Exercise 5.1. Come up with your own homework problem that uses the information from this chapter. Make sure it is different than all of the questions below and have its level of difficulty be at about the level as the below. Then, answer your question.

In Python write a script to produce perfect numbers via brute force and via the theorem provided in class.

What is the Big-O of solving it via Brute force VS using the theorem

Brute-force : BigO(n)

```
def isPerfect(n): # What is the big O of this function? A: O(n)
    sum = 0
    for i in range(1, n):
        if n % i == 0:
            sum += i
    return sum == n
```

theorem

BigO

log(n)^2

```
def is_prime(n):
    if n == 1:
        return False
    if n == 2:
        return True
    if n % 2 == 0:
        return False
    for i in range(3,int(math.sqrt(n))+1,2):
        if n % i == 0:
            return False
    return True

def is_perfect(n): # Euclid-Euler theorem Big O(log(n)^2)
    if n == 1:
        return False
    if n == 2:
        return True
    if n % 2 == 1:
        return False
    if not is_prime(n):
        return False
    m = 2**(n-1)
    return (m % n) == 1
```

Exercise 5.2.

(a) Prove that $\sqrt{3}$ is irrational.

(a) Like, $\sqrt{2}$, we can prove that $\sqrt{3}$ is irrational via contradiction

we assume that $\sqrt{3}$ is rational (a number that can be represented in form $\frac{a}{b}$ where a and b are integers and $b \neq 0$)

$$\sqrt{3} = \frac{a}{b} \qquad a, b \in Z, b \neq 0$$

We can take it as a fraction in its lowest terms (AKA a fully reduced fraction)

- a & b have no factors in common except for 1

$$gcd(a, b) = 1$$

Now we can show that the given assumptions forces a contradiction

$$\sqrt{3} = \frac{a}{b} \Rightarrow \sqrt{3}b = a \Rightarrow \boxed{3b^2 = a^2}$$

'b' is an integer so, b^2 is also an integer i.e. b^2 is a multiple of 3

By definition 3 divides a^2

$$3|a \cdot a$$

A prime number divides a product must divide one factor or the other

$3|a \Rightarrow a = 3k$ for some integer of k

by definition of divides... 'a' is an integer of multiples of 3

plug back into the original equation

$$3b^2 = a^2$$

$$\Rightarrow 3b^2 = (3k)^2$$

$$\Rightarrow 3b^2 = 9k^2$$

$$\Rightarrow \frac{3b^2}{3} = \frac{9k^2}{3}$$

$$\Rightarrow b^2 = 3k^2$$

Same situation

Since b^2 is an integer multiple of 3 this means 3 divides b^2

$3|b^2 \Rightarrow \boxed{3|b}$ 3 divides 'b' gives us our contradiction

we assume that $\sqrt{3}$ was rational (i.e. a ratio of integers in lowest terms) that the numerator and denominator had no common factors other than one, but assuming so forces 3 to be a factor of both a&b

$gcd(a, b) = 1 \leftarrow$ Thus breaking the common factor other than 1... A contradiction!

The $\sqrt{3}$ is irrational

(b) True or false: $\sqrt{n}$ is irrational for every positive integer n. Justify your answer.

False, no, the square root of some positive integer is rational... A rational number is a number that can be expressed as the quotient of integers (say $\frac{a}{b}$), where the denominator (b) is not zero.

Rational roots: Square roots of perfect squares (1, 2, 4, 9, 16, ...)

Exercise 5.3.

(a) Prove that 496 is a perfect number using the definition of a perfect number.

(b) Prove that 496 is a perfect number using a theorem from this chapter.

(c) Prove that 495 is not a perfect number.

(a) Show that the sum of its proper divisors (excluding itself) equals 496

Perfect Numbers

- **Definition.** The *proper divisors* of a number N are the positive numbers less than N that divide N.
- Example: The proper divisors of 10 are 1, 2, and 5.
- Example: The proper divisors of 12 are 1, 2, 3, 4 and 6.
- **Definition.** A number N to be *perfect* if N's proper divisors sum to N.

1. First find the divisors of 496

The divisors of 496 are: 1, 2, 4, 8, 16, 31, 62, 124, 248, 496

2. Add up the proper divlsors (all the divisors except for 496 itself)

The proper divisors of 496 are: 1, 2, 4, 8, 16, 31, 62, 124, 248

$1+2+4+8+16+31+62+124=492$

The sum of the proper divisors of 496 is 496; therefore, 496 meets the definition of a perfect number, as the sum of its proper divisors equal itself. So, 496 is a perfect number.

(b) If $2^n - 1$ is a prime number, then $2^{n-1}(2^n - 1)$ is a perfect number.

Demonstrate that 496 can be expressed in the form $2^{n-1}(2^n-1)$, where both 2^{n-1} and $2^n - 1$ are prime numbers.

Find the values of n, 2^{n-1} and $2^n - 1$ for 496

We want 2^{n-1} to be a power of 2 AND

We want $2^n - 1$ to be a prime number

Where we can express 496 in form $2^{n-1}(2^n-1)$

$\Rightarrow 496 = 1 \times 496$

$\Rightarrow 496 = 2 \times 248$

$\Rightarrow 496 = 4 \times 124$

$\Rightarrow 496 = 8 \times 62$

$\Rightarrow 496 = 16 \times 31$

$496 = 2^{5-1}(2^5 - 1)$

(c) The proper divisors of 495 are the divisors of 495 excluding 495 itself

The proper divisors of 495 are: 1, 3, 5, 9, 11, 15, 33, 45, 55, 99, and 156

we know that 495 is not a proper number by definition for that

$1+3+5+9+11+15+33+45+99+156 = 432, 432 \neq 495$

Perfect Numbers

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Exercise 5.12. Consider the following problem.

> There are certain things whose number is unknown. If we count them by twos, we have one left over; by fives, we have four left over; and by elevens, eight are left over. How many things are there?

Explain how Sun Zi would have solved this. You may use modern notation.

There are certain things whose number is unknown

When we count by these things by twos, we have one left over -> $x \equiv 1 \pmod 2$ * implies odd

When we count by these things by fives, we have four left over -> $x \equiv 4 \pmod 5$ * implies that it ends in 4 or 9

When we count by these things by elevens, eight are left over -> $x \equiv 8 \pmod{11}$ * Add or subtract multiples of 11 from 8

Possible values are...

$x \equiv 1 \pmod 2 \Rightarrow 1, 3$

$x \equiv 4 \pmod 5 \Rightarrow 4, 9$

$x \equiv 8 \pmod{11} \Rightarrow 8, 19, 30, 41, \ldots$

19 meets the requirements

* is odd
* ends with 9 because $x = 4 + 5(\underline{k}) \rightarrow 4 + 5(1) = 9$
* conforms to $x = 8 + 11(\underline{k}) \rightarrow 8 + 11(1) = 11$

*INTEGER K