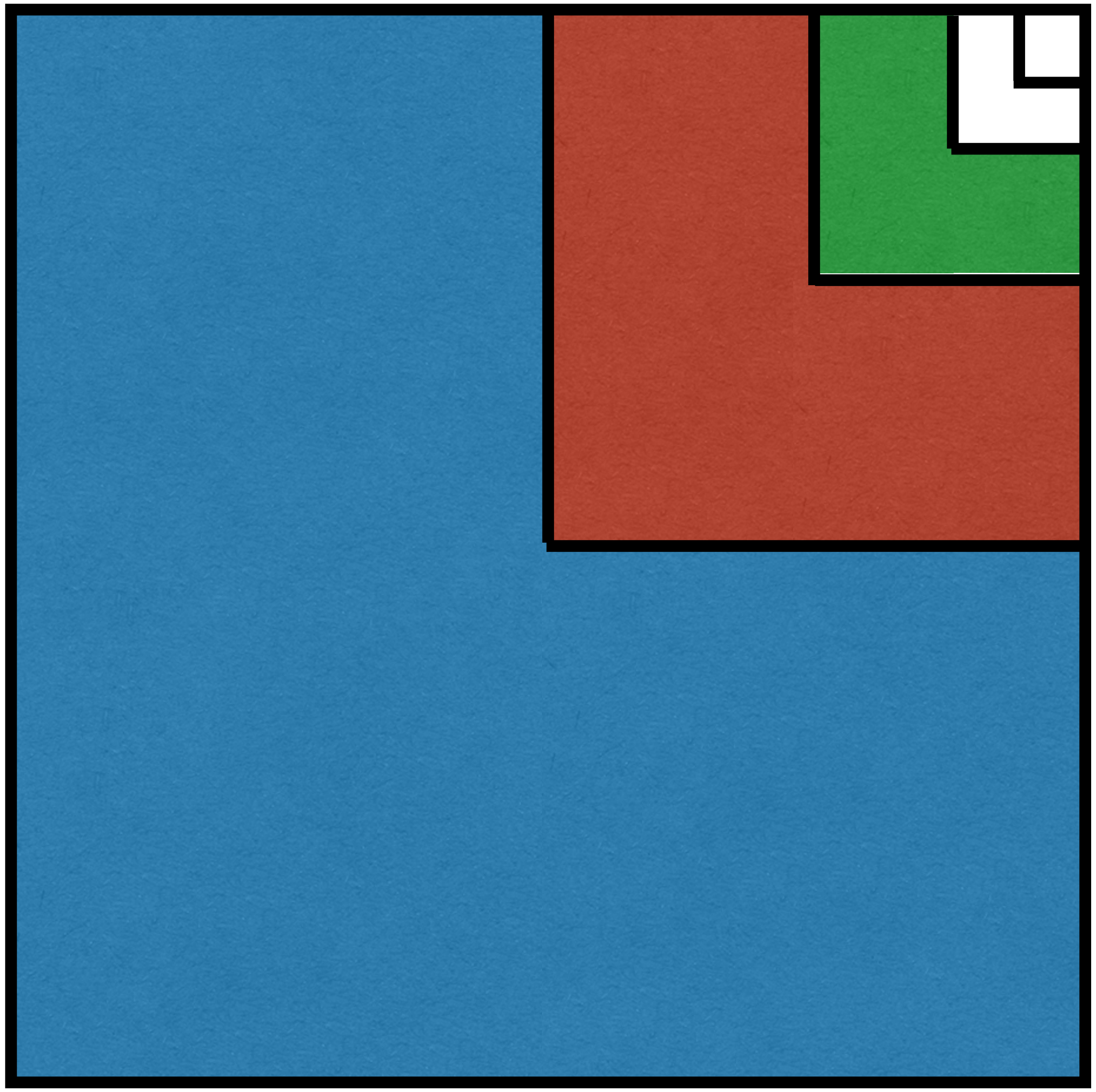

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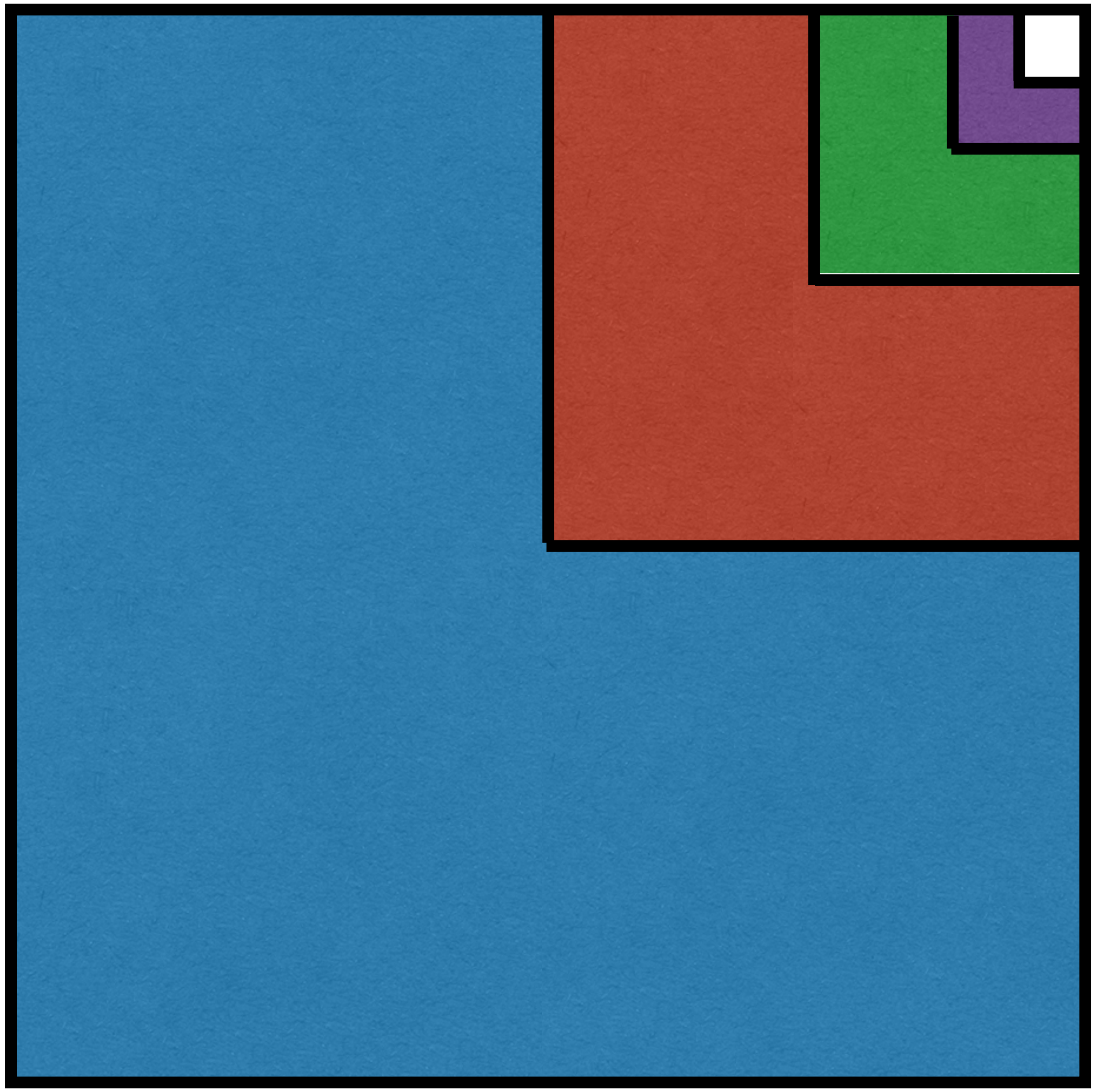

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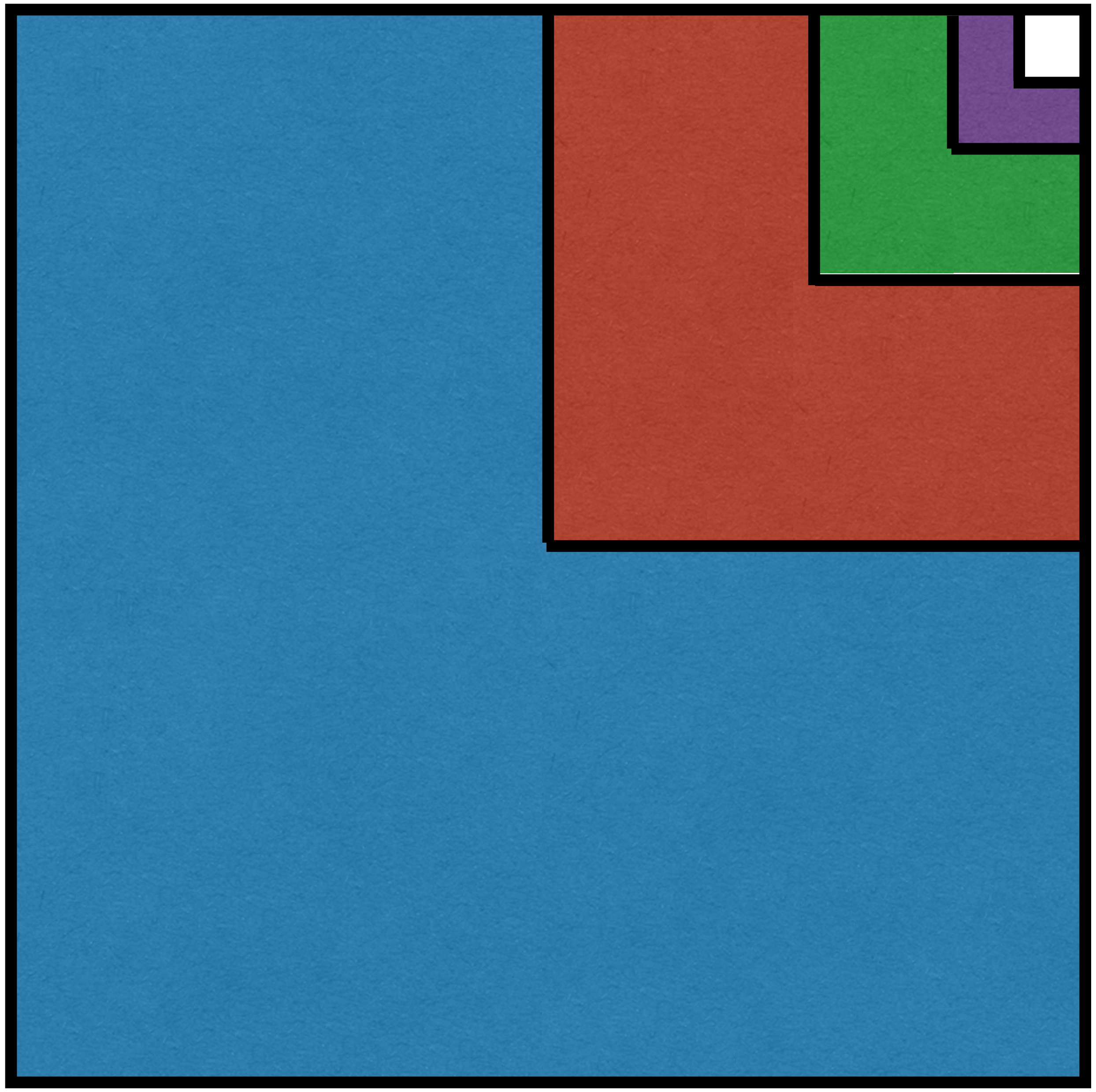

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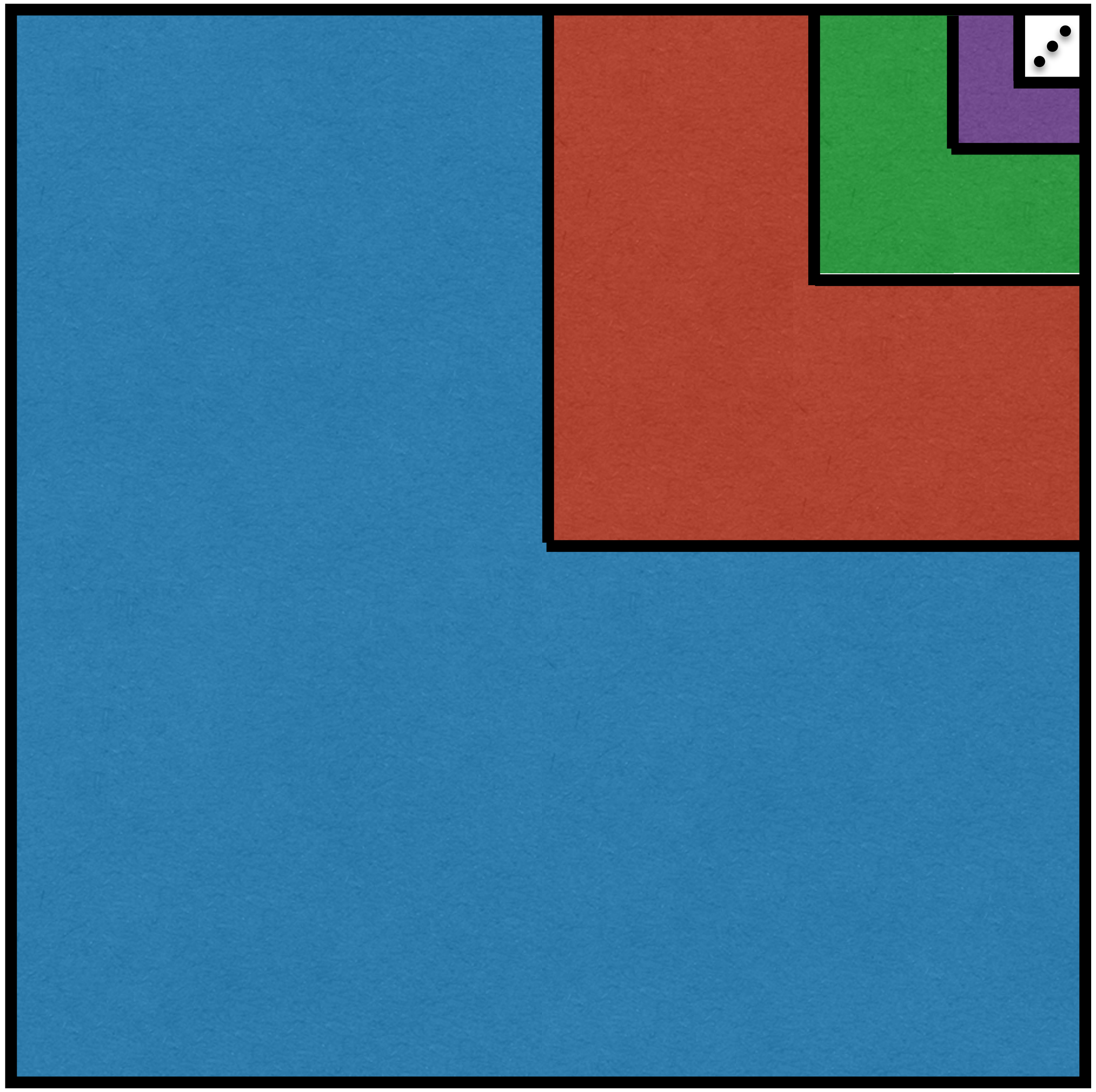

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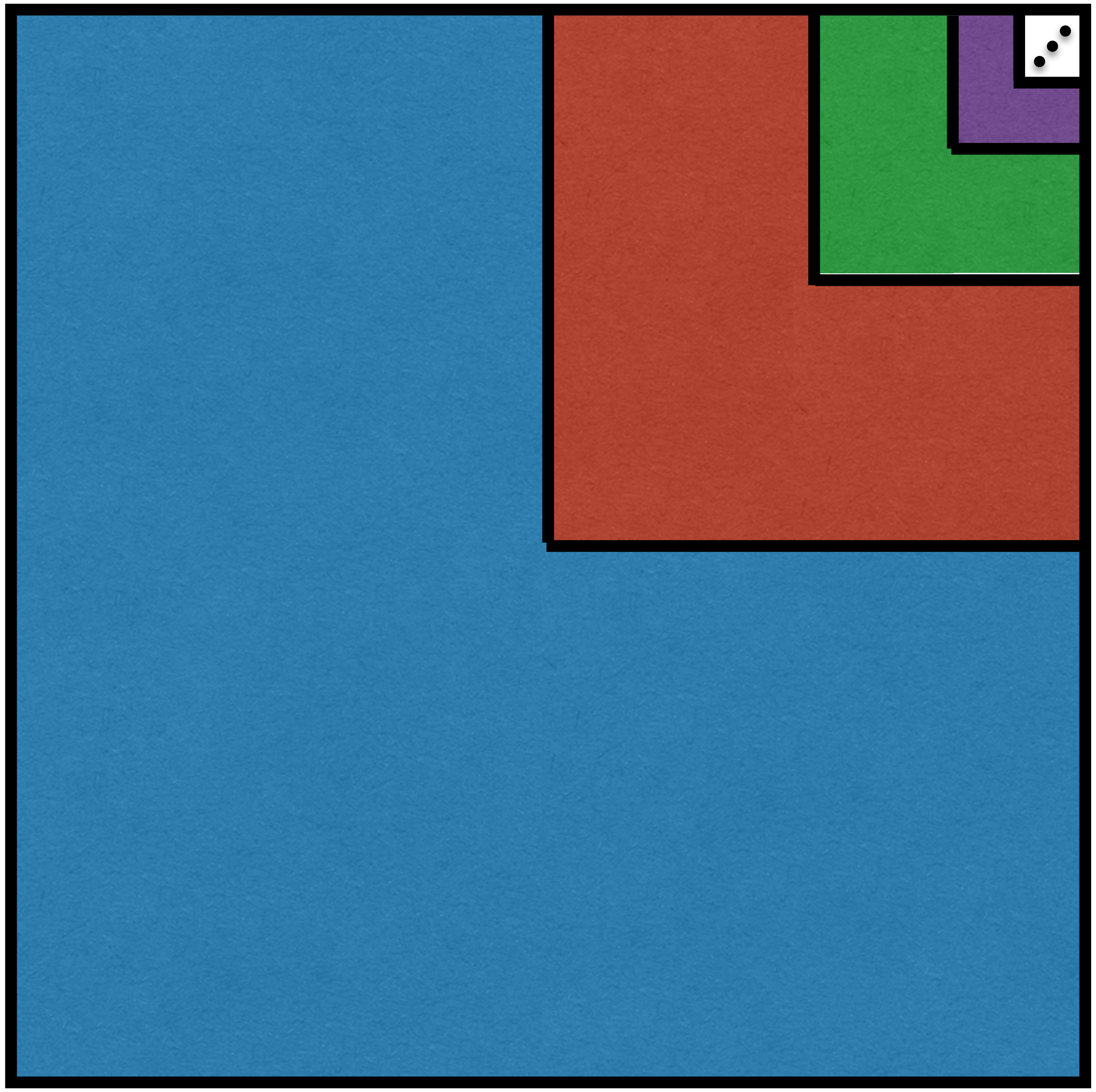

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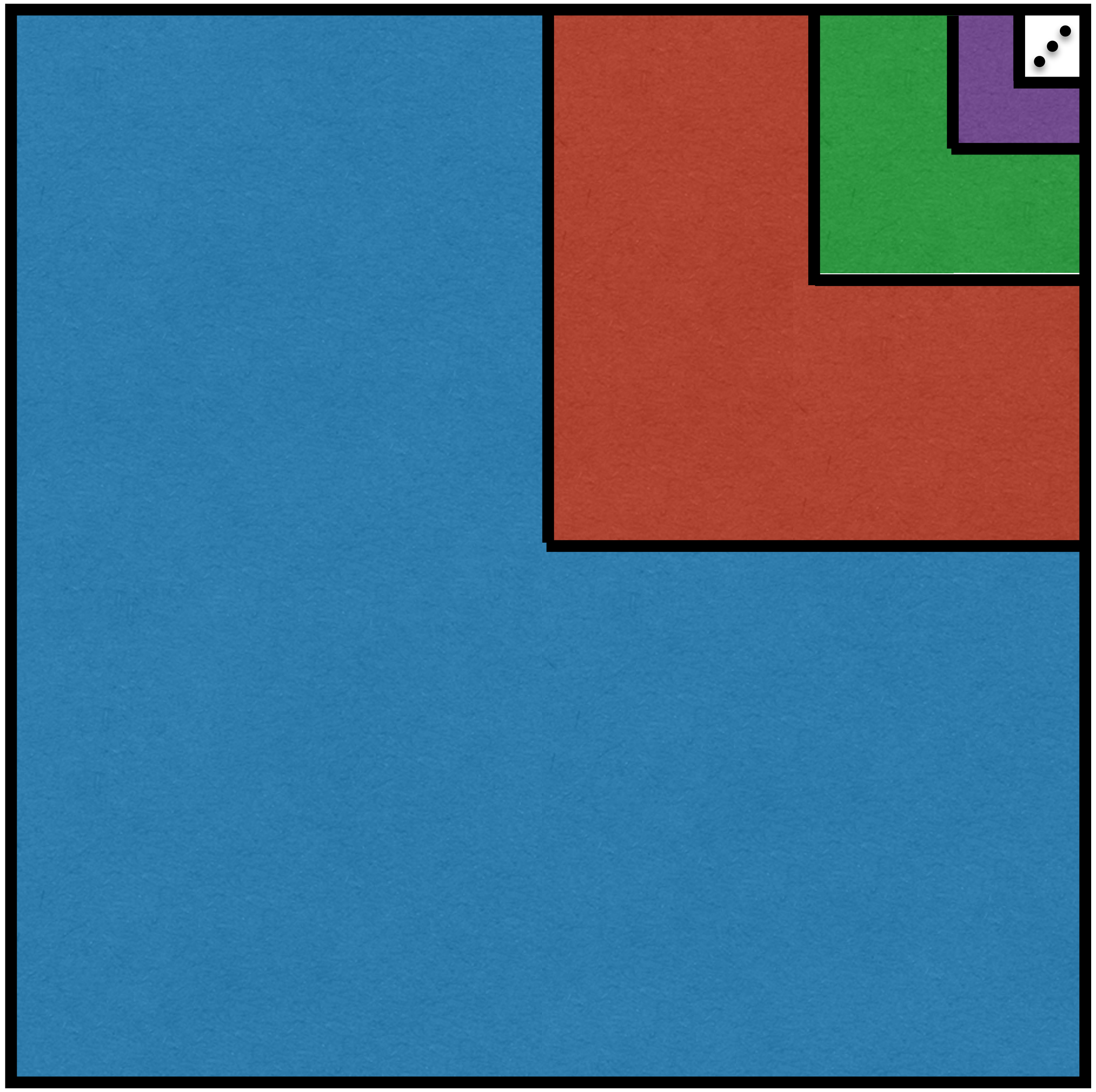
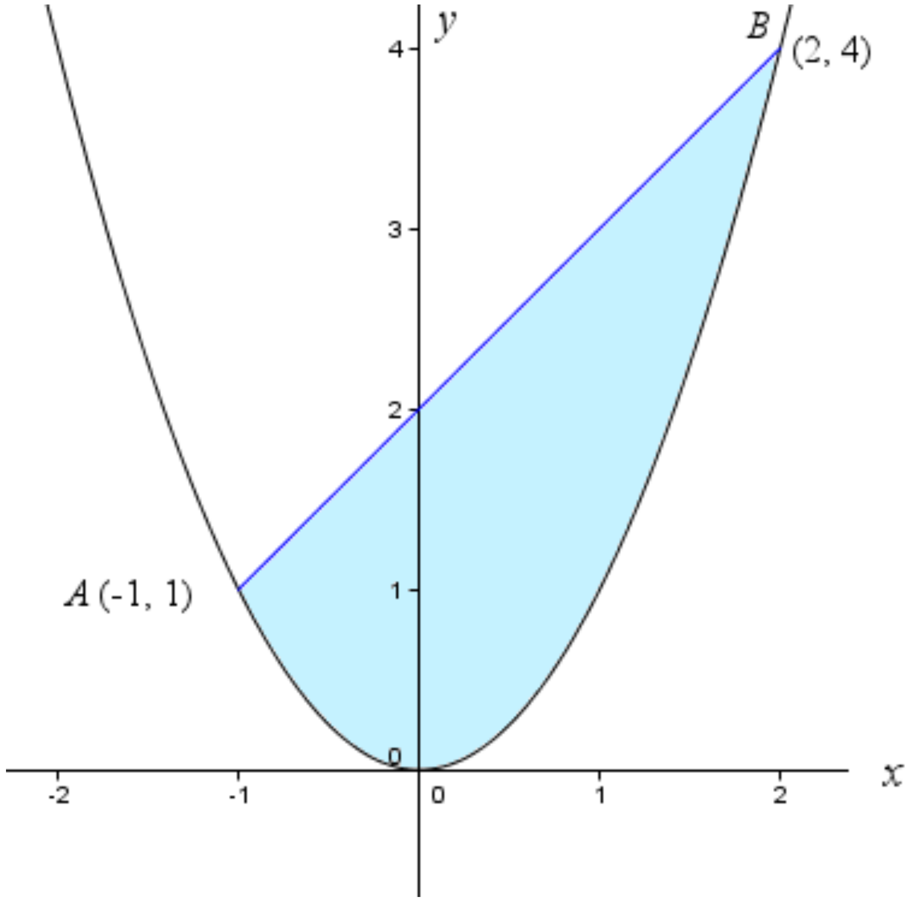
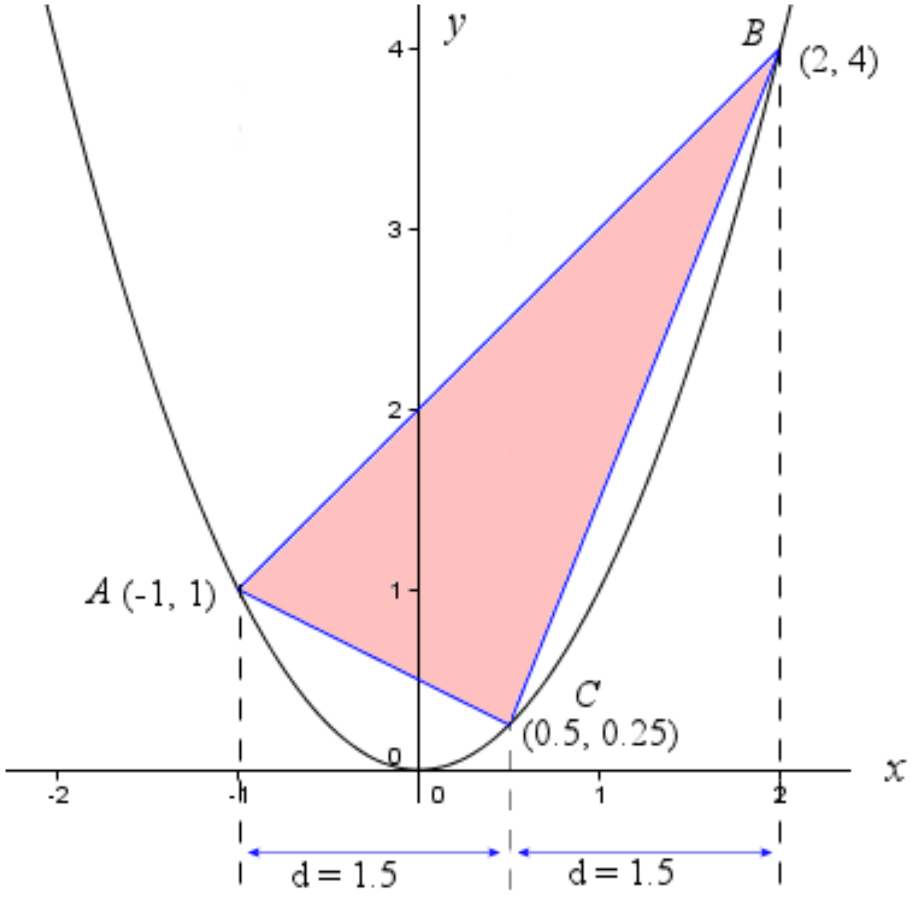


Divide both sides by 3
to get the result.

Archimedes

- Thus, the area between the line a parabola is
$$A + A\left(\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots\right) = \frac{4}{3}A,$$
where A is the area of that inscribed triangle.





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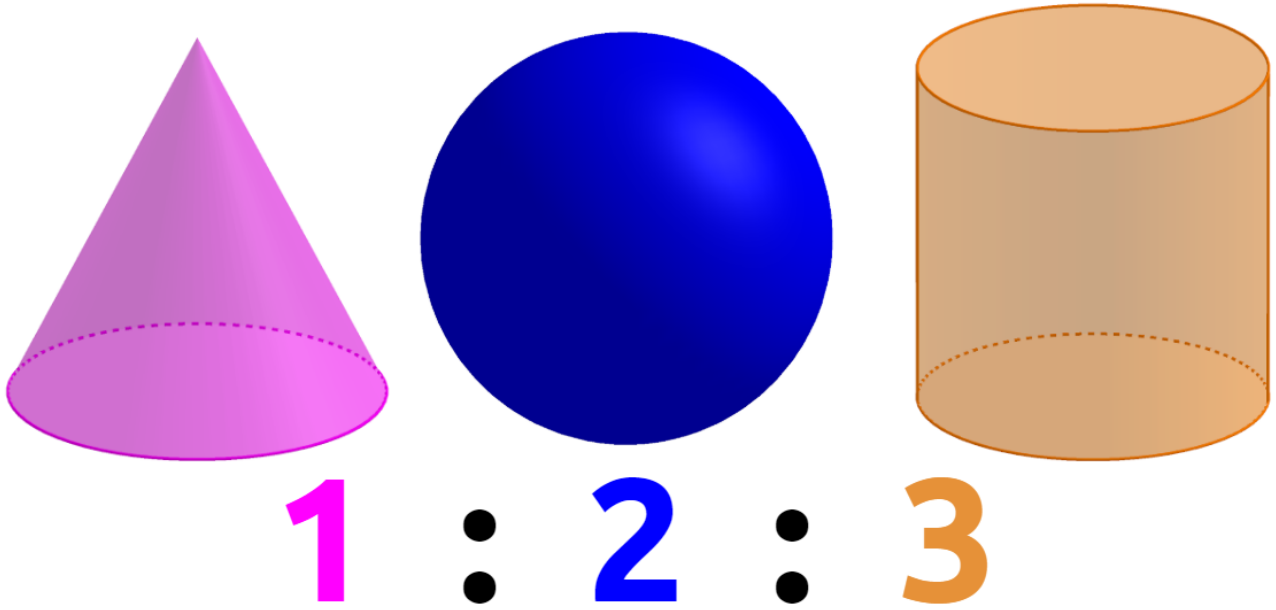

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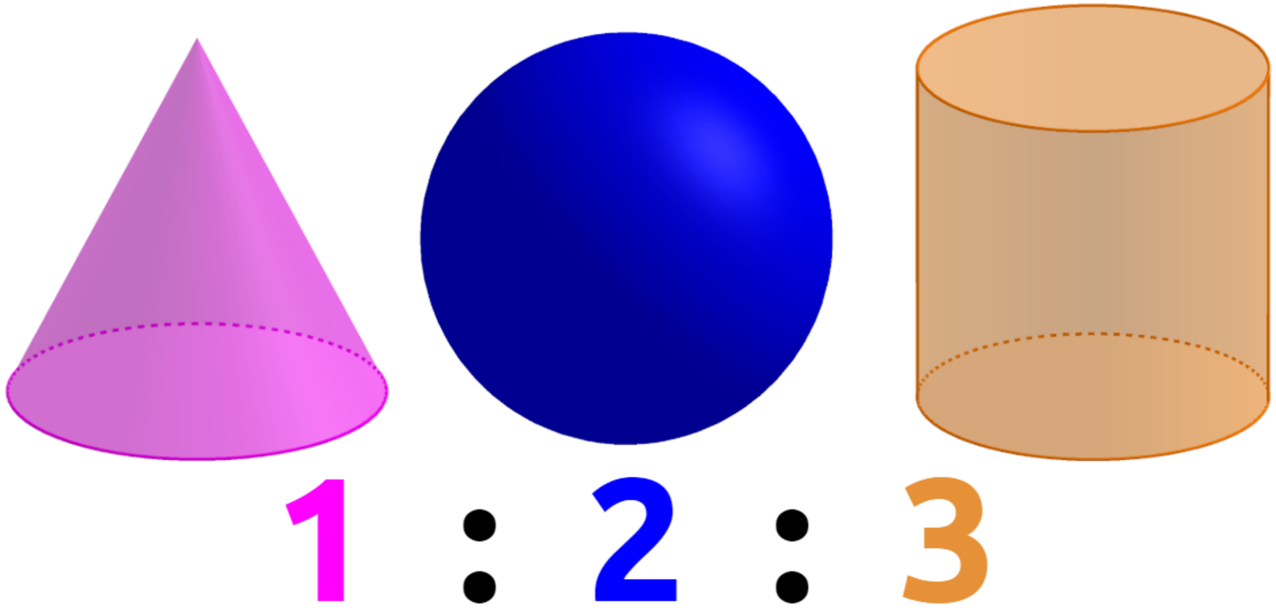
- He proved it using his "mechanical method."



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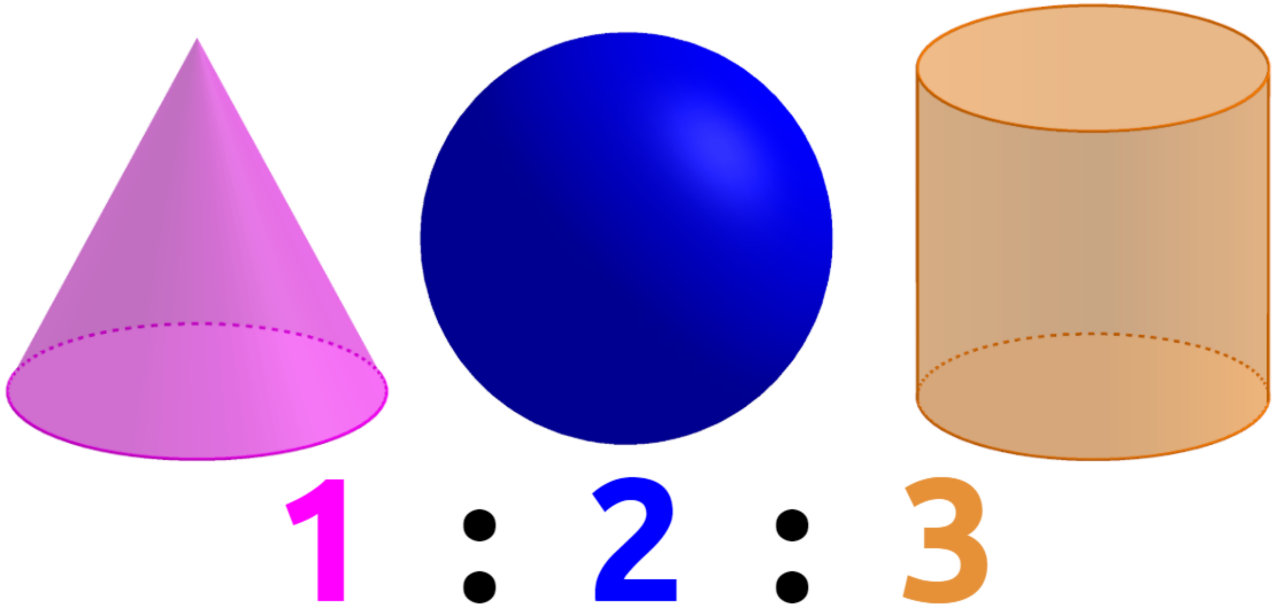

- He proved it using his “mechanical method.”

- He asked that it be carved onto his tombstone, and it was.

Archimedes

Archimedes

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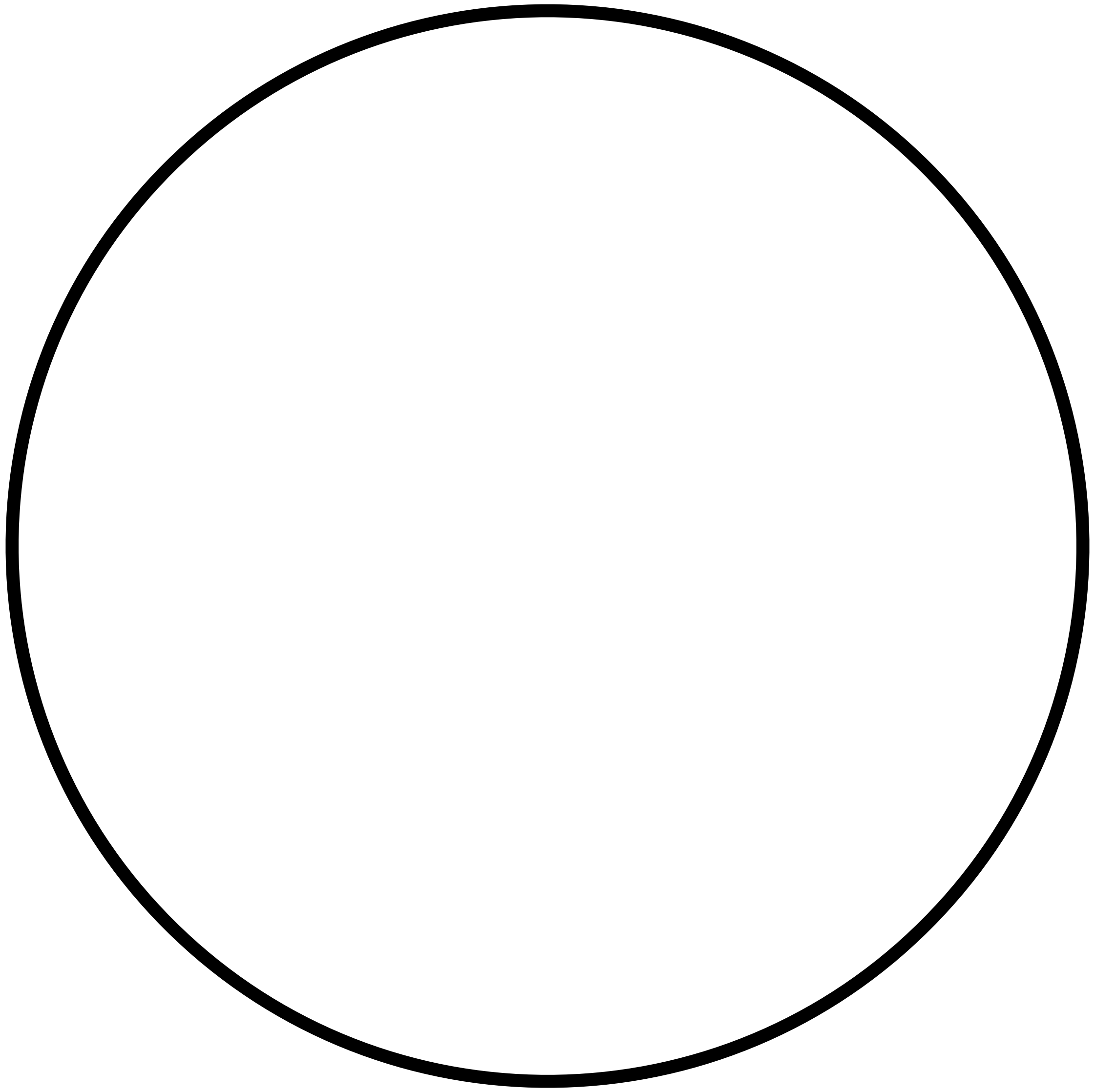
- After Archimedes it was developed independently in China by Liu Hui.

- But Archimedes truly mastered it and used it to find:
  —Area of a circle, ellipse, spiral and parabolic region;
  —Volume of a sphere, and some solids of revolution;
  —The surface area of a sphere.

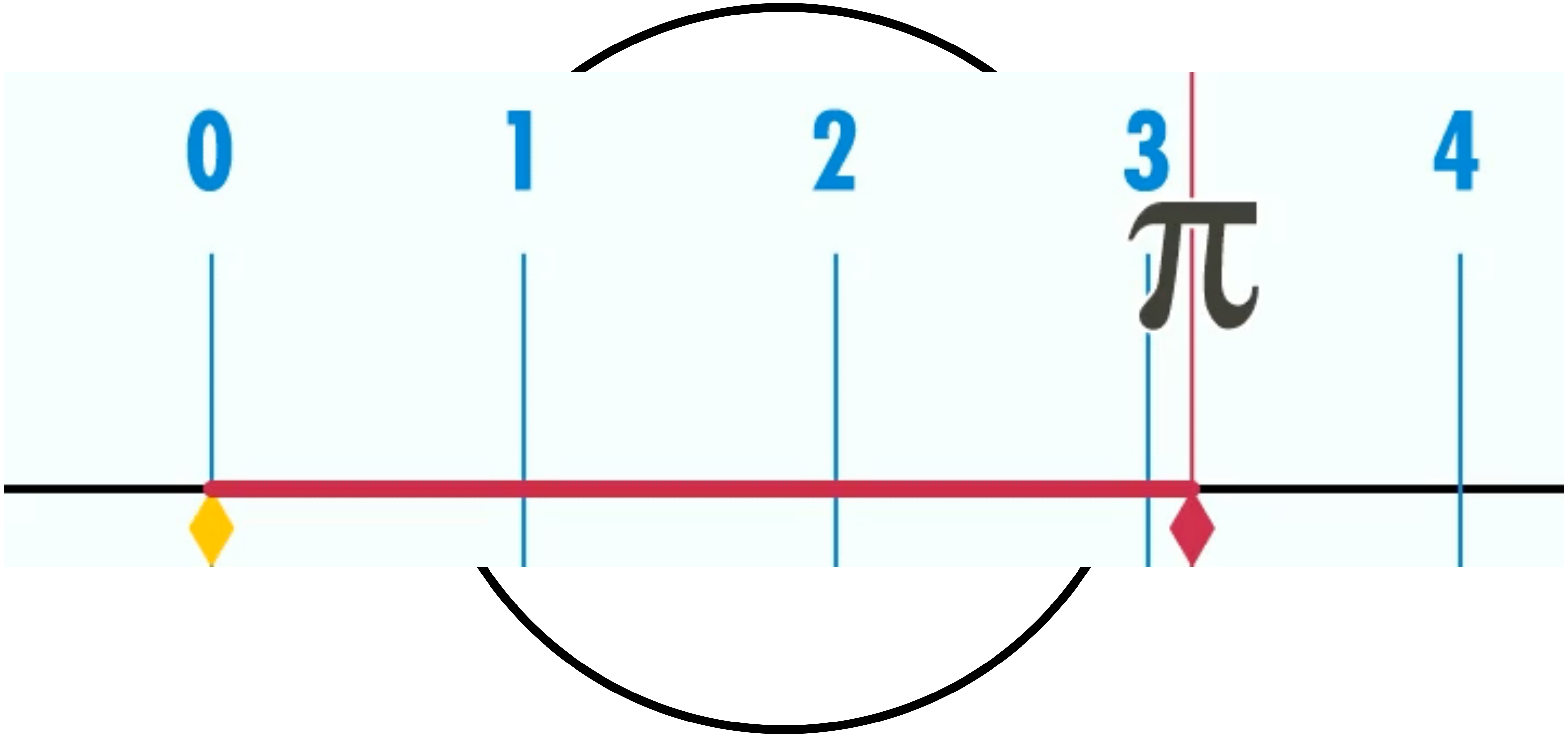
Archimedes

Finding $\boldsymbol{\pi}$: The circumference of a circle of diameter 1



Archimedes

Finding π



Archimedes

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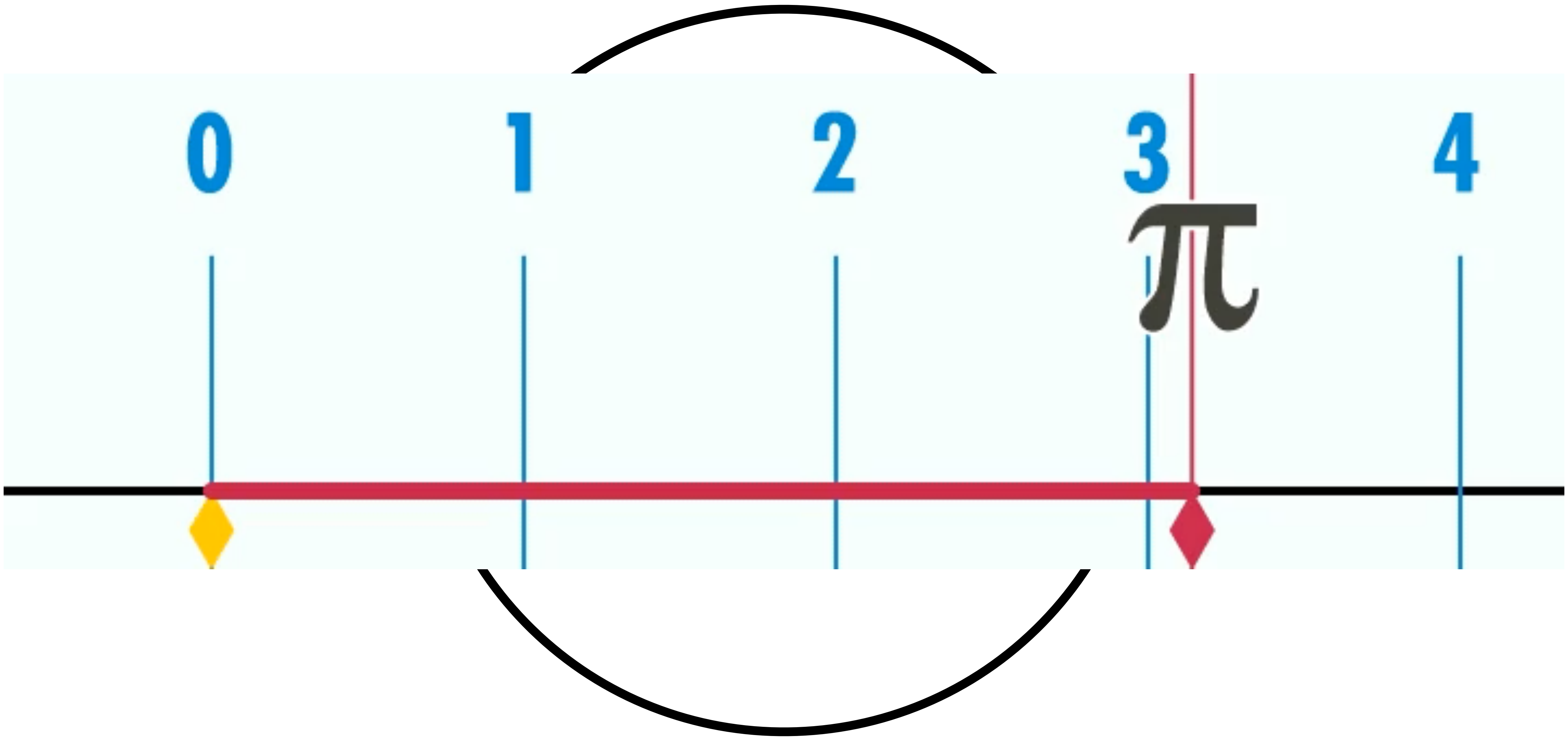

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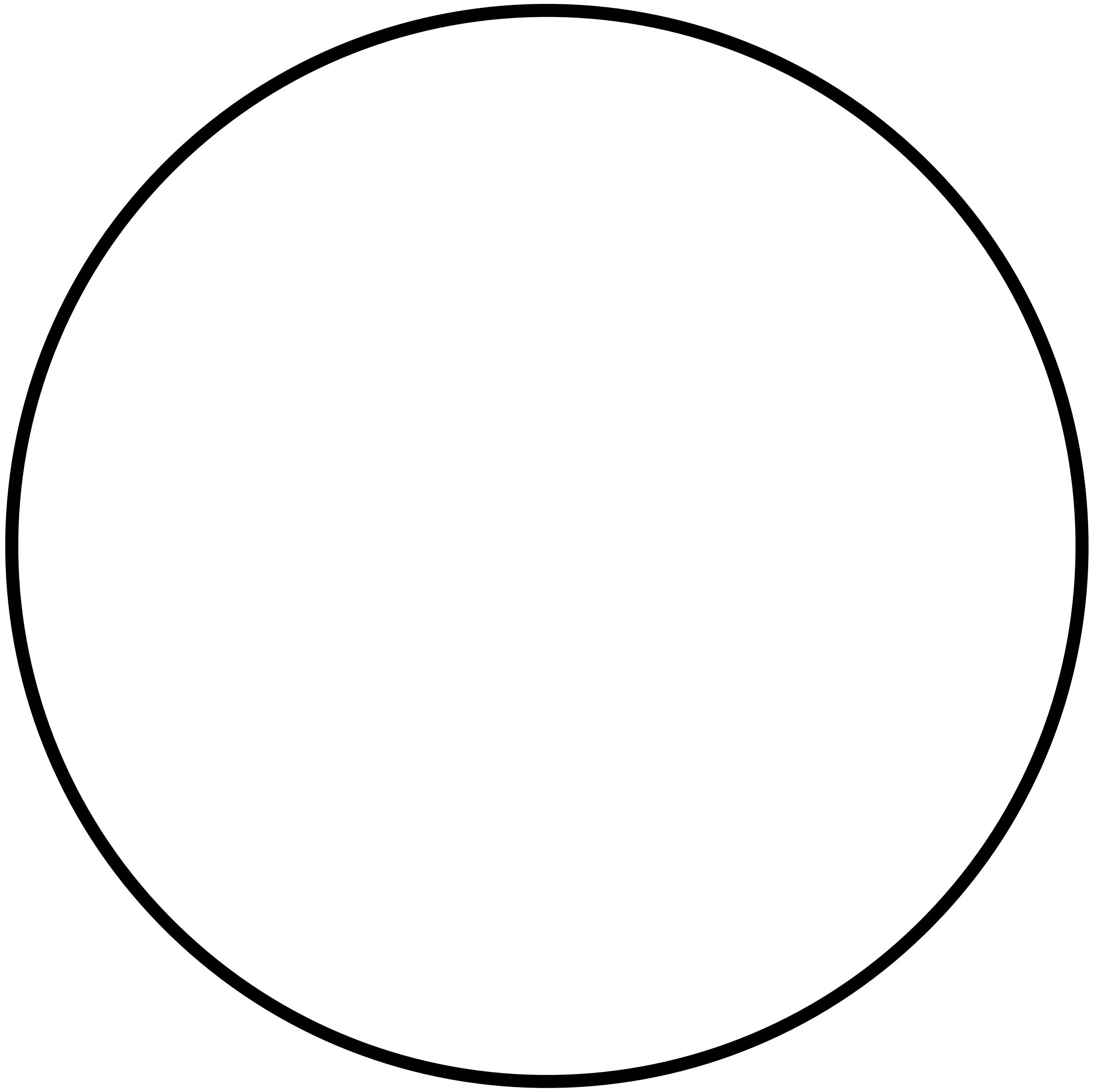
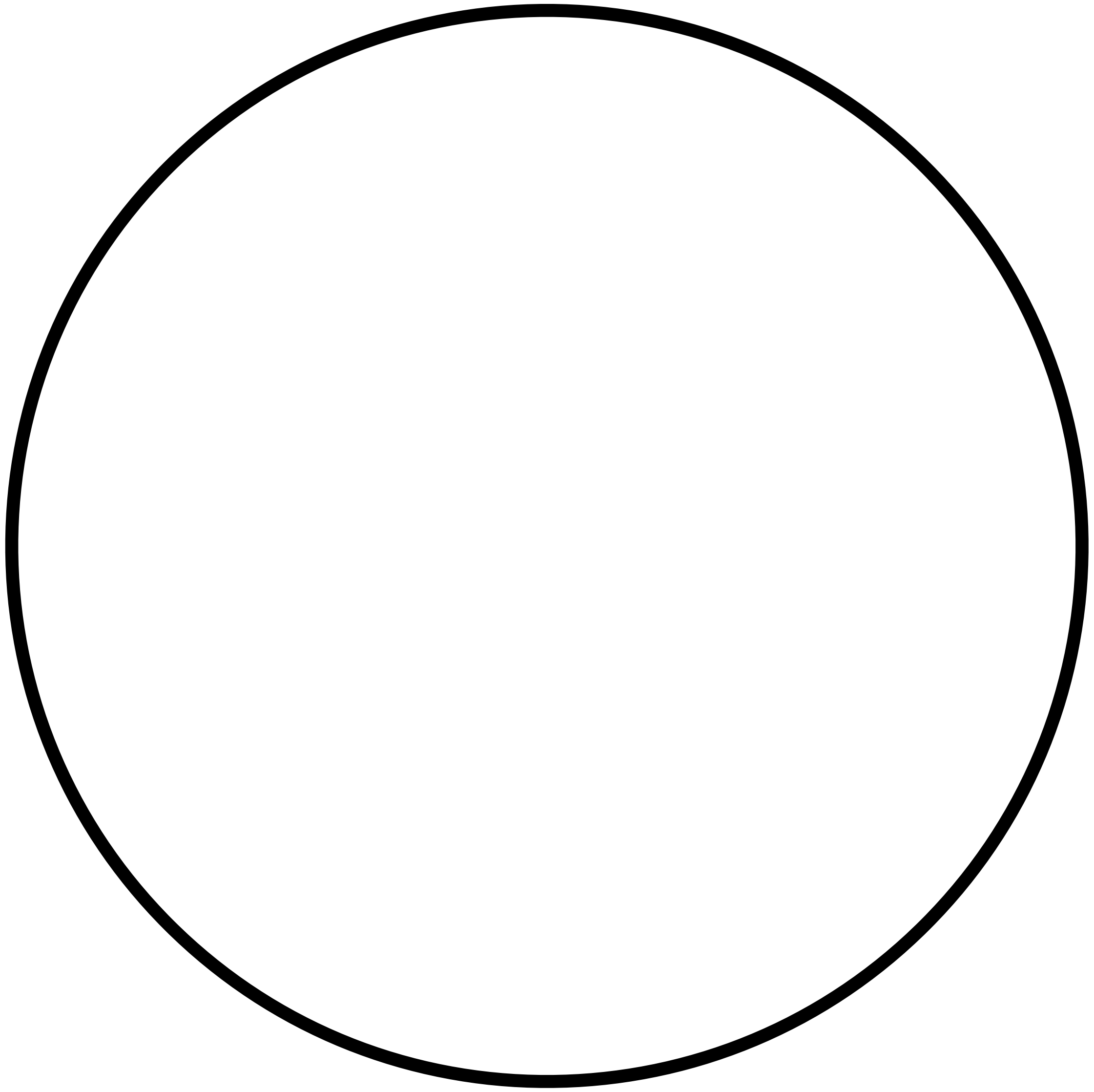
Finding π



Archimedes

Finding $\boldsymbol{\pi}$ =3.14159…



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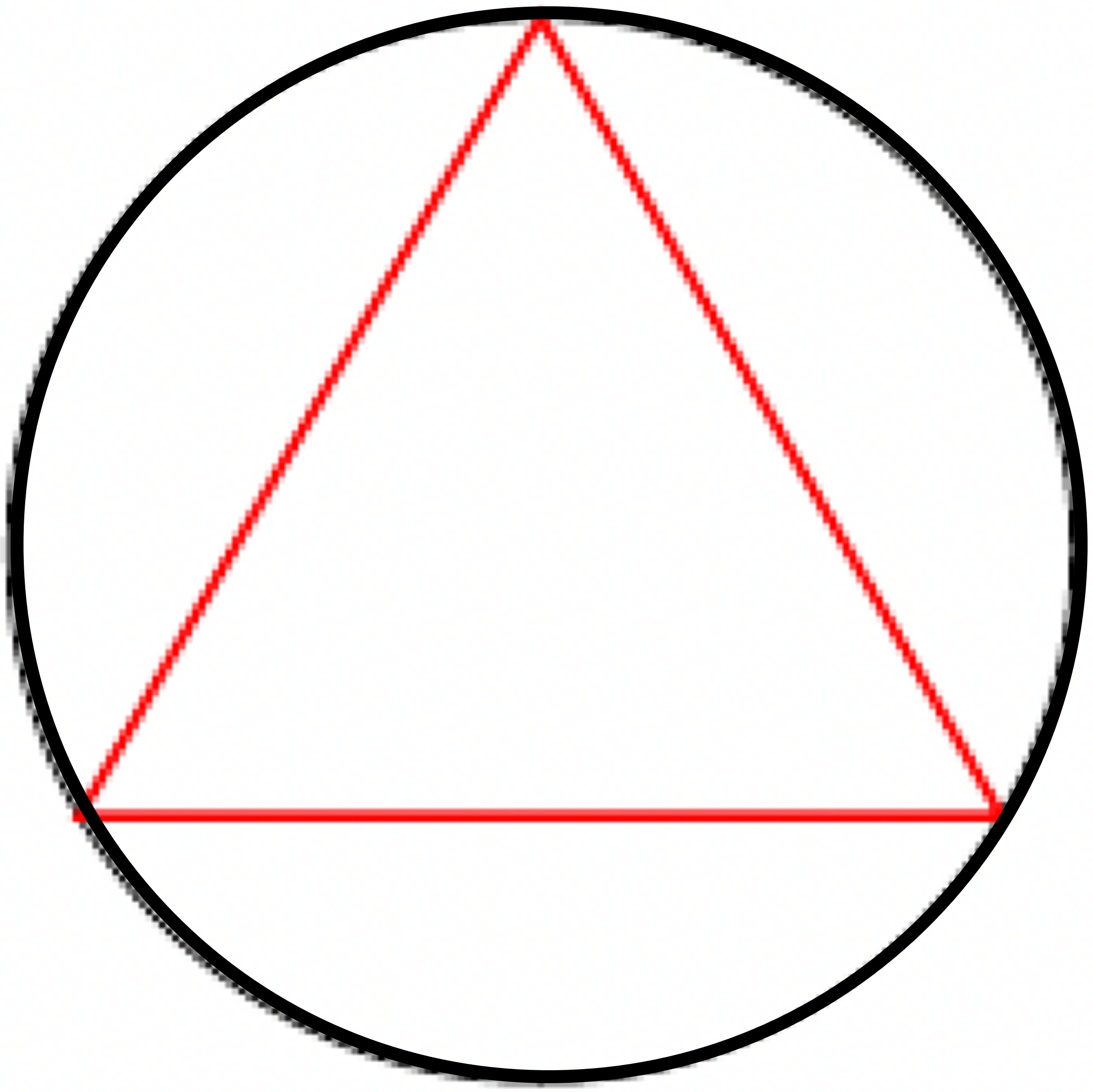
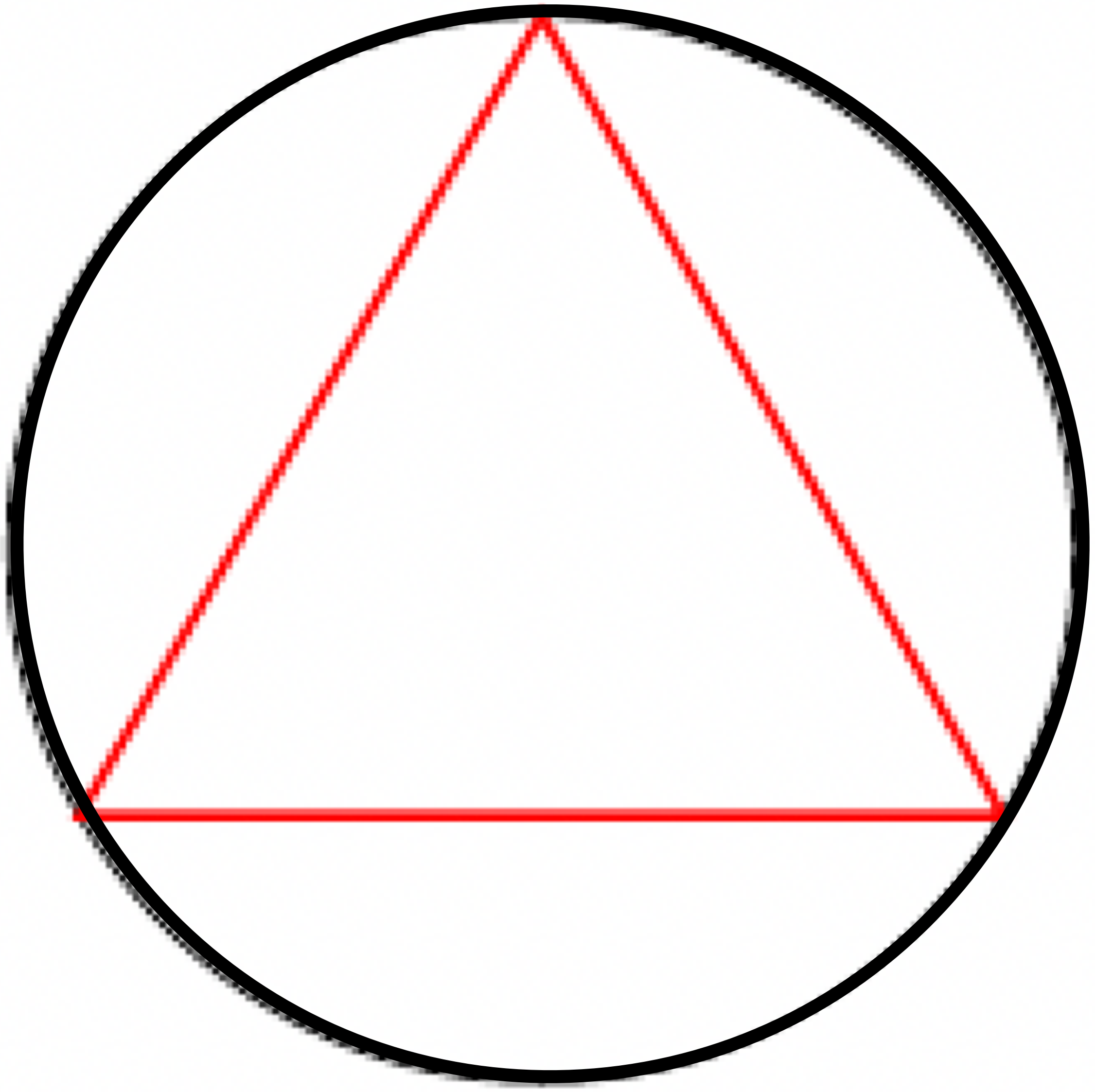


Sides = 3

Perimeter =

Archimedes

Finding π =3.14159…



Sides = 3

Perimeter = 2.598

Archimedes

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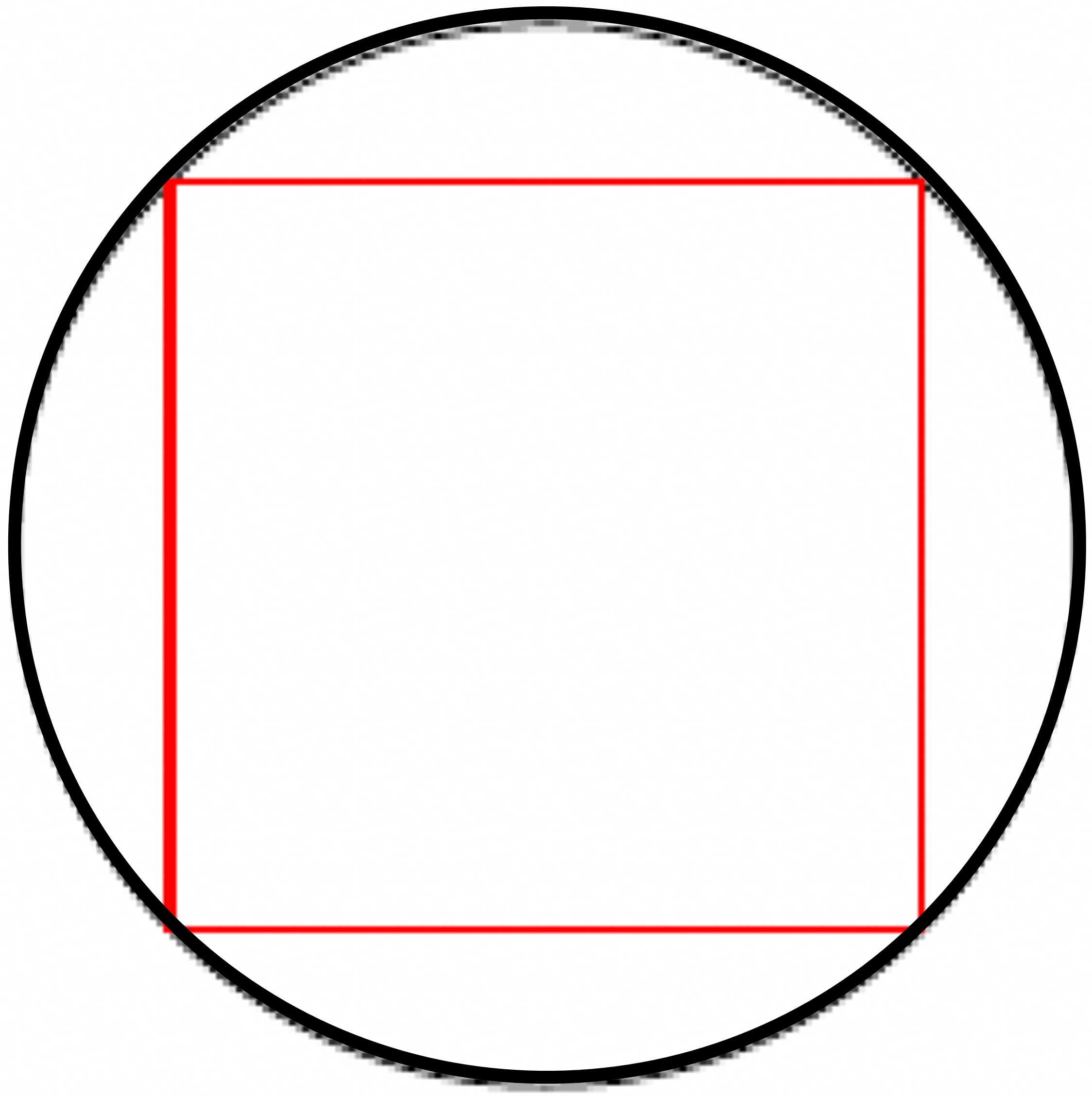
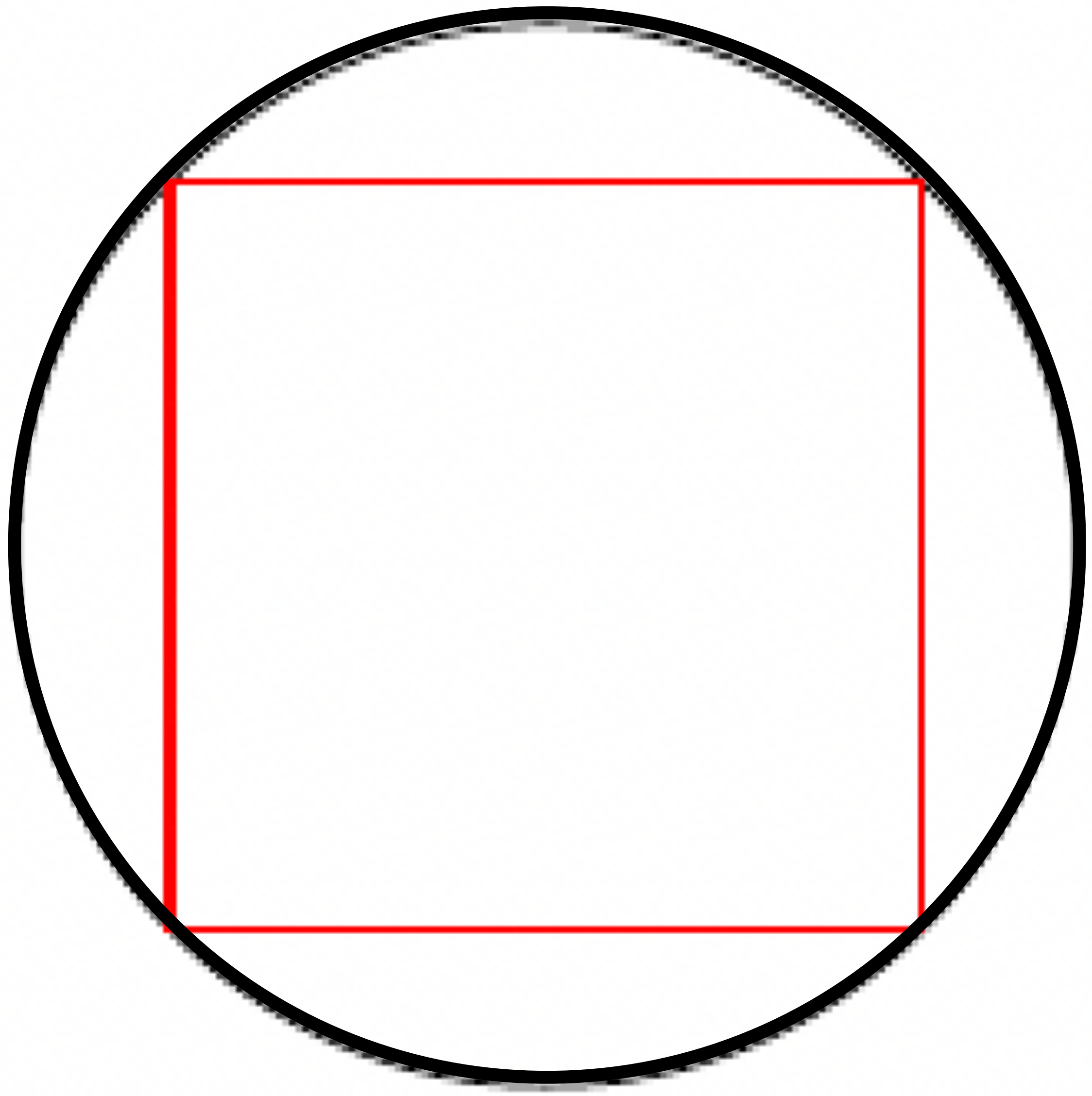


Sides = 4

Perimeter =

Archimedes

Finding π =3.14159...

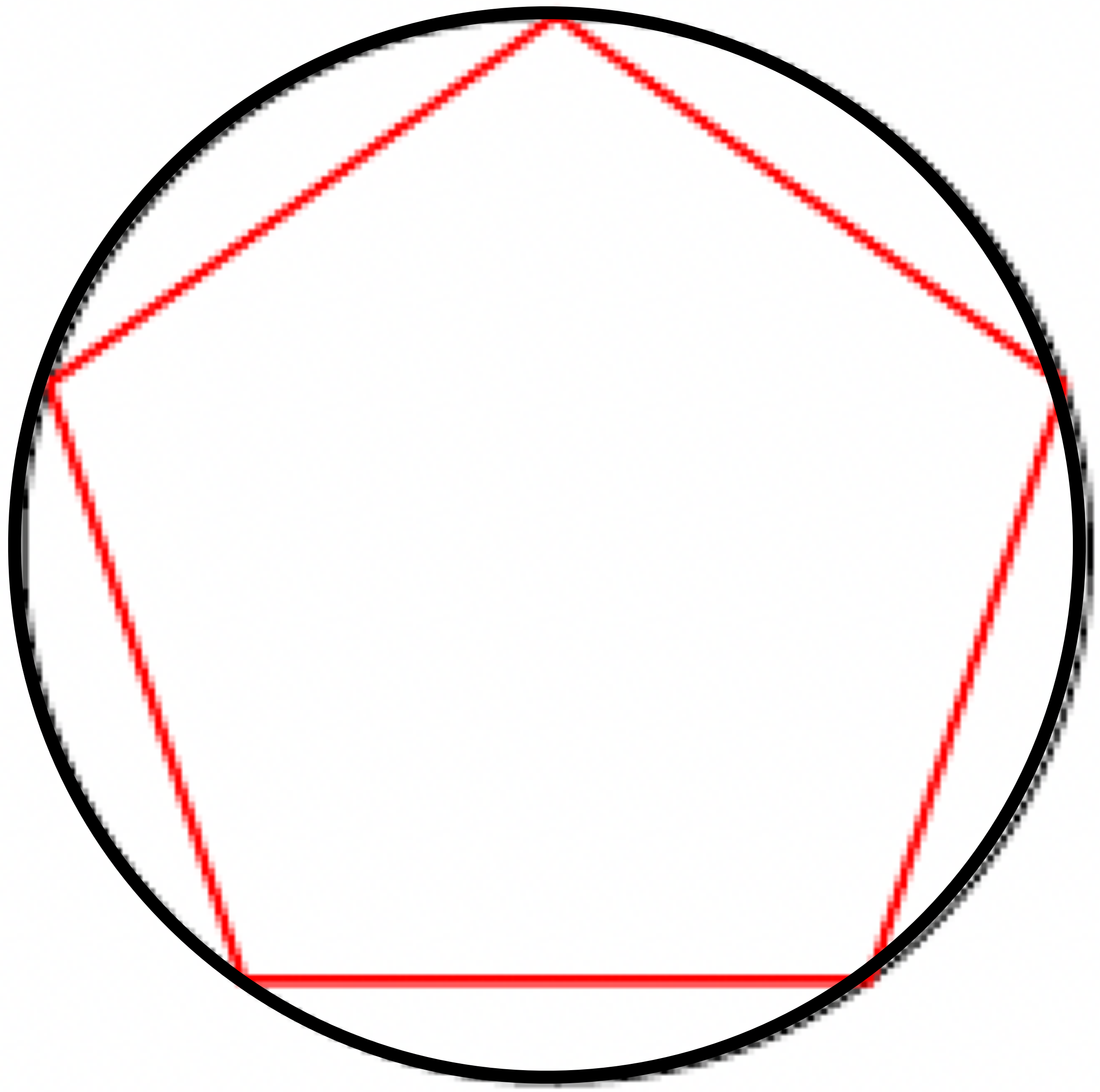


Sides = 4

Perimeter = 2.828

Archimedes

Finding π =3.14159…

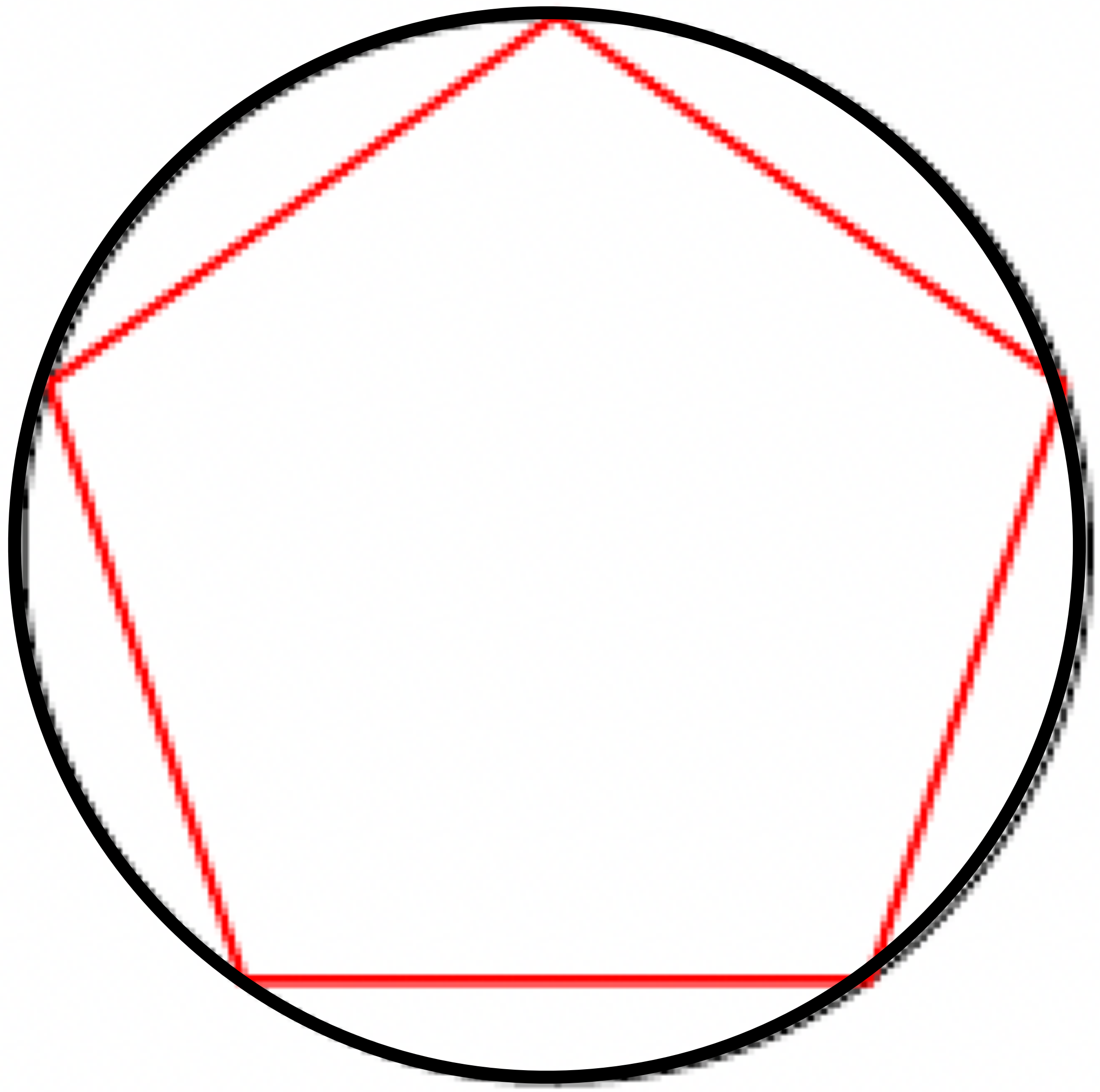


Sides = 5

Perimeter =

Archimedes

Finding π $=3.14159...$



Sides = 5

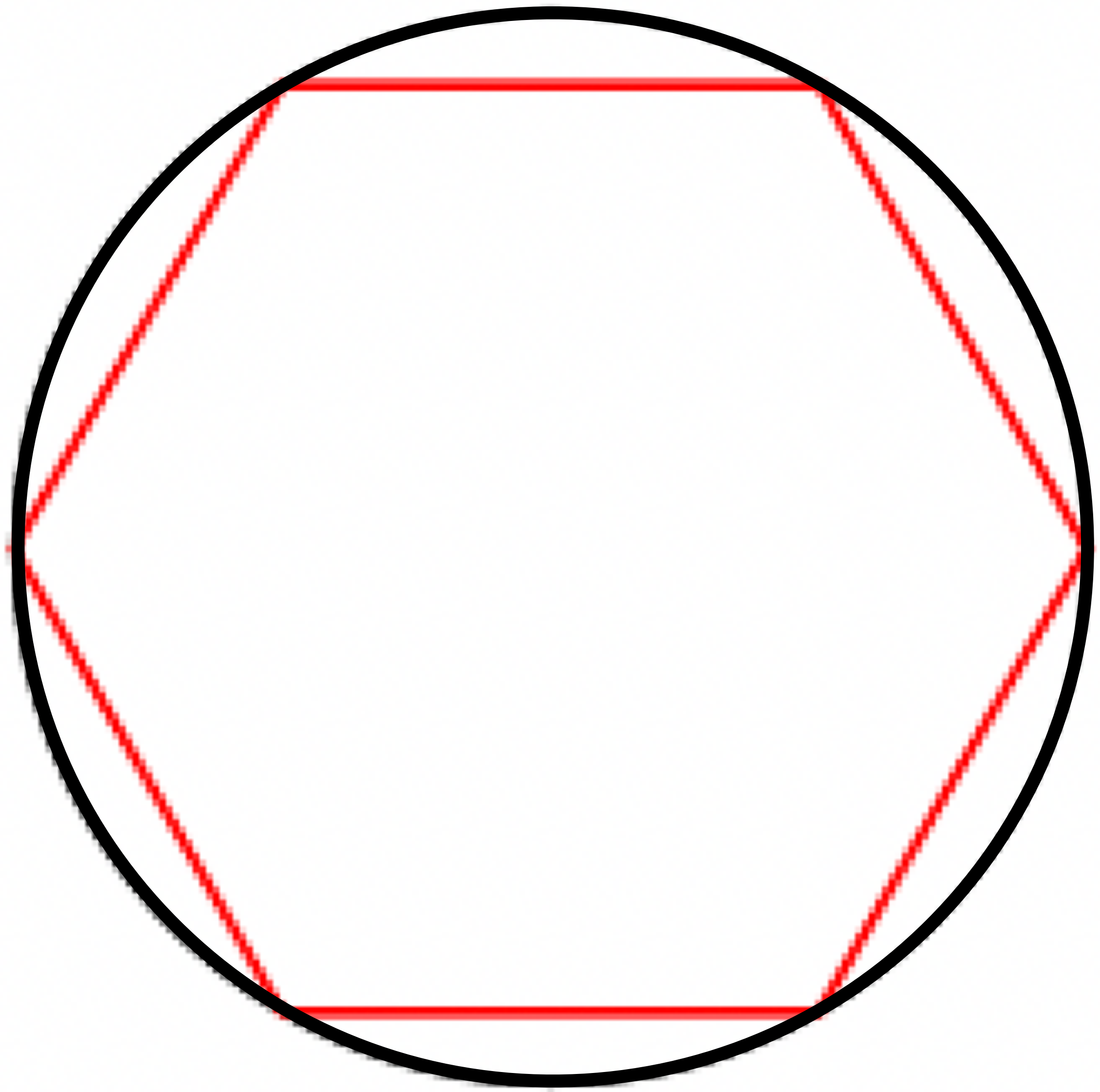
Perimeter = 2.939

Archimedes

Finding π =3.14159…

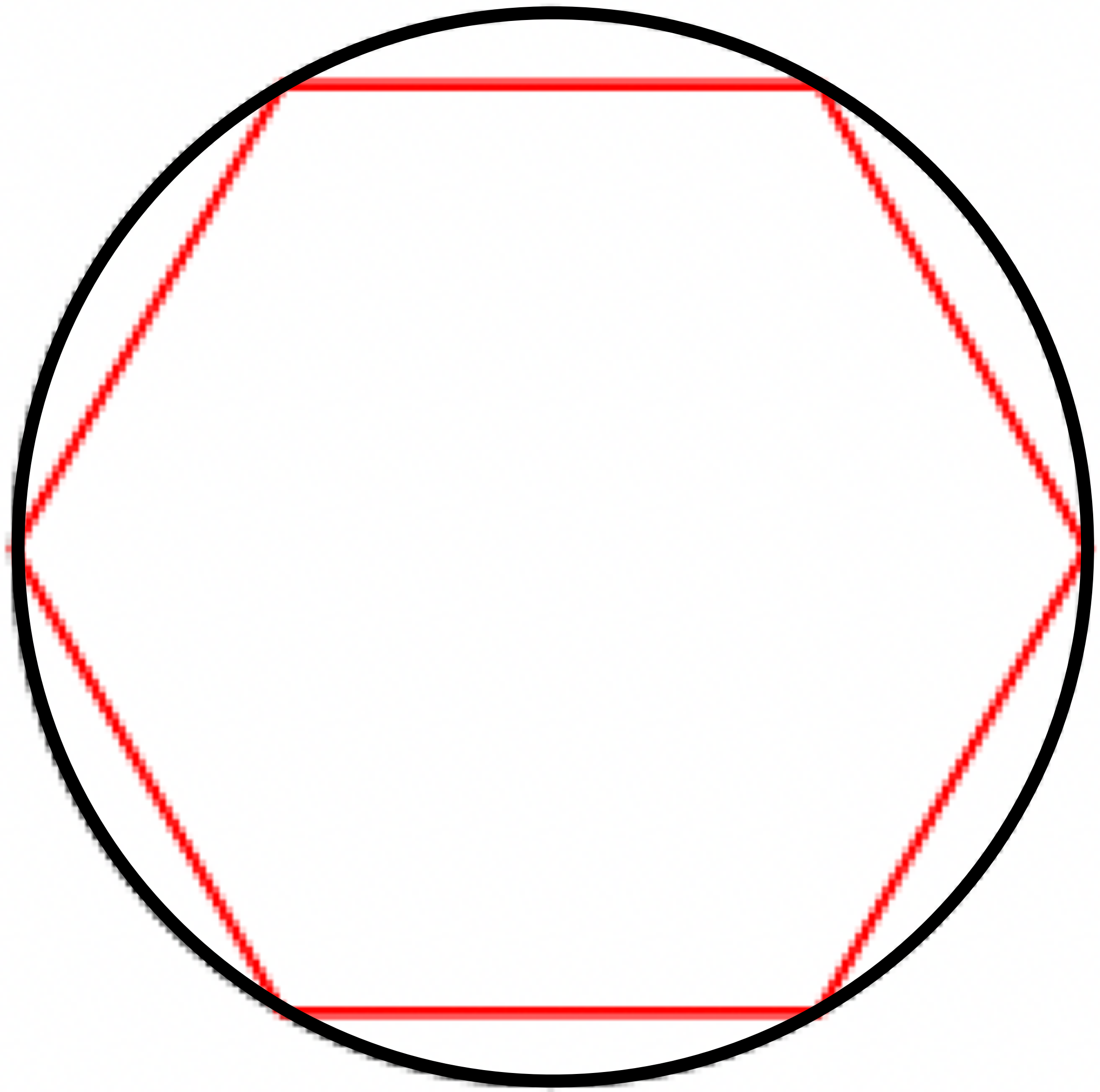
Sides = 6

Perimeter =



Archimedes

Finding π =3.14159…



Sides = 6

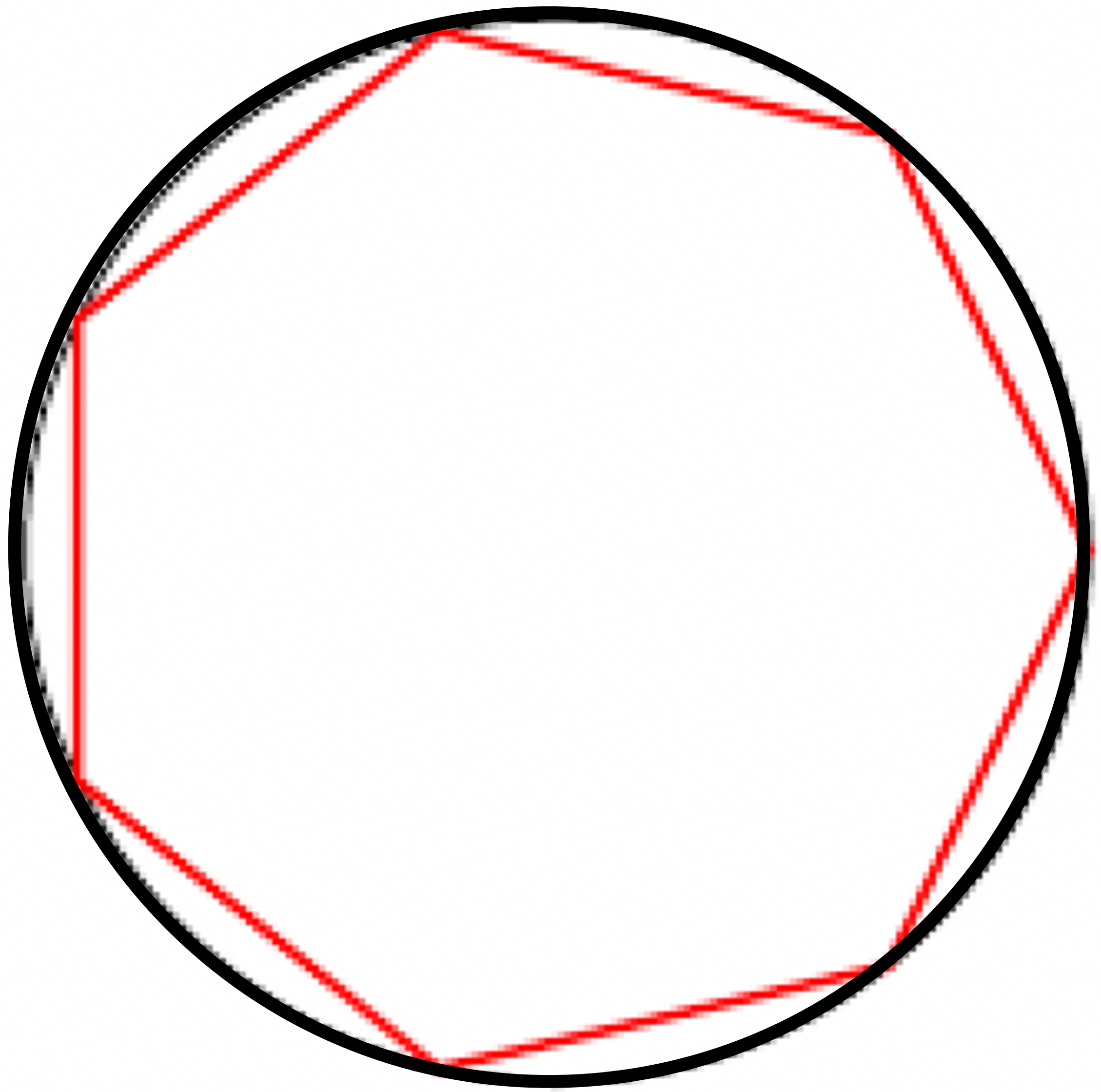
Perimeter = 3.000

Archimedes

Finding π =3.14159…

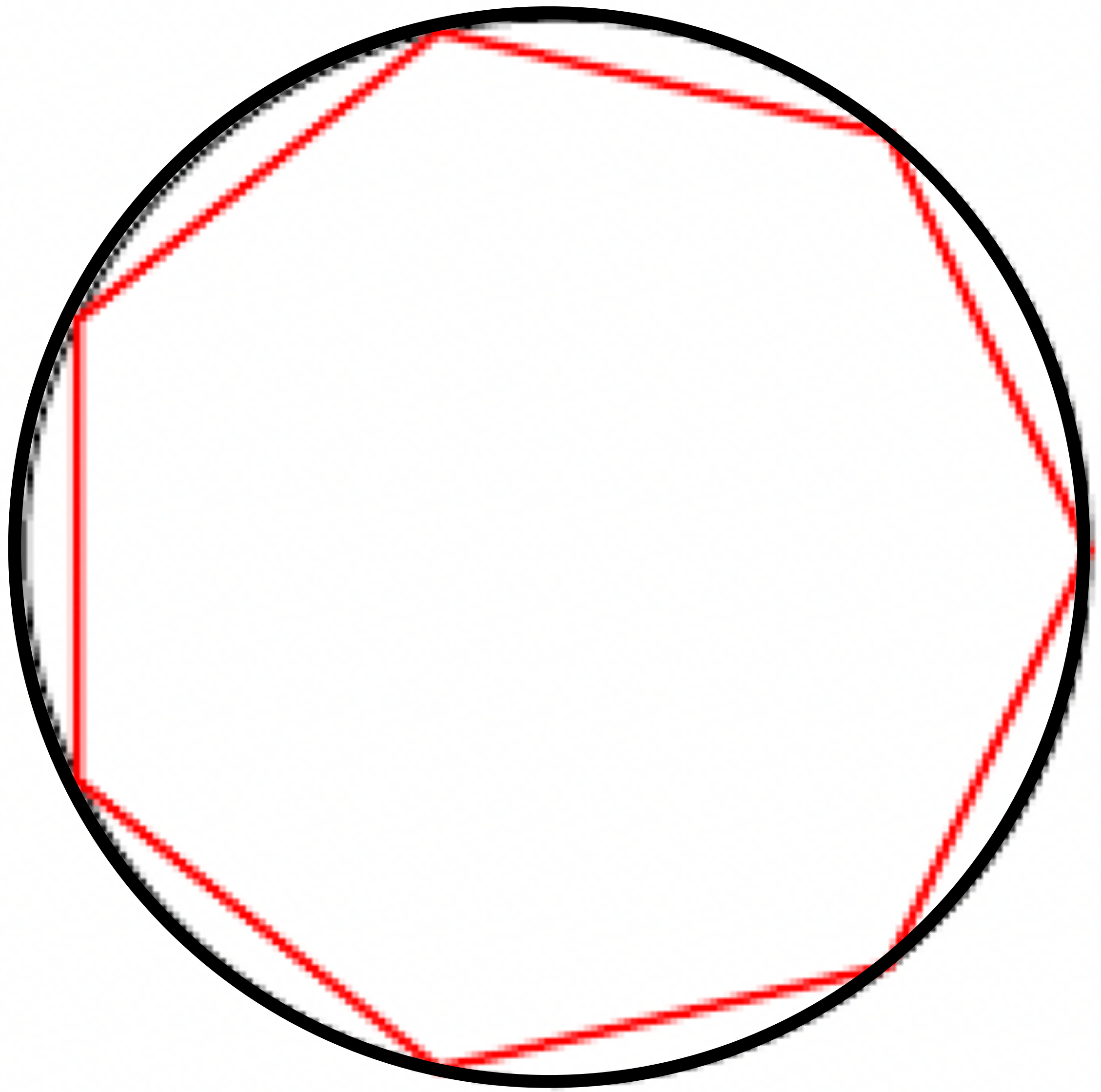
Sides = 7

Perimeter =



Archimedes

Finding π $=3.14159...$



Sides = 7

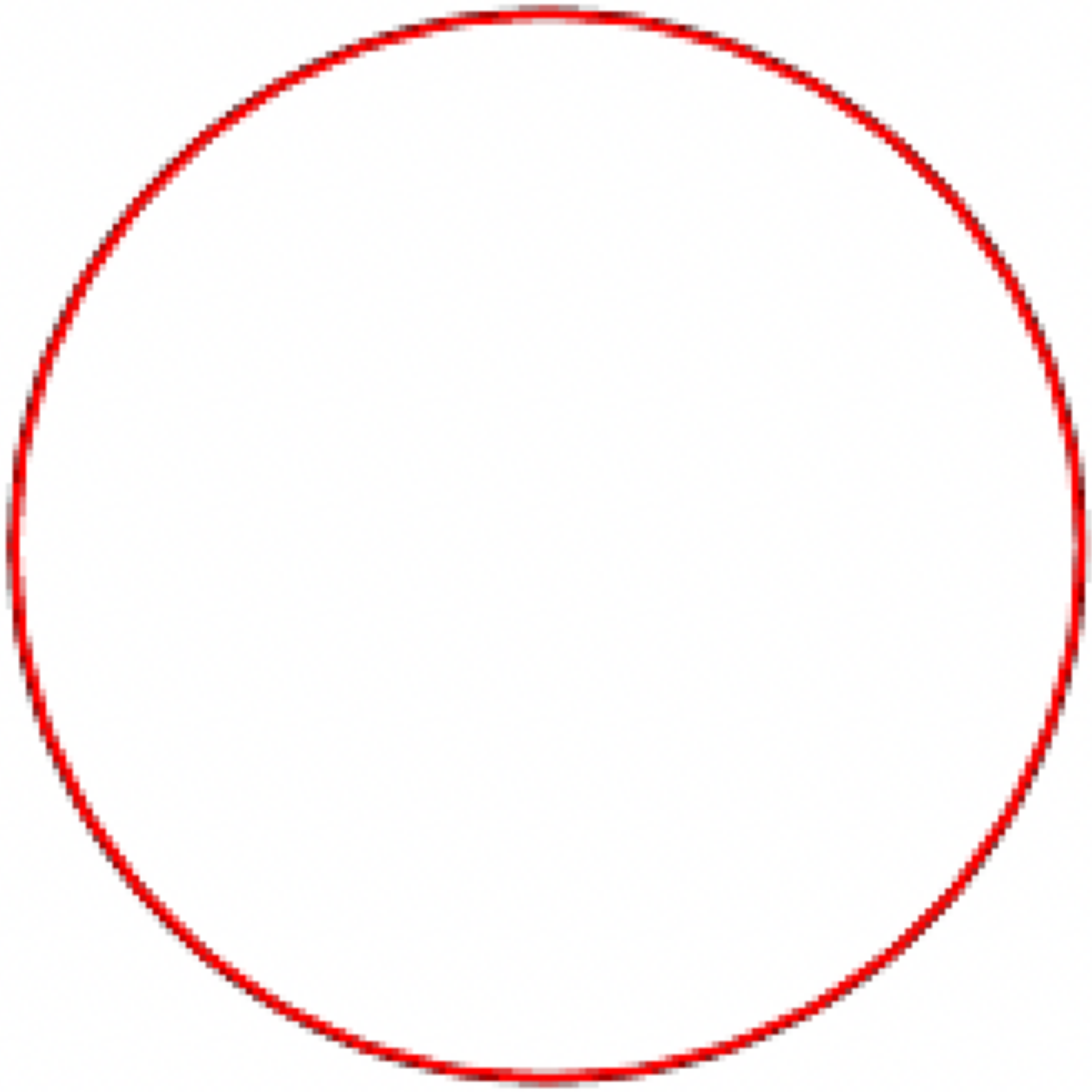
Perimeter = 3.037

Archimedes

Finding π $=3.14159...$

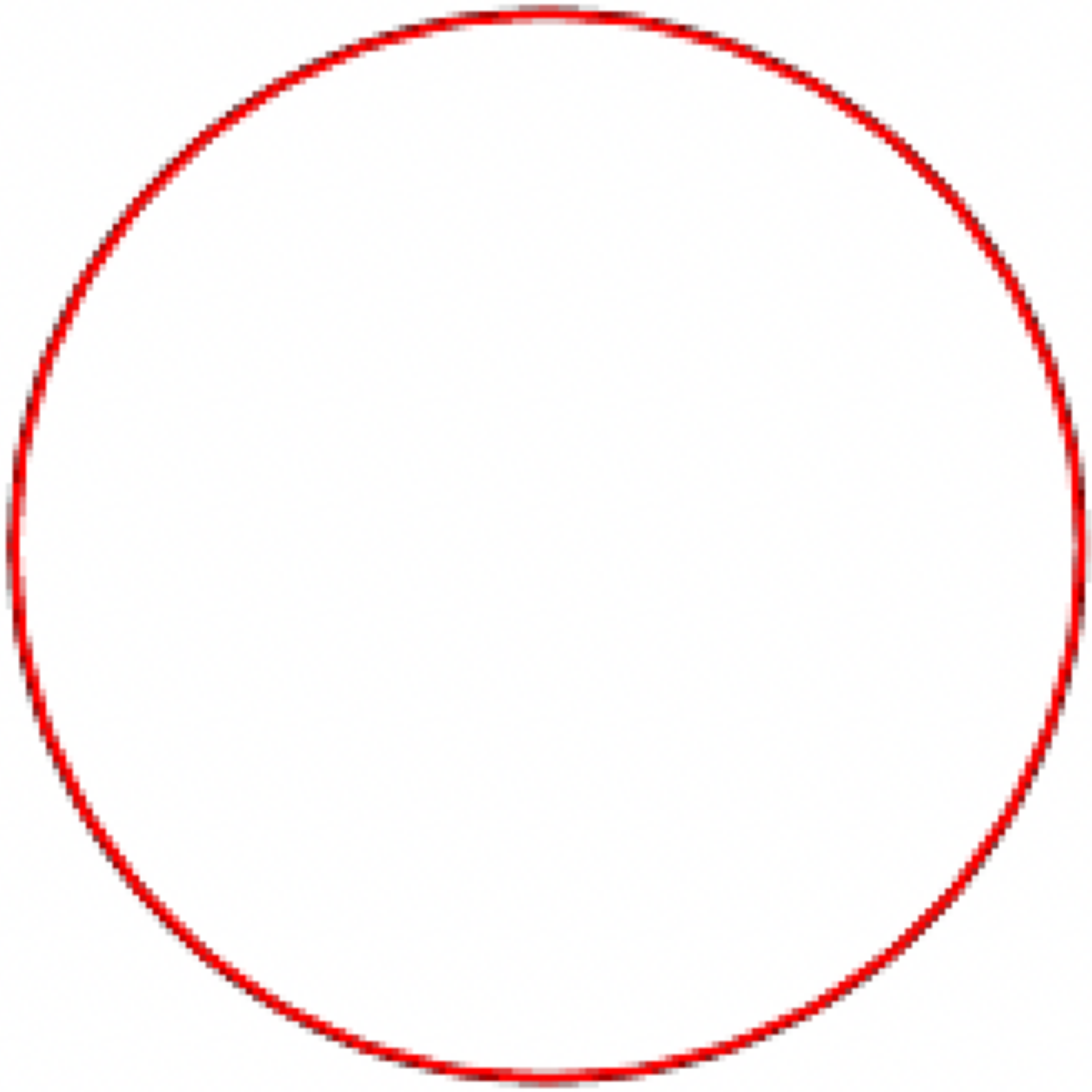
Sides = 10

Perimeter =



Archimedes

Finding π =3.14159…



Sides = 10

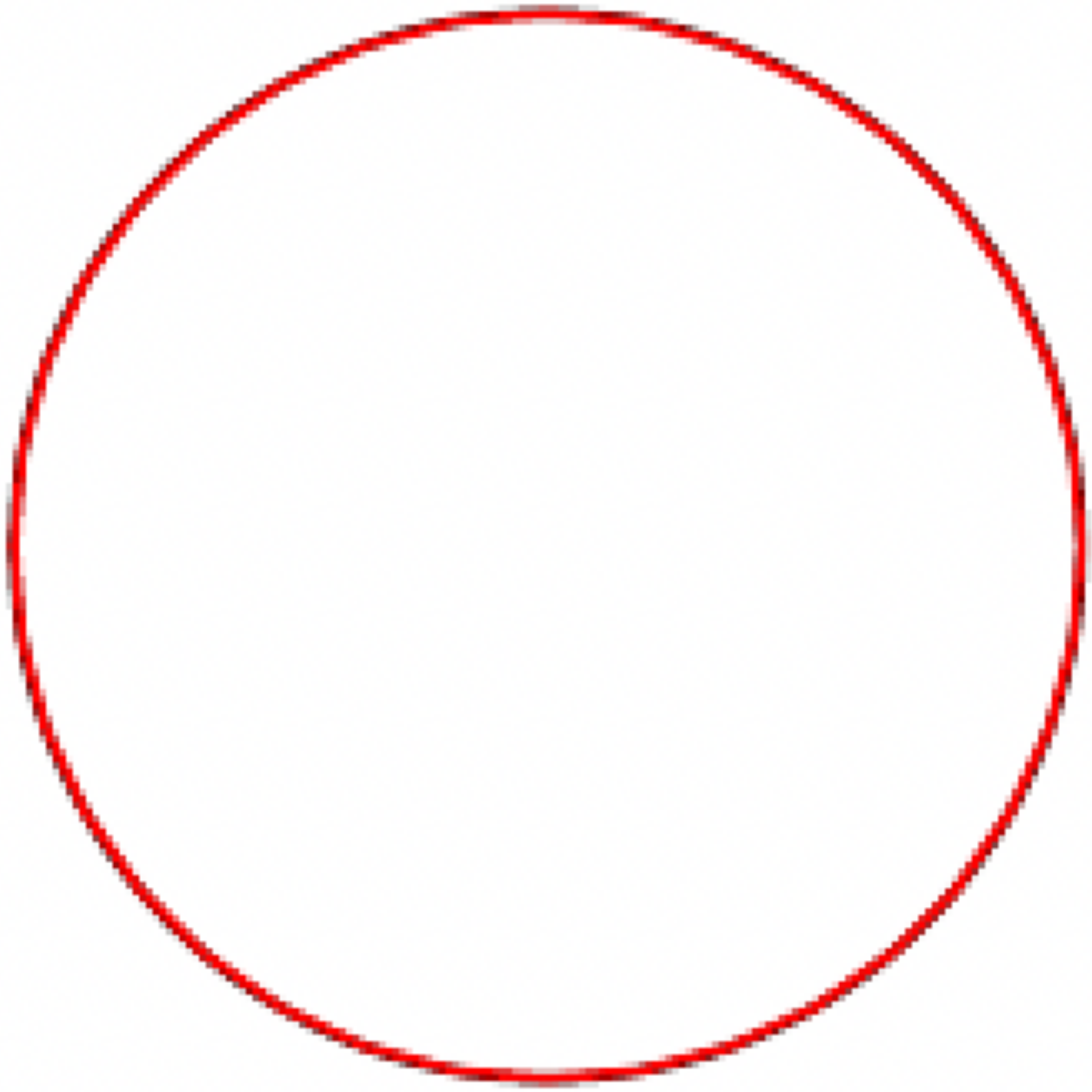
Perimeter = 3.090

Archimedes

Finding π $=3.14159...$

Sides = 50

Perimeter =

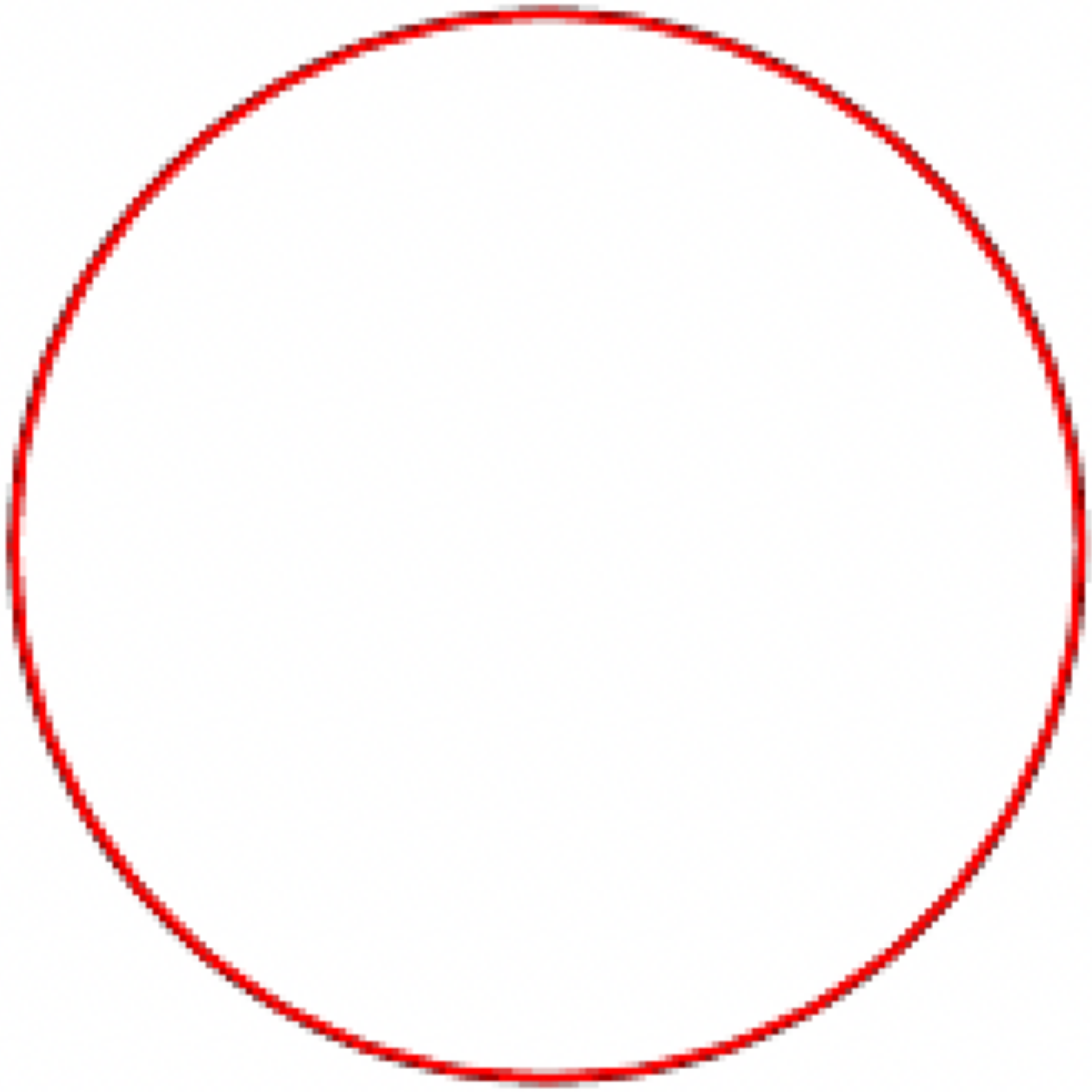


Archimedes

Finding π $=3.14159\ldots$

Sides = 50

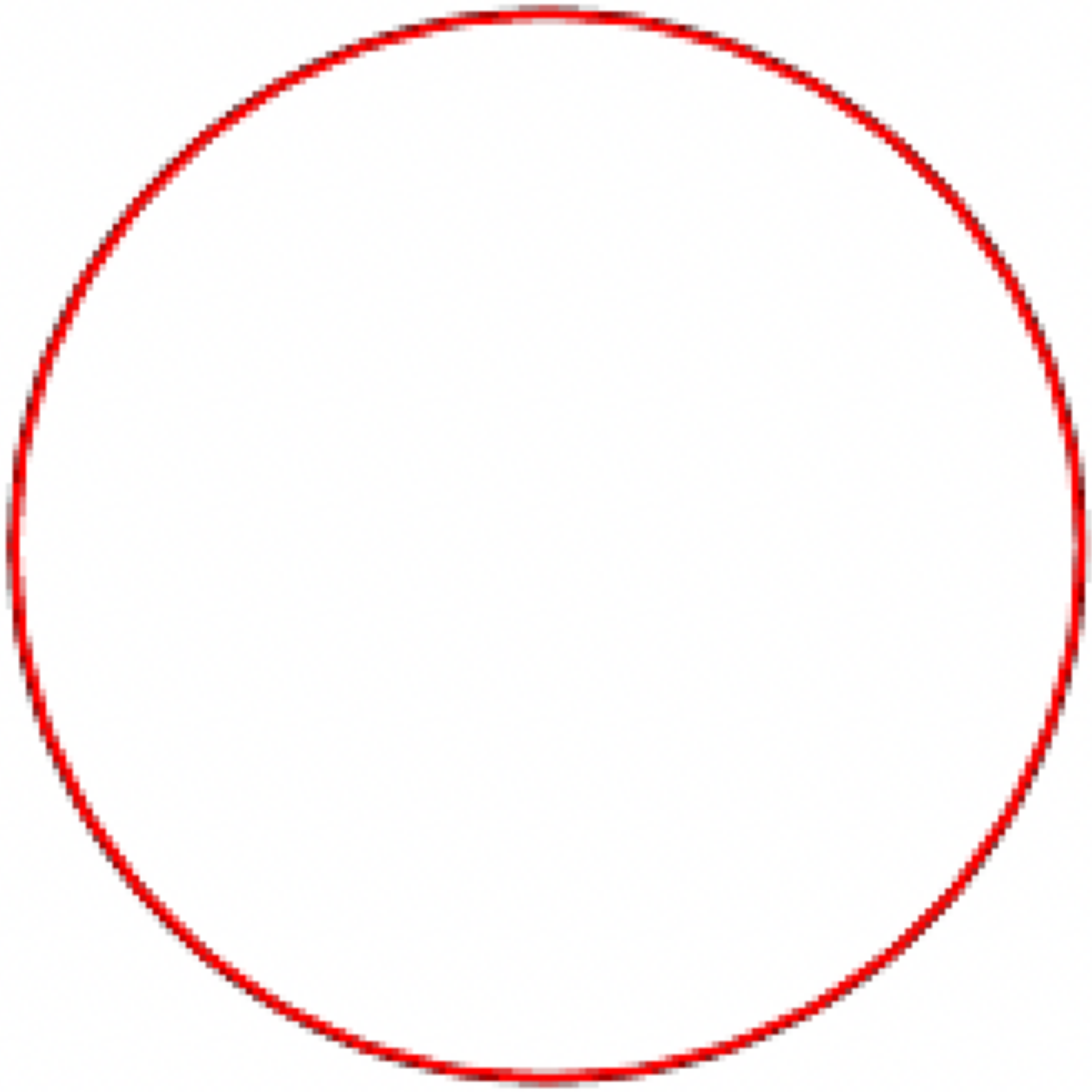
Perimeter = 3.140



Archimedes

Finding π =3.14159…

Sides = 100



Perimeter =

Archimedes

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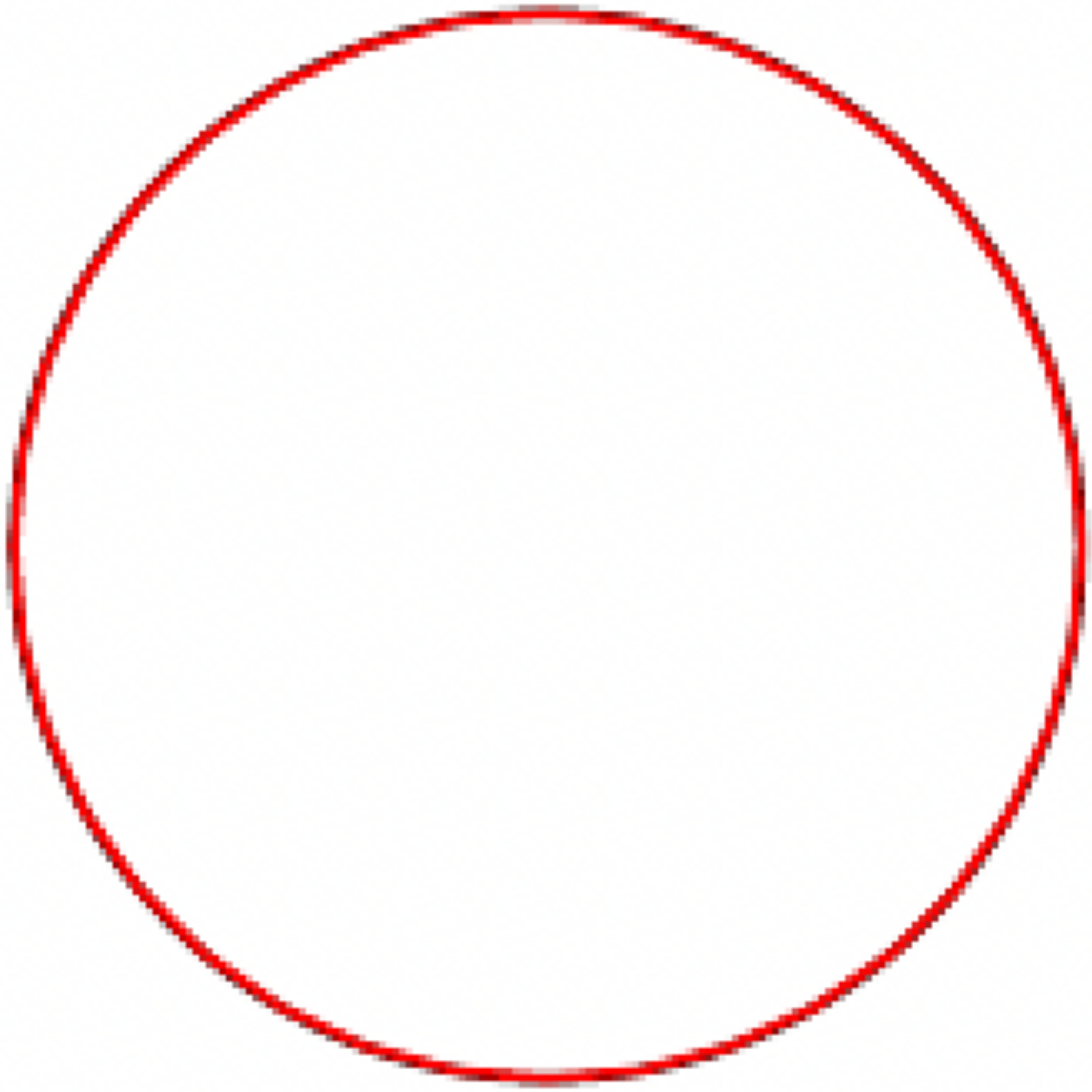

Sides = 100

Perimeter = 3.141

Archimedes

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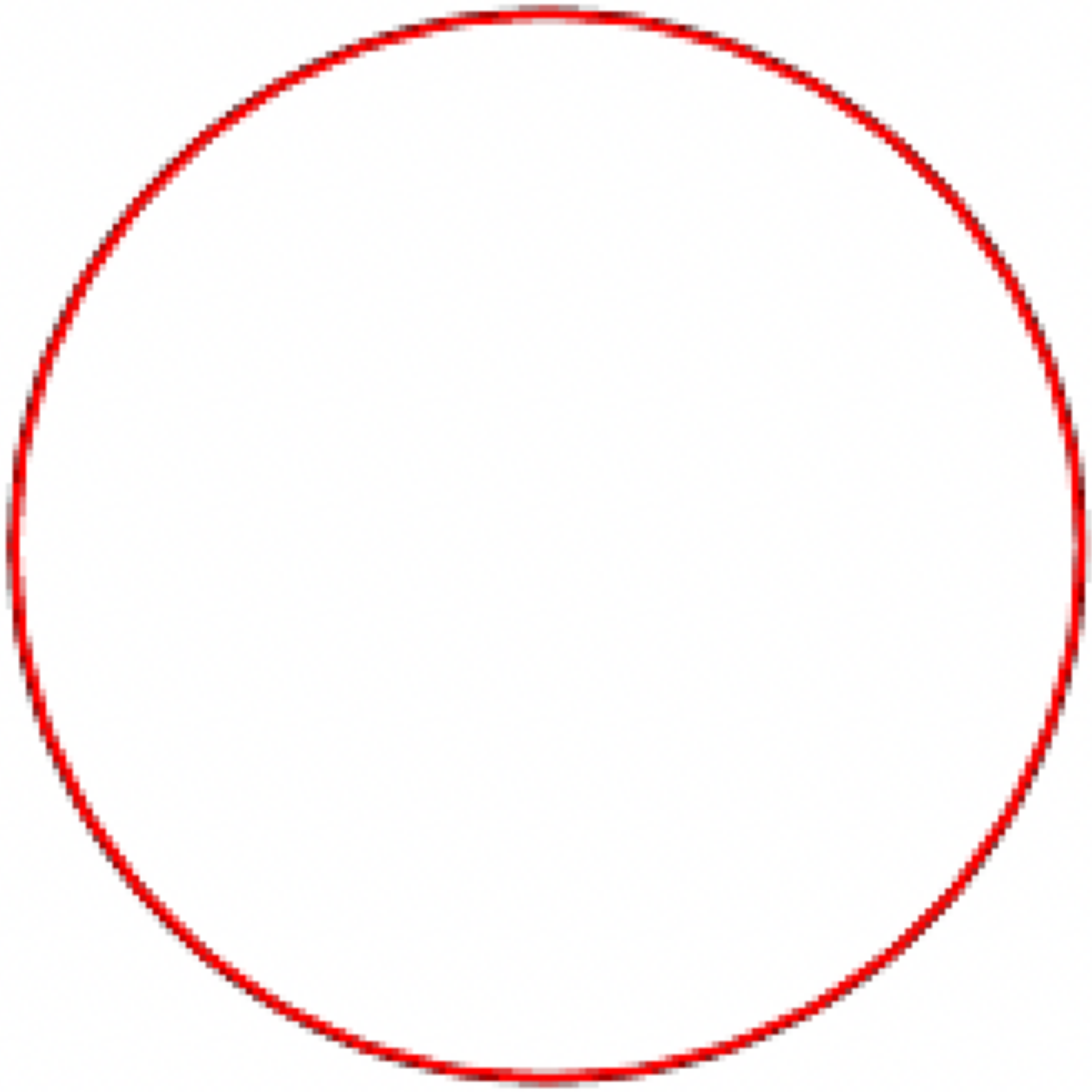

Sides = 100

Perimeter = 3.141

Using a 96-gon, Archimedes proved

$$3\frac{10}{71} < \pi < 3\frac{10}{70}$$

Archimedes

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Why is π also in the circle's area formula? $A = \pi r^2$.

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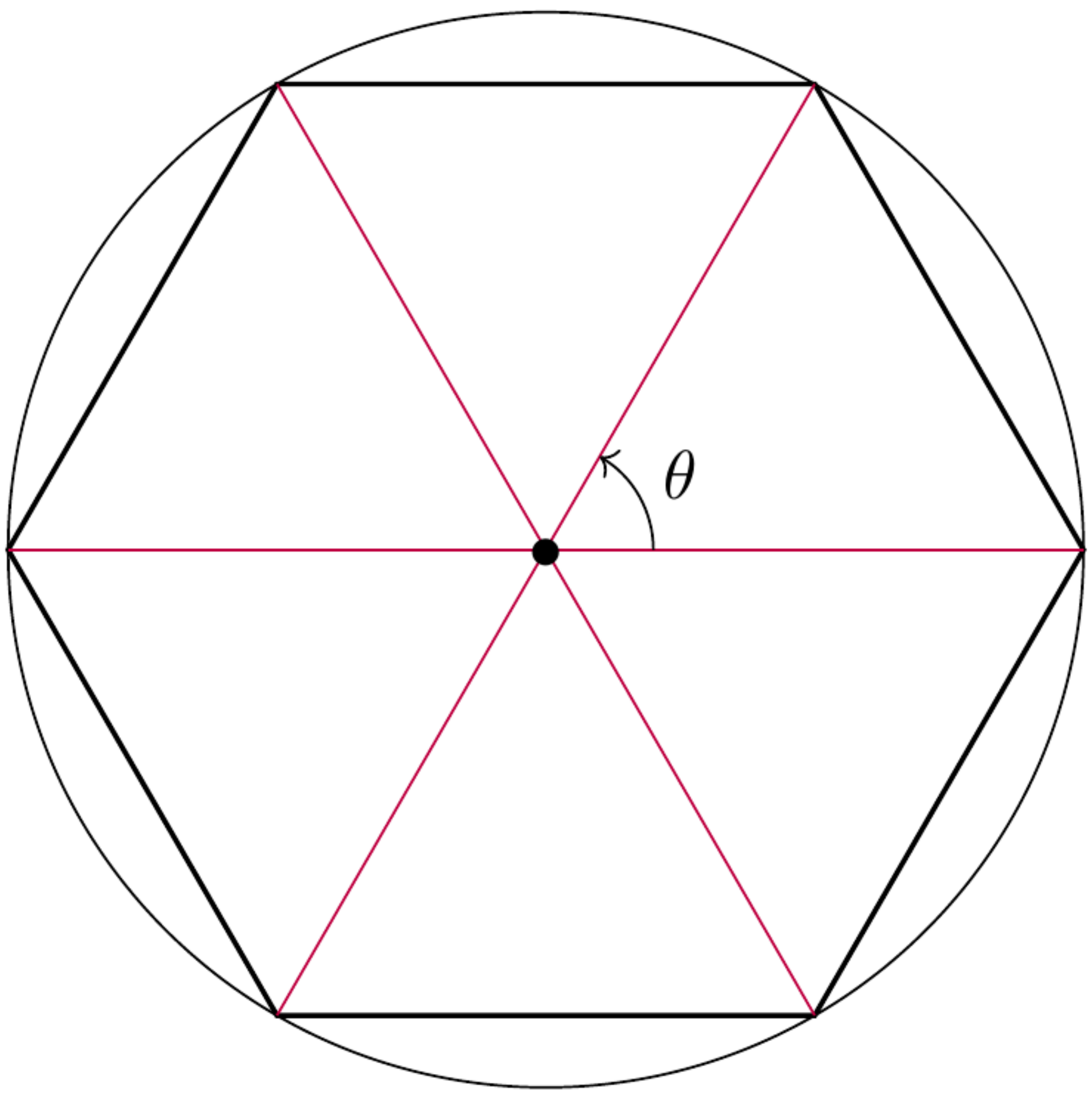

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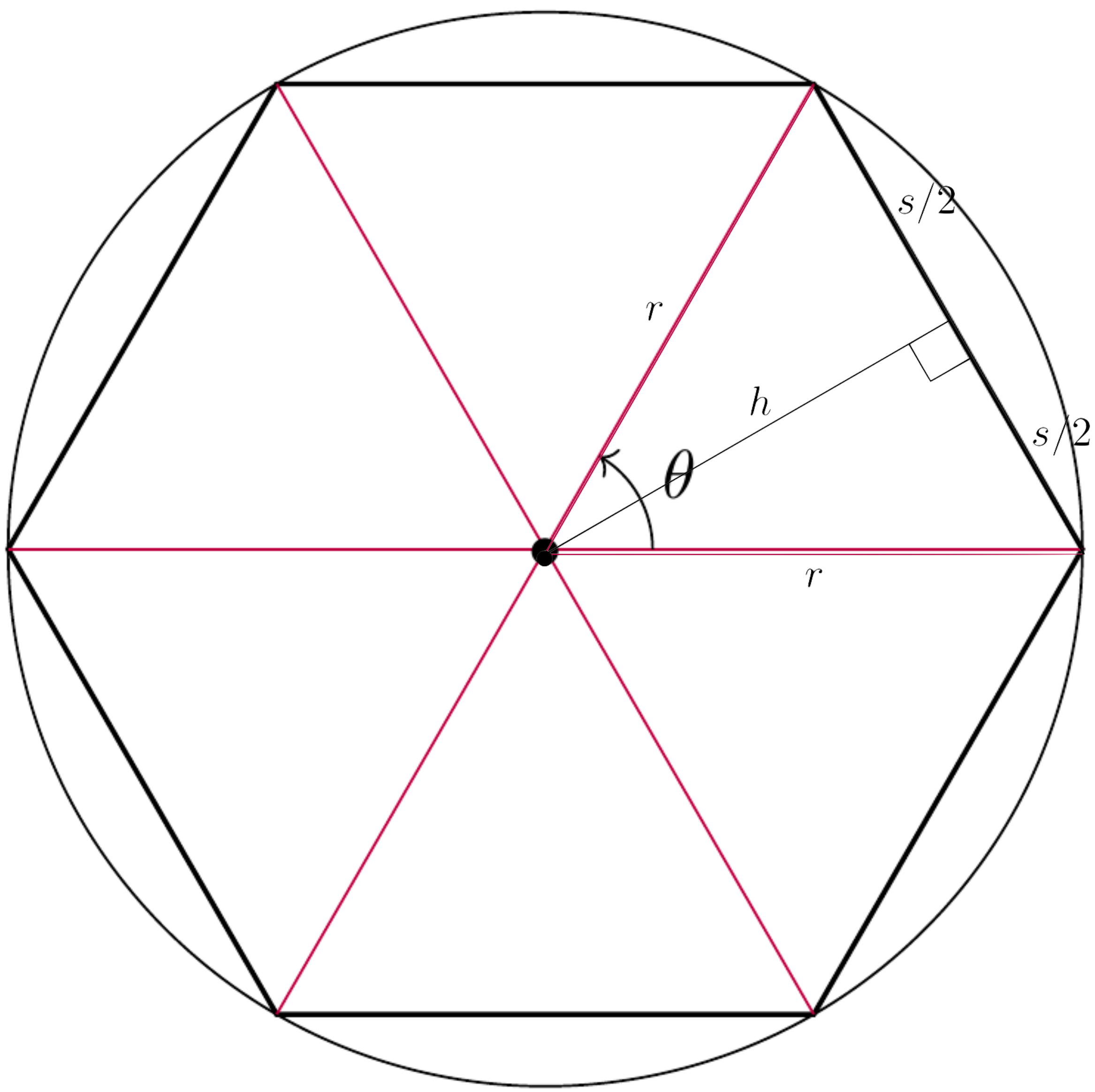

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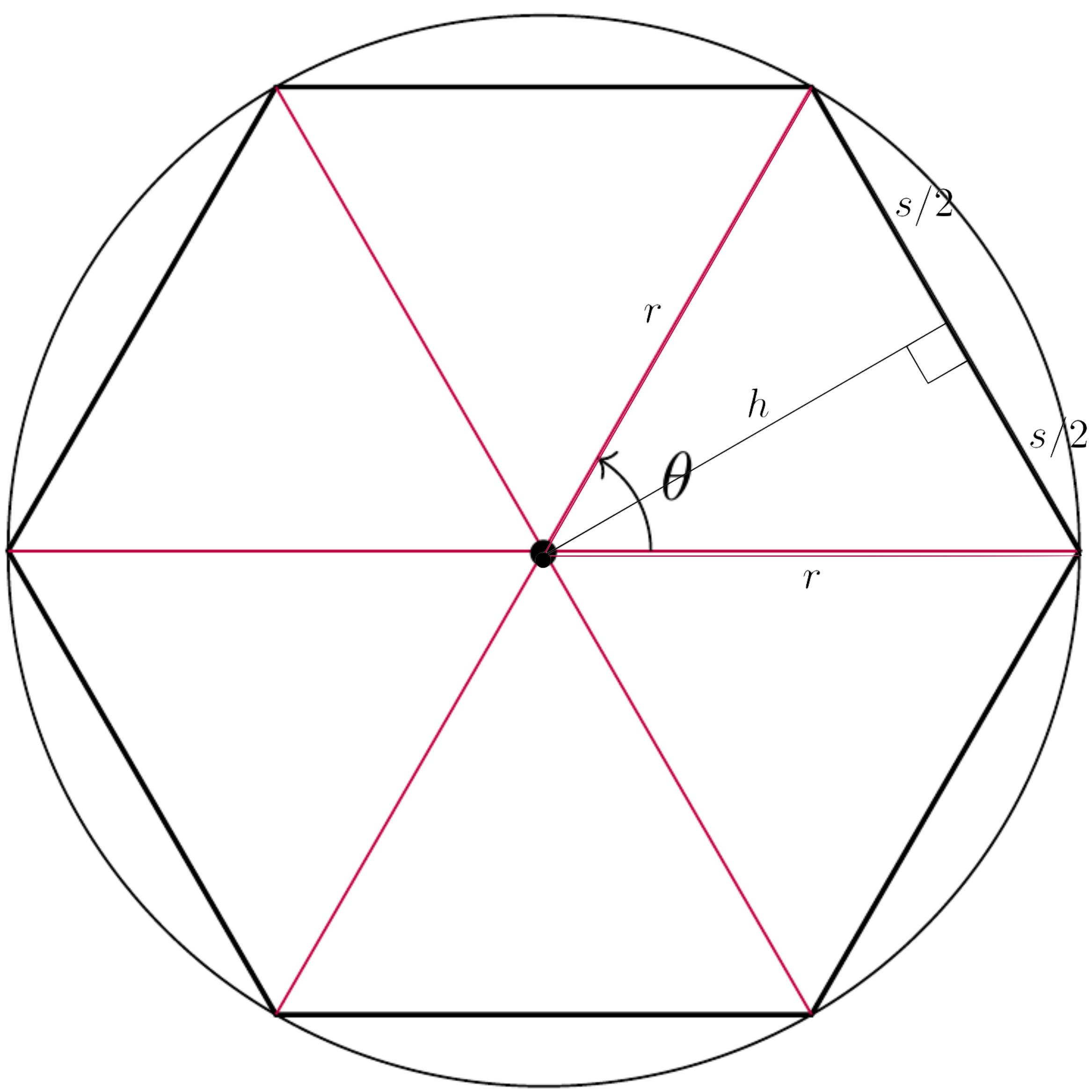

Area of inscribed n-gon $= \dfrac{1}{2} shn$

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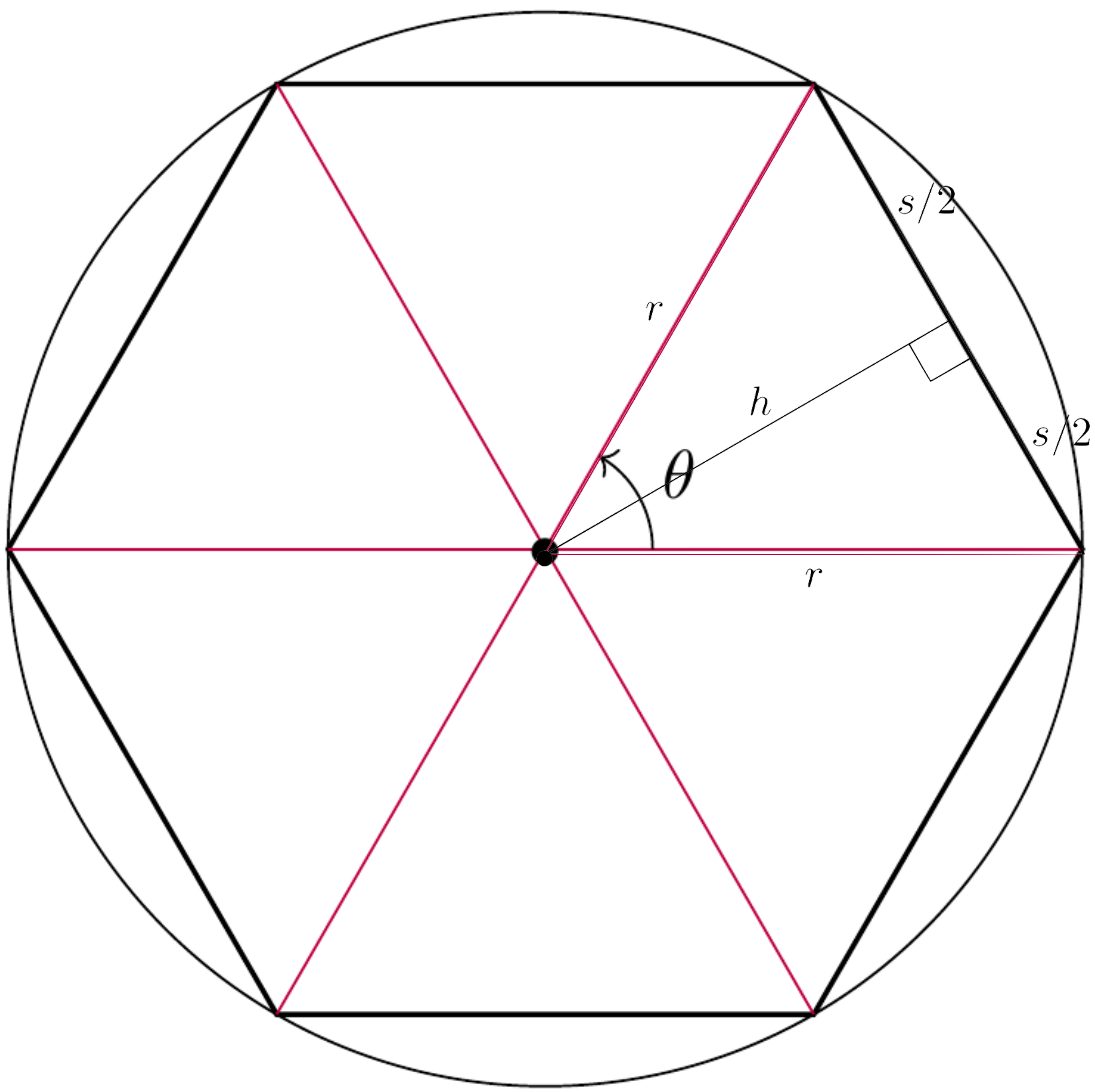

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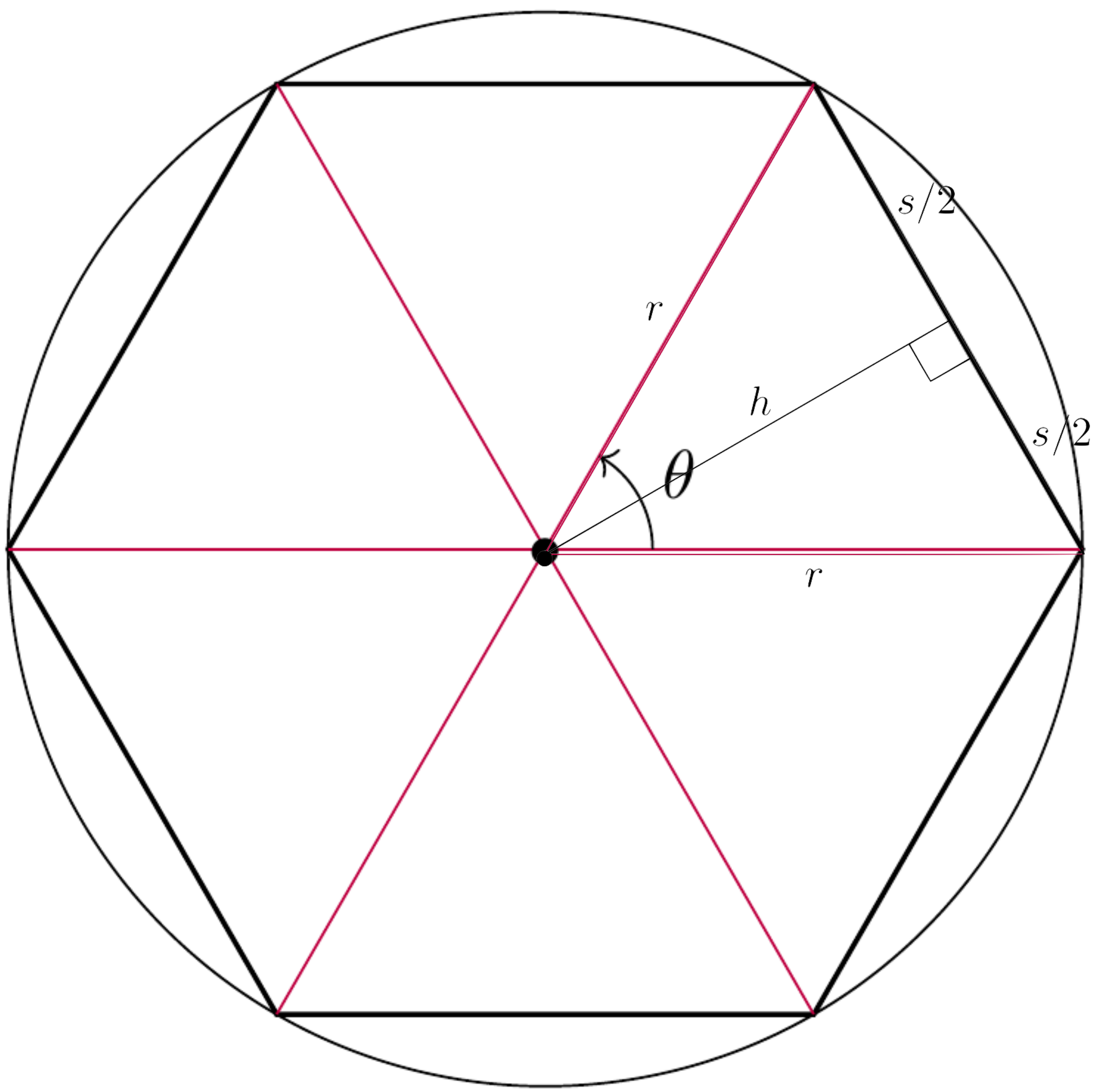

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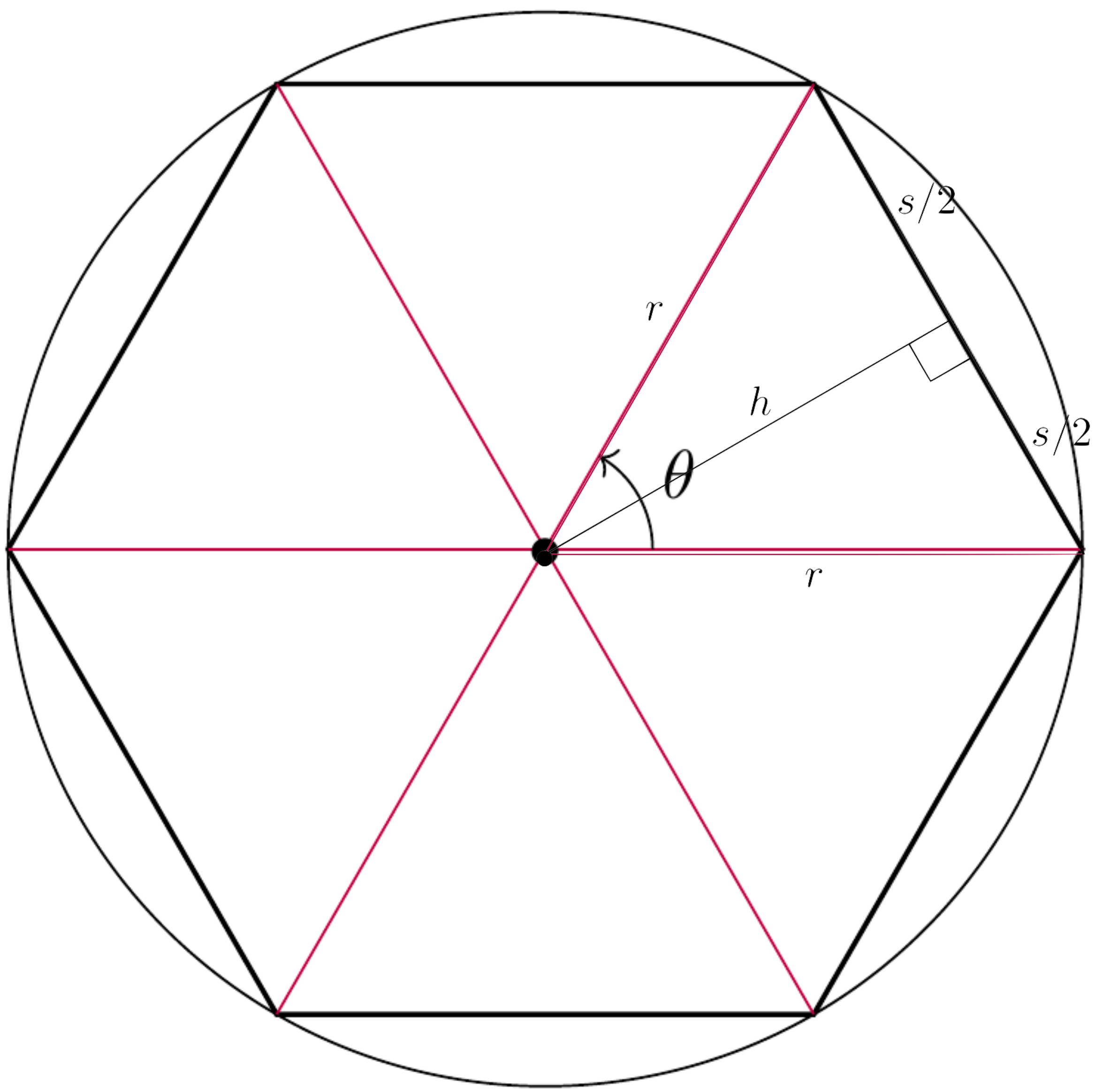

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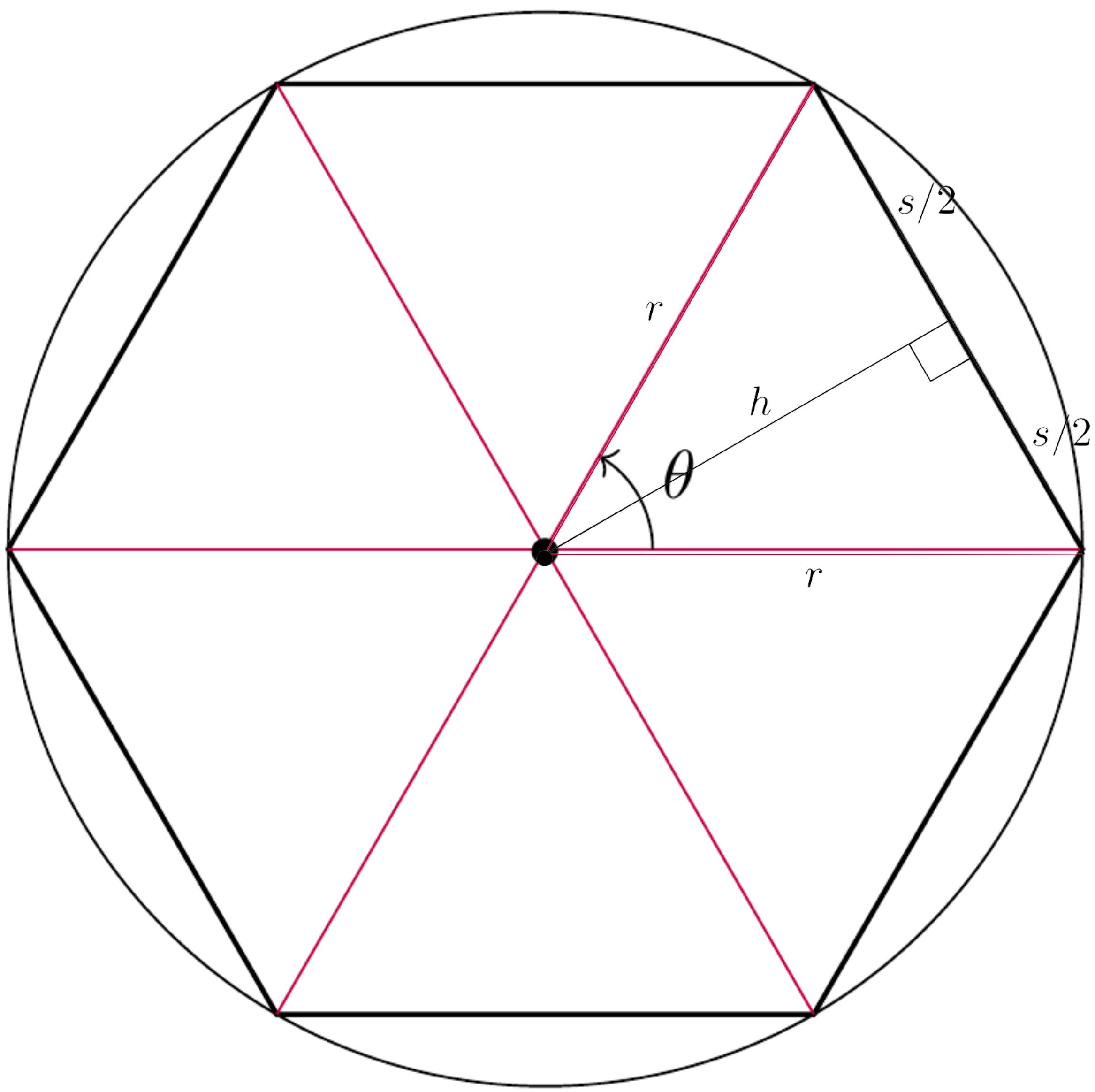

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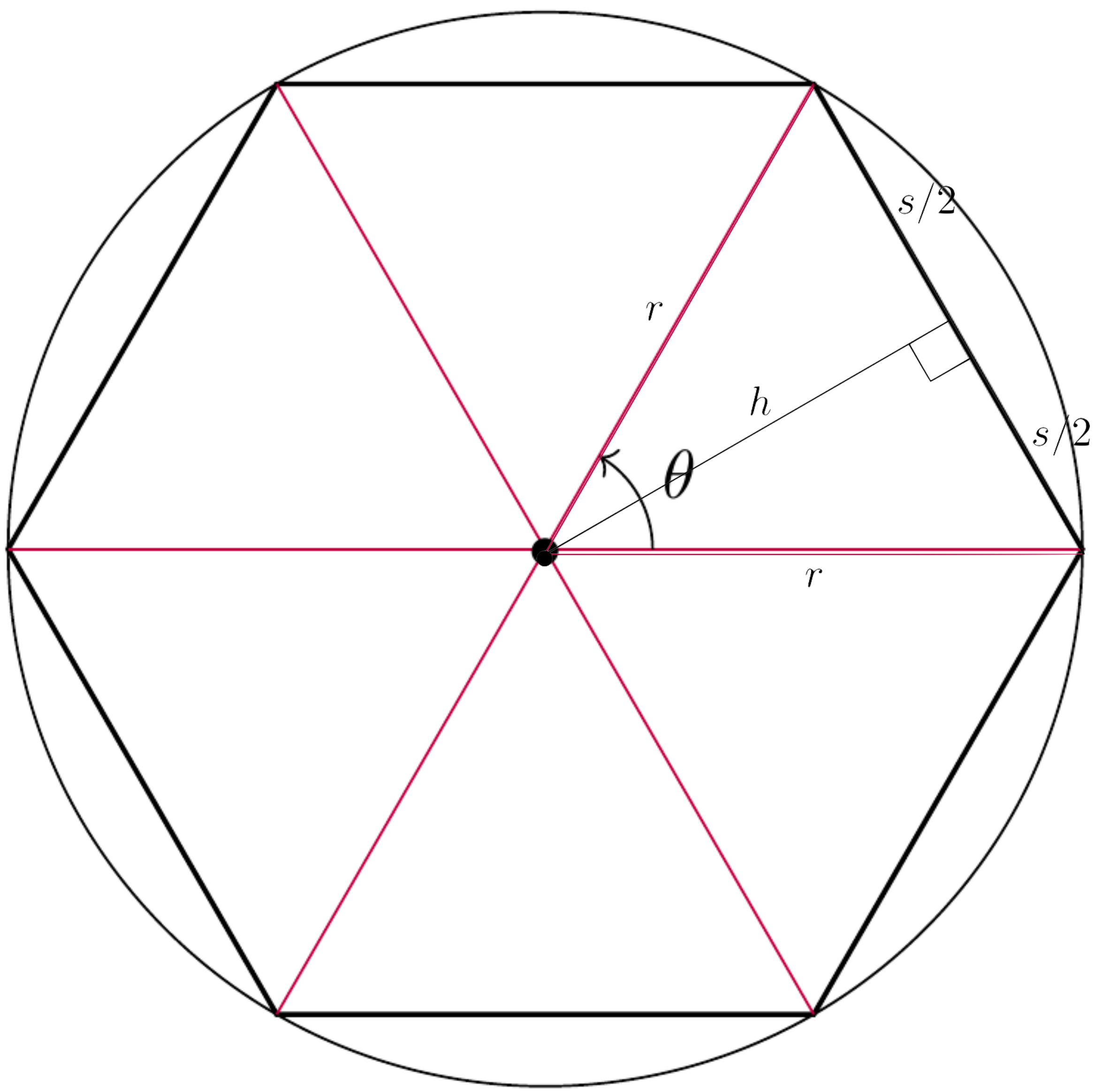

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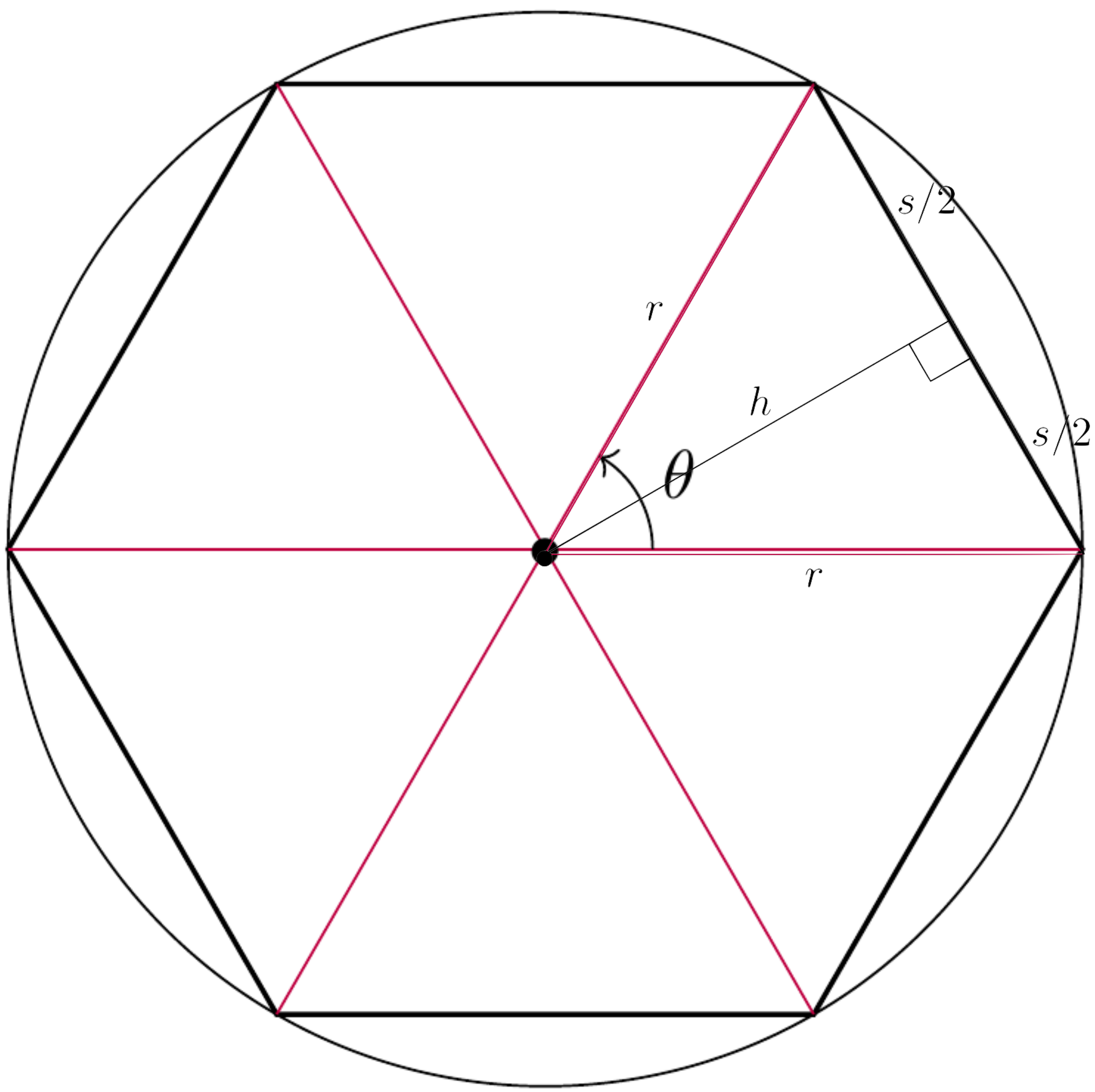

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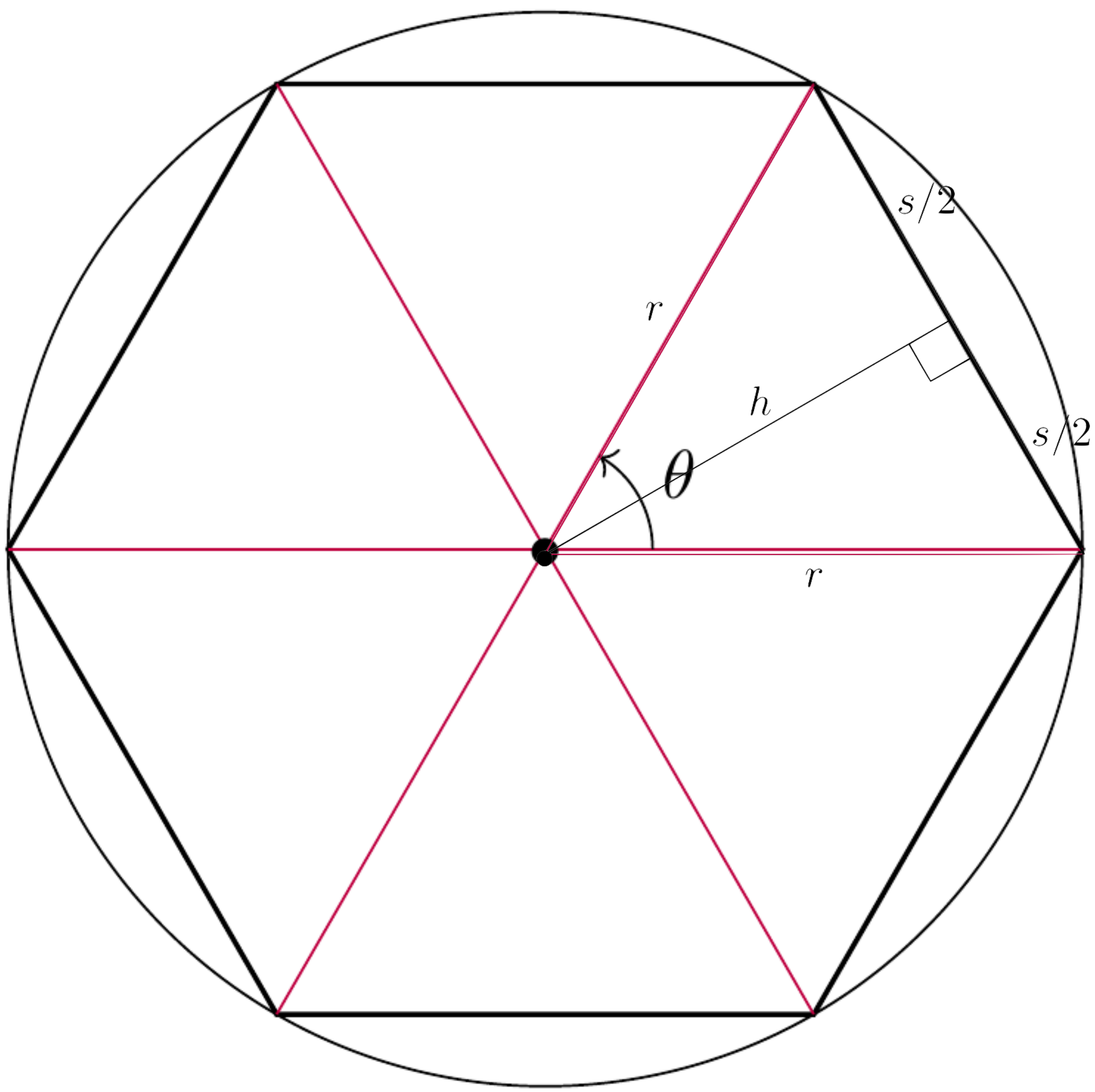

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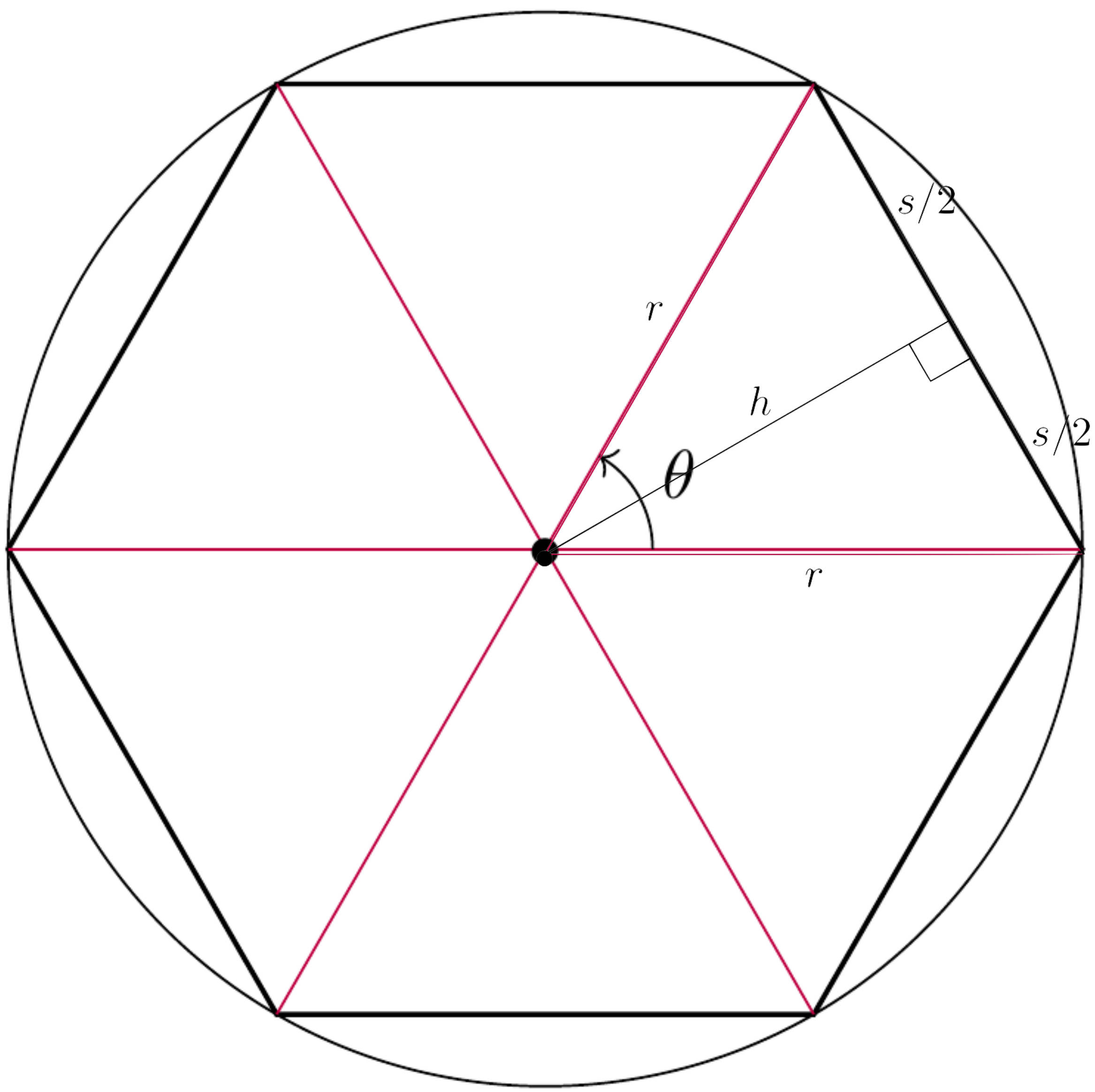

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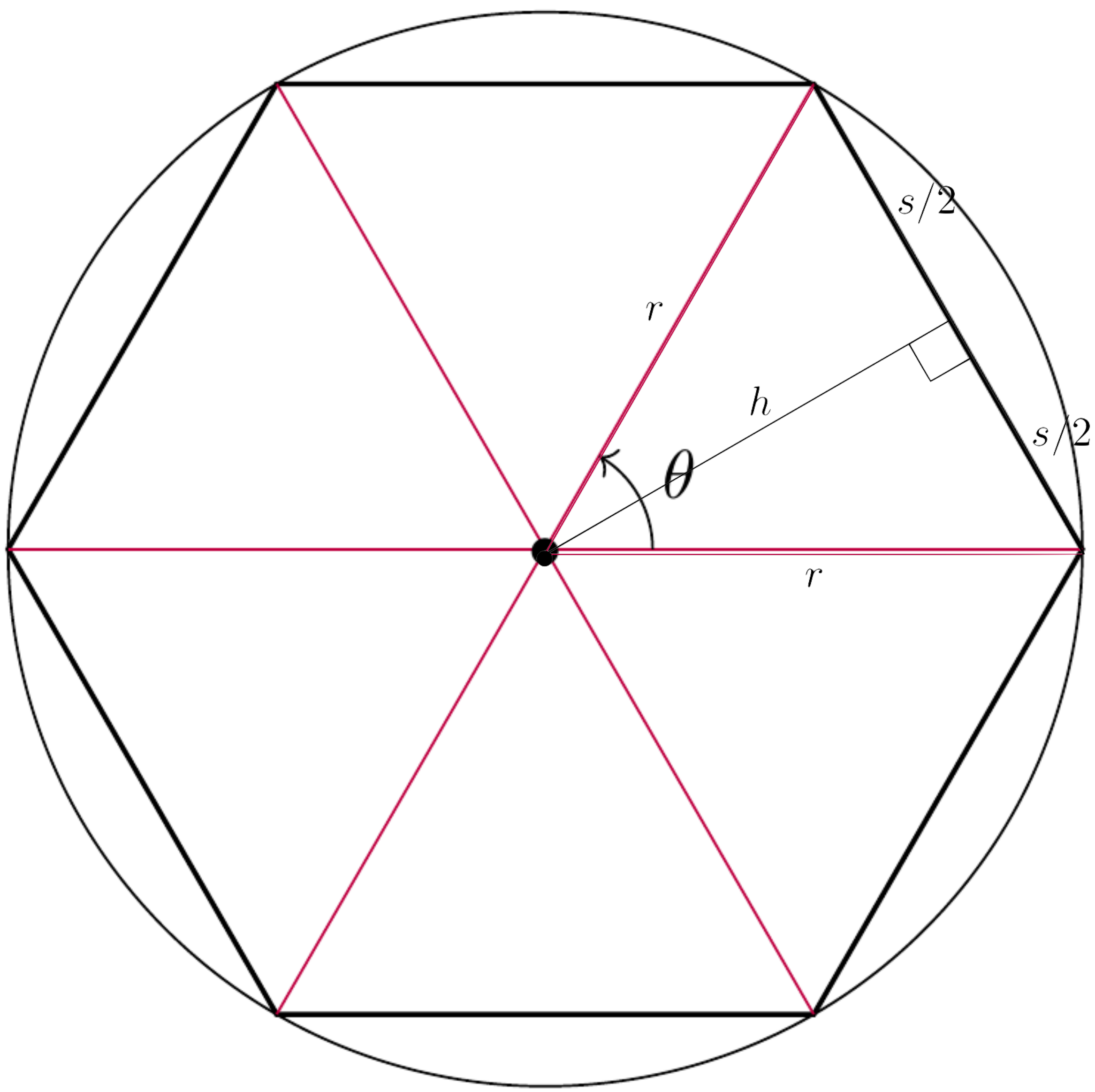

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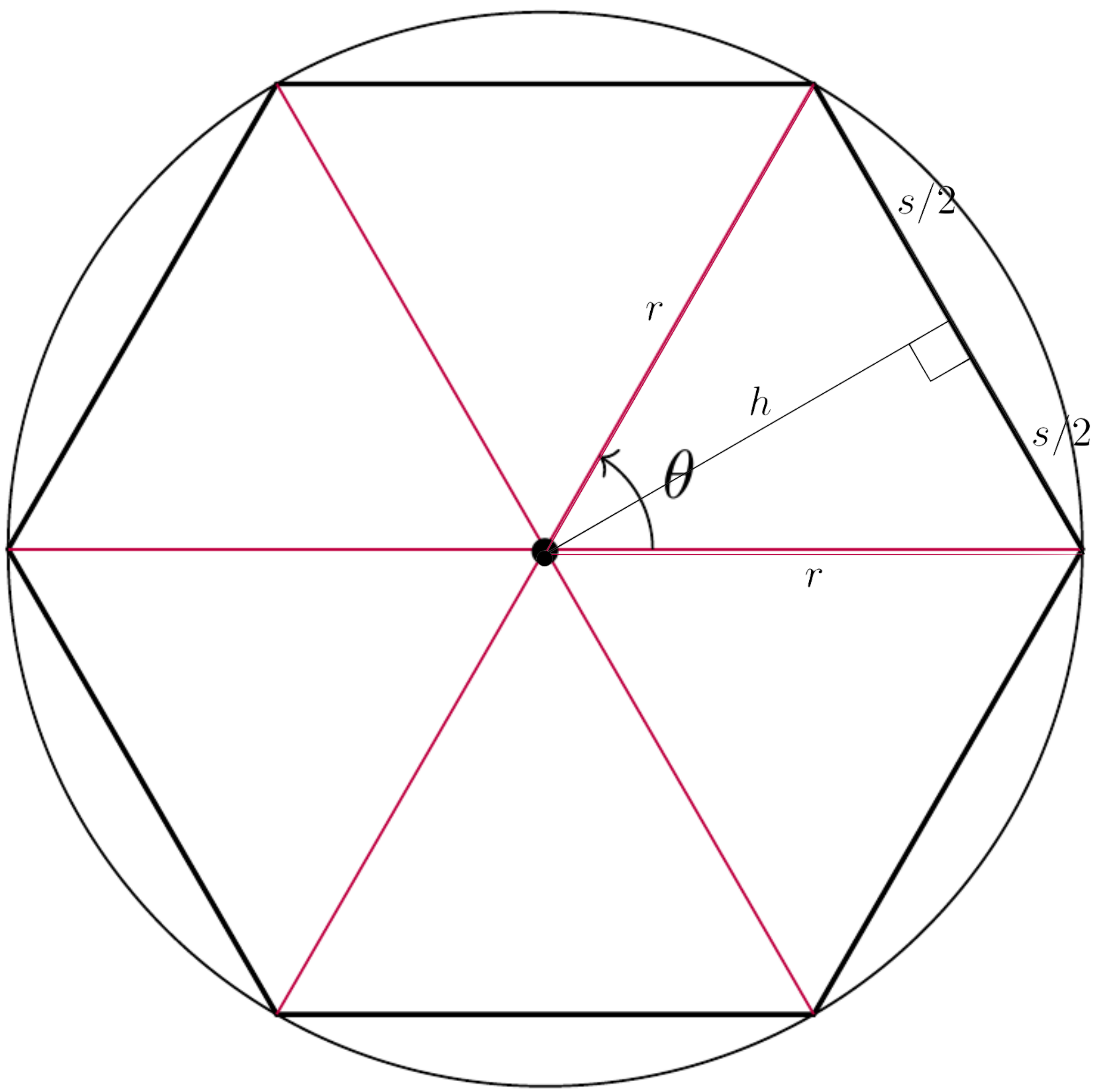

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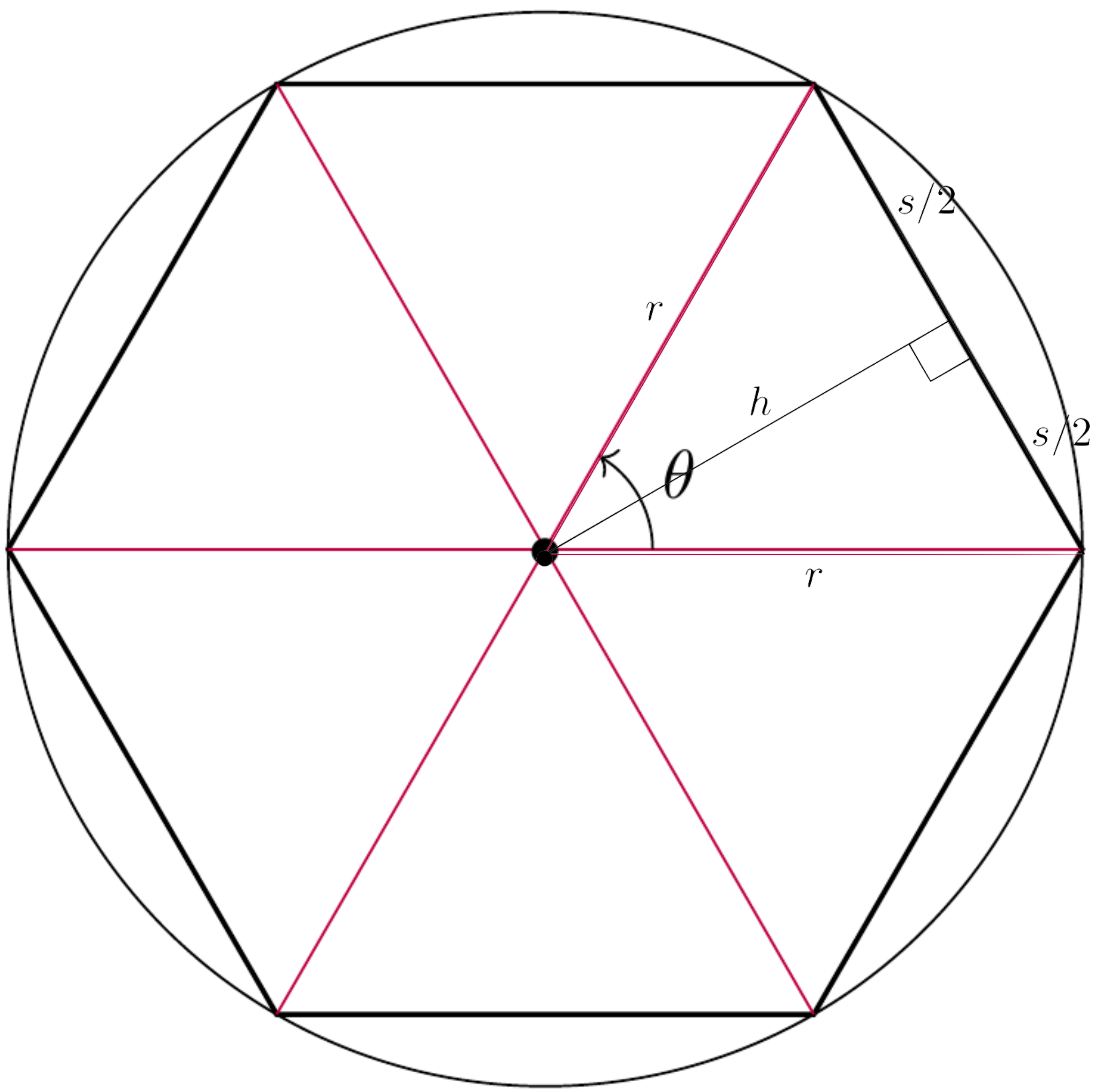

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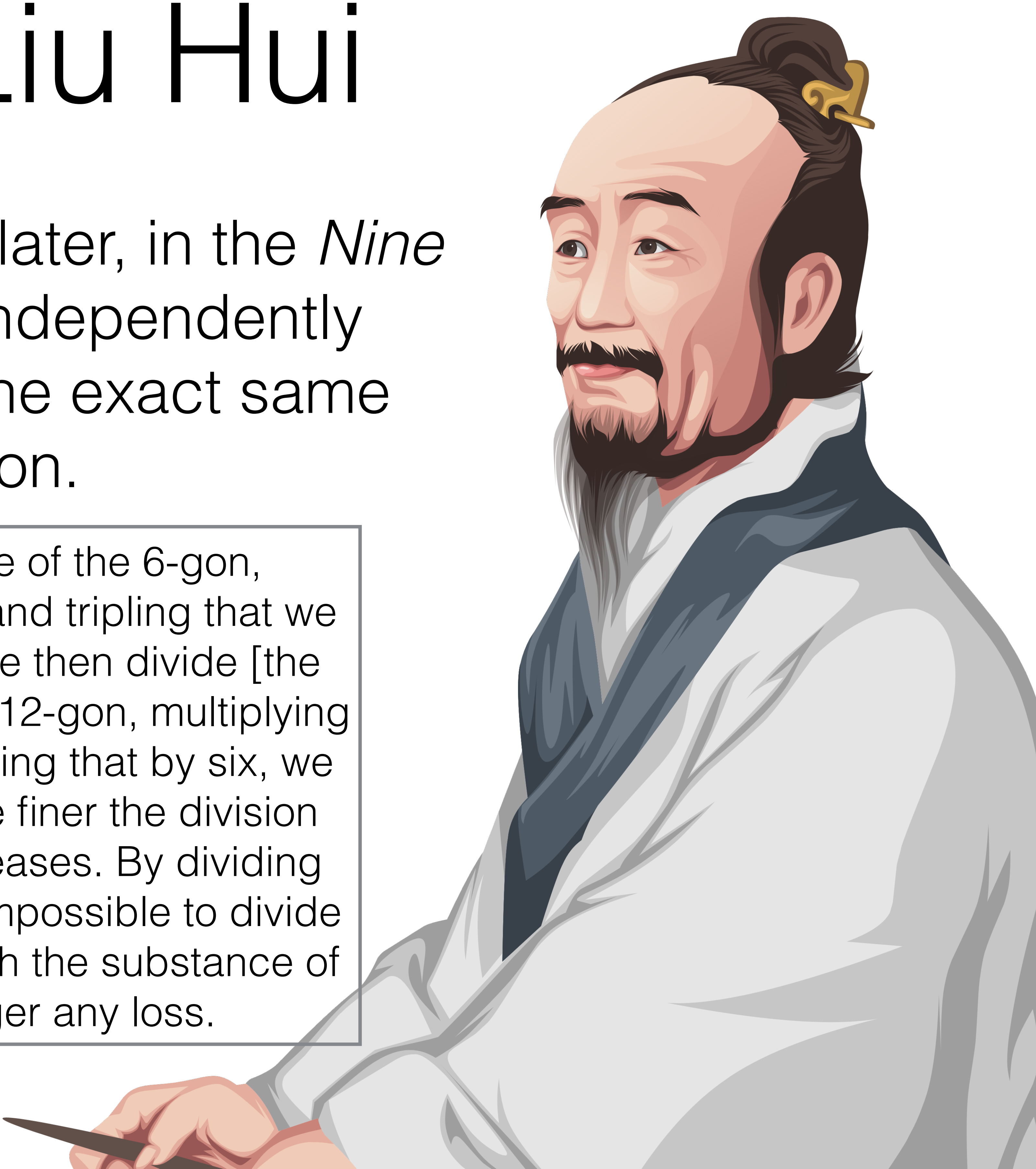
Making a figure, taking the side of the 6-gon, multiplying it by half the diameter and tripling that we obtain the area of the 12-gon. If we then divide [the circle] again, taking the side of the 12-gon, multiplying it by half the diameter and multiplying that by six, we obtain the area of the 24-gon. The finer the division becomes, the more the loss decreases. By dividing again and again until it becomes impossible to divide (further) we obtain coincidence with the substance of the disk and there is no longer any loss.



Madhava of Sangamagrama

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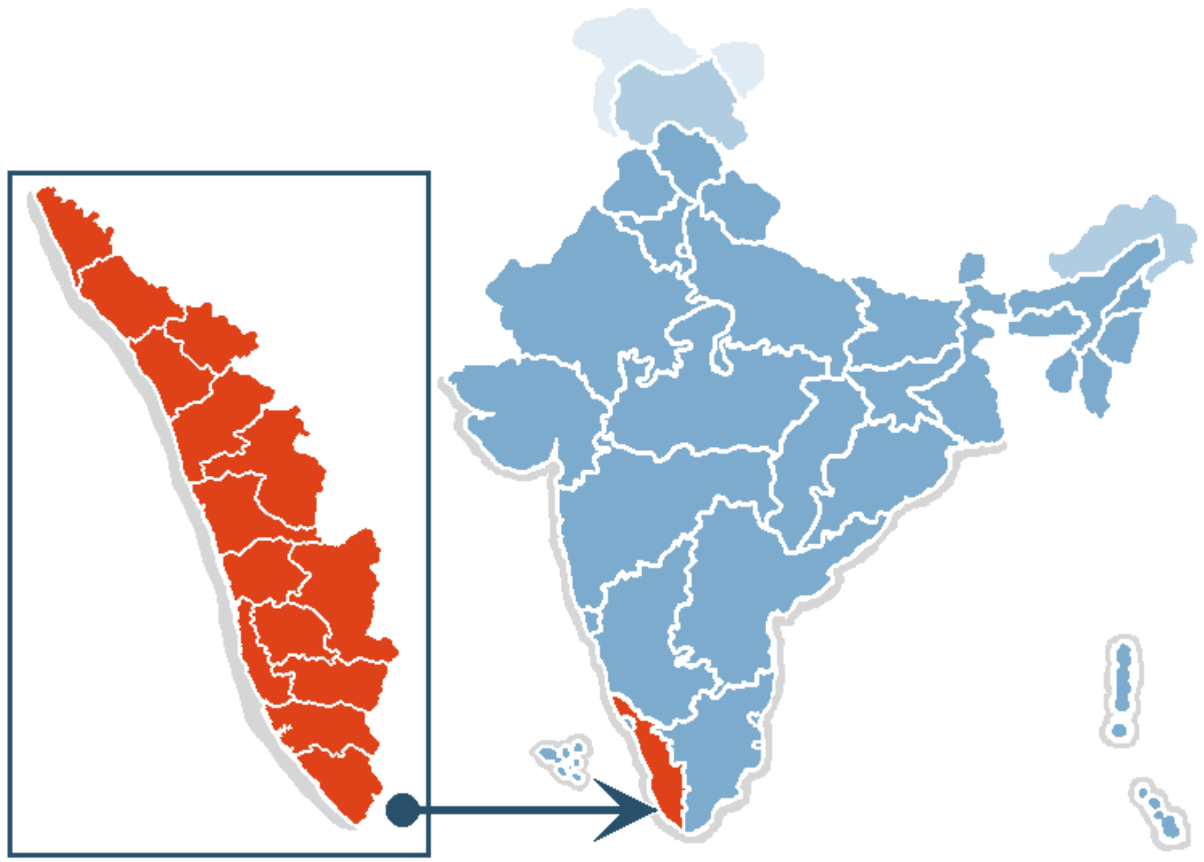

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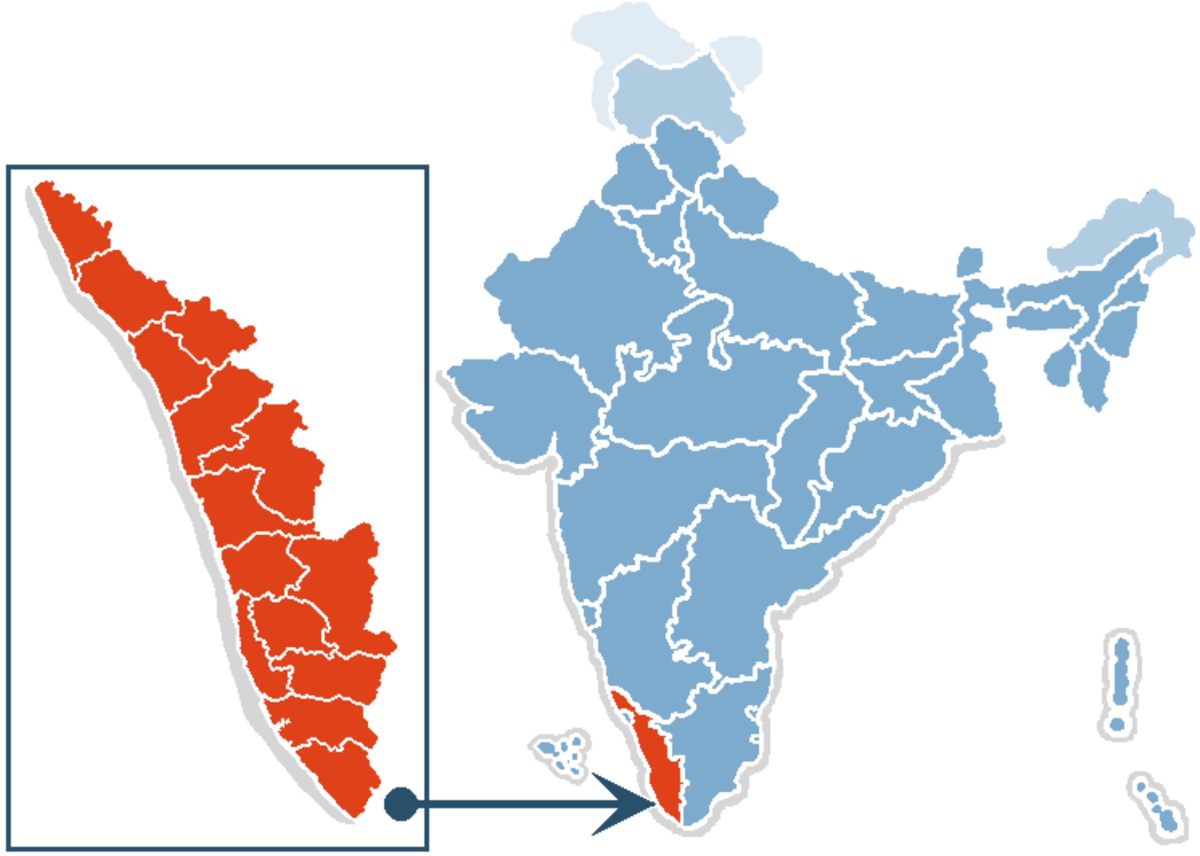

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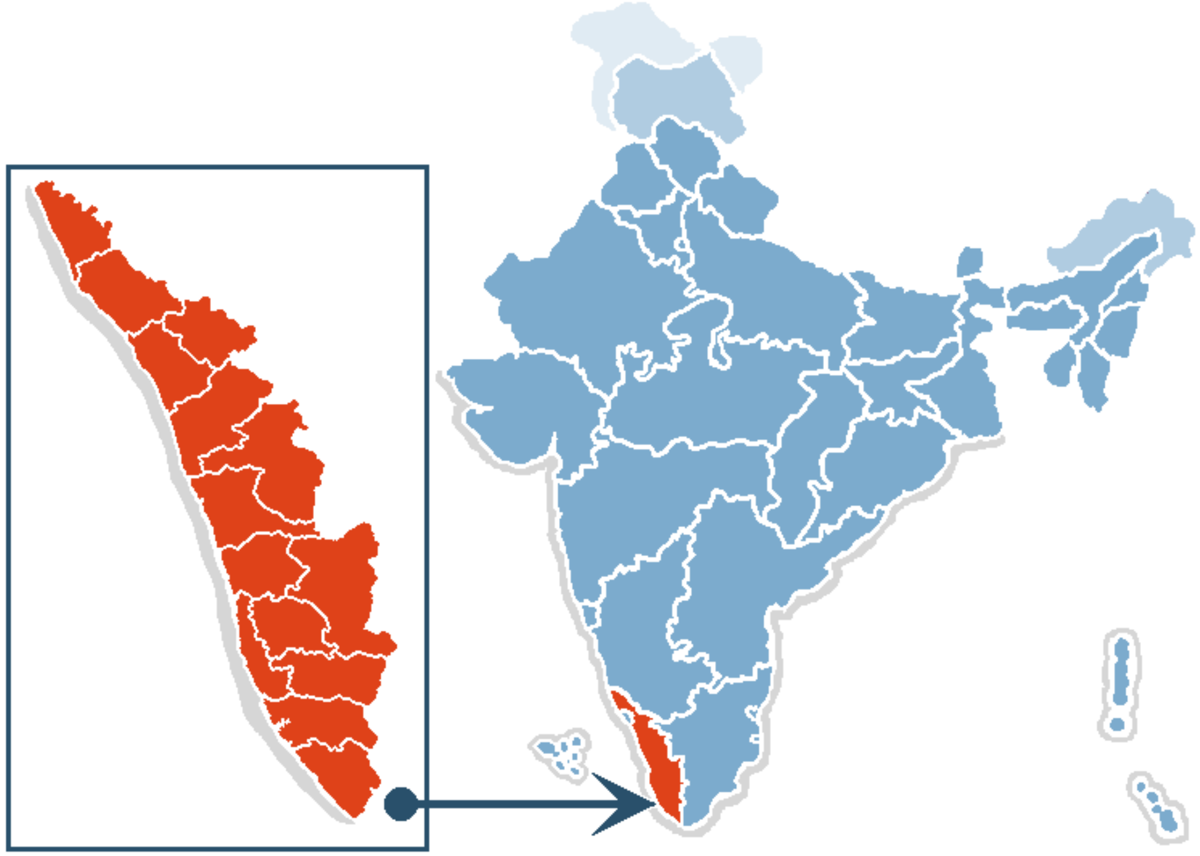

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- First, though, we discuss one result from another mathematician at his Kerala School.

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Theorem. Let k be a fixed positive integer. For large positive integers n,

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Note the resemblance to $\displaystyle\int_0^n x^k dx = \frac{n^{k+1}}{k+1}$.

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recall $s_1 \approx \frac{1}{2}n^2$

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The Kerala School

Theorem. $1^k + 2^k + 3^k + \ldots + n^k \approx \dfrac{n^{k+1}}{k+1}$.

Proof. Let $s_k = 1^k + 2^k + 3^k + \ldots + n^k$.

Next they proved the $k = 2$ case. Note that

$$s_1 \cdot n - s_2 = 1(n-1) + 2(n-2) + 3(n-3) + \ldots + (n-1)(1) + 0$$

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Solving, $\boxed{s_2 \approx \dfrac{n^3}{3}}$

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(Not rigorous by today's standards, but they had the main ideas and clearly understood induction.)

Madhava of Sangamagrama

Madhava of Sangamagrama

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In this case, note that $r, r^2, r^3, r^4, \ldots$ are all between 0 and 1.

Madhava of Sangamagrama

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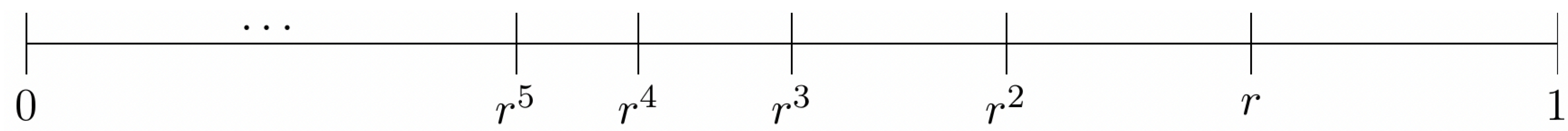

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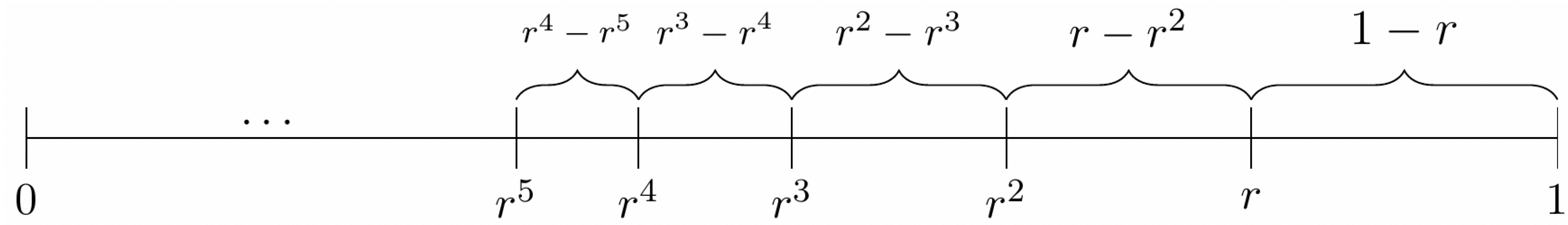

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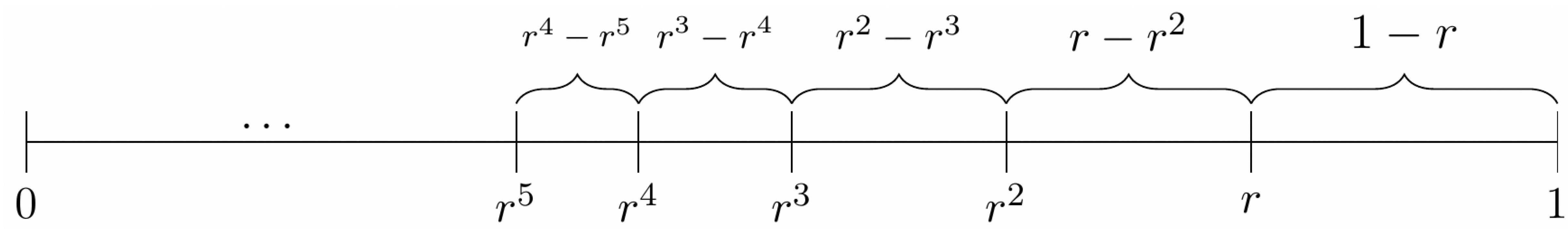

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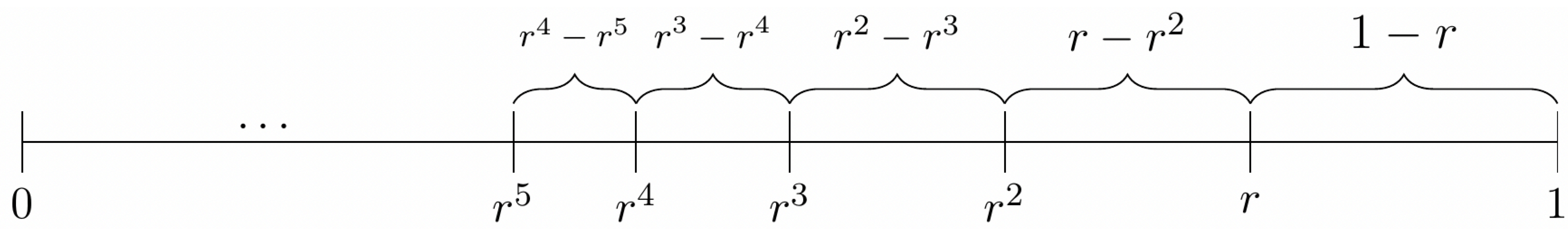

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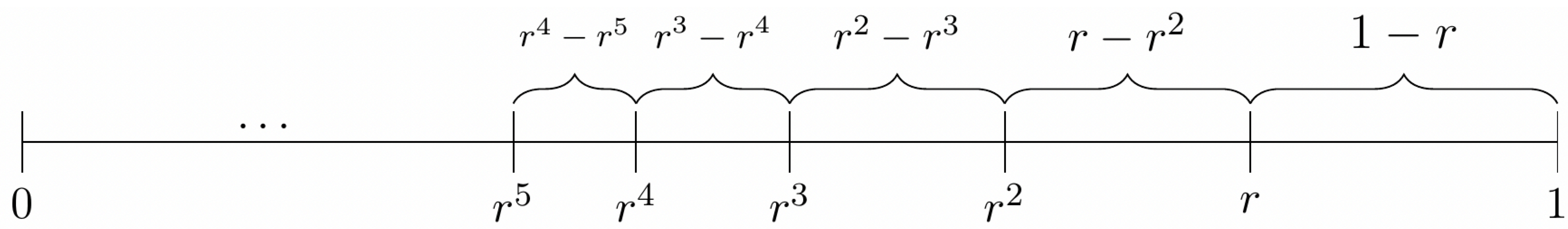

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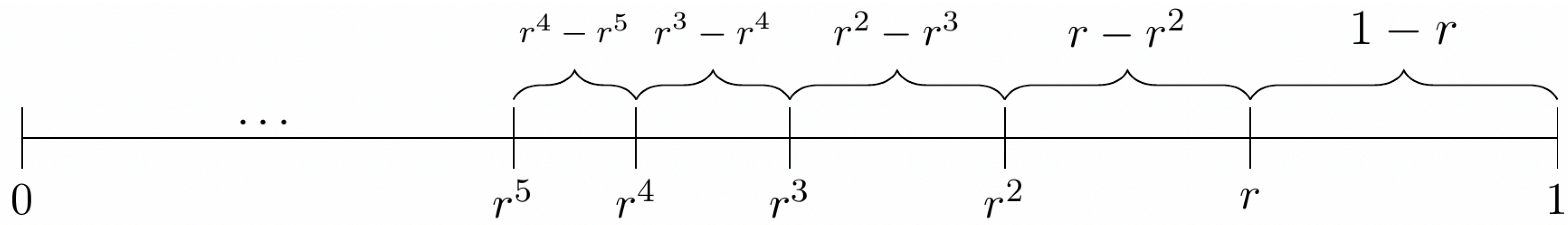

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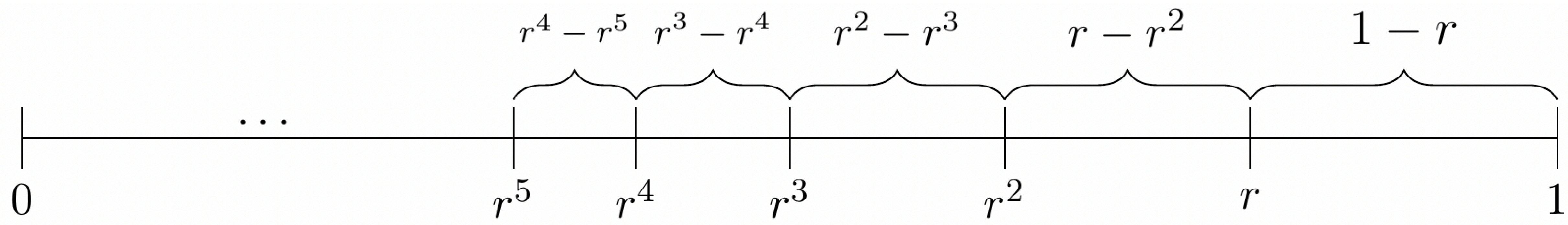

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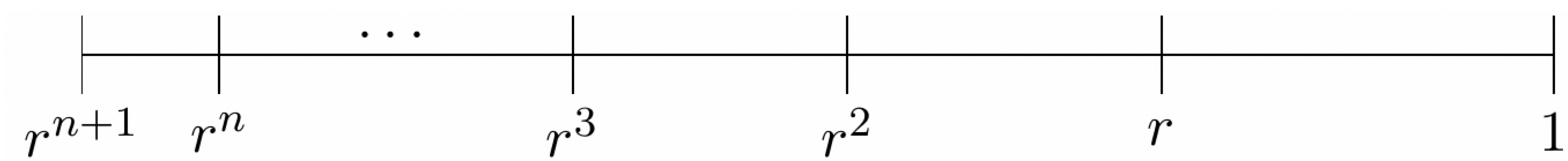

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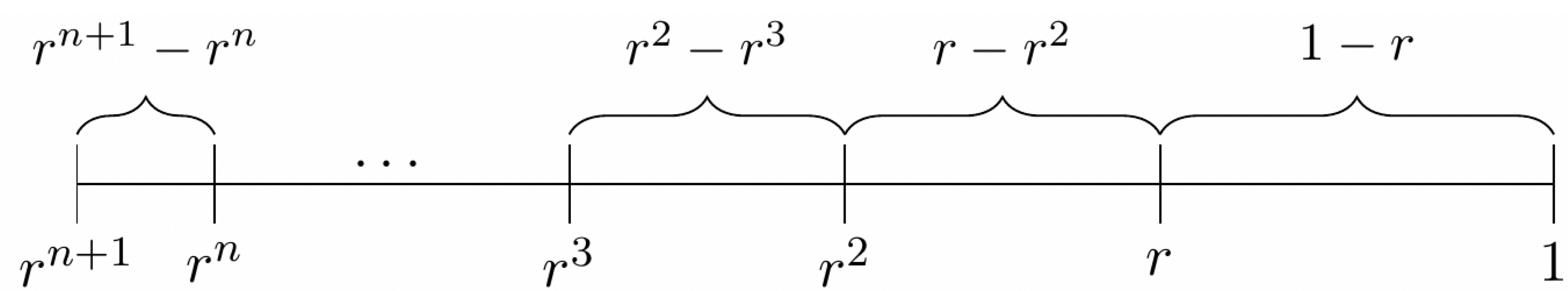

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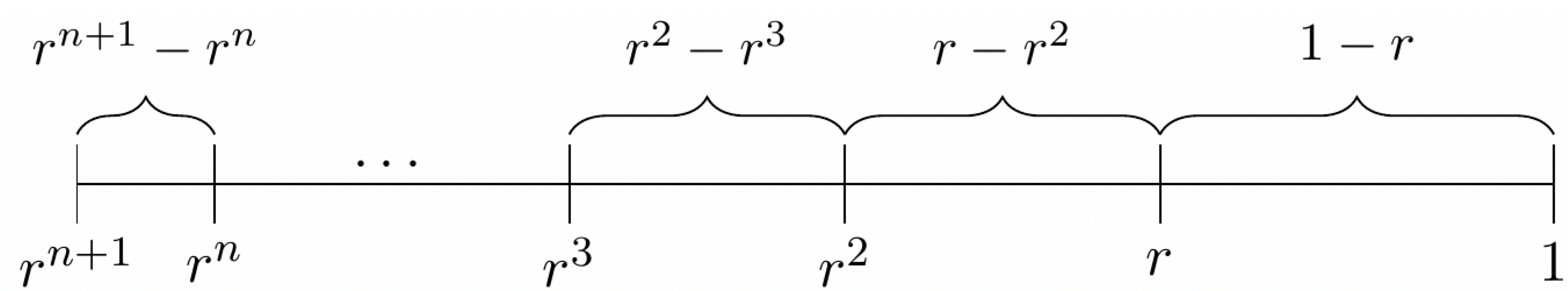

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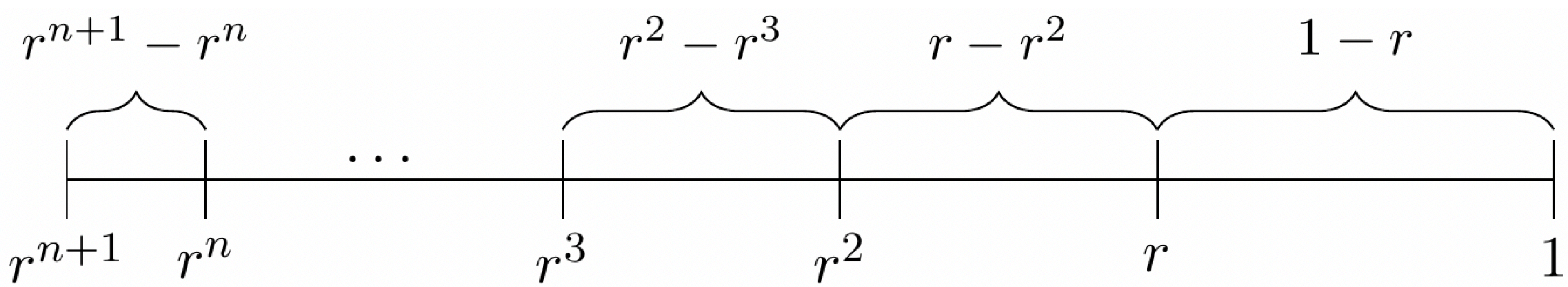

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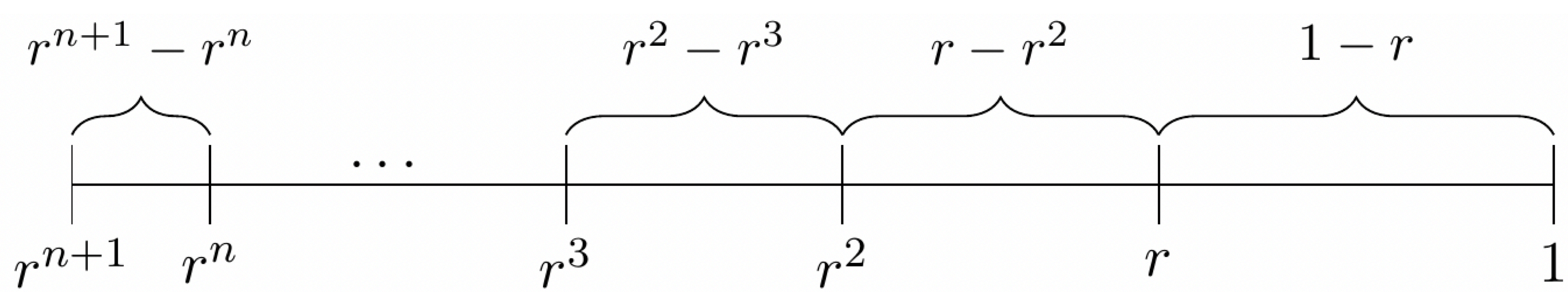

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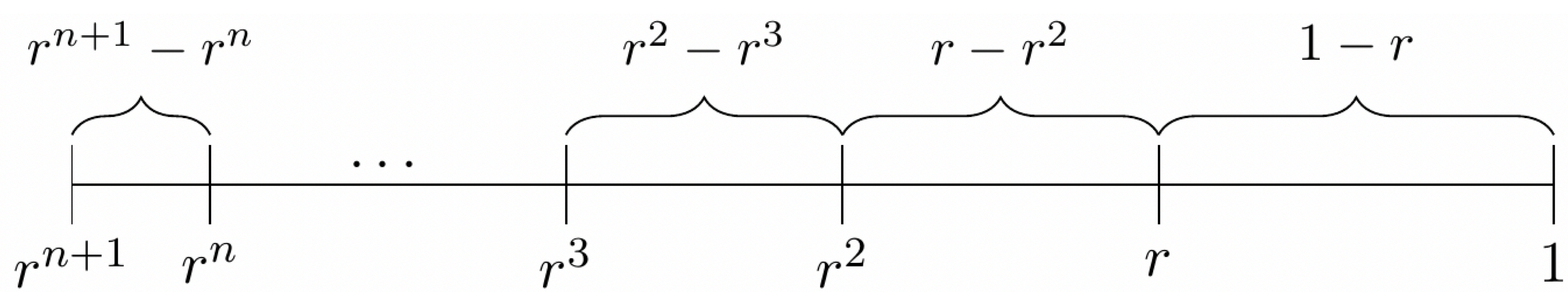

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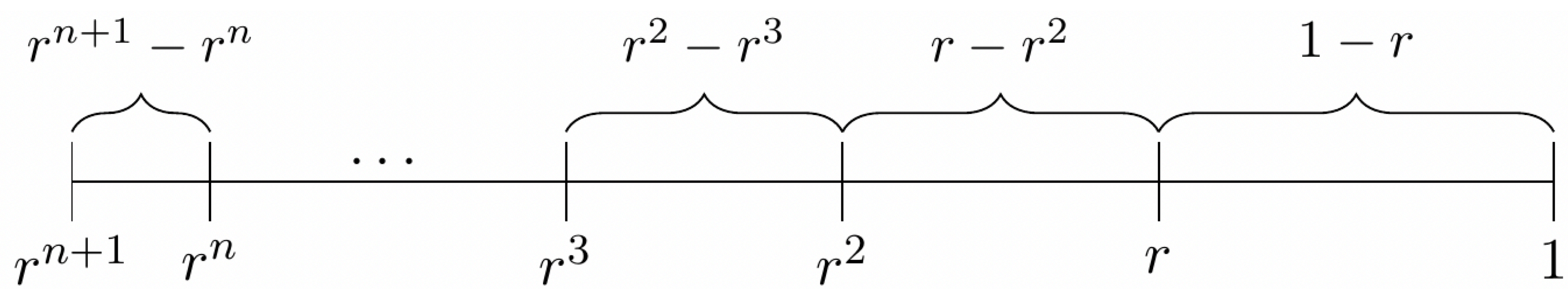

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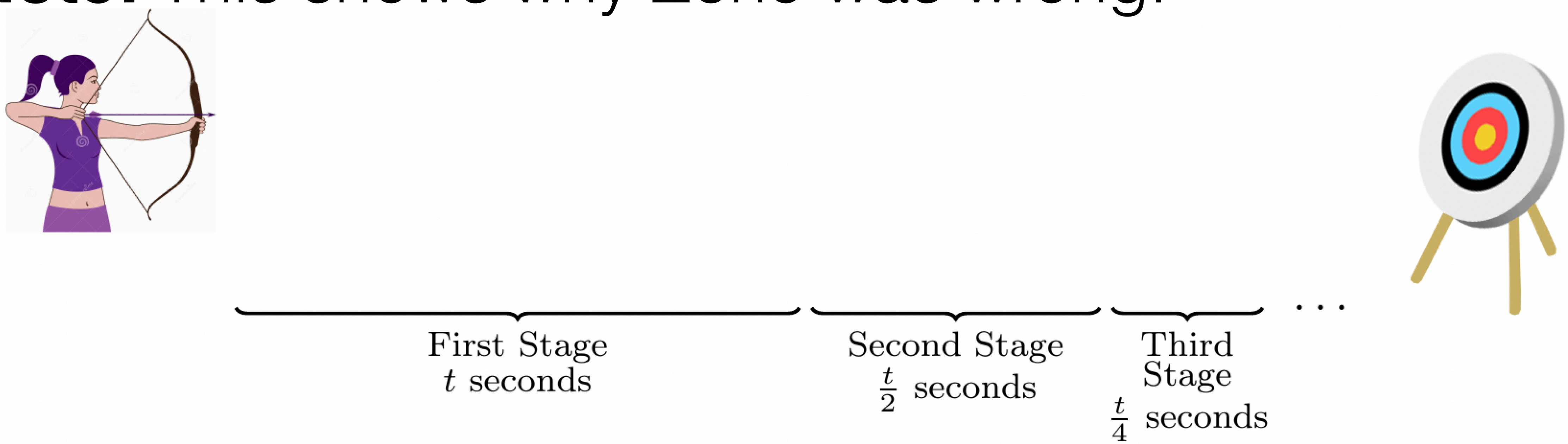

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First Stage
t seconds

Second Stage
$\frac{t}{2}$ seconds

Third Stage
$\frac{t}{4}$ seconds

. . .

Then the total amount of time (in seconds) to complete <u>all</u> the stages is

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So the the arrow completes all the stages in just 2t seconds.

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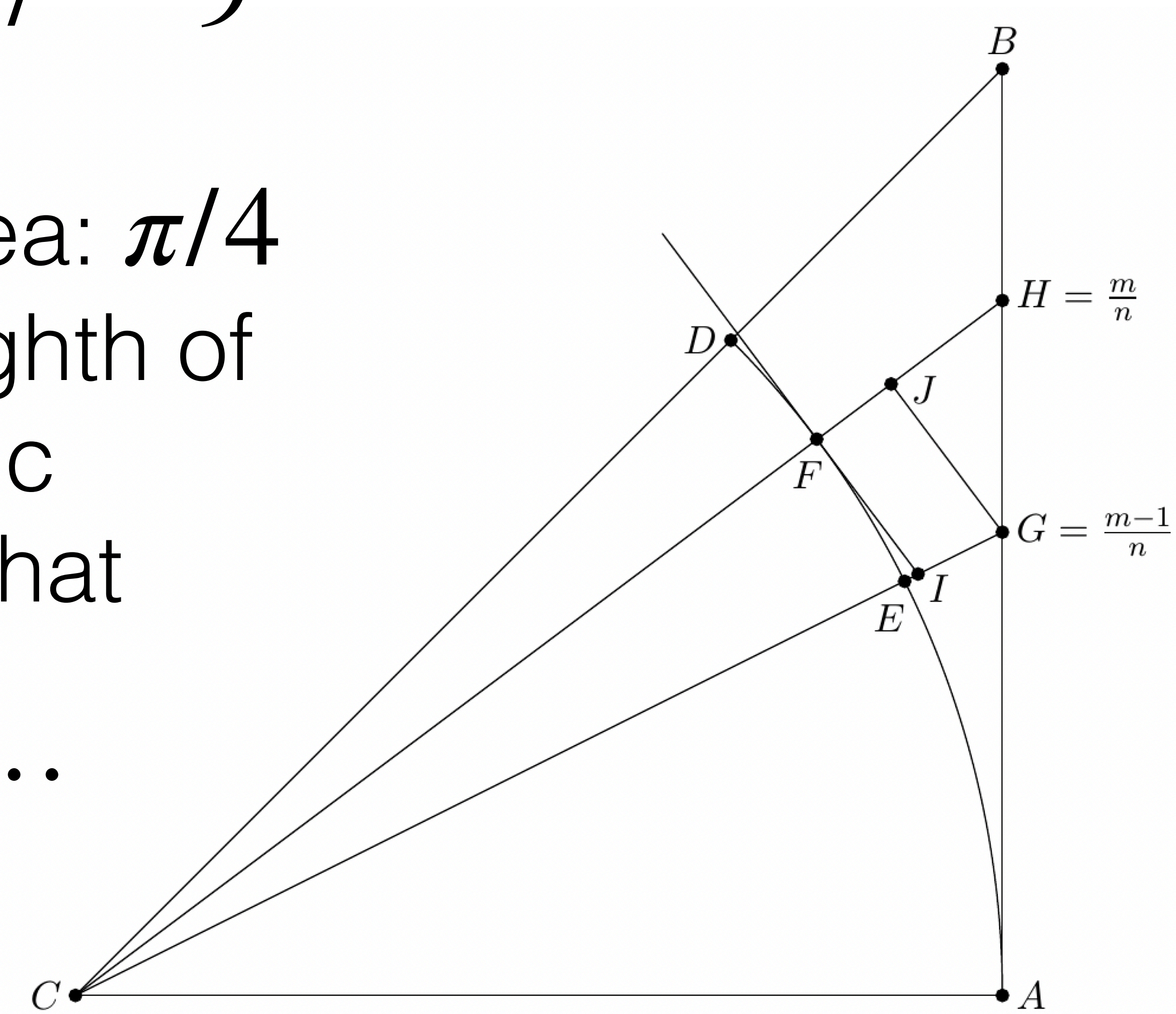
Theorem. $$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

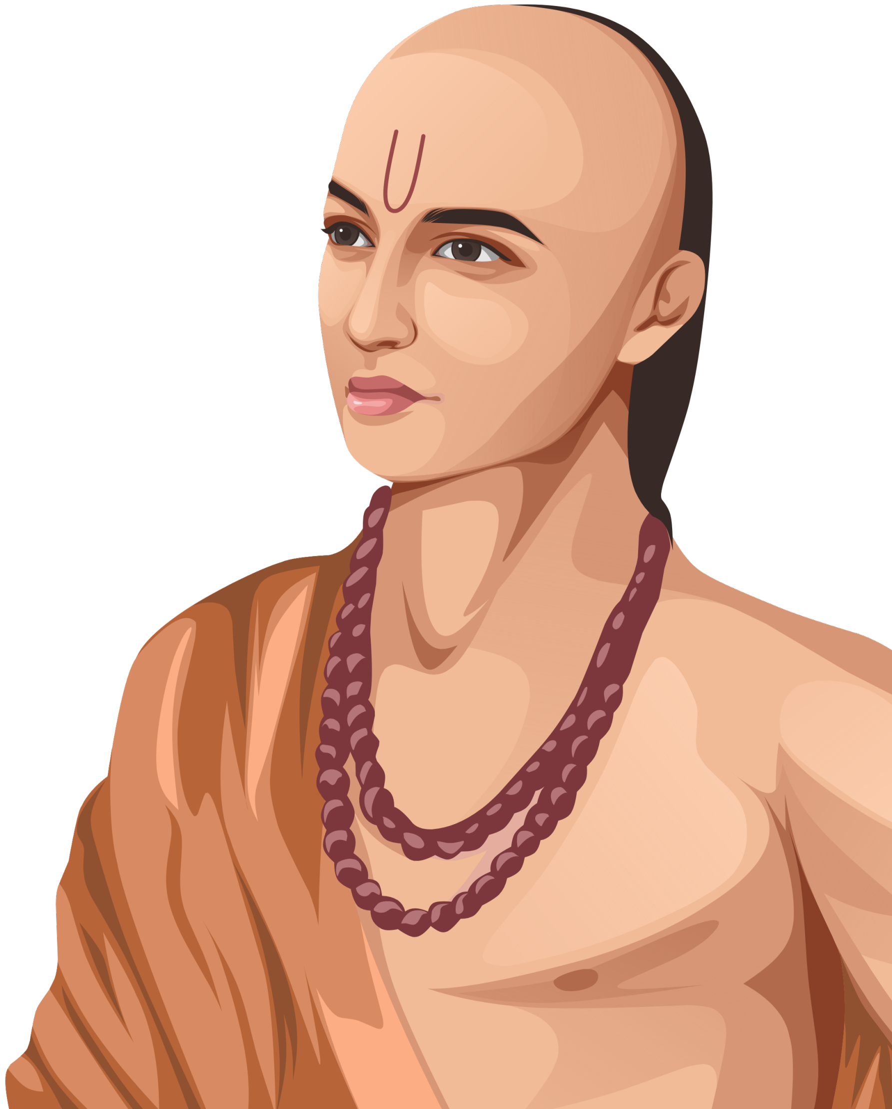
Proof. It's complicated. Basic idea: $\pi/4$ equals the length of an eighth of a unit circle. Use geometric series in reverse to show that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

also equals this.



Bhāskara II



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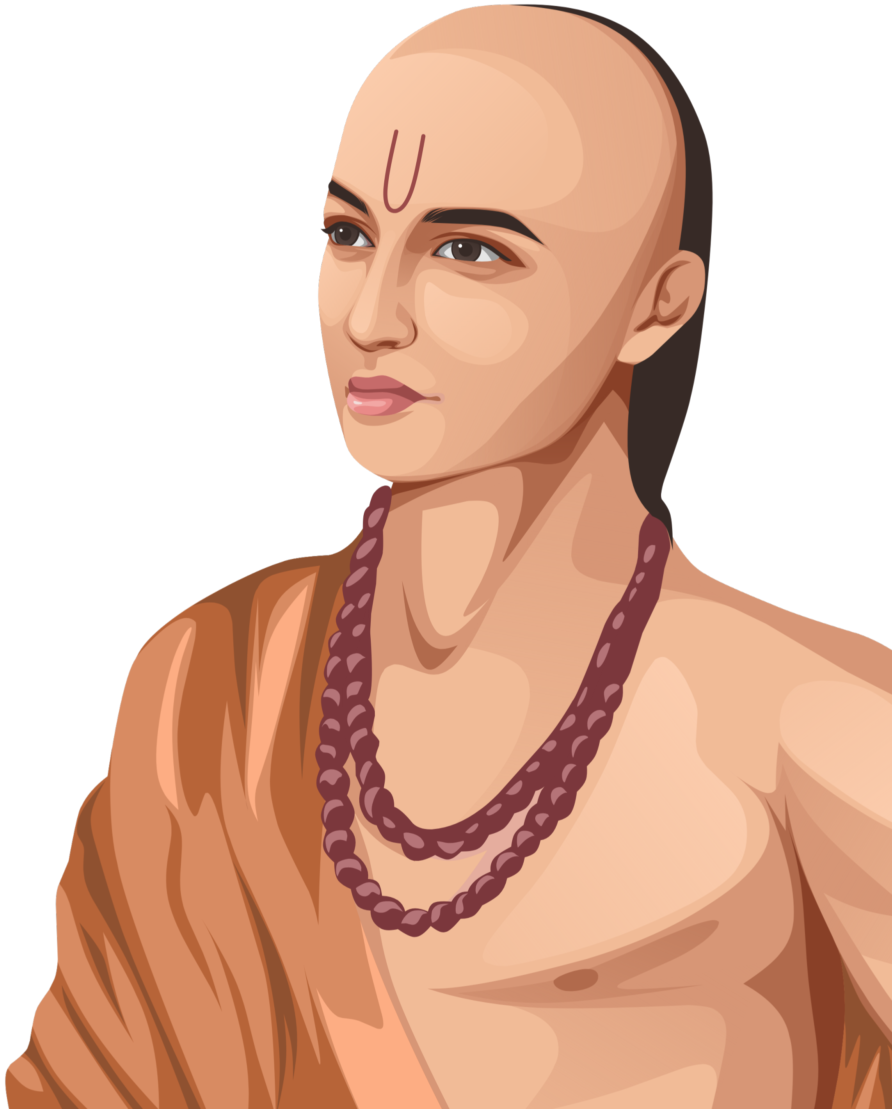
Bhāskara II was a prominent mathematician-astronomer.



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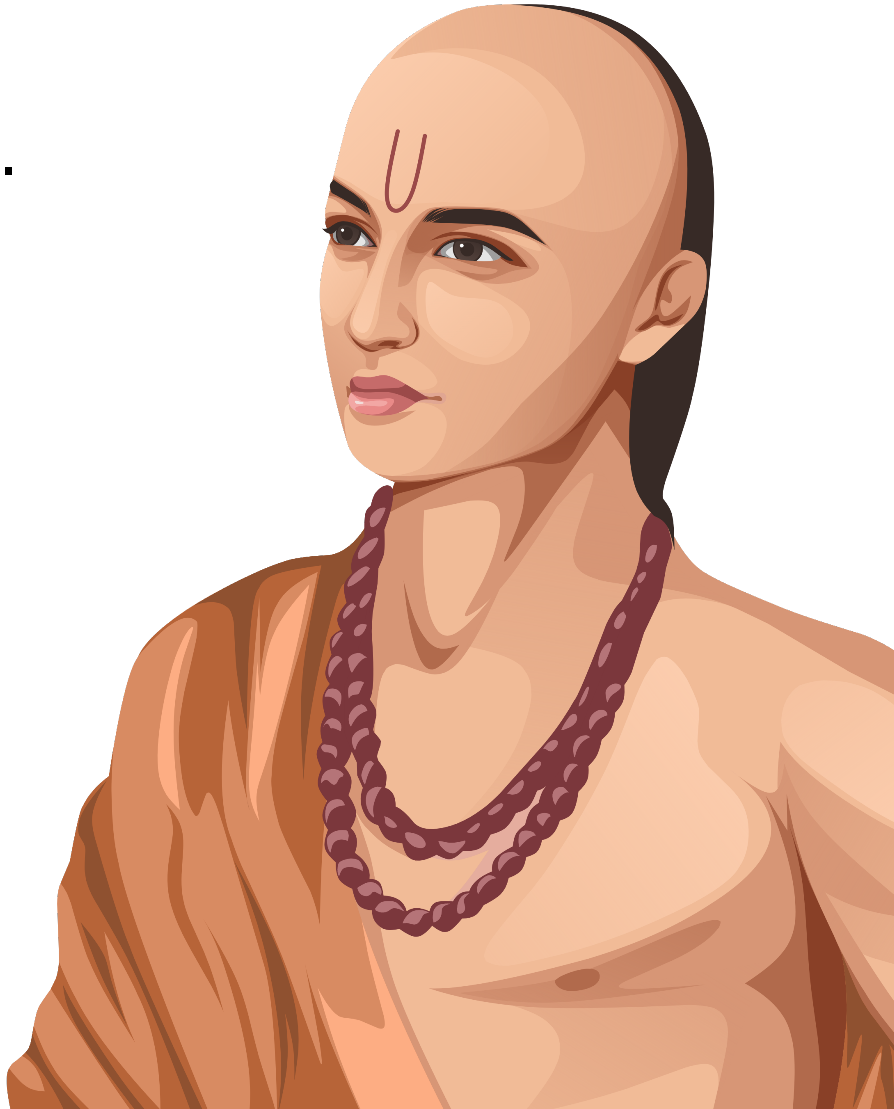
  This essentially says $\dfrac{d}{dx}\sin(x) = \cos(x)$.



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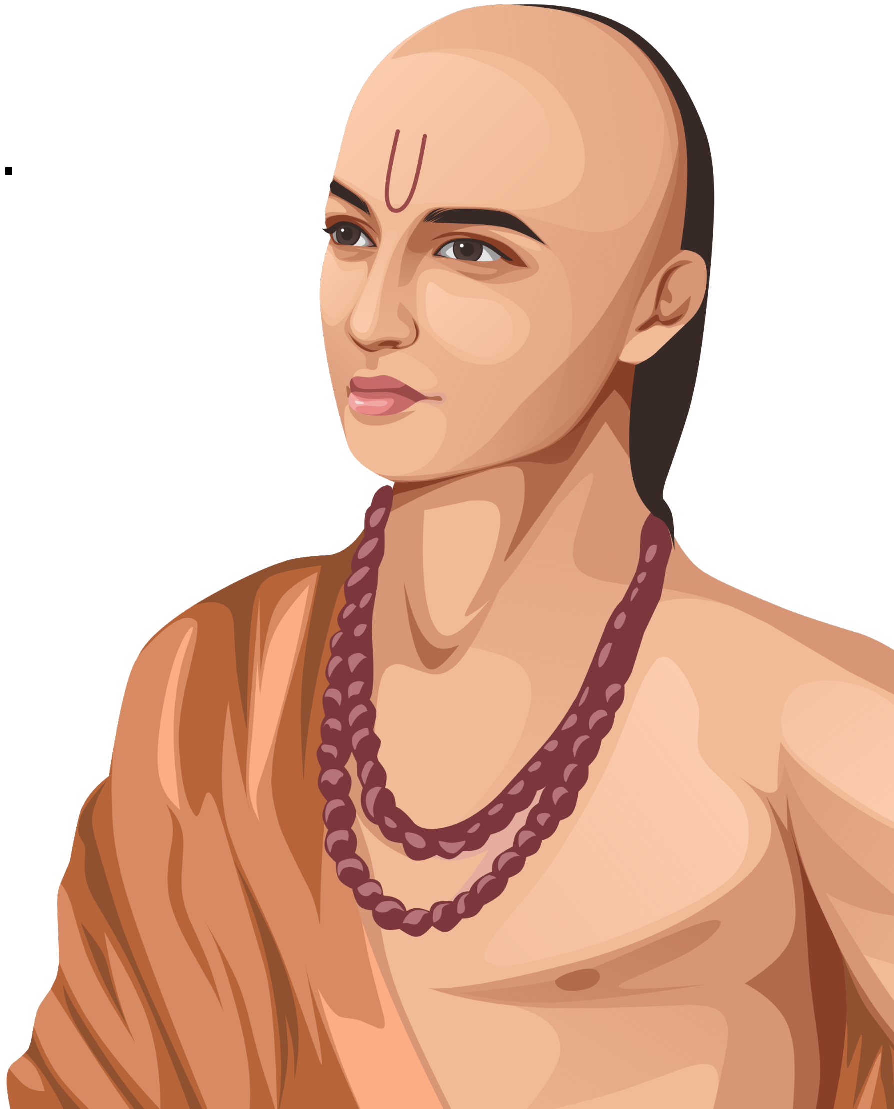

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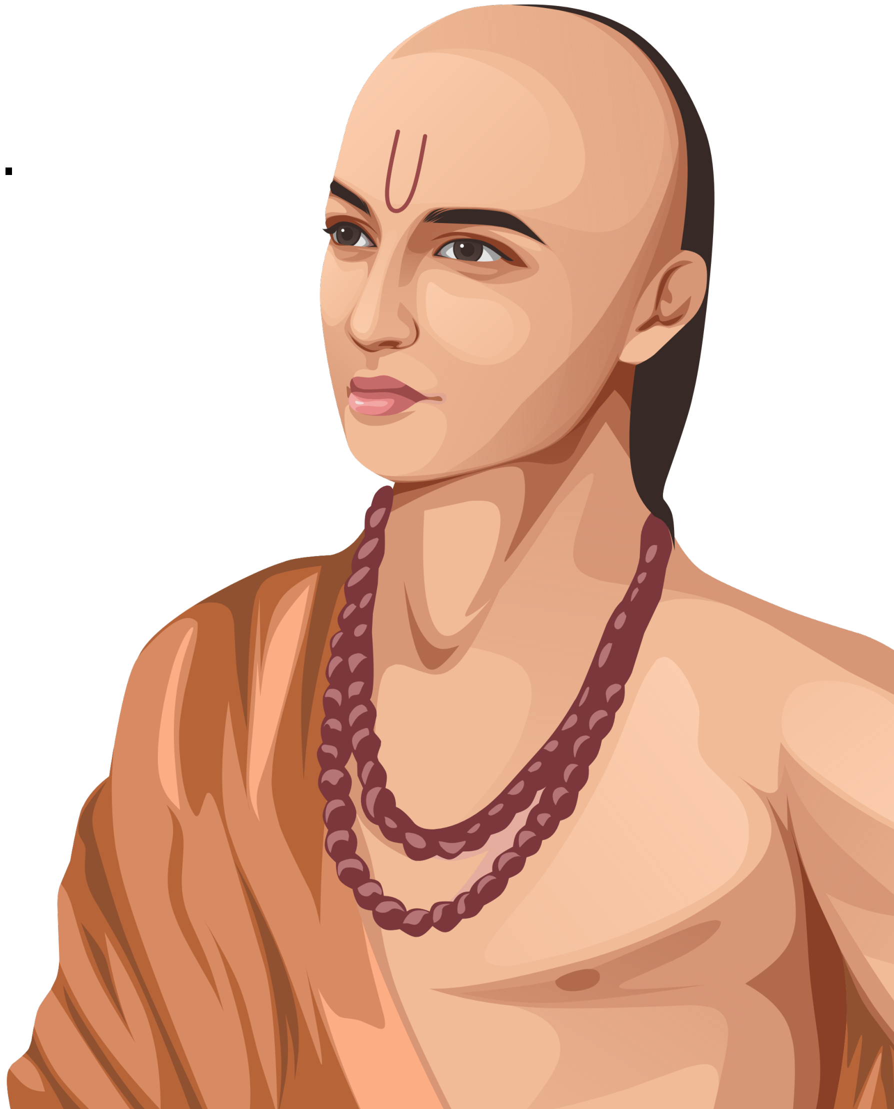

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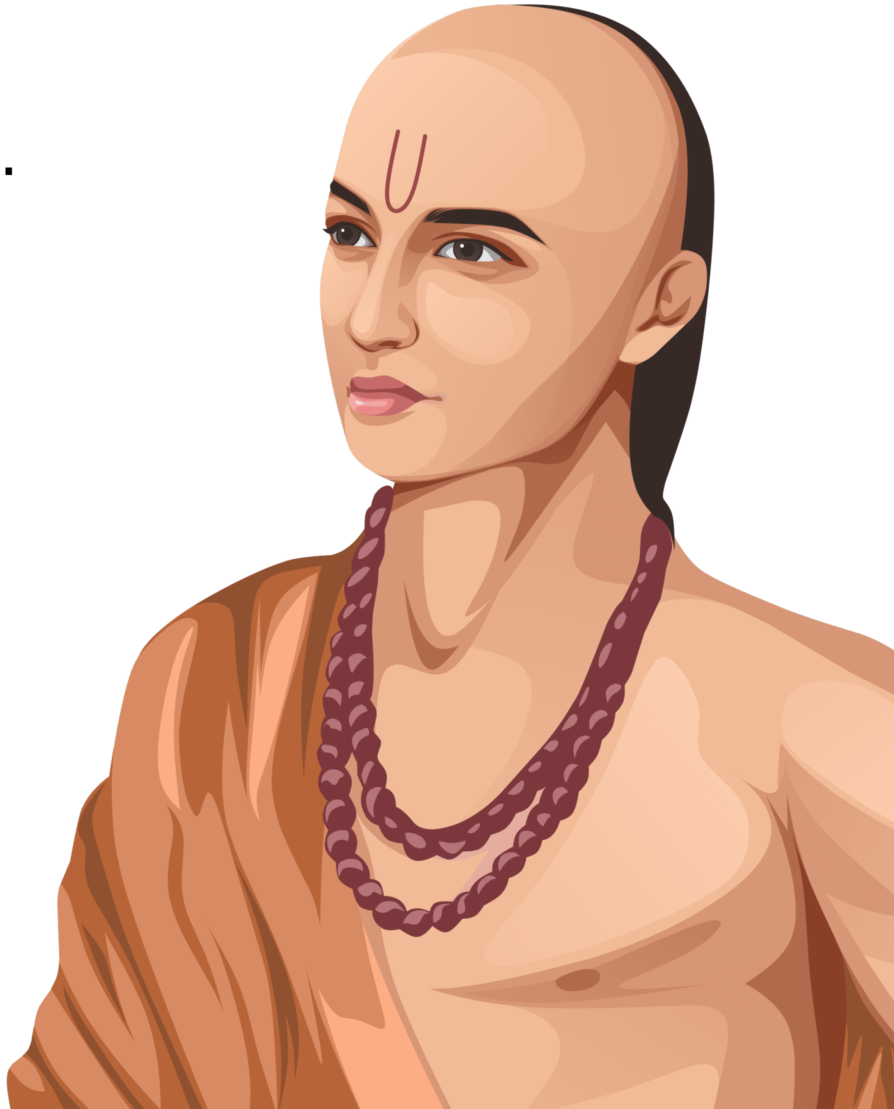

Cavalieri’s Principle

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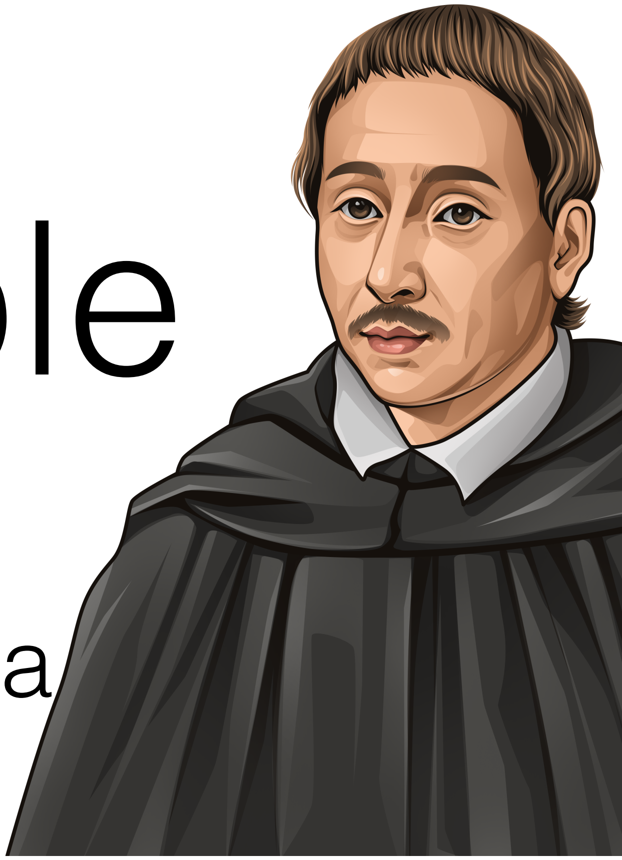

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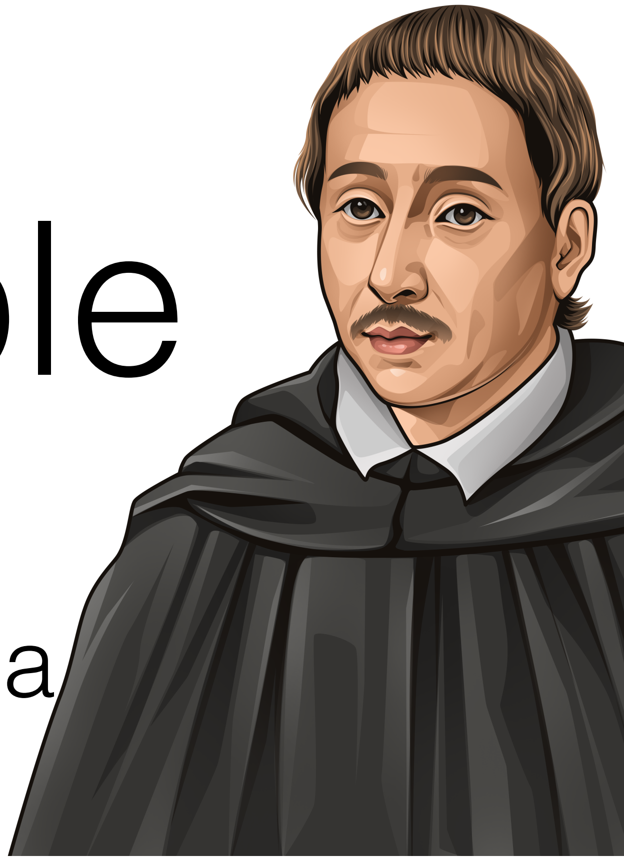
Cavalieri's Principle



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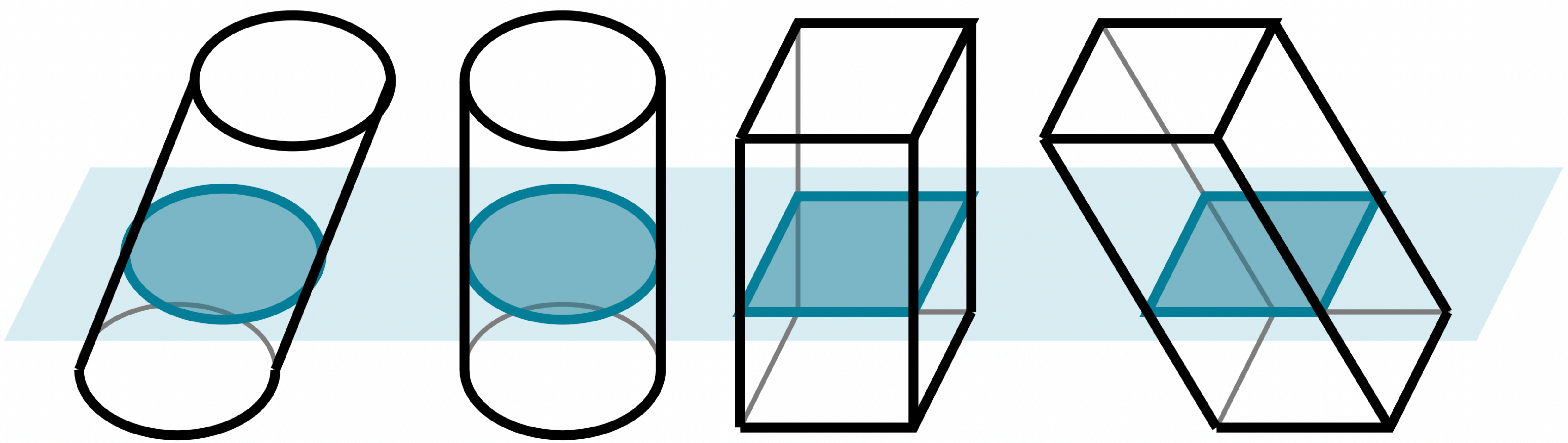

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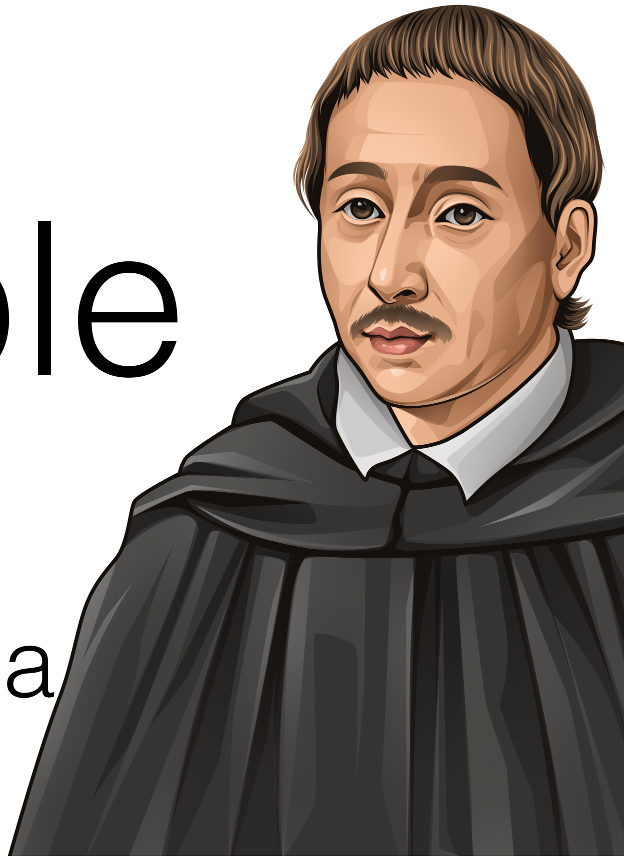

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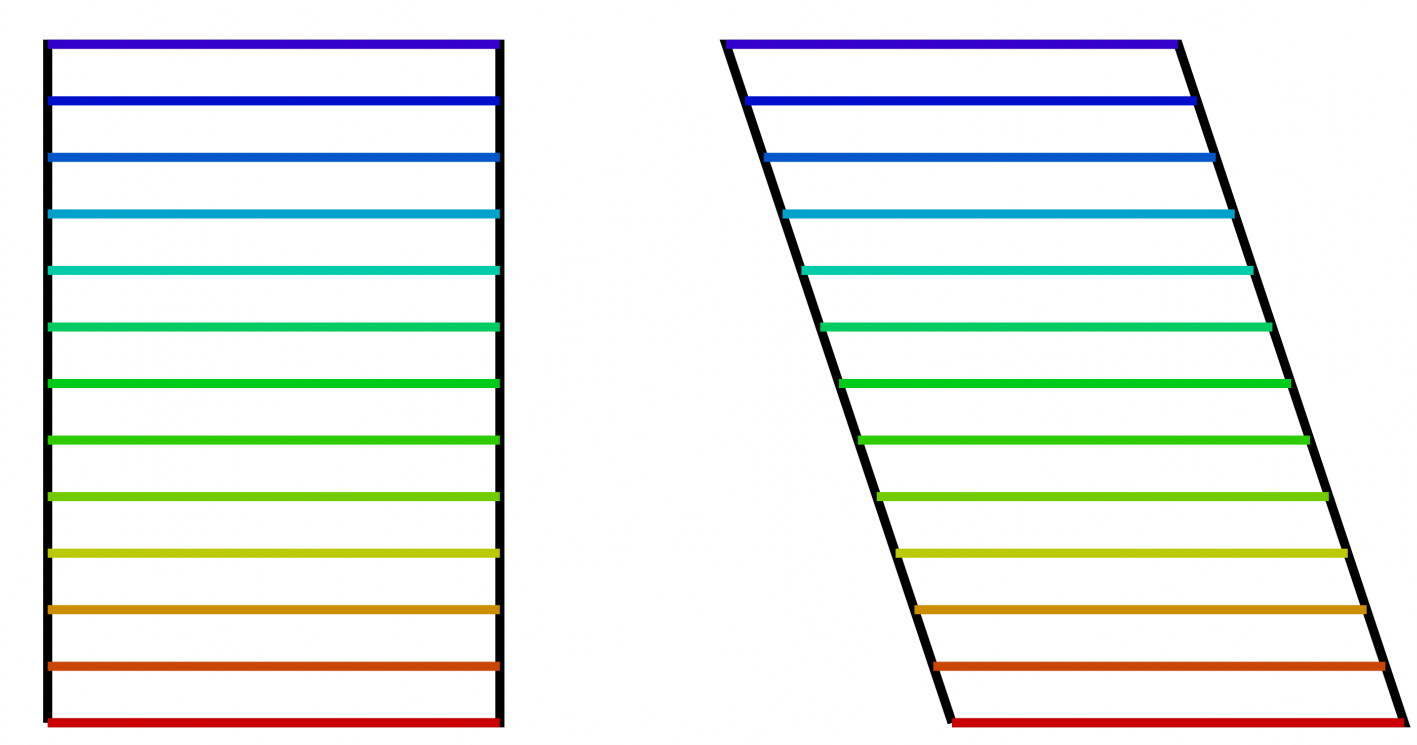

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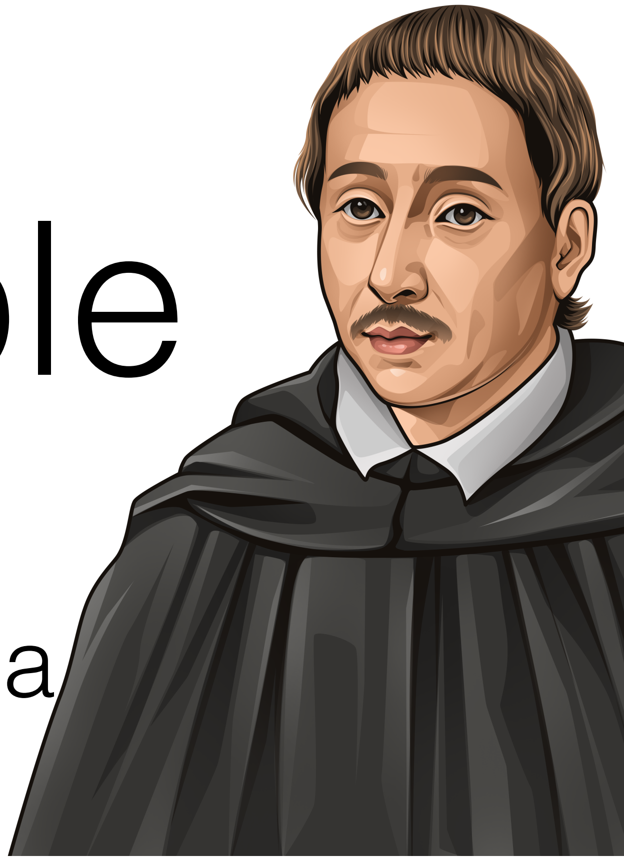

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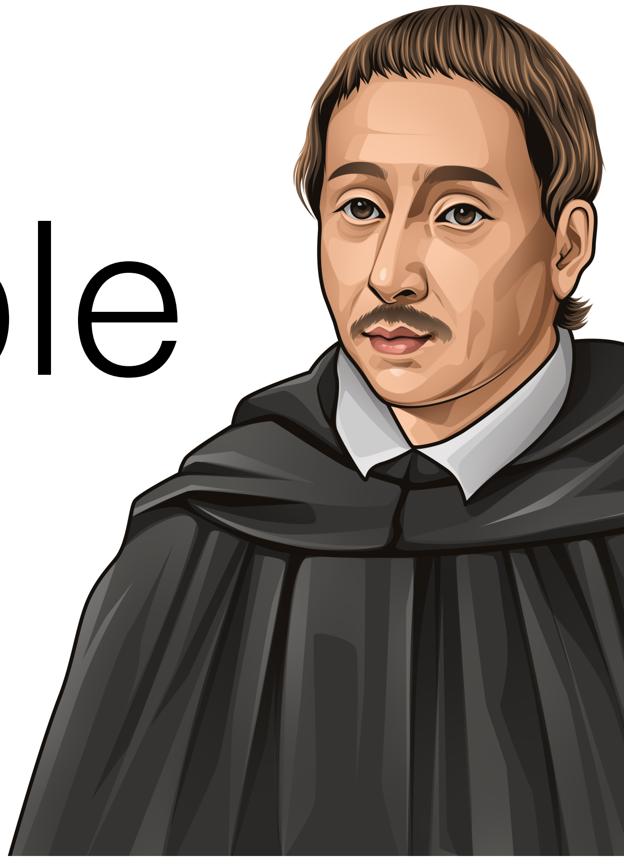

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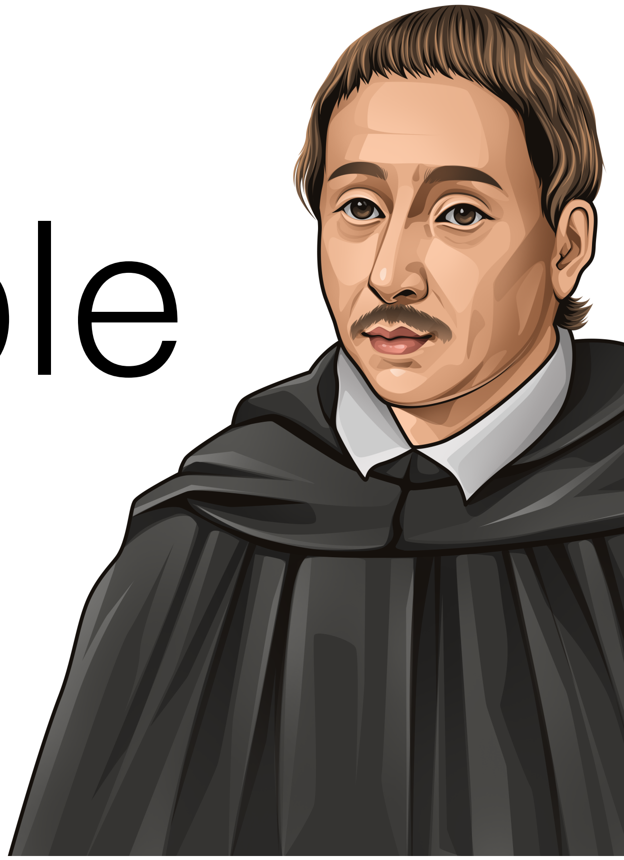
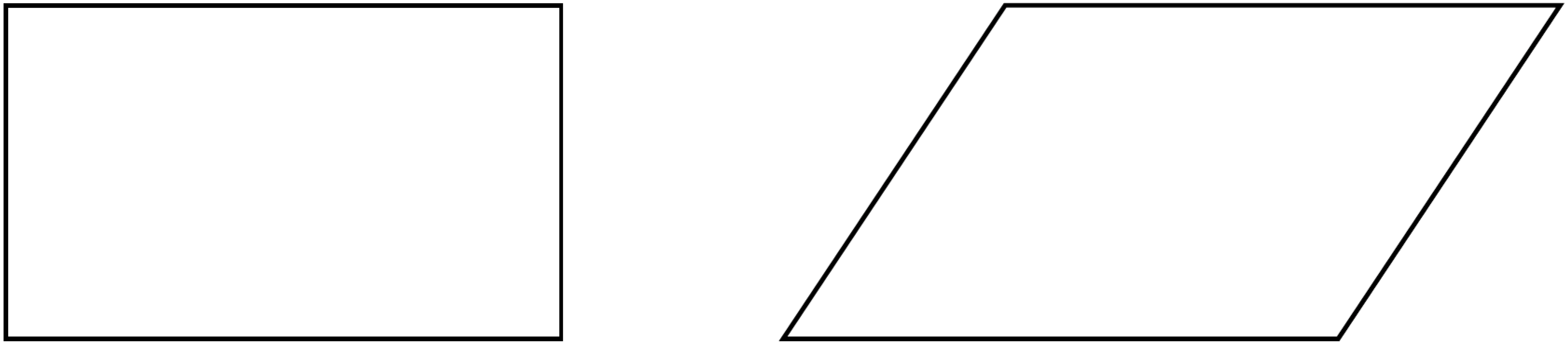


- Classic example: This rectangle and parallelogram have the same area.



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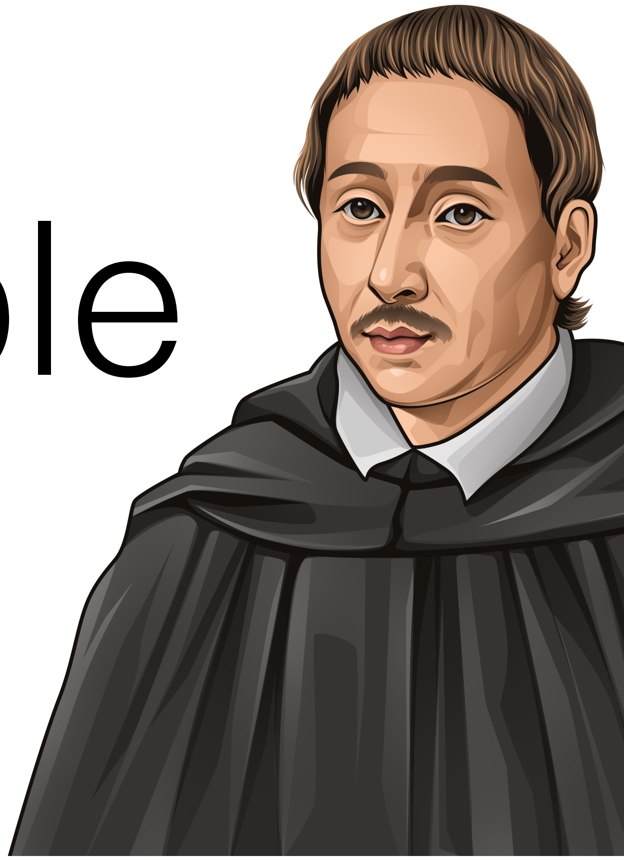
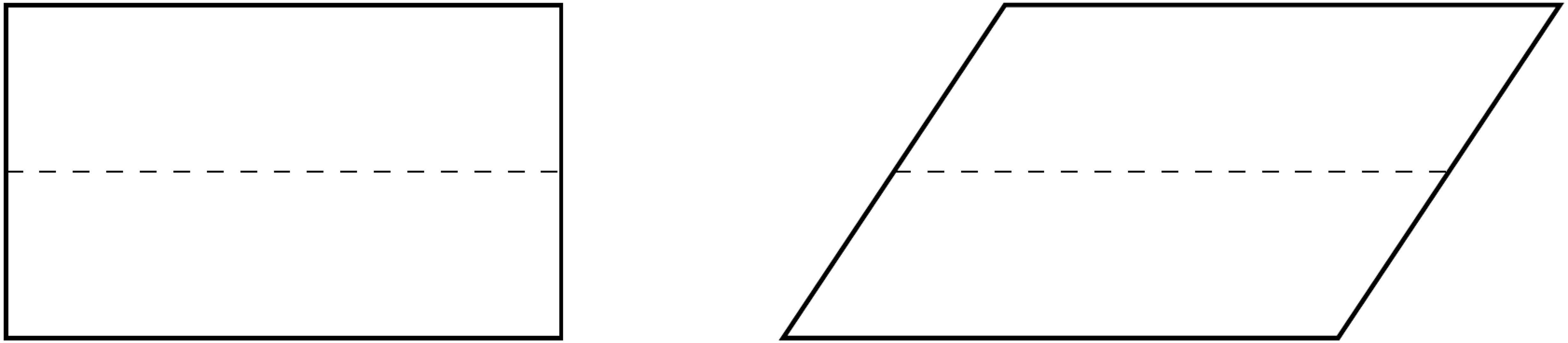


- They have the same base and height, and every horizontal slice produces cross-sections of the same width.



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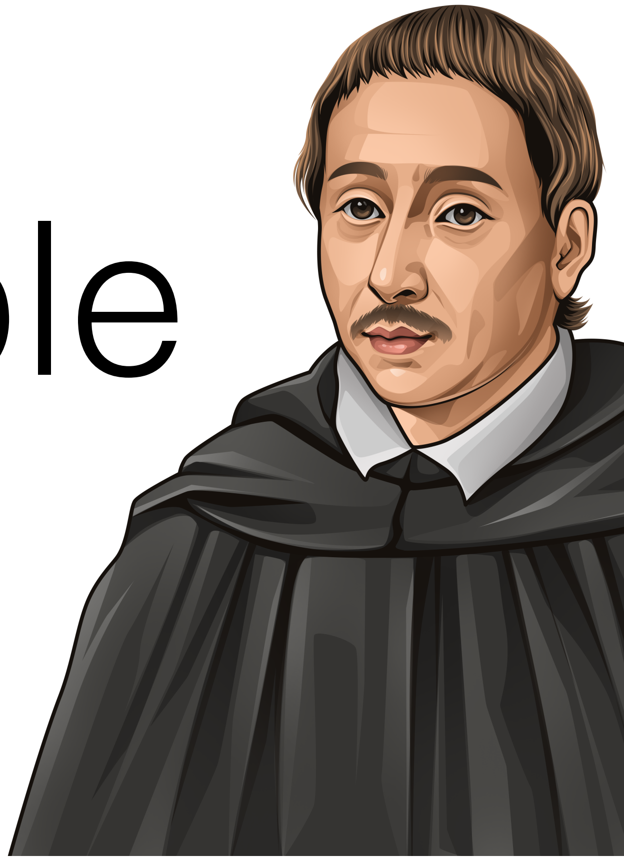
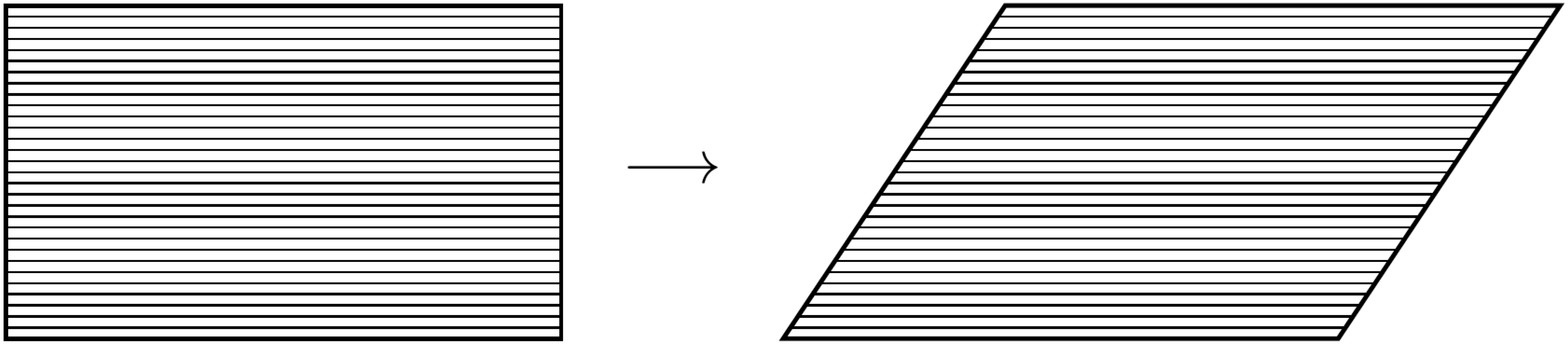


- Therefore, Cavalieri’s principle says that they must have the same area, because their areas are made up of the same infinite set of lines.



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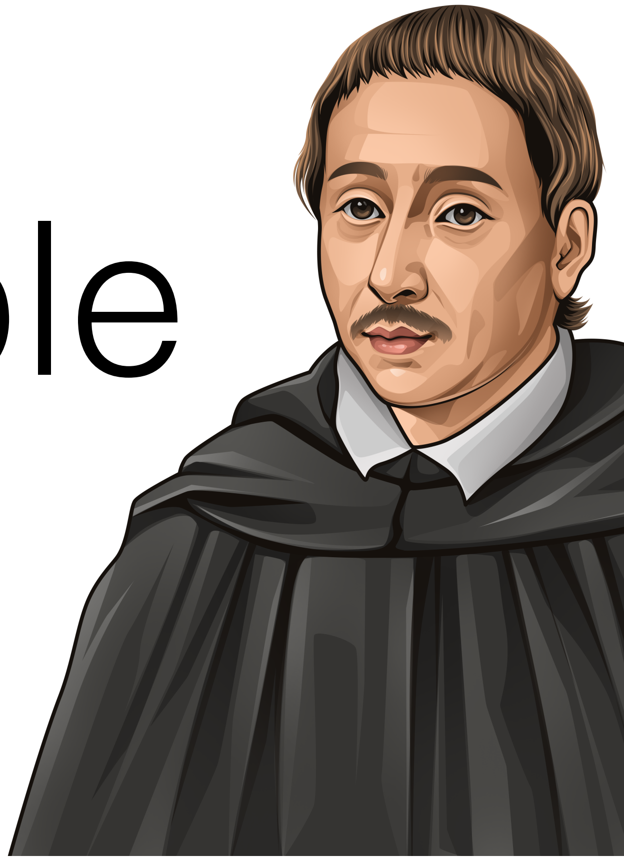

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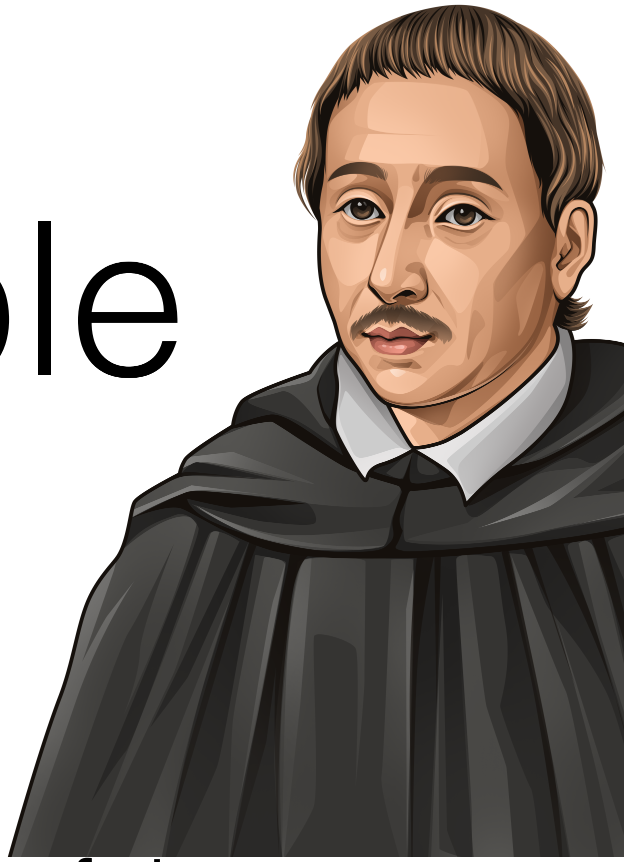

- Cavalieri big theorem, in modern notation, is that $$\int_0^a x^n \, dx = \frac{a^{n+1}}{n+1}$$ for $n = 1,2,3,\ldots,9$. This is a special case of the power rule for integrals.

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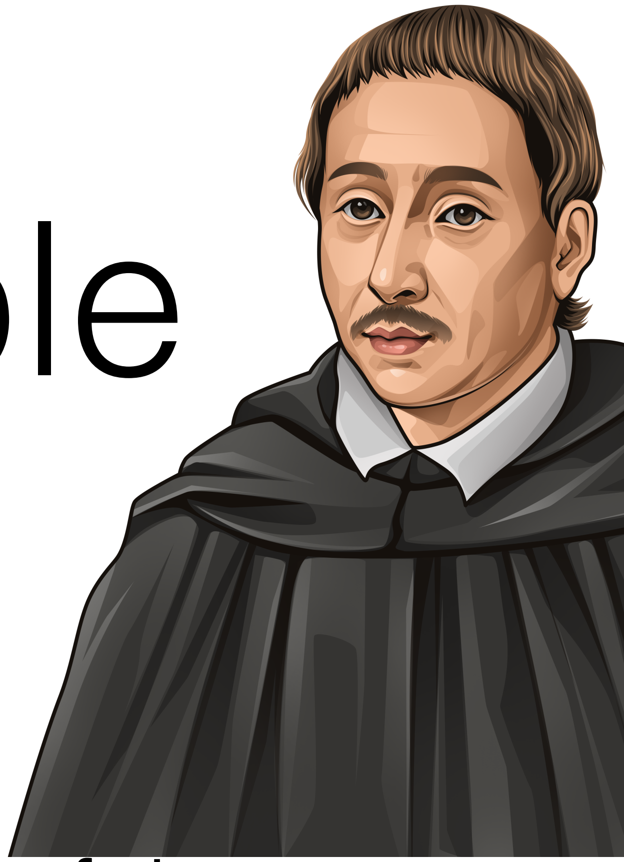

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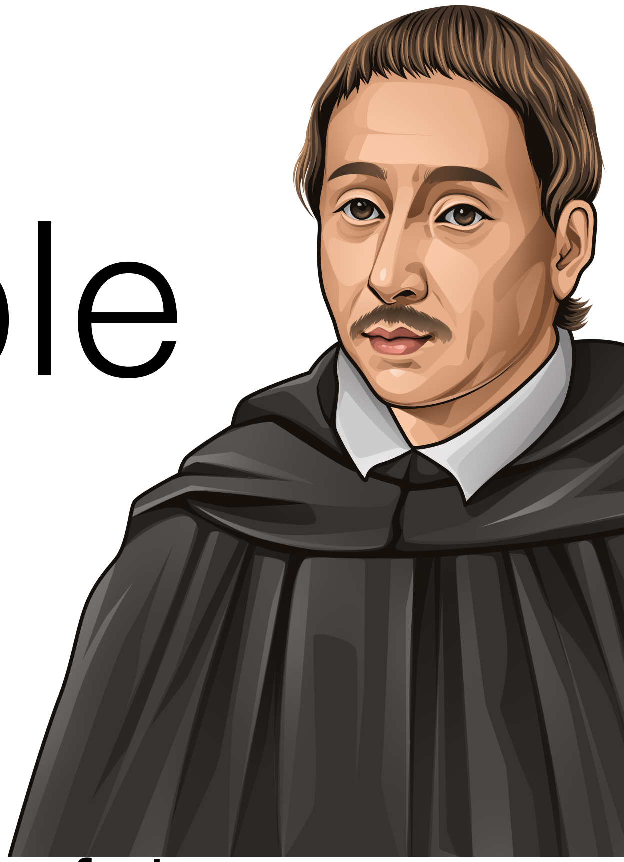

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- To do this, we will instead think about the area from $x = 0$ to $x = 2a$.

Cavalieri’s Principle

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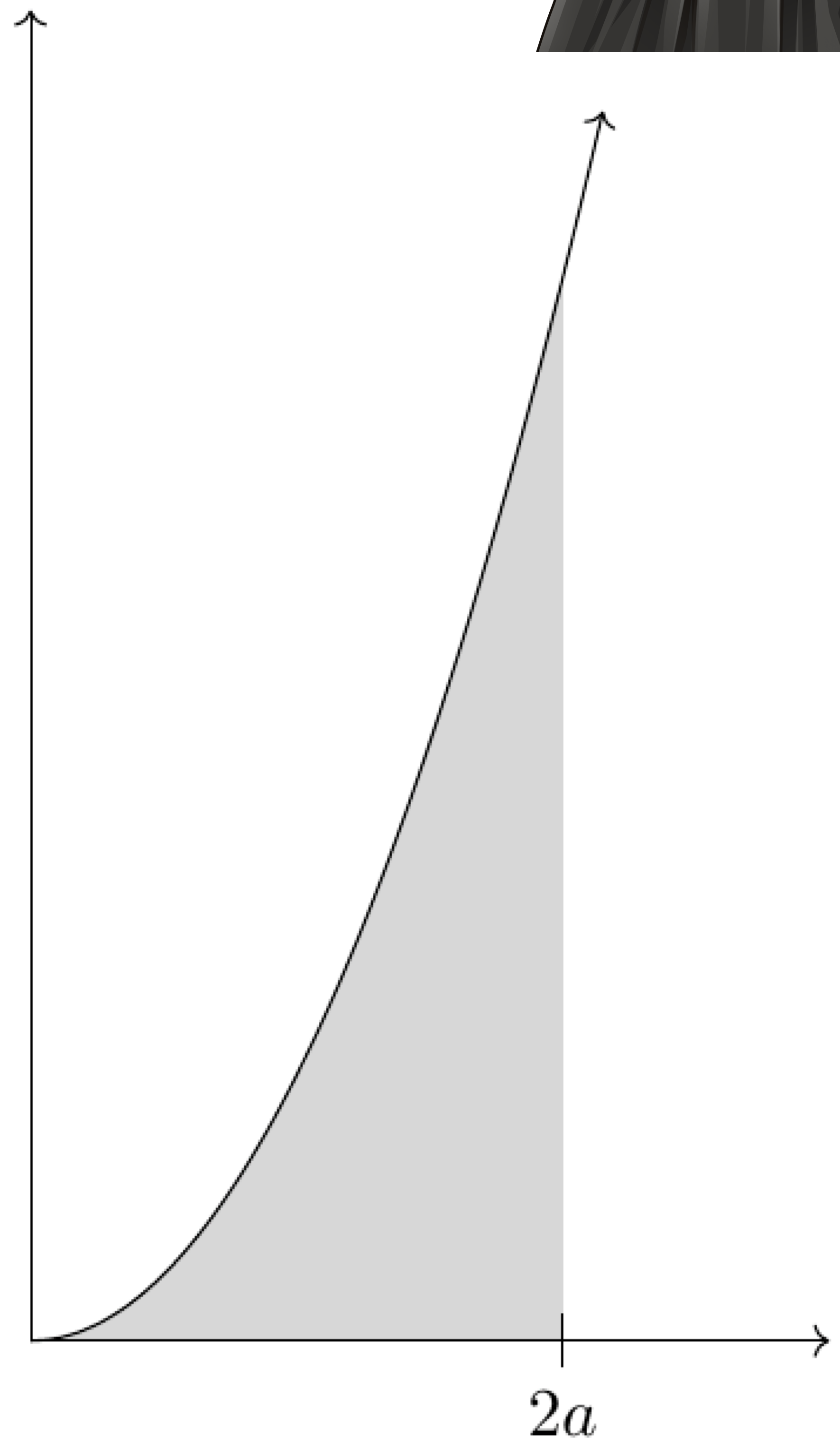

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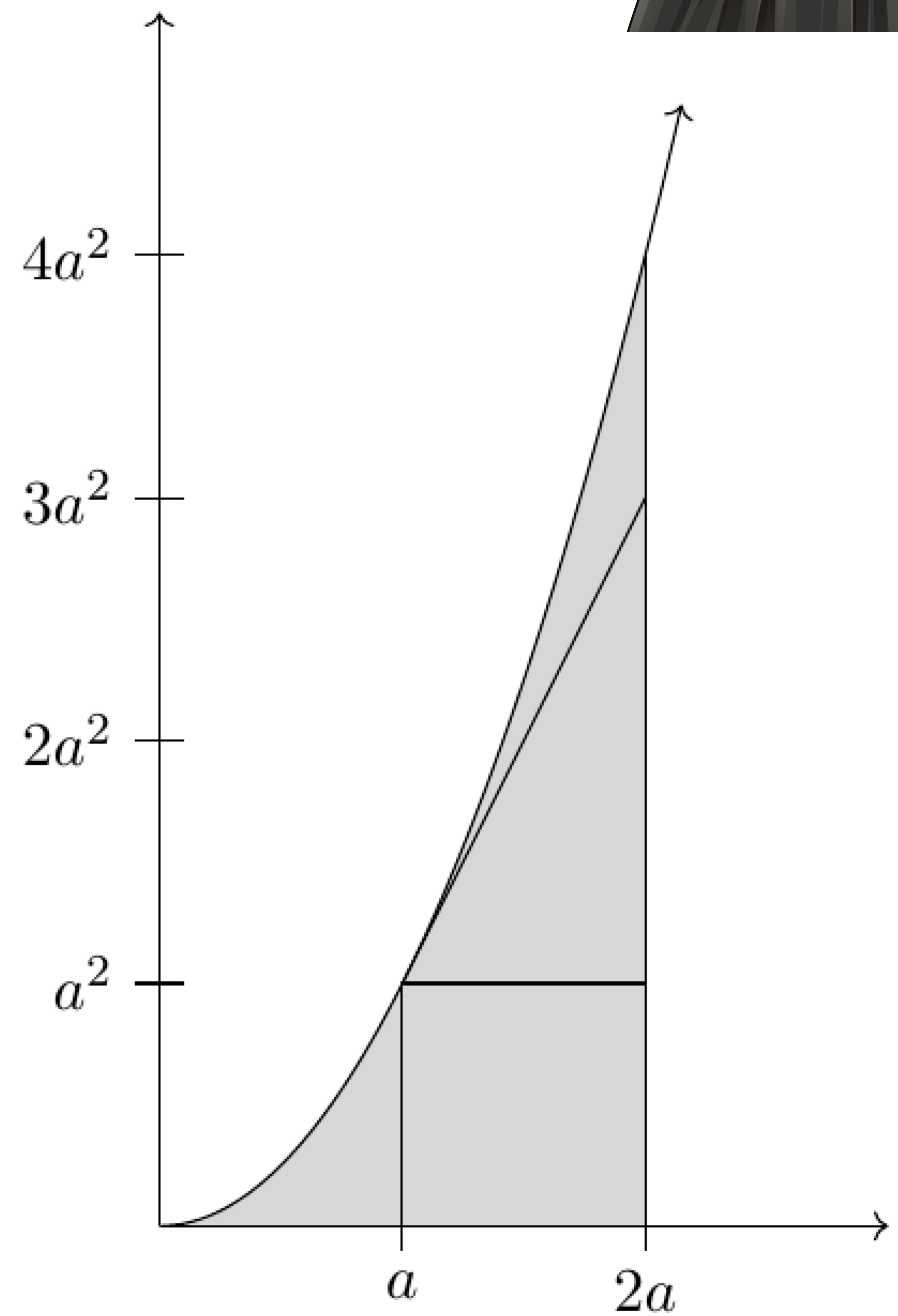

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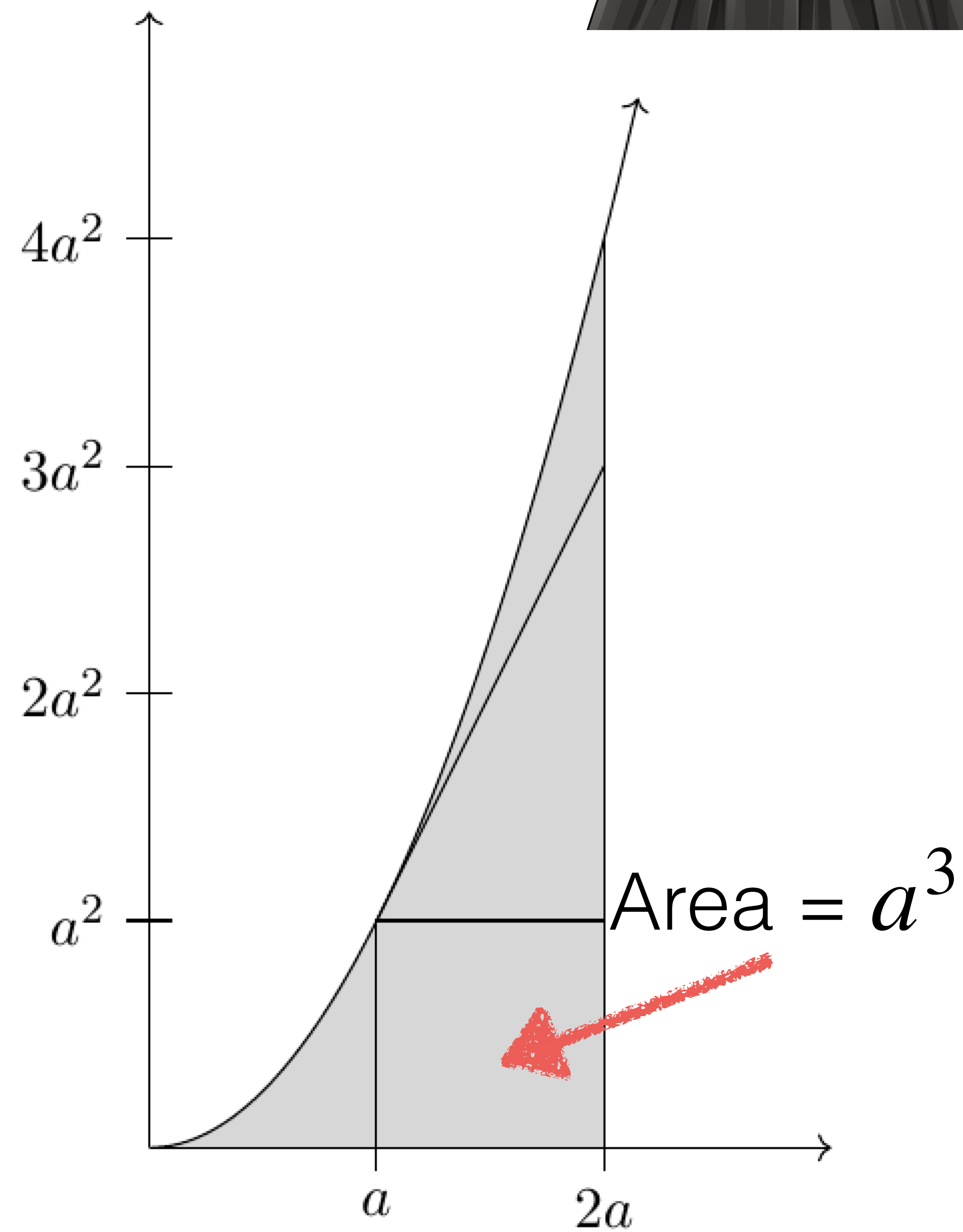

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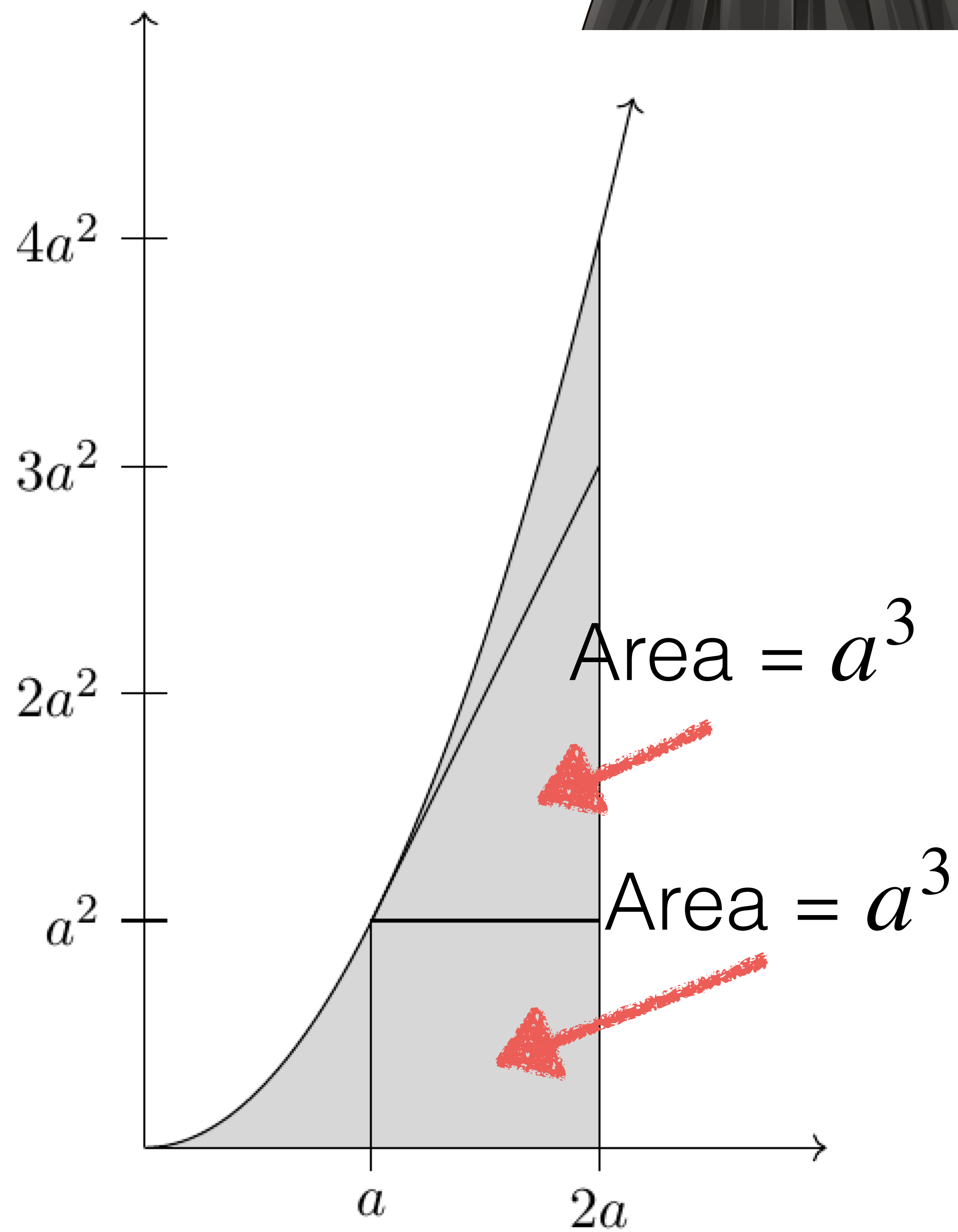

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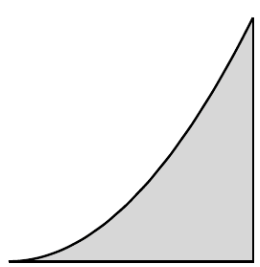

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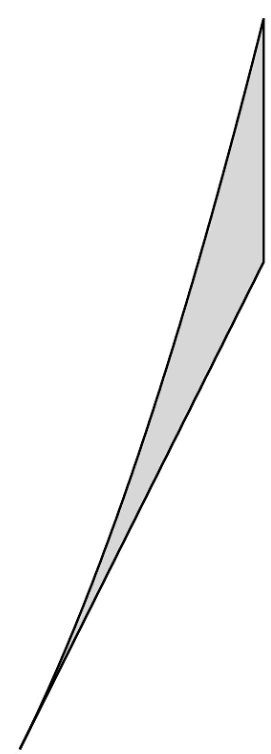

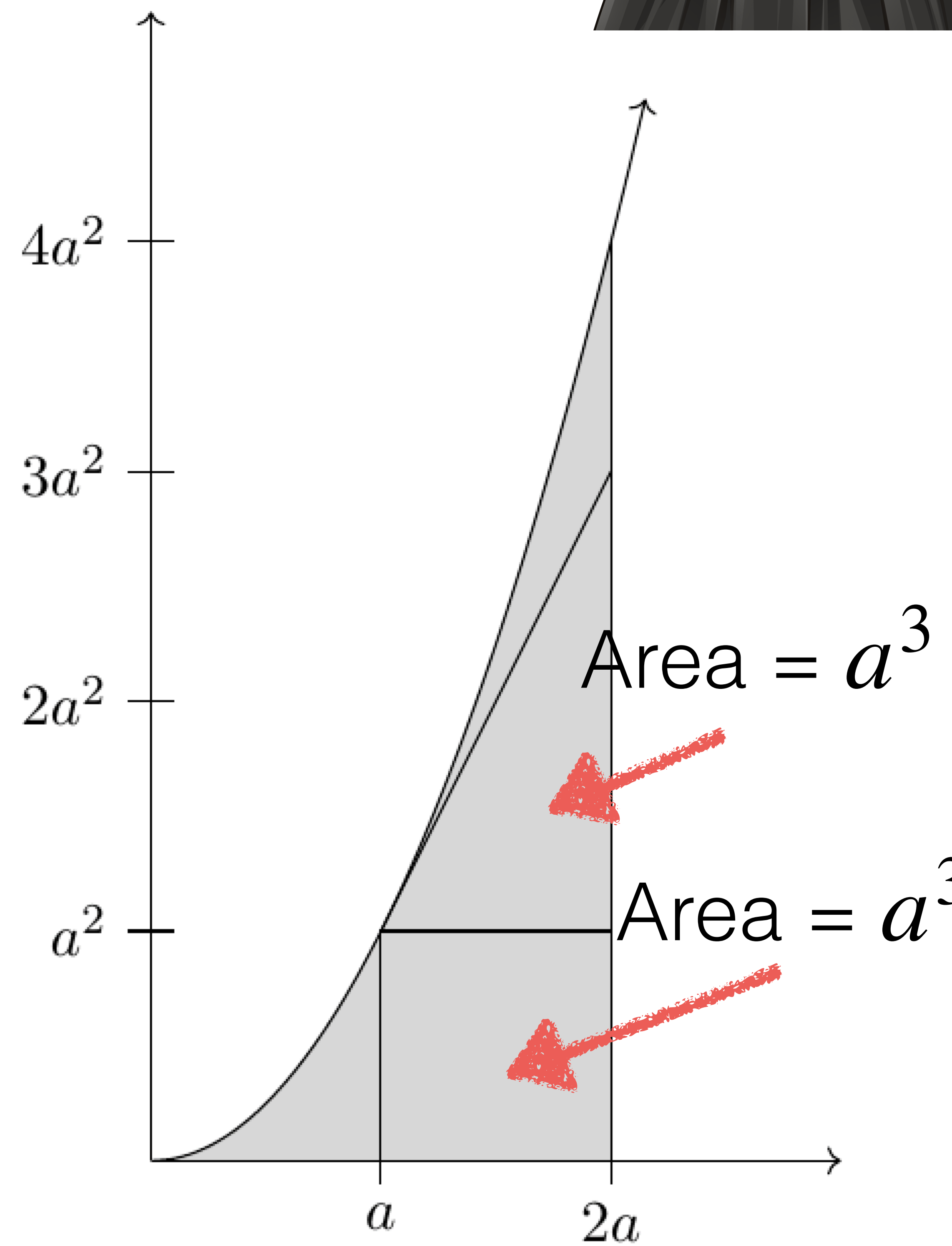

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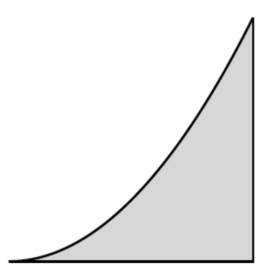

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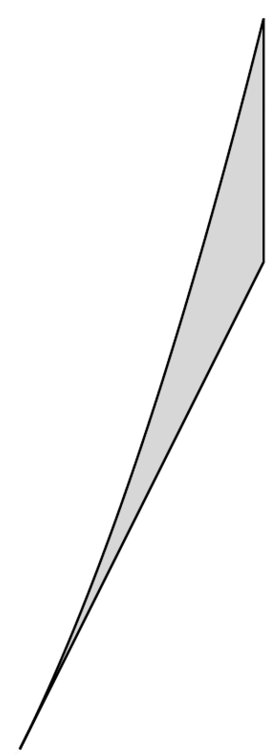
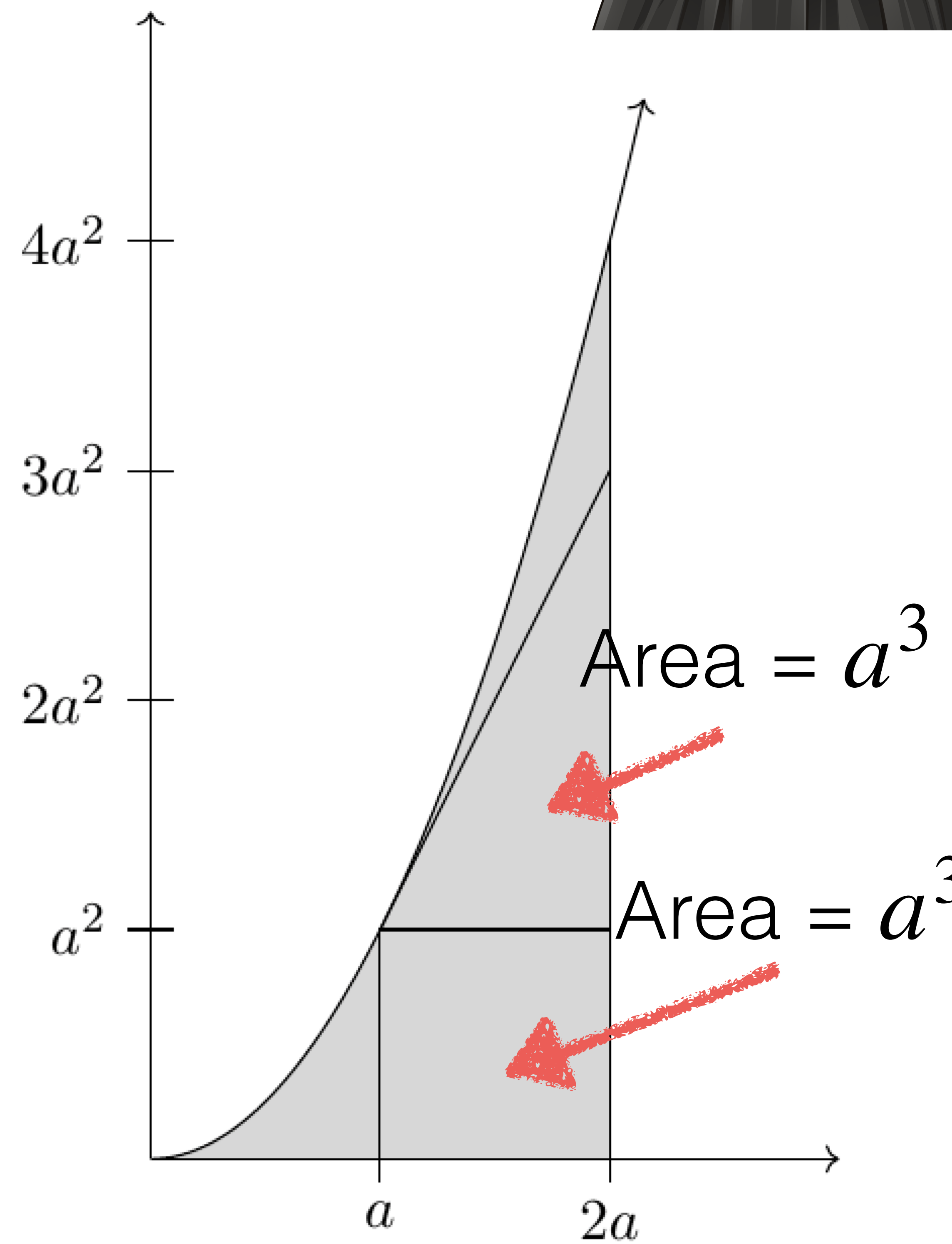


By Cavalieri's principle, they have the same area!



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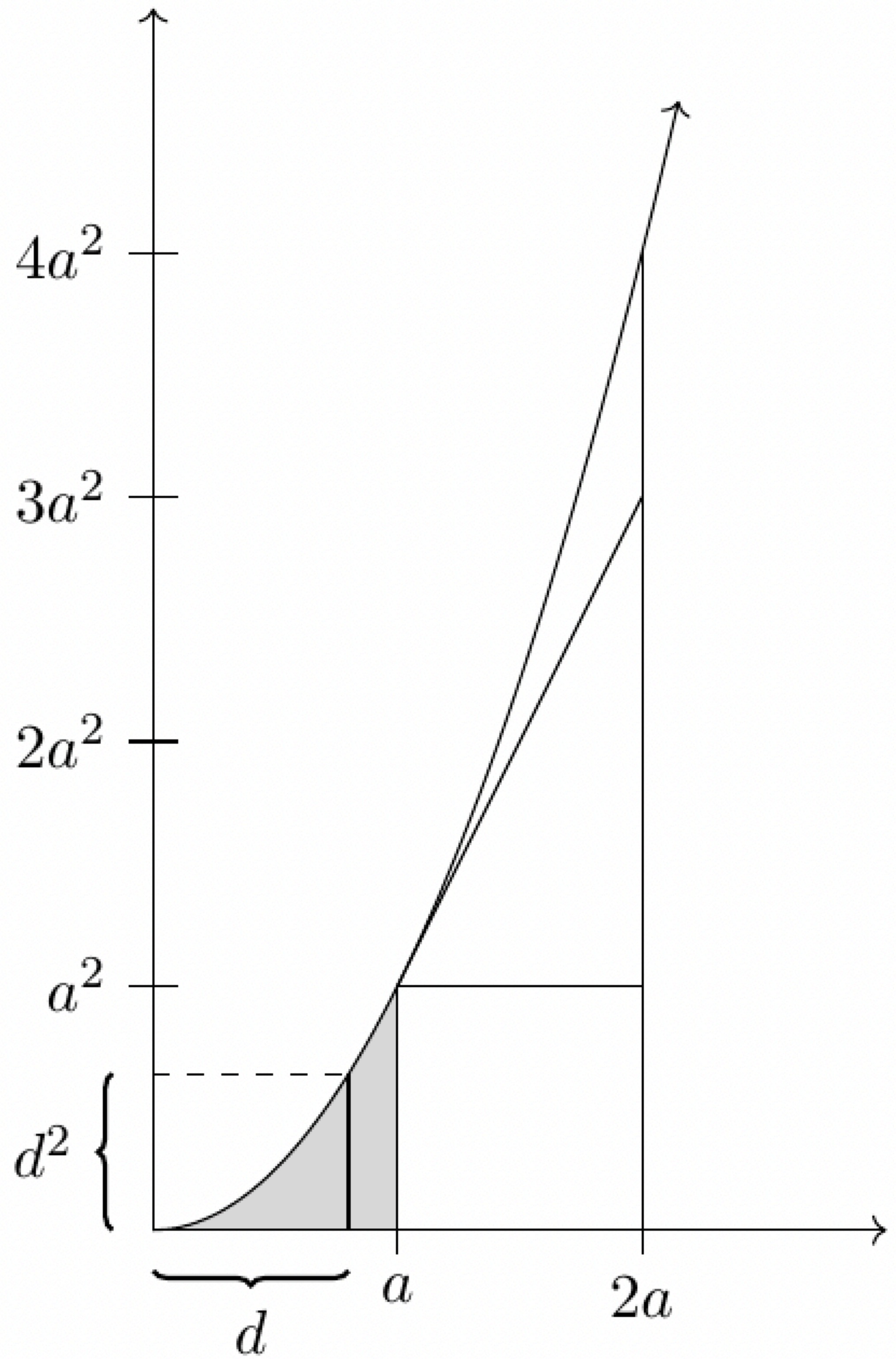

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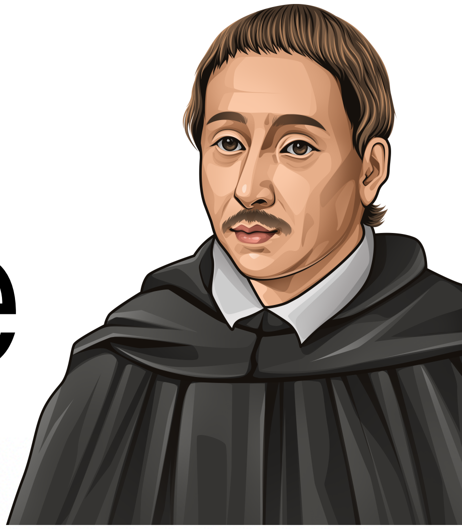

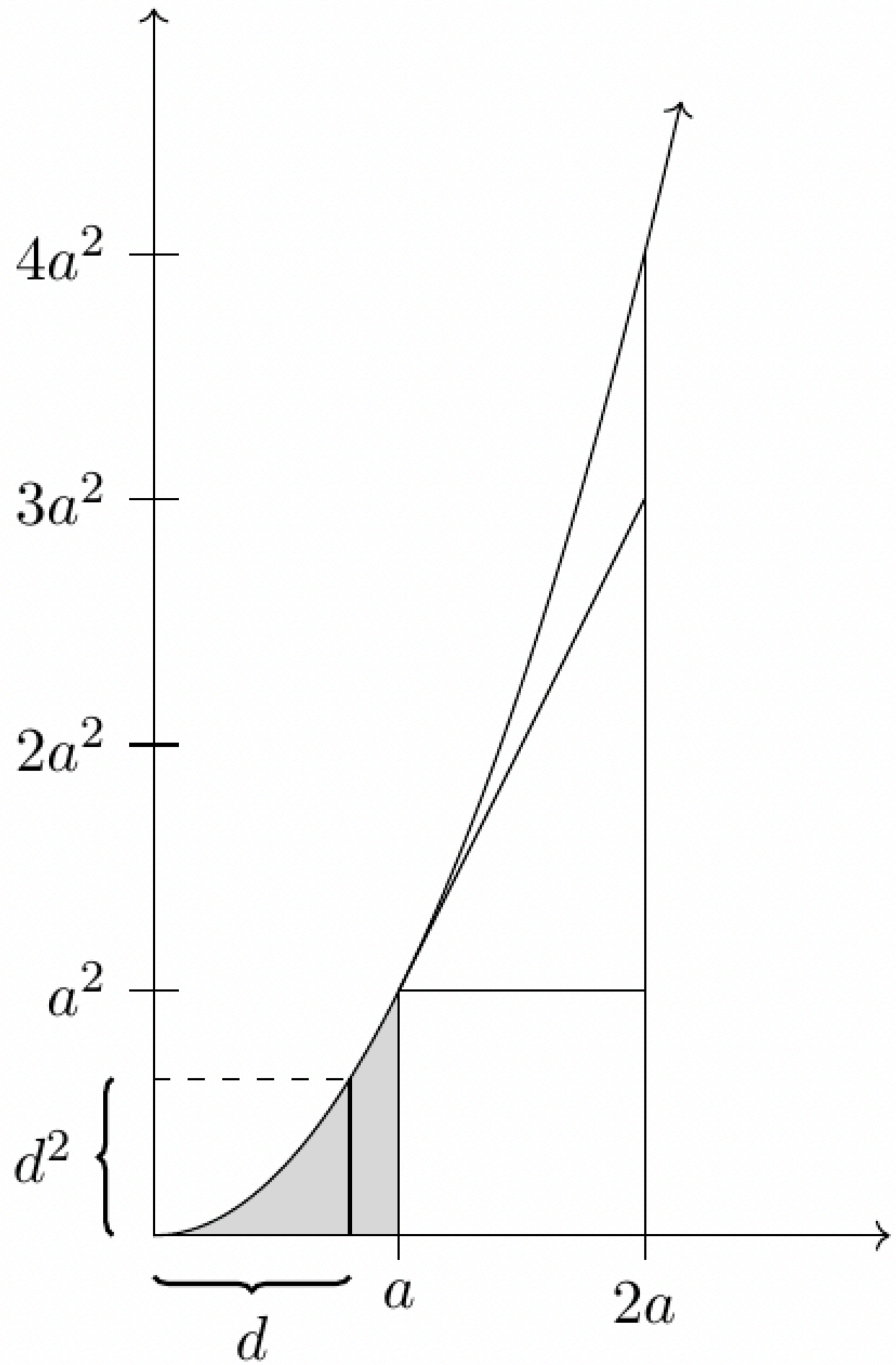

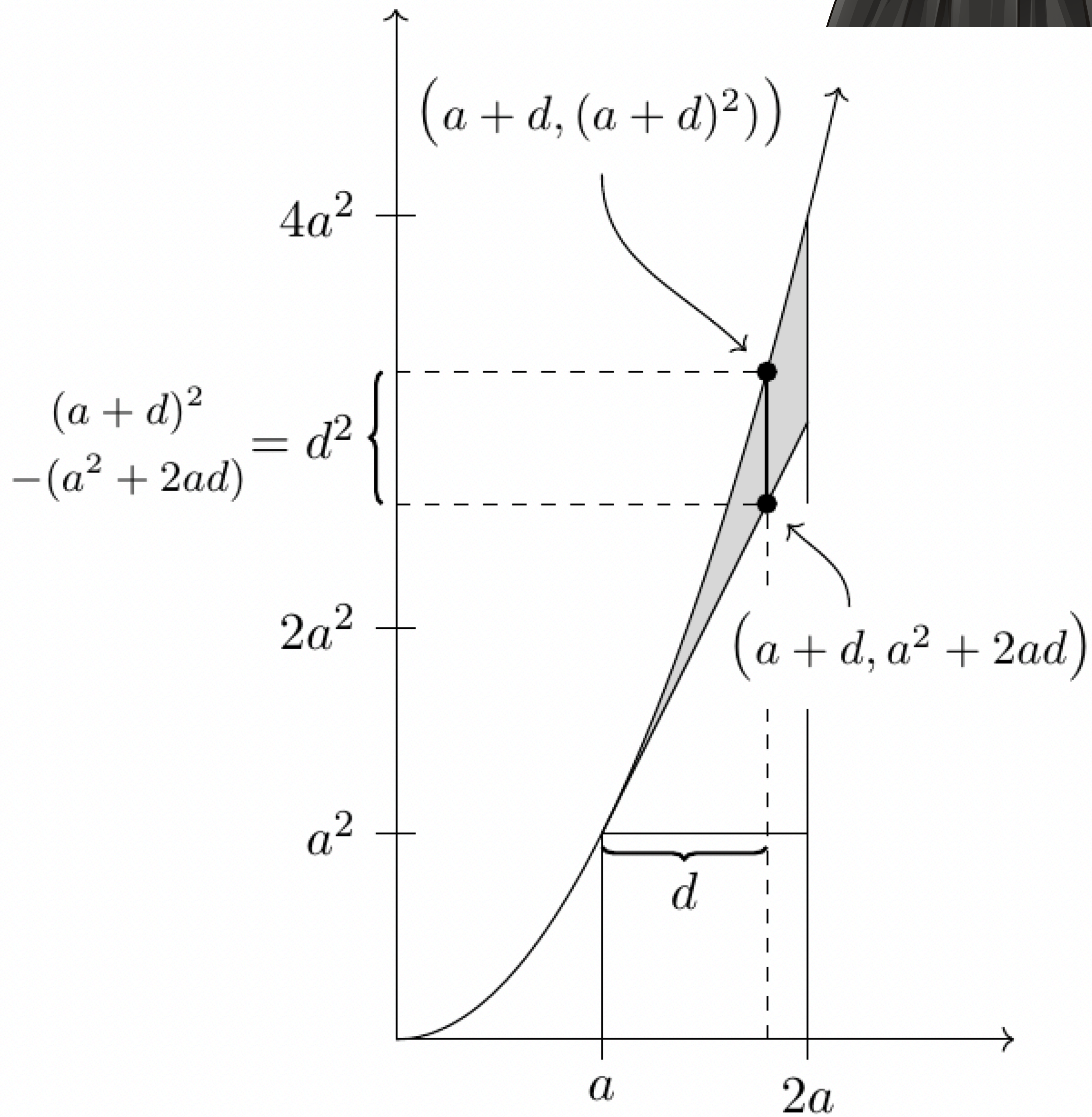

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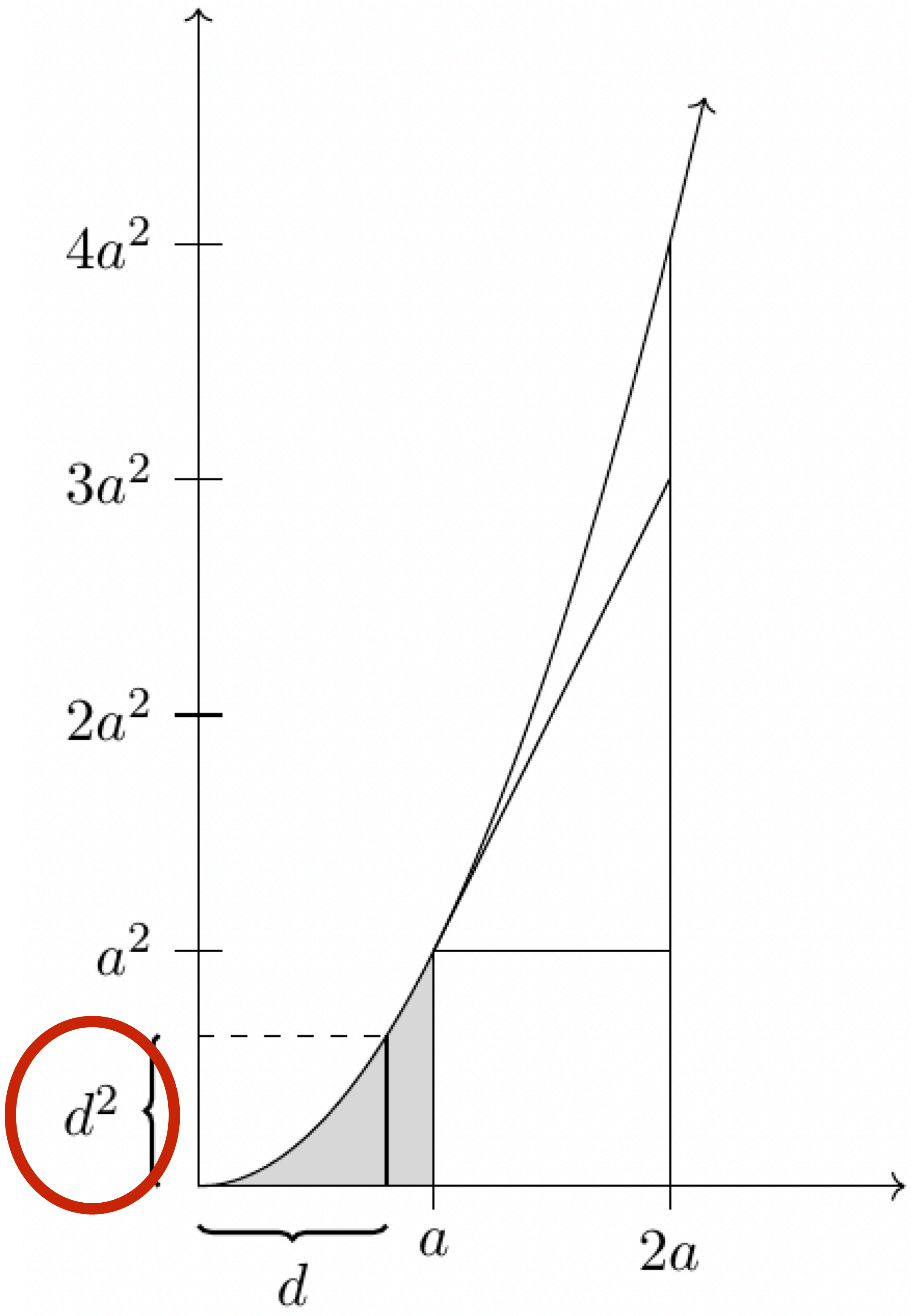

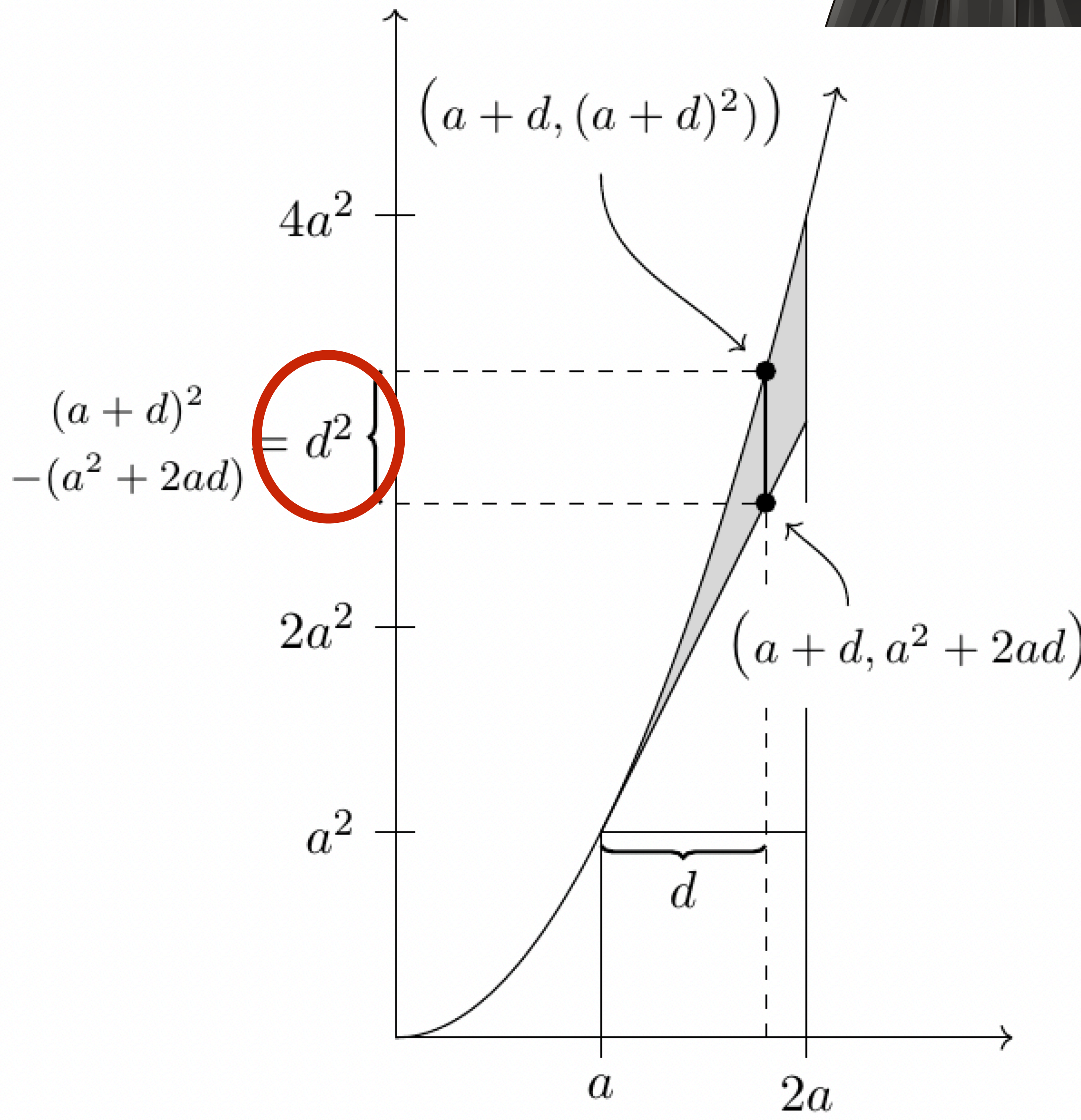

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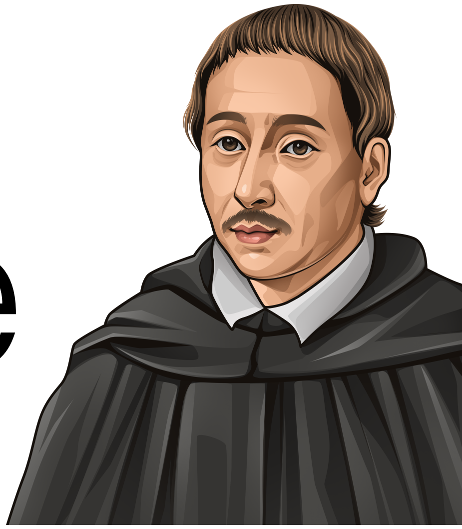

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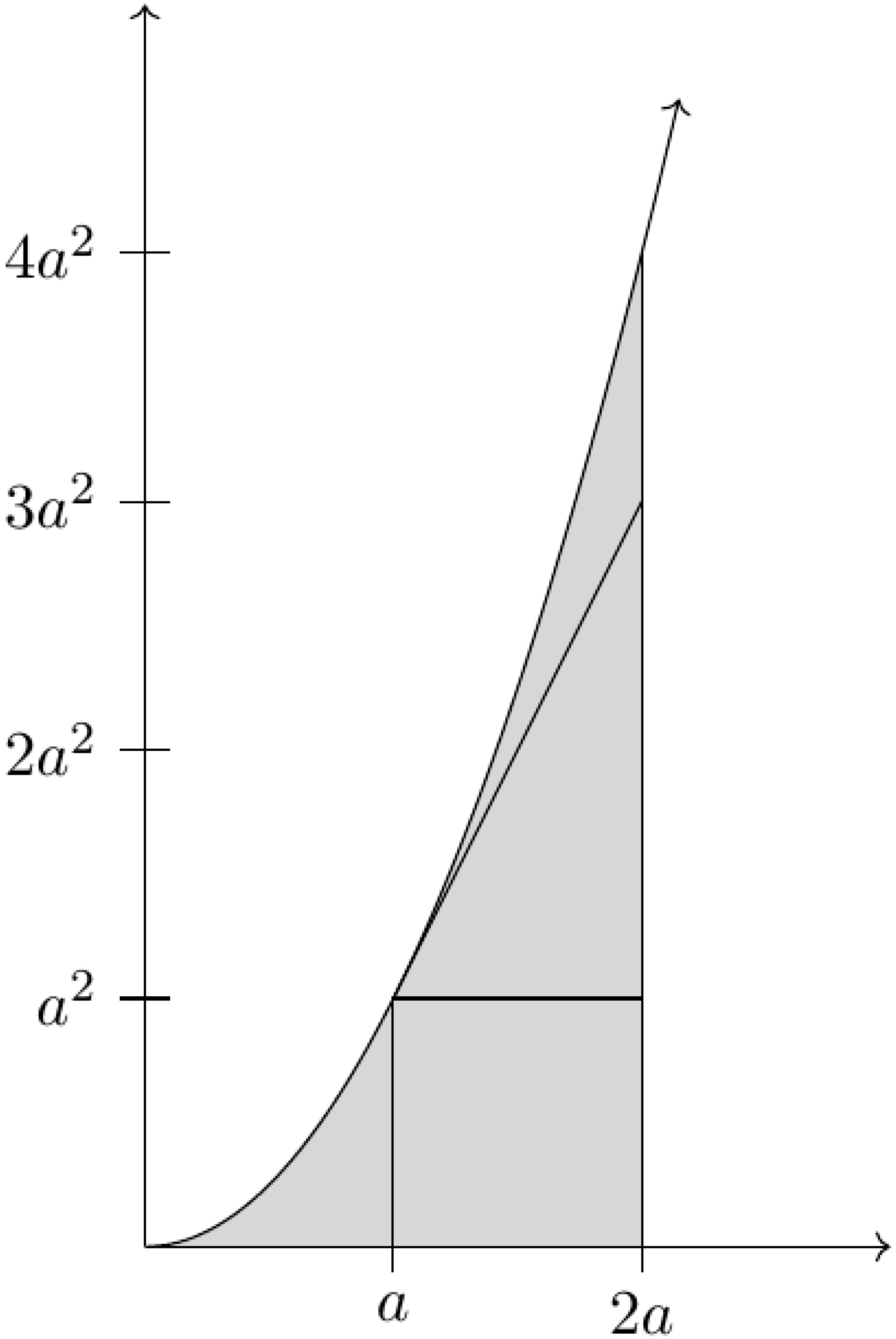

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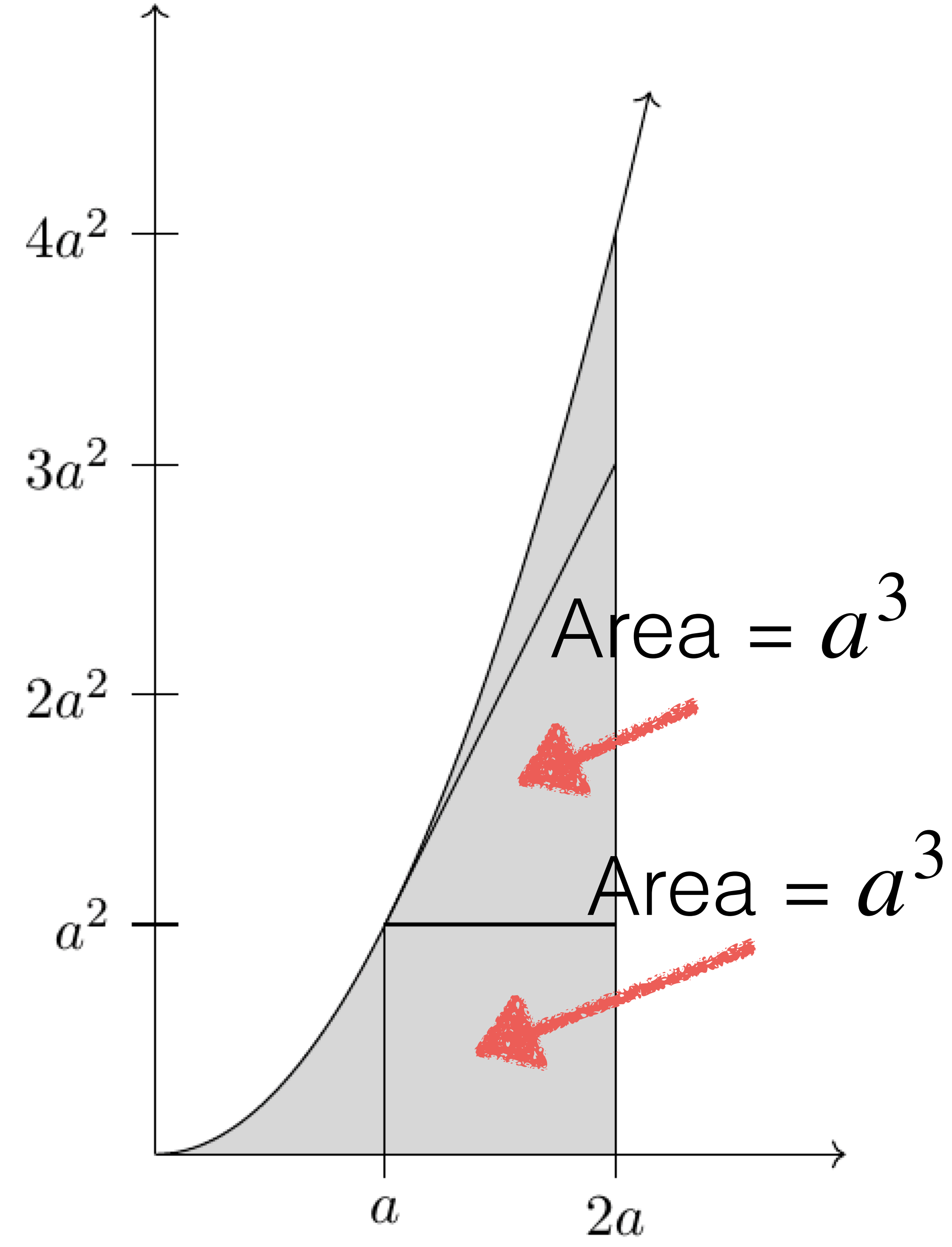

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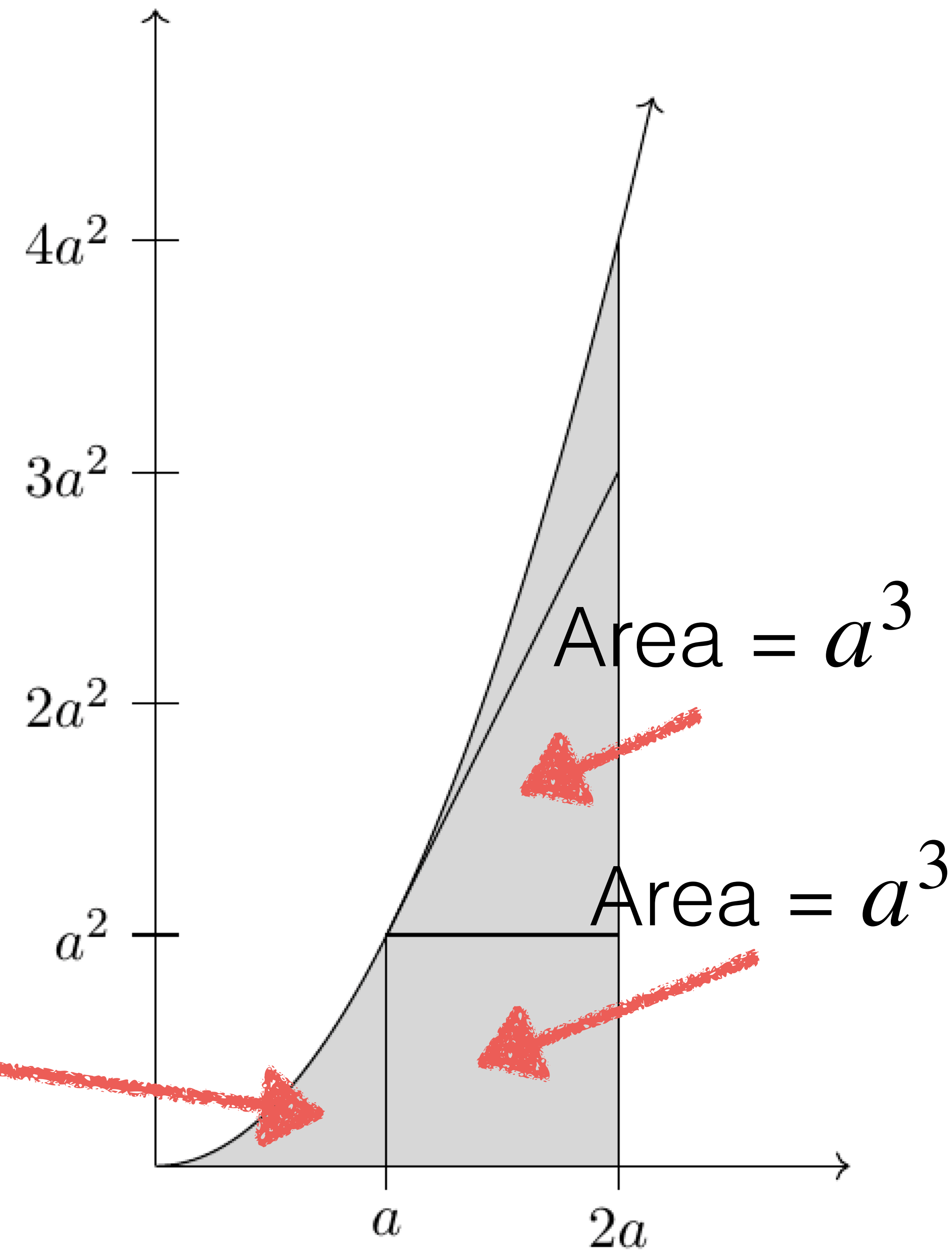

Area = $\int_0^a x^2 \, dx$

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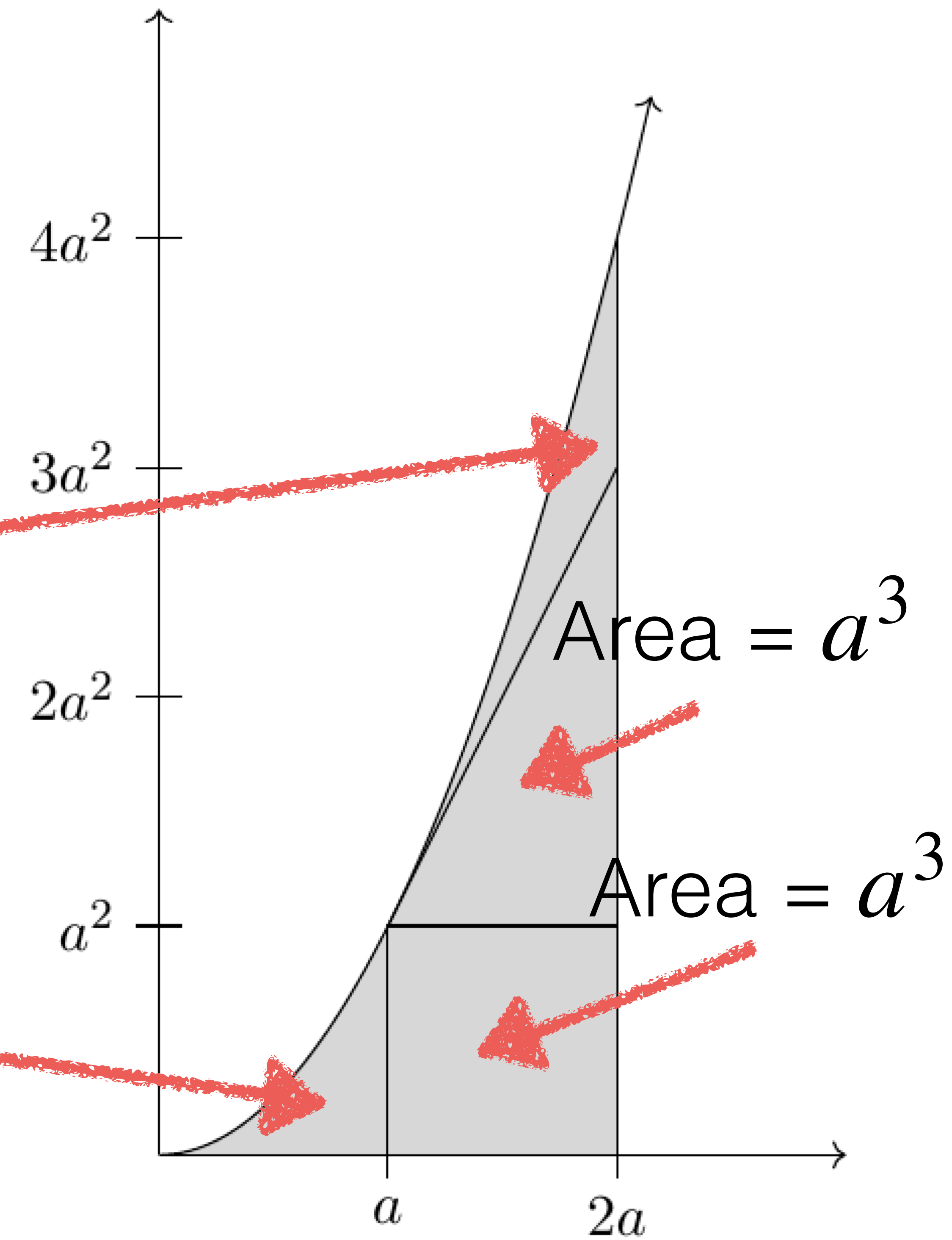

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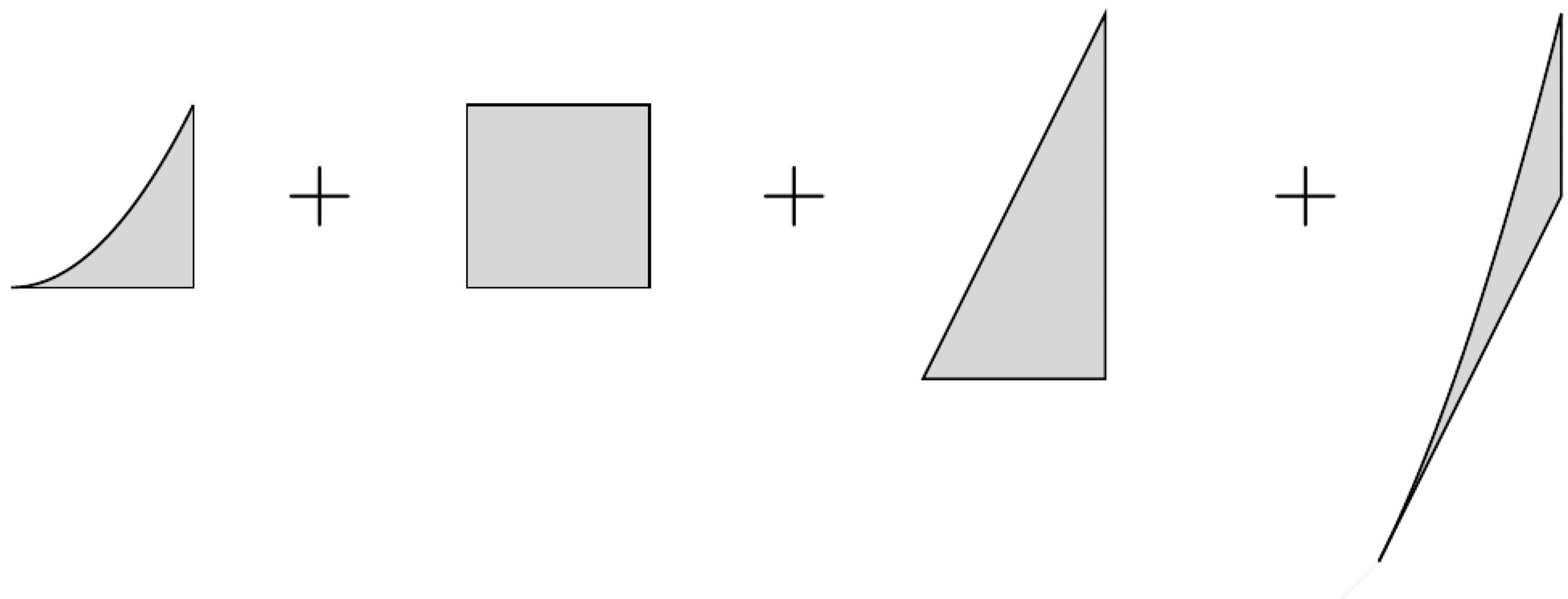

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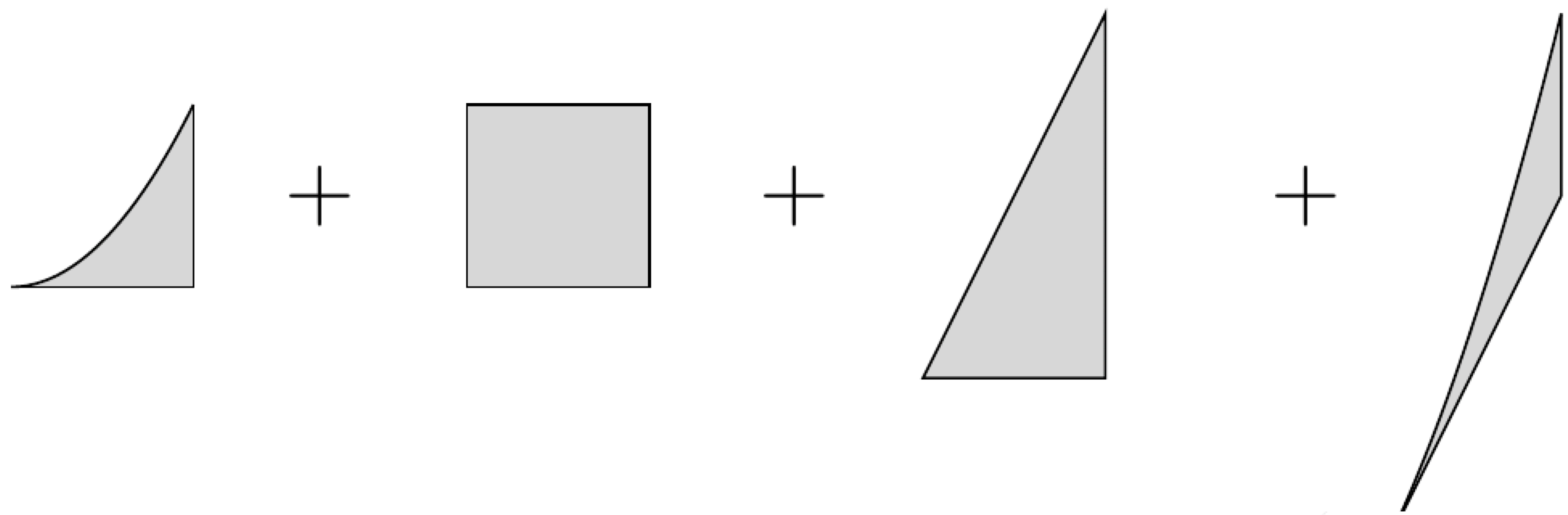

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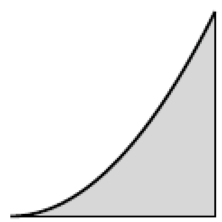
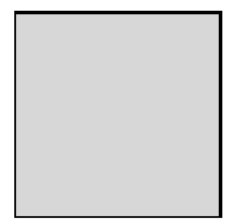
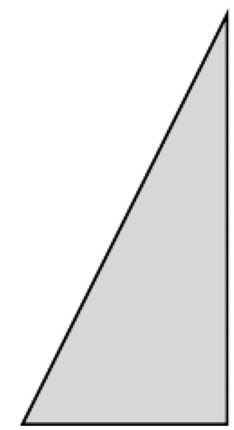
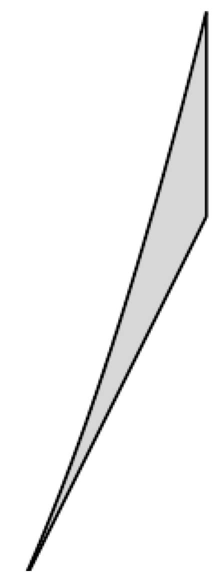
$\int_0^{2a} x^2\,dx =$  $+$  $+$  $+$ 

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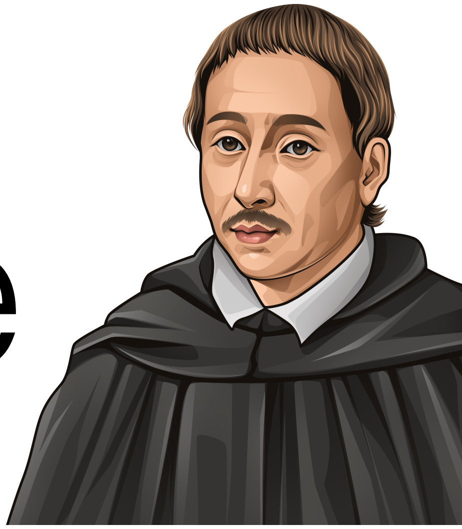

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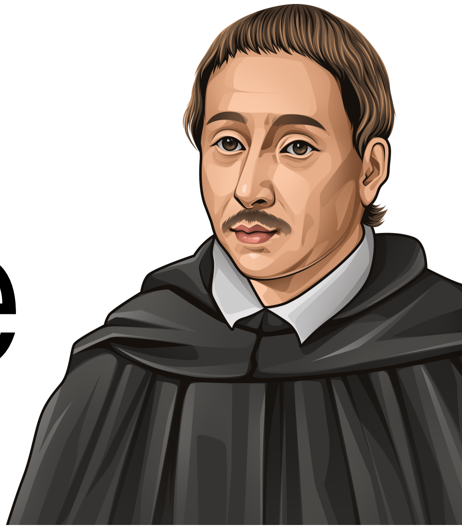

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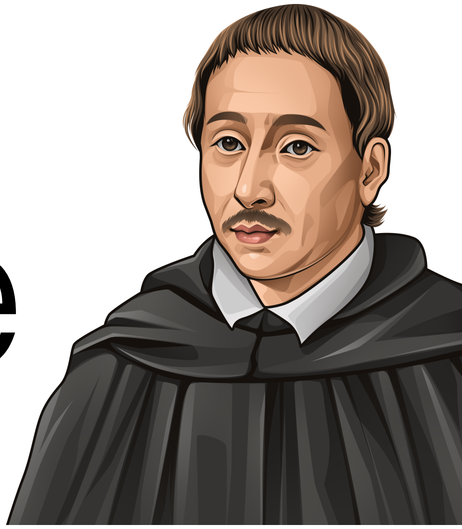

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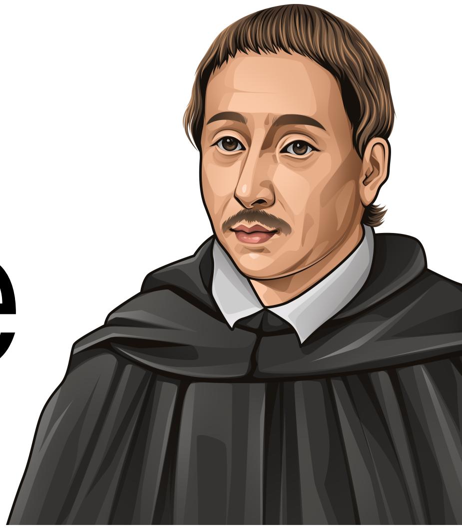

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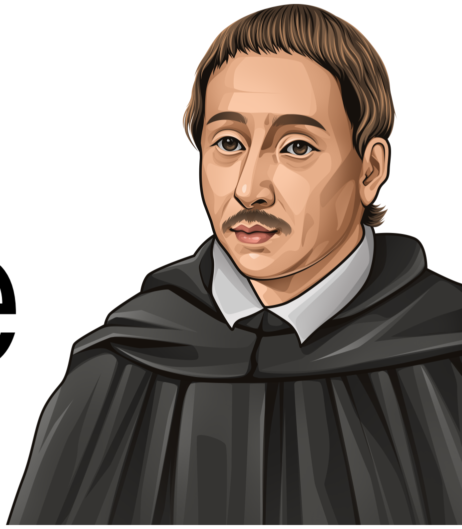

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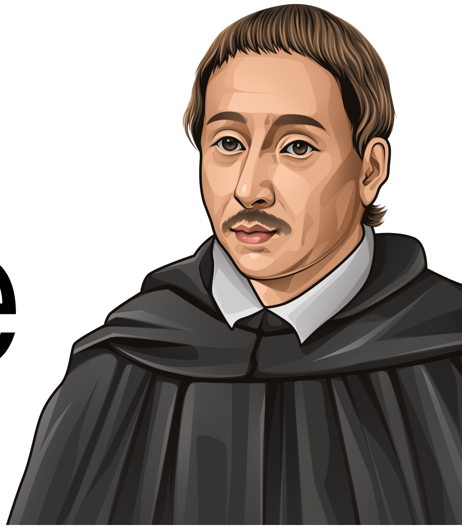

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$$\boxed{8\int_0^a x^2\,dx = \int_0^{2a} x^2\,dx}$$

$$\Rightarrow 8\int_0^a x^2\,dx = 2a^3 + 2\int_0^a x^2\,dx$$

$$\Rightarrow 6\int_0^a x^2\,dx = 2a^3 \quad \Rightarrow \int_0^a x^2\,dx = \frac{a^3}{3}.$$

Q.E.D.

Shout-outs!

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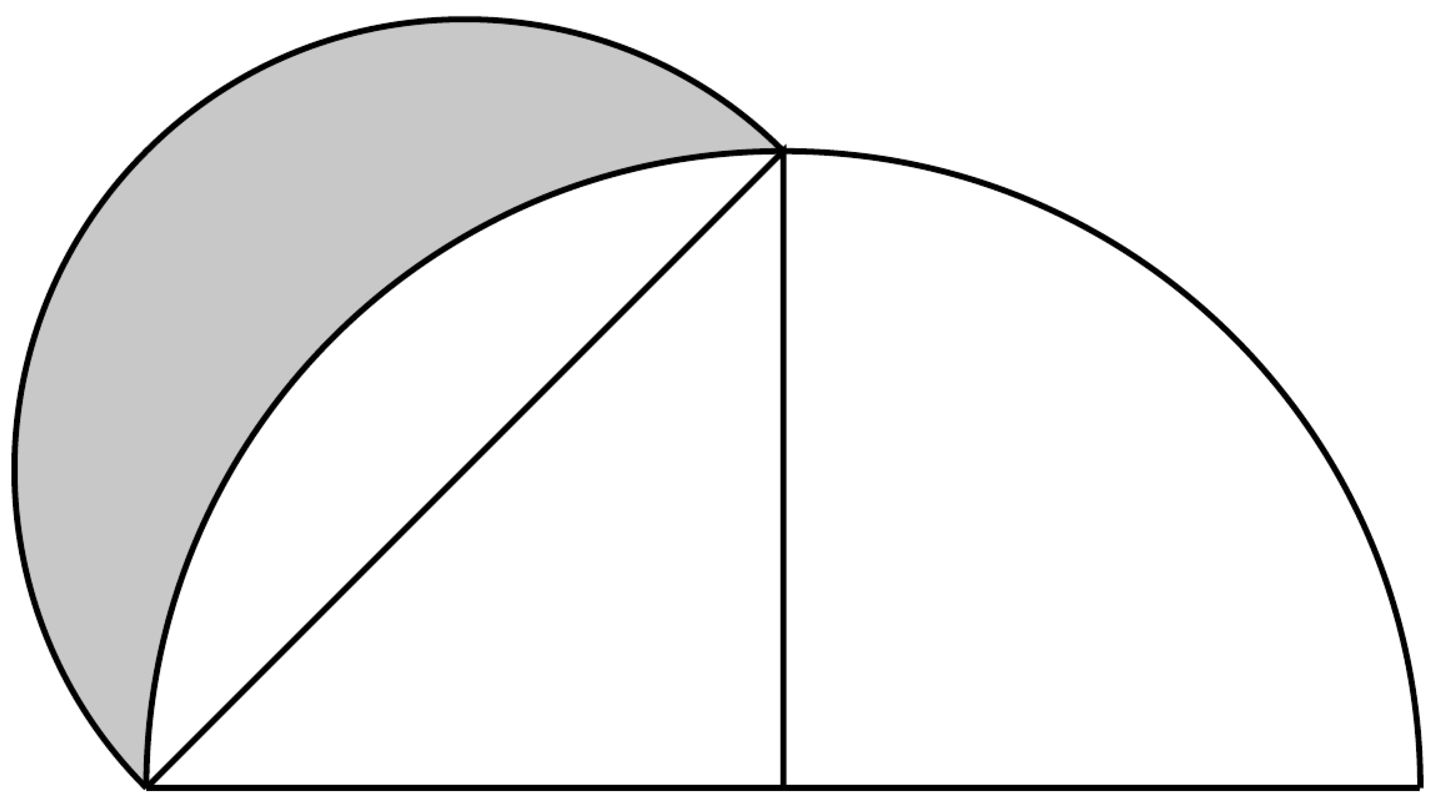
- The first known person to determine the exact area of a region with curved sides was the Greek Hippocrates of Chio. In the mid-fifth century BC, he found the exact area of a *lune*, which is made with two semicircles as shown below.



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  - The slices are *very* weird. Not “normal” slices.