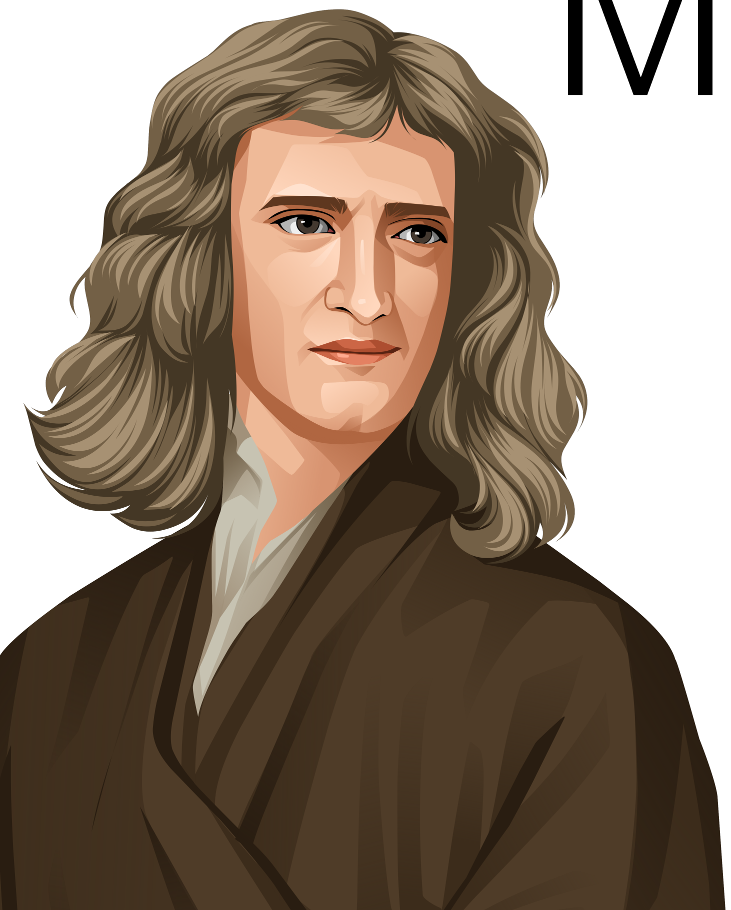


Chapter 8: Modern Calculus



Two Early Contributors To The Mathematization of Science

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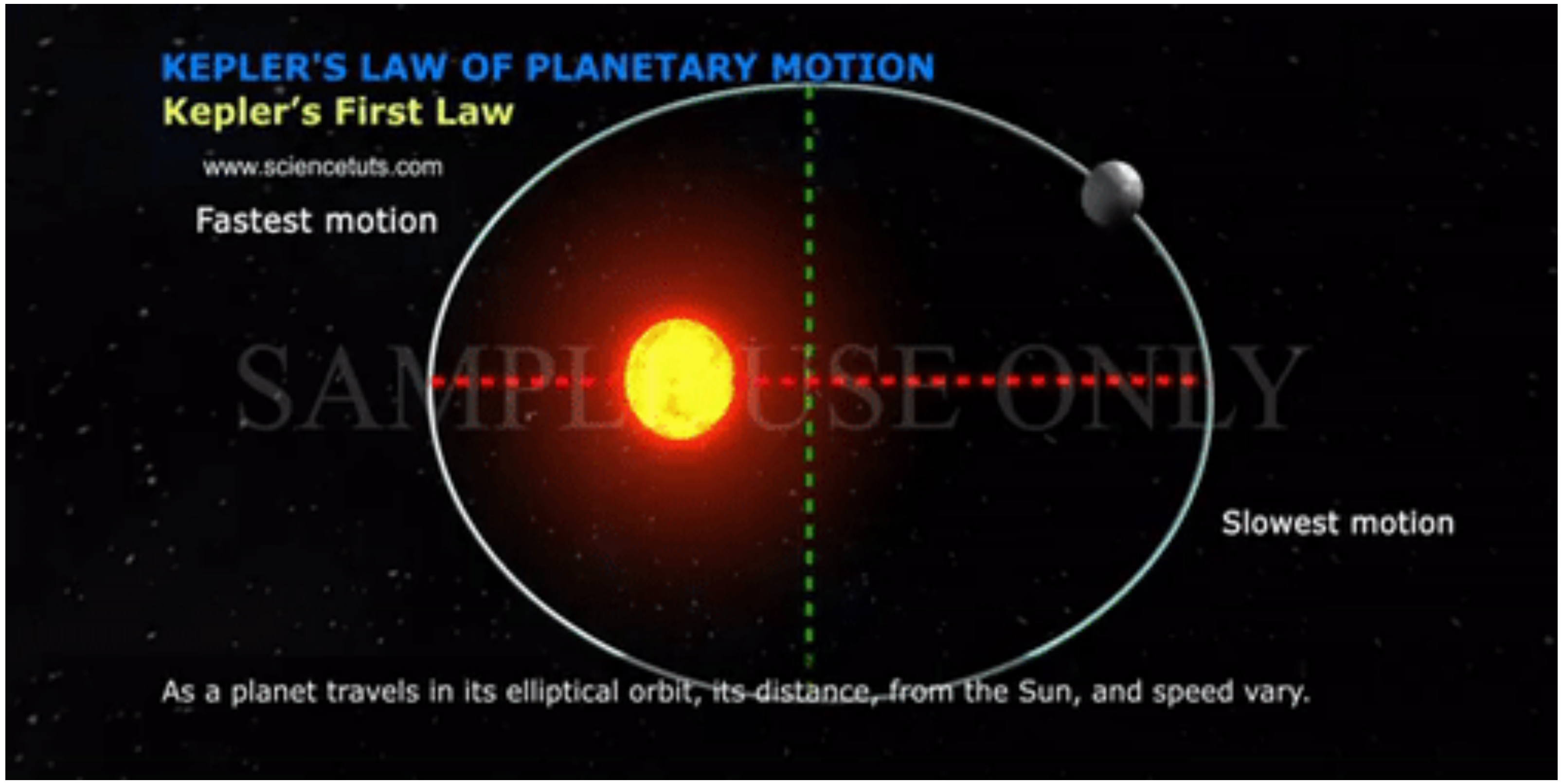

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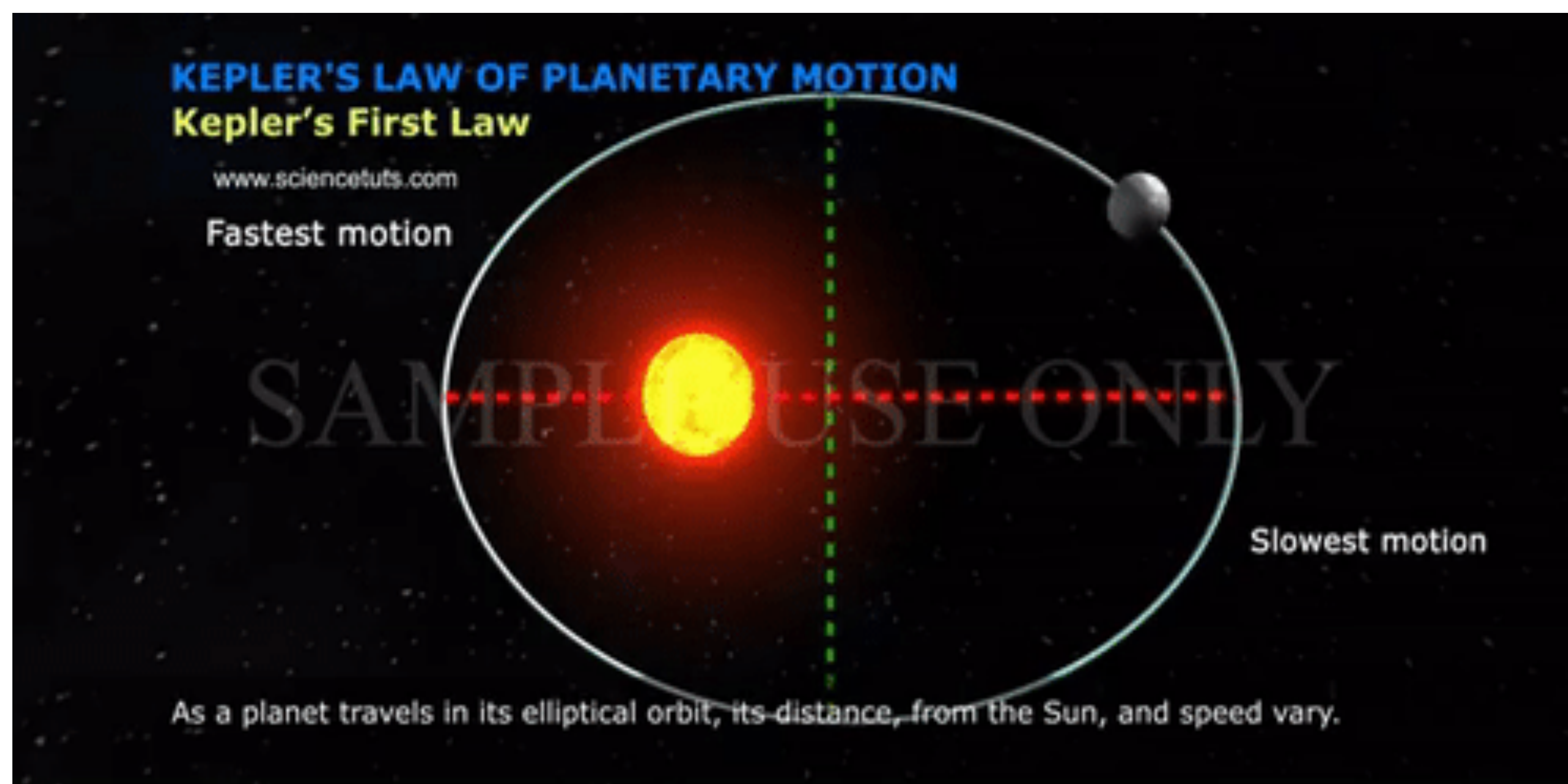

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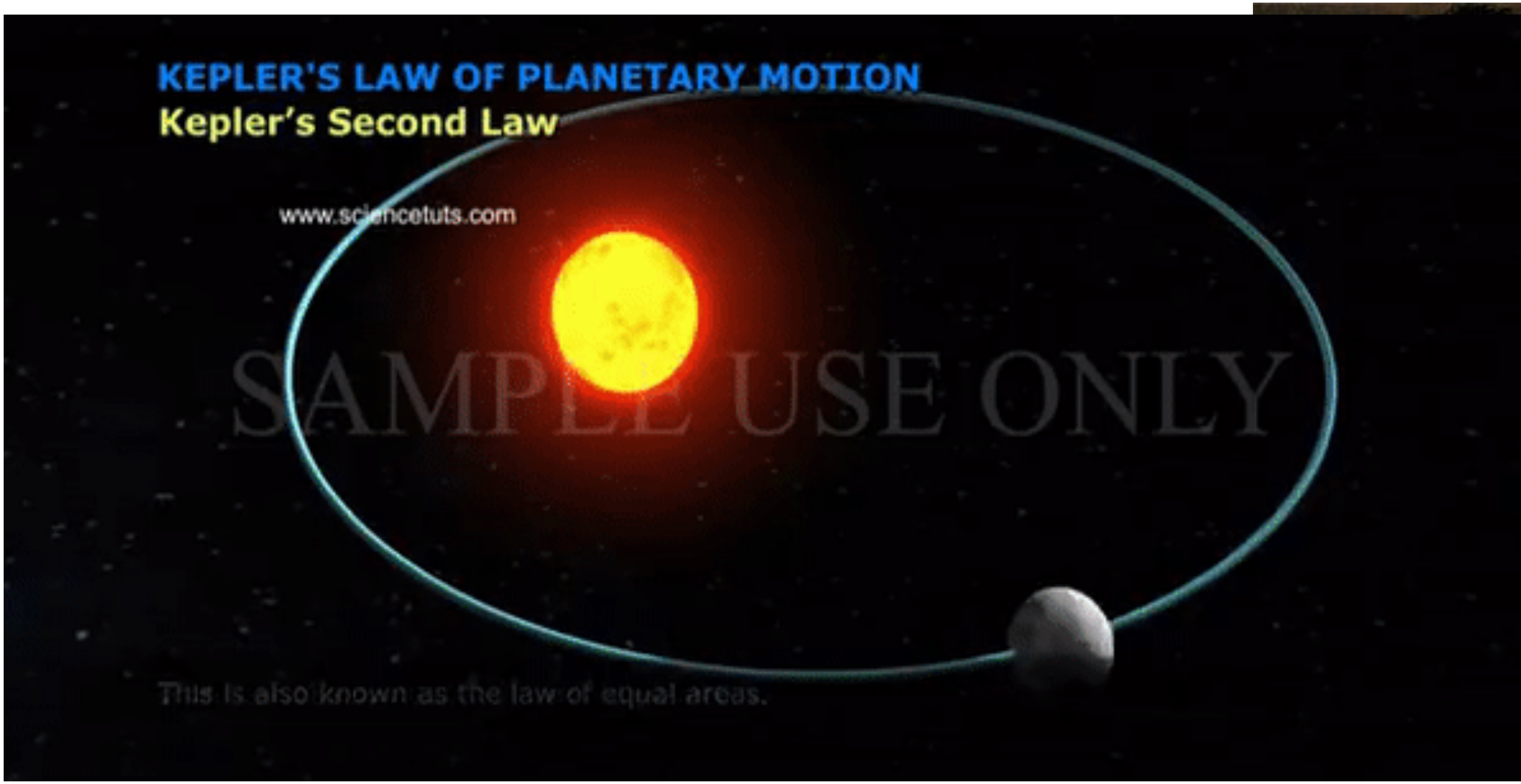
2. A line segment joining. A planet and the Sun sweeps out equal areas during equal intervals of time.



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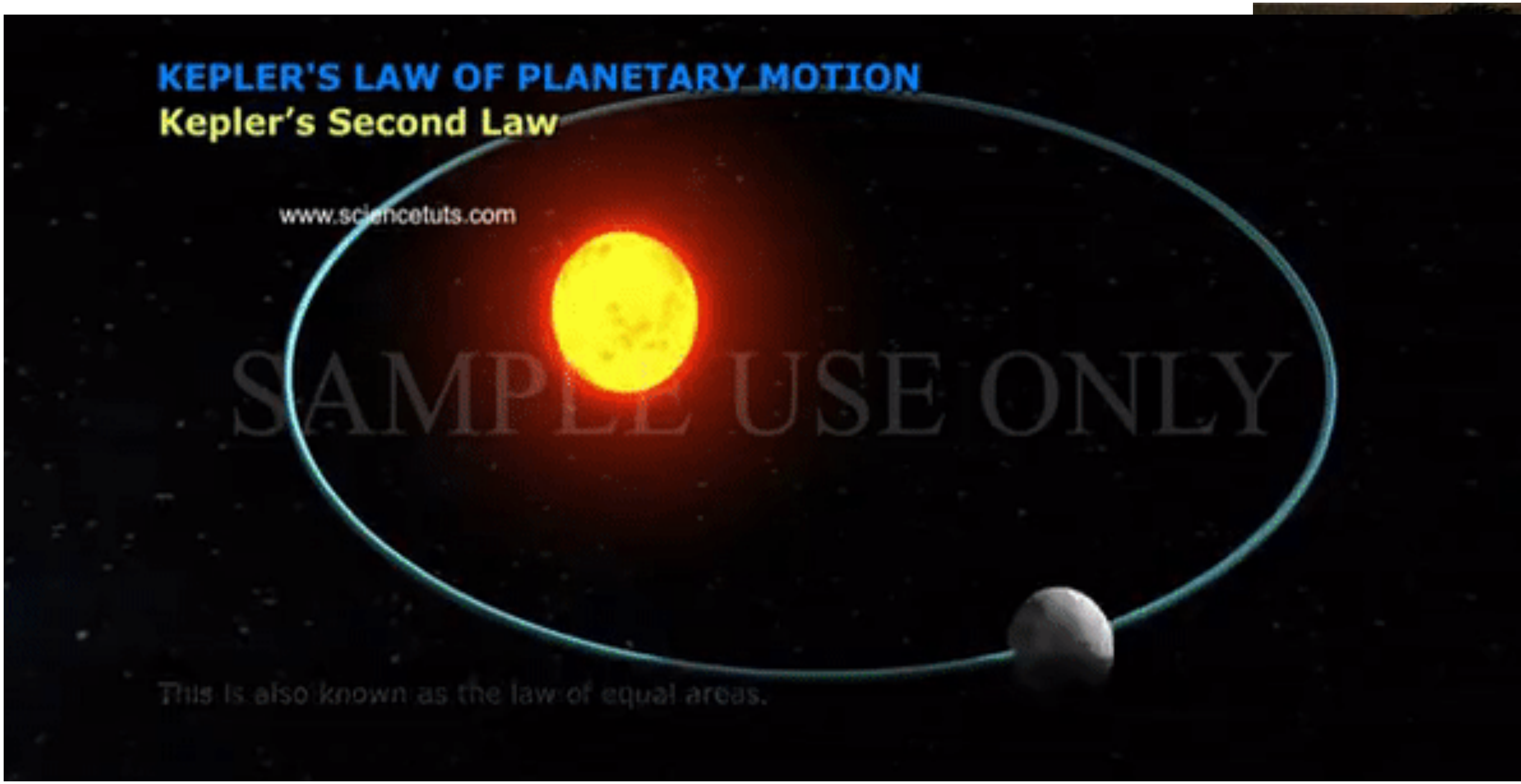

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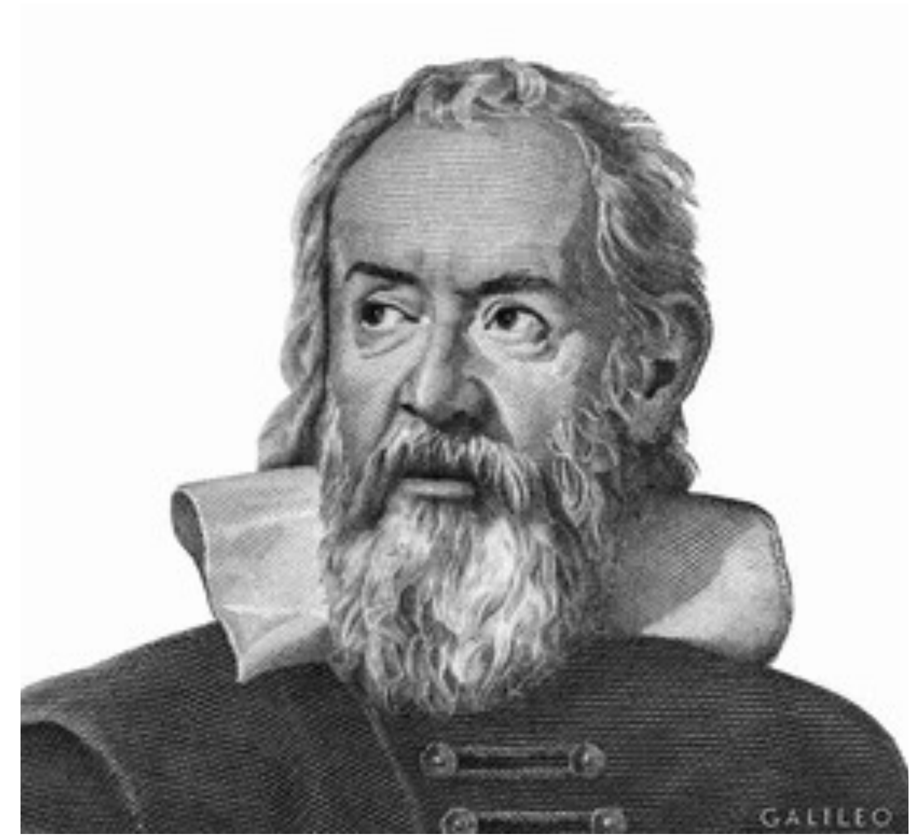
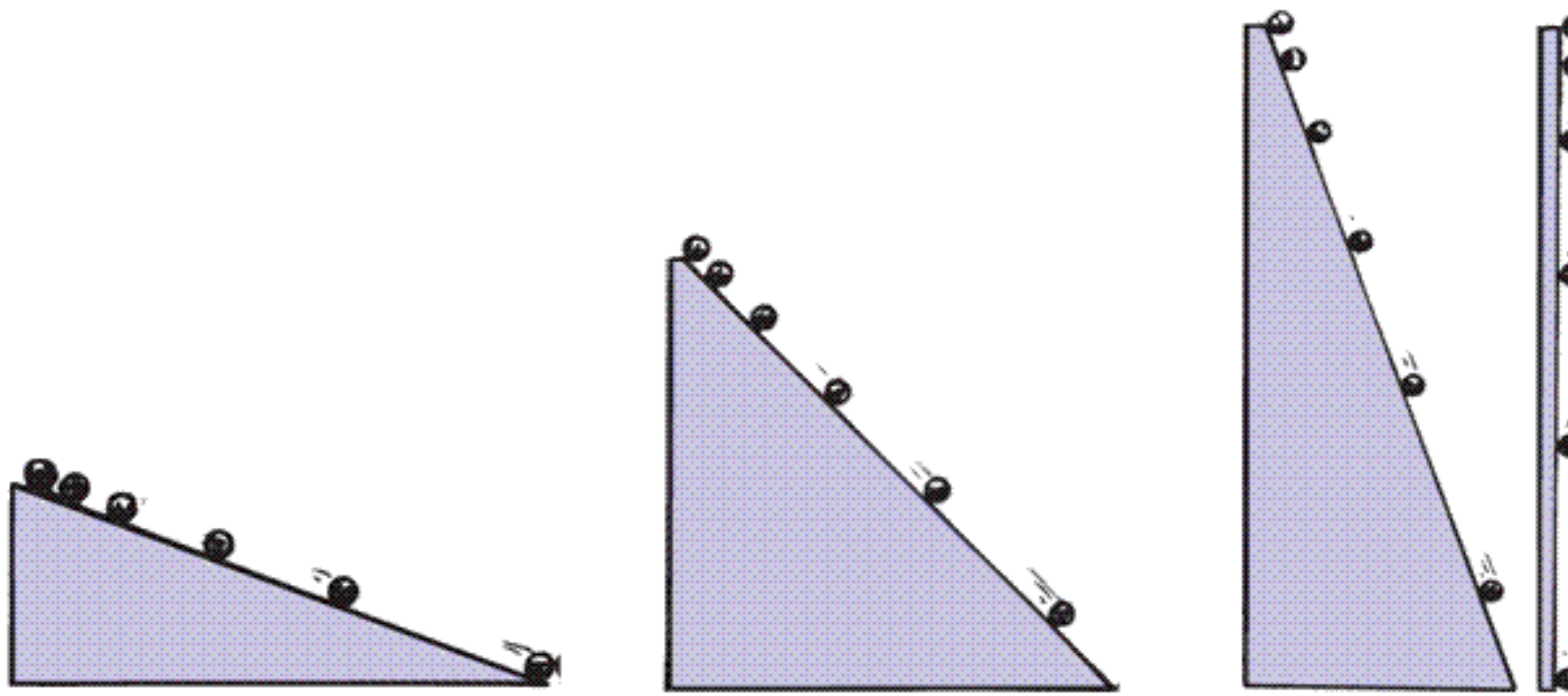
3. The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit.

Galileo





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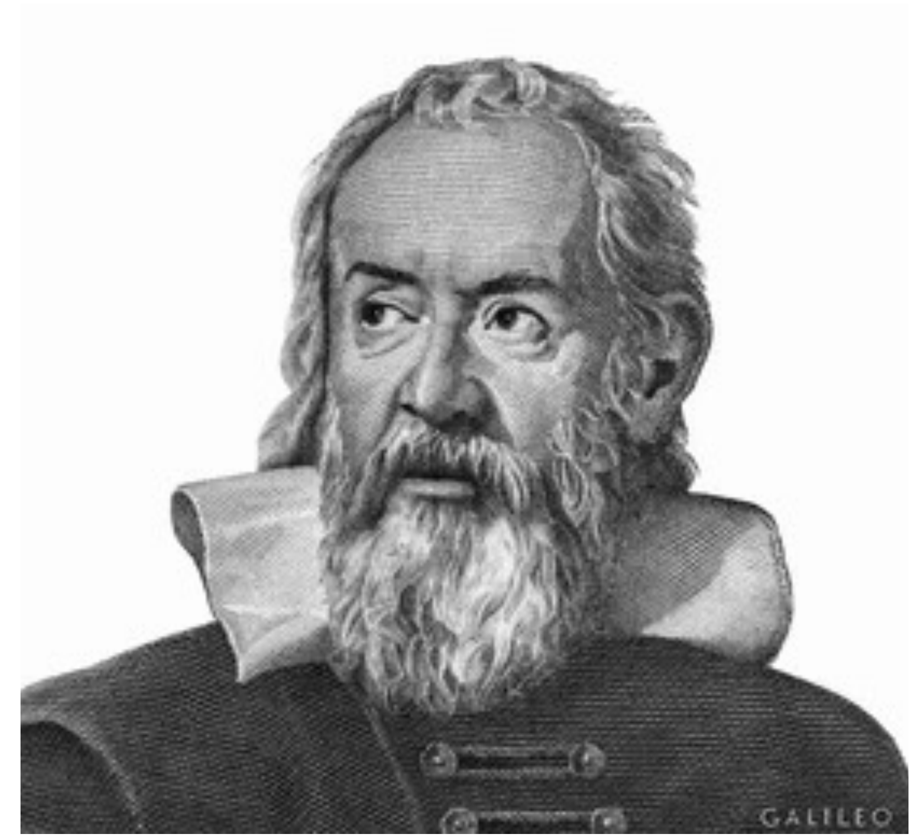

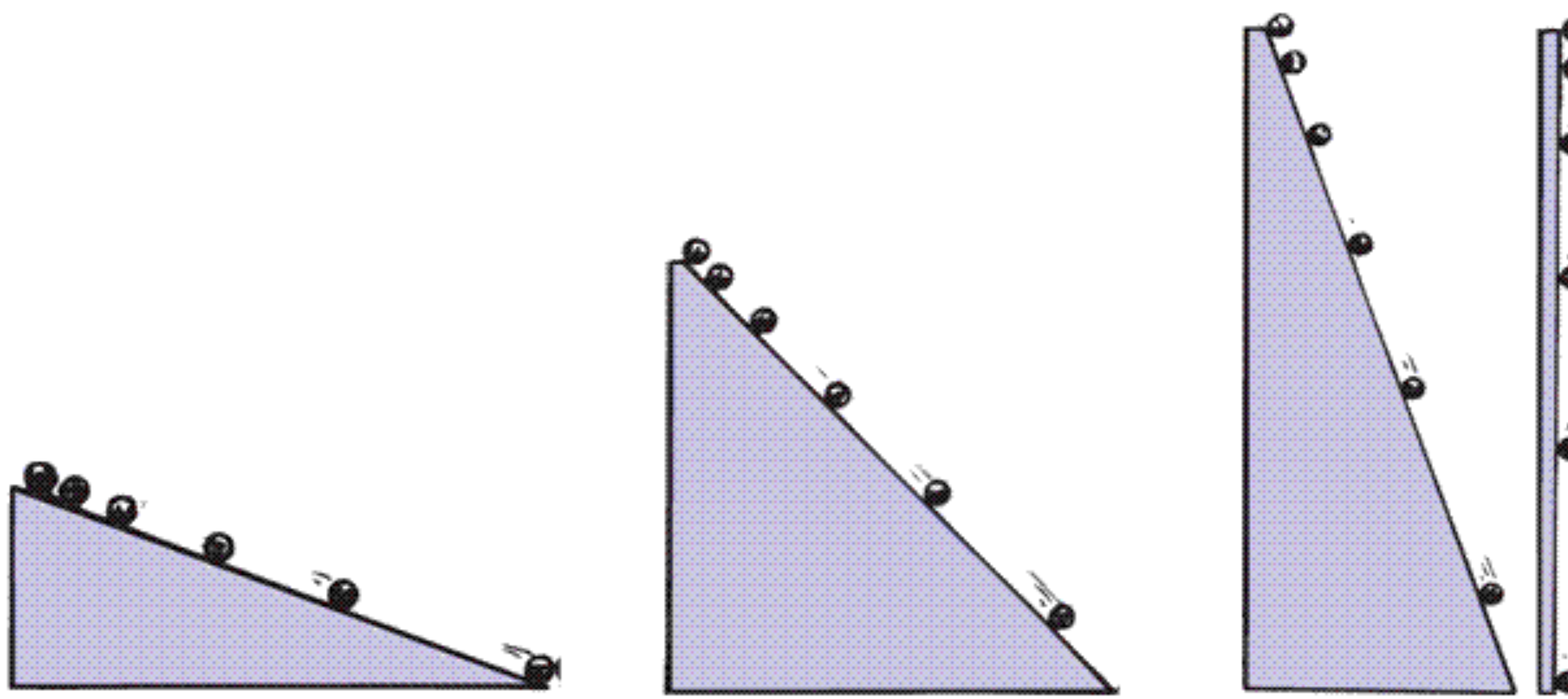

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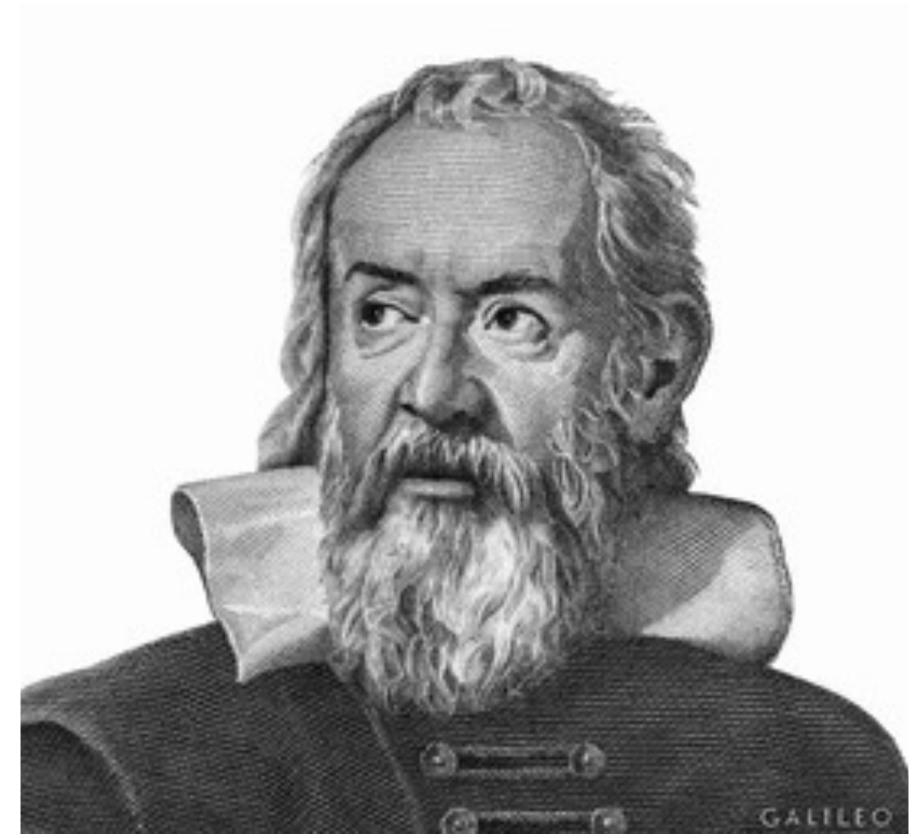

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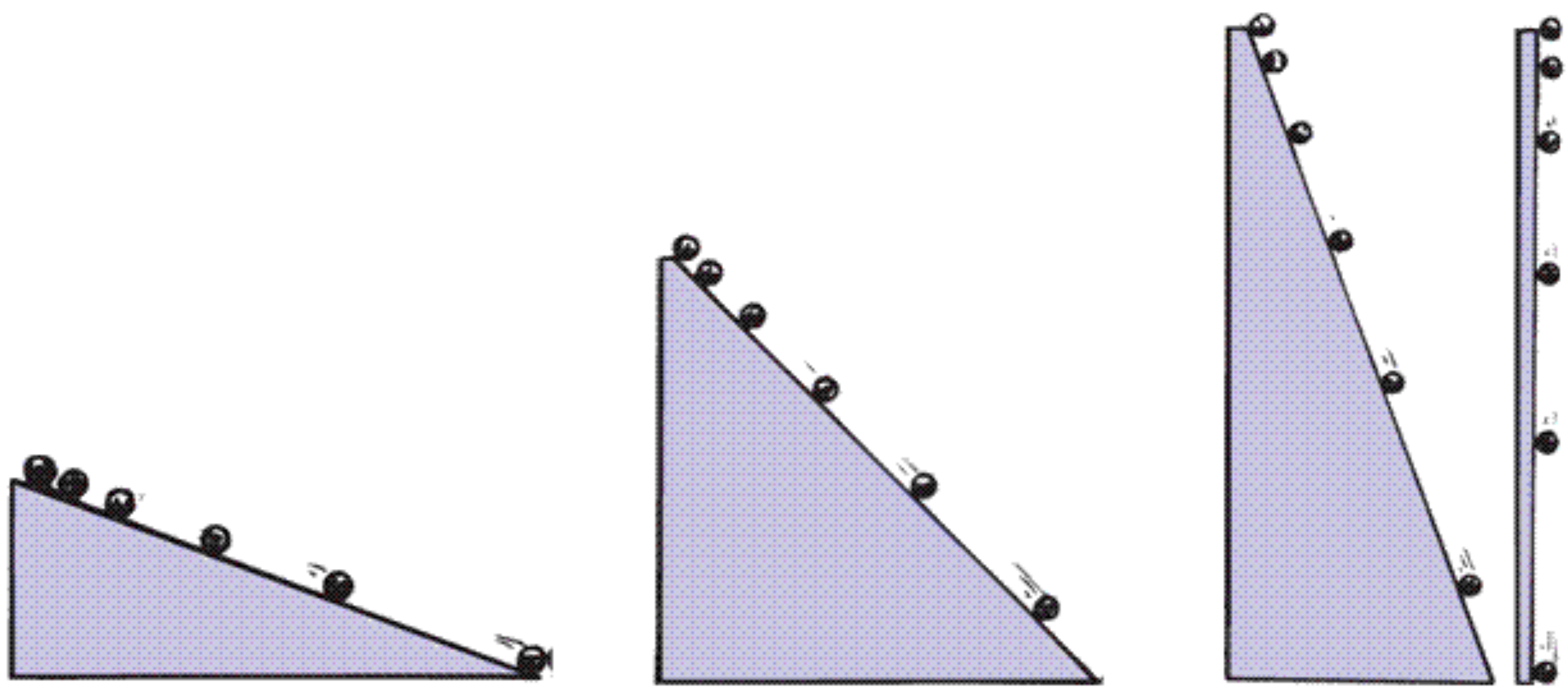

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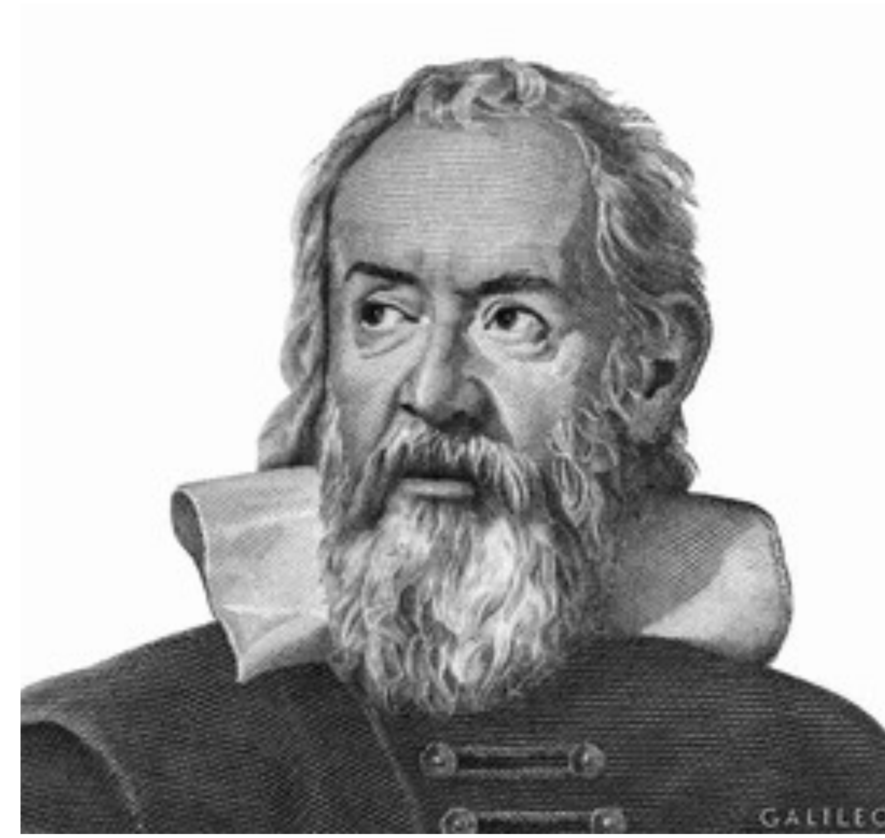

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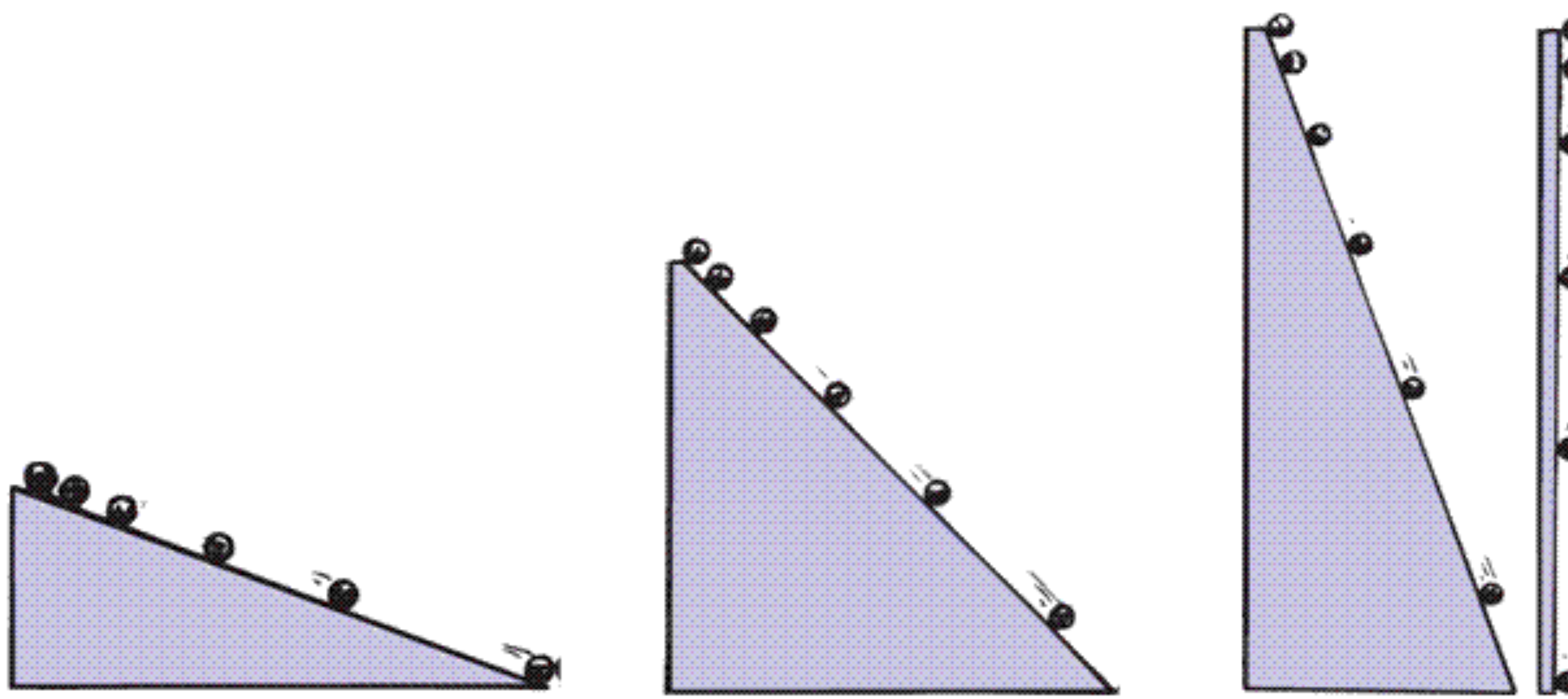

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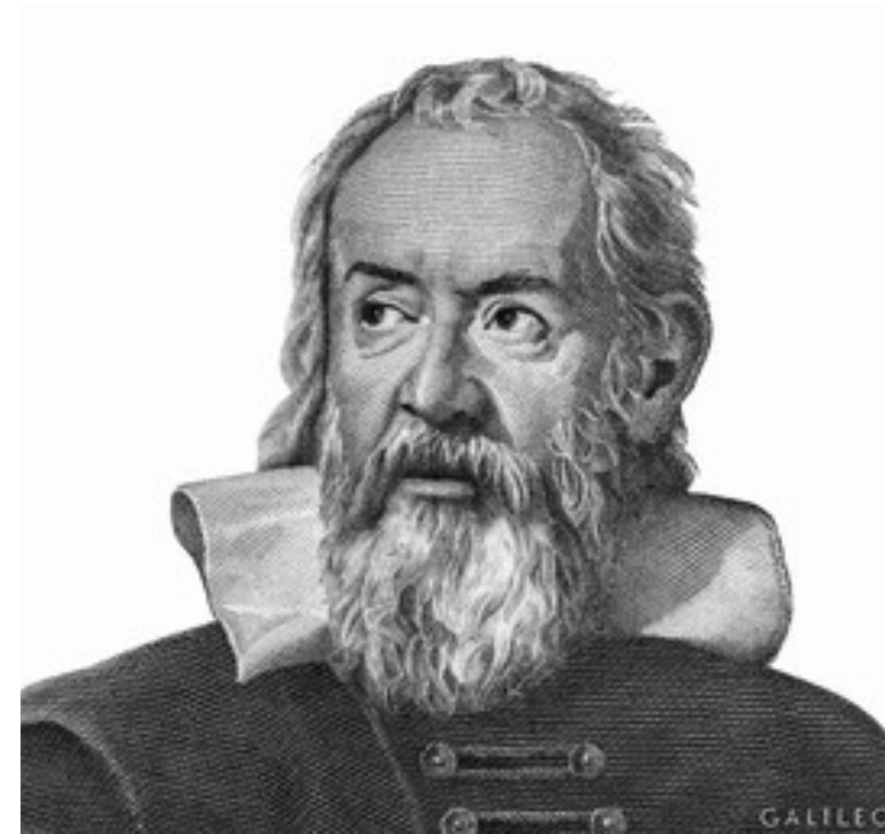

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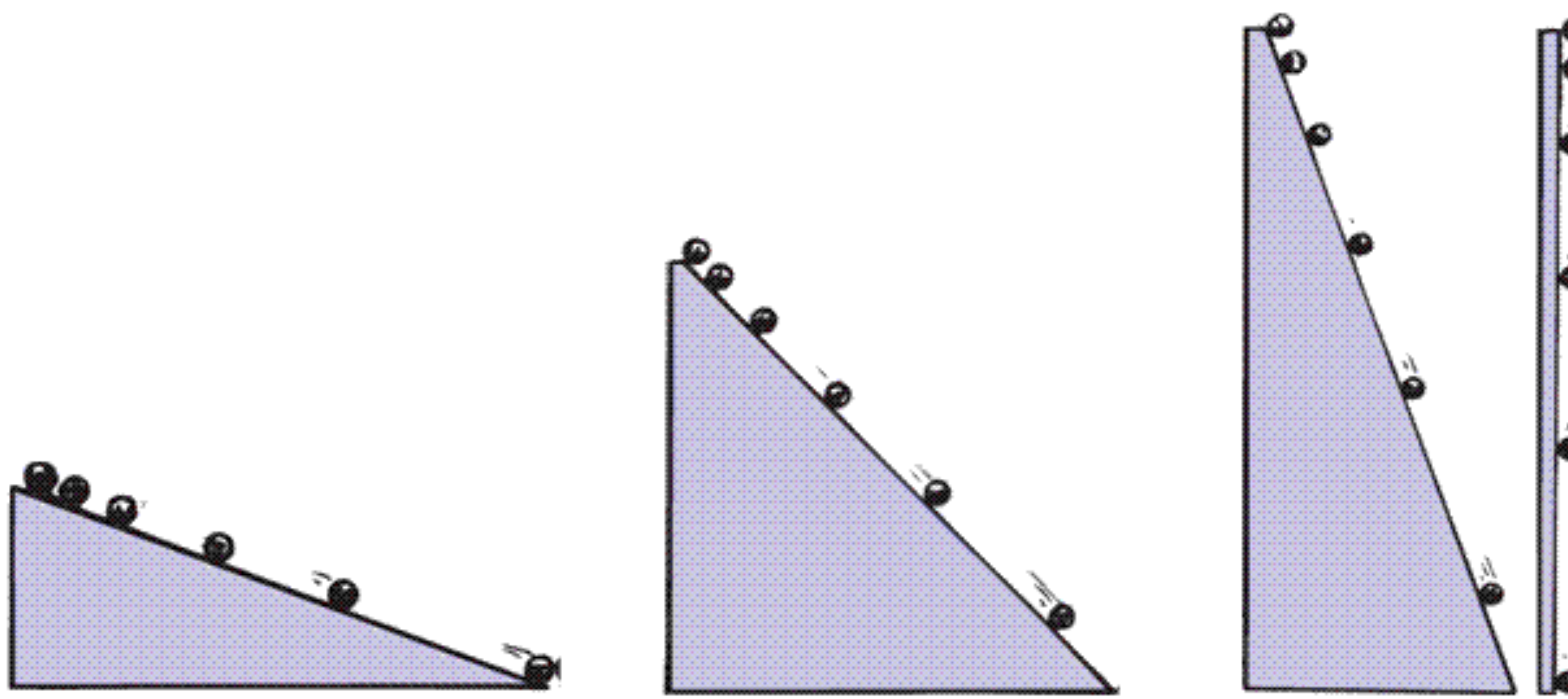

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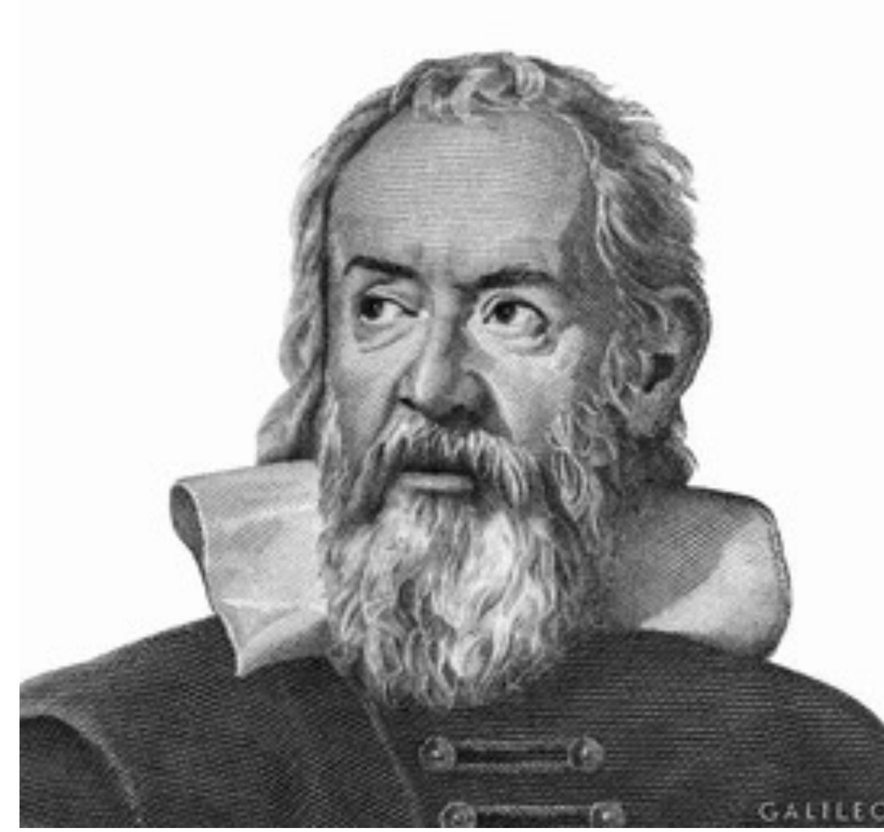
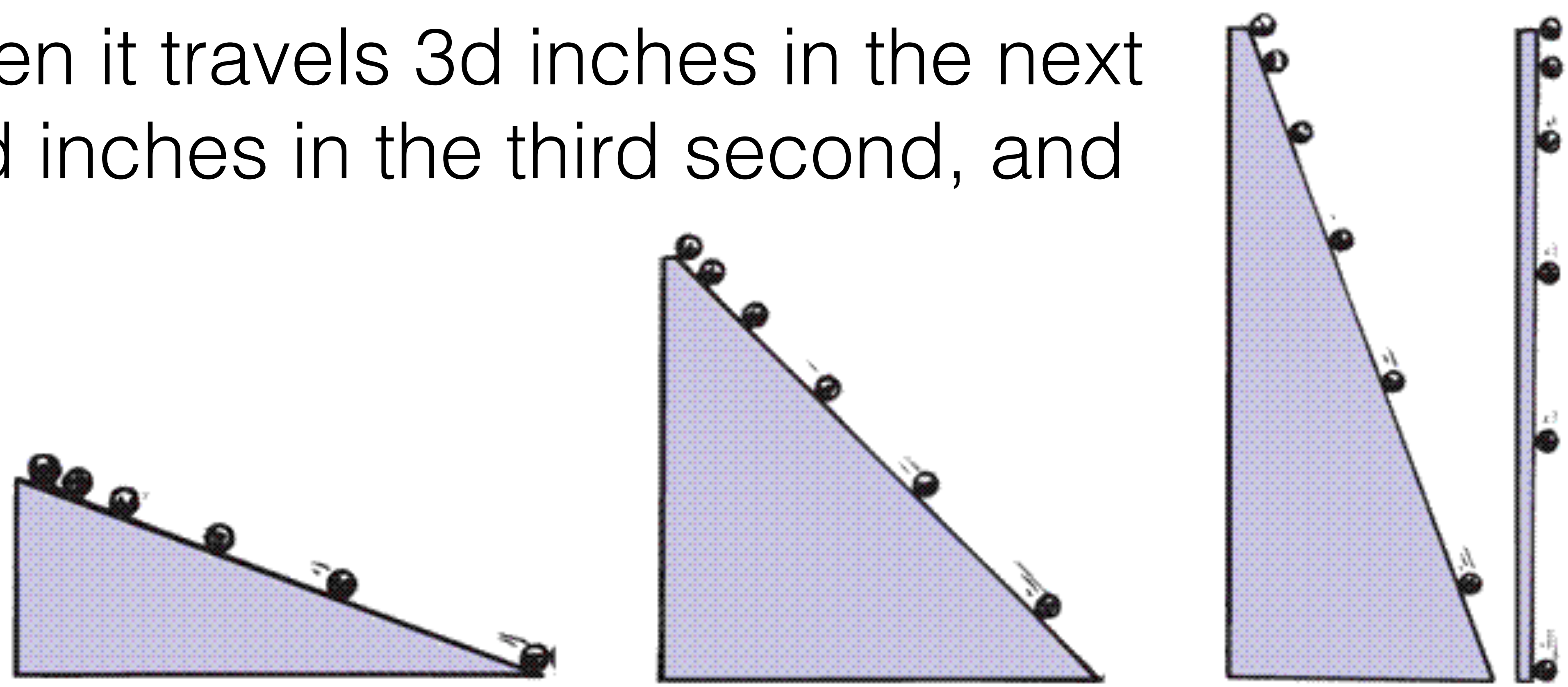


- Galileo Galilei ran experiments to investigate objects in motion.
- Example: He rolled objects down an incline.
- He discovered that if it travels d inches in the first second, then it travels 3d inches in the next second, 5d inches in the third second, and so on.



Isaac
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Wilhelm Leibniz

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Apple Tree!

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- Newton took the chance to study advanced math, and invented more. Spent his money on math books, and borrowed more.

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- The 24 year old discovered his fundamental theories on gravity, motion and optics. He invented differential and integral calculus. He told no one.

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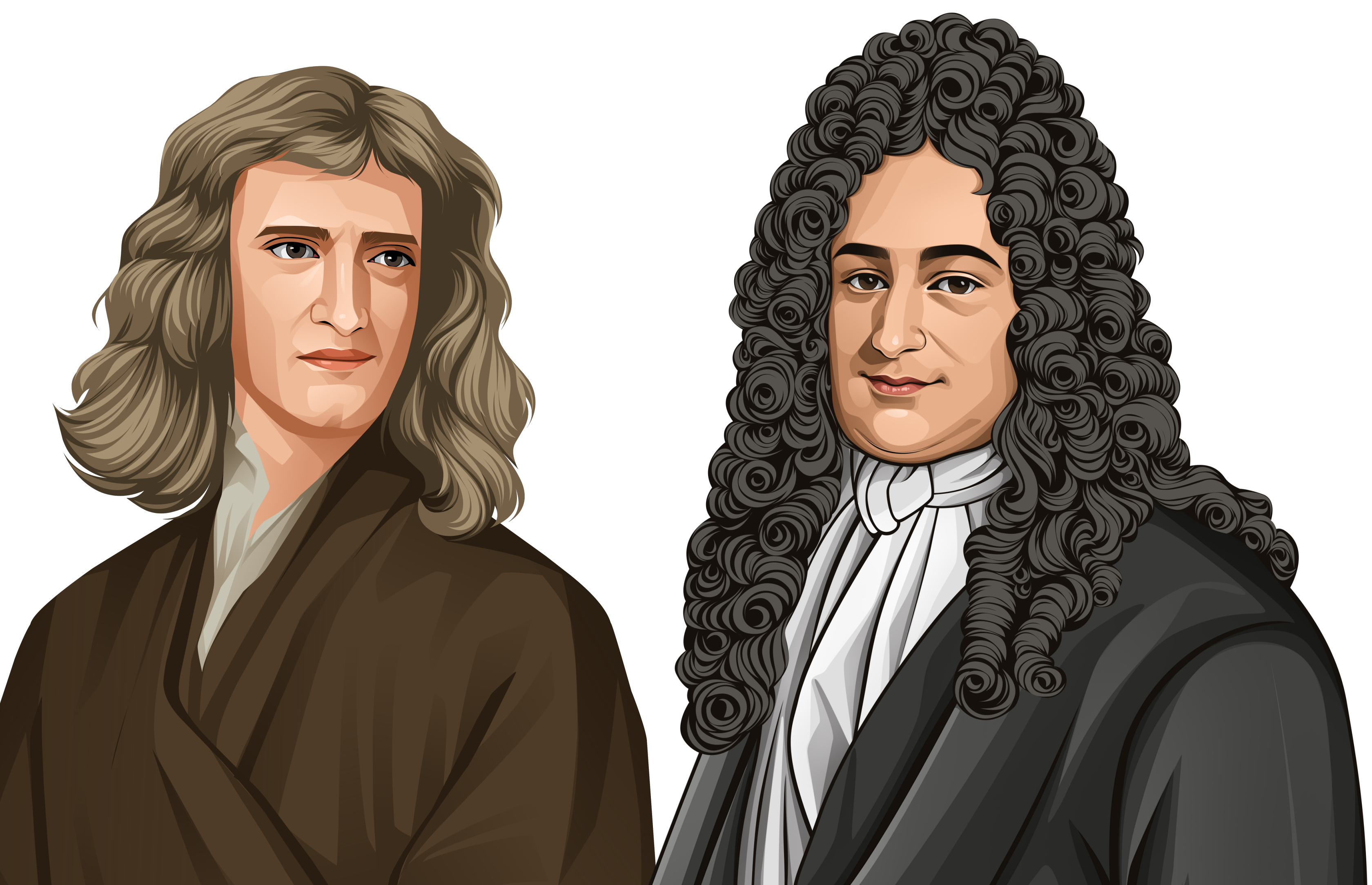
Link to scan

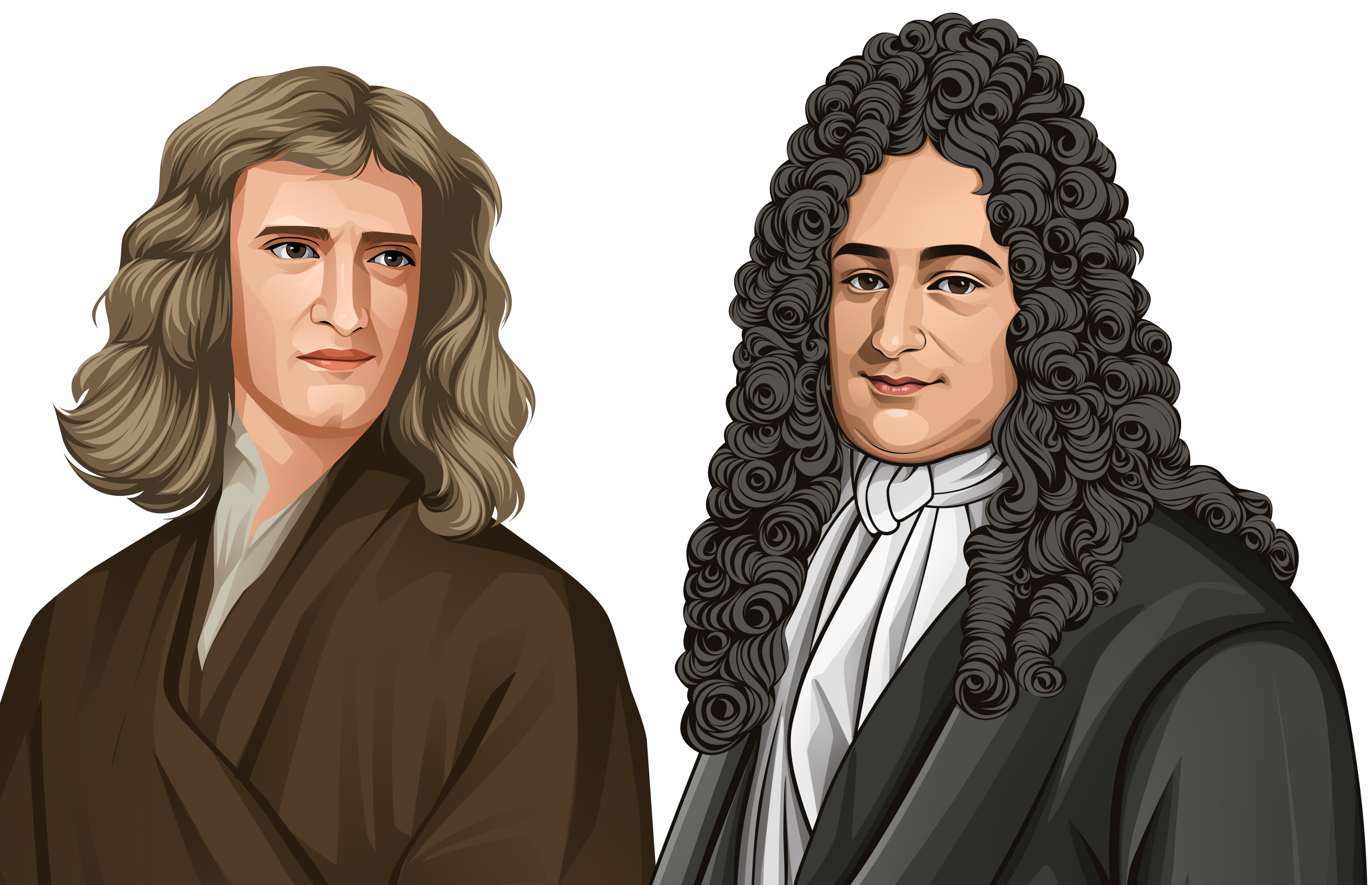
On the Board:
Isaac Newton’s
Fluxions and Fluents

Newton vs. Leibniz



Newton vs. Leibniz

- **Theorem (Newton).** Suppose $y = x^r$ where r is a rational number. Then, $\dfrac{\dot{y}}{\dot{x}} = rx^{r-1}$.



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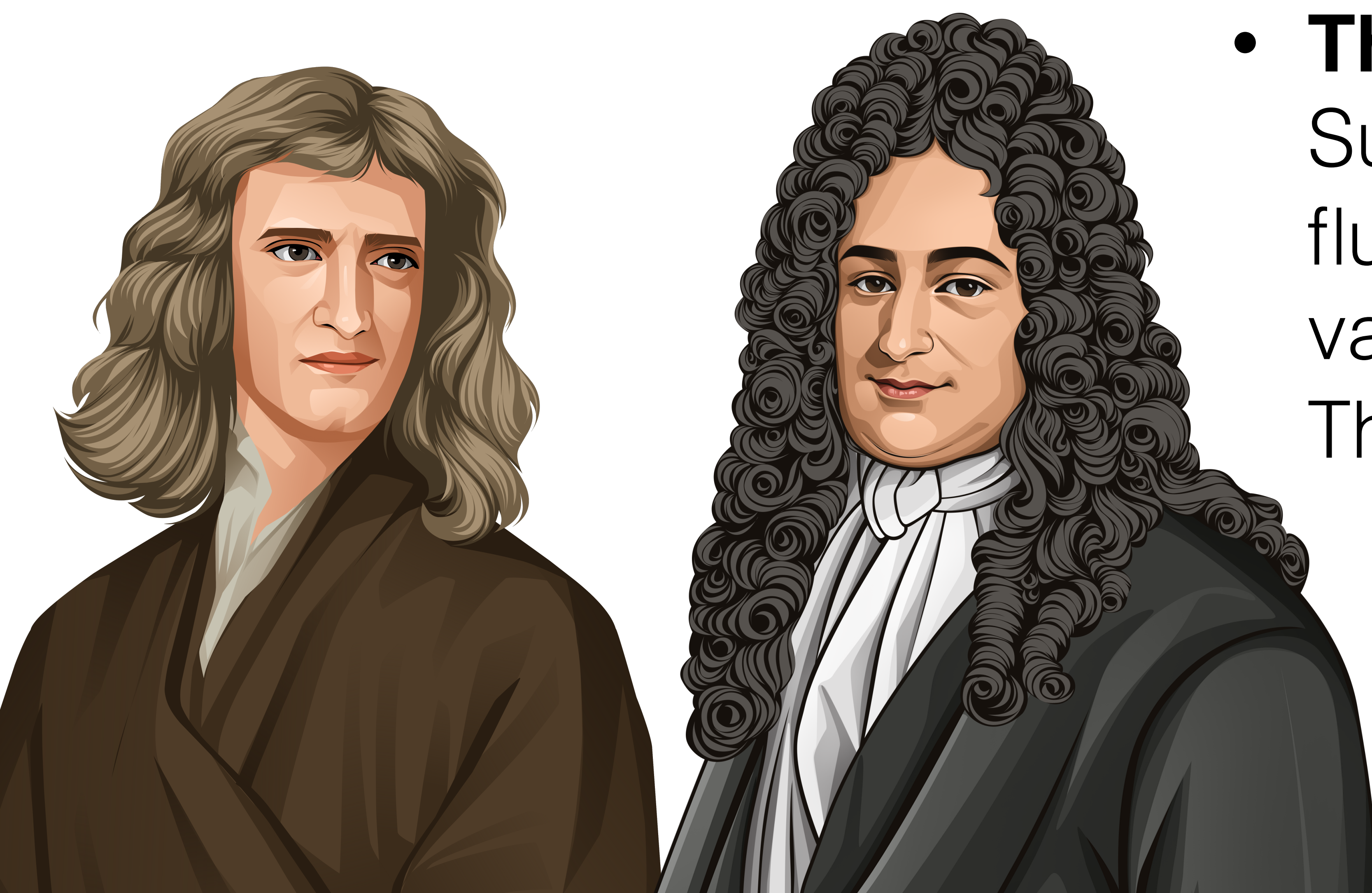

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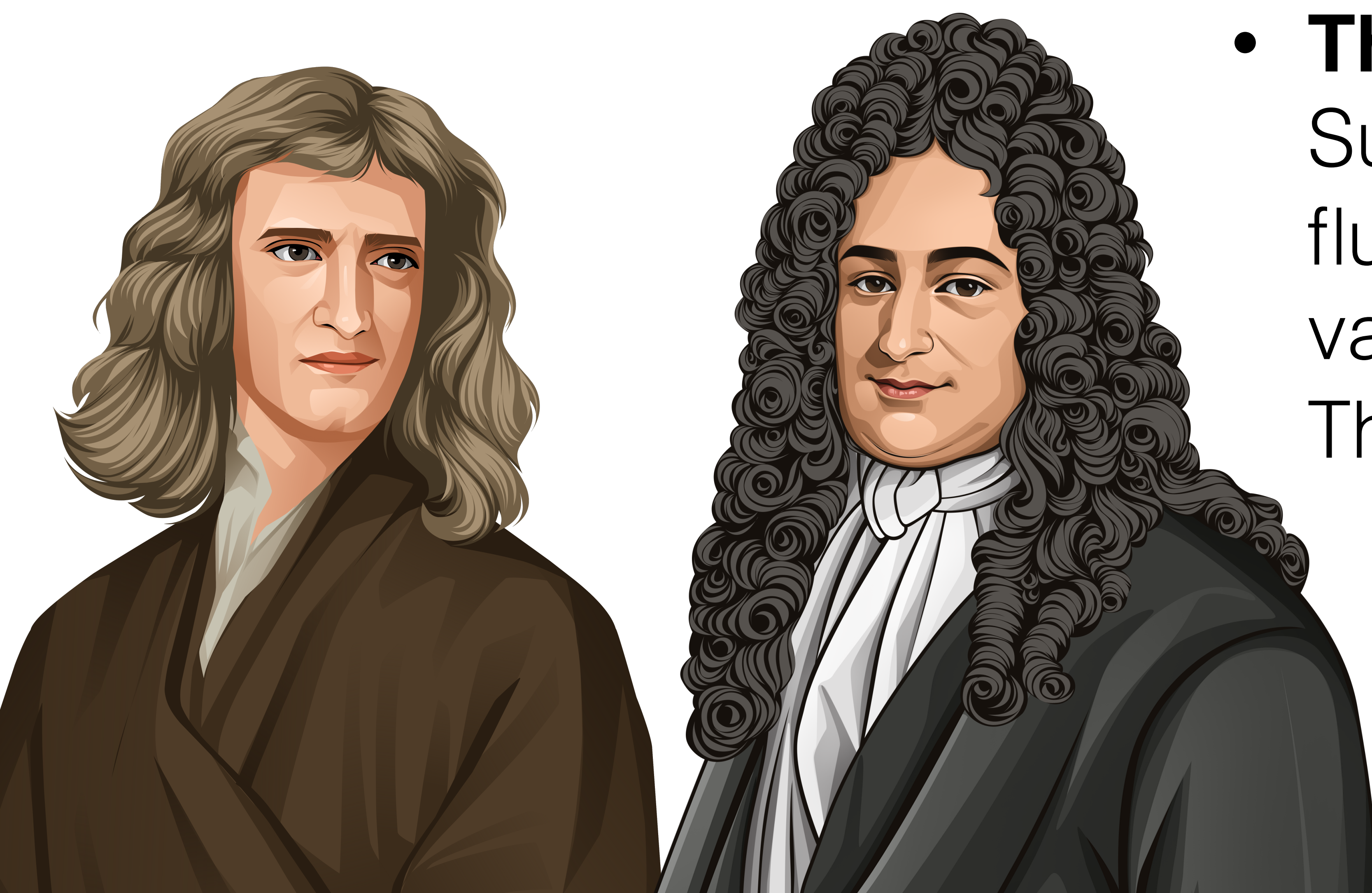

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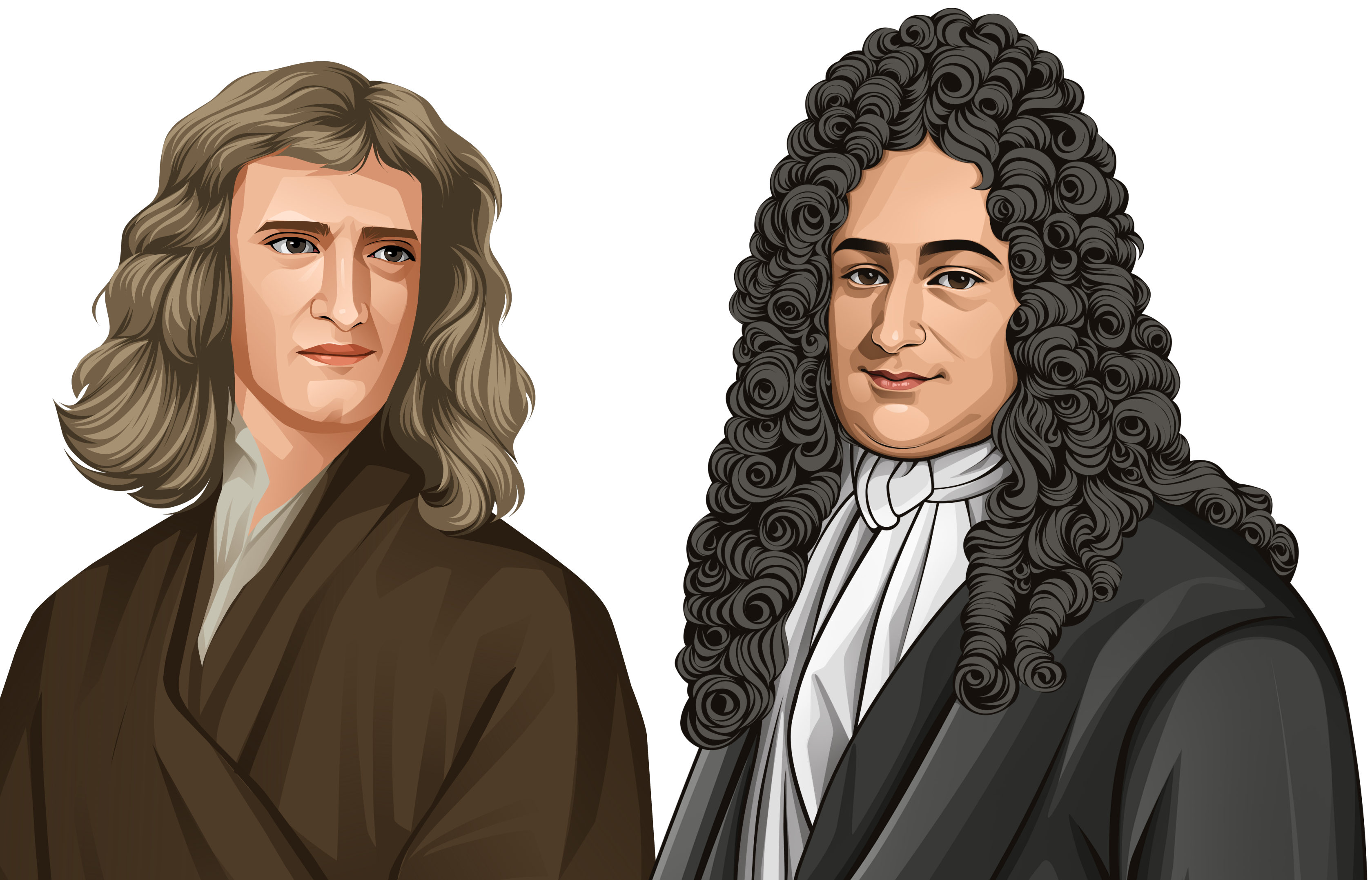

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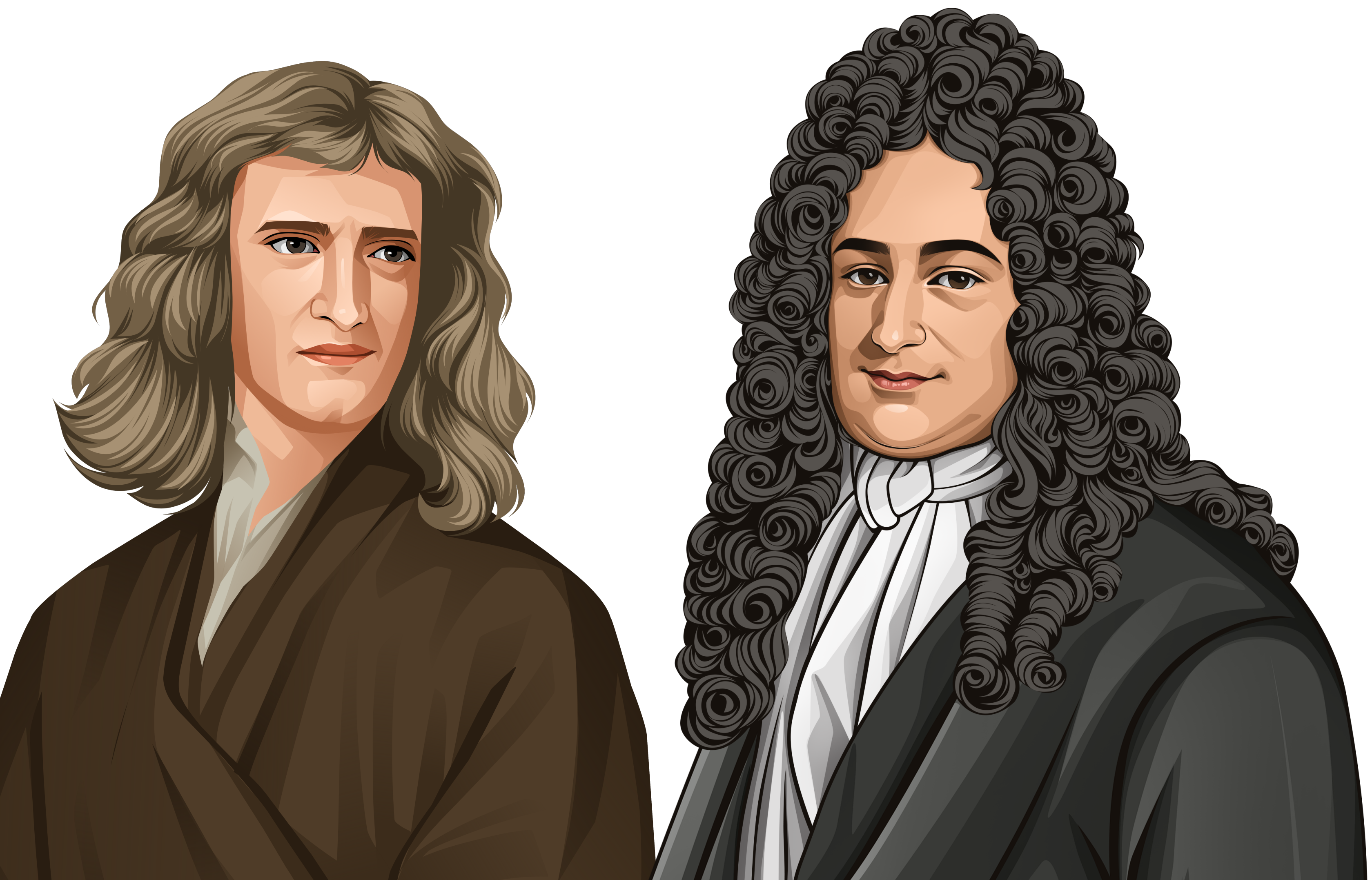
- It is likely that Leibniz independently invented the calculus. Their work was fundamentally the same, but they developed it in different ways. Their work *looked* quite different. Today, we use Leibniz's notation.



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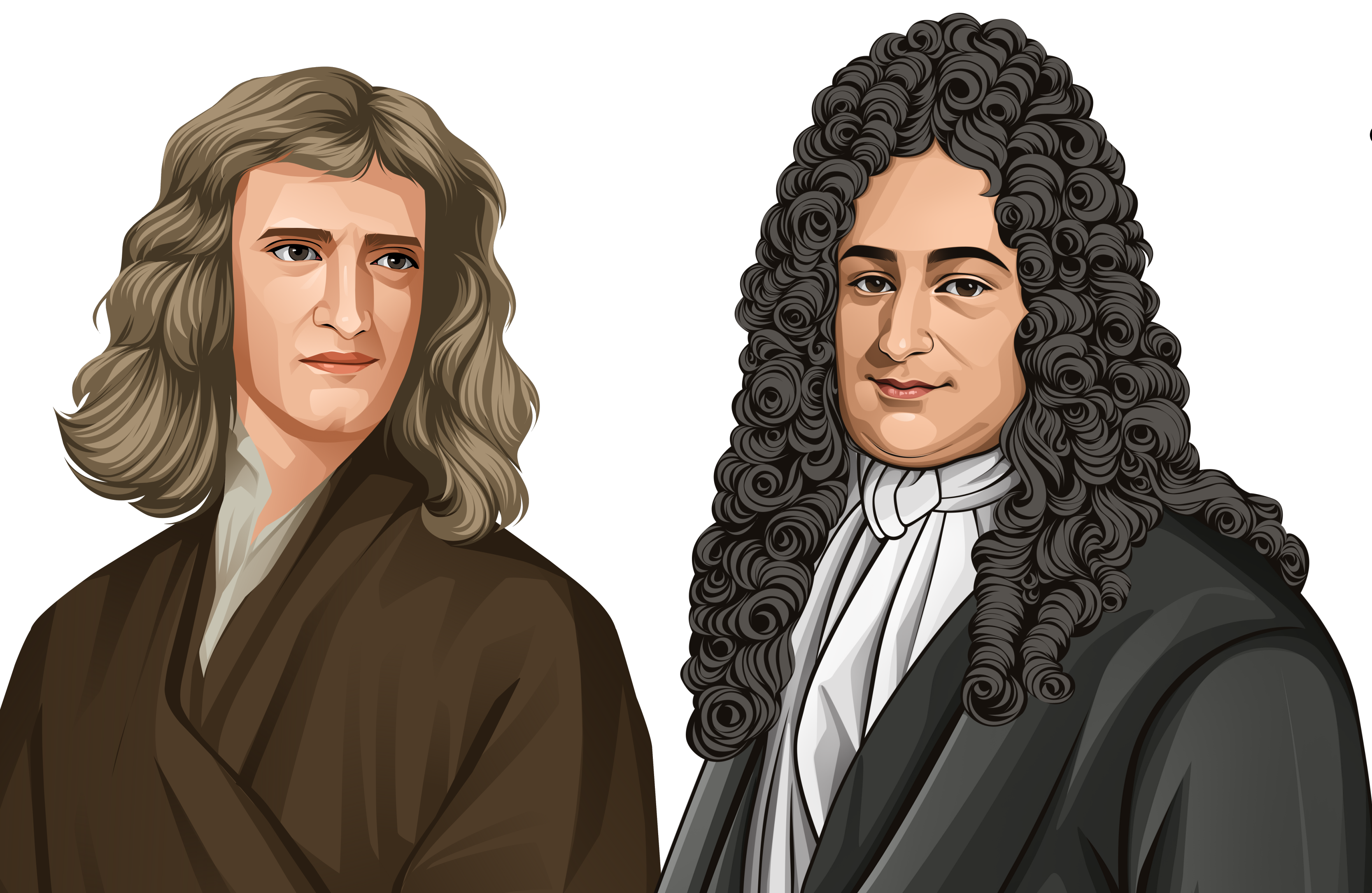

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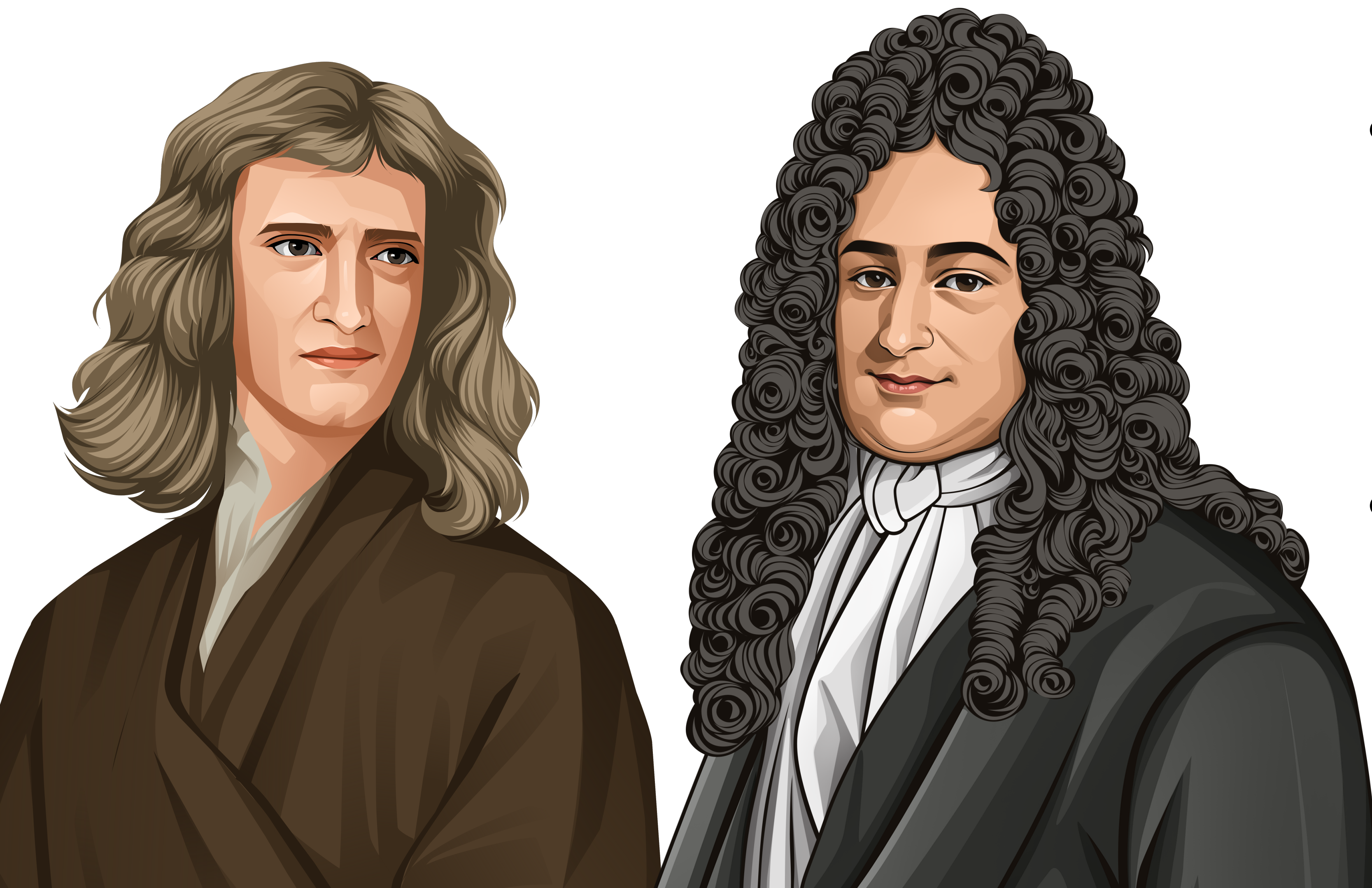

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- Newton's work was better; Liebniz's notation was better. Both were superb.

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- In 1684, Halley asked Newton how the planets moved. This set off his second intense period of inquiry, leading to his masterpiece: *Principia*.

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Albert Einstein regarded Newton with highest regard.

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> All the time he was with him, he never observed him to laugh; but once. He remembers, it was on this occasion. He asked a friend, to whom he had lent a copy of Euclid's *Elements* to read, what progress he had made and how he liked it. [The friend] answered by desiring to know what use and benefit in life is that kind of study to him? Upon which Sr. Isaac was very merry.

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Poem:
Isaac Newton Alone

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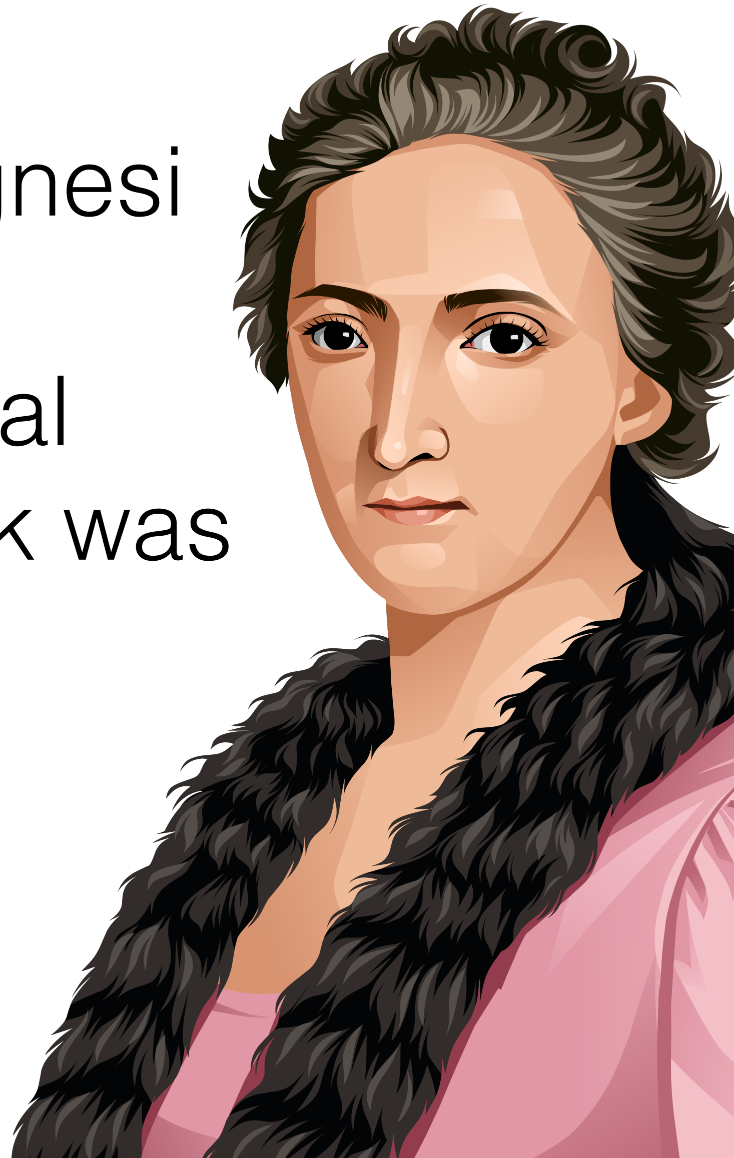

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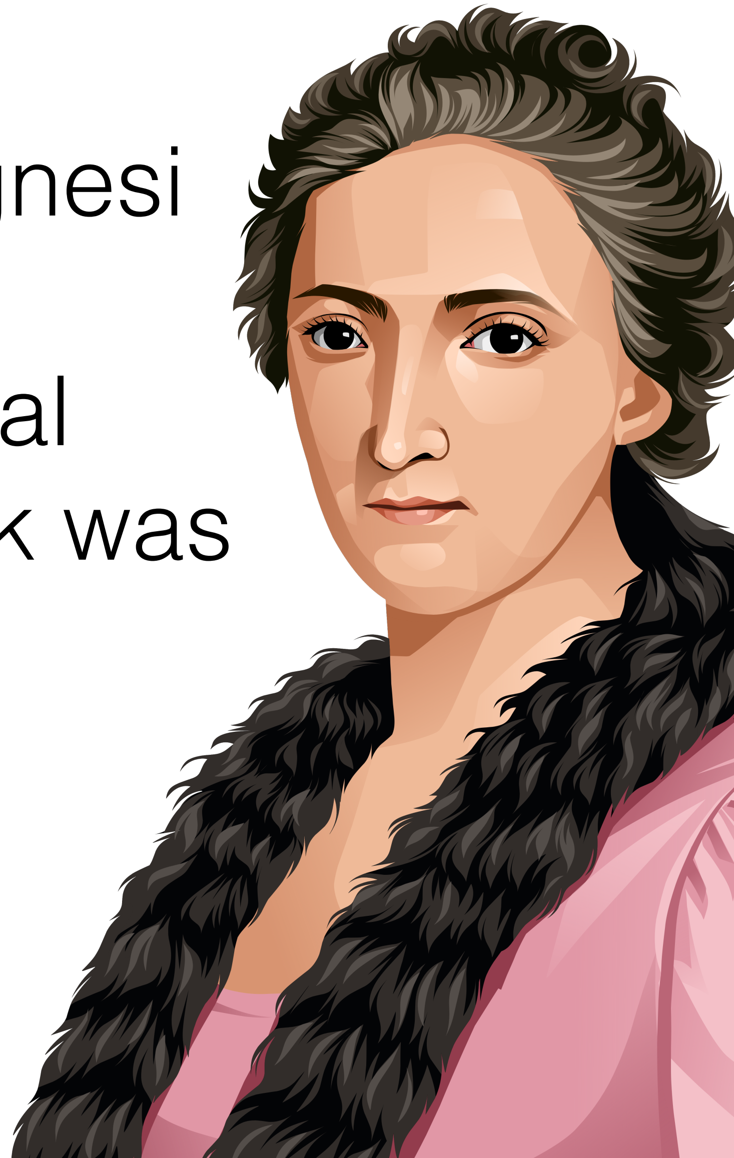

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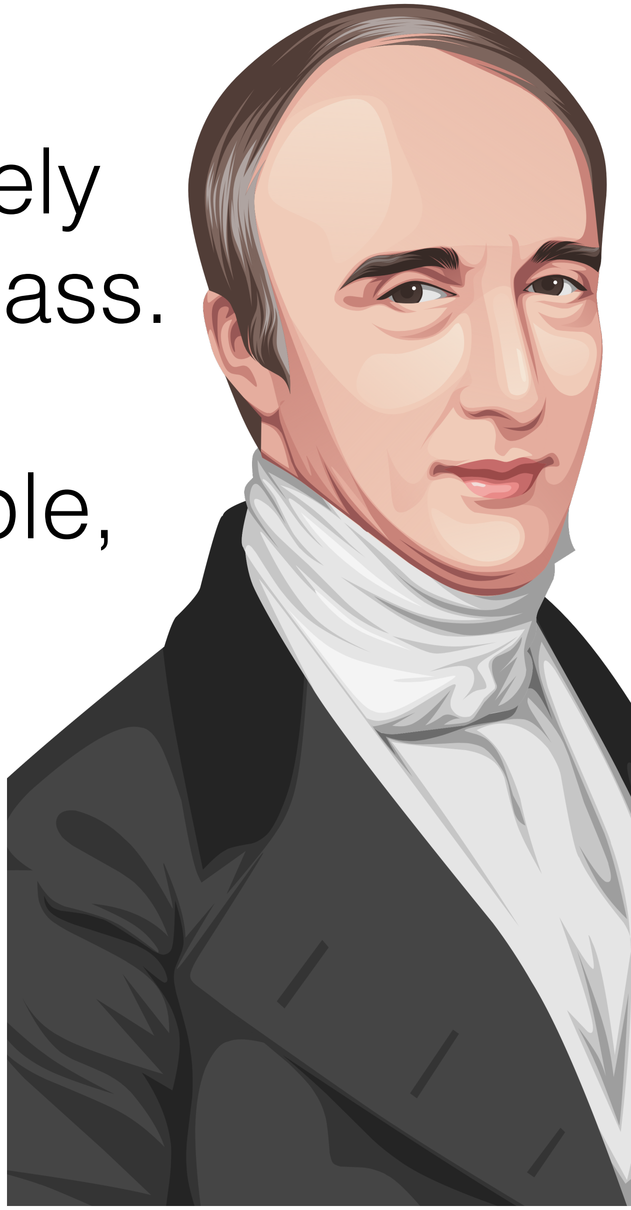

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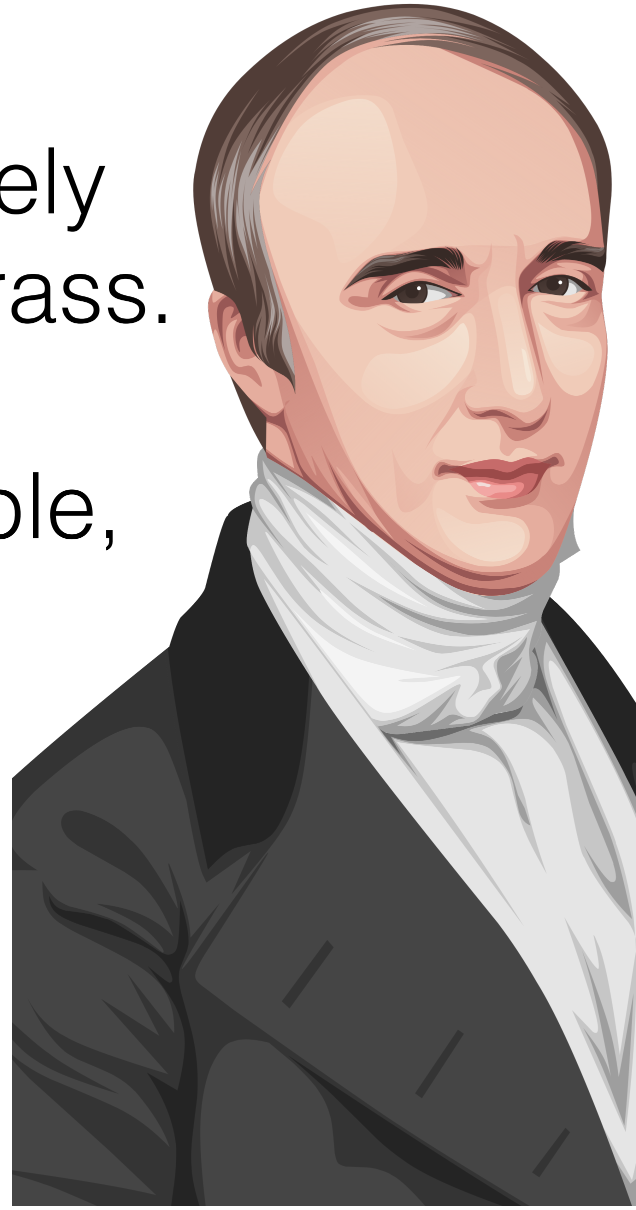

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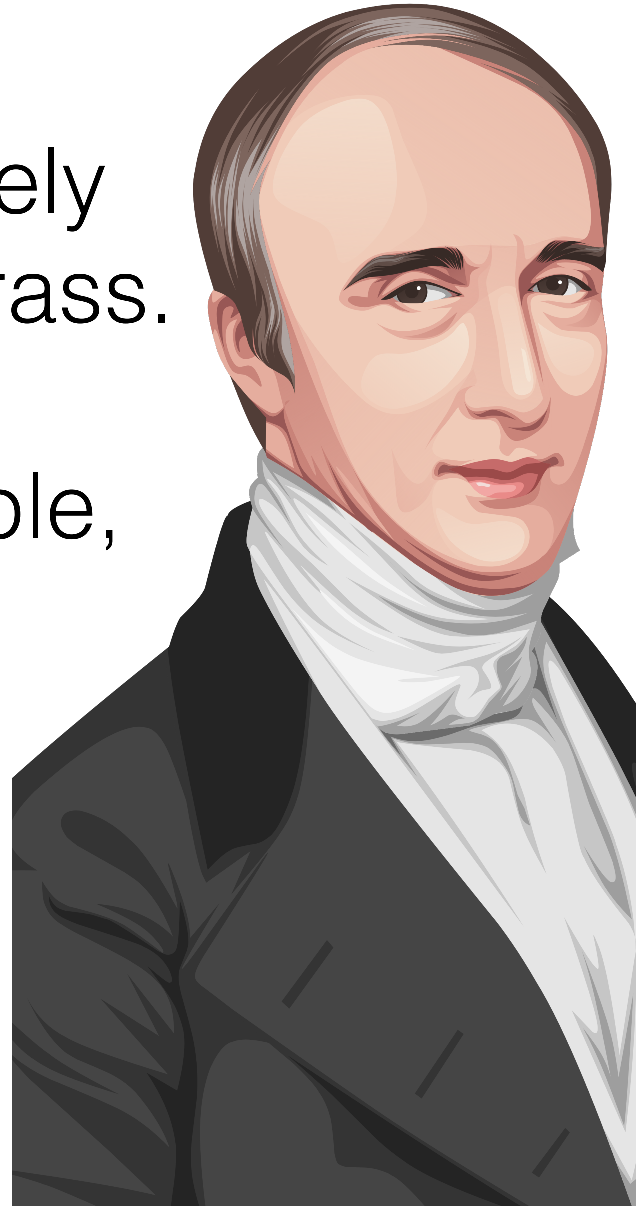

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- Weierstrass: For all $\epsilon > 0$, there exists a $\delta > 0$ such that, if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

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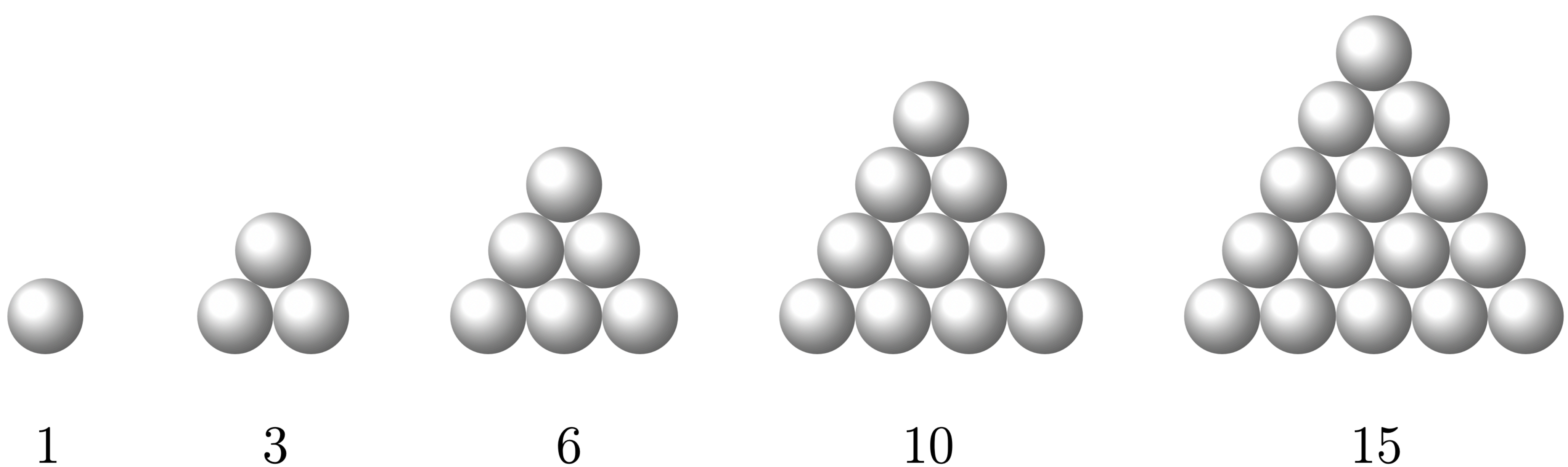

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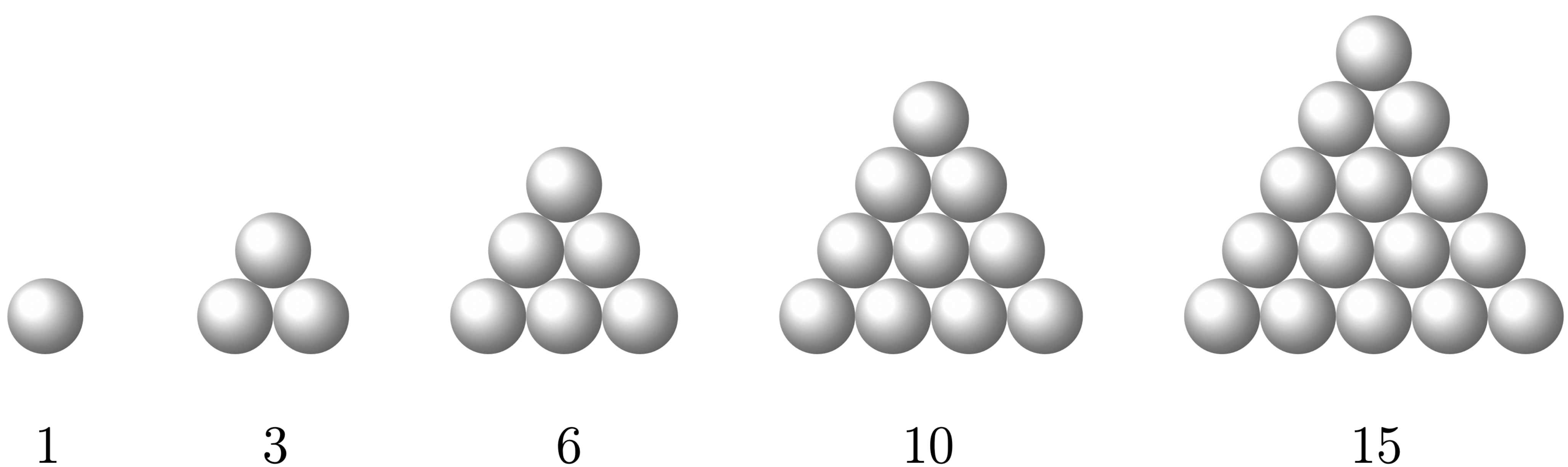

- Leibniz wanted to find $\dfrac{1}{1}+\dfrac{1}{3}+\dfrac{1}{6}+\dfrac{1}{10}+\dfrac{1}{15}+\dfrac{1}{21}+\ldots$

Infinite Series

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$$\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \frac{1}{21} + \cdots$$

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$$\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}+\dots$$

$$=(2-1)+\left(1-\frac{2}{3}\right)+\left(\frac{2}{3}-\frac{2}{4}\right)+\left(\frac{2}{4}-\frac{2}{5}\right)+\left(\frac{2}{5}-\frac{2}{6}\right)+\dots$$

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$$=2-(1-1)-\left(\frac{2}{3}-\frac{2}{3}\right)-\left(\frac{2}{4}-\frac{2}{4}\right)-\left(\frac{2}{5}-\frac{2}{5}\right)-\left(\frac{2}{6}-\frac{2}{6}\right)-\dots$$

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$$=2-(1-1)-\left(\frac{2}{3}-\frac{2}{3}\right)-\left(\frac{2}{4}-\frac{2}{4}\right)-\left(\frac{2}{5}-\frac{2}{5}\right)-\left(\frac{2}{6}-\frac{2}{6}\right)-\ldots$$

$$=2.$$

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- This sum diverges suuuuuper slowly. In 1968, J.W. Wrench Jr. showed that $\displaystyle\sum_{k=1}^{n} \frac{1}{k} > 100$ for the first time when

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This is called the *harmonic series*. Details in notes.

- This sum diverges suuuuuper slowly. In 1968, J.W. Wrench Jr. showed that $\sum_{k=1}^{n} \frac{1}{k} > 100$ for the first time when

$n =$ 15,092,688,622,113,788,323,693,563,264,538,101,449,859,497.

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- Jacob never solved it. But his younger brother Johann mentored the person who would.

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- Euler was amazing at determining how far his techniques would go.

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- *When he went blind, he became more productive.*

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Plugging $x = i\theta$ into the e^x Taylor series:

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- *Polyhedra* are 3D objects comprised of faces which meet at edges, and edges which meet at vertices. Euler noticed a property in their counts:

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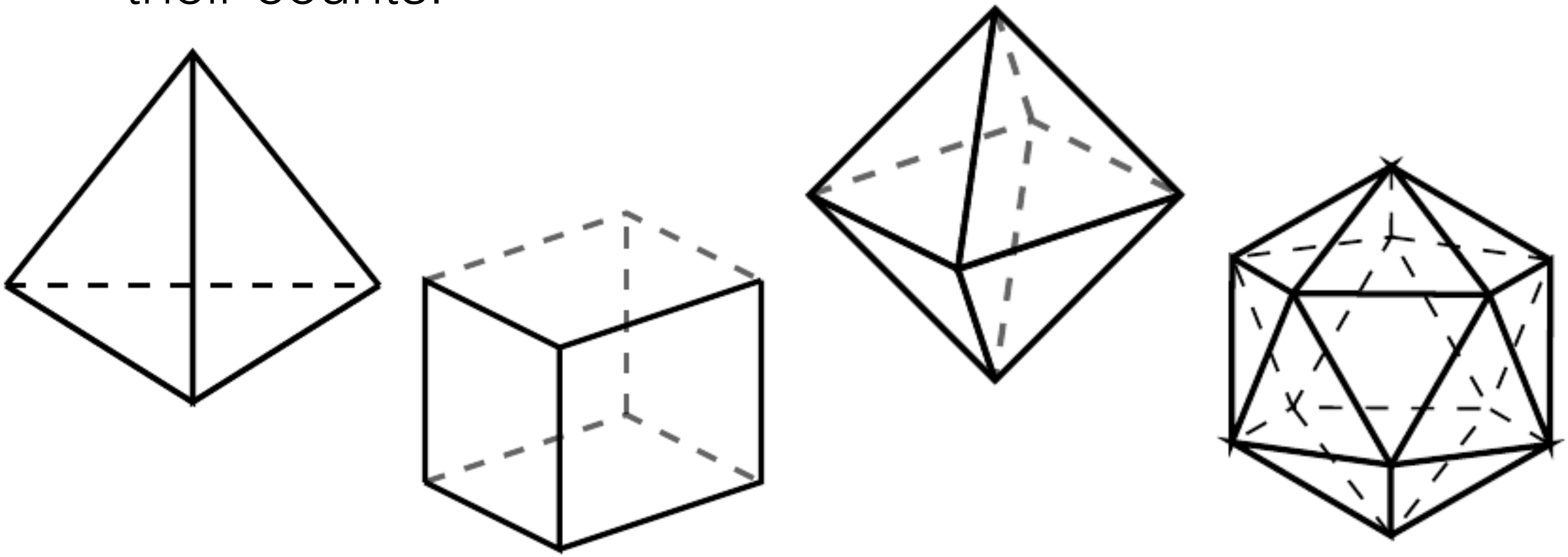

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	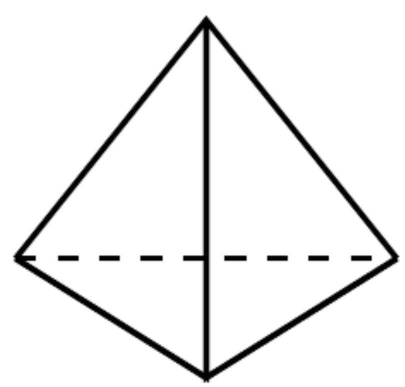	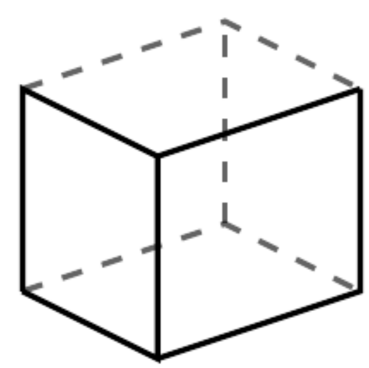	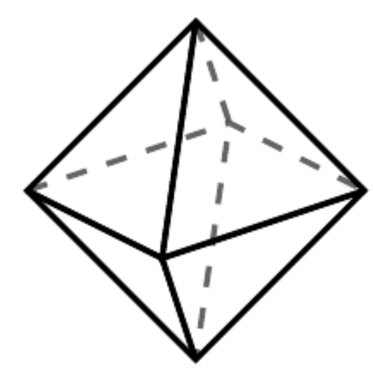	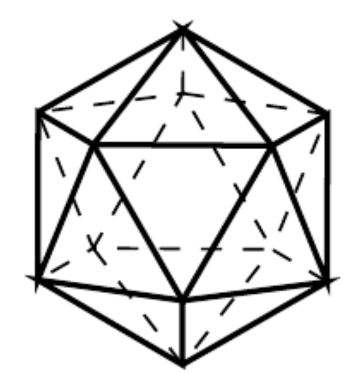
Vertices				
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	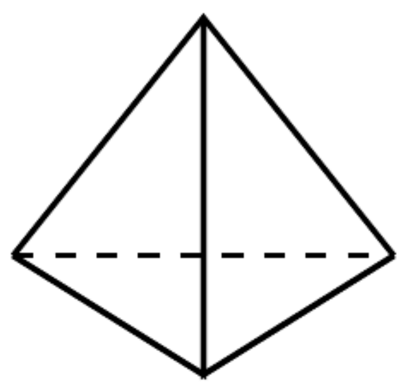	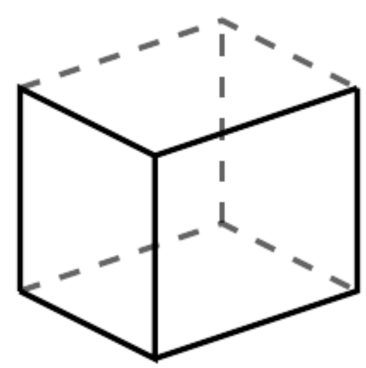	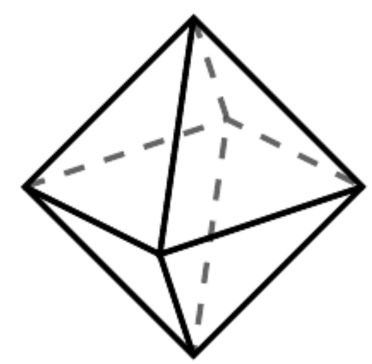	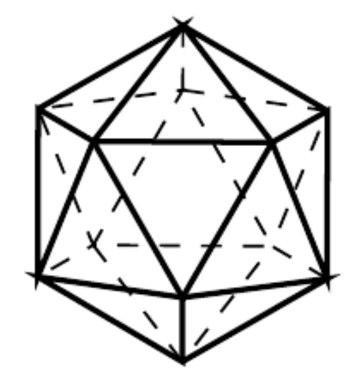
Vertices	4			
Edges				
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	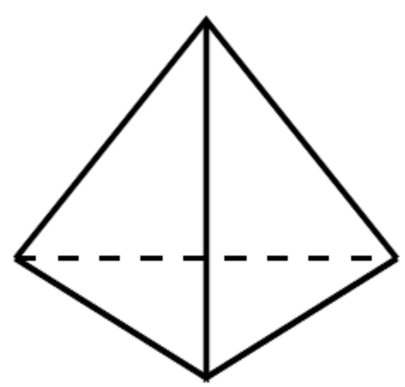	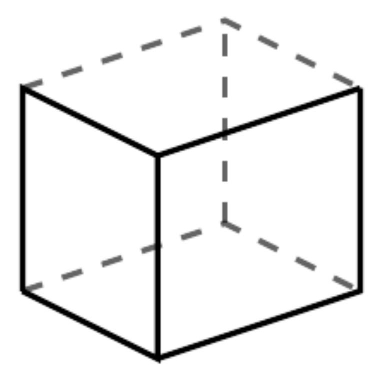	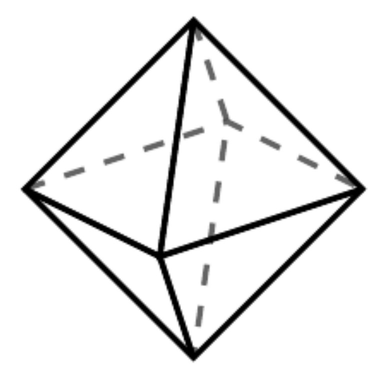	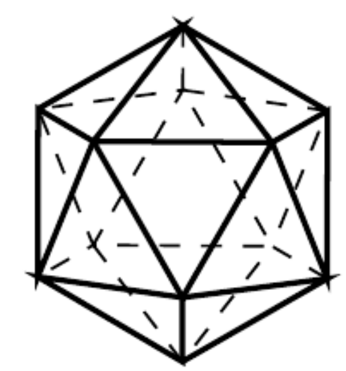
Vertices	4			
Edges	6			
Faces				

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	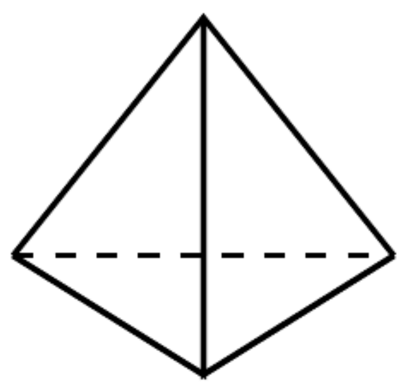	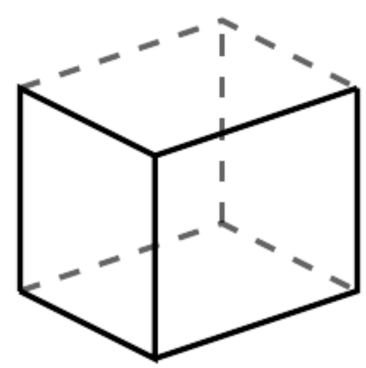	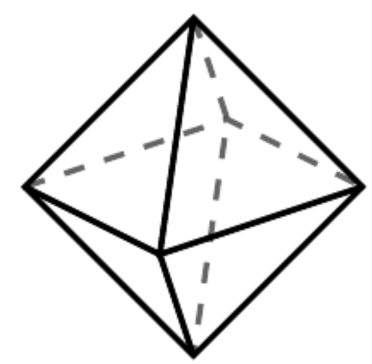	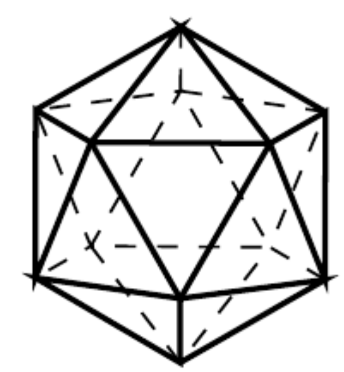
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	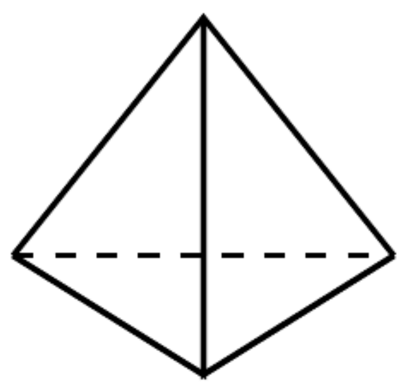	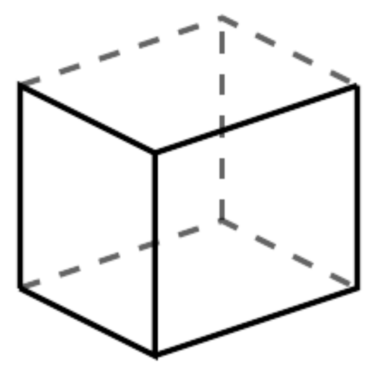	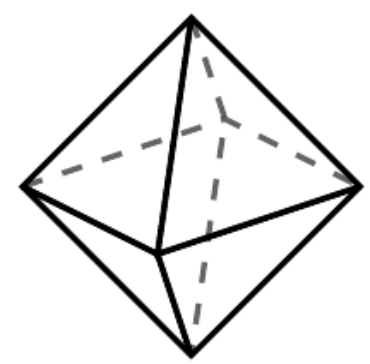	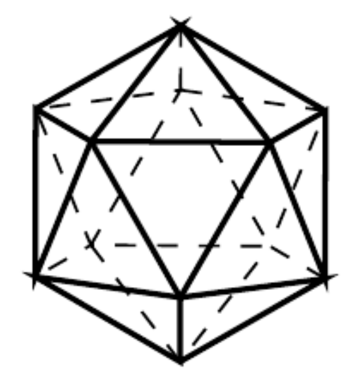
Vertices	4	8		
Edges	6			
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	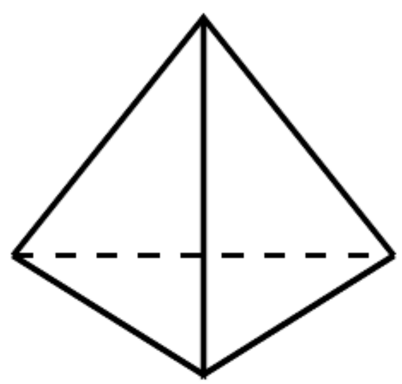	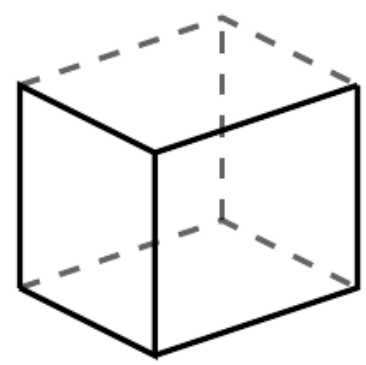	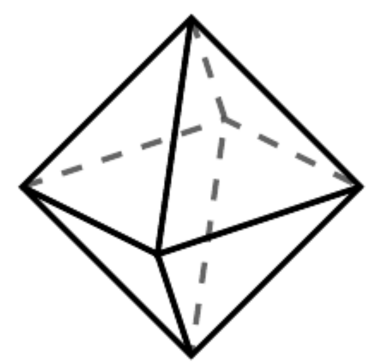	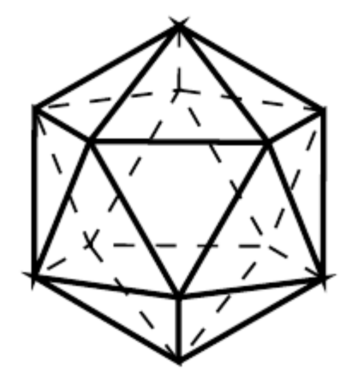
Vertices	4	8		
Edges	6	12		
Faces	4			

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	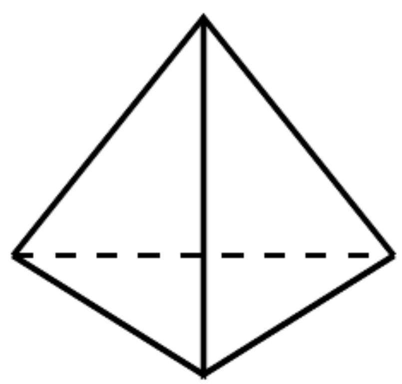	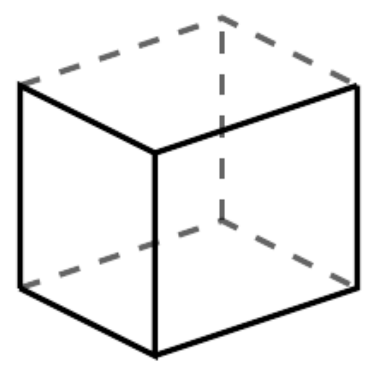	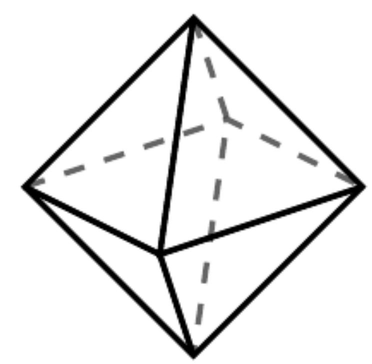	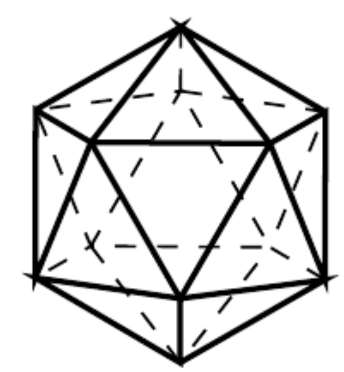
Vertices	4	8		
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	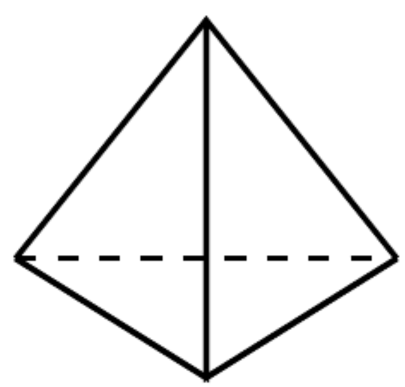	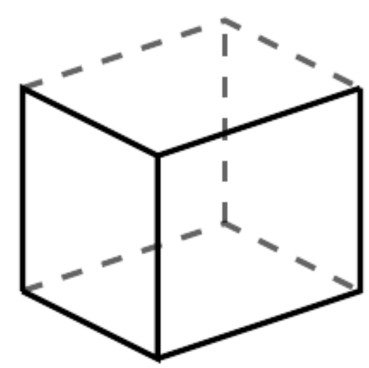	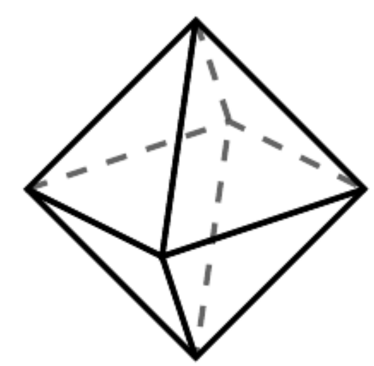	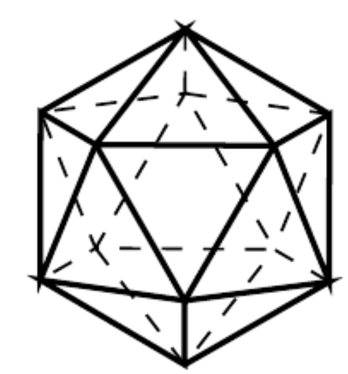
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	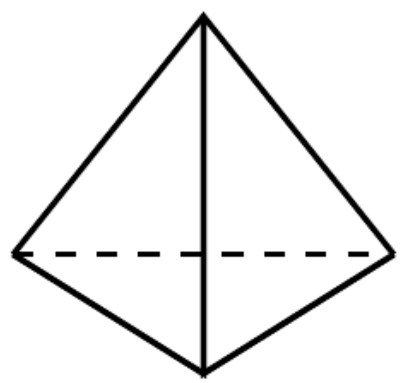	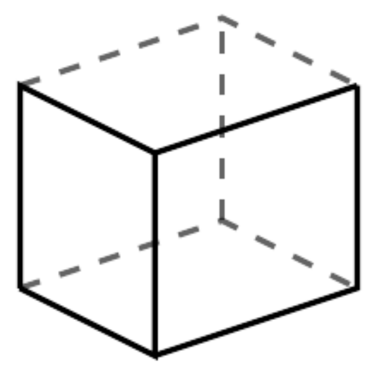	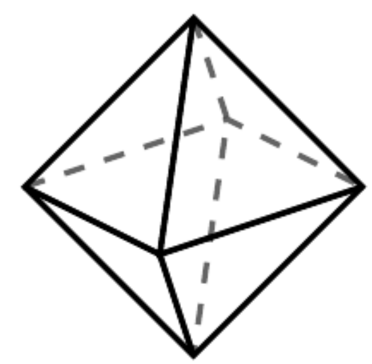	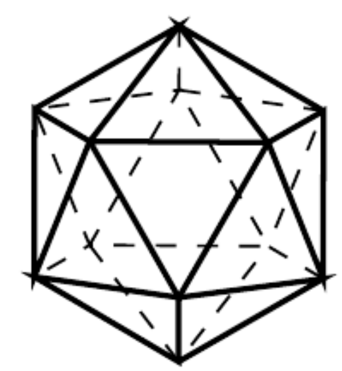
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	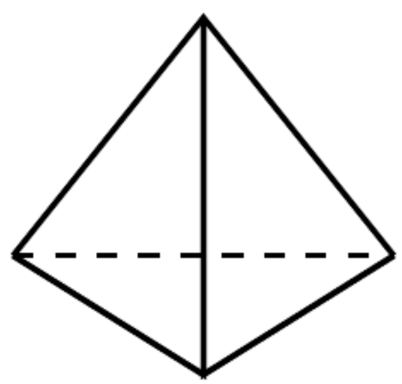	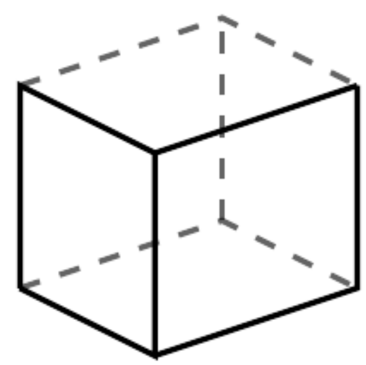	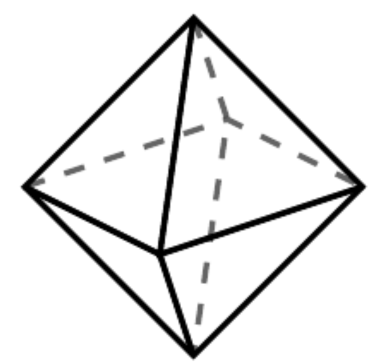	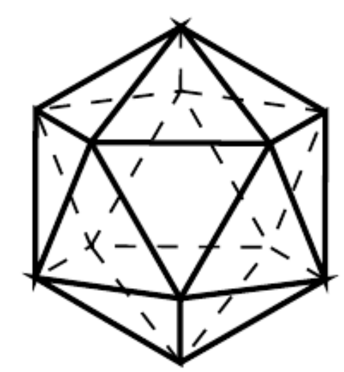
Vertices	4	8	6	
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	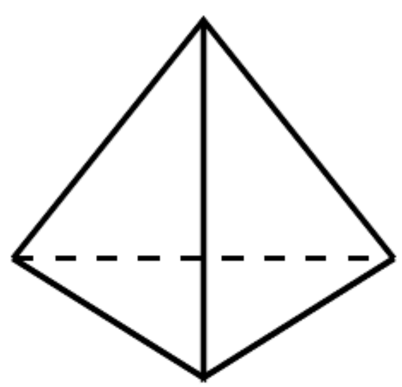	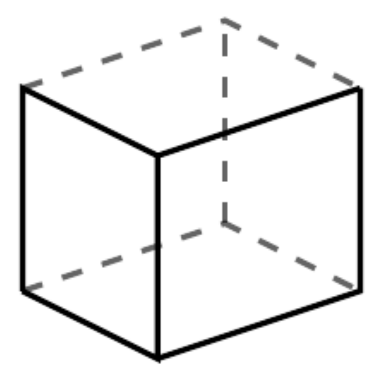	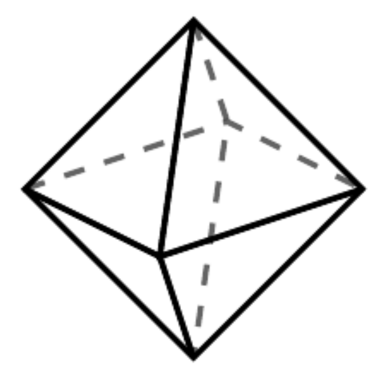	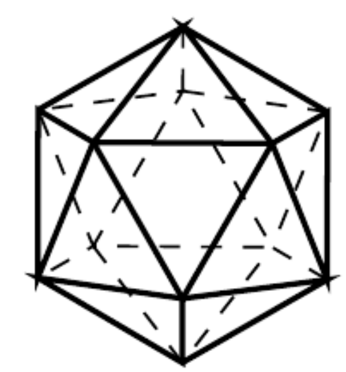
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	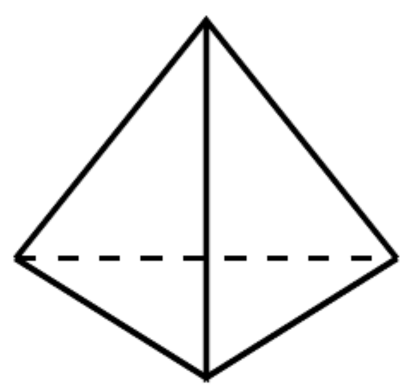	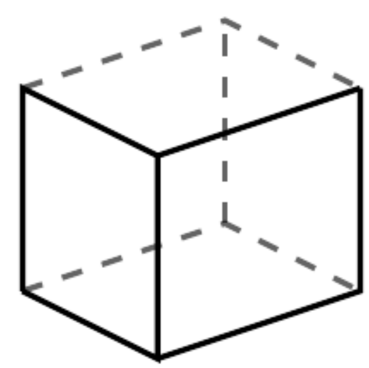	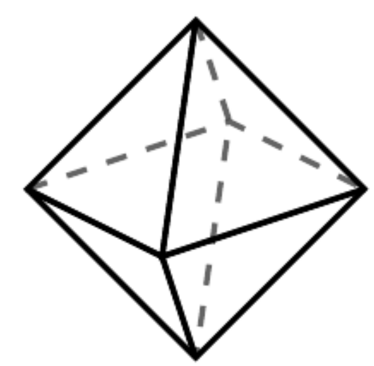	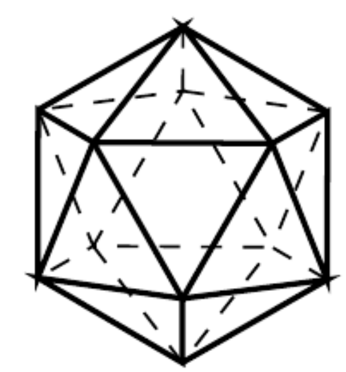
Vertices	4	8	6	12
Edges	6	12	12	30
Faces	4	6	8	

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	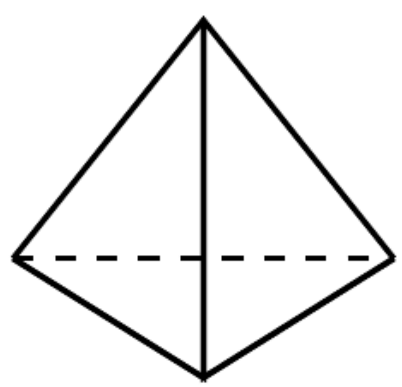	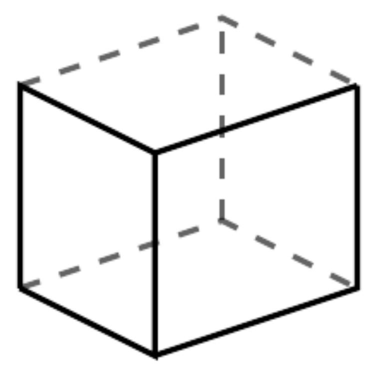	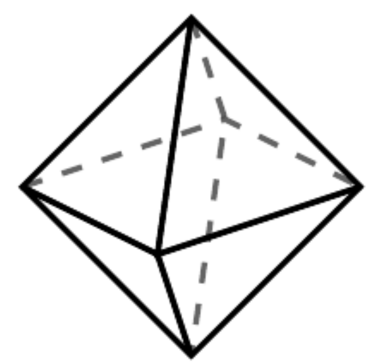	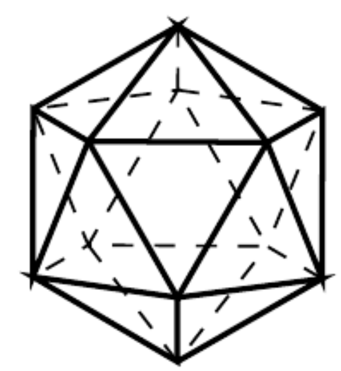
Vertices	4	8	6	12
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	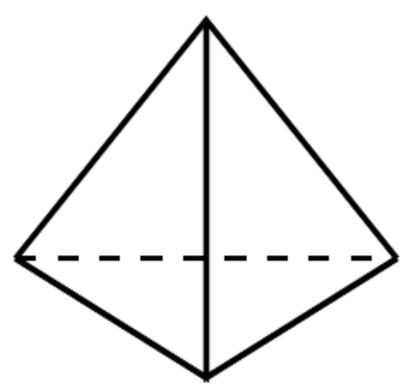	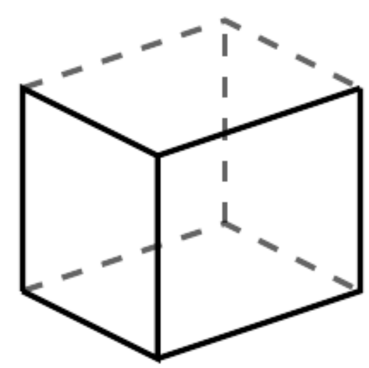	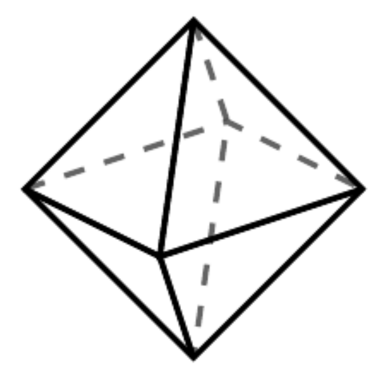	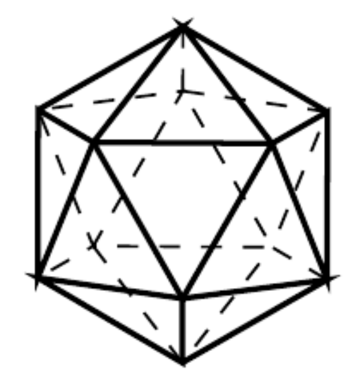
Vertices	4	8	6	12
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$V - E + F$				

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	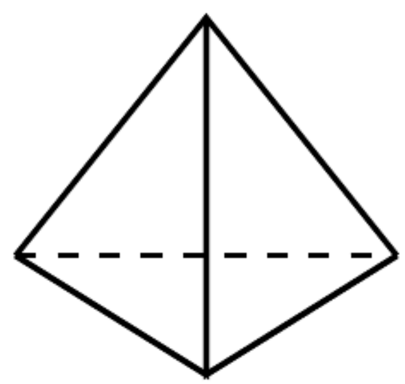	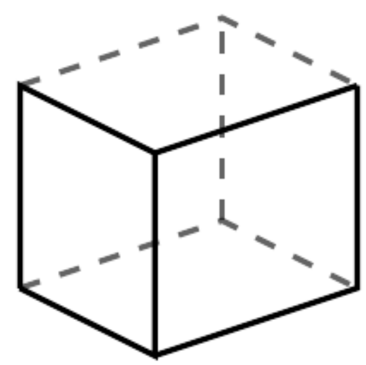	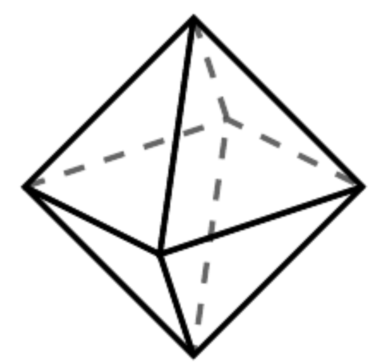	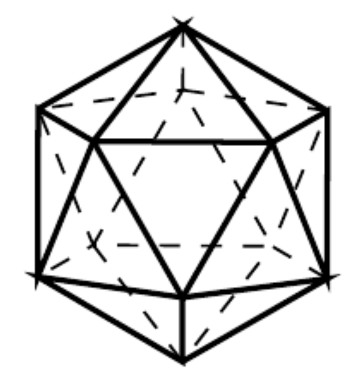
Vertices	4	8	6	12
Edges	6	12	12	30
Faces	4	6	8	20
$V - E + F$	2			

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	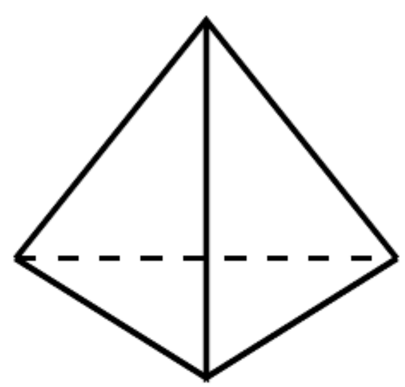	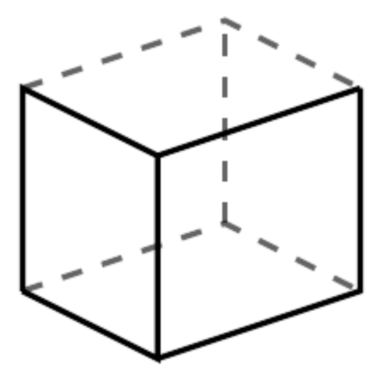	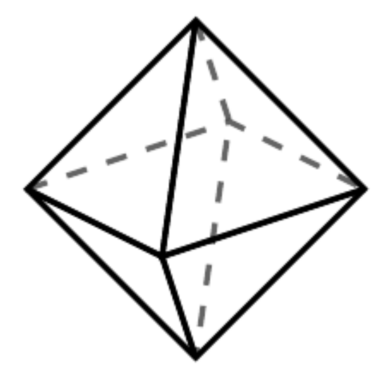	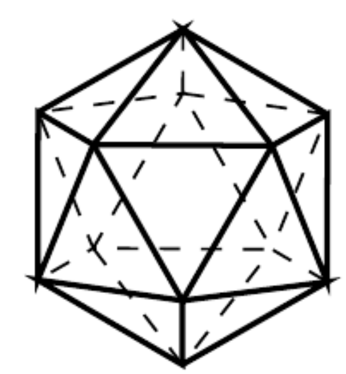
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$V - E + F$	2	2		

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	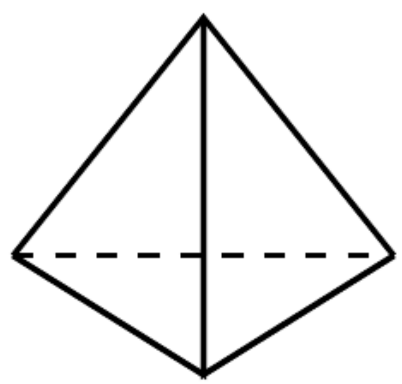	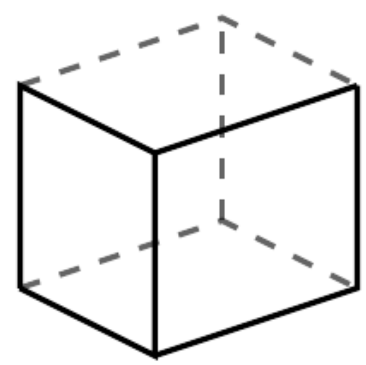	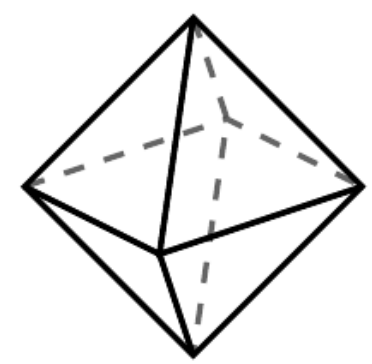	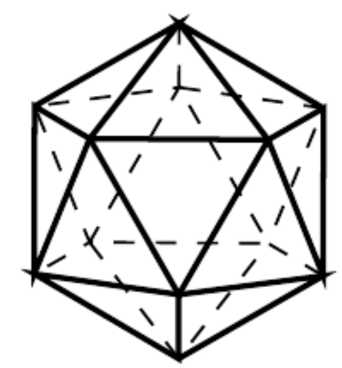
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$V - E + F$	2	2	2	

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	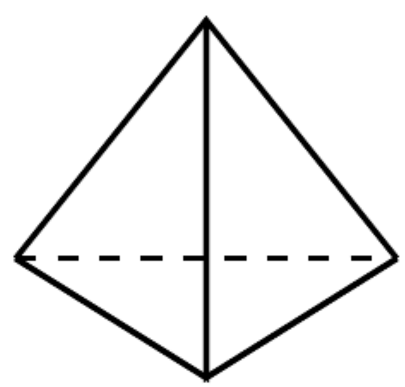	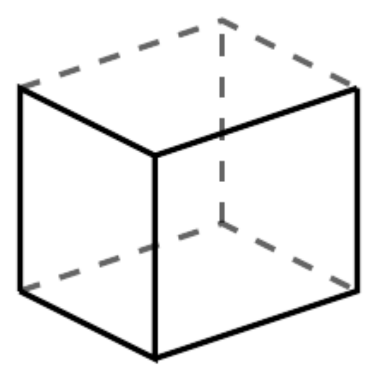	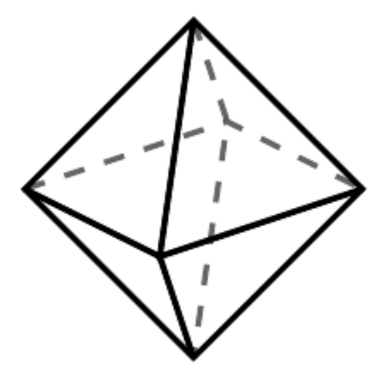	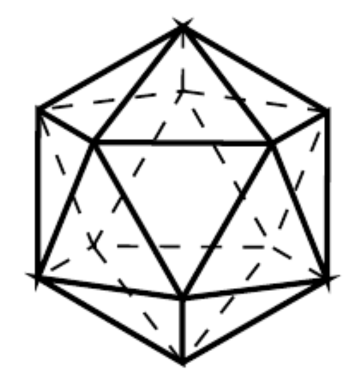
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Bernoulli Family

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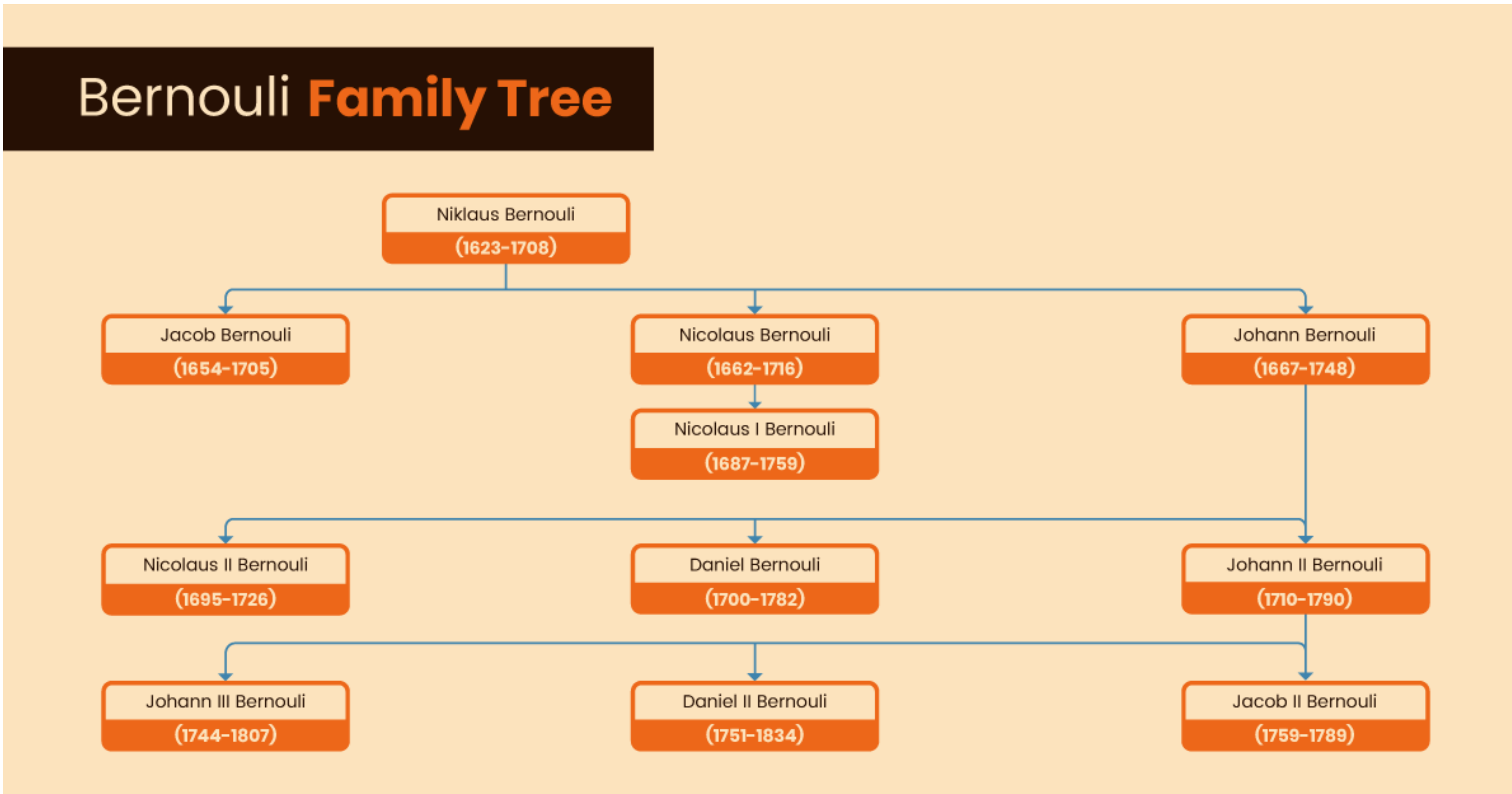

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- At age 16, he got a copy of *A Synopsis of Elementary Results in Pure and Applied Mathematics*, which is a book of ~5,000 math facts/theorems, with essentially no proofs or justification.

Srinivasa Ramanujan

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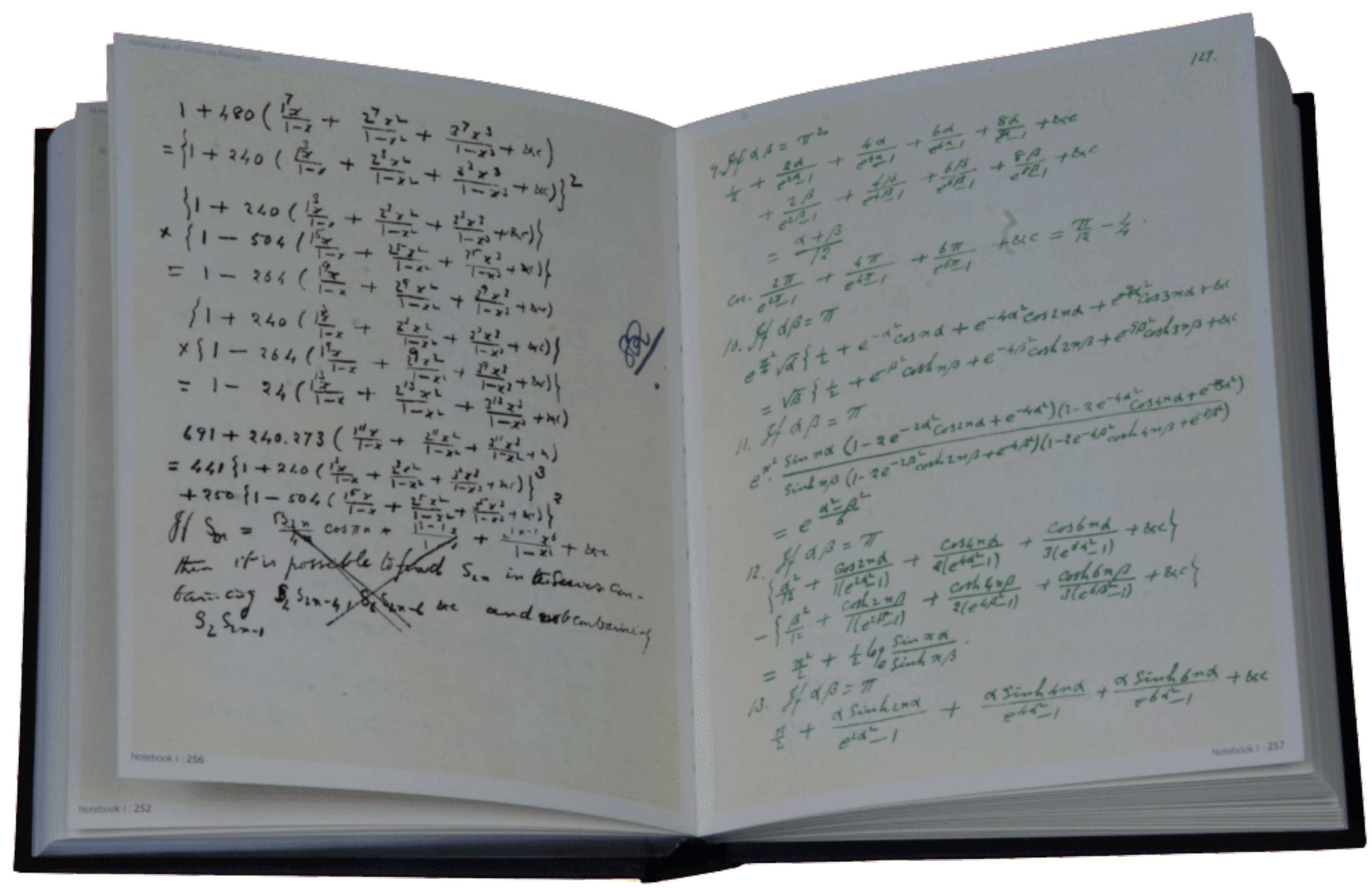
- He neglected his non-math classes and flunked out of college. Instead, he worked on his *notebooks*—in which he wrote his mathematical discoveries.



Srinivasa Ramanujan

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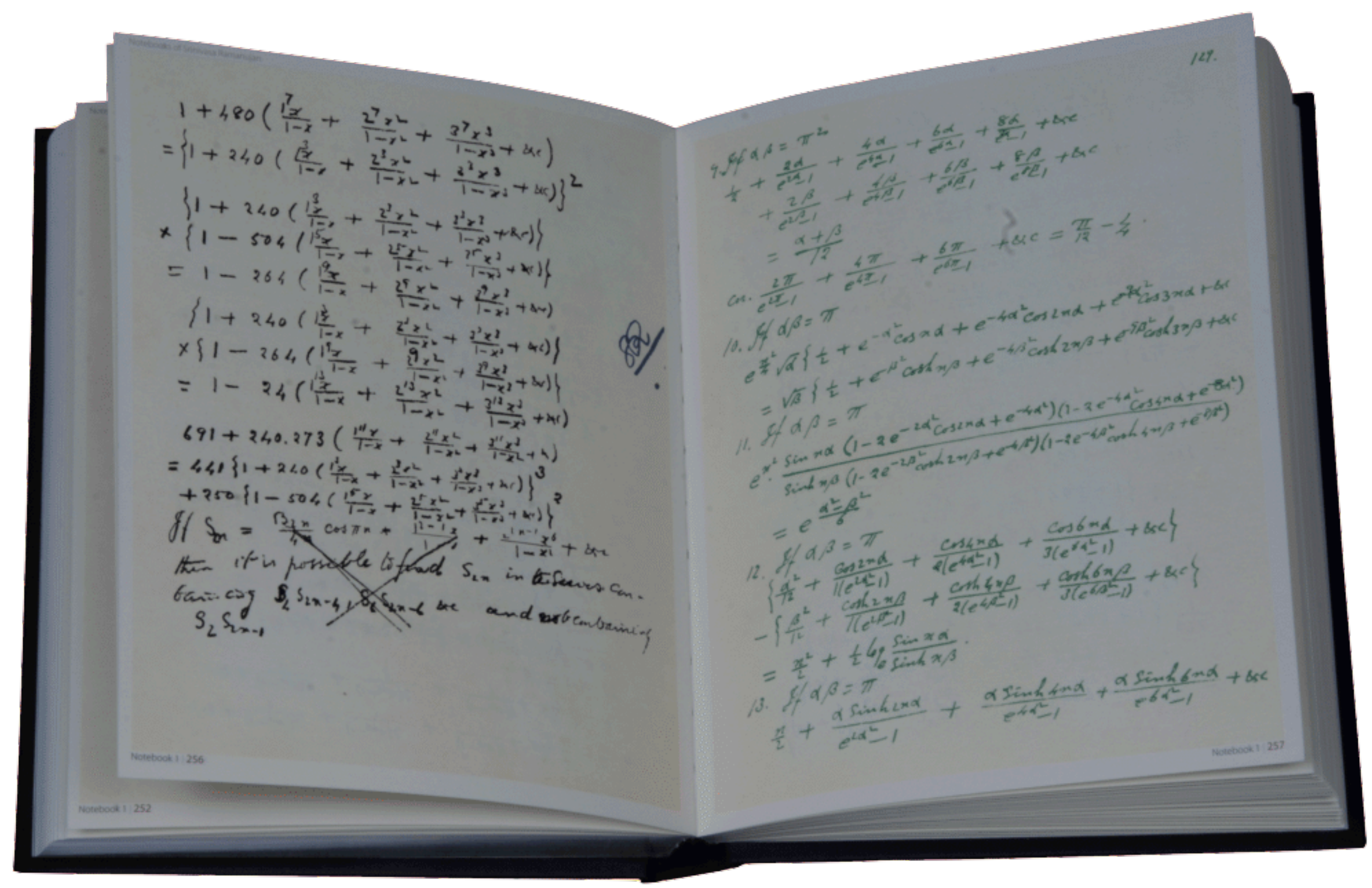
- For 5 years he had lots of time to work and no collaborators or math teachers. They allowed him to blaze his own trail in math.



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- Ramanujan traveled to Madras where a patron supported him while he did math, including by publishing his first paper.

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$$3 = \sqrt{9} = \sqrt{1+8} = \sqrt{1+2\cdot 4} = \sqrt{1+2\sqrt{16}}$$

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$$=\sqrt{1+2\sqrt{1+3\cdot 5}}=\sqrt{1+2\sqrt{1+3\sqrt{25}}}$$

Srinivasa Ramanujan

- He also published a problem for readers of the Journal of the Indian Mathematical Society. He asked for the value of $\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+\ldots}}}}$
- His answer: 3. Reasoning:

$$3=\sqrt{9}=\sqrt{1+8}=\sqrt{1+2\cdot 4}=\sqrt{1+2\sqrt{16}}=\sqrt{1+2\sqrt{1+15}}$$

$$=\sqrt{1+2\sqrt{1+3\cdot 5}}=\sqrt{1+2\sqrt{1+3\sqrt{25}}}=\sqrt{1+2\sqrt{1+3\sqrt{1+24}}}$$

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$$= \sqrt{1+2\sqrt{1+3\sqrt{1+4\cdot 6}}}$$

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- Because:

$$\sqrt{n^2}=\sqrt{1+(n^2-1)}=\sqrt{1+(n-1)(n+1)}=\sqrt{1+(n-1)\sqrt{(n+1)^2}}$$

Srinivasa Ramanujan

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Srinivasa Ramanujan

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Madras, 16th January 1913,
Dear Sir,

I beg to introduce myself to you as a clerk in the Accounts Department of the Port Trust Office at Madras on a salary of only £20 per annum. I am now about 23 years of age. I have no University education but I have undergone the ordinary school course. After leaving school I have been employing the spare time at my disposal to work at Mathematics. I have not trodden through the conventional regular course followed in a University course, but I am striking out a new path for myself. I have made special investigation of divergent series in general and the results I get are termed by the local mathematicians as “startling."

Srinivasa Ramanujan

Srinivasa Ramanujan

- The letter also contained 9 pages of theorems, but no proofs. For example, Ramanujan claimed that if

$$u = \cfrac{x}{1+\cfrac{x^5}{1+\cfrac{x^{10}}{1+\cfrac{x^{15}}{1+\cdots}}}} \quad \text{and} \quad v = \cfrac{x^{1/5}}{1+\cfrac{x}{1+\cfrac{x^2}{1+\cfrac{x^3}{1+\cdots}}}},$$

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then $$v^5 = u\frac{1-2u+4u^2-3u^3+u^4}{1+3u+4u^2+2u^3+u^4}.$$

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then $v^5 = u\dfrac{1-2u+4u^2-3u^3+u^4}{1+3u+4u^2+2u^3+u^4}$.

- Hardy: “They must be true because, if they were not true, no one would have had the imagination to invent them.”

Srinivasa Ramanujan

Srinivasa Ramanujan

- Hardy wanted to bring Ramanujan to Cambridge. Initially Ramanujan resisted for religious reasons, but in 1914 he went.

Srinivasa Ramanujan

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- They had a very productive few years, makings advances on highly composite numbers, the partition function, and much more.

Srinivasa Ramanujan

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- When WWI started, Ramanujan's health deteriorated. Was treated for tuberculosis. Once tried to commit suicide.

Srinivasa Ramanujan

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- He published 9 papers in 1915, earning the reputation he desired.

Srinivasa Ramanujan

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- He published 9 papers in 1915, earning the reputation he desired.

- He was elected a Fellow of the Royal Society, a tremendous honor.

Srinivasa Ramanujan

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Srinivasa Ramanujan

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- The following year he died, at the age of 32.
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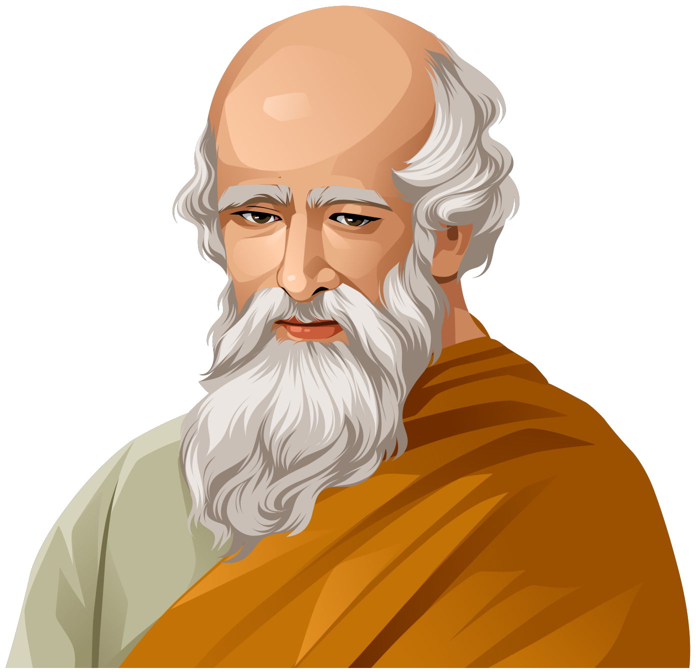
$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 396^{4k}}$$

Poem: Joined By Math

Who Is The GMOAT?

The Contenders

The Contenders



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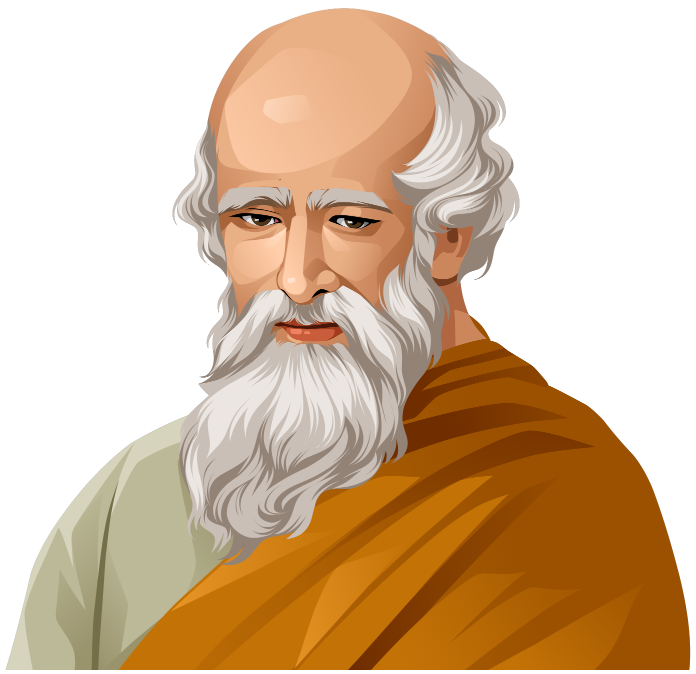
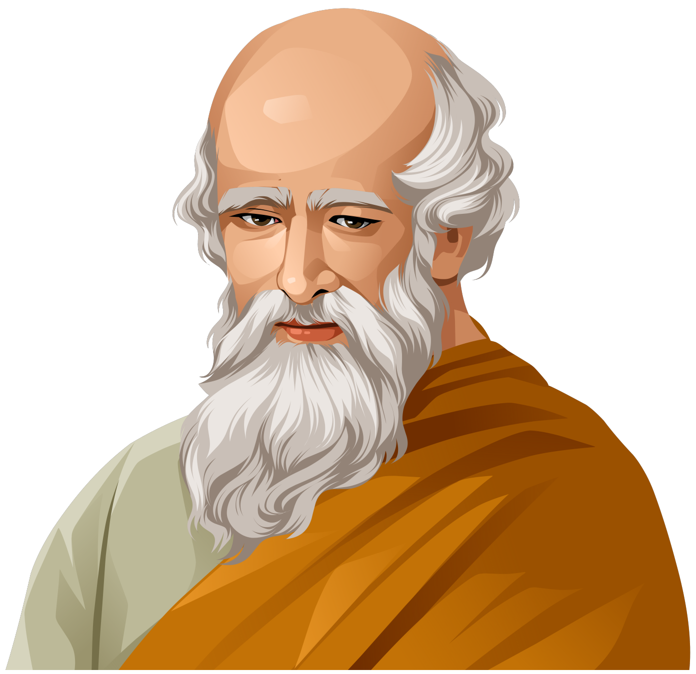


The Contenders



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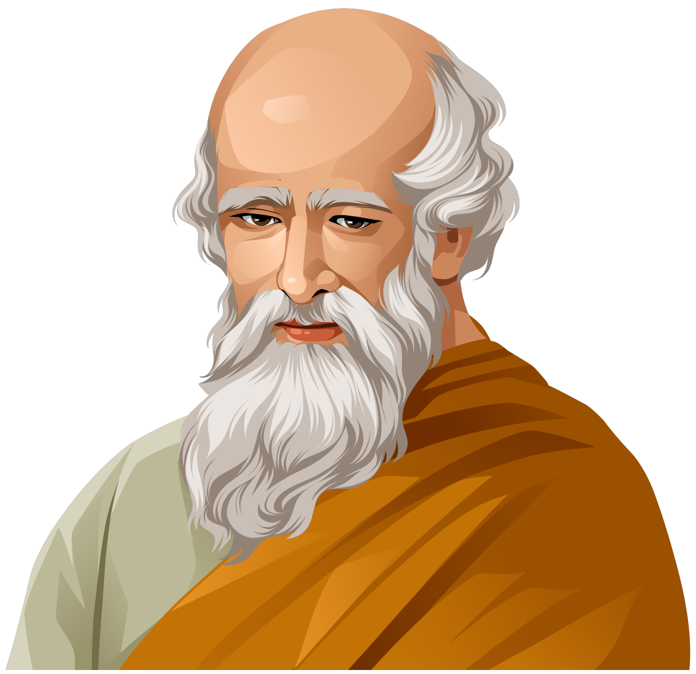

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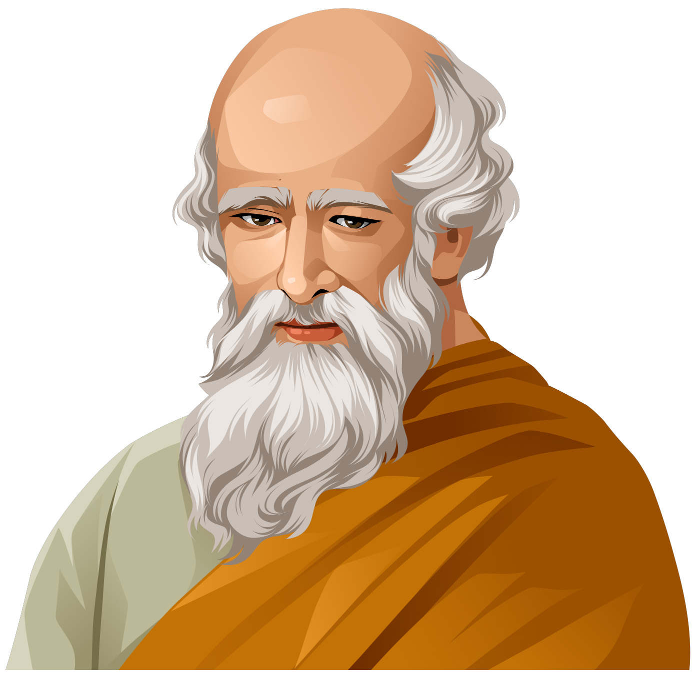

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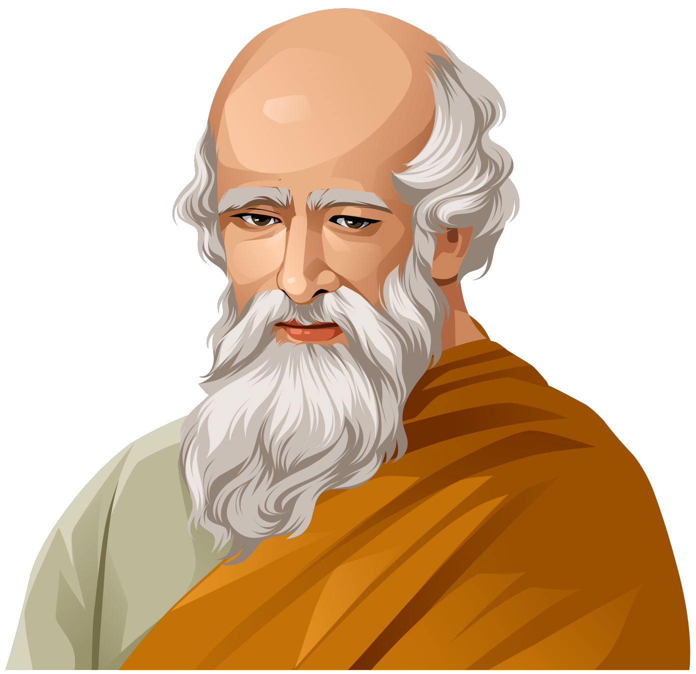
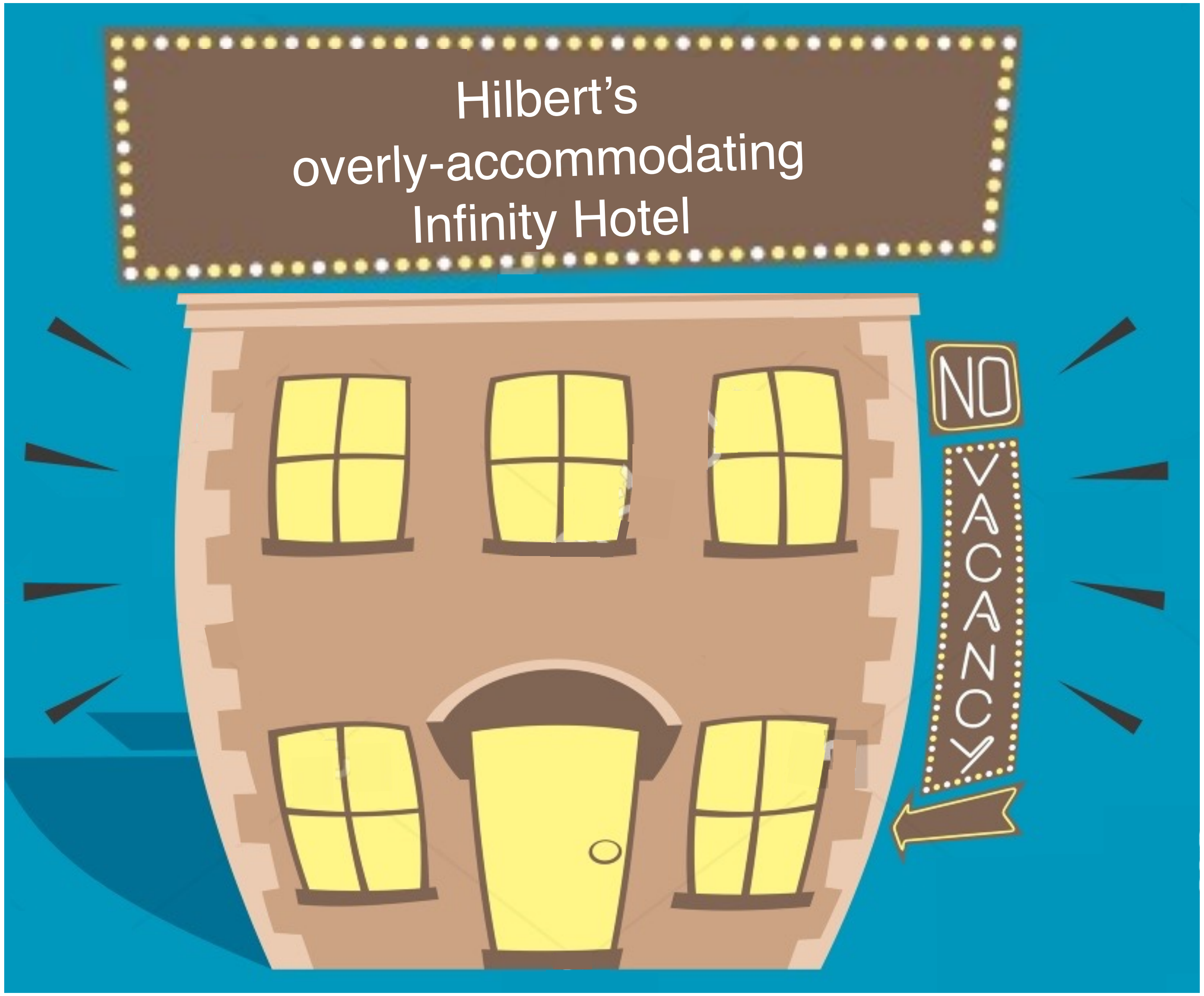


Infinities

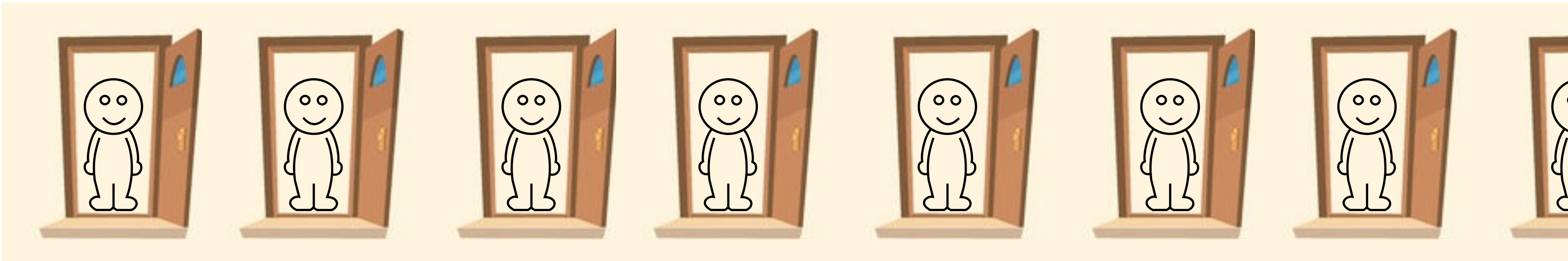

Hilbert's
overly-accommodating
Infinity Hotel
NO
VACANCY

"No other question has ever moved so profoundly the spirit of mankind; no other idea has so fruitfully stimulated our intellect; yet no other concept stands in greater need of clarification than that of the infinite."

—David Hilbert

1 2 3 4 5 6 7 …

1 2 3 4 5 6 7 …



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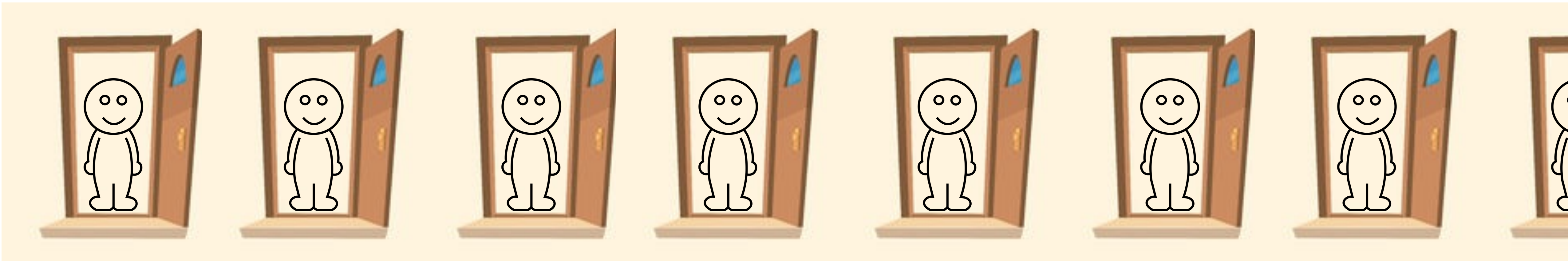

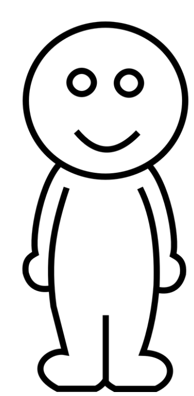

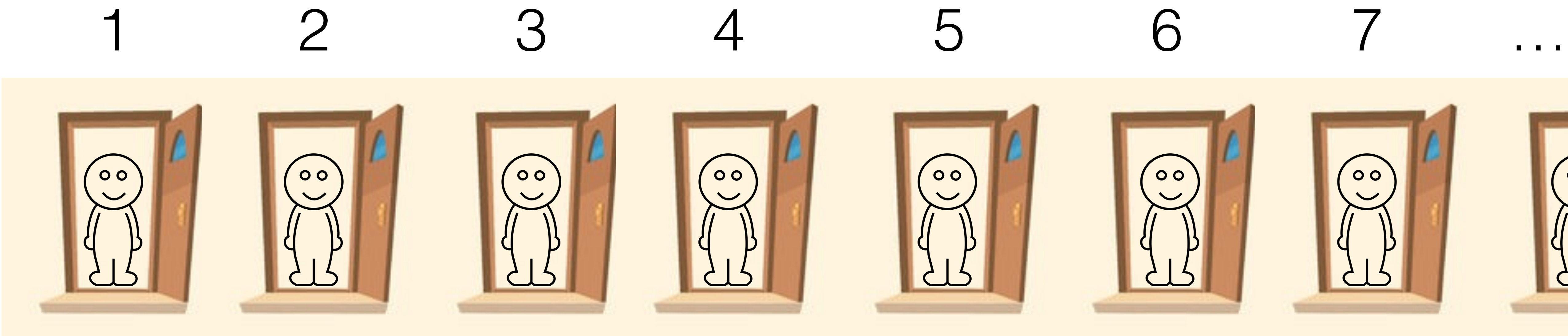
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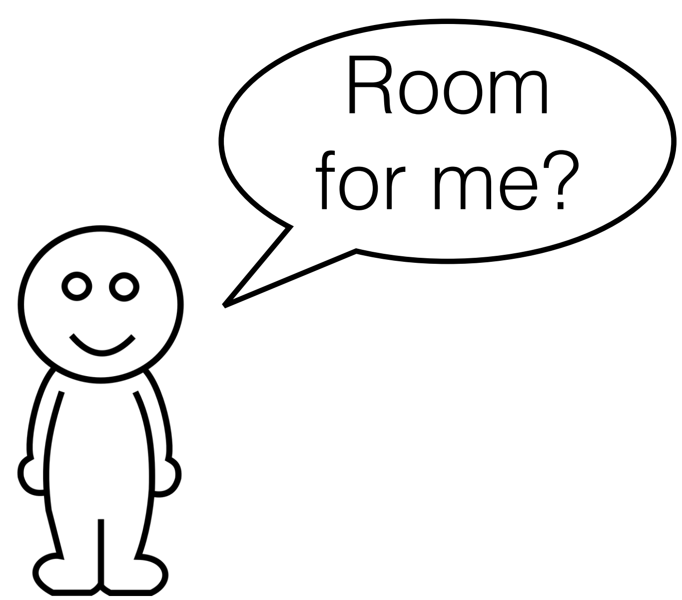
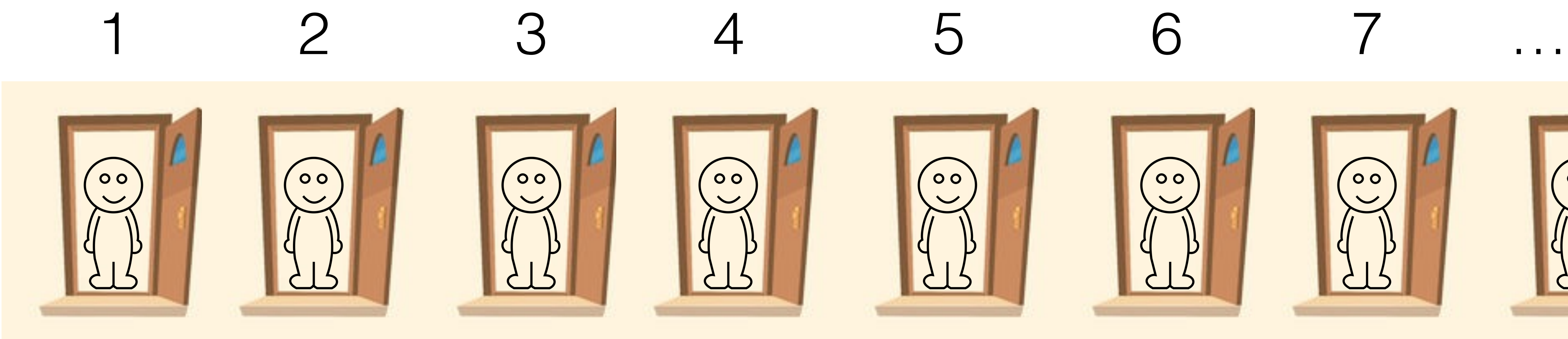
Hilbert's
overly-accommodating
Infinity Hotel
NO
VACANCY


Room
for me?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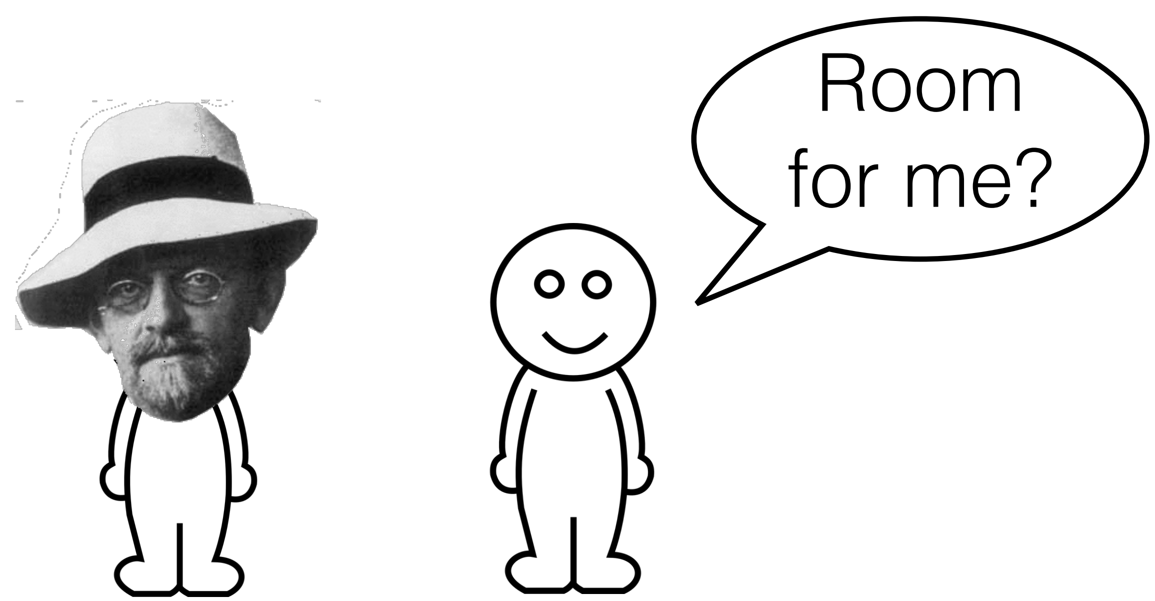
Hilbert's
overly-accommodating
Infinity Hotel
NO
VACANCY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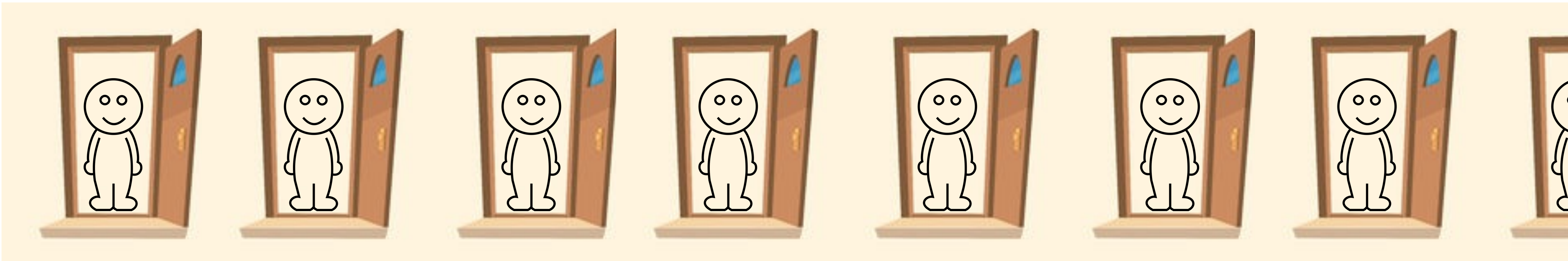

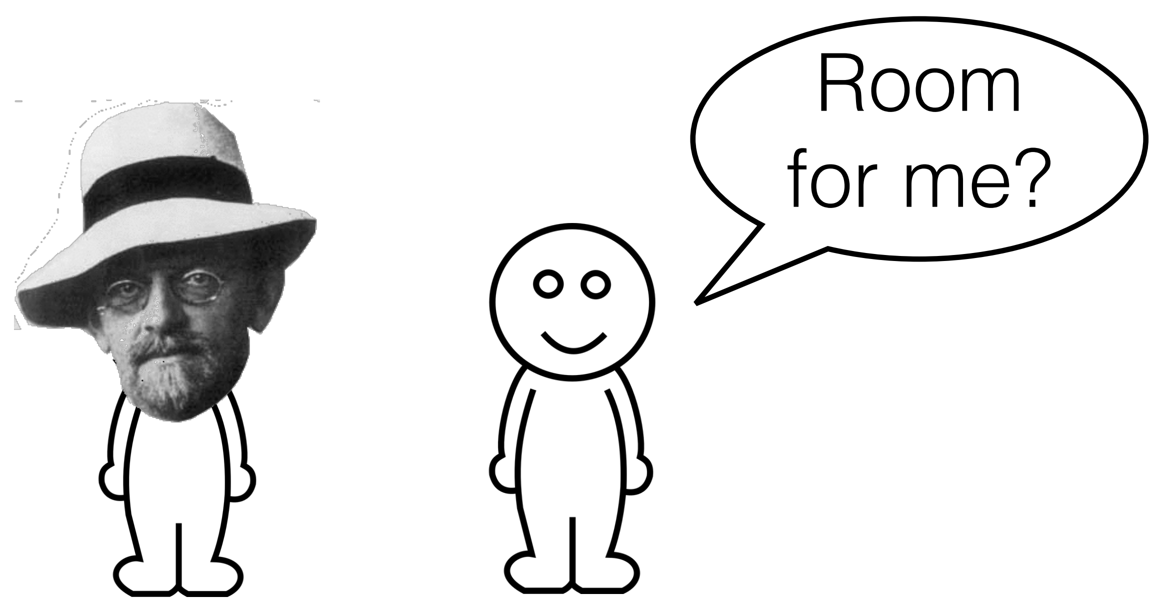

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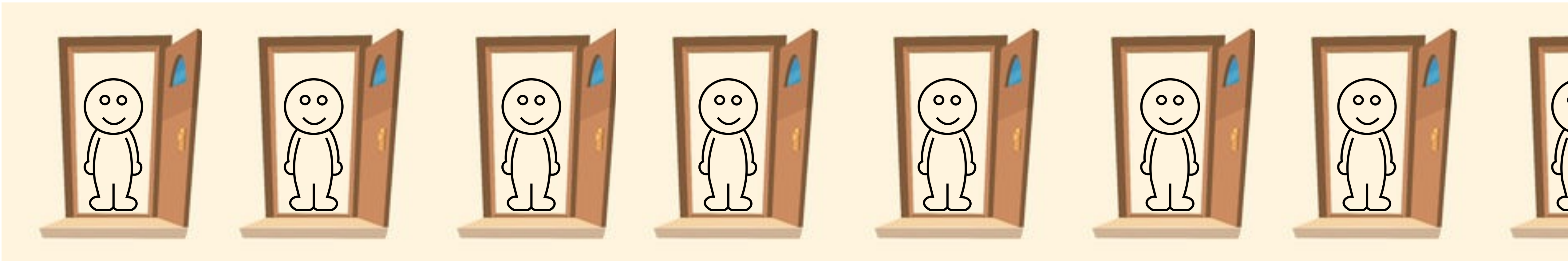
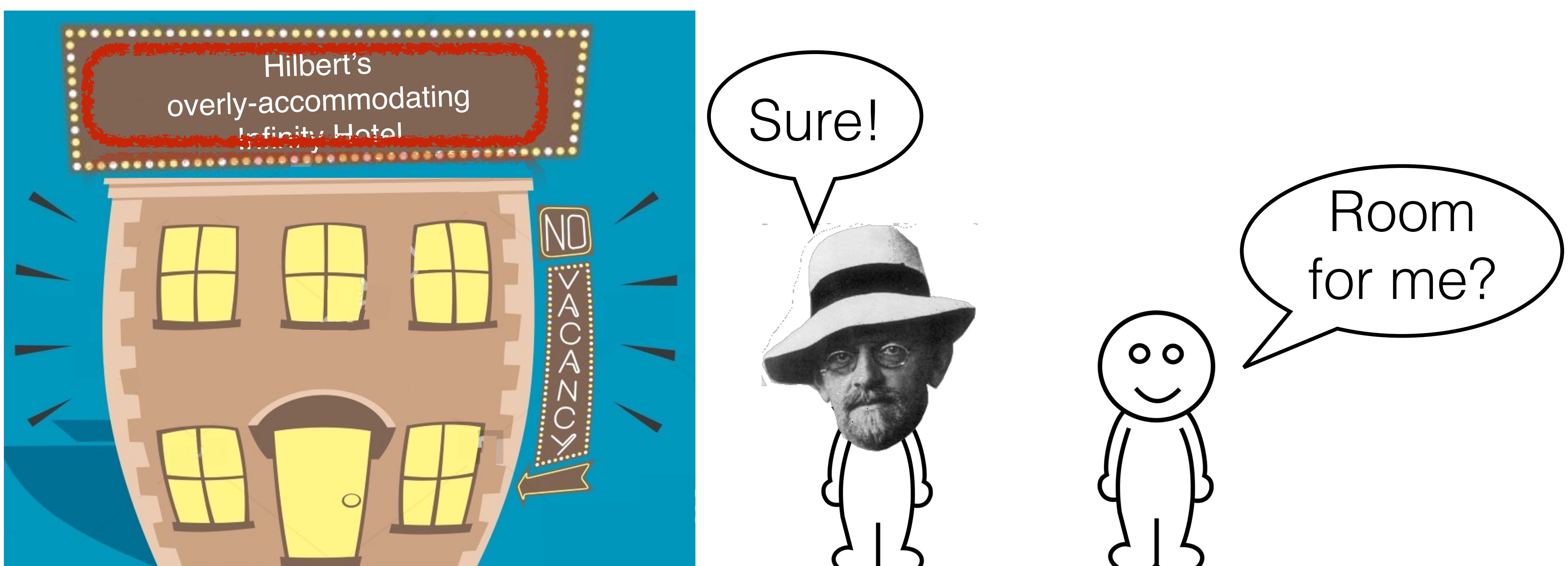
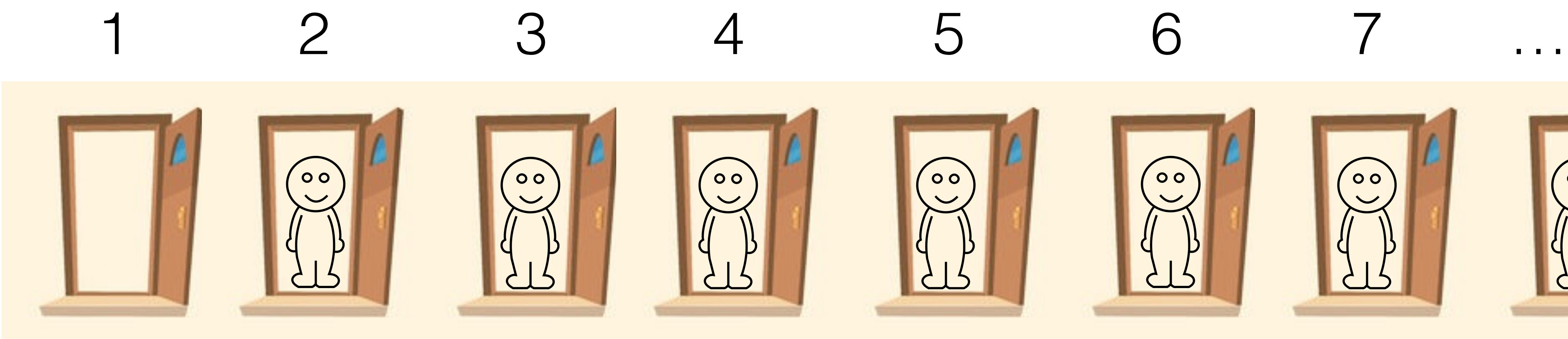




Hilbert's
overly-accommodating
Infinity Hotel
NO
VACANCY
Sure!
Room
for me?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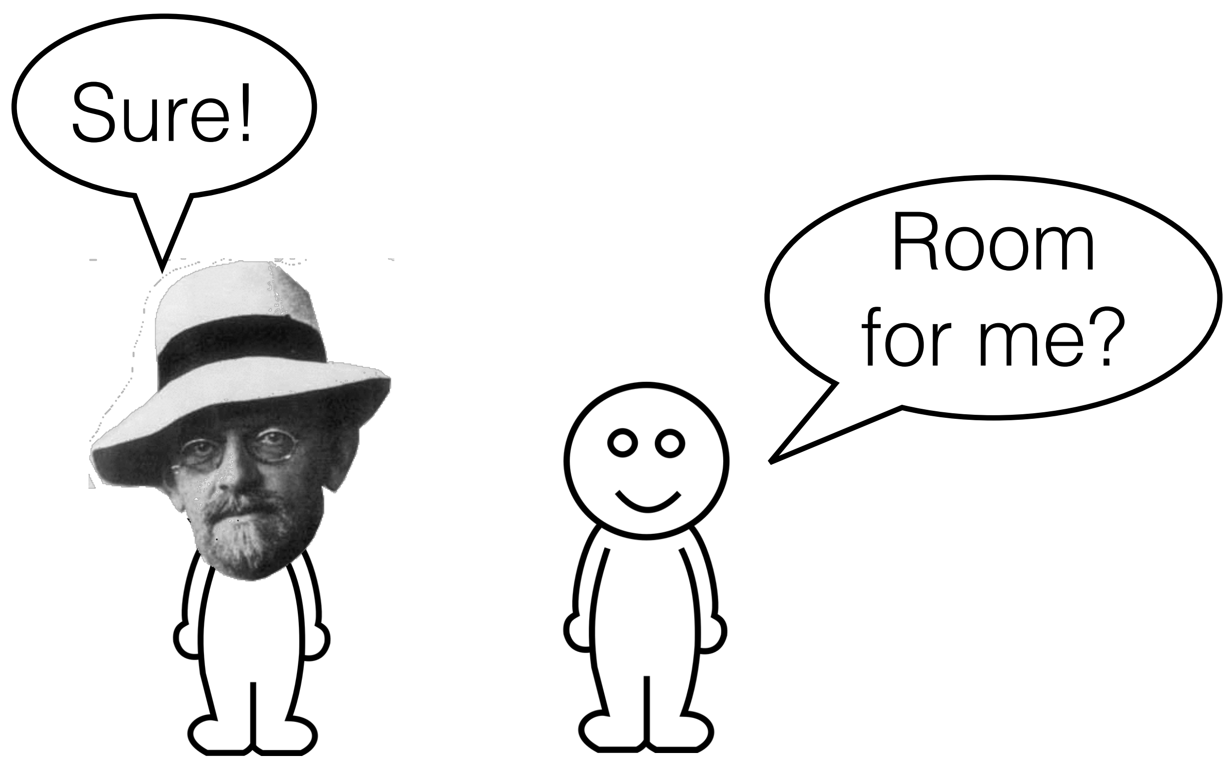
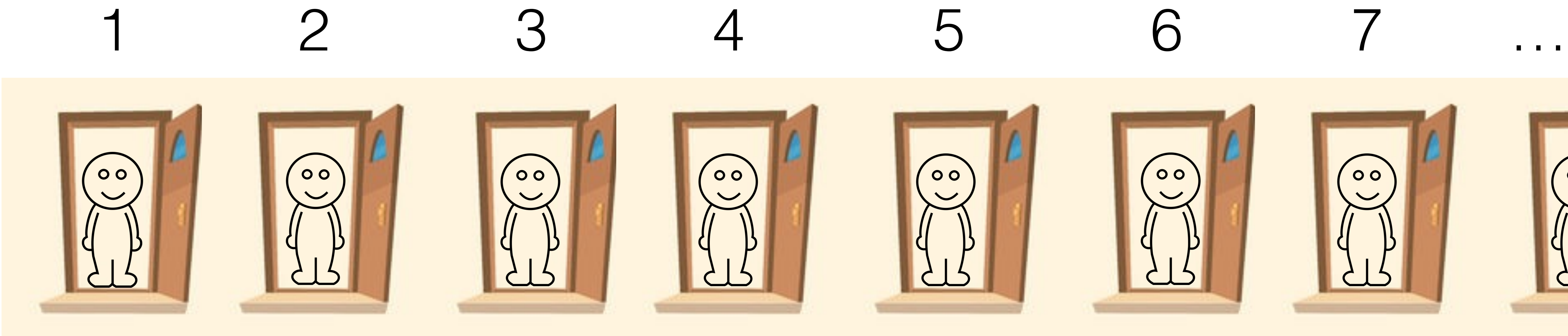
Hilbert's
overly-accommodating
NO
VACANCY


Sure!
Room for me?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Hilbert's
overly-accommodating
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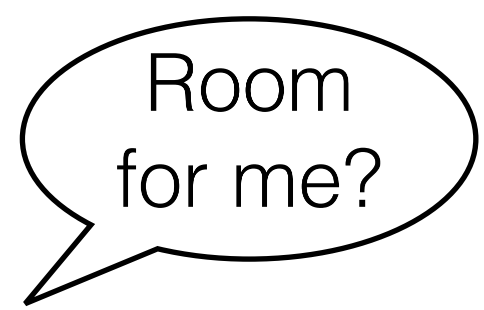
Sure!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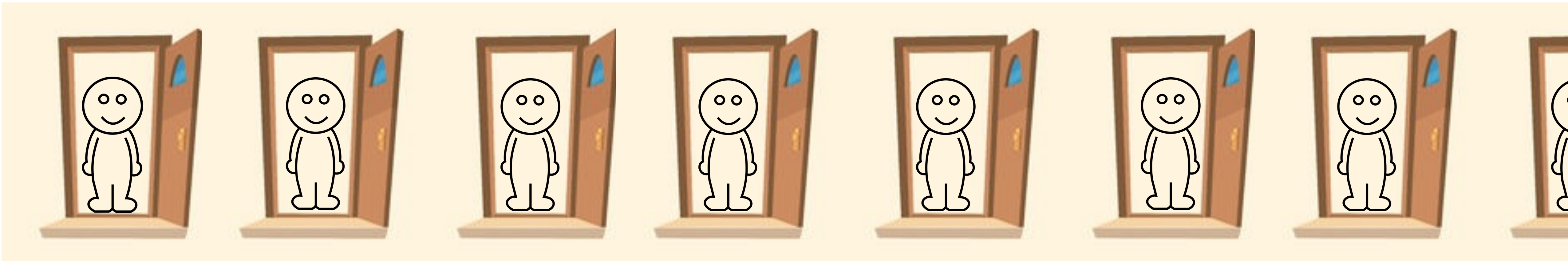

Person in room $n \rightarrow$ room $n + 1$

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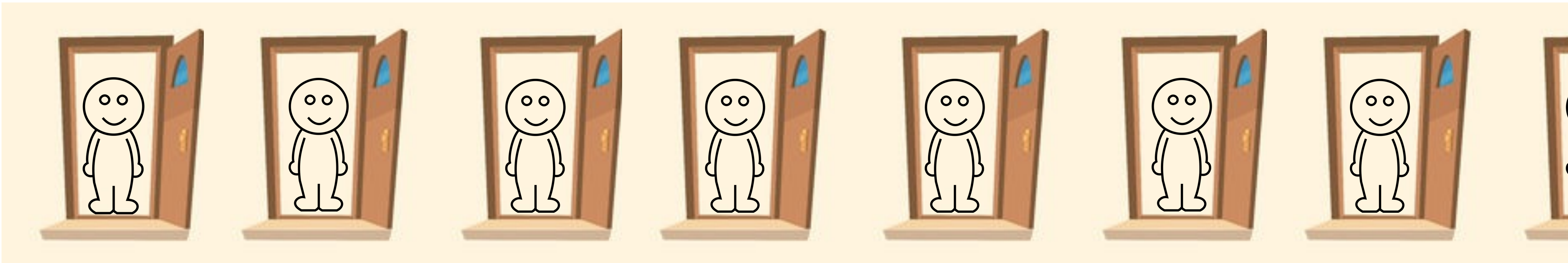
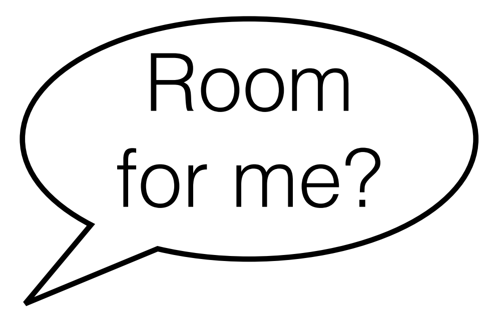
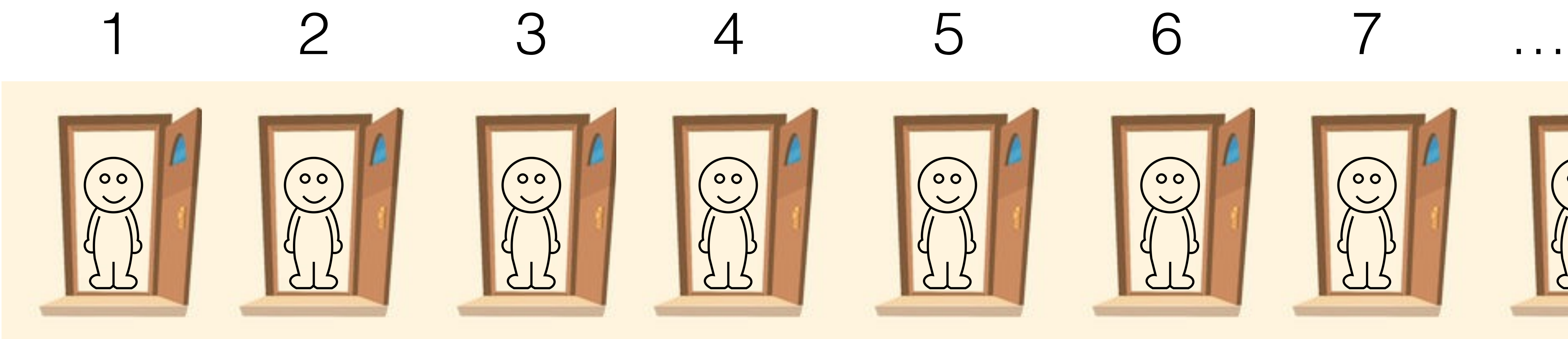


Person in room $n \to$ room $n+1$

$$n \to n+1$$

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Hilbert's
overly-accommodating
Infinity Hotel
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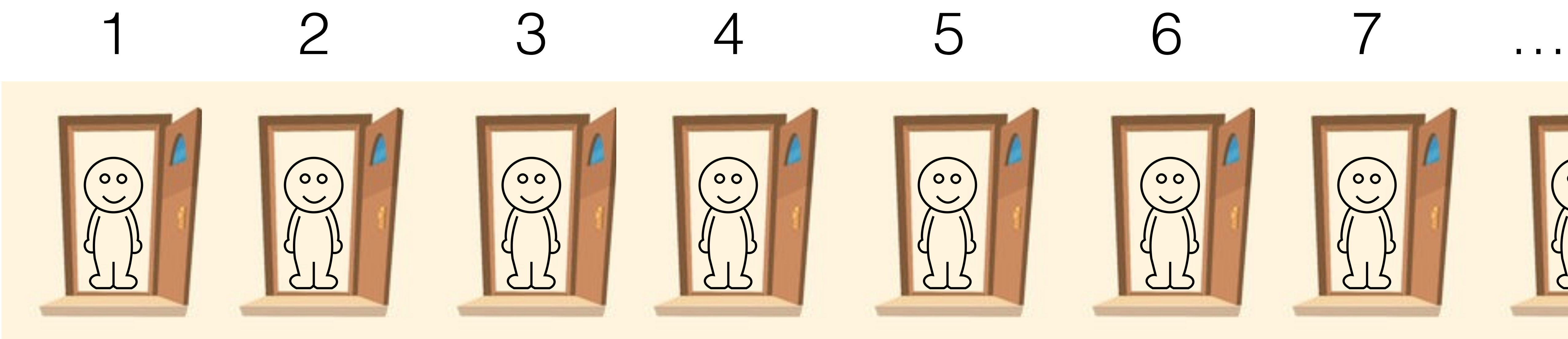
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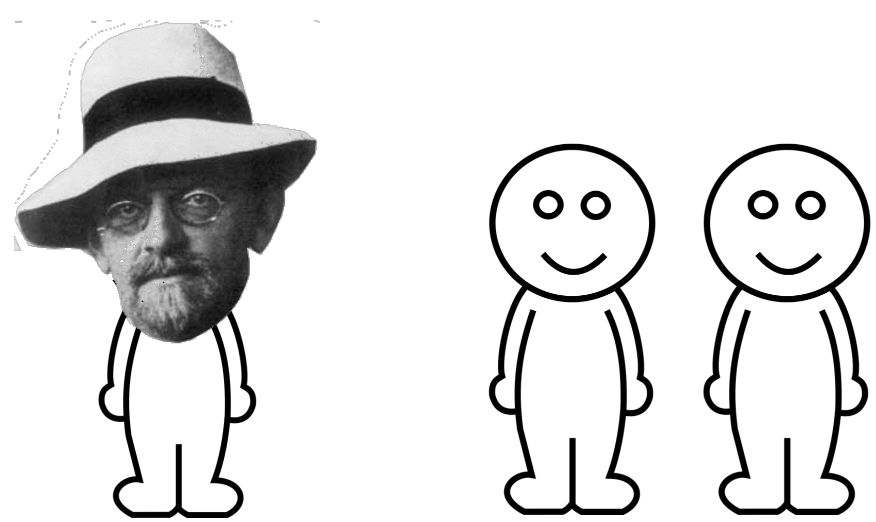

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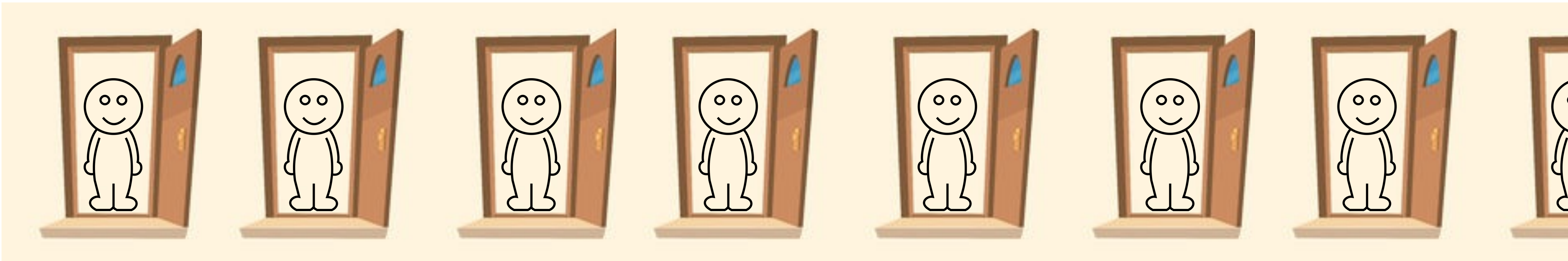


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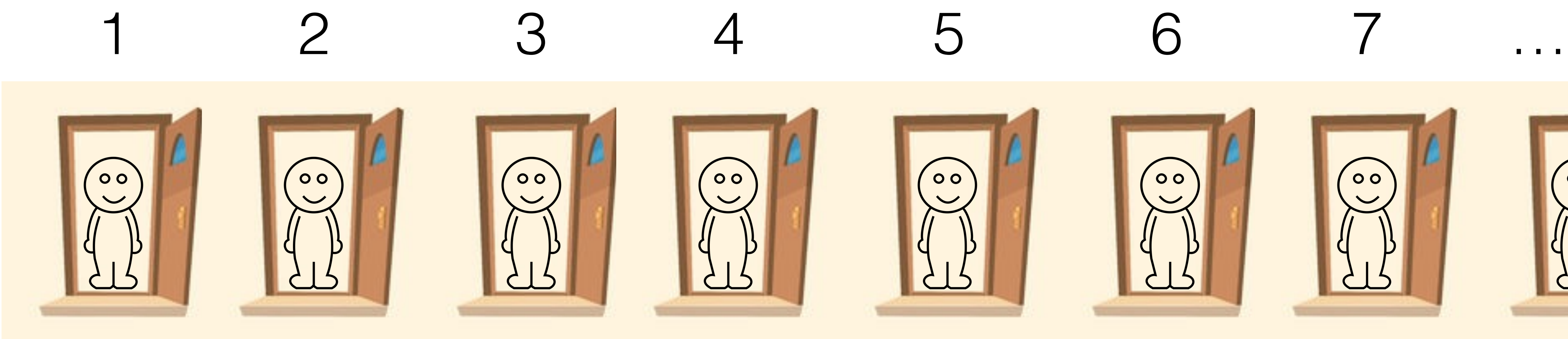
Infinity Hotel
NO
VACANCY


Room
for us?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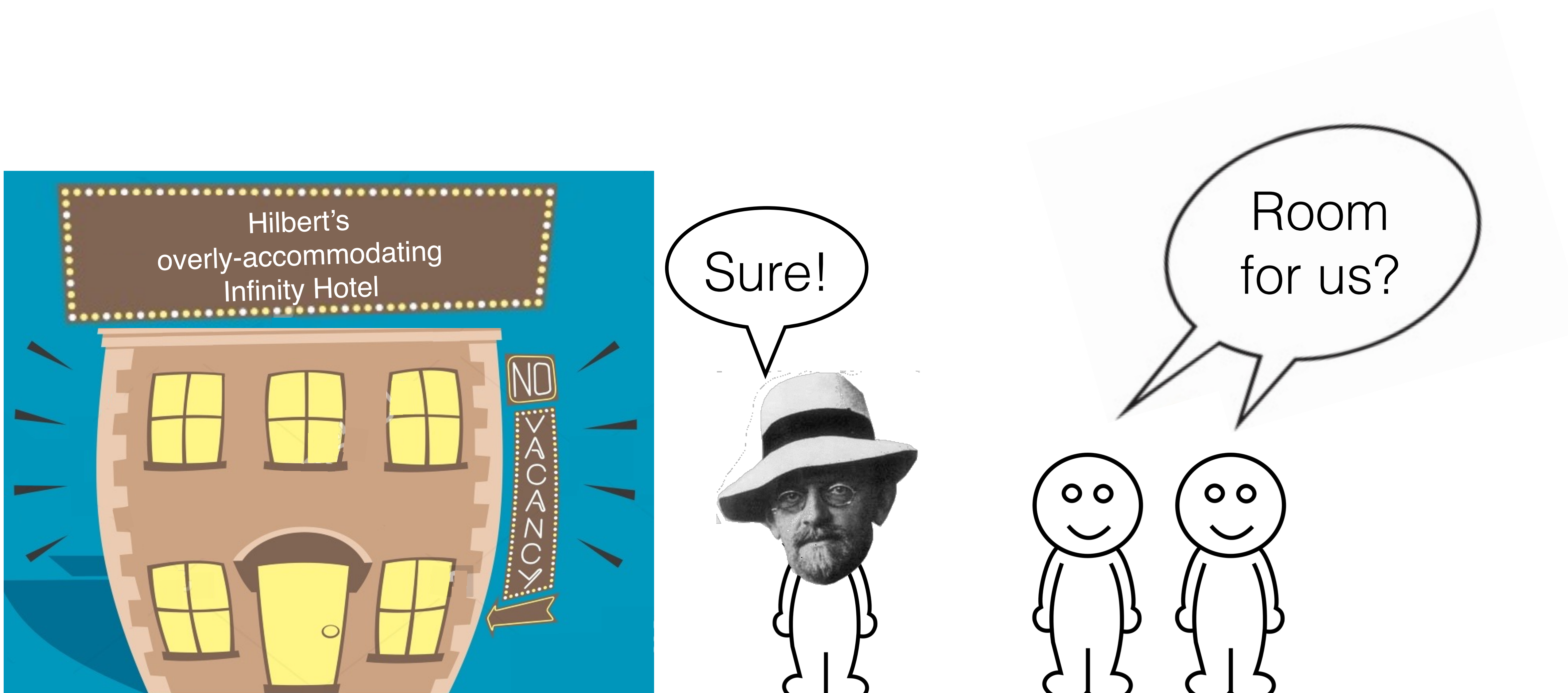
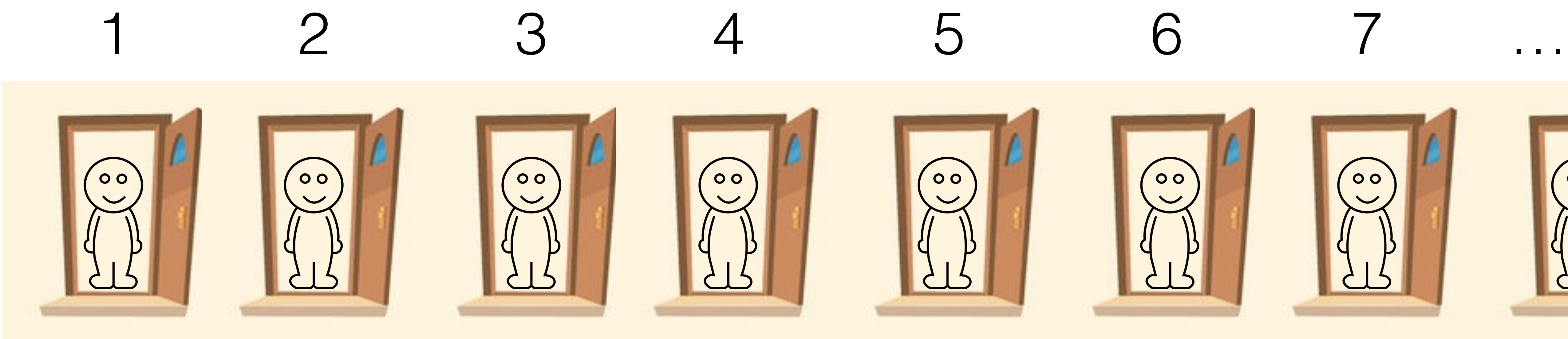

Hilbert's overly-accommodating Infinity Hotel
NO
VACANCY
Sure!
Room for us?



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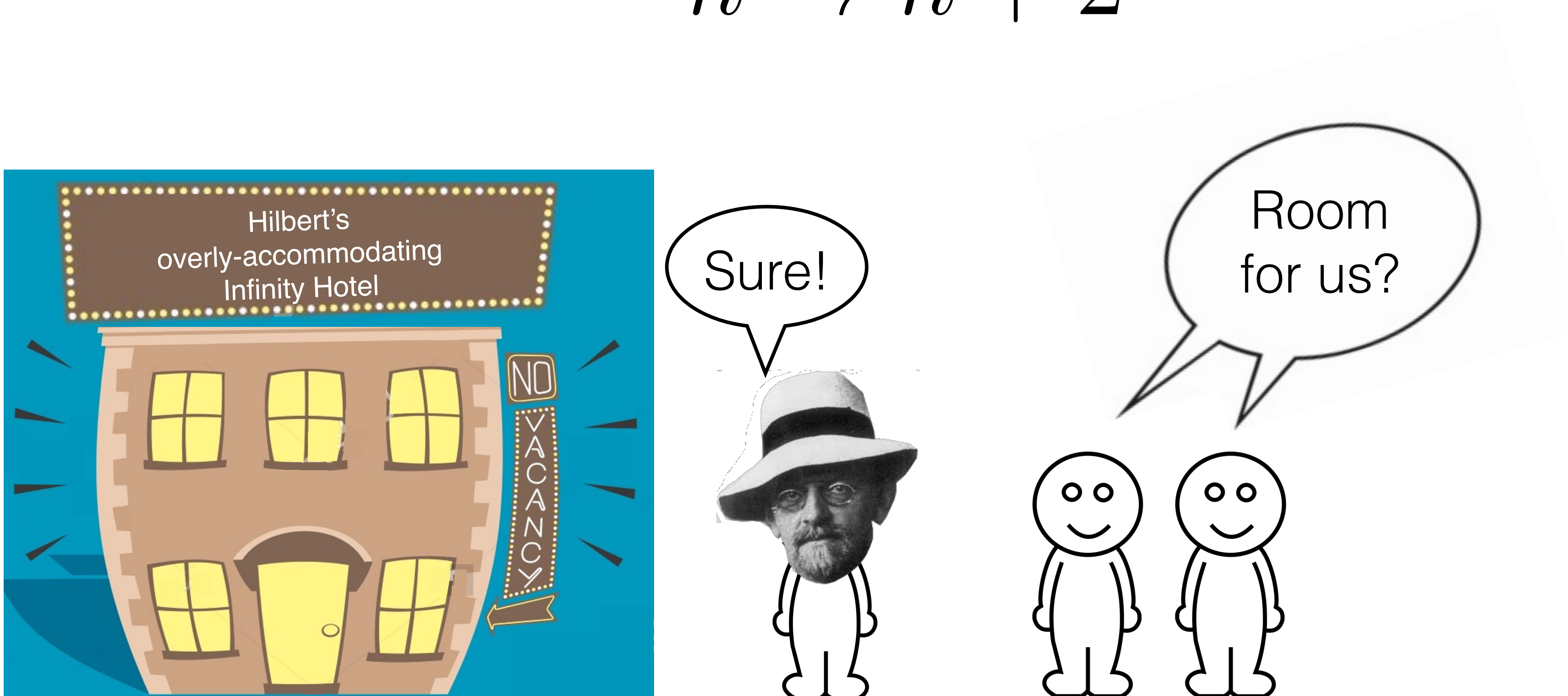
$$n \to n + 2$$



Hilbert's
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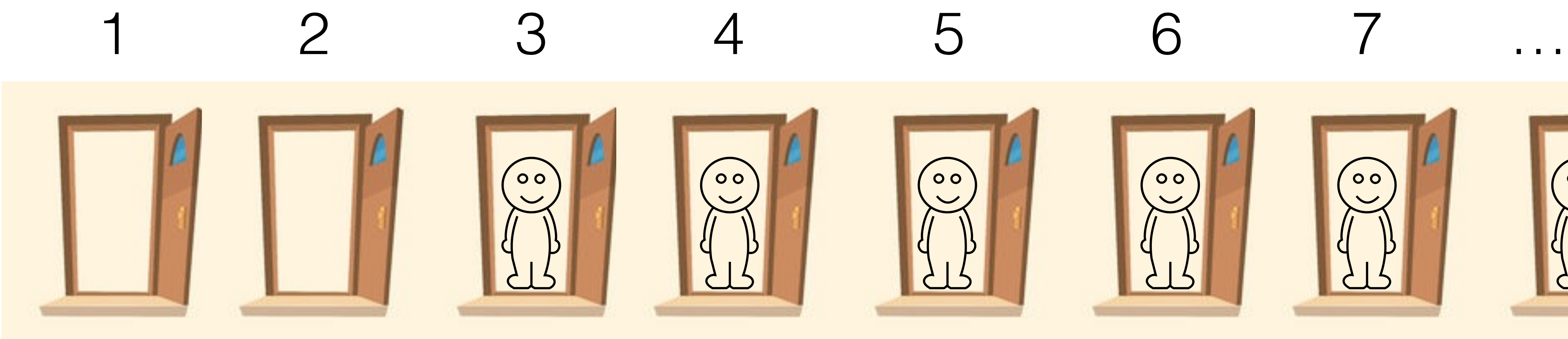
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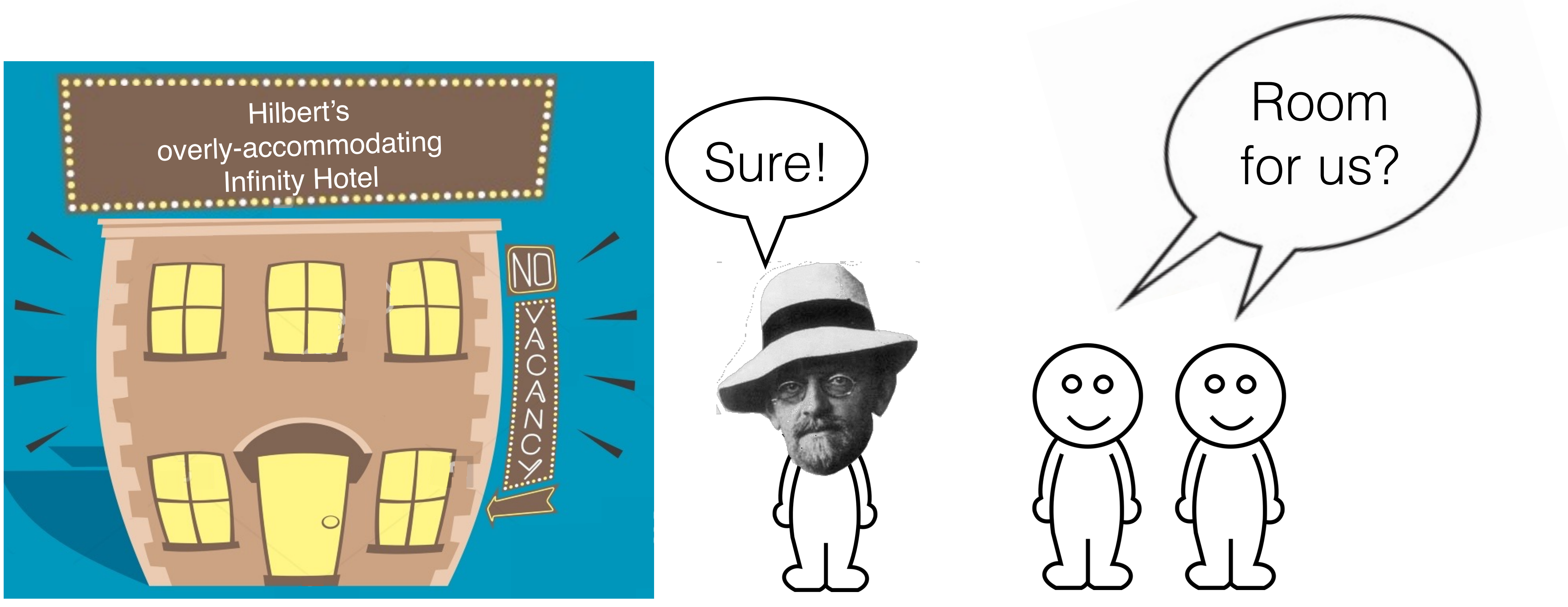
$$n \to n + 2$$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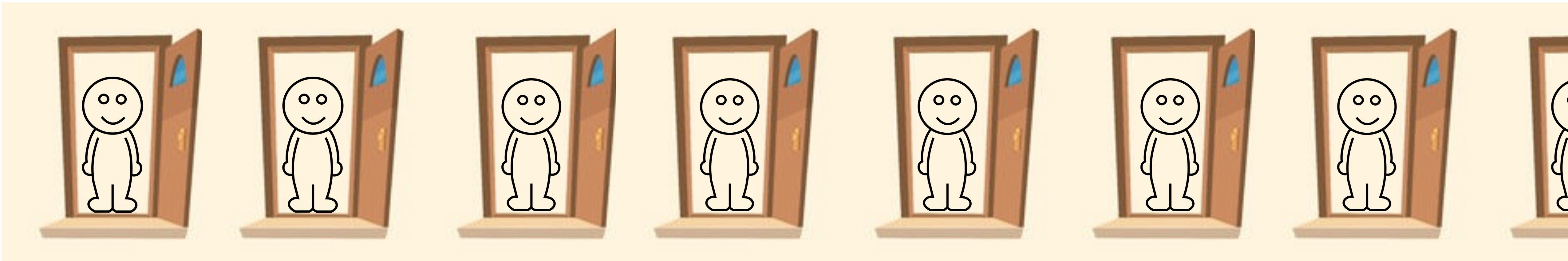

$$n \to n + 2$$

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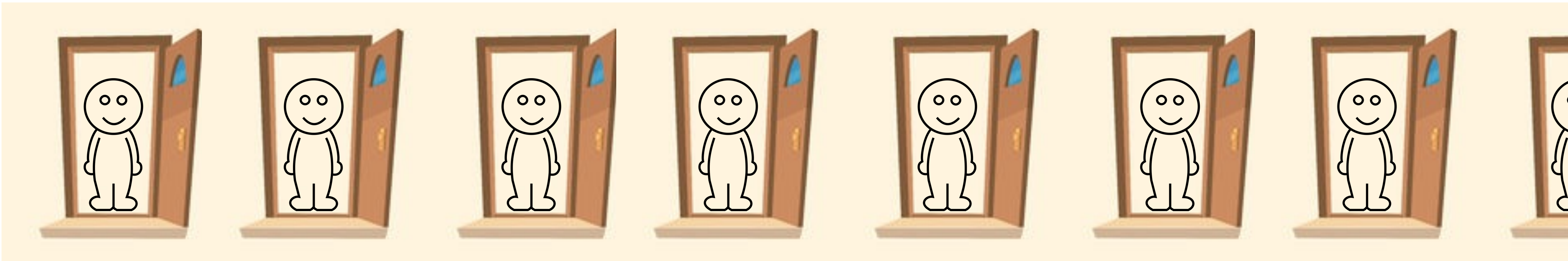

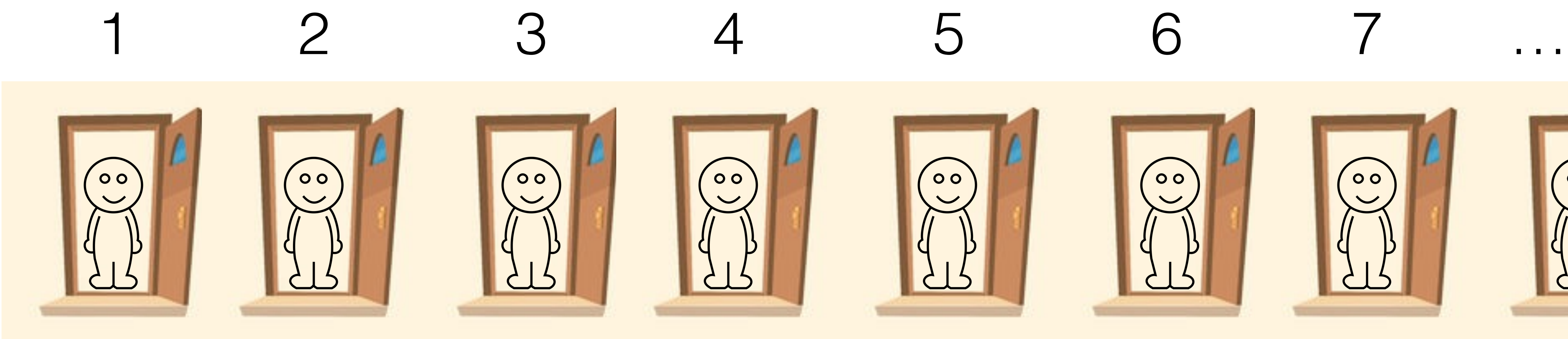
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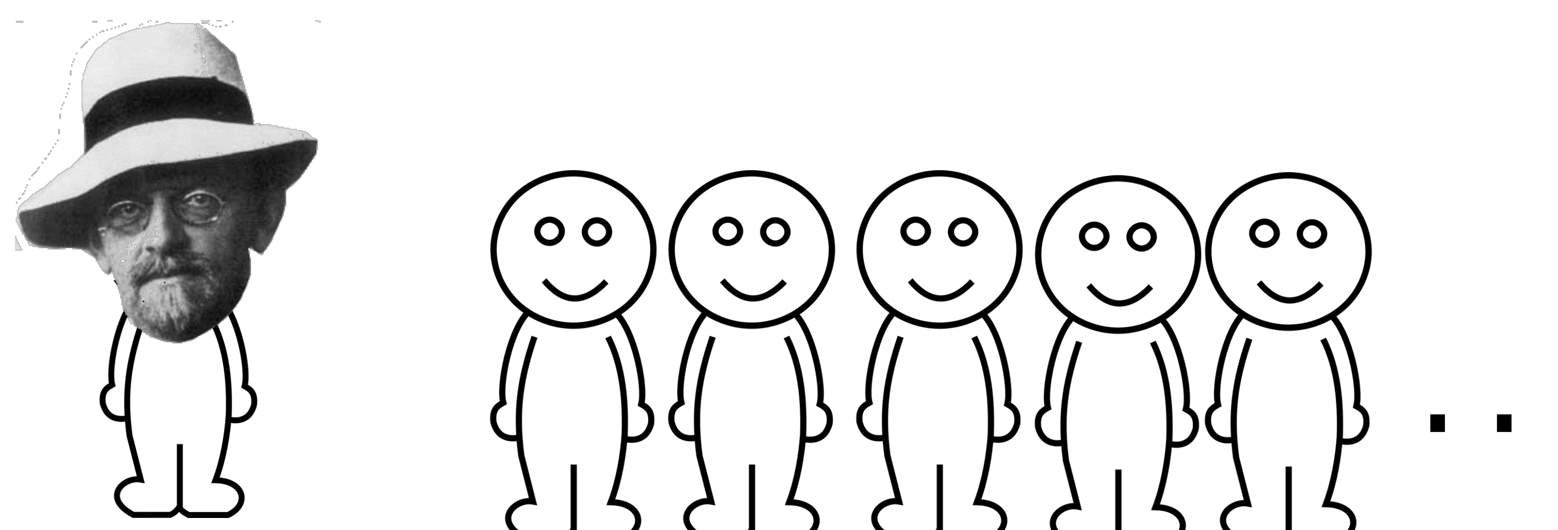

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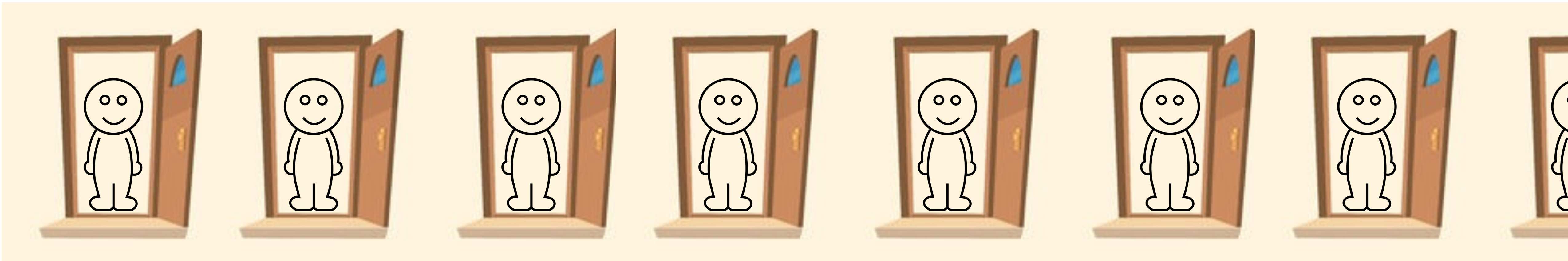


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Infinity Hotel
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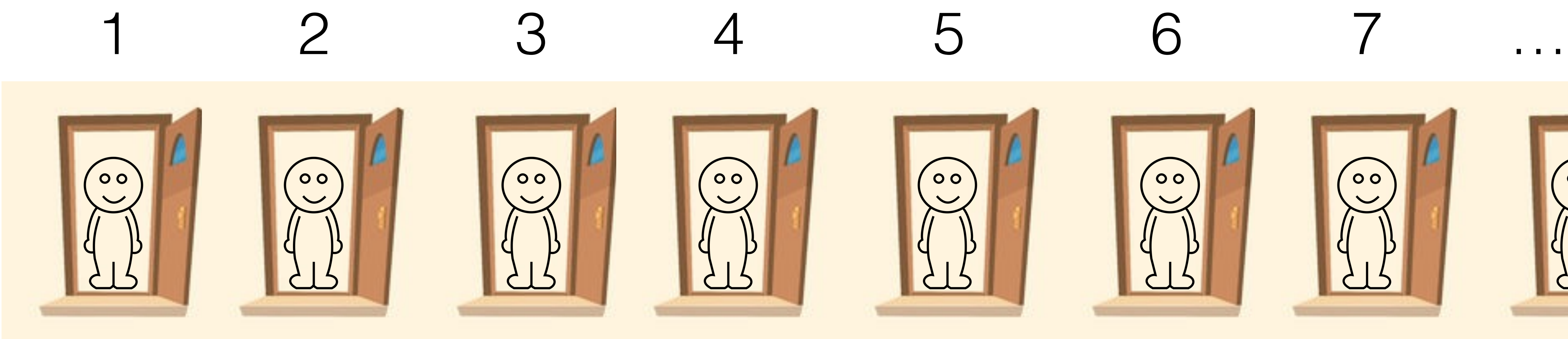
Room for
all of us?

. . .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Hilbert's
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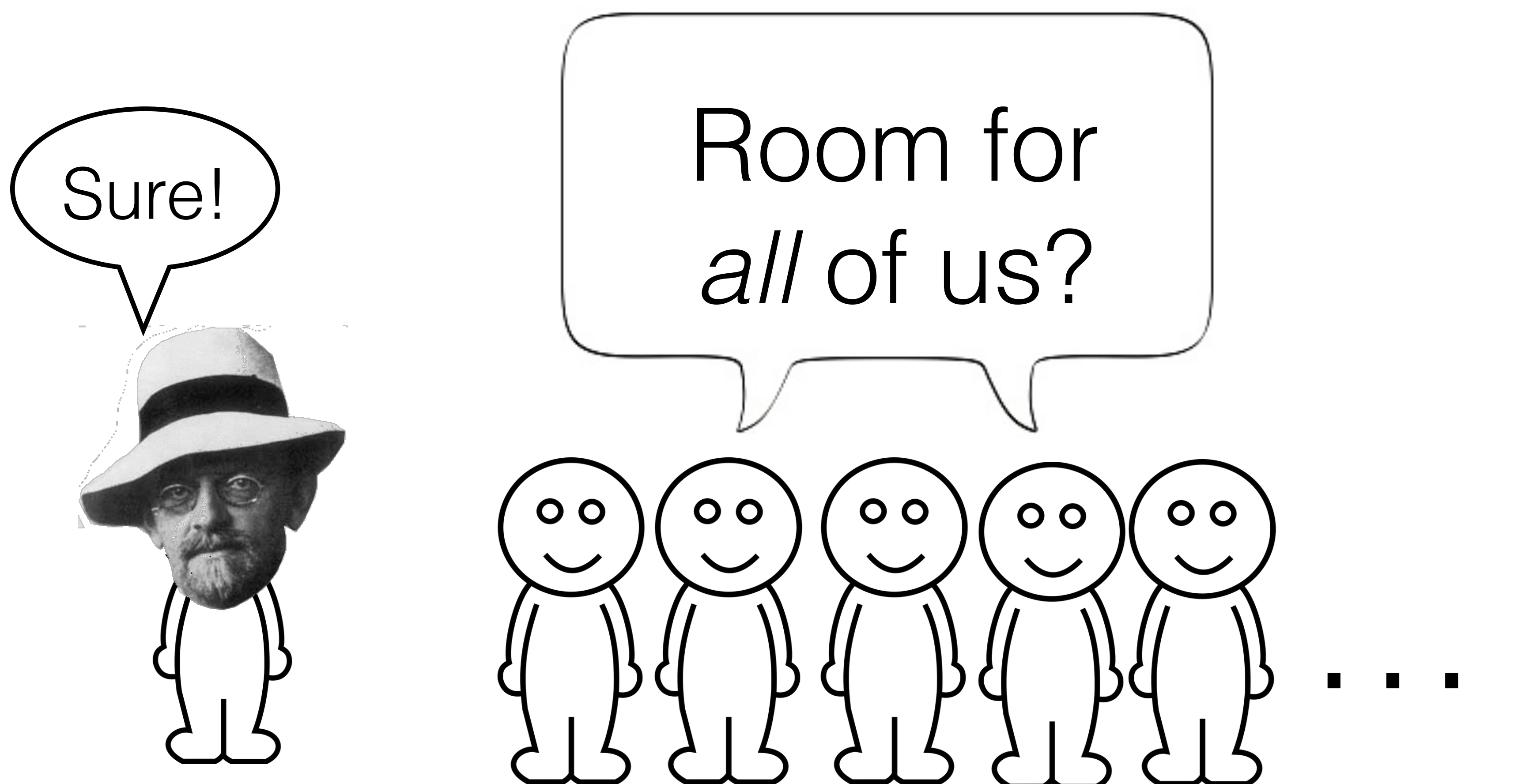
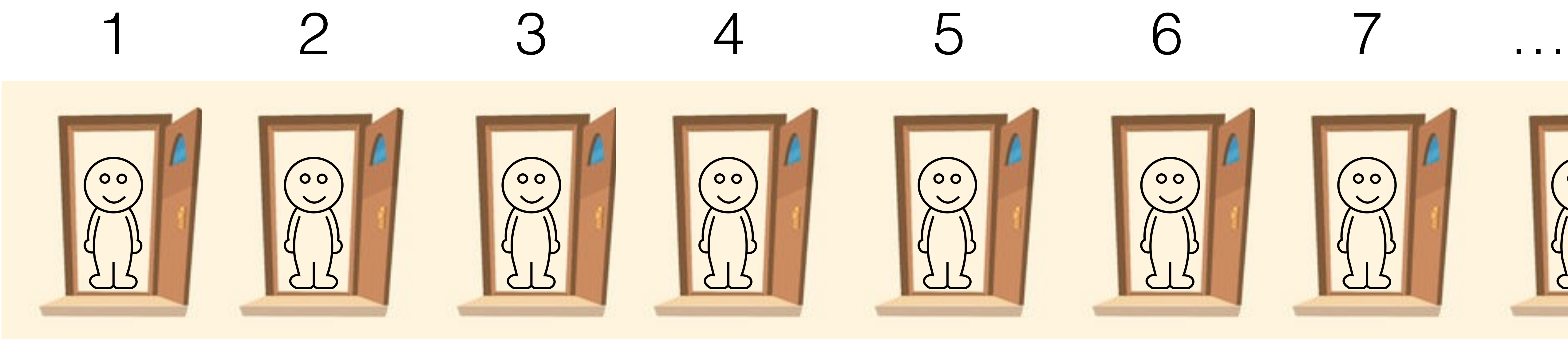

Sure!
Room for *all* of us?
...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$$n \to 2n$$


Hilbert's
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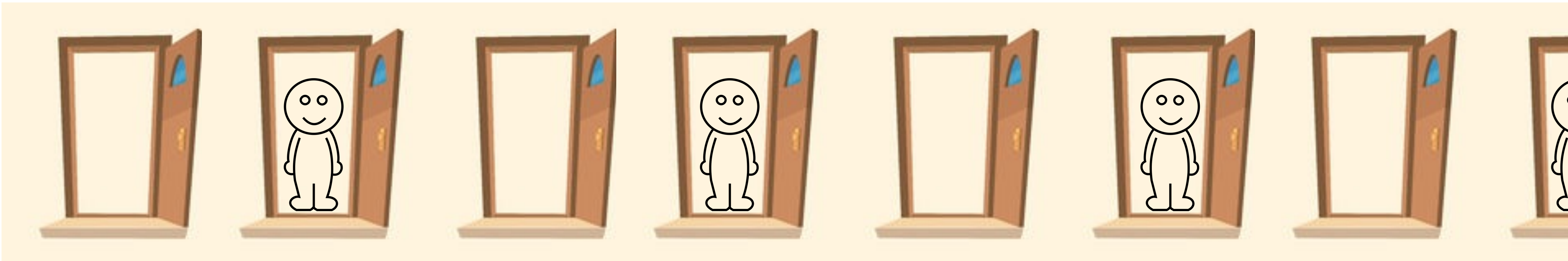
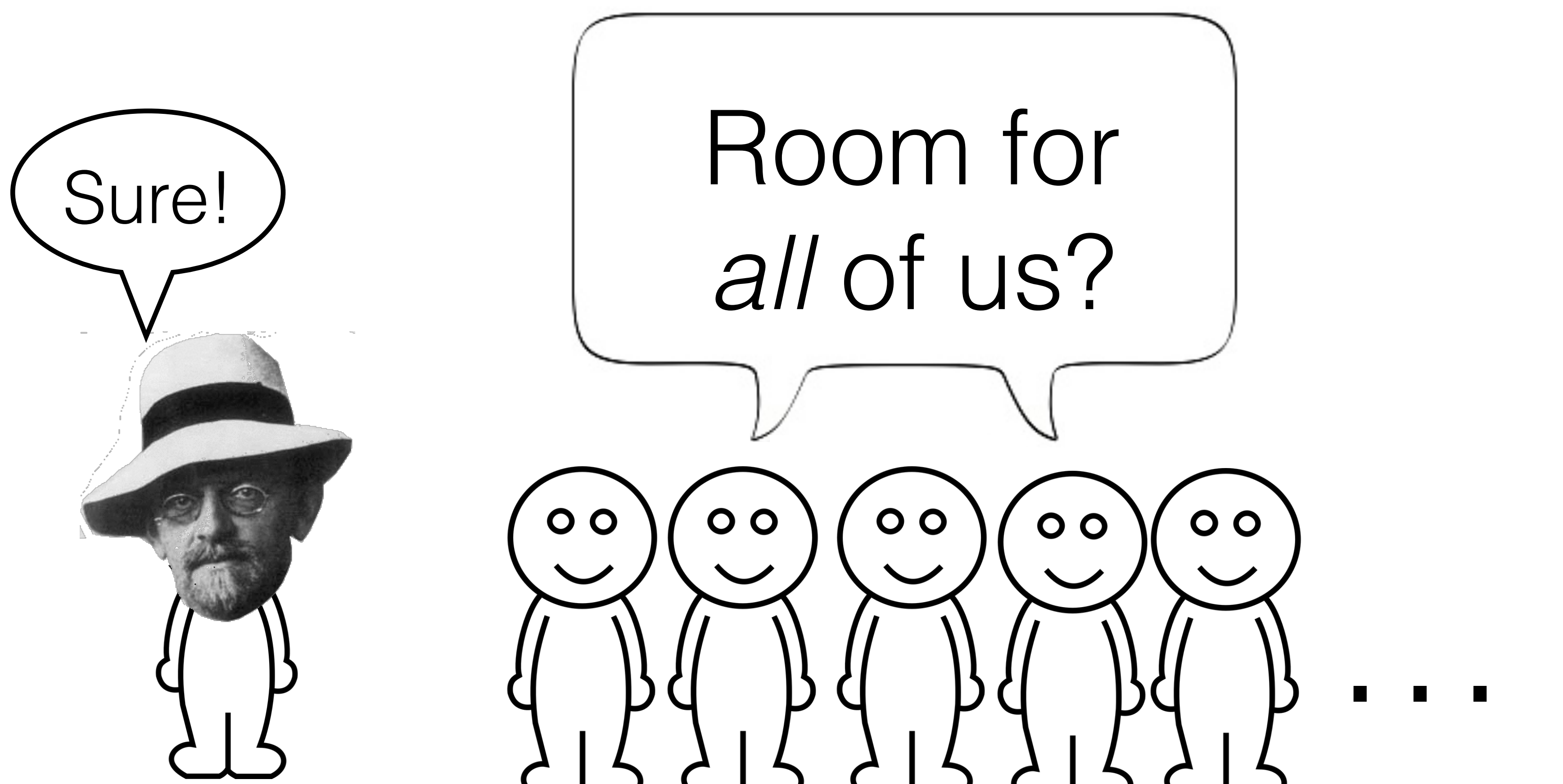
Sure!
Room for
all of us?
. . .

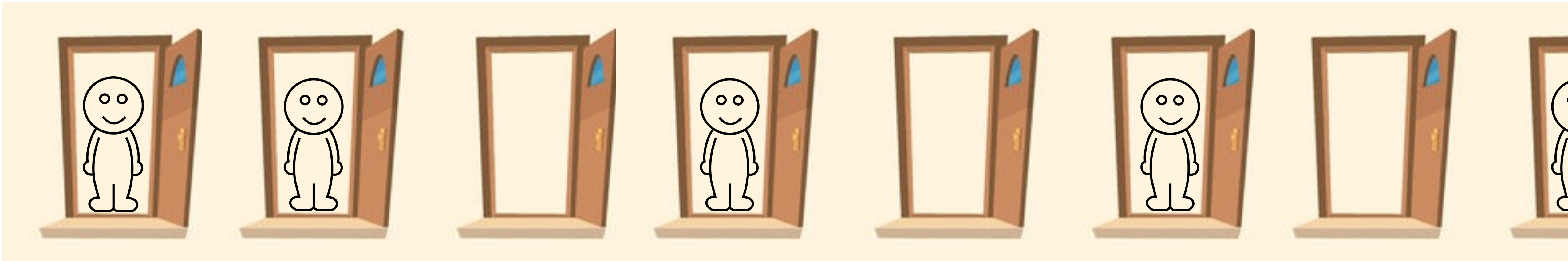
1 2 3 4 5 6 7 …



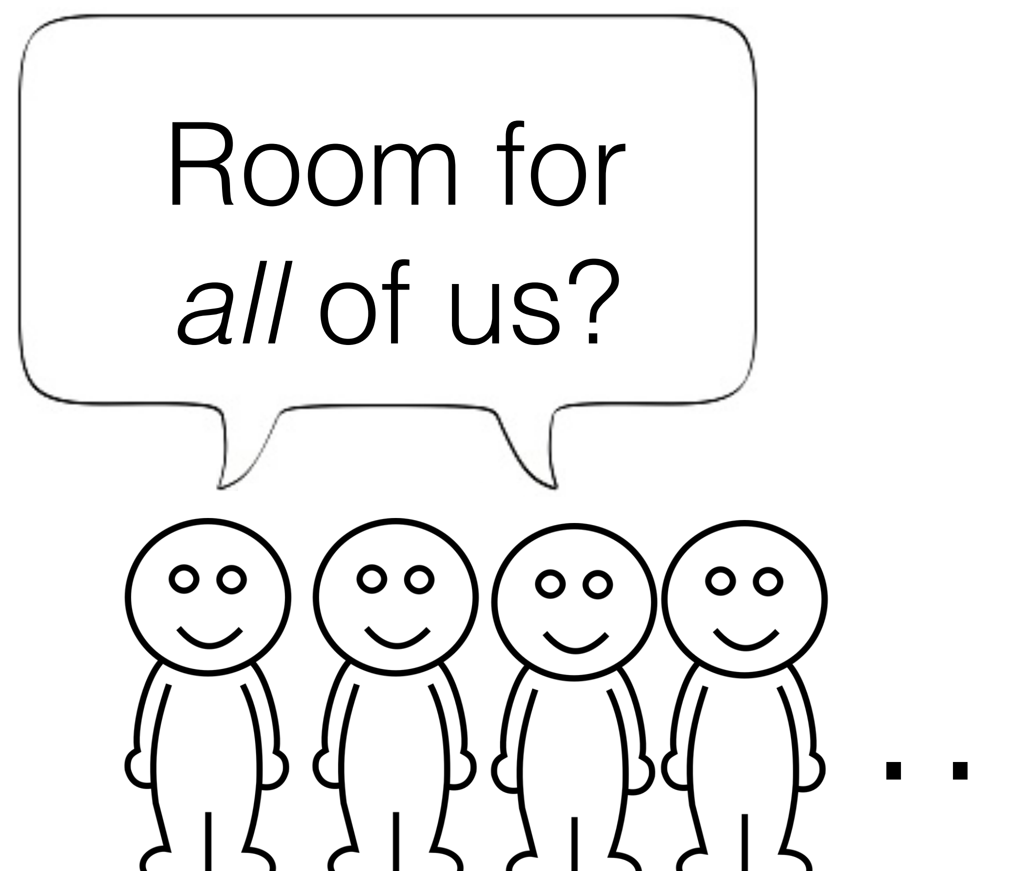
$$n \to 2n$$



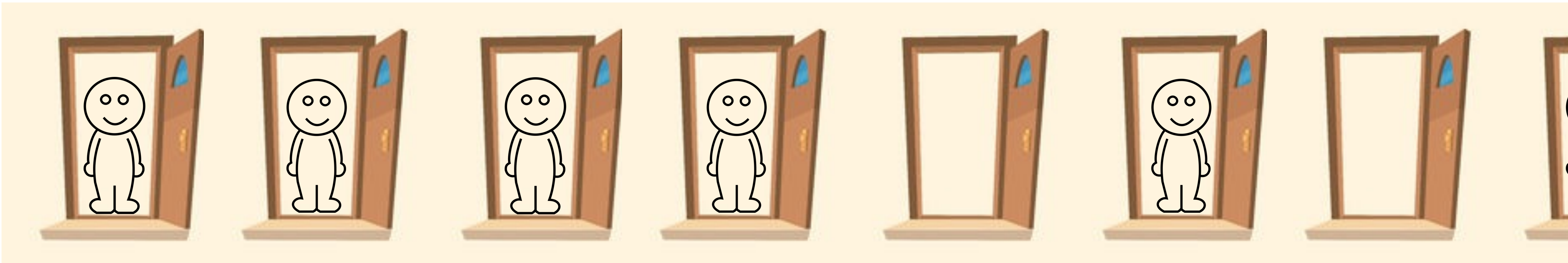
1 2 3 4 5 6 7 …



$$n \to 2n$$



1 2 3 4 5 6 7 …



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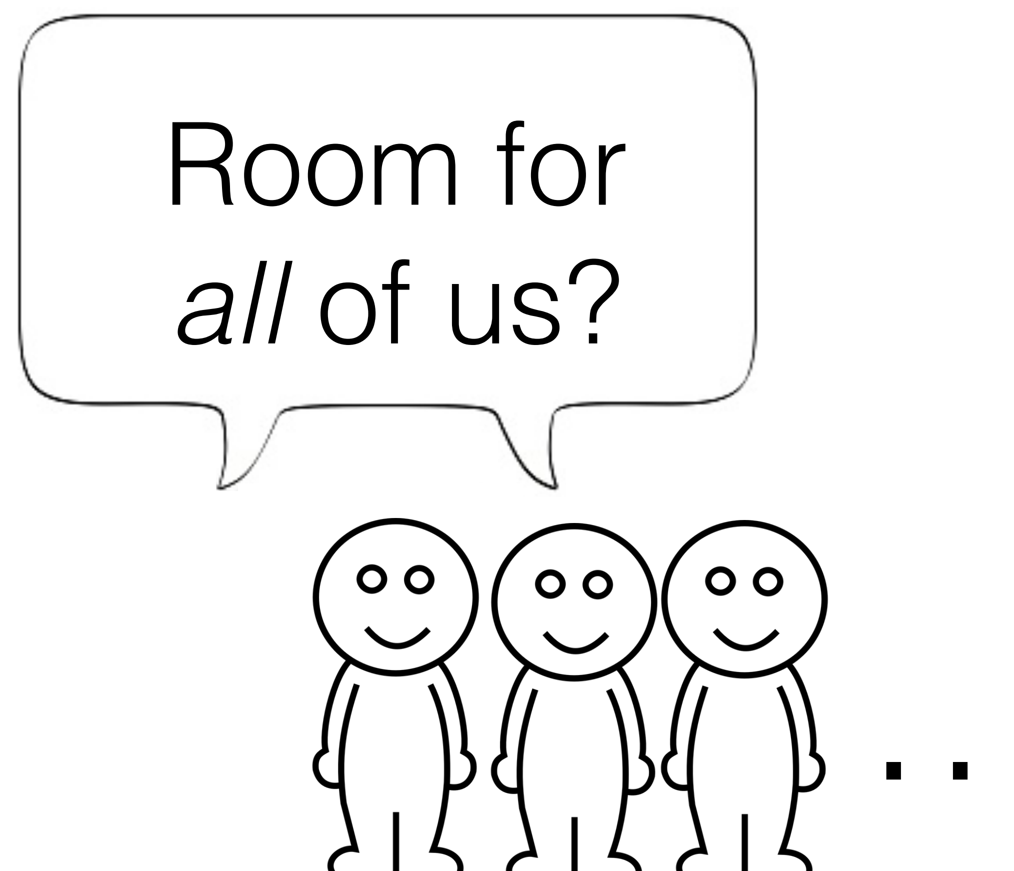

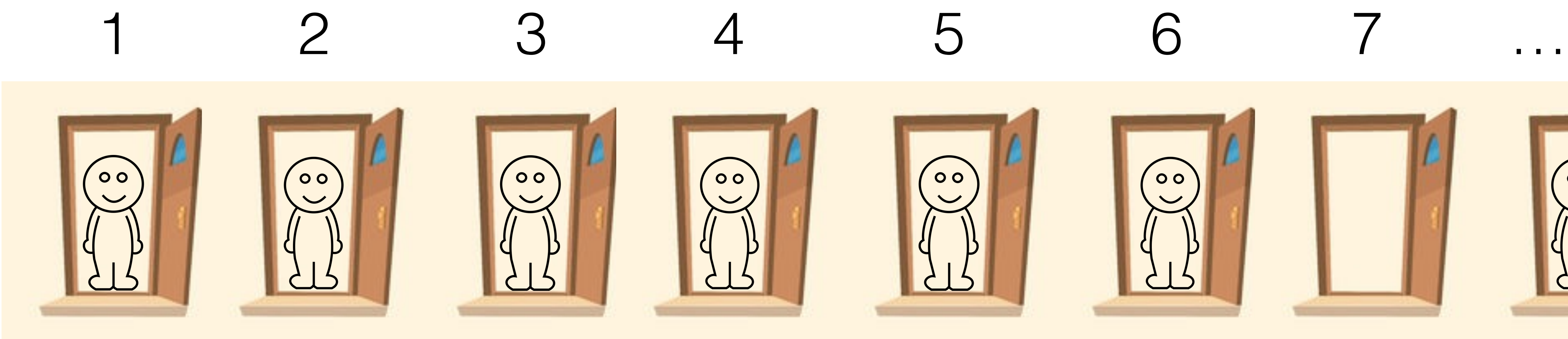
1
2
3
4
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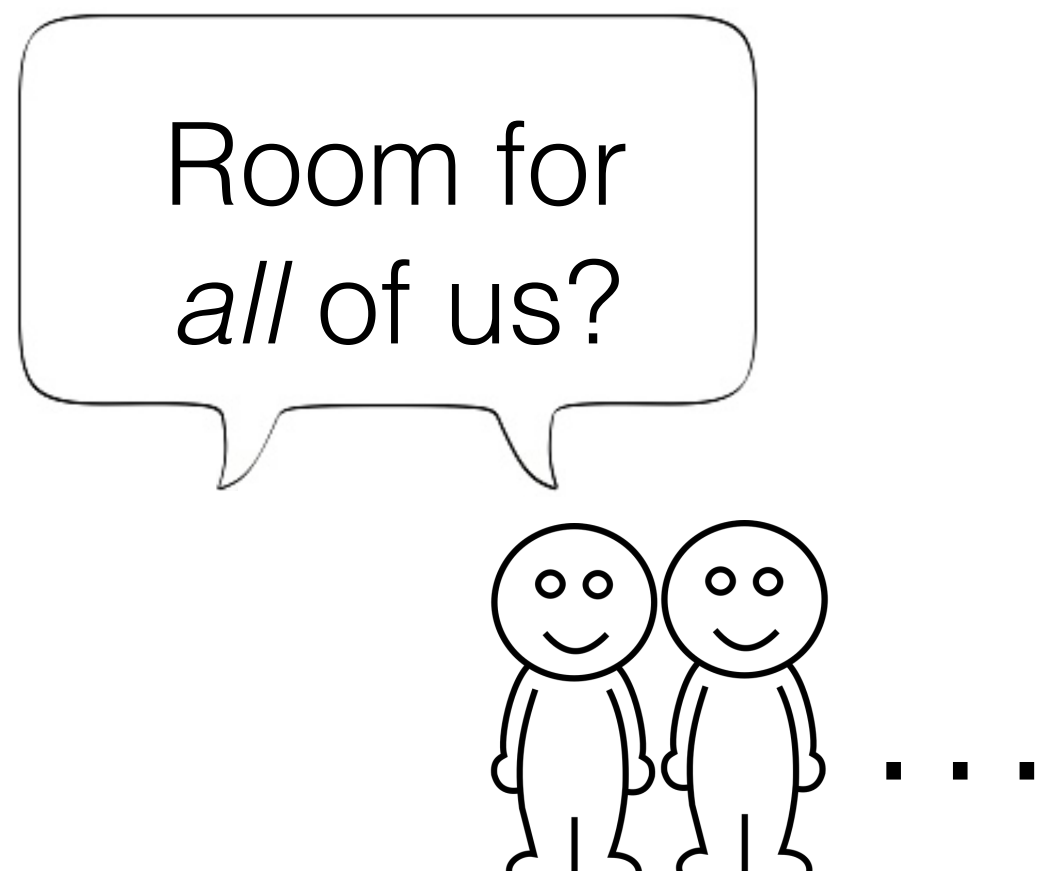
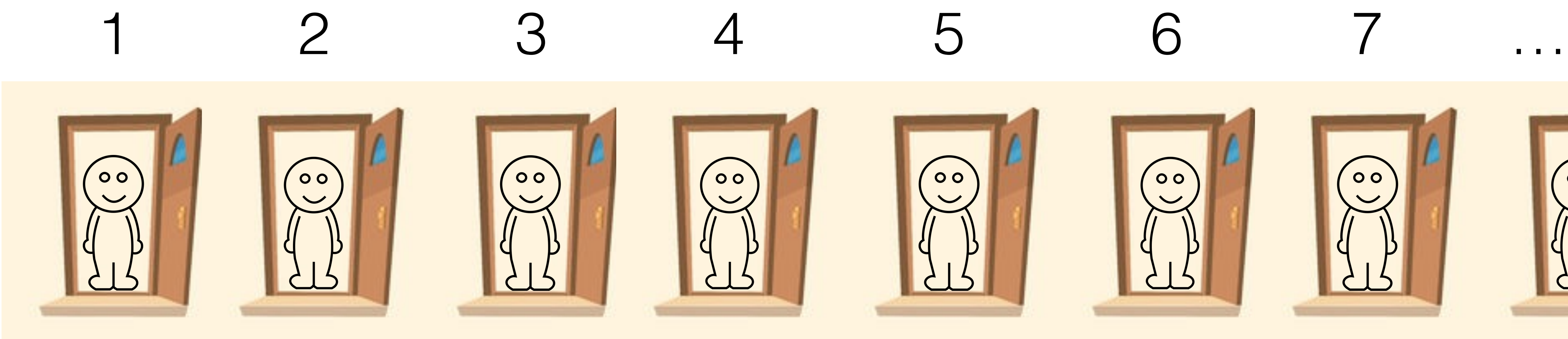
7
…

$$n \to 2n$$


Hilbert's
overly-accommodating
Infinity Hotel
NO
VACANCY


Sure!


Room for
all of us?
…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$$n \to 2n$$


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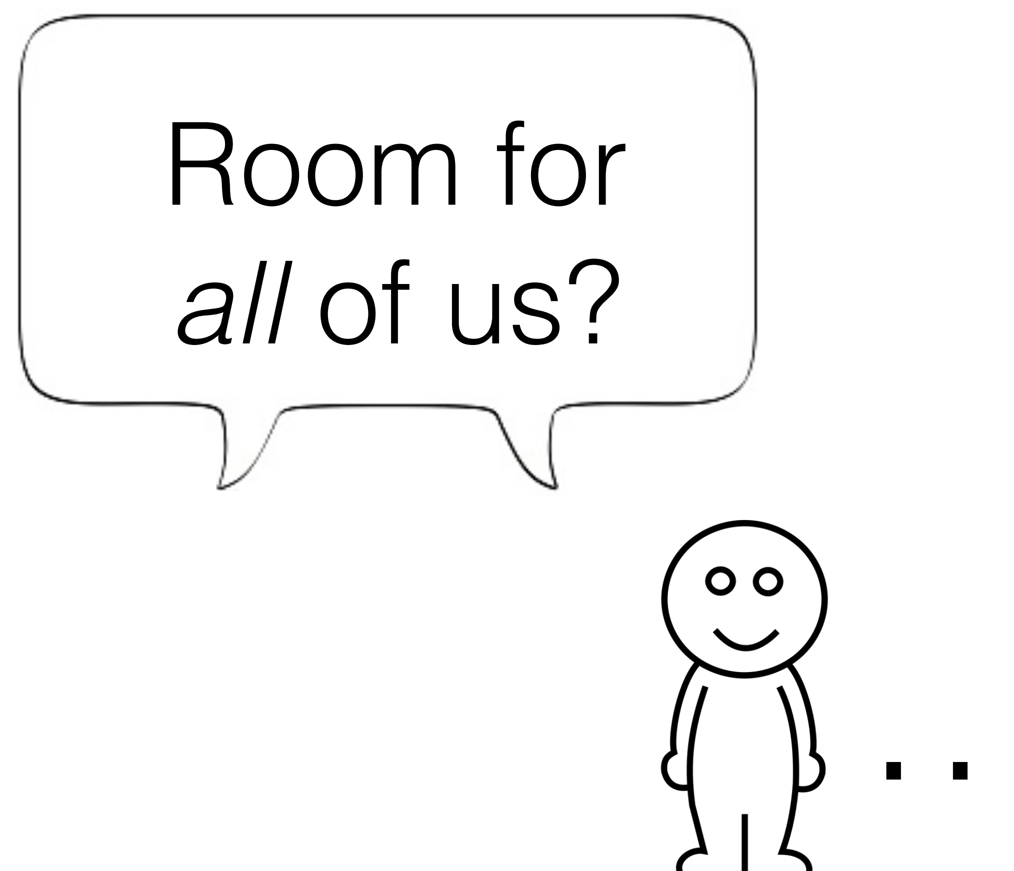
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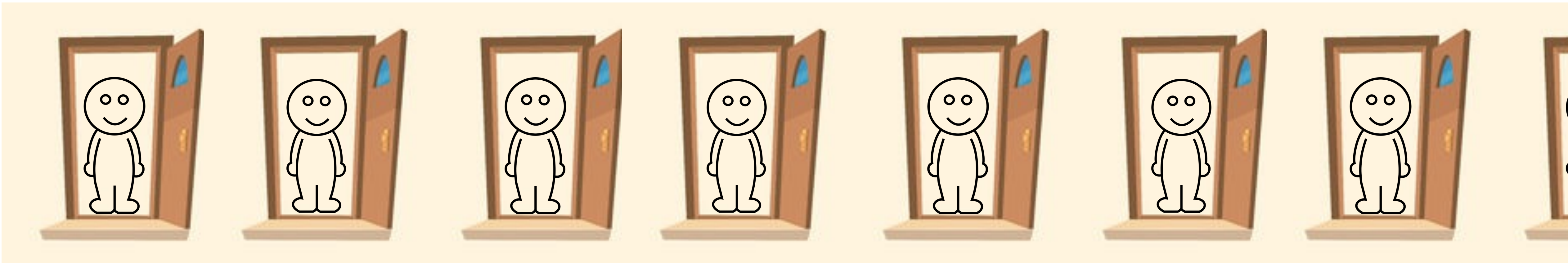
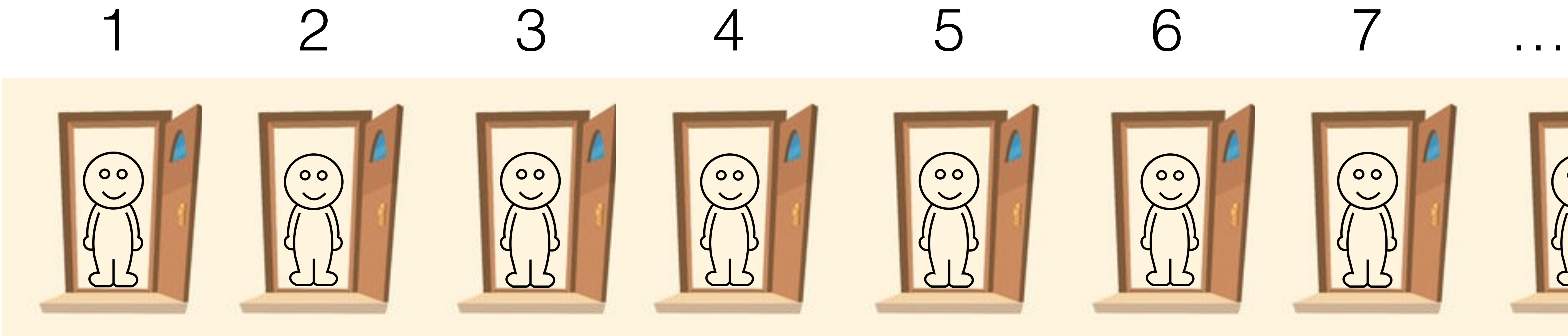


$$n \rightarrow 2n$$

Room for
all of us?

…

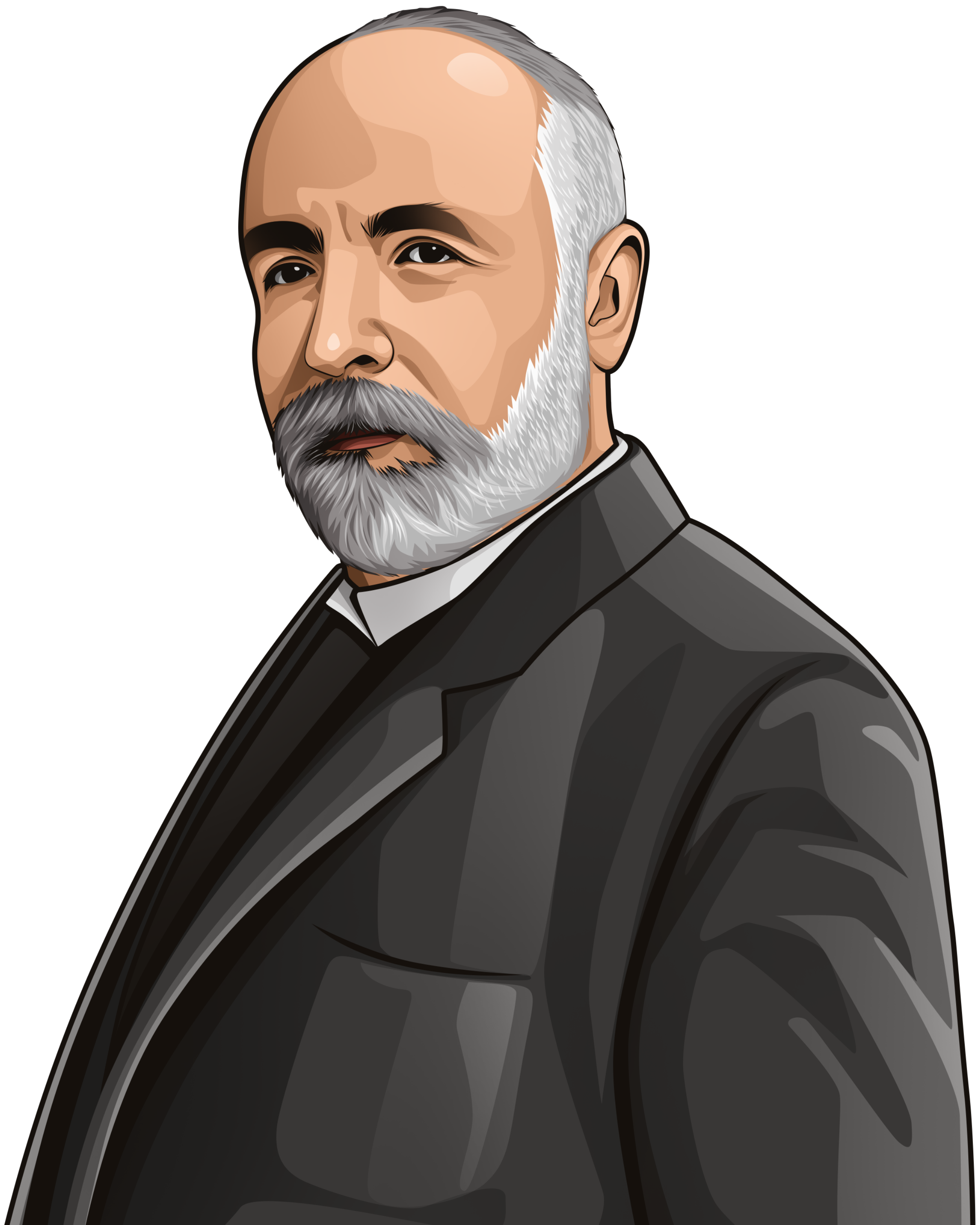


$$n \to 2n$$

Room for *all* of us?

Moral: ∞ is weird

What does size mean?



"In mathematics the art of proposing a question must be held of higher value than solving it."

—Georg Cantor

What does size mean?

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- Until the mid 1800s, infinity was widely treated carelessly or with suspicion.

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- It was made precise Georg Cantor (1845-1918). Who was a PhD student of Weierstrass.

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{ ,2,3}   { ,b,c}

1 $\longleftrightarrow$ a

$\longleftrightarrow$

$\longleftrightarrow$

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{ , ,3}   { , ,c}

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$\longleftrightarrow$

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$$\{\ ,\ ,\ \} \qquad \{\ ,\ ,\ \}$$

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$$1 \longleftrightarrow b$$
$$2 \longleftrightarrow a$$
$$3 \longleftrightarrow c$$

- Sets are the same size if we can "pair up" the elements

Fact. There are the same number of positive integers as negative integers

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Proof.

Fact. There are the same number of positive integers as negative integers

Proof. We need to find a way to pair them all up.

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$$\{-1,-2,-3,-4,-5,\ldots\} \qquad \{1,2,3,4,5,\ldots\}$$

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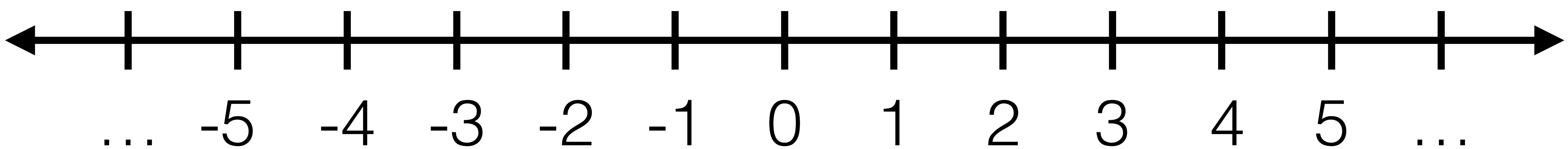

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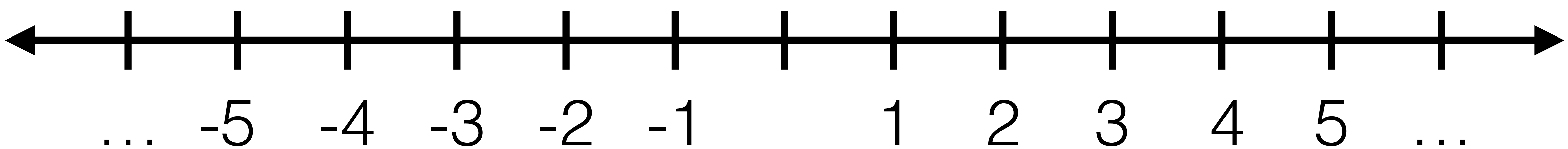

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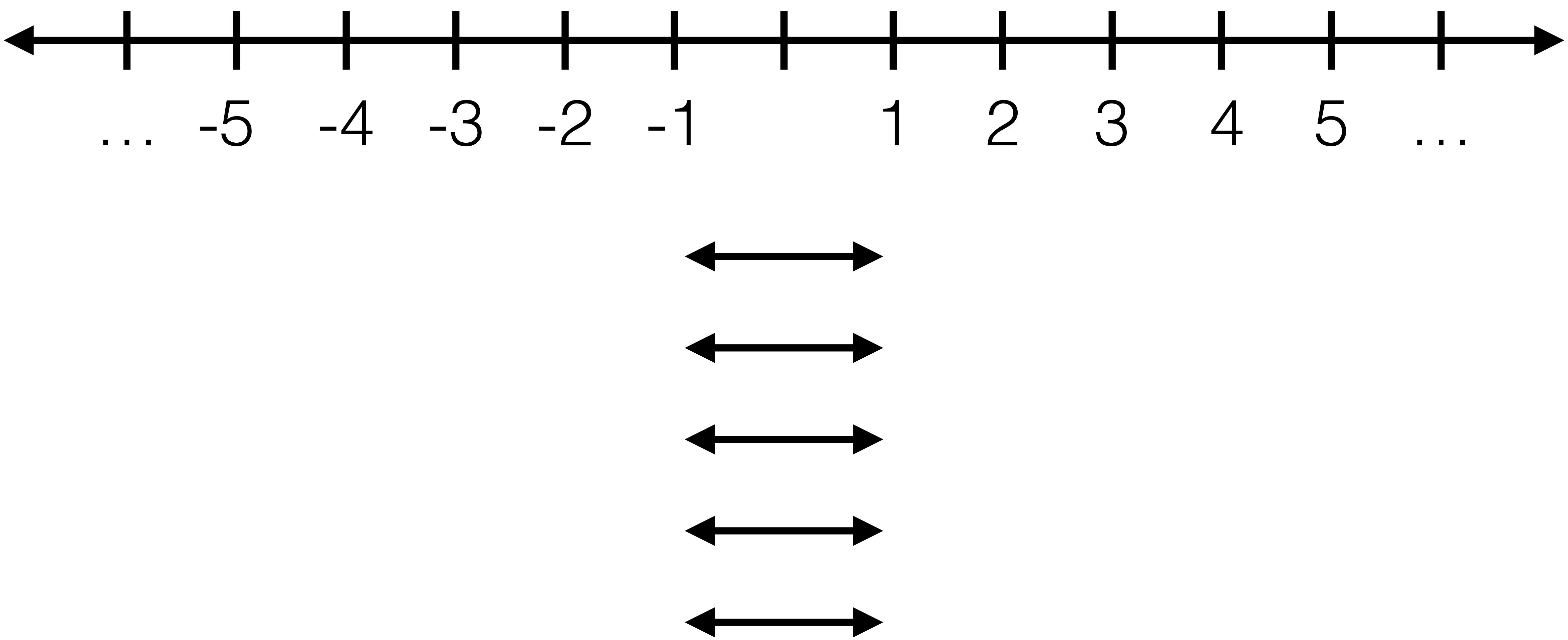

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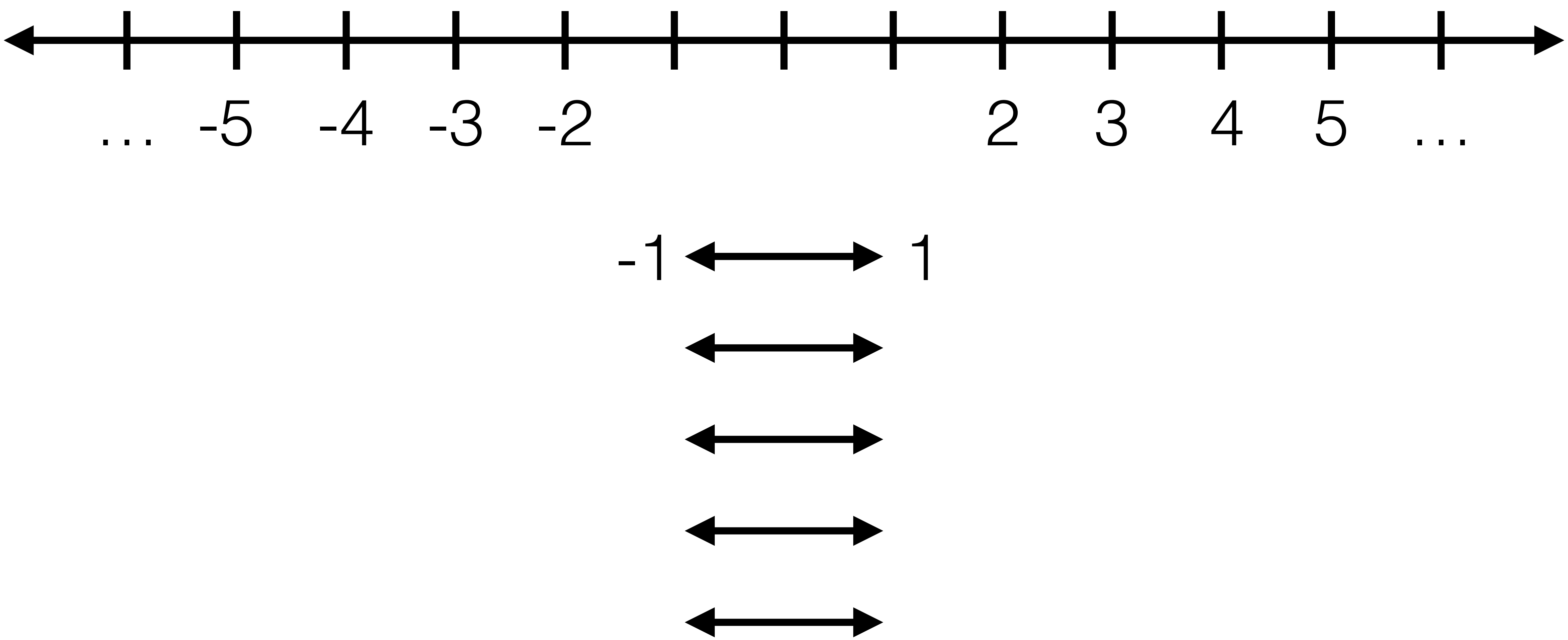

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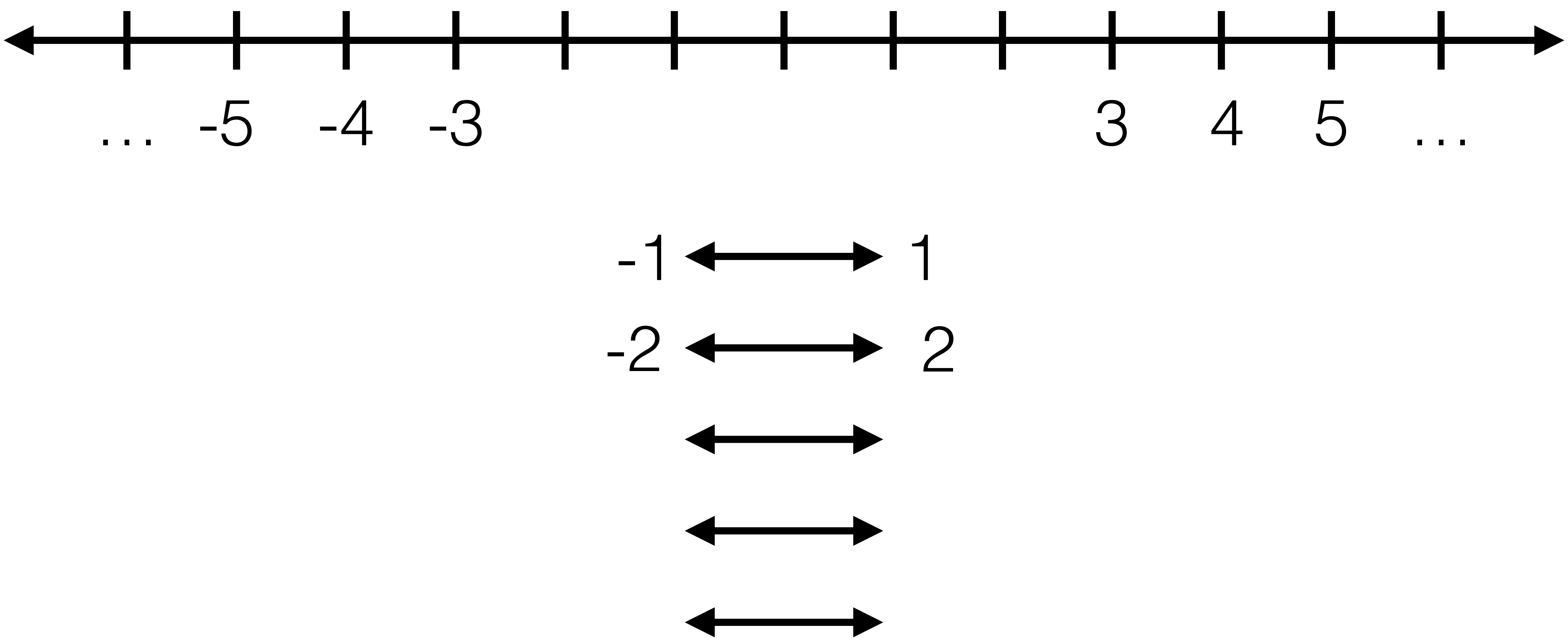

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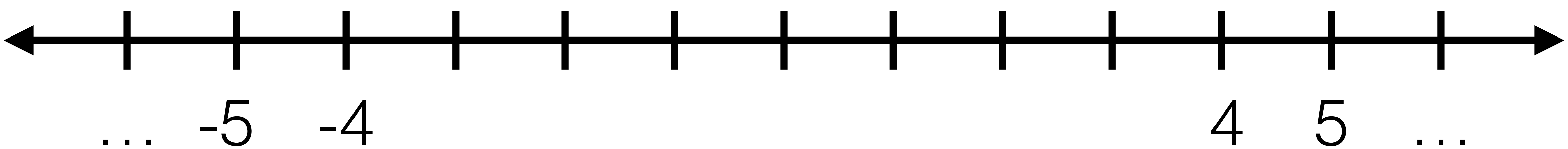

$$-1 \longleftrightarrow 1$$

$$-2 \longleftrightarrow 2$$

$$-3 \longleftrightarrow 3$$

$$\longleftrightarrow$$

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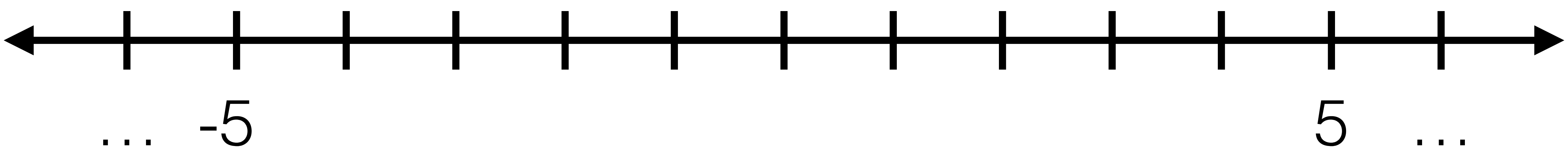

$-1 \longleftrightarrow 1$

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$-3 \longleftrightarrow 3$

$-4 \longleftrightarrow 4$

$\longleftrightarrow$

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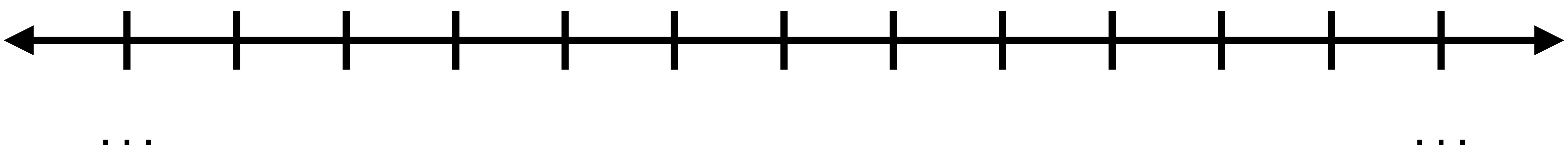

$-1 \longleftrightarrow 1$

$-2 \longleftrightarrow 2$

$-3 \longleftrightarrow 3$

$-4 \longleftrightarrow 4$

$-5 \longleftrightarrow 5$

Fact. There are the same number of positive integers as negative integers

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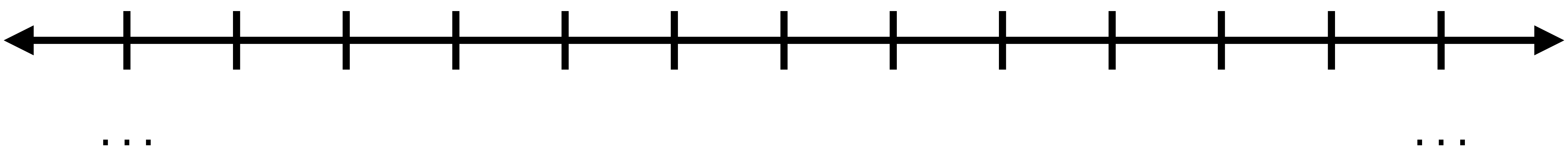

$$
\begin{array}{c}
-1 \longleftrightarrow 1 \\
-2 \longleftrightarrow 2 \\
-3 \longleftrightarrow 3 \\
-4 \longleftrightarrow 4 \\
-5 \longleftrightarrow 5 \\
\vdots
\end{array}
$$

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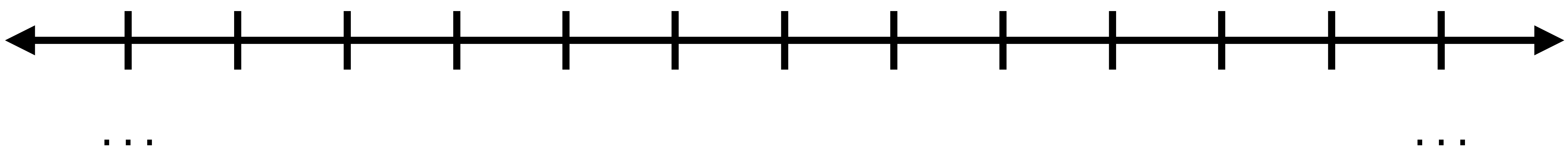

$$-1 \longleftrightarrow 1$$
$$-2 \longleftrightarrow 2$$
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$$\vdots$$
$$-n \longleftrightarrow n$$

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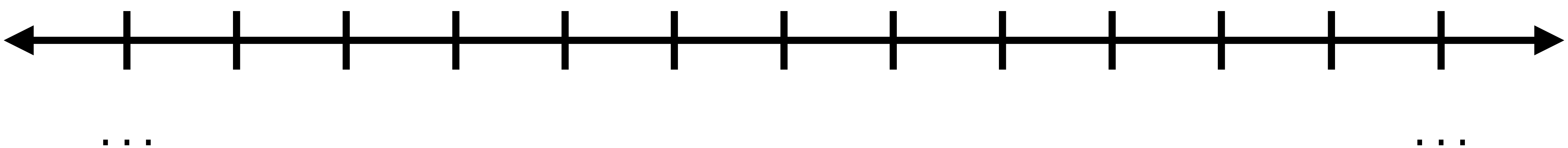

$$
\begin{array}{ccc}
-1 & \longleftrightarrow & 1 \\
-2 & \longleftrightarrow & 2 \\
-3 & \longleftrightarrow & 3 \\
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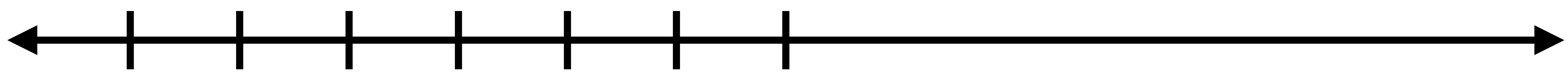

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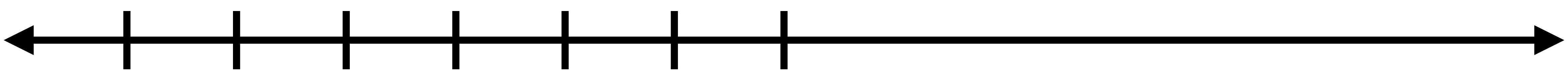

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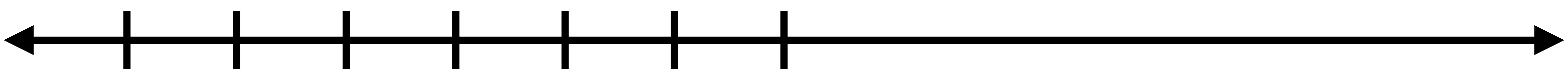

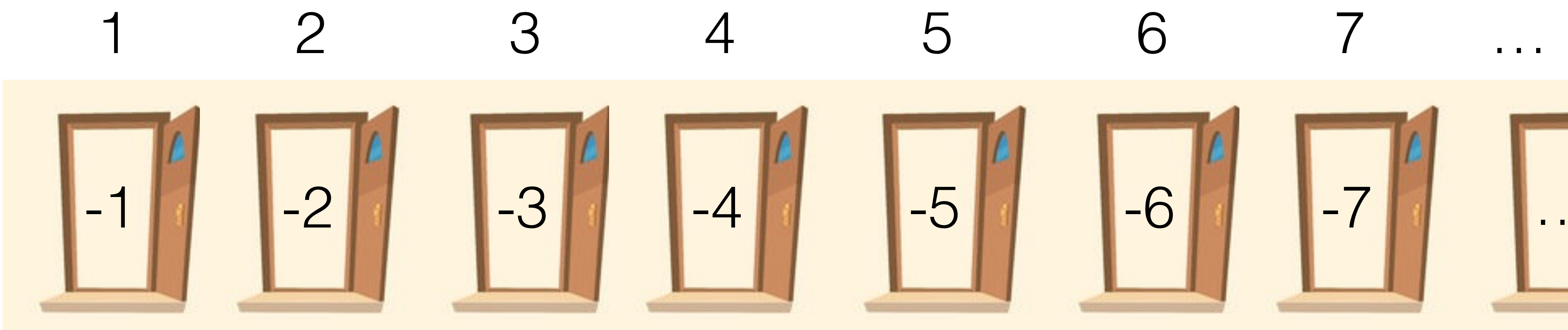

Fact. There are the same number of positive integers as positive *even* integers

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$$\{1, 2, 3, 4, 5, \ldots\} \quad \{2, 4, 6, 8, 10, \ldots\}$$

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$$\{1,2,3,4,5,\ldots\} \quad \{2,4,6,8,10,\ldots\}$$

$$\longleftrightarrow$$

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$$\longleftrightarrow$$

$$\longleftrightarrow$$

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$$\{\ \ ,2,3,4,5,\ldots\} \quad \{\ \ ,4,6,8,10,\ldots\}$$

$$1 \longleftrightarrow 2$$

$$\longleftrightarrow$$

$$\longleftrightarrow$$

$$\longleftrightarrow$$

$$\longleftrightarrow$$

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$$\{\ ,\ ,3,4,5,\ldots\} \quad \{\ ,\ ,6,8,10,\ldots\}$$

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$$2 \longleftrightarrow 4$$
$$\longleftrightarrow$$
$$\longleftrightarrow$$
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$$\{\ ,\ ,\ ,4,5,\ldots\} \quad \{\ ,\ ,\ ,8,10,\ldots\}$$

$$\begin{array}{ccc} 1 & \longleftrightarrow & 2 \\ 2 & \longleftrightarrow & 4 \\ 3 & \longleftrightarrow & 6 \\ & \longleftrightarrow & \\ & \longleftrightarrow & \end{array}$$

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\end{array}$$

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 & \vdots & \\
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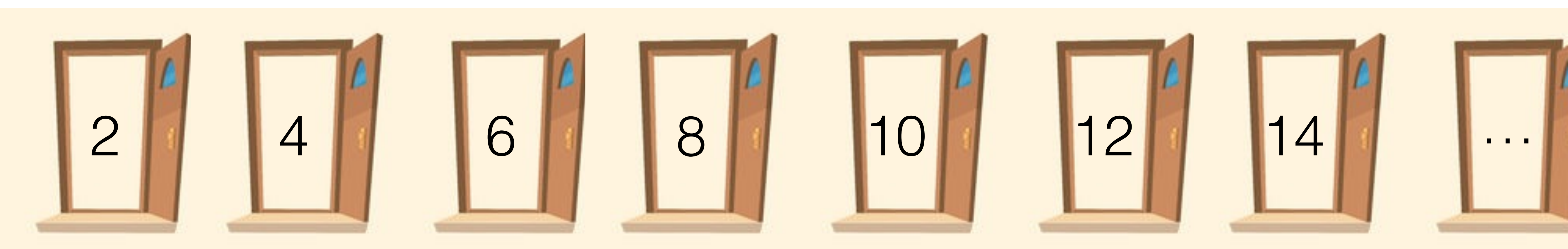

Different sizes of ∞?

<u>Different</u> sizes of ∞?

Theorem (Cantor).
There are the same number of positive integers as rational numbers.

Different sizes of ∞?

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Proof. Pair up the nth positive integer with the nth new rational number in this order:

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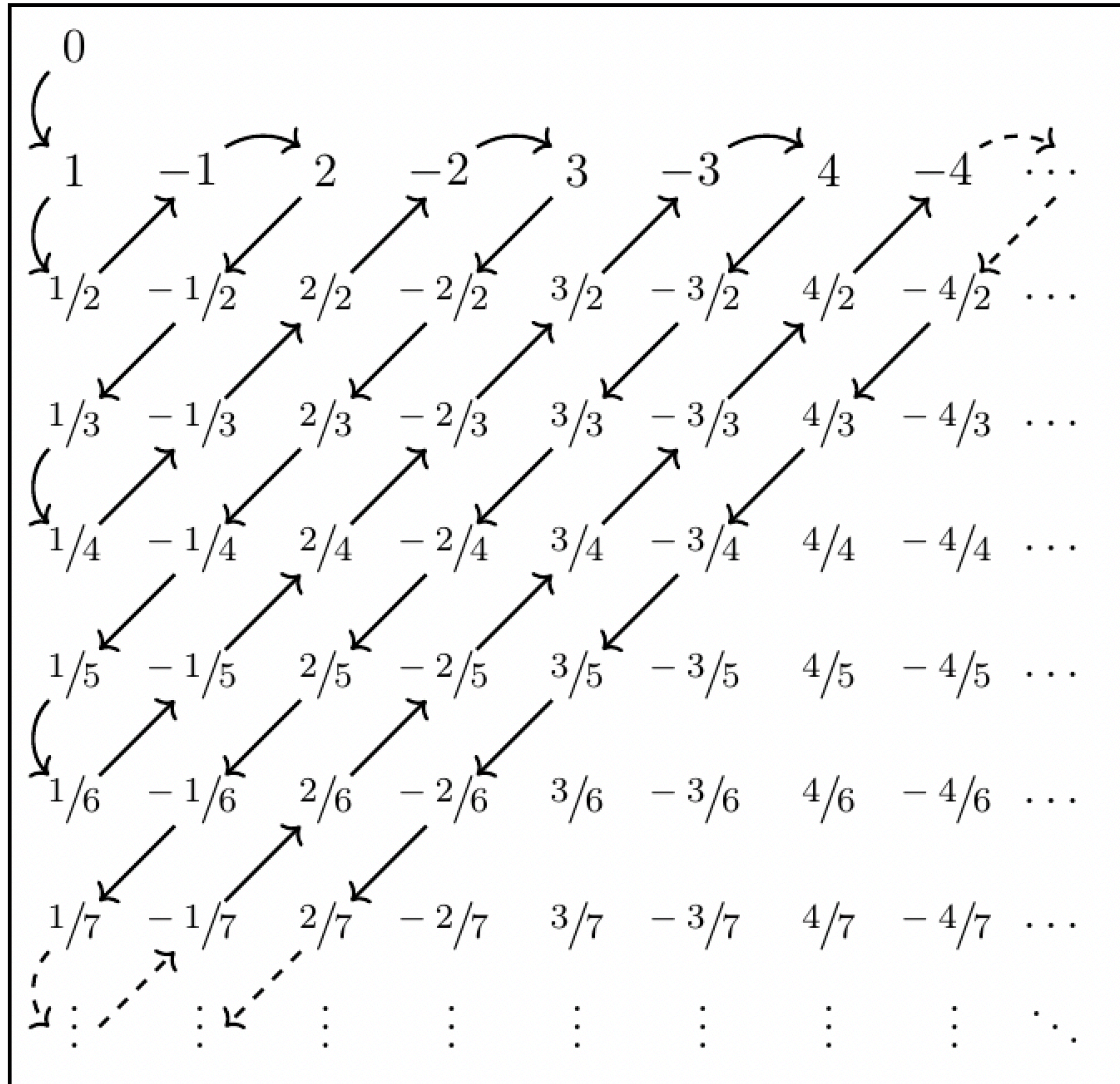

<u>Different</u> sizes of ∞?

- Sets are *different* sizes if we *can't* pair up the elements

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Proof. By contradiction — we assume we can pair them up.

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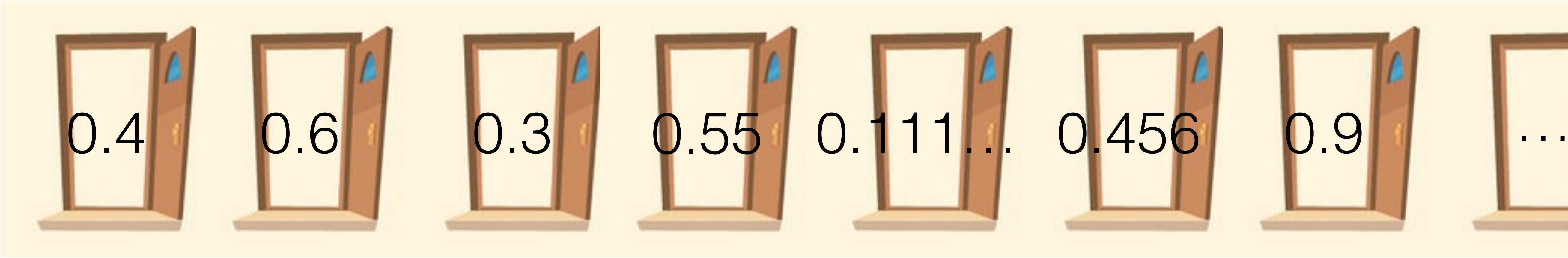

- Sets are *different* sizes if we *can't* pair up the elements

Theorem (Cantor). There is a greater number of real numbers between 0 and 1 than positive integers.

Proof. Assume for a contradiction that we *can* pair these up.

- Sets are *different* sizes if we *can't* pair up the elements

Theorem (Cantor). There is a greater number of real numbers between 0 and 1 than positive integers.

1	0.7 4 9 6 8 2 6 5 9 …
2	0.4 8 6 8 0 3 6 1 3 …
3	0.9 4 6 3 2 6 4 6 8 …
4	0.1 3 2 4 5 6 4 6 7 …
5	0.2 4 5 3 5 7 8 4 5 …
6	0.3 3 4 5 4 3 8 8 3 …
7	0.8 4 6 6 4 2 5 3 7 …
8	0.3 3 5 1 3 7 4 7 9 …
9	0.0 0 9 8 3 1 4 0 3 …
⋮	⋮

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5	0.2 4 5 3 5 7 8 4 5 …
6	0.3 3 4 5 4 3 8 8 3 …
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8	0.3 3 5 1 3 7 4 7 9 …
9	0.0 0 9 8 3 1 4 0 3 …
⋮	⋮

Proof. Assume for a contradiction that we *can* pair these up.

Build a new number between 0 and 1. Its nth digit is anything besides 0, 9 or the nth digit of the nth number in this table

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1	0.7 4 9 6 8 2 6 5 9 …
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4	0.1 3 2 4 5 6 4 6 7 …
5	0.2 4 5 3 5 7 8 4 5 …
6	0.3 3 4 5 4 3 8 8 3 …
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⋮	⋮

0.

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4	0.1 3 2 4 5 6 4 6 7 ...
5	0.2 4 5 3 5 7 8 4 5 ...
6	0.3 3 4 5 4 3 8 8 3 ...
7	0.8 4 6 6 4 2 5 3 7 ...
8	0.3 3 5 1 3 7 4 7 9 ...
9	0.0 0 9 8 3 1 4 0 3 ...
⋮	⋮

0.3

Proof. Assume for a contradiction that we *can* pair these up.

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$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$$

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- If n is unselfish then n is not in D — same contradiction!

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$$\mathbb{N}$$

$$\downarrow$$

Same size as $\mathbb{R}$ $\longrightarrow$ $\text{PowerSet}(\mathbb{N})$

$$\downarrow$$

$$\text{PowerSet}(\text{PowerSet}(\mathbb{N}))$$

$$\downarrow$$

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Smallest infinity

$$\mathbb{N}$$

$$\downarrow$$

Same size as $\mathbb{R}$

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$$\downarrow$$

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Smallest infinity

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$$\downarrow$$

Same size as $\mathbb{R}$

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Second smallest infinity?

$$\downarrow$$

$$\text{PowerSet}(\text{PowerSet}(\mathbb{N}))$$

$$\downarrow$$

$$\text{PowerSet}(\text{PowerSet}(\text{PowerSet}(\text{PowerSet}(\mathbb{N}))))$$

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The continuum hypothesis claims that the answer is ‘no’

- The continuum hypothesis is undecidable in ZFC. That is, its truth value is independent of the standard axioms of math.

Cantor

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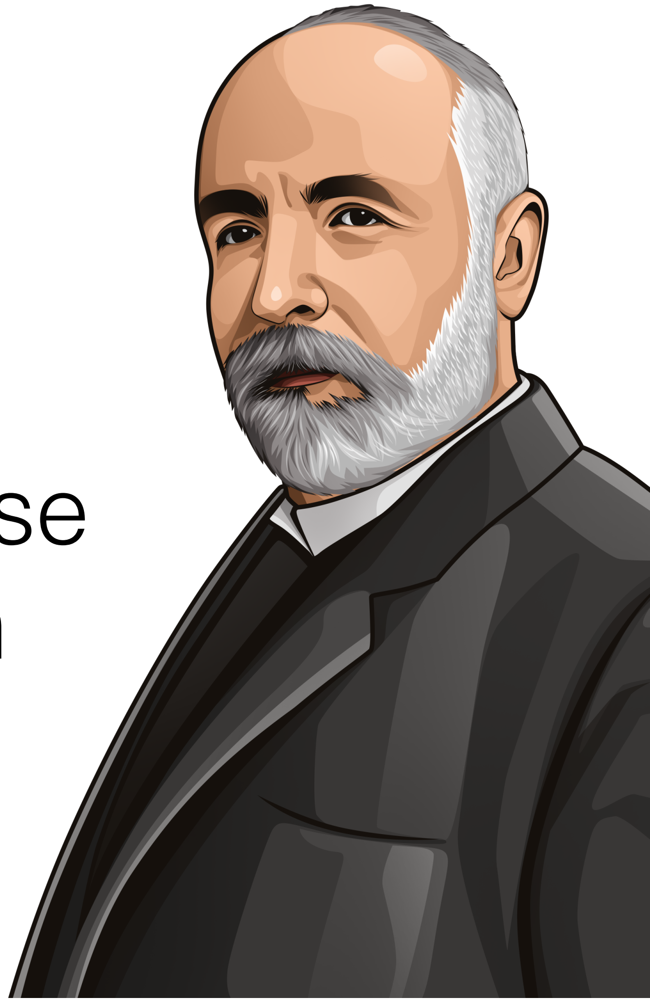
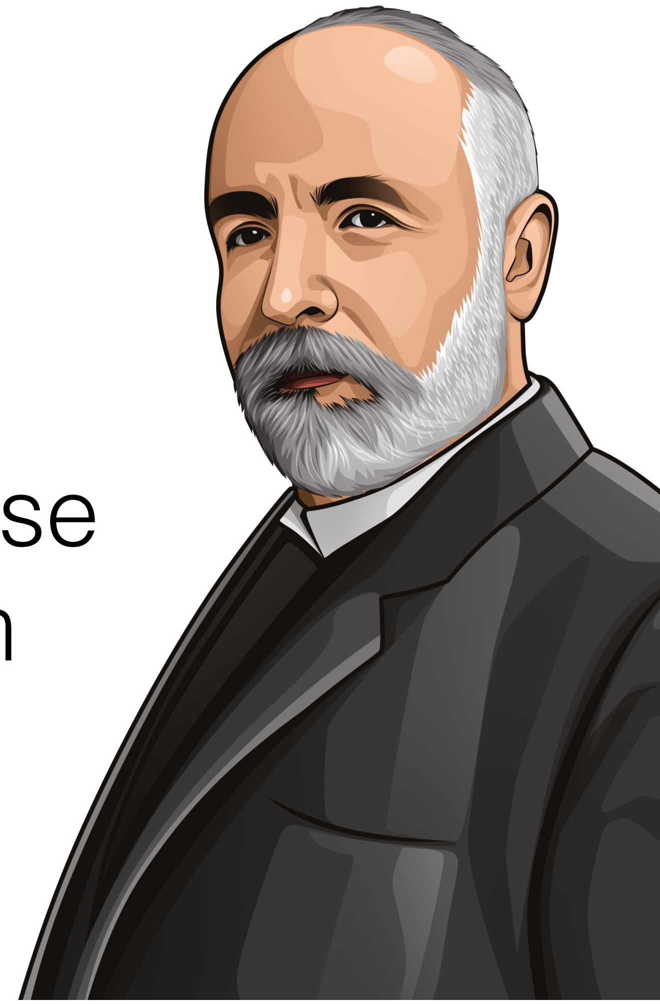


- Cantor struggled with mental illness. These were worsened due to the harsh criticism he received for his work on infinities.

Cantor



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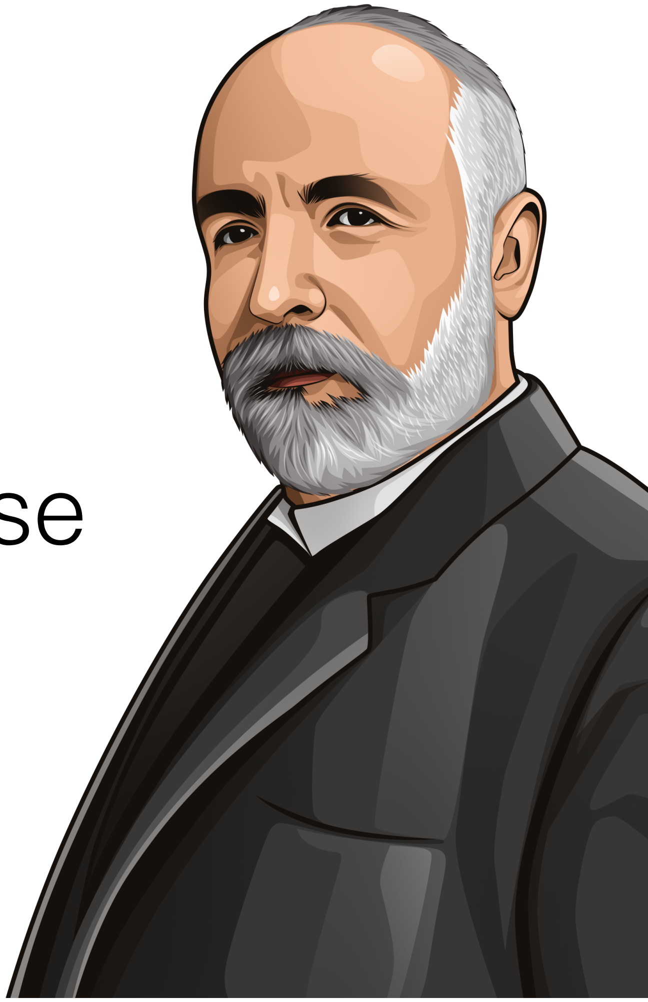

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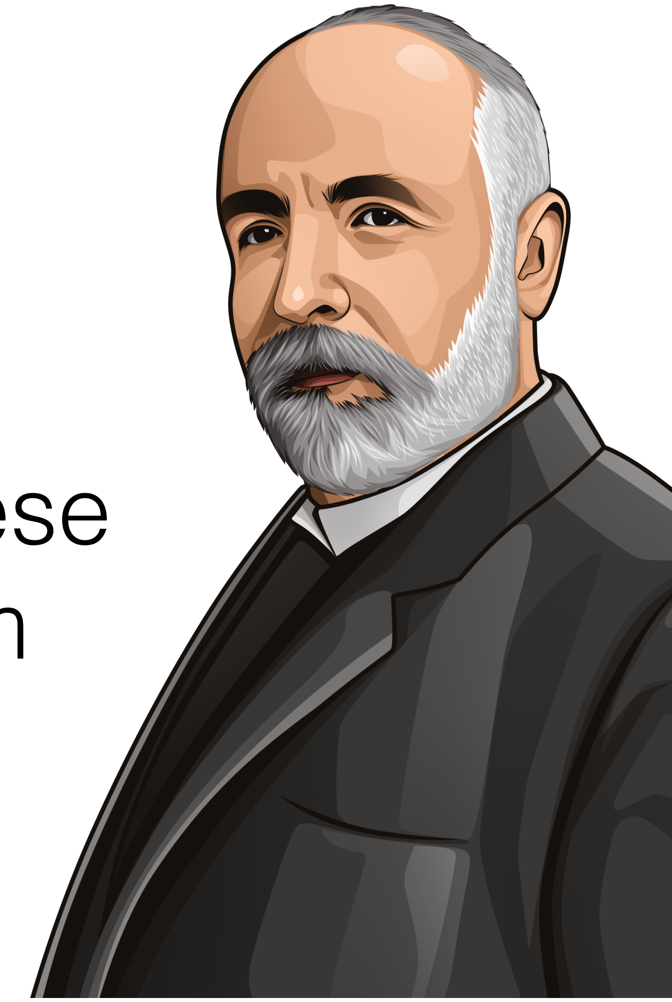

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> “No one shall expel us from the paradise that Cantor has created.”

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- She also included in it the addition of a new law of physics: The law of conservation of total energy.

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> Paul Dirac quote: "The mathematician plays a game in which he himself invents the rules while the physicist plays a game in which the rules are provided by nature, but as time goes on it becomes increasingly evident that the rules which the mathematician finds interesting are the same as those which nature has chosen."

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- An important problem in real analysis is the study of how heat diffuses through a region. Jean-Baptiste Joseph Fourier developed *Fourier analysis* to study this.

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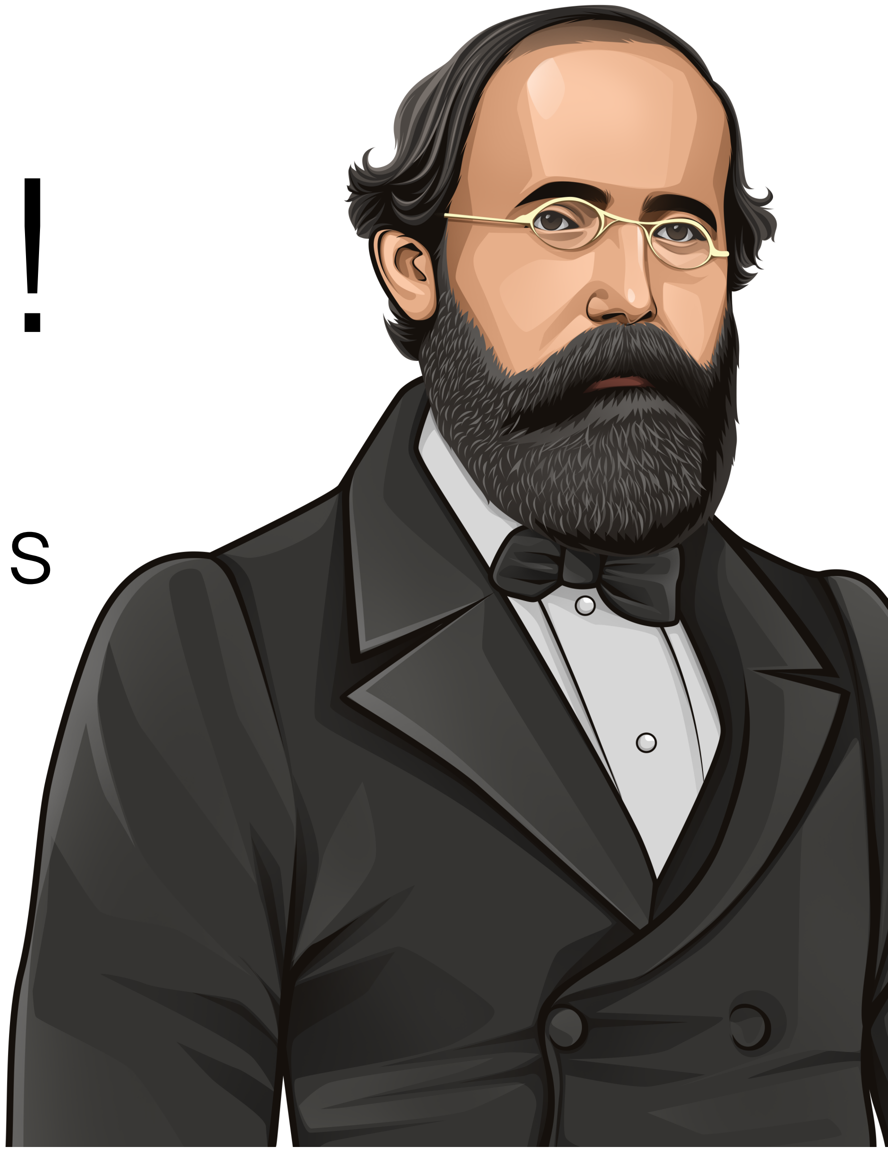

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- Bernhard Riemann (1826-1846) also made important contributions to Fourier analysis, but he is best remembered in real analysis for being the first to construct the modern, rigorous formulation of the integral, which today is called the *Riemann integral*.

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- In his 1902 PhD dissertation, he used measure theory to develop a theory of integration that is central to modern real analysis and modern probability.
- All these people helped to give a rigorous foundation to one of the biggest accomplishments in history: the creation of calculus.

A People's History of Infinity

Cultural Influence

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> I read the Nine Chapters as a boy, and studied it in full detail when I was older. I observed the division between the dual natures of Yin and Yang [the positive and negative aspects] which sum up the fundamentals of mathematics.

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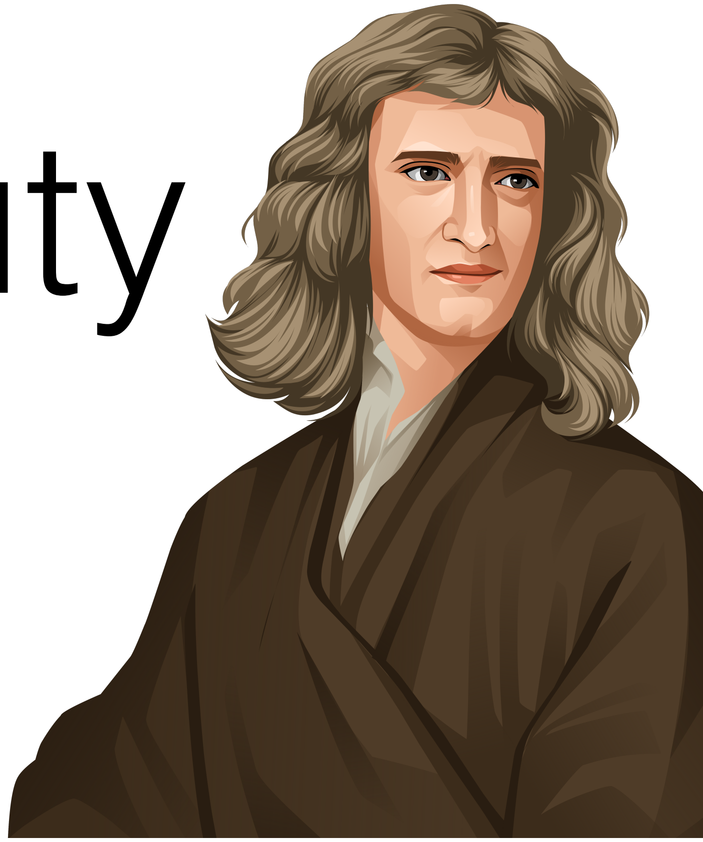

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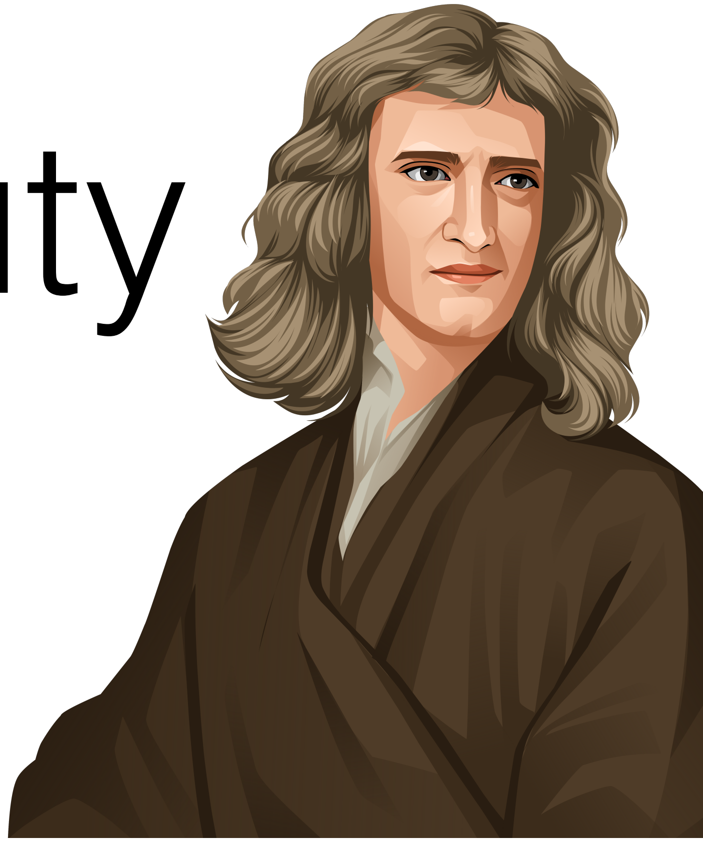

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- Srinivasa Ramanujan: “An equation has no meaning unless it expresses a thought of God.”

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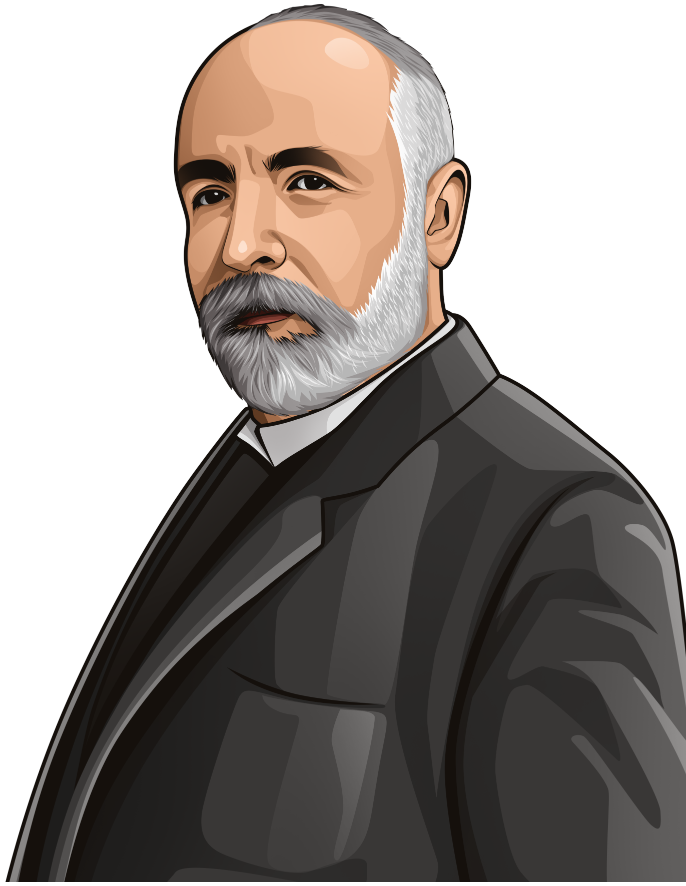

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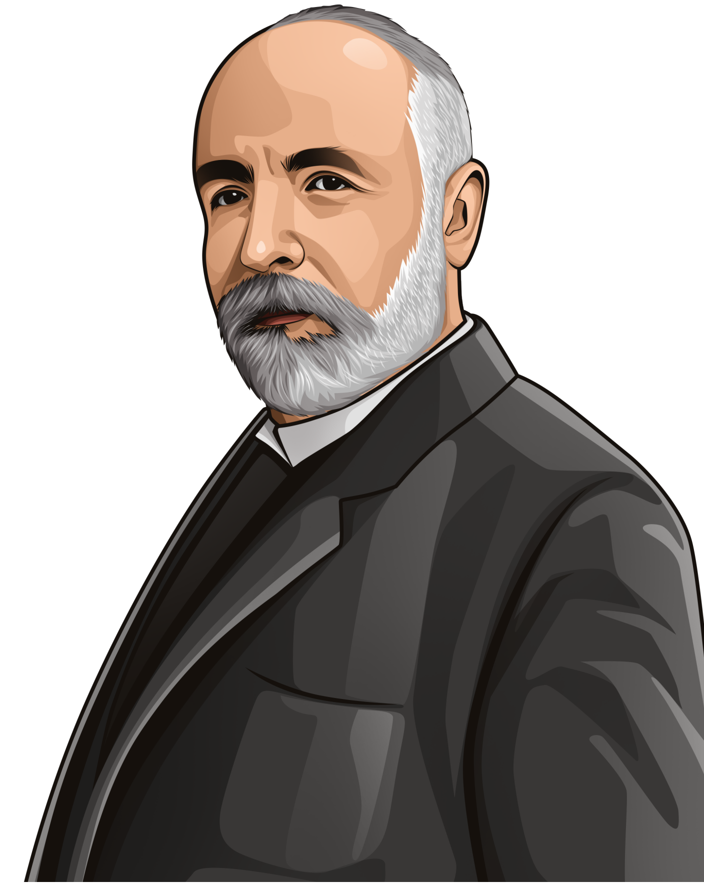

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> The fear of infinity is a form of myopia that destroys the possibility of seeing the actual infinite, even though it in its highest form has created and sustains us, and in its secondary transfinite forms occurs all around us and even inhabits our minds.

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- Would these people have cared about and discovered all this without being religious? It is impossible to say.
- But culture influencing math is undeniable.
- It is still happening today, and it will likely happen forever. It's worth being aware of it.