Do any 4 problems from the end of Chapter 5 on Perusall.

— Exercises —

Exercise 5.1. Come up with your own homework problem that uses the information from this chapter. Make sure it is different than all of the questions below and have its level of difficulty be at about the level as the below. Then, answer your question.

Exercise 5.2.

(a) Prove that $\sqrt{3}$ is irrational.

(b) True or false: $\sqrt{n}$ is irrational for every positive integer n. Justify your answer.

Exercise 5.3.

(a) Prove that 496 is a perfect number using the definition of a perfect number.

(b) Prove that 496 is a perfect number using a theorem from this chapter.

(c) Prove that 495 is not a perfect number.

Exercise 5.4. Prove that every even perfect number is a triangular number, and hence has the form

$$\begin{aligned} 6 &= 1+2+3 \\ 28 &= 1+2+3+4+5+6+7 \\ 496 &= 1+2+3+\cdots+31 \\ &\text{etc.} \end{aligned}$$

Exercise 5.5. In 1575, perfect number enthusiast Pietro Cataldi proved that every even perfect number ends in a '6' or an '8'. Prove this with the following steps.

(a) Prove that for every $n \in \mathbb{N}$, we have $6^n \equiv 6 \pmod{10}$ and $16^n \equiv 6 \pmod{10}$.

(b) Using Euler's theorem on even perfect numbers, as well as the observation that every odd prime must be of the form $4m+1$ or $4m+3$, prove Cataldi's result.

Exercise 5.6. Verify that 17,296 and 18,416 form a pair of amicable numbers. It may be helpful to note that their prime factorizations are $17296 = 2^4 \cdot 23 \cdot 47$, and $18416 = 2^4 \cdot 1151$.

Exercise 5.7. As we discussed in this chapter, Euclid and Euler combined to prove that $2^n - 1$ is prime if and only if $2^{n-1}(2^n - 1)$ is a perfect number. To prove this, Euler used the function $\sigma(n)$, which he defined to be the sum of *all* positive divisors of a number n. For example, $\sigma(3) = 1+3 = 4$, and $\sigma(6) = 1+2+3+6 = 12$.

Exercise 5.7. As we discussed in this chapter, Euclid and Euler combined to prove that $2^n - 1$ is prime if and only if $2^{n-1}(2^n - 1)$ is a perfect number. To prove this, Euler used the function $\sigma(n)$, which he defined to be the sum of *all* positive divisors of a number n. For example, $\sigma(3) = 1+3 = 4$, and $\sigma(6) = 1+2+3+6 = 12$.

This function was useful as it has some nice properties, which you will investigate in this exercise. Prove the following.

(a) p is prime if and only if $\sigma(p) = p + 1$.

(b) N is perfect if and only if $\sigma(N) = 2N$.

(c) If p is prime, then $\sigma(p^k) = \dfrac{p^{k+1} - 1}{p - 1}$.

(d) If p and q are different primes, then $\sigma(pq) = \sigma(p)\sigma(q)$.

(e) If a and b are relatively prime, then $\sigma(ab) = \sigma(a)\sigma(b)$.

Exercise 5.8. In Exercise 5.7 the σ function was defined and some properties of it were proved. The following is a sketch of Euler's original proof of Theorem 5.2. Fill in the details to write a complete proof of this theorem.
Proof sketch. Let $n = 2^k m$ where m is odd and $k > 0$. If n is perfect, then its divisors sum to $2n$. That is,

$$2n = \left(2^{k+1} - 1\right) \cdot \sigma(m) \qquad \Rightarrow \qquad \frac{m}{\sigma(m)} = \frac{2^{k+1} - 1}{2^{k+1}},$$

and this latter fraction is in lowest terms. Thus, $m = \left(2^{k+1} - 1\right) \cdot L$ for some integer L. If $L = 1$, then $\sigma(m) = 2^{k+1}$ implying $m = 2^{k+1} - 1$ is prime. And this is the only option, because $L > 1$ would lead to a contradiction:

$$\sigma(m) \geq m + (2^{k+1} - 1) + L + 1,$$

and hence

$$\frac{\sigma(m)}{m} \geq \frac{2^{k+1}(L+1)}{m} > \frac{2^{k+1}}{2^{k+1} - 1}.$$

Exercise 5.9. Let n be an even perfect number and $n \neq 6$. In this exercise you will prove that the sum of n's digits is congruent to 1 (mod 9).

(a) Let m be a positive integer and let S be the sum of the digits of m. Prove that $m \equiv S \pmod 9$.

(b) Show that

$$2^{p-1}\left(2^p - 1\right) = \left(2^{p-1} - 1\right)\left(2^p + 1\right) + 1.$$

(c) Using parts (a) and (b) and Theorem 5.2, prove the result.

Exercise 5.10. Using what was proven in Exercise 5.7, determine $\sigma(4320)$. Begin by writing the prime factorization of 4320. (Make sure you use Exercise 5.7; you should not solve this problem by writing out all of the divisors of 4320.)

Exercise 5.11. It is unknown whether there is an odd perfect number. However, it is a theorem of James Joseph Sylvester that if a number N *is* an odd perfect number, then N must have at least 3 different prime divisors. You will prove this theorem by showing why having one or two prime divisors is impossible.

In the following, you may make use of what was proven in Exercise 5.7.

(a) Prove that N cannot have exactly one prime divisor.

(b) Prove that N cannot have exactly two prime divisors.

Exercise 5.12. Consider the following problem.

> There are certain things whose number is unknown. If we count them by twos, we have one left over; by fives, we have four left over; and by elevens, eight are left over. How many things are there?

Explain how Sun Zi would have solved this. You may use modern notation.

Exercise 5.13. Suppose that a comet has an orbital period of 5 years and another has an orbital period of 18 years. If the first comet reached its perihelion (the moment when the comet is closest to the sun) in the year 502, and the second comet reached its perihelion in the year 506, when is the next year in which they will both reach their perihelia in same year?

Exercise 5.14. In his astronomical treatise *Aryabhatiya*, published around 510, Indian mathematician Aryabhata included the following problem.

> Find the [smallest positive] number that if divided by 8 is known to leave 5, that if divided by 9 leaves a remainder 4, and that if divided by 7 leaves a remainder 1.

Solve this problem.

Exercise 5.15. During the 7^{th} century, Indian mathematician Brahmagupta proposed the following problem. In it, he refers to the *Kalpa*, which you can take to mean the point when all of the celestial bodies mentioned below began their revolutions together.

Suppose at a certain time since the *Kalpa*, the Sun, Moon, Mars, Mercury, Jupiter and Saturn have traveled for the following number of days after completing their most recent full revolutions:[39]

Sun	Moon	Mars	Mercury	Jupiter	Saturn
1000	41	315	1000	1000	1000

[39]The numbers Brahmagupta used are very close to the exact numbers based on modern data, showing the precise work of Indian astronomers.

Given that the sun completes 3 revolutions in 1096 days, the moon completes 5 revolutions in 137 days, Mars completes 1 revolution in 185 days, Mercury completes 13 revolutions in 1096 days, Jupiter completes 3 revolutions in 10960 days, and Saturn completes 1 revolution in 10960 days, find the number of days since the *Kalpa*.

Given that the sun completes 3 revolutions in 1096 days, the moon completes 5 revolutions in 137 days, Mars completes 1 revolution in 185 days, Mercury completes 13 revolutions in 1096 days, Jupiter completes 3 revolutions in 10960 days, and Saturn completes 1 revolution in 10960 days, find the number of days since the *Kalpa*.

Exercise 5.16. There have been many incorrect conjectures over the years regarding primes. Prove that the following claims are all wrong.

(a) In 1556, Italian mathematician Tartaglia claimed that the sums

$$1+2+4 \quad , \quad 1+2+4+8 \quad , \quad 1+2+4+8+16 \quad , \quad \ldots$$

alternate between being prime and being composite.

(b) In 1509, French mathematician Charles de Bovelles claimed that for every positive integer n, one or both of $6n-1$ and $6n+1$ is prime.

Exercise 5.17. In 1723, Leonhard Euler wrote,

> I derived these results from the elegant theorem, of whose truth I am certain, although I have no proof: $a^n - b^n$ is divisible by the prime $n+1$, if neither a nor b is.

Shortly afterward, Euler discovered a proof himself. Show how Fermat's little theorem implies this result.

Exercise 5.18. In 1644, Marin Mersenne asked what the smallest natural number is that has 60 positive divisors. Answer his question.

Exercise 5.19. Show that the converse to Fermat's little theorem is false. Hint: Consider $2^{340} \pmod{341}$.

Exercise 5.20. In this exercise, you will investigate a generalization to Fermat's little theorem.

(a) Use the Internet to look up *Euler's theorem*.[40] State the theorem and explain why it is a generalization of Fermat's little theorem.

(b) Let φ be the Euler totient function. Determine $\varphi(6)$, $\varphi(7)$ and $\varphi(8)$.

(c) Euler's theorem claims that $a^{\varphi(8)} \equiv 1 \pmod 8$ for $a = 1$, 3 and 5. Verify each of these without invoking the theorem.

(d) Prove that if p is a prime and n is a positive integer, then

$$\varphi(p^n) = p^{n-1}(p-1).$$

[40] Note: It is sometimes called *Euler's totient theorem* to distinguish it from the dozens of other "Euler's theorem"s.

Exercise 5.21. For a positive integer n, the function $d(n)$ counts the number of positive divisors of n. For example, there are four positive divisors of 6 (1, 2, 3 and 6), and hence $d(6) = 4$.

(a) Determine $d(7)$, $d(8)$ and $d(24)$.

(b) This function was first introduced by Italian mathematician Gerolamo Cardano in 1537. He noted that if $p_1, p_2, \ldots, p_k$ are distinct primes and $n = p_1 p_2 \cdots p_k$, then

$$d(n) - 1 = 1 + 2 + 2^2 + 2^3 + \cdots + 2^{k-1}.$$

Prove this result.

Exercise 5.22. Come up with your own homework problem that uses the information from this chapter. Make sure it is different than all of the questions above and have its level of difficulty be at about the level as the above. Then, answer your question.

Exercise 5.23. Show how Fermat used his method of infinite descent to prove the $n = 4$ case of Fermat's last theorem.

Exercise 5.24. Write a 250 word essay on the work of Indian mathematician Aryabhata I.

Exercise 5.25. Write a 250 word essay on how unsolved problems affect the direction of mathematics and of math history.