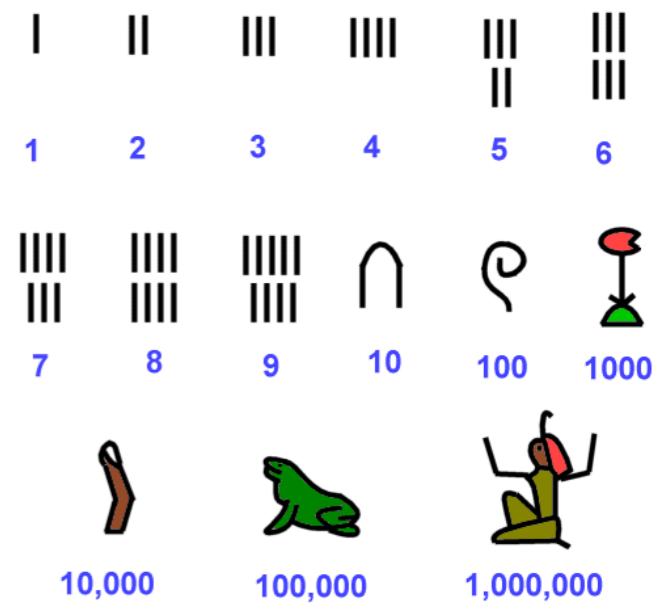
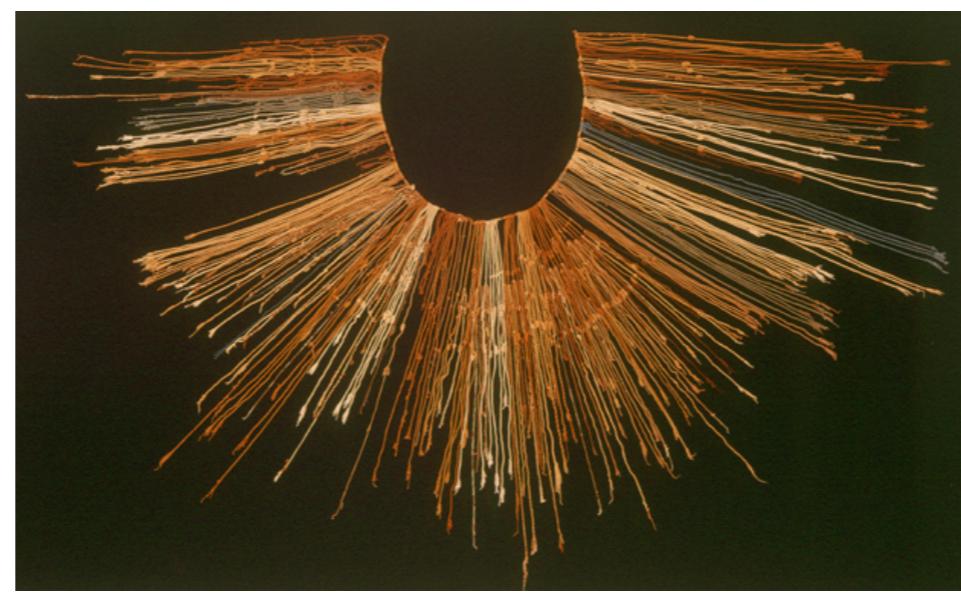


The History of Mathematics

Math 190, Professor Jay Cummings



Syllabus Key Points

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- Name: Jay Cummings

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- Name: Jay Cummings
- Email: Jay.Cummings@csus.edu

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- Name: Jay Cummings
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- Office: Shasta 253

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- Name: Jay Cummings
- Email: Jay.Cummings@csus.edu
- Office: Shasta 253
- Office Hours: Monday 10–11:30, Thursday 12-1:20

Syllabus Key Points

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 - Written Homeworks (25%). Due on Thursdays about every other week

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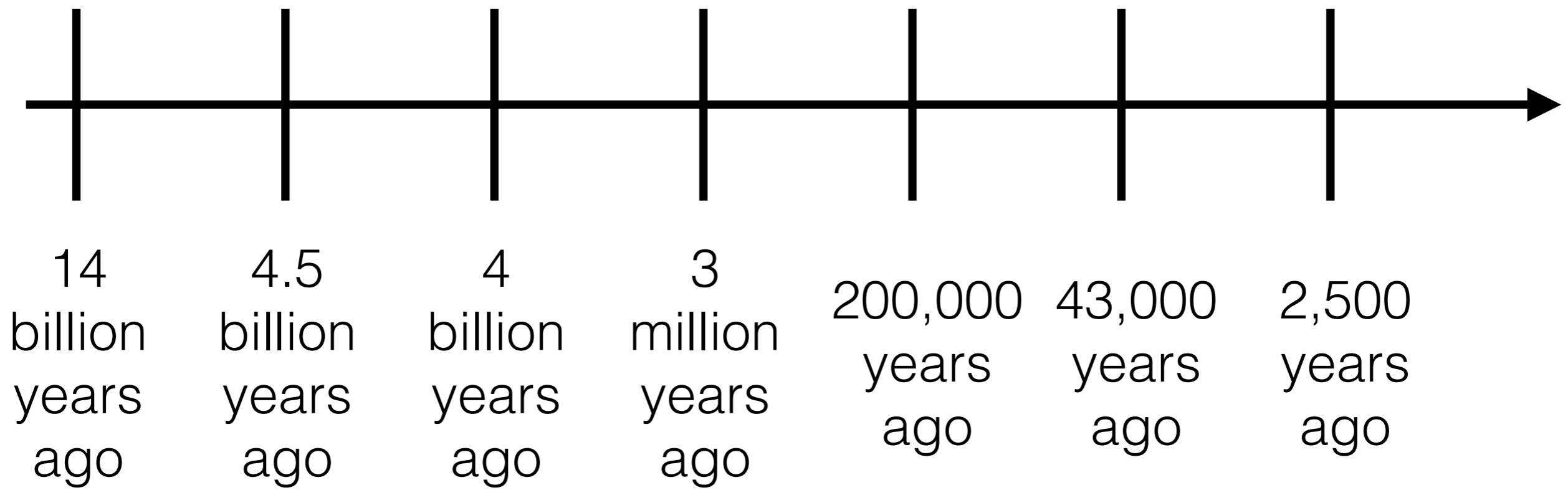
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- Attendance/Participation (10%). Mostly graded on showing up and paying attention.

When to Begin?

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- Talk to a classmate about when you think the history of math began.

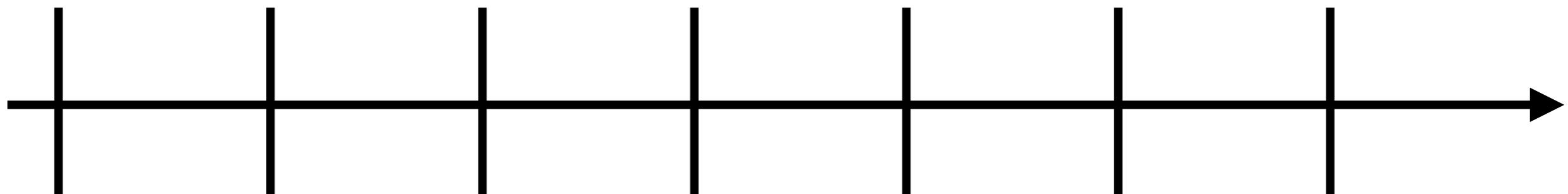
A Brief History of Time



A Brief History of Time



Universe
born



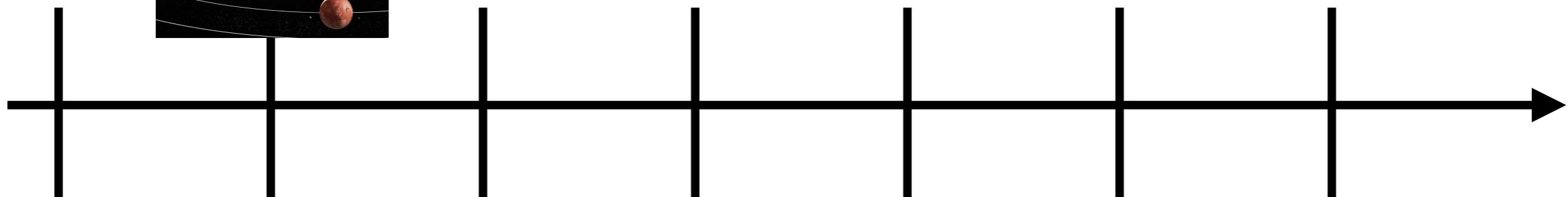
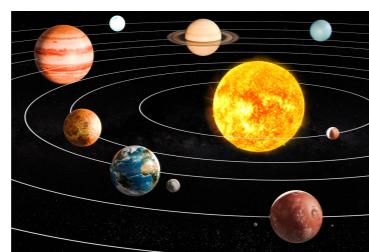
14 billion years ago	4.5 billion years ago	4 billion years ago	3 million years ago	200,000 years ago	43,000 years ago	2,500 years ago
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A Brief History of Time



Solar
system

Universe
born
born



14
billion
years
ago

4.5
billion
years
ago

4
billion
years
ago

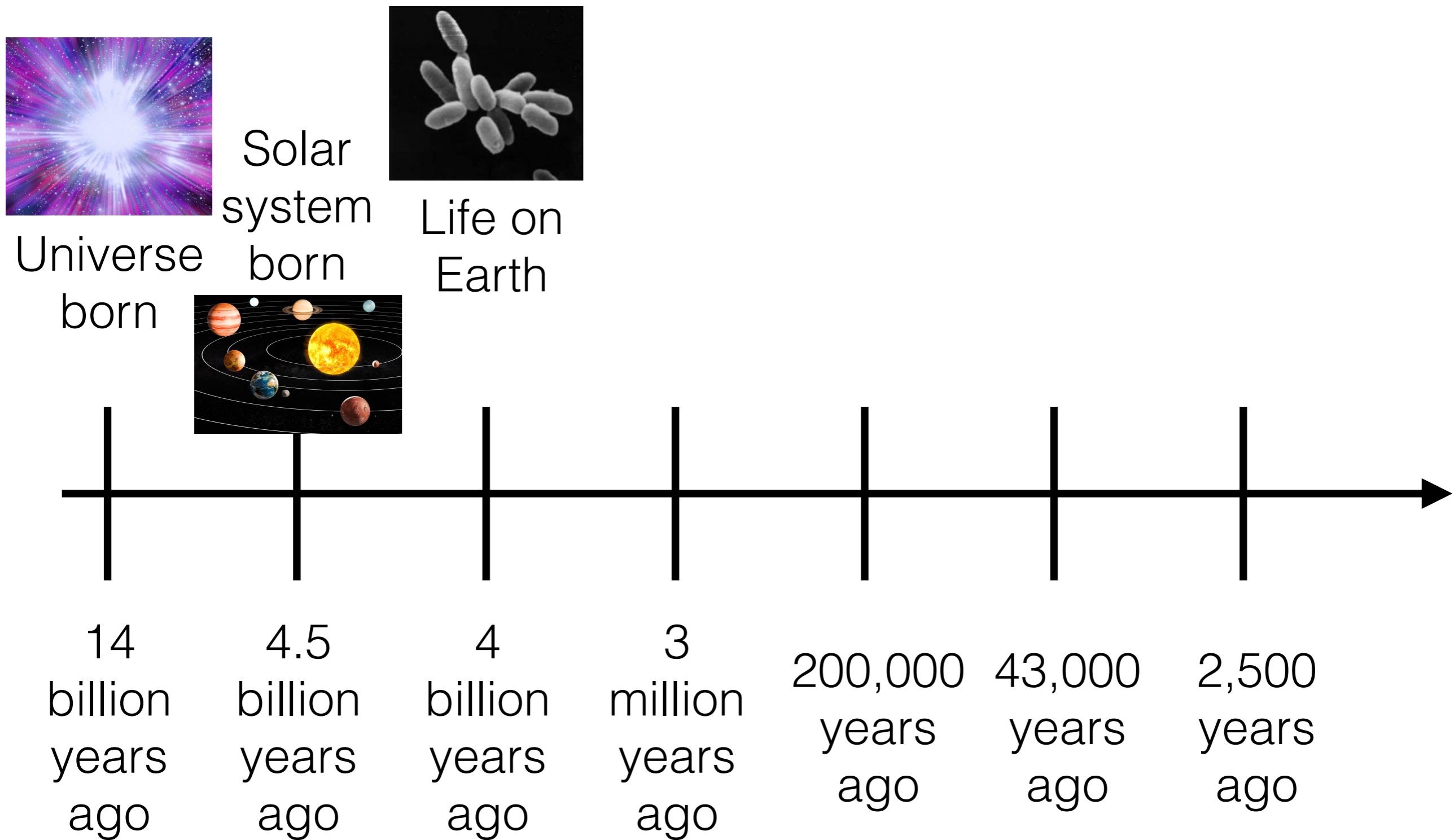
3
million
years
ago

200,000
years
ago

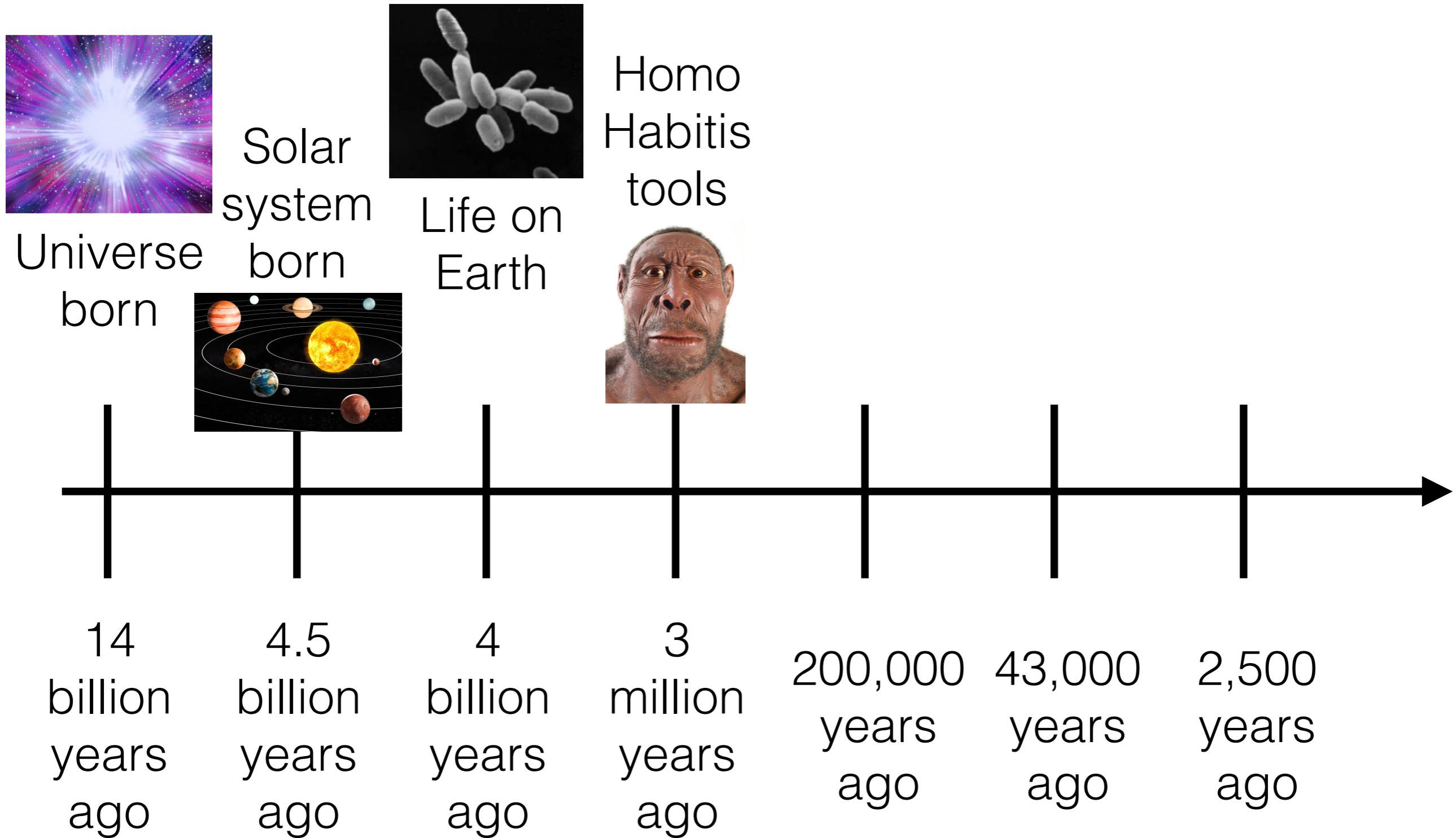
43,000
years
ago

2,500
years
ago

A Brief History of Time



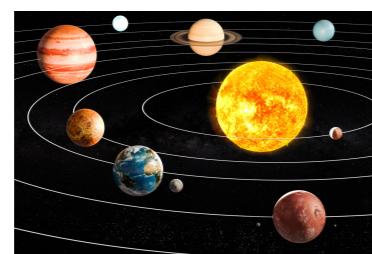
A Brief History of Time



A Brief History of Time



Universe
born



Solar
system
born



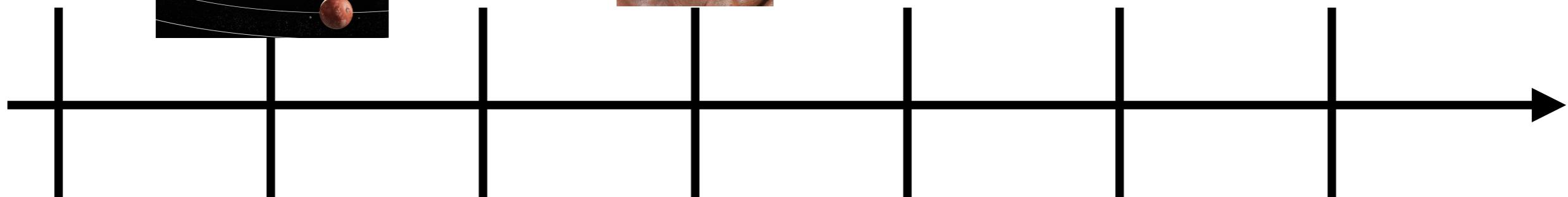
Life on
Earth



Homo
Habilis
tools



Homo
Sapiens,
fire



14
billion
years
ago

4.5
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ago

4
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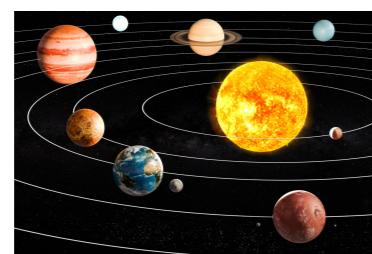
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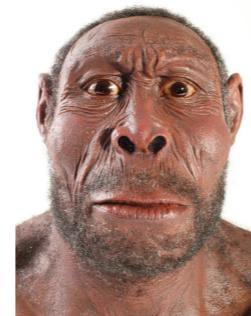


Solar
system
born



Life on
Earth

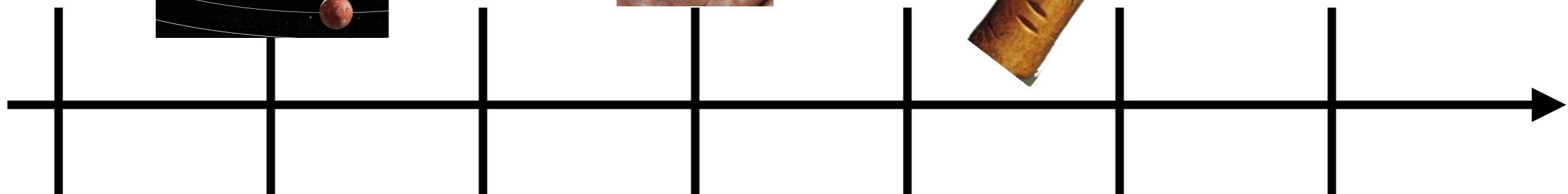
Homo
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Tally
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on bone



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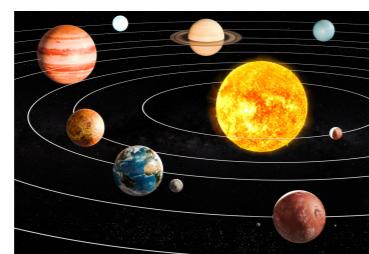
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A Brief History of Time



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Solar system born



Life on Earth

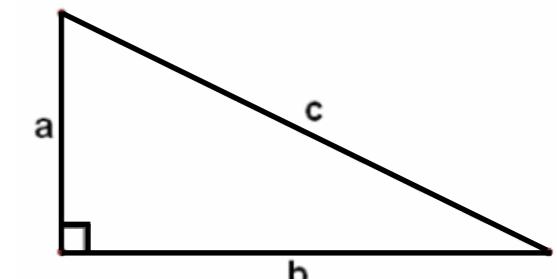
Homo Habitus tools



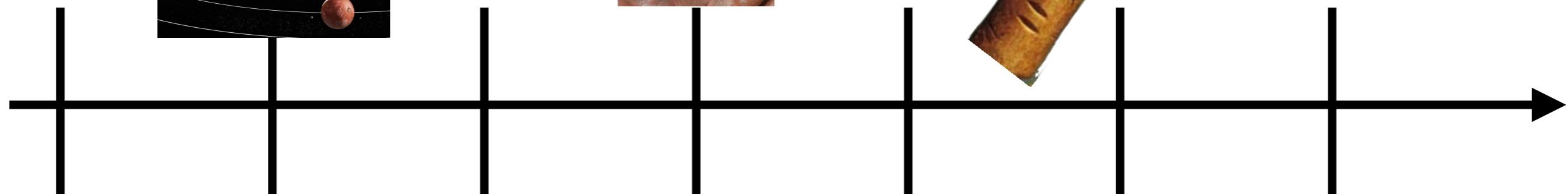
Homo Sapiens, fire



Tally marks on bone



Pythag theorem proved



14 billion years ago

4.5 billion years ago

4 billion years ago

3 million years ago

200,000 years ago

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2,500 years ago

Lebombo Bone



Lebombo Bone

- From ~41,000 BC.
- 29 clear notches.



Lebombo Bone



Ishango Bone

Ishango Bone

- From ~18,000 BC.

Ishango Bone

- From ~18,000 BC.
- Has 3 columns, with 48, 60 and 60 notches.

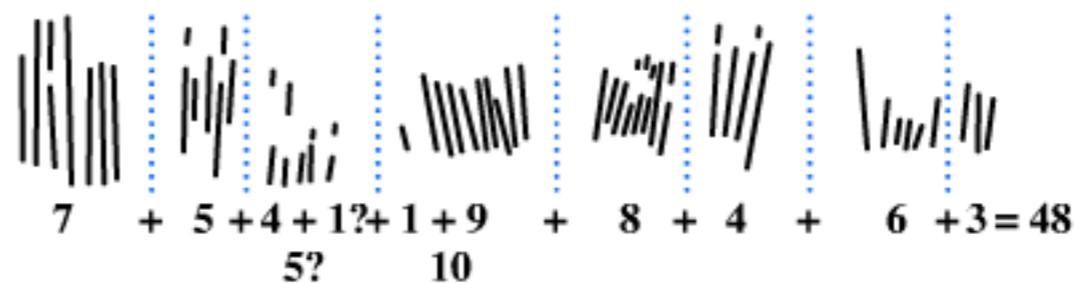
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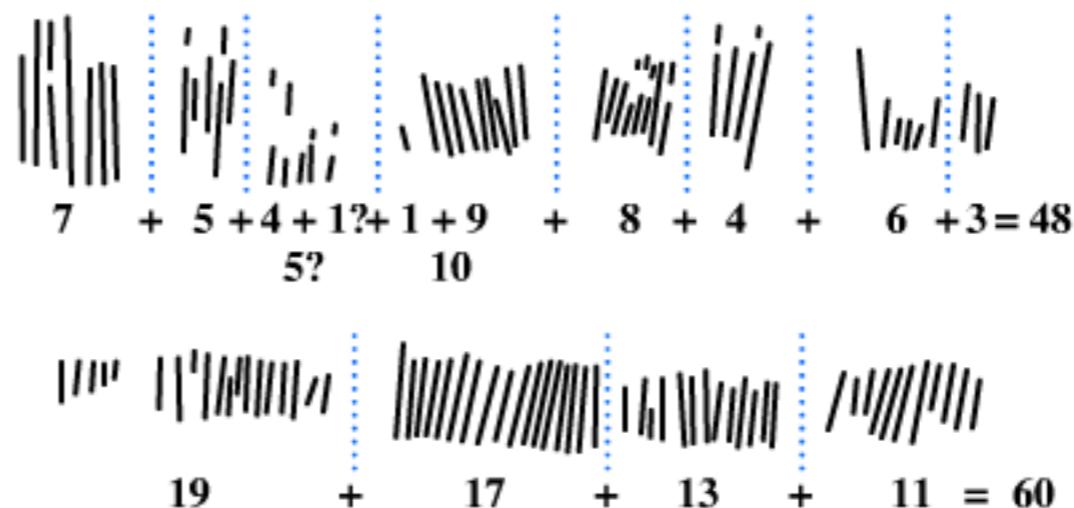
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$$\begin{array}{r} \text{7} \quad + \quad 5 + 4 + 1? + 1 + 9 \quad + \quad 8 \quad + \quad 4 \quad + \quad 6 \quad + 3 = 48 \\ \text{5?} \qquad \qquad \qquad \text{10} \qquad \end{array}$$
$$\begin{array}{r} \text{19} \quad + \quad 17 \quad + \quad 13 \quad + \quad 11 = 60 \\ \text{19} \qquad \qquad \qquad \text{17} \qquad \qquad \qquad \text{13} \qquad \qquad \qquad \text{11} \end{array}$$
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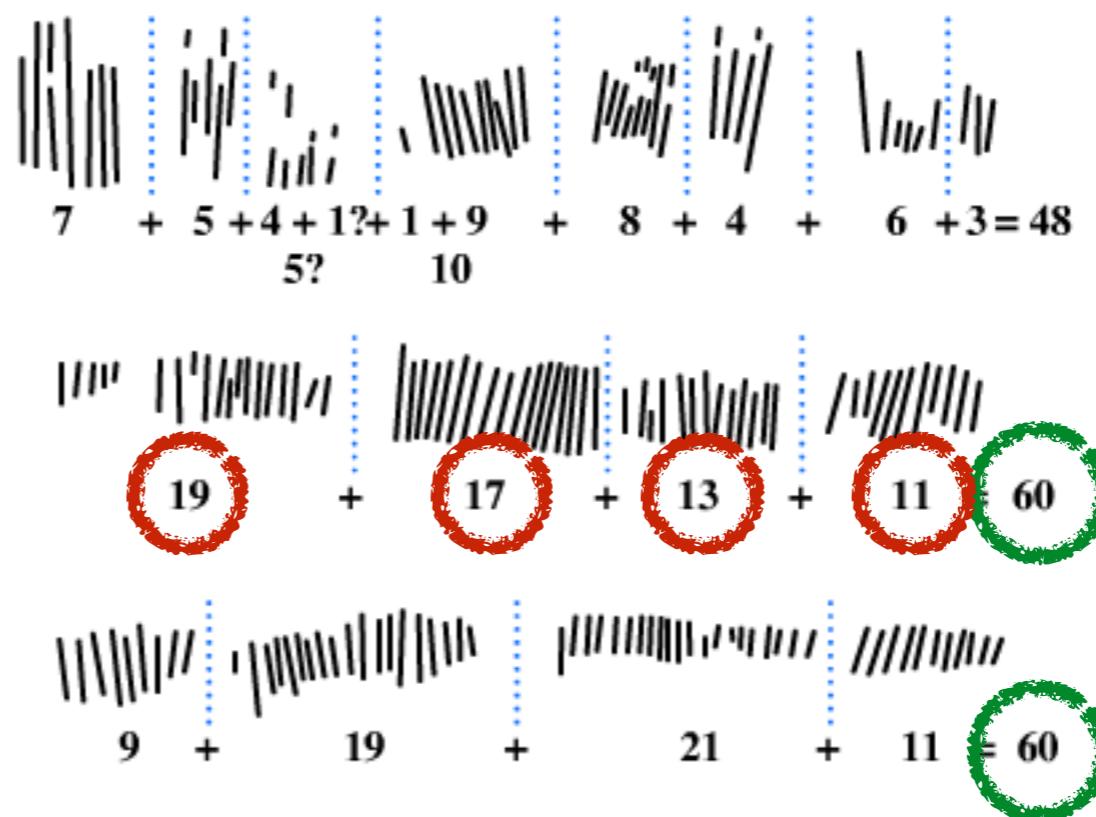
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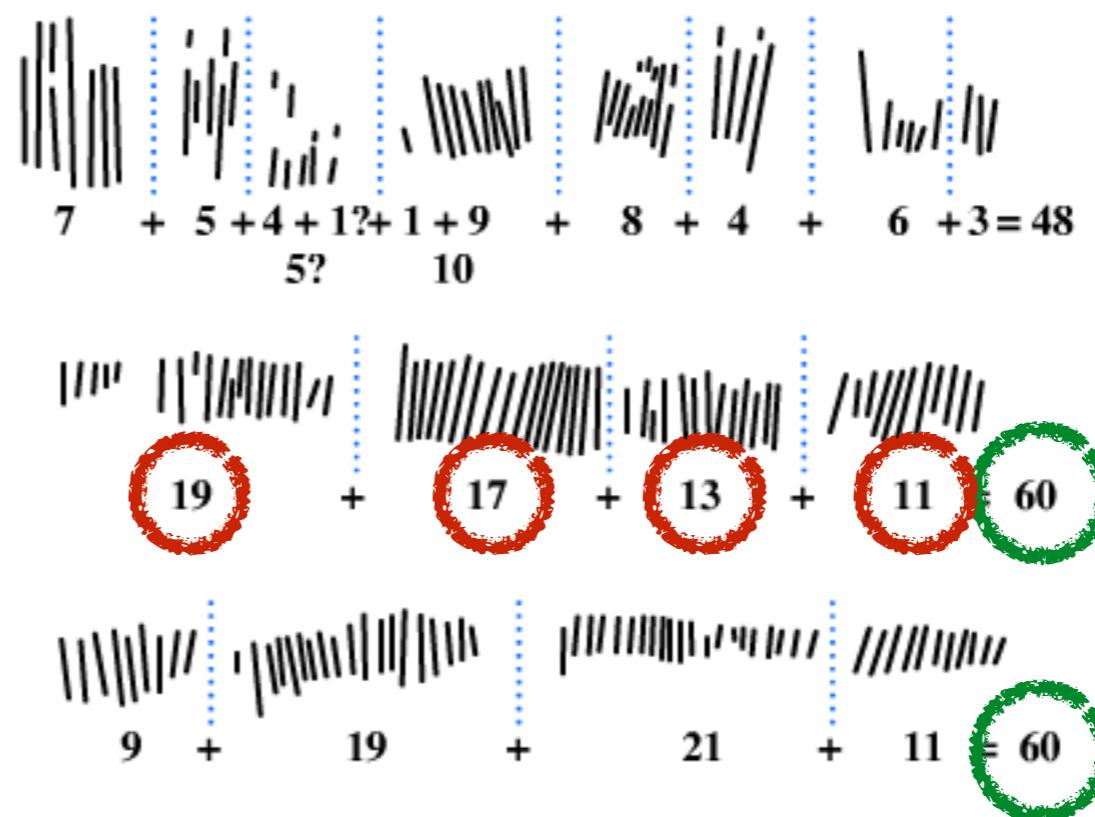
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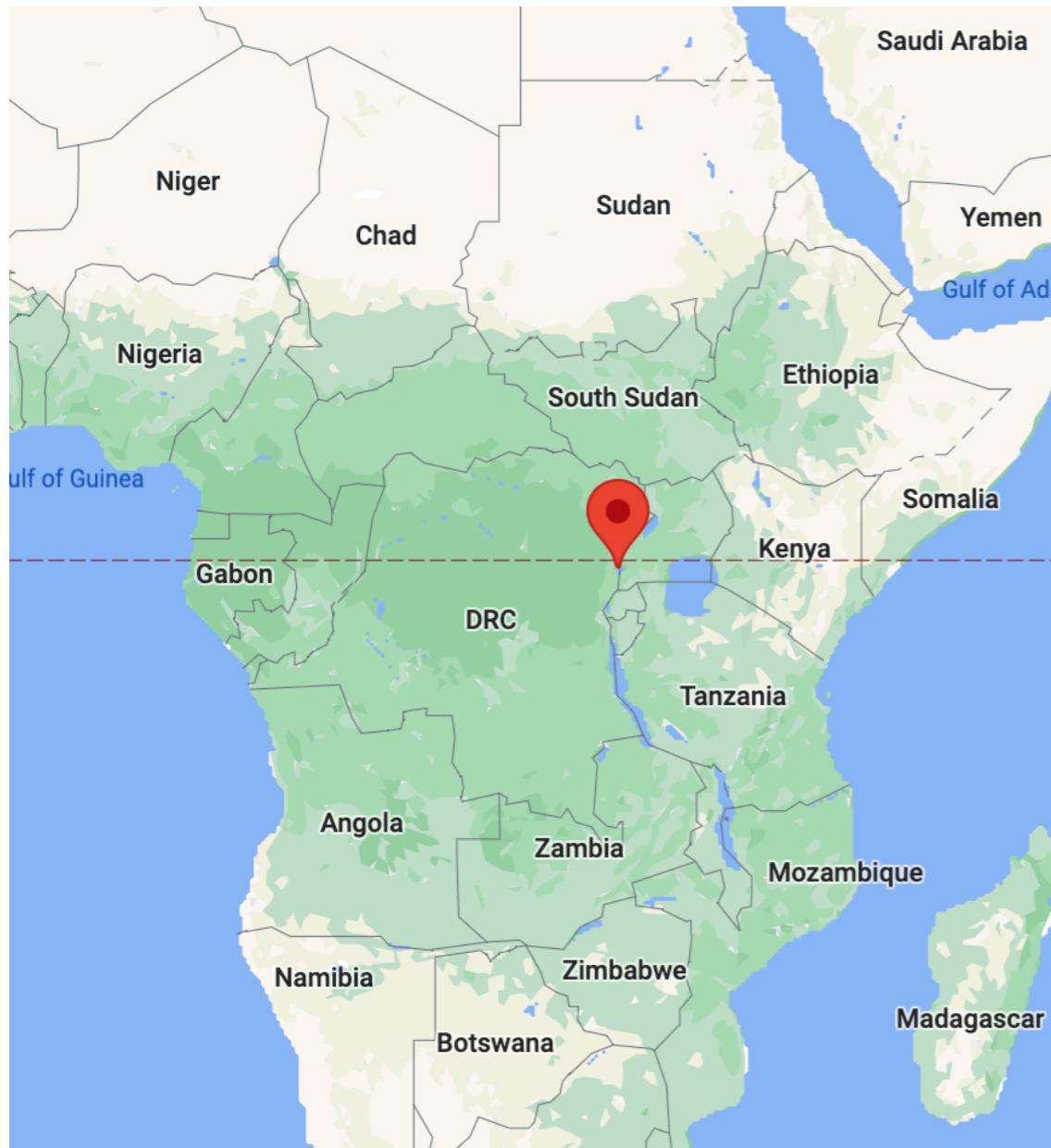
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Both bones are the fibula of baboons!

Ishango Bone



How Humans Count

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Question: If counting is the start of math history, does it have to be humans counting?

Number Bases

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- Base 4: Papua New Guinea, my son at 22 months
- Base 2 (binary system): Computers

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Converting the number $(a_k a_{k-1} \dots a_2 a_1 a_0)_b$ to base 10 gives

$$a_k b^k + a_{k-1} b^{k-1} + \dots + a_2 b^2 + a_1 b^1 + a_0 \cdot b^0.$$

Practice

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Example. $(123)_4$ is a base 4 number.

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Solution.

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$$= 16 + 8 + 3$$

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$$= 27$$

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So,

$$(34)_{10} = (1021)_3.$$

4 Ways People Write Numbers

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Example: Chinese-Japanese numeral system

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1 = 一	10 = 十	100 = 一百	1,000 = 一千
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3 = 三	30 = 三十	300 = 三百	3,000 = 三千
4 = 四	40 = 四十	400 = 四百	4,000 = 四千
5 = 五	50 = 五十	500 = 五百	5,000 = 五千
6 = 六	60 = 六十	600 = 六百	6,000 = 六千
7 = 七	70 = 七十	700 = 七百	7,000 = 七千
8 = 八	80 = 八十	800 = 八百	8,000 = 八千
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$$5,062 = 5 \cdot 1,000 + 6 \cdot 10 + 2$$

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Example: Chinese-Japanese numeral system

$$5,062 = 5 \cdot 1,000 + 6 \cdot 10 + 2$$

= 五千六十二

4 Ways People Write Numbers

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- Simple Grouping System: Add everything up.

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Example: Egyptian hieroglyphics

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$10,000 = \emptyset$, $100,000 = \text{B}$, $1,000,000 = \text{A}$

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321

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$10,000 = \text{J}$, $100,000 = \text{B}$, $1,000,000 = \text{H}$

$321 = \wp\wp\wp \cap\cap |$

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$1 = |$, $10 = \cap$, $100 = \text{ꝛ}$, $1,000 = \text{ҝ}$

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$321 = \text{ꝛꝛꝛ} \cap\cap |$
 $= \text{ꝛꝛꝛ} | \cap\cap$

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$$321 = \text{ꝛꝛꝛ} \cap\cap |$$

$$= \text{ꝛꝛꝛ} | \cap\cap$$

$$= \text{ꝛ}\cap\text{ꝛ} | \text{ꝛ}\cap$$

4 Ways People Write Numbers

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- Ciphered systems: It's... complicated. For a base b , we need symbols for

4 Ways People Write Numbers

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$$1, 2, \dots, b - 1$$

$$b, 2b, \dots, (b - 1)b$$

$$b^2, 2b^2, \dots, (b - 1)b^2$$

$$b^3, 2b^3, \dots, (b - 1)b^3$$

⋮

4 Ways People Write Numbers

- Ciphered systems: It's... complicated. For a base b , we need symbols for

$$1, 2, \dots, b - 1$$
$$b, 2b, \dots, (b - 1)b$$
$$b^2, 2b^2, \dots, (b - 1)b^2$$
$$b^3, 2b^3, \dots, (b - 1)b^3$$
$$\vdots$$

Example: Ionic Greek system

4 Ways People Write Numbers

$1 = \alpha$ (alpha)	$10 = \iota$ (iota)	$100 = \rho$ (rho)
$2 = \beta$ (beta)	$20 = \kappa$ (kappa)	$200 = \sigma$ (sigma)
$3 = \gamma$ (gamma)	$30 = \lambda$ (lambda)	$300 = \tau$ (tau)
$4 = \delta$ (delta)	$40 = \mu$ (mu)	$400 = \upsilon$ (upsilon)
$5 = \varepsilon$ (epsilon)	$50 = \nu$ (nu)	$500 = \phi$ (phi)
$6 = \varsigma$ (vau)	$60 = \xi$ (xi)	$600 = \chi$ (chi)
$7 = \zeta$ (zeta)	$70 = \omicron$ (omicron)	$700 = \psi$ (psi)
$8 = \eta$ (eta)	$80 = \pi$ (pi)	$800 = \omega$ (omega)
$9 = \theta$ (theta)	$90 = \kappa$ (koppa)	$900 = \lambda$ (sampi)

Example: Ionic Greek system

Question: Which
system is best?

Zero



Zero



The Indians and the Mayans were the two civilizations to discover zero and recognize its deep potential as:



Zero



The Indians and the Mayans were the two civilizations to discover zero and recognize its deep potential as:

- A symbol



Zero



The Indians and the Mayans were the two civilizations to discover zero and recognize its deep potential as:

- A symbol
- A number



Zero



The Indians and the Mayans were the two civilizations to discover zero and recognize its deep potential as:

- A symbol
- A number
- A magnitude



Zero



The Indians and the Mayans were the two civilizations to discover zero and recognize its deep potential as:

- A symbol
- A number
- A magnitude
- A direction separator



Zero



The Indians and the Mayans were the two civilizations to discover zero and recognize its deep potential as:

- A symbol
- A number
- A magnitude
- A direction separator
- A place-holder



Zero



The Indians and the Mayans were the two civilizations to discover zero and recognize its deep potential as:

- A symbol
- A number
- A magnitude
- A direction separator
- A place-holder
- An idea



Zero



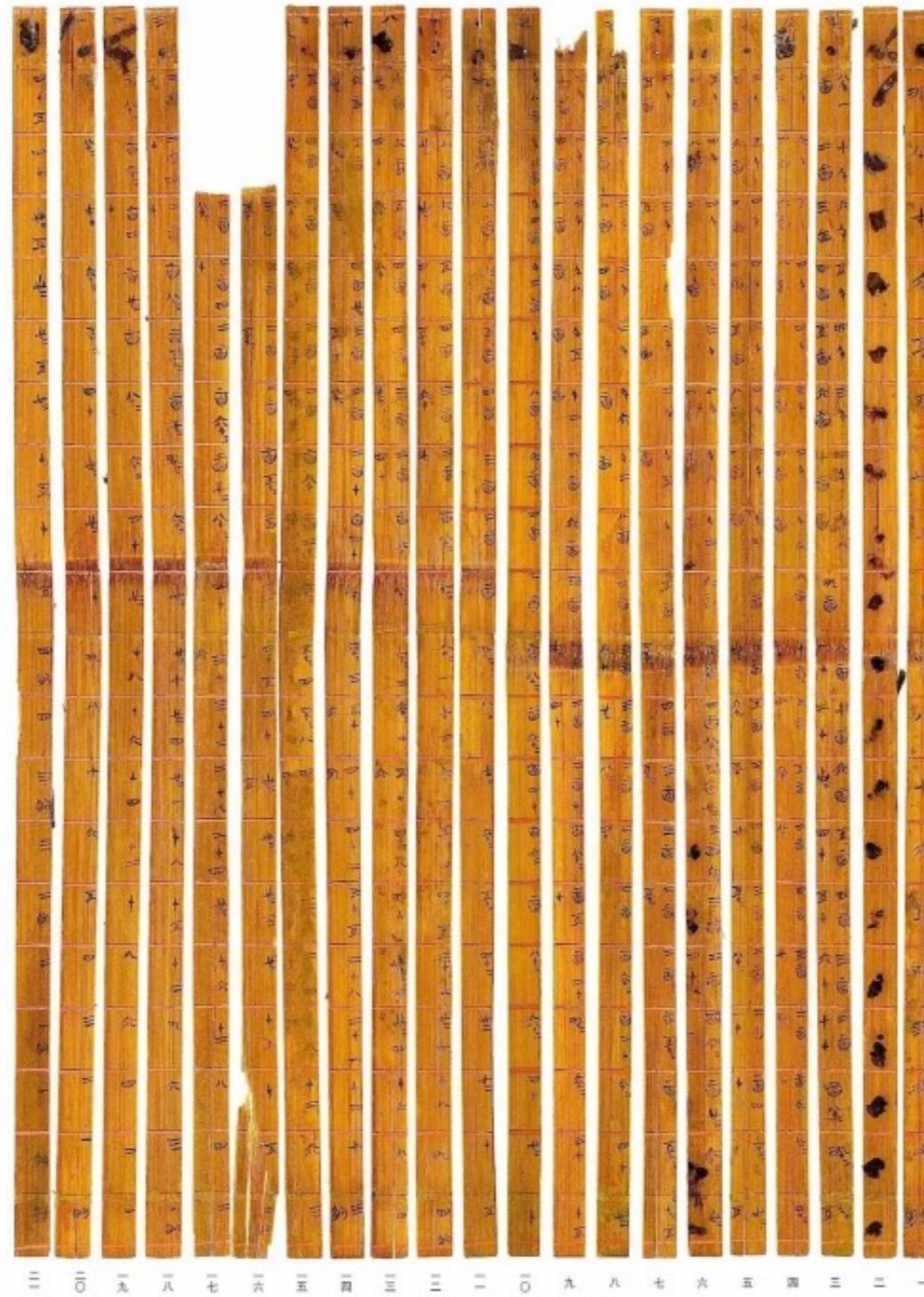
The Indians and the Mayans were the two civilizations to discover zero and recognize its deep potential as:

- A symbol
- A number
- A magnitude
- A direction separator
- A place-holder
- An idea

This was a big deal.

Think Like A
Math Historian

Think Like A Math Historian



Think Like A Math Historian

8	5	4	20	9	10	40	60	2	90	60	5400	4800	180
720	450	360	1800	810	900	3600	5400	2	900	5400	4800	4200	160
640	400	320	1600	720	800	3200	4200	180	800	5400	4800	4200	160
560	350	360	1400	630	700	2800	3600	160	700	5400	4800	4200	160
480	300	320	1200	540	600	2400	3600	140	600	5400	4800	4200	140
400	250	320	1000	450	500	2000	3000	120	500	5400	4800	4200	120
320	200	280	800	360	400	1600	2400	100	400	5400	4800	4200	100
240	150	240	600	270	300	1200	2000	80	300	5400	4800	4200	80
160	100	240	400	180	200	800	1600	60	200	5400	4800	4200	60
80	50	200	200	90	100	400	1200	40	100	5400	4800	4200	40
72	45	160	180	10	120	360	1800	80	80	5400	4800	4200	20
64	40	120	180	10	120	320	1500	60	70	5400	4800	4200	18
56	35	80	120	8	120	280	1200	40	60	5400	4800	4200	60
48	30	40	120	7	90	210	180	30	50	5400	4800	4200	40
40	25	36	100	6	80	140	1500	20	40	5400	4800	4200	30
32	20	32	80	5	70	120	1200	12	30	5400	4800	4200	20
24	15	28	60	4	60	100	1000	10	20	5400	4800	4200	18
16	10	32	60	3	60	80	800	8	10	5400	4800	4200	16
8	5	27	40	2	40	60	600	6	9	5400	4800	4200	14
4	21/2	20	27	10	56	40	400	4	8	5400	4800	4200	12
		16	27	10	49	36	300	3	7	5400	4800	4200	10
		12	27	20	63	30	200	2	6	5400	4800	4200	8
		8	27	20	56	27	180	1	5	5400	4800	4200	6
		4	27	10	49	20	160	1	4	5400	4800	4200	4
		1	27	10	42	18	140	1	3	5400	4800	4200	2
			12	15	35	10	120		2	5400	4800	4200	1
			9	15	41/2	5	100			5400	4800	4200	
			6	12	28	1	80			5400	4800	4200	
			3	9	35	1/2	60			5400	4800	4200	
			1	6	41/2	1	40			5400	4800	4200	
			1/2	3	12	1/2	20			5400	4800	4200	
			1/2	1	28	1/2	180			5400	4800	4200	
			1/2	1/2	28	1/2	160			5400	4800	4200	
			1/2	1/2	21	1/2	140			5400	4800	4200	
			1/2	1/2	49	1/2	120			5400	4800	4200	
			1/2	1/2	42	1/2	100			5400	4800	4200	
			1/2	1/2	35	1/2	80			5400	4800	4200	
			1/2	1/2	41/2	1/2	60			5400	4800	4200	
			1/2	1/2	10	1/2	40			5400	4800	4200	
			1/2	1/2	10	1/2	20			5400	4800	4200	
			1/2	1/2	10	1/2	180			5400	4800	4200	
			1/2	1/2	10	1/2	160			5400	4800	4200	
			1/2	1/2	10	1/2	140			5400	4800	4200	
			1/2	1/2	10	1/2	120			5400	4800	4200	
			1/2	1/2	10	1/2	100			5400	4800	4200	
			1/2	1/2	10	1/2	80			5400	4800	4200	
			1/2	1/2	10	1/2	60			5400	4800	4200	
			1/2	1/2	10	1/2	40			5400	4800	4200	
			1/2	1/2	10	1/2	20			5400	4800	4200	
			1/2	1/2	10	1/2	180			5400	4800	4200	
			1/2	1/2	10	1/2	160			5400	4800	4200	
			1/2	1/2	10	1/2	140			5400	4800	4200	
			1/2	1/2	10	1/2	120			5400	4800	4200	
			1/2	1/2	10	1/2	100			5400	4800	4200	
			1/2	1/2	10	1/2	80			5400	4800	4200	
			1/2	1/2	10	1/2	60			5400	4800	4200	
			1/2	1/2	10	1/2	40			5400	4800	4200	
			1/2	1/2	10	1/2	20			5400	4800	4200	
			1/2	1/2	10	1/2	180			5400	4800	4200	
			1/2	1/2	10	1/2	160			5400	4800	4200	
			1/2	1/2	10	1/2	140			5400	4800	4200	
			1/2	1/2	10	1/2	120			5400	4800	4200	
			1/2	1/2	10	1/2	100			5400	4800	4200	
			1/2	1/2	10	1/2	80			5400	4800	4200	
			1/2	1/2	10	1/2	60			5400	4800	4200	
			1/2	1/2	10	1/2	40			5400	4800	4200	
			1/2	1/2	10	1/2	20			5400	4800	4200	
			1/2	1/2	10	1/2	180			5400	4800	4200	
			1/2	1/2	10	1/2	160			5400	4800	4200	
			1/2	1/2	10	1/2	140			5400	4800	4200	
			1/2	1/2	10	1/2	120			5400	4800	4200	
			1/2	1/2	10	1/2	100			5400	4800	4200	
			1/2	1/2	10	1/2	80			5400	4800	4200	
			1/2	1/2	10	1/2	60			5400	4800	4200	
			1/2	1/2	10	1/2	40			5400	4800	4200	
			1/2	1/2	10	1/2	20			5400	4800	4200	
			1/2	1/2	10	1/2	180			5400	4800	4200	
			1/2	1/2	10	1/2	160			5400	4800	4200	
			1/2	1/2	10	1/2	140			5400	4800	4200	
			1/2	1/2	10	1/2	120			5400	4800	4200	
			1/2	1/2	10	1/2	100			5400	4800	4200	
			1/2	1/2	10	1/2	80			5400	4800	4200	
			1/2	1/2	10	1/2	60			5400	4800	4200	

Think Like A Math Historian

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
4 ^{1/2}	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
3 ^{1/2}	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
2 ^{1/2}	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
1 ^{1/2}	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
1 ^{1/2}	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
1/4	1/2	1	1 ^{1/2}	2	2 ^{1/2}	3	3 ^{1/2}	4	4 ^{1/2}	5	10	15	20	25	30	35	40	45	1/2

Think Like A Math Historian

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
4 ^{1/2}	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
3 ^{1/2}	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
2 ^{1/2}	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
1 ^{1/2}	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
1 ^{1/2}	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
1/4	1/2	1	1 ^{1/2}	2	2 ^{1/2}	3	3 ^{1/2}	4	4 ^{1/2}	5	10	15	20	25	30	35	40	45	1/2

We will come back to this soon.

Arithmetic

Multiplication Tables

Multiplication Tables

- After counting comes arithmetic. Simple addition and subtraction could be done in head, on fingers, or with simple tools. Multiplication was probably much harder.

Multiplication Tables

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- First known multiplication table is a 4,000 year old Babylonian (base 60) multiplication table.

Multiplication Tables

- After counting and subtracting or with simple multiplication was much harder.
- First known multiplication table from Babylonian (but not the first multiplication table)
- c. Simple addition
• in head, on fingers, on fingers
ion was probably
is a 4,000 year old
ation table.



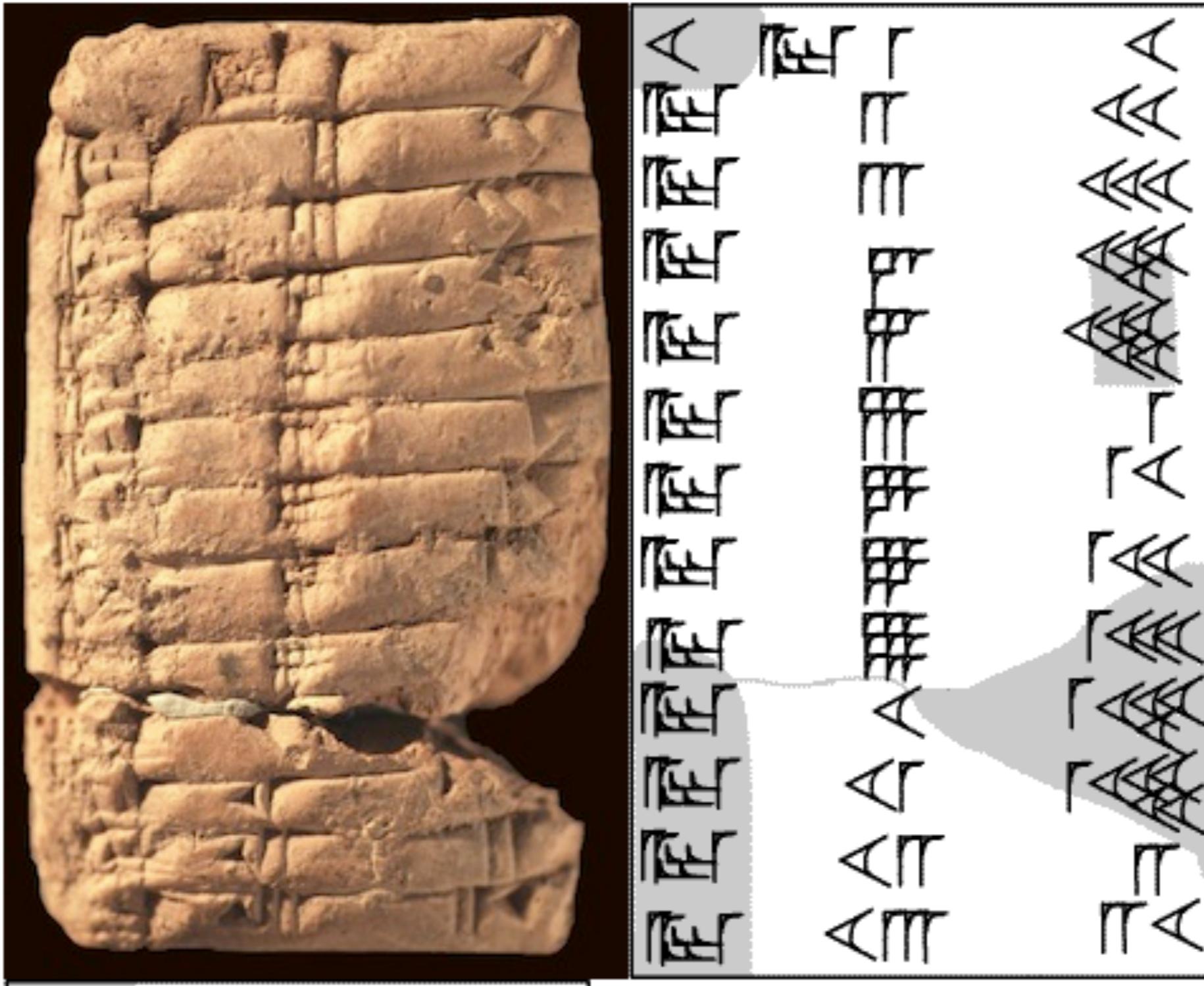
MS 3866
Multiplication table for $1.12(=72)$.
Babylonia, 19th c. BC

Multiplication Tables

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Multiplication Tables

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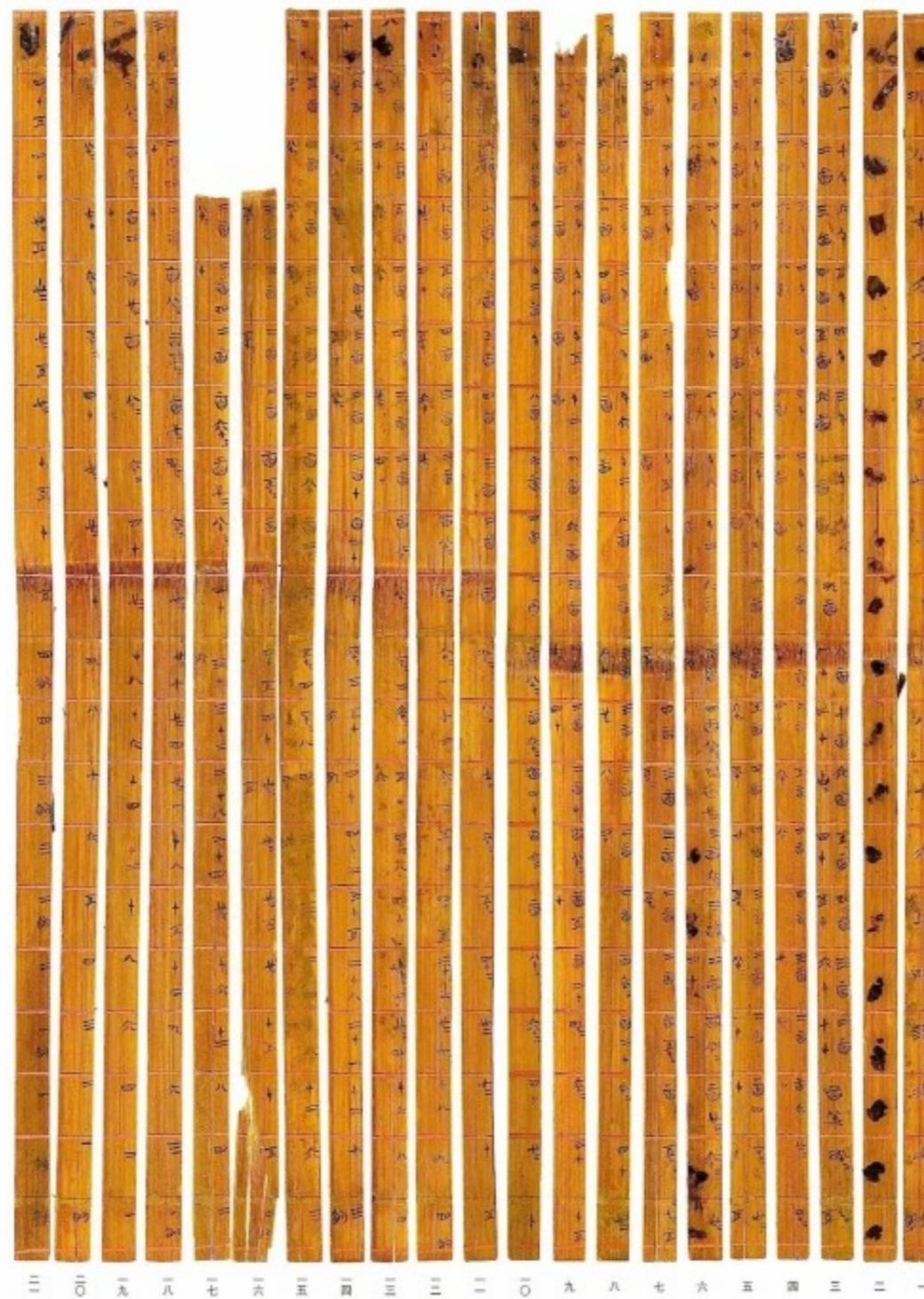
Multiplication Tables

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- First known *decimal* multiplication table is a 2,300 year old Chinese multiplication table.

Multiplication Tables

- After counting comes arithmetic. Simple addition and subtraction could be done in head, on fingers, or with simple tools. Multiplication was probably much harder.
- First known multiplication table is a 4,000 year old Babylonian (base 60) multiplication table.
- First known *decimal* multiplication table is a 2,300 year old Chinese multiplication table.
- It was discovered in 2009!

Tsinghua Bamboo Strips



Tsinghua Bamboo Slips

- An ancient table, written on bamboo strips and found in China, contains what is essentially a multiplication table.
- With a little extra work, one can use it to compute the product of any two integers or half integers from 0.5 to 99.5.
That is, you can multiply together any two numbers from this list: 0.5, 1, 1.5, 2, 2.5, ..., 99.5.
- This is the first known decimal multiplication table.

Tsinghua Bamboo Slips

- This multiplication table can compute the product of any two numbers from the set $\{0.5, 1, 1.5, 2, 2.5, 3, 3.5, \dots, 99.5\}$. How?

$1/2$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4\frac{1}{2}$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3\frac{1}{2}$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2\frac{1}{2}$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$

Think Like A
Math Historian

Tsinghua Bamboo Slips

- This multiplication table can compute the product of any two numbers from the set $\{0.5, 1, 1.5, 2, 2.5, 3, 3.5, \dots, 99.5\}$. How?

$1/2$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4\frac{1}{2}$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3\frac{1}{2}$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2\frac{1}{2}$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$

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- Example: $24 \cdot 36.5 = ?$

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4\frac{1}{2}$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3\frac{1}{2}$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2\frac{1}{2}$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$

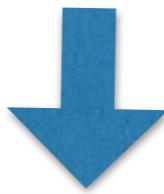
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- Example: $24 \cdot 36.5 = ?$

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4\frac{1}{2}$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3\frac{1}{2}$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2\frac{1}{2}$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$

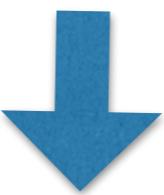
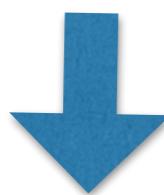
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• Example: $24 \cdot 36.5 = ?$

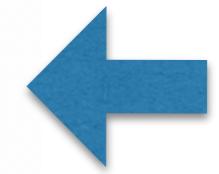
1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4\frac{1}{2}$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3\frac{1}{2}$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2\frac{1}{2}$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$

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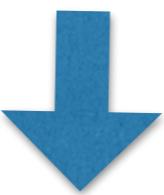
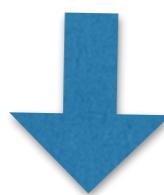


• Example: $24 \cdot 36.5 = ?$

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4\frac{1}{2}$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3\frac{1}{2}$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2\frac{1}{2}$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$

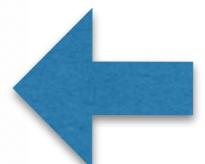
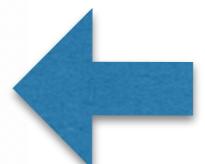


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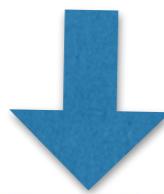


• Example: $24 \cdot 36.5 = ?$

$1/2$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4\frac{1}{2}$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3\frac{1}{2}$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2\frac{1}{2}$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$

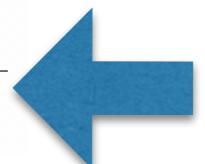
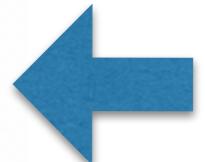
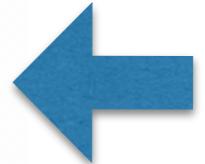


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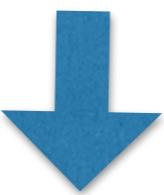
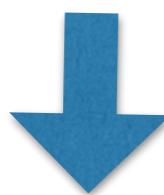


• Example: $24 \cdot 36.5 = ?$

$1/2$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4\frac{1}{2}$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3\frac{1}{2}$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2\frac{1}{2}$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$

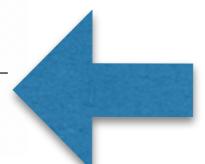
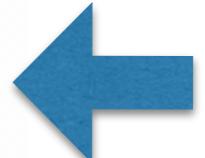


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• Example: $24 \cdot 36.5 = ?$

$1/2$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4\frac{1}{2}$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3\frac{1}{2}$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2\frac{1}{2}$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$



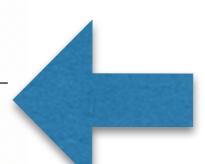
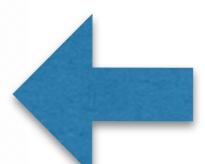
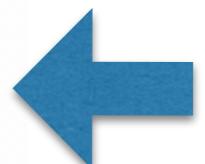
$$24 \cdot 36.5 = (20 + 4) \cdot (30 + 6 + 0.5)$$

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- Example: $24 \cdot 36.5 = ?$

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4\frac{1}{2}$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3\frac{1}{2}$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2\frac{1}{2}$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$



$$24 \cdot 36.5 = (20 + 4) \cdot (30 + 6 + 0.5)$$

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- Example: $24 \cdot 36.5 = ?$

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4\frac{1}{2}$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3\frac{1}{2}$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2\frac{1}{2}$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$

$$24 \cdot 36.5 = (20 + 4) \cdot (30 + 6 + 0.5)$$

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- Example: $24 \cdot 36.5 = ?$

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4\frac{1}{2}$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3\frac{1}{2}$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2\frac{1}{2}$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$

$$24 \cdot 36.5 = (20 + 4) \cdot (30 + 6 + 0.5)$$

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- Example: $24 \cdot 36.5 = ?$

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
4½	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
3½	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
2½	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
1½	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
½	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
¼	½	1	1½	2	2½	3	3½	4	4½	5	10	15	20	25	30	35	40	45	½

$$24 \cdot 36.5 = (20 + 4) \cdot (30 + 6 + 0.5)$$

Tsinghua Bamboo Slips

- Example: $24 \cdot 36.5 = ?$

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
4½	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
3½	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
2½	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
1½	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
½	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
¼	½	1	1½	2	2½	3	3½	4	4½	5	10	15	20	25	30	35	40	45	½

$$24 \cdot 36.5 = (20 + 4) \cdot (30 + 6 + 0.5)$$

Tsinghua Bamboo Slips

- Example: $24 \cdot 36.5 = ?$

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4\frac{1}{2}$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3\frac{1}{2}$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2\frac{1}{2}$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$\frac{1}{2}$

$$24 \cdot 36.5 = (20 + 4) \cdot (30 + 6 + 0.5)$$

$$= 20 \cdot 30 + 20 \cdot 6 + 20 \cdot 0.5 + 4 \cdot 30 + 4 \cdot 6 + 4 \cdot 0.5$$

Tsinghua Bamboo Slips

- Example: $24 \cdot 36.5 = ?$

1/2	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	
45	90	180	270	360	450	540	630	720	810	900	1800	2700	3600	4500	5400	6300	7200	8100	90
40	80	160	240	320	400	480	560	640	720	800	1600	2400	3200	4000	4800	5600	6400	7200	80
35	70	140	210	280	350	420	490	560	630	700	1400	2100	2800	3500	4200	4900	5600	6300	70
30	60	120	180	240	300	360	420	480	540	600	1200	1800	2400	3000	3600	4200	4800	5400	60
25	50	100	150	200	250	300	350	400	450	500	1000	1500	2000	2500	3000	3500	4000	4500	50
20	40	80	120	160	200	240	280	320	360	400	800	1200	1600	2000	2400	2800	3200	3600	40
15	30	60	90	120	150	180	210	240	270	300	600	900	1200	1500	1800	2100	2400	2700	30
10	20	40	60	80	100	120	140	160	180	200	400	600	800	1000	1200	1400	1600	1800	20
5	10	20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	10
$4\frac{1}{2}$	9	18	27	36	45	54	63	72	81	90	180	270	360	450	540	630	720	810	9
4	8	16	24	32	40	48	56	64	72	80	160	240	320	400	480	560	640	720	8
$3\frac{1}{2}$	7	14	21	28	35	42	49	56	63	70	140	210	280	350	420	490	560	630	7
3	6	12	18	24	30	36	42	48	54	60	120	180	240	300	360	420	480	540	6
$2\frac{1}{2}$	5	10	15	20	25	30	35	40	45	50	100	150	200	250	300	350	400	450	5
2	4	8	12	16	20	24	28	32	36	40	80	120	160	200	240	280	320	360	4
$1\frac{1}{2}$	3	6	9	12	15	18	21	24	27	30	60	90	120	150	180	210	240	270	3
1	2	4	6	8	10	12	14	16	18	20	40	60	80	100	120	140	160	180	2
$1\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	1
$1\frac{1}{4}$	$1\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	10	15	20	25	30	35	40	45	$1\frac{1}{2}$

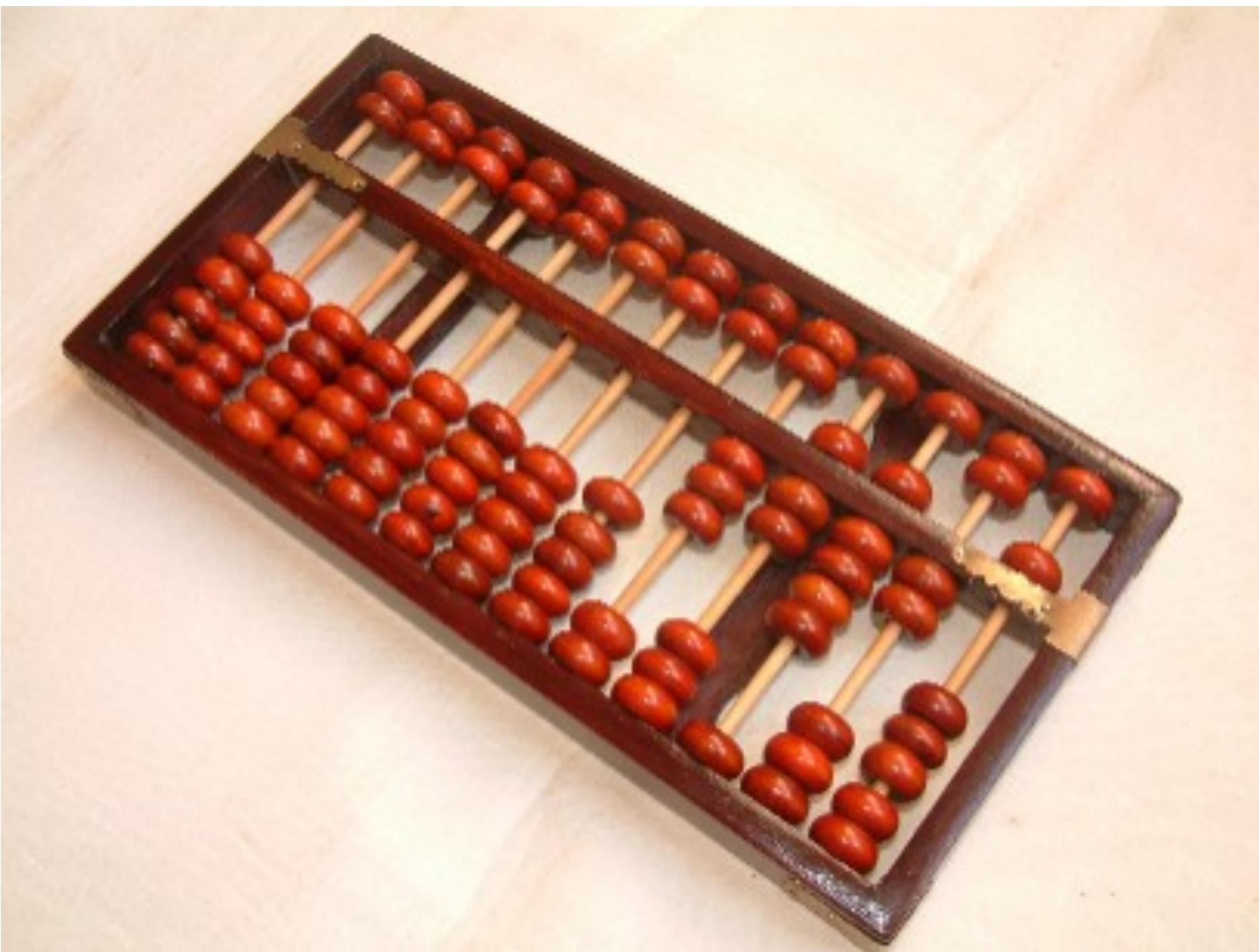
$$24 \cdot 36.5 = (20 + 4) \cdot (30 + 6 + 0.5)$$

$$= 20 \cdot 30 + 20 \cdot 6 + 20 \cdot 0.5 + 4 \cdot 30 + 4 \cdot 6 + 4 \cdot 0.5$$

= Look up on table, then add together

Ancient Calculators

Abacus



The Incan Empire



The Incan Empire



The Incan Empire



The Incan Empire

The Incan Empire

- The Incans were a great but relatively short-lived empire.

The Incan Empire

- The Incans were a great but relatively short-lived empire.
- They did not have a written language.

The Incan Empire

- The Incans were a great but relatively short-lived empire.
- They did not have a written language.
- How do you record numbers and do math without writing?

The Incan Empire

- The Incans were a great but relatively short-lived empire.
- They did not have a written language.
- How do you record numbers and do math without writing?
- The Incans used what is called a *quipu*.

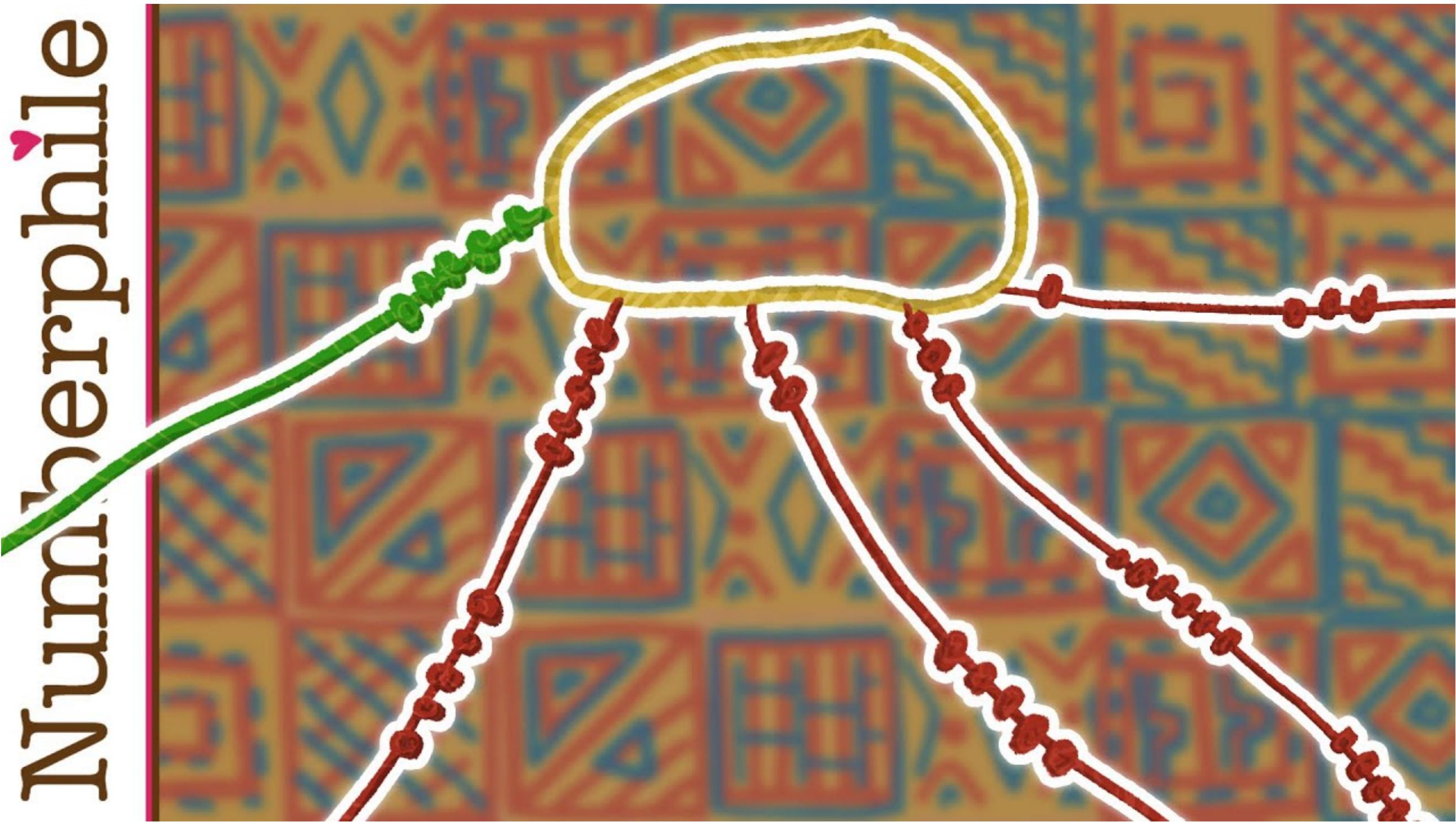
Devices



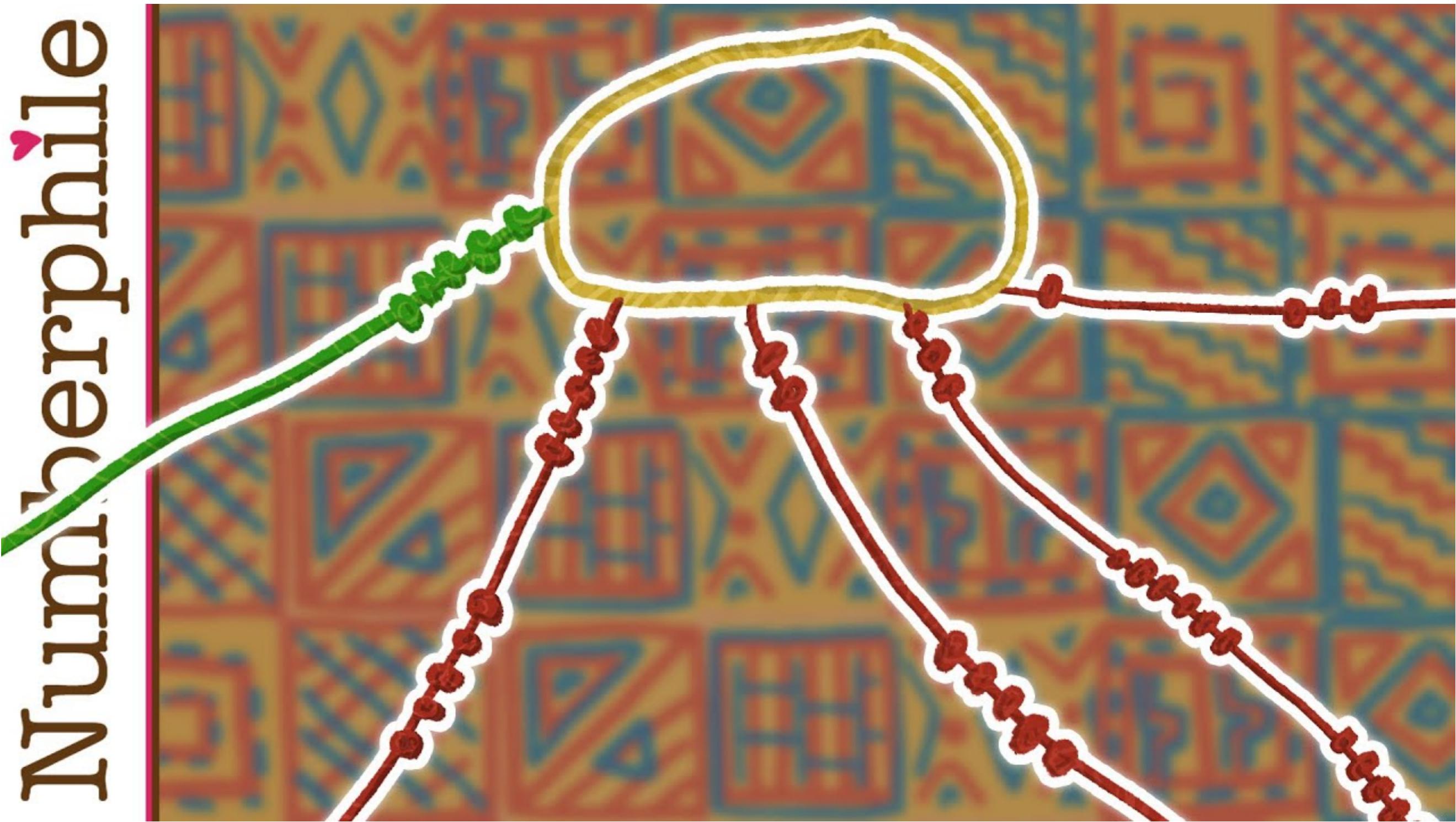
Devices



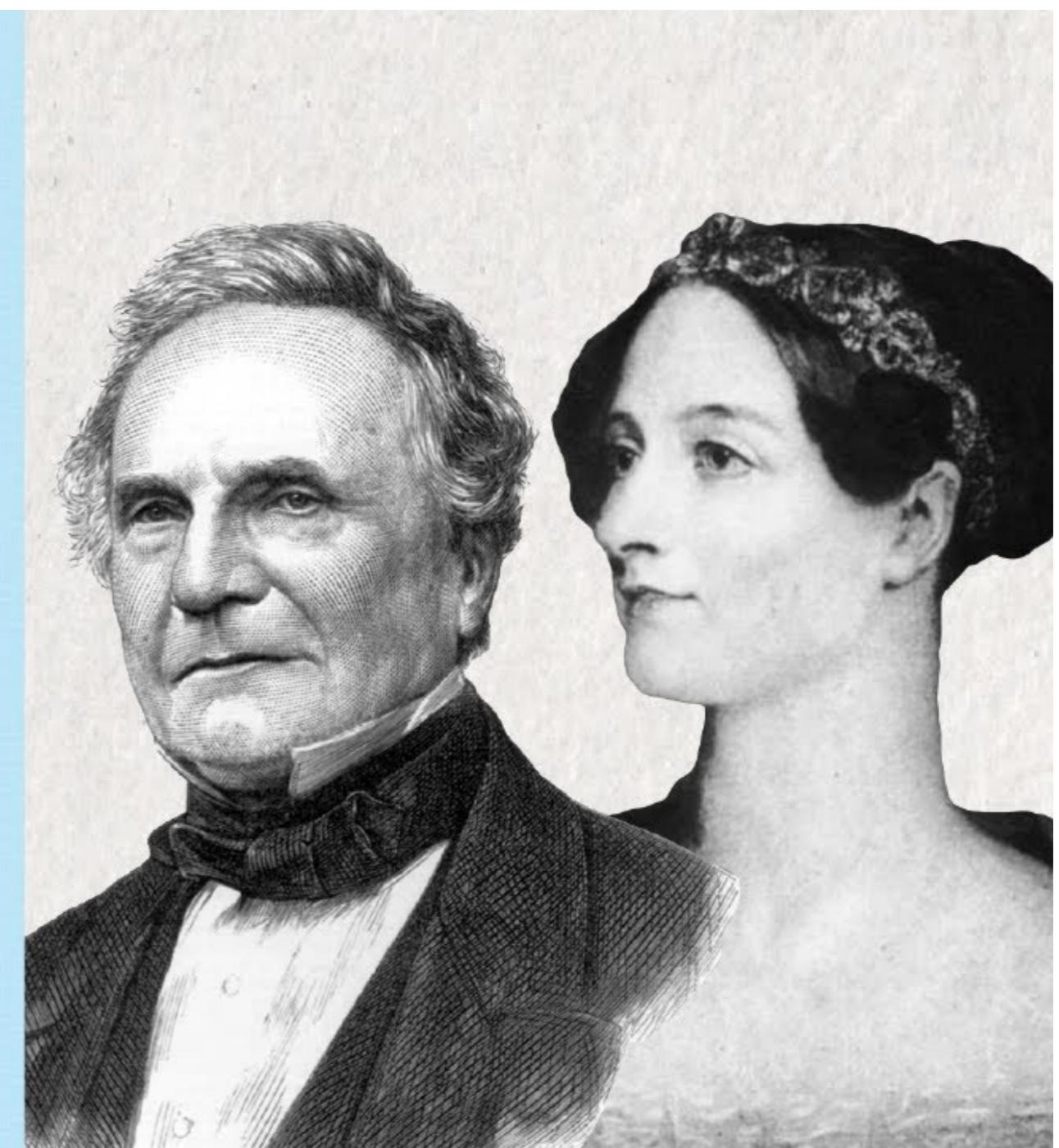
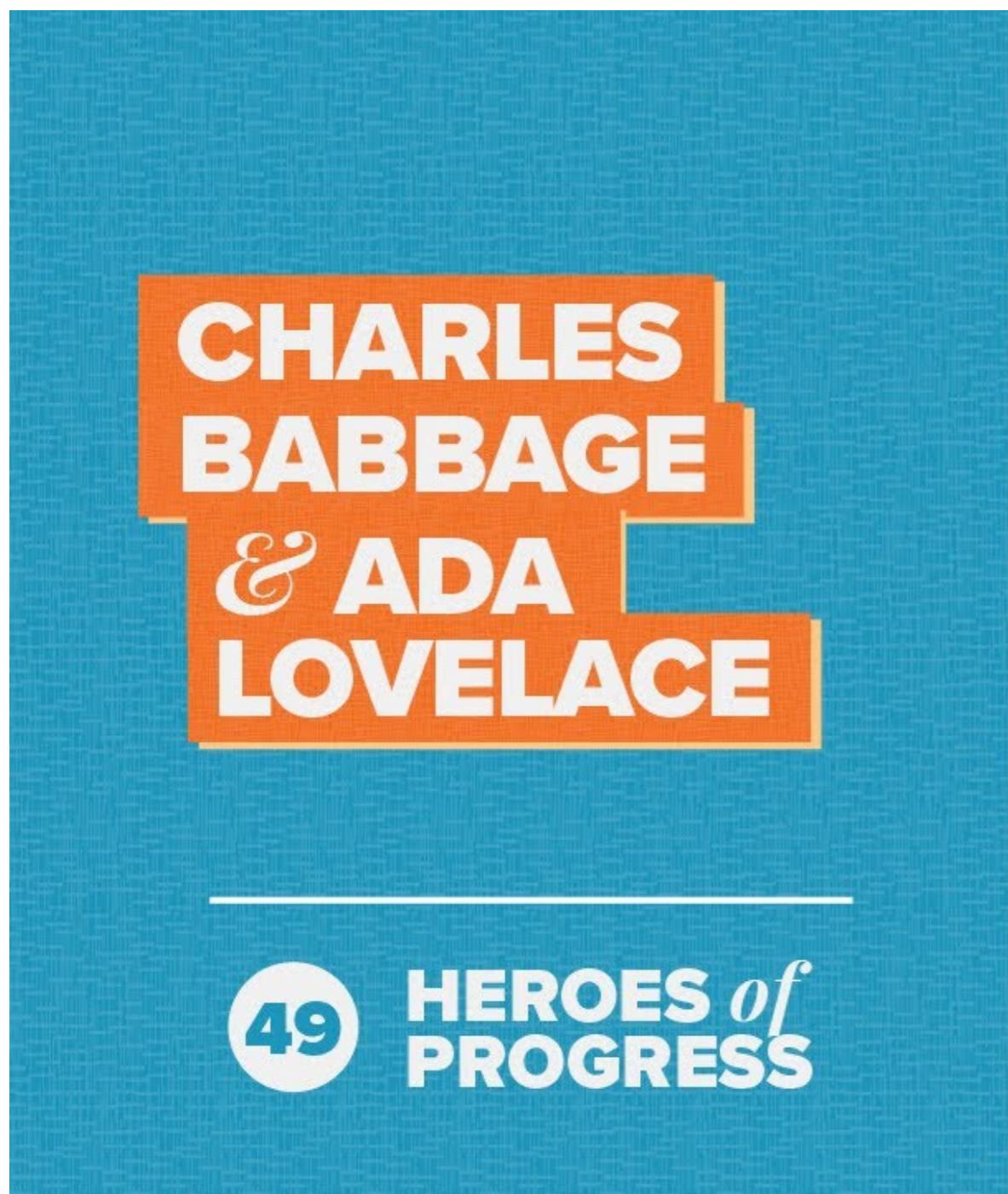
Devices



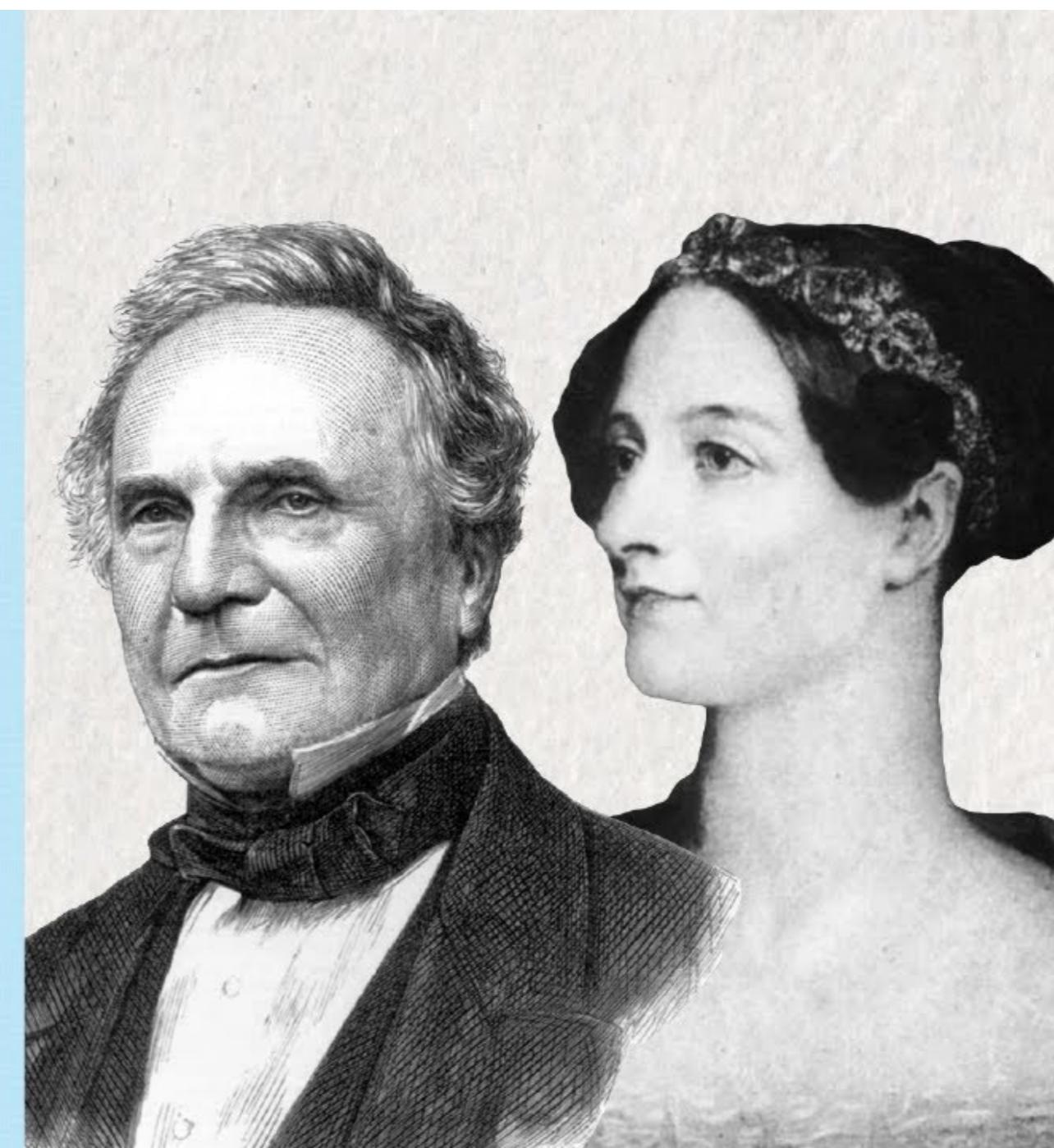
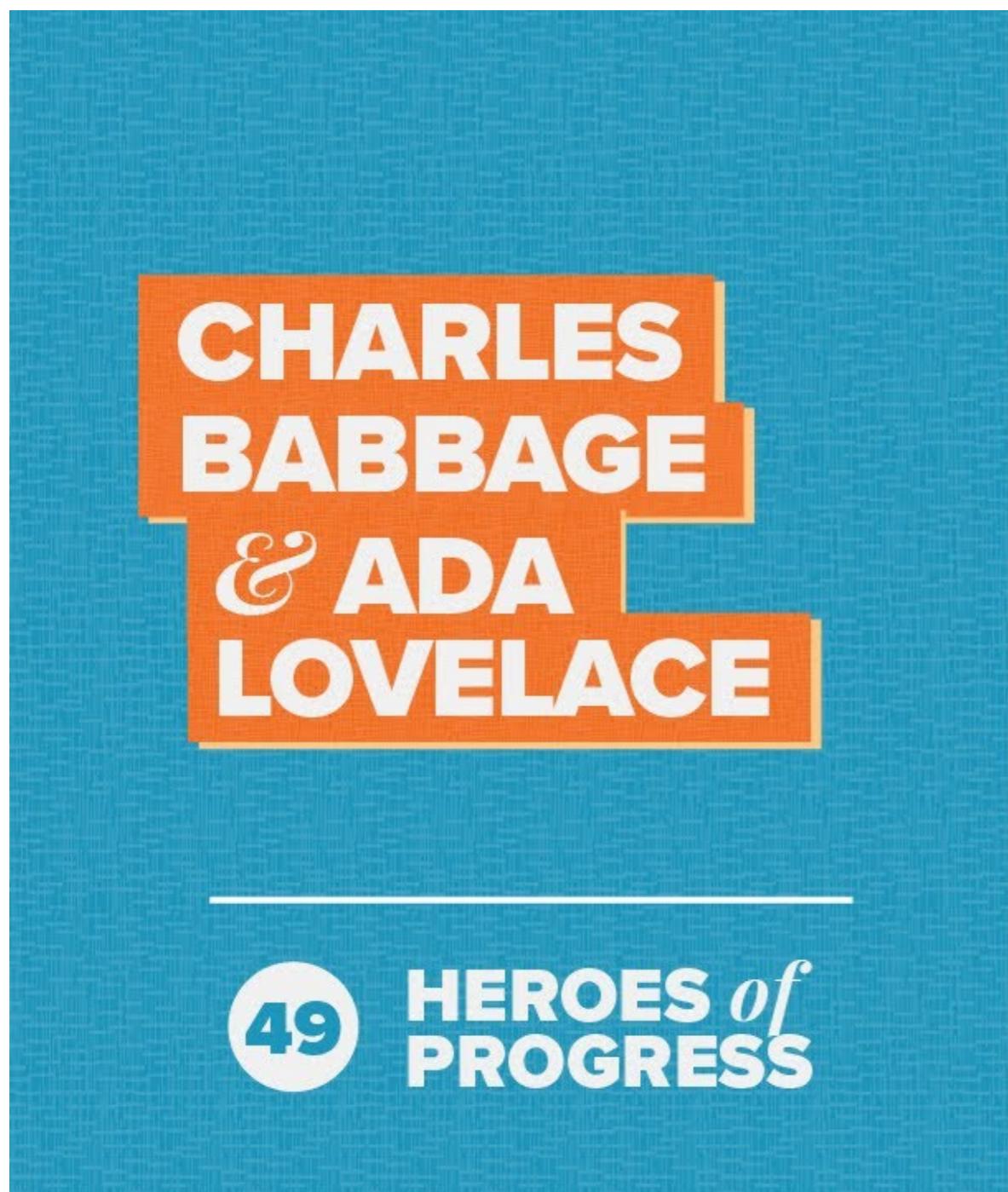
Devices



Aftermath: Computers



Aftermath: Computers

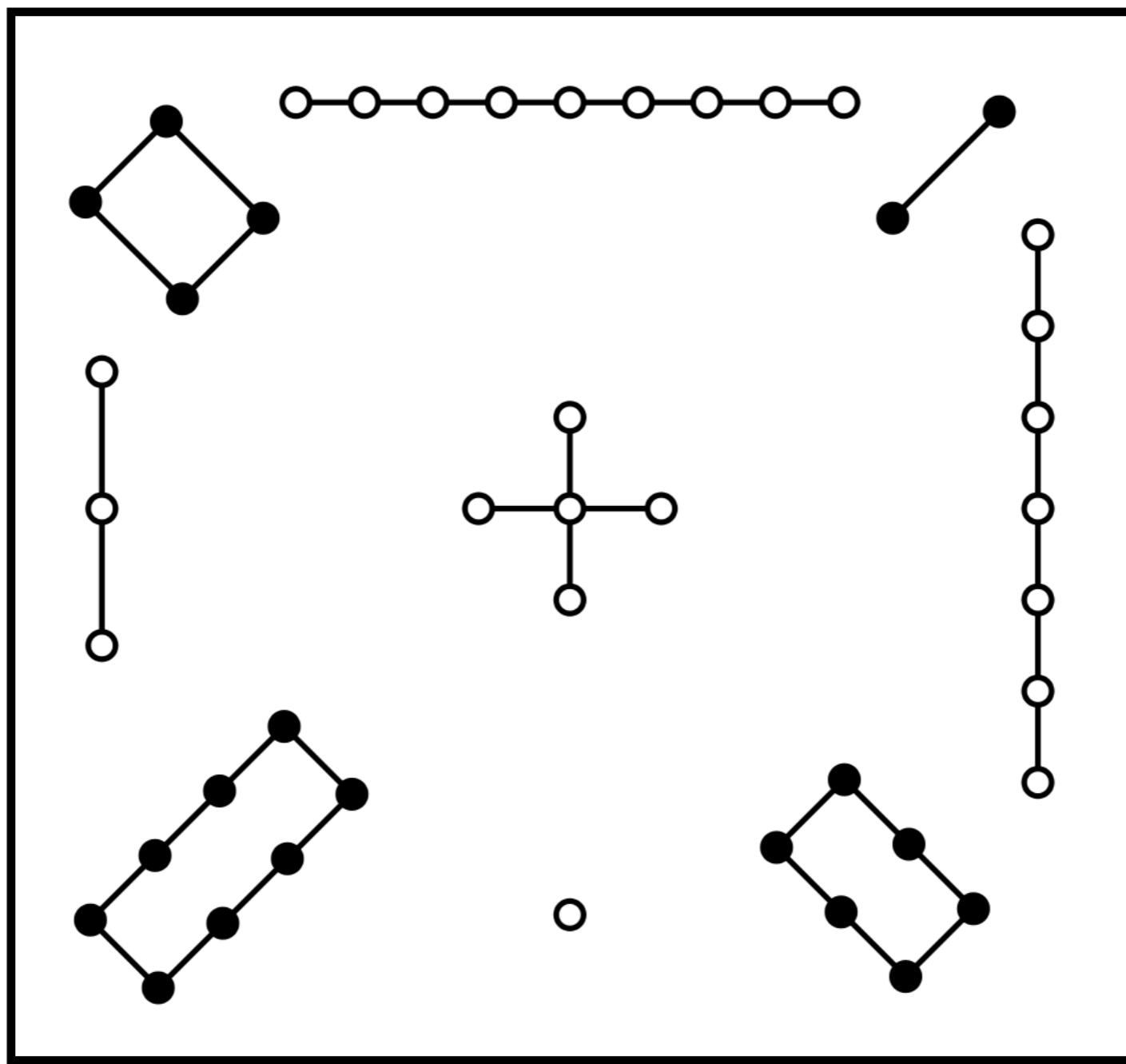


Think Like A
Math Historian

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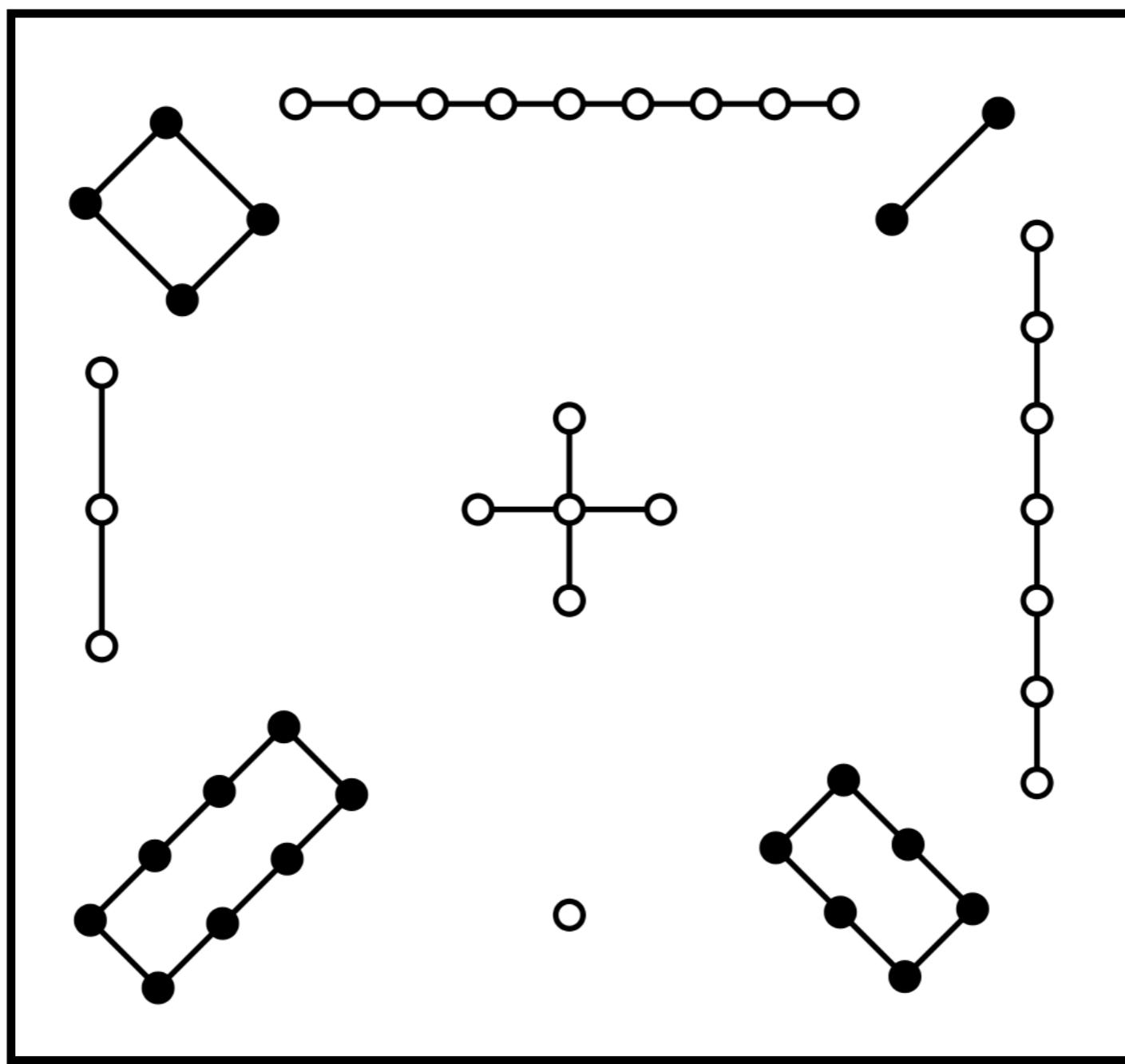
Think Like a Math Historian

- What does this mean?



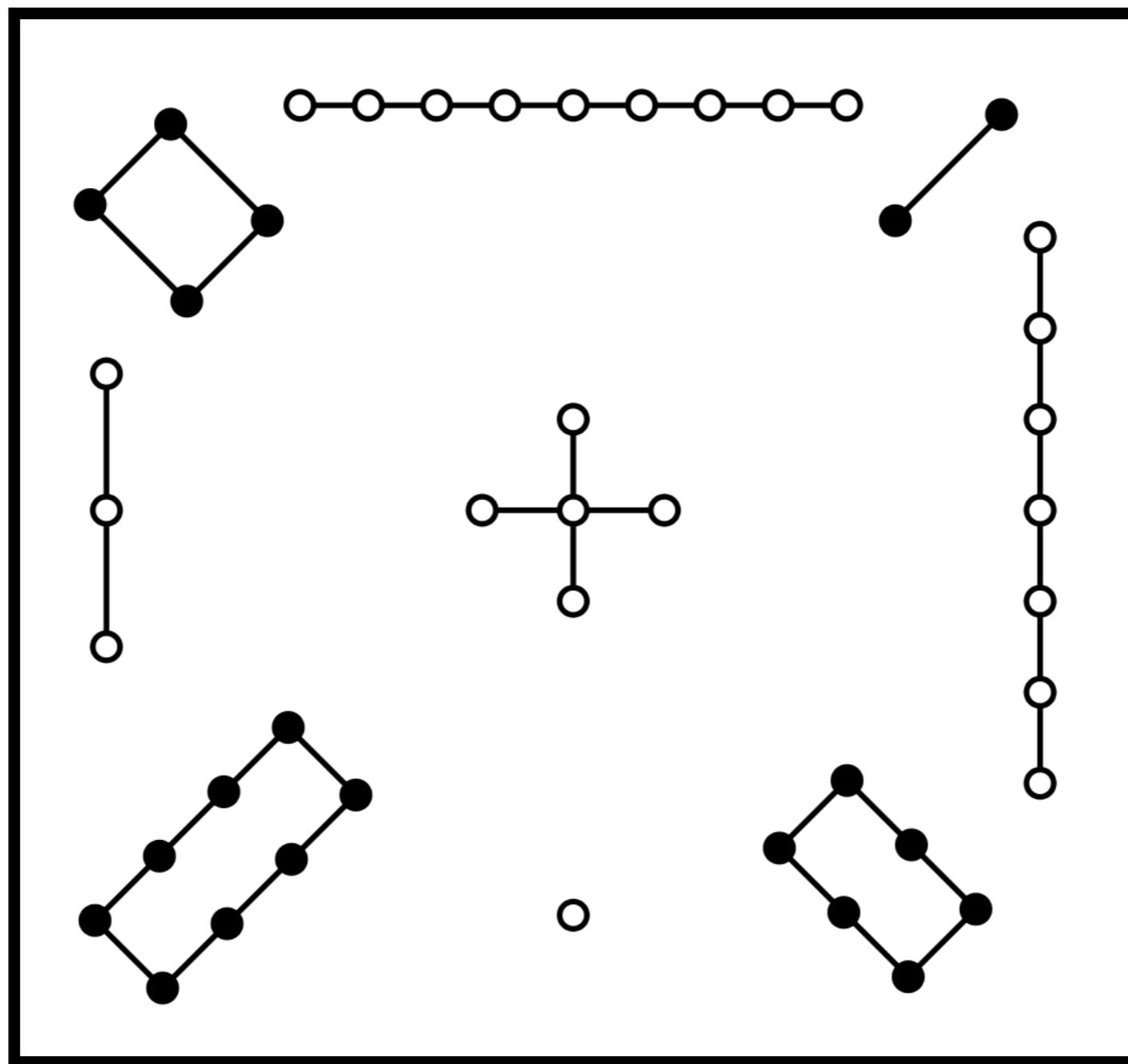
Think Like a Math Historian

- Hint: Focus on how many dots are in each group.



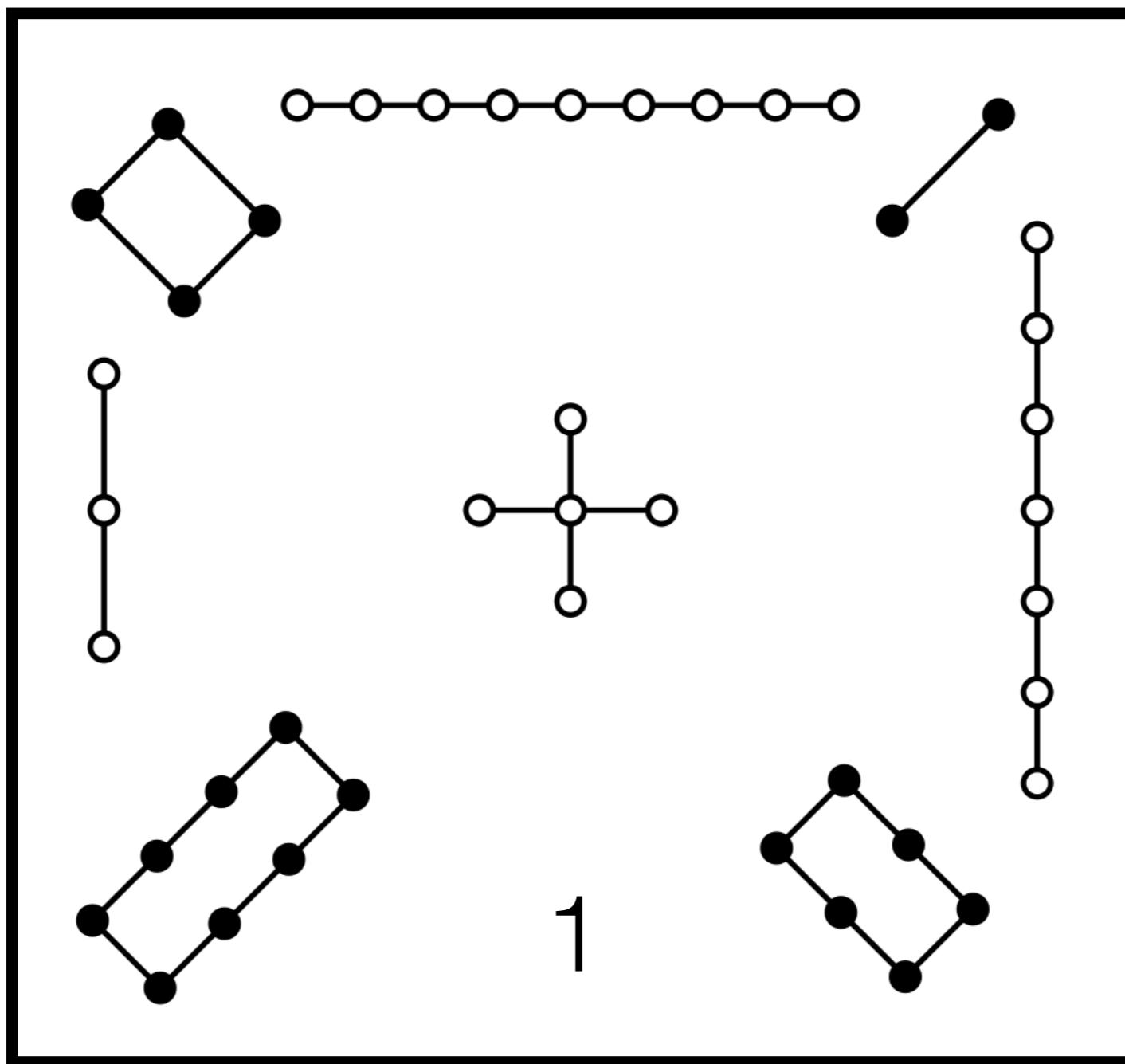
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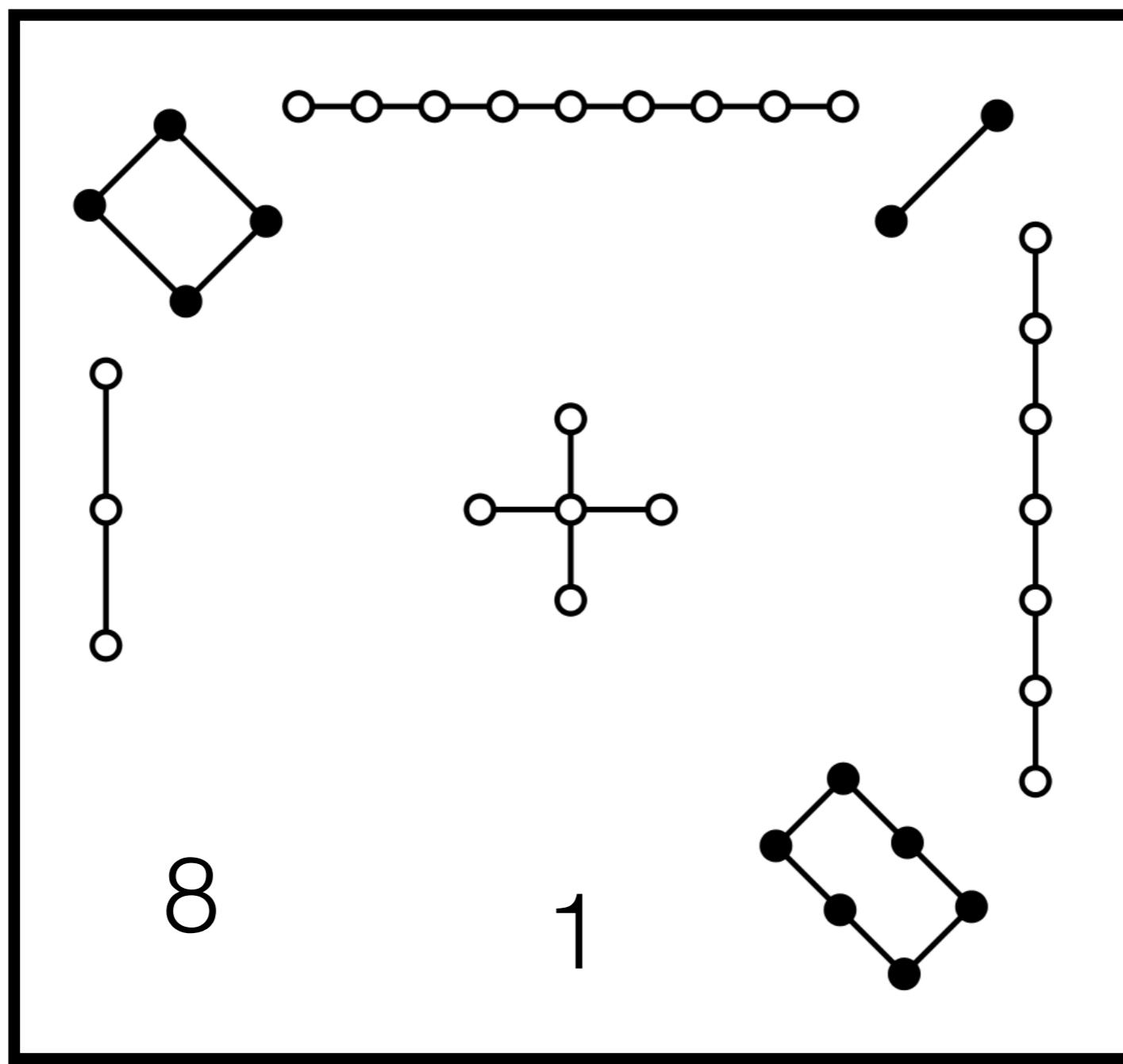
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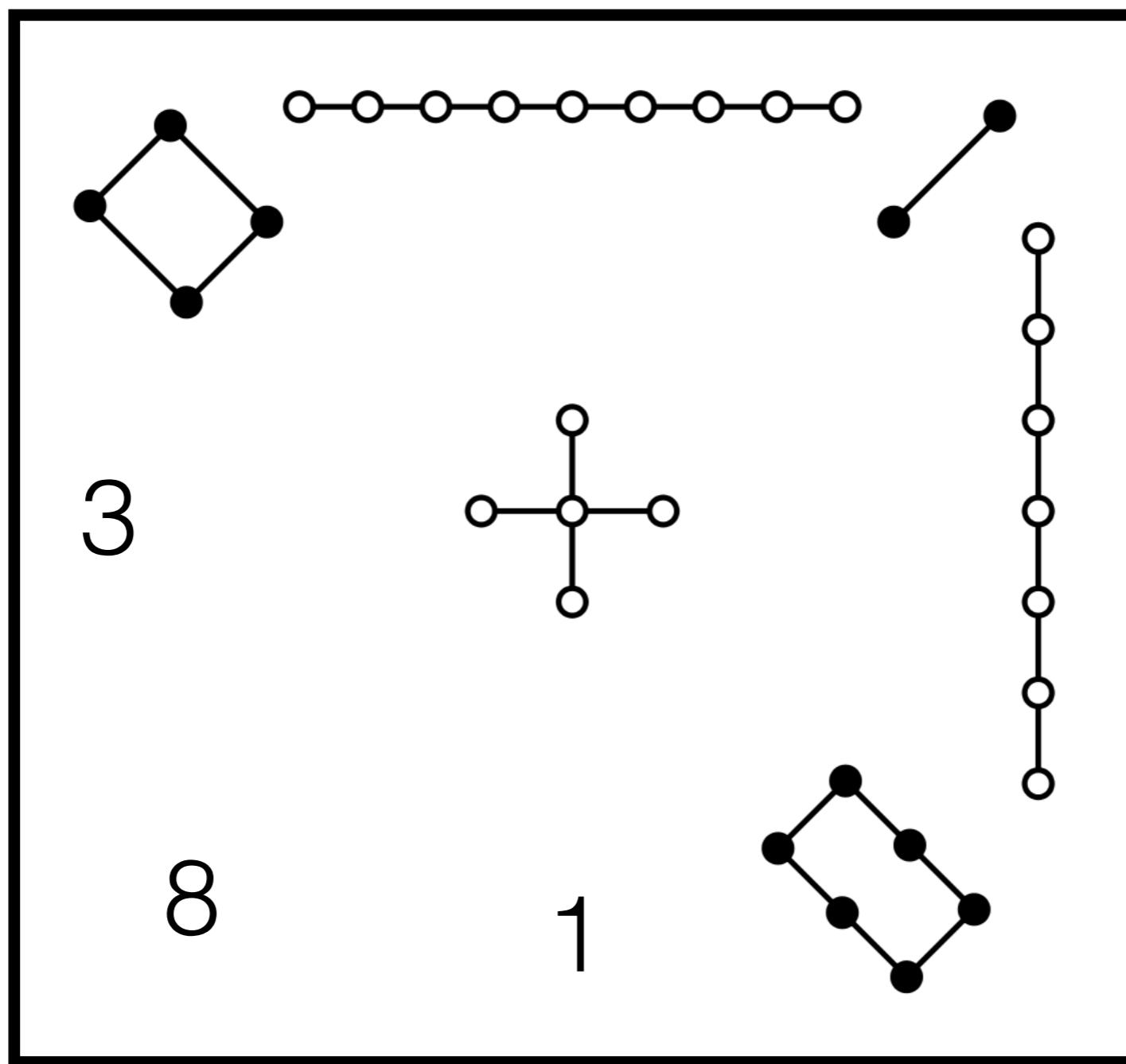
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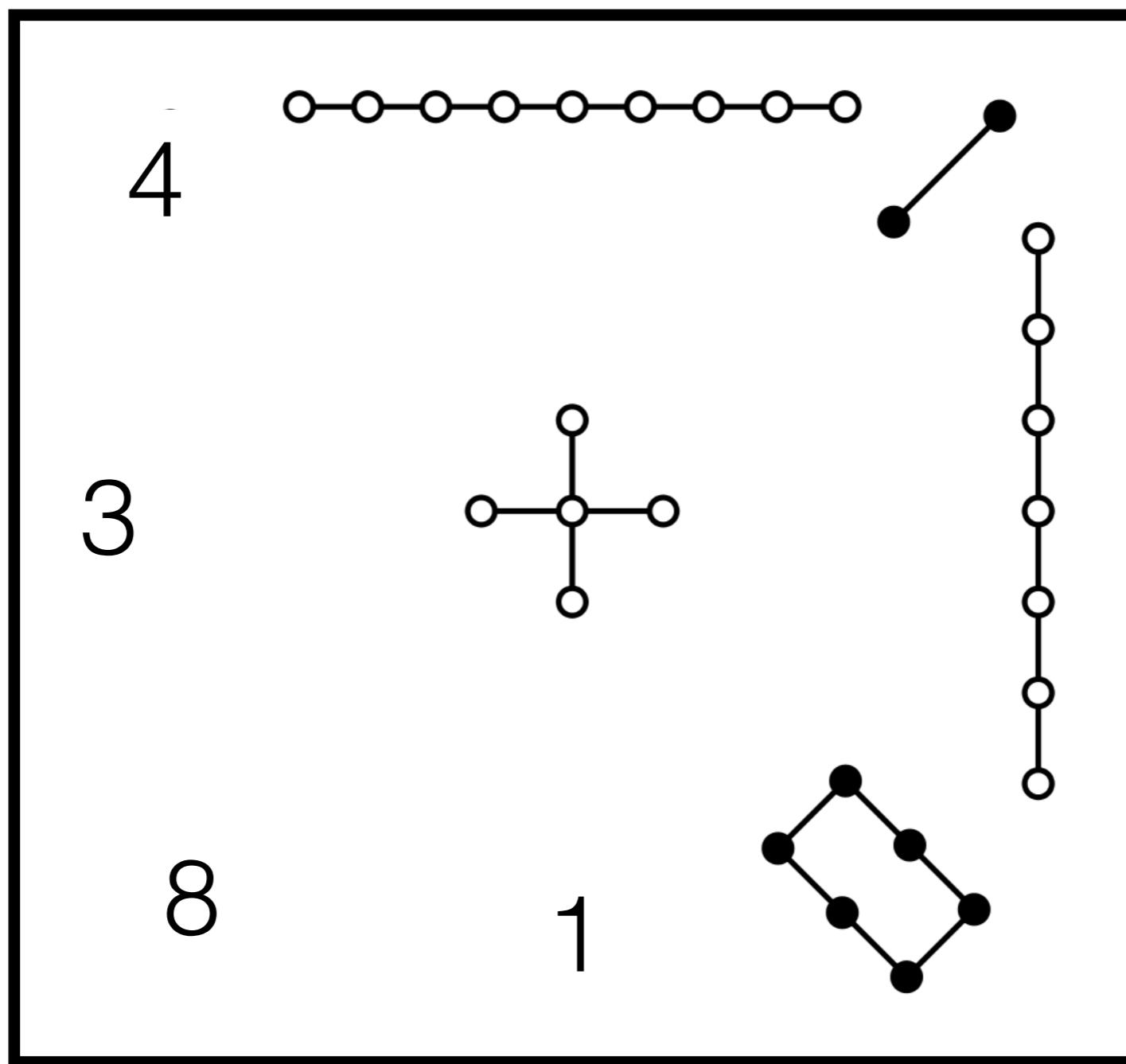
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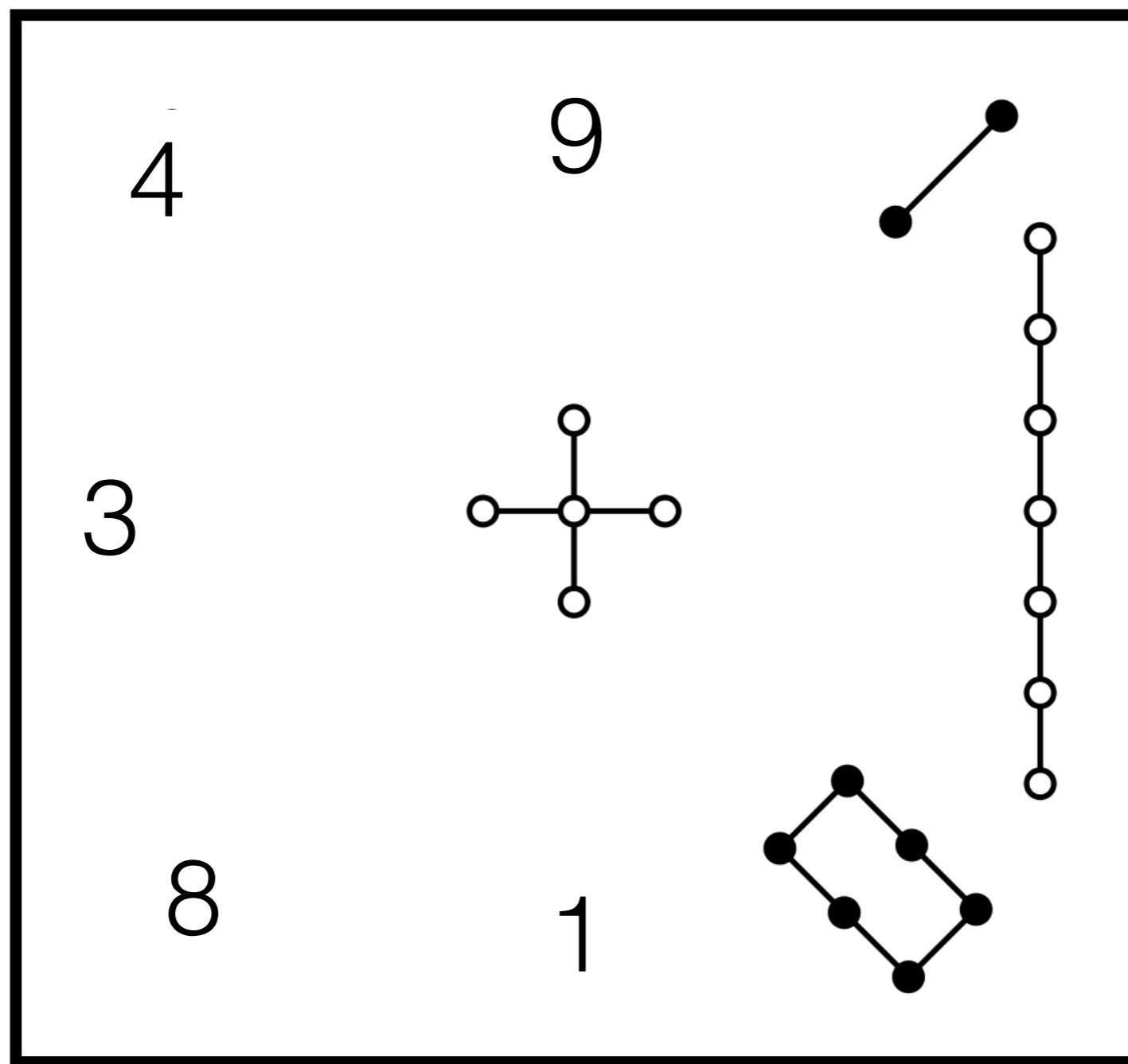
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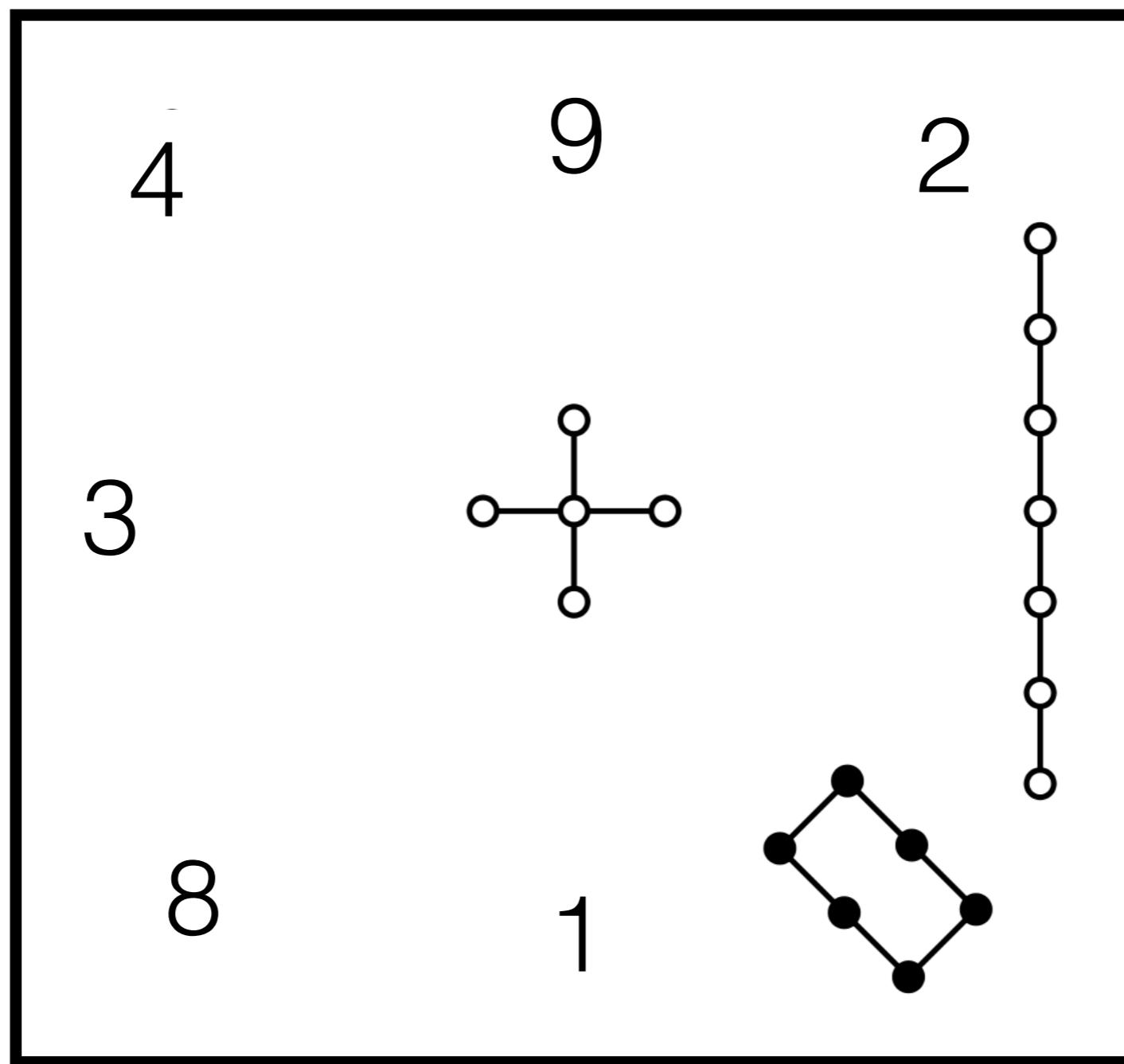
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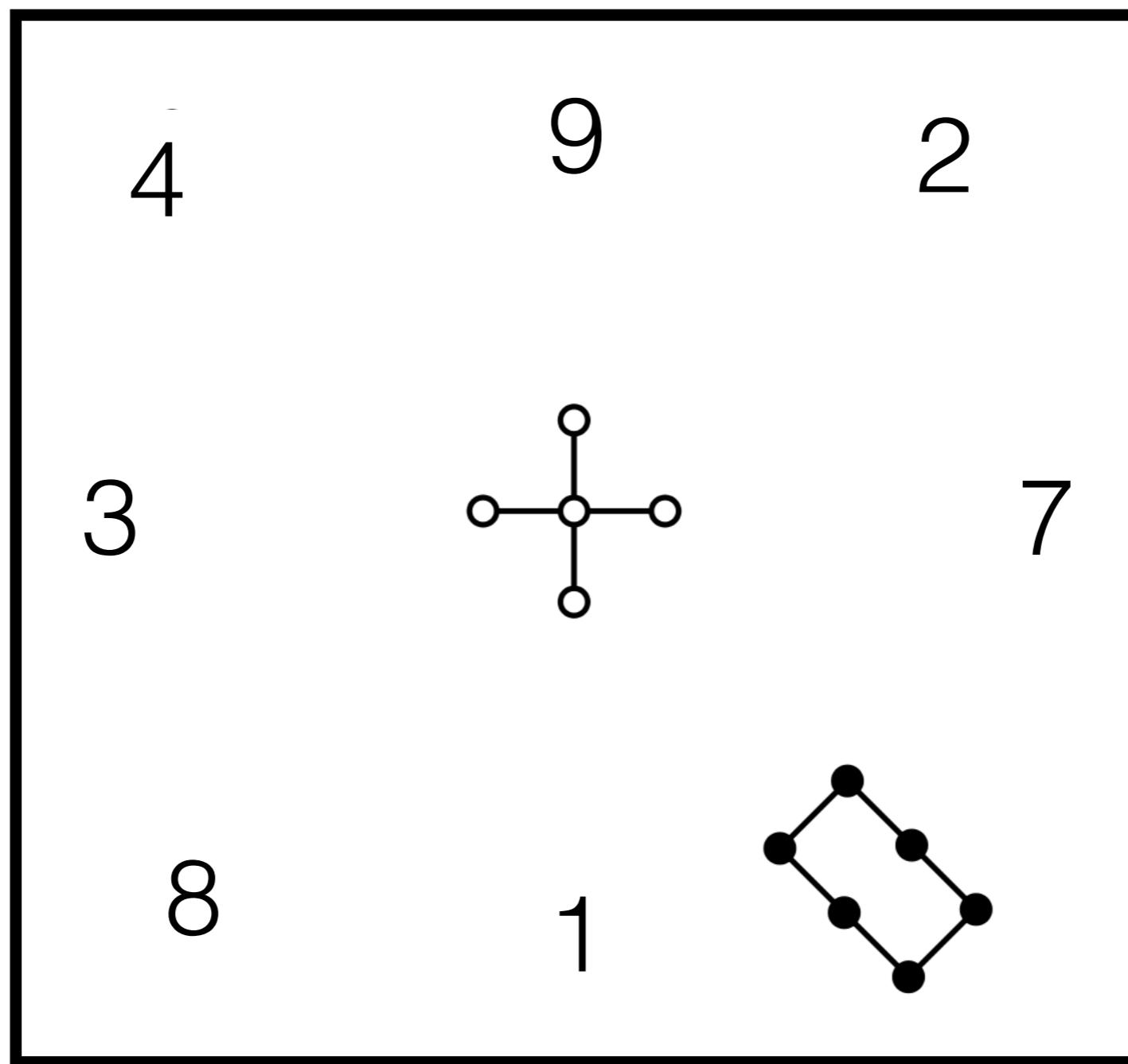
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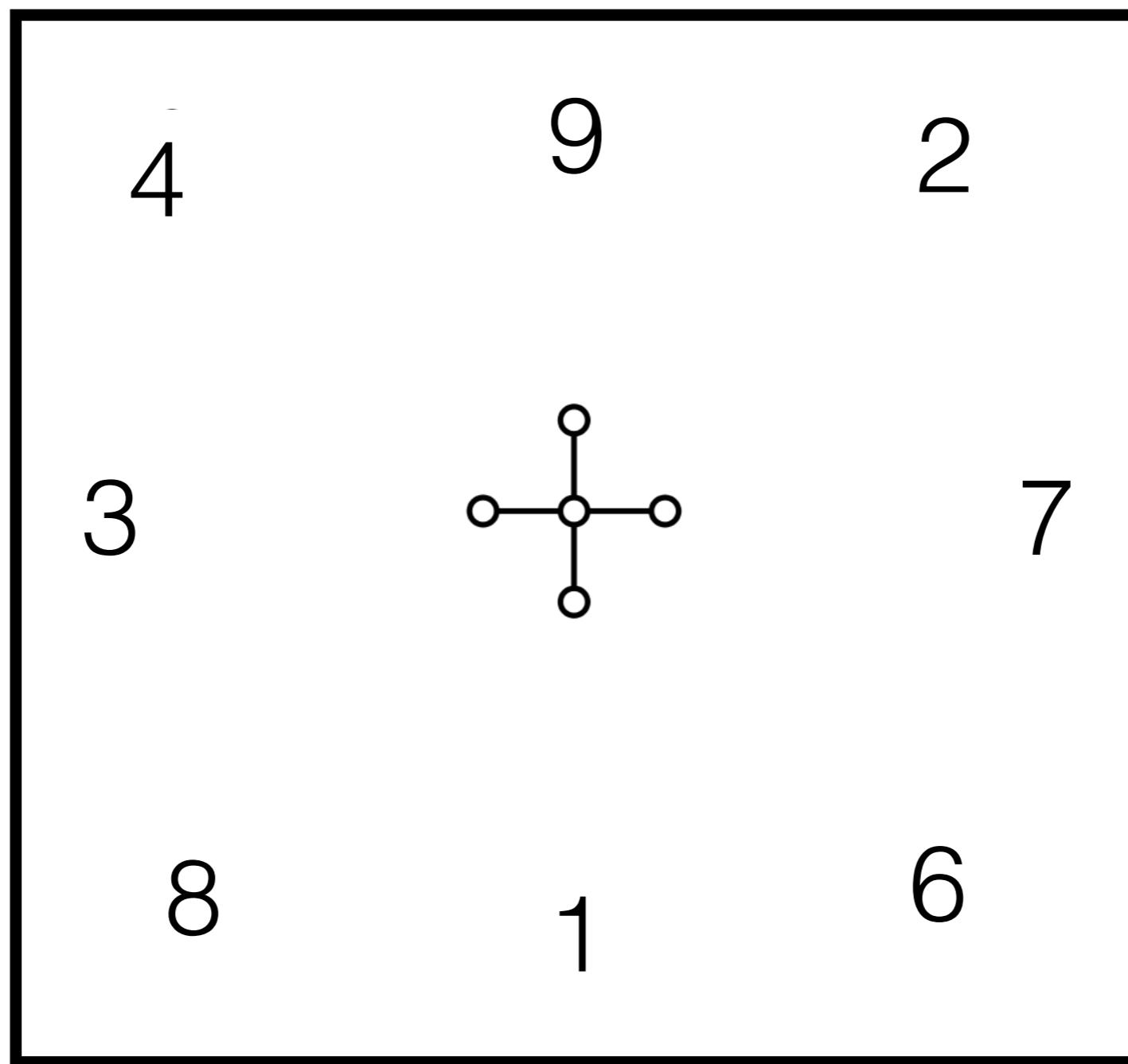
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Think Like a Math Historian

- Magic Square

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3	5	7
8	1	6

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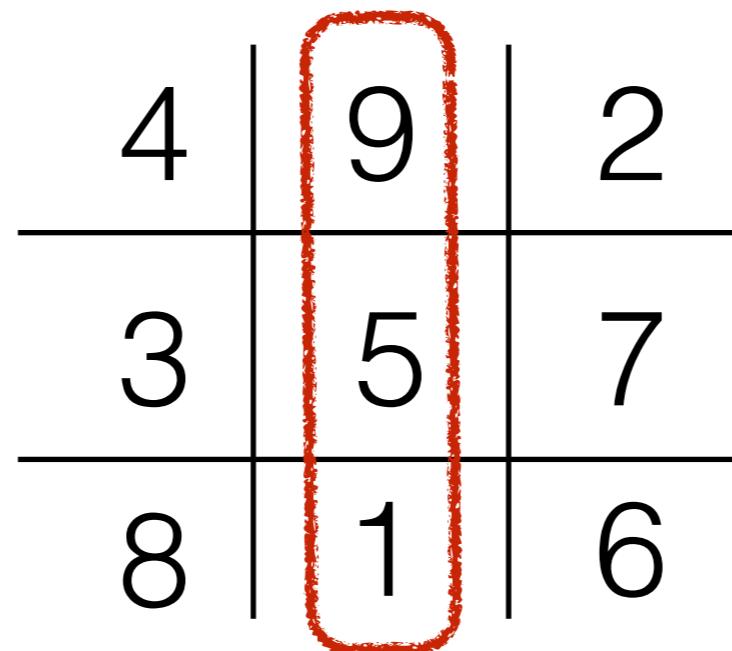
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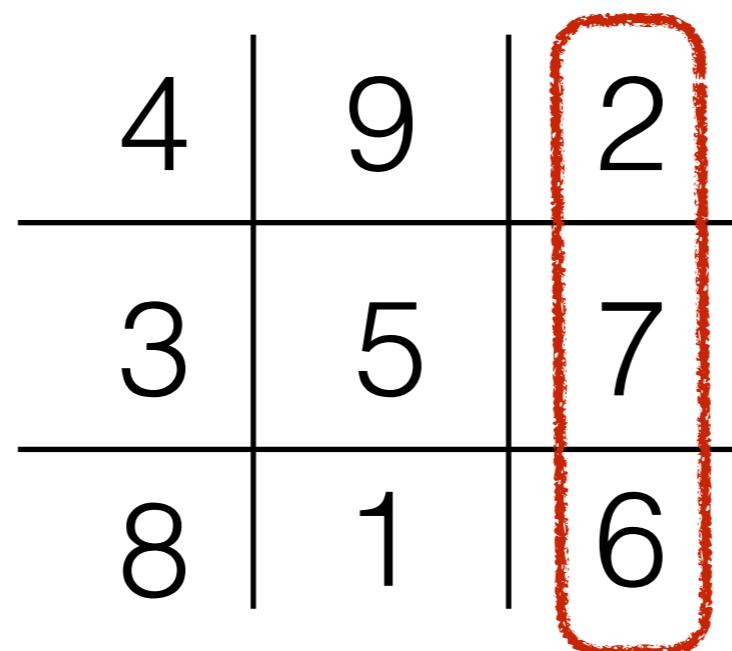
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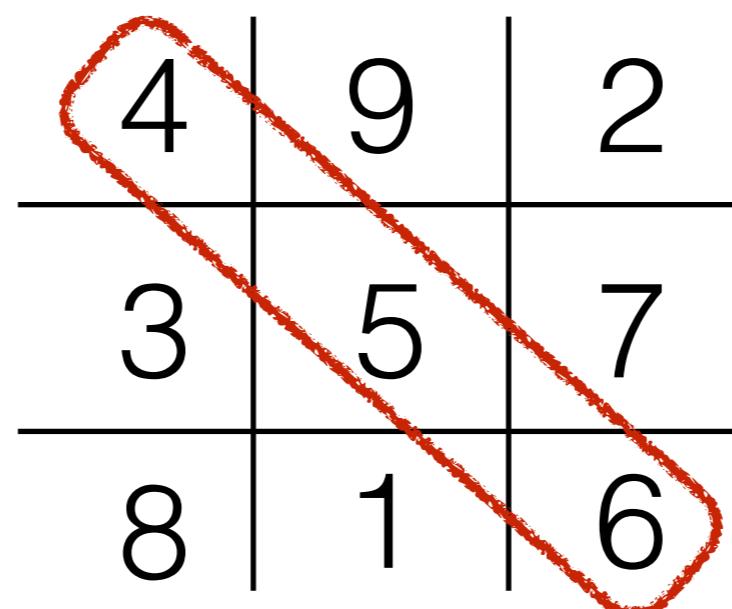
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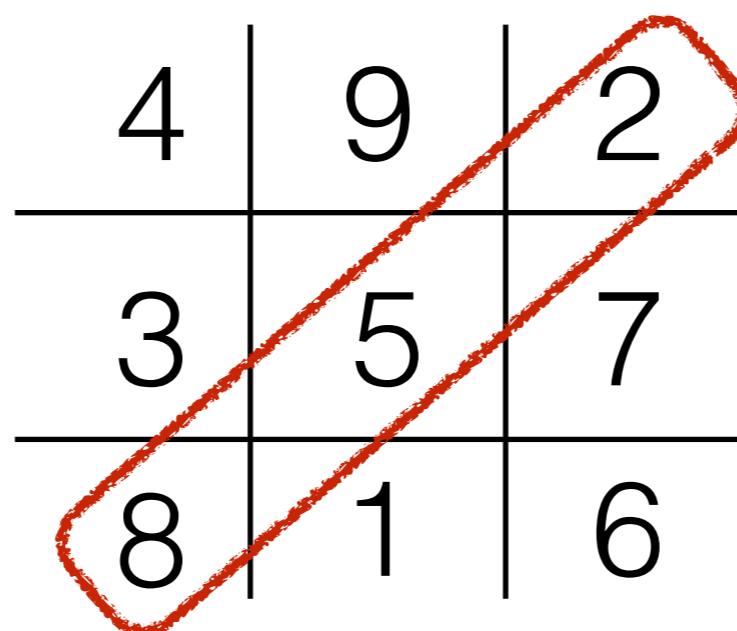
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The Aftermath

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- Yang Hui, a much more modern Chinese mathematician (~1238-1298 AD) had a 3-step procedure to create this square.



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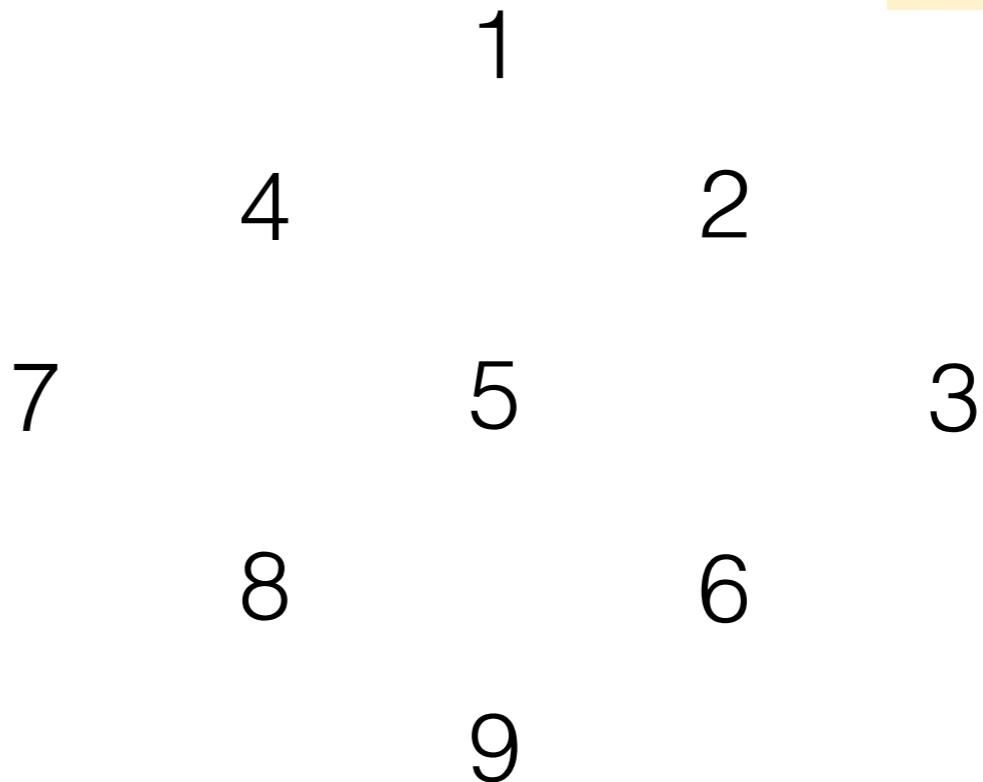
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Step 2: Swap corners.

		1
	4	2
7		5
	8	6
		9

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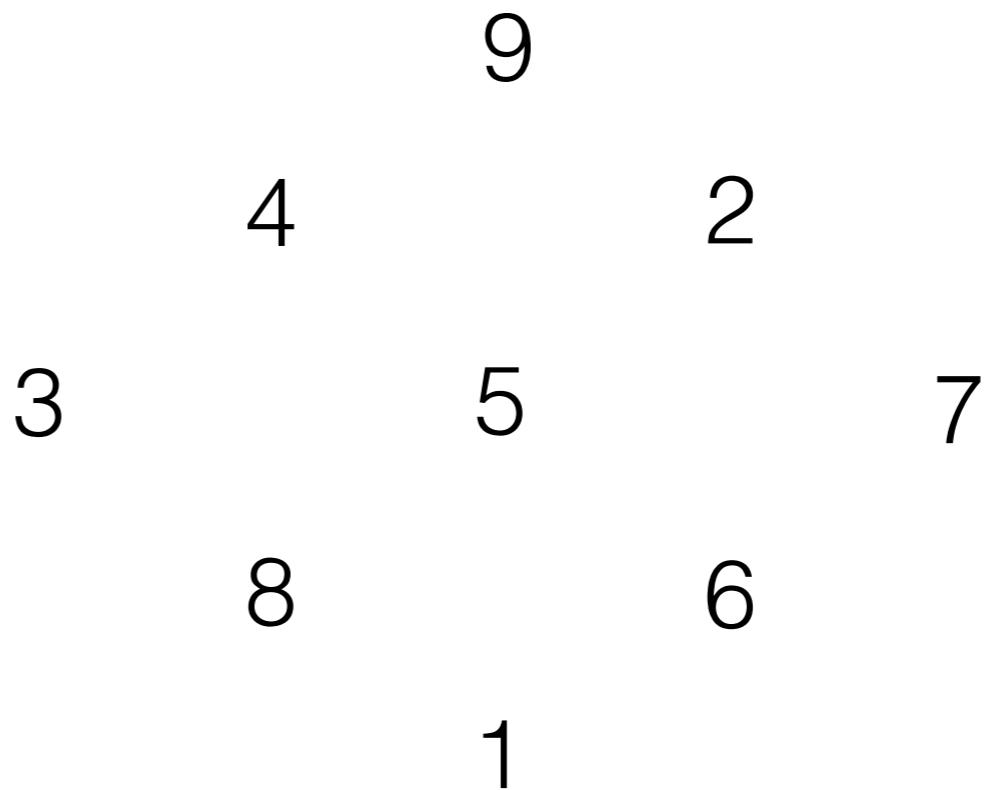
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		9	
	4		2
7		5	3
	8		6
		1	

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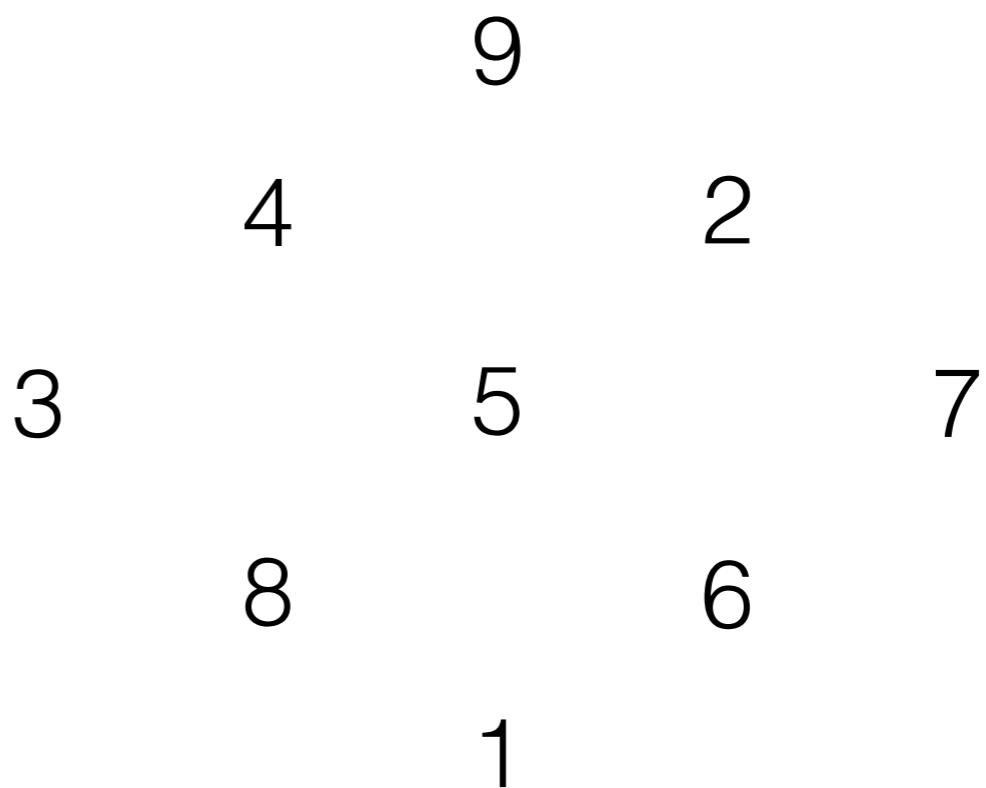
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Hui also came up with a similar algorithm to construct a 4x4 magic square.

The Aftermath

- Yang Hui constructed a magic square of every size up to 10x10.

The Aftermath

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一	二	二	四	四	六	六	八	八	一百
九	八	七	六	五	四	三	二	一	二
九	九	九	二	九	二	九	二	九	九
三	十八	二	三	三	五	六	七	八	八
九	八	七	六	五	四	三	二	一	四
七	四	七	四	七	四	七	四	七	九
五	十六	二	三	四	五	六	七	八	六
九	六	五	六	五	六	五	六	五	十
五	八	七	六	五	四	三	二	一	六
十	六	五	六	五	六	五	六	五	九
四	七	三	二	五	四	三	二	一	八
八	九	六	七	四	五	六	七	八	七
八	三	八	三	八	三	八	三	三	十三
十二	九	三	二	五	二	四	七	二	九
九	九	七	七	五	五	三	三	三	八
一	十	一	七十	一	五十	三	三十	十一	十

The Aftermath

- Yang Hui constructed a magic square of every size up to 10x10.

一	二	二	四	四	六	六	八	八	一百
九	八	七	六	五	四	三	二	一	二
九	九	九	二	九	二	九	二	九	二
三	十	二	三	八	四	五	六	七	八
九	八	七	六	五	四	三	七	二	四
七	四	七	四	七	四	七	二	四	七
五	十	二	三	四	五	六	七	八	九
六	六	五	六	五	六	五	六	五	六
九	八	七	六	五	四	三	五	二	十
五	六	五	六	五	六	五	六	五	六
十四	七	三	二	五	四	七	四	六	九
八	九	六	七	四	五	三	二	三	八
八	三	八	三	八	三	八	二	三	十三
十二	九	三	二	五	二	九	二	六	九
九	九	七	七	五	五	三	三	三	八
一	十	一	七十	一	五十	三	三十	十一	十

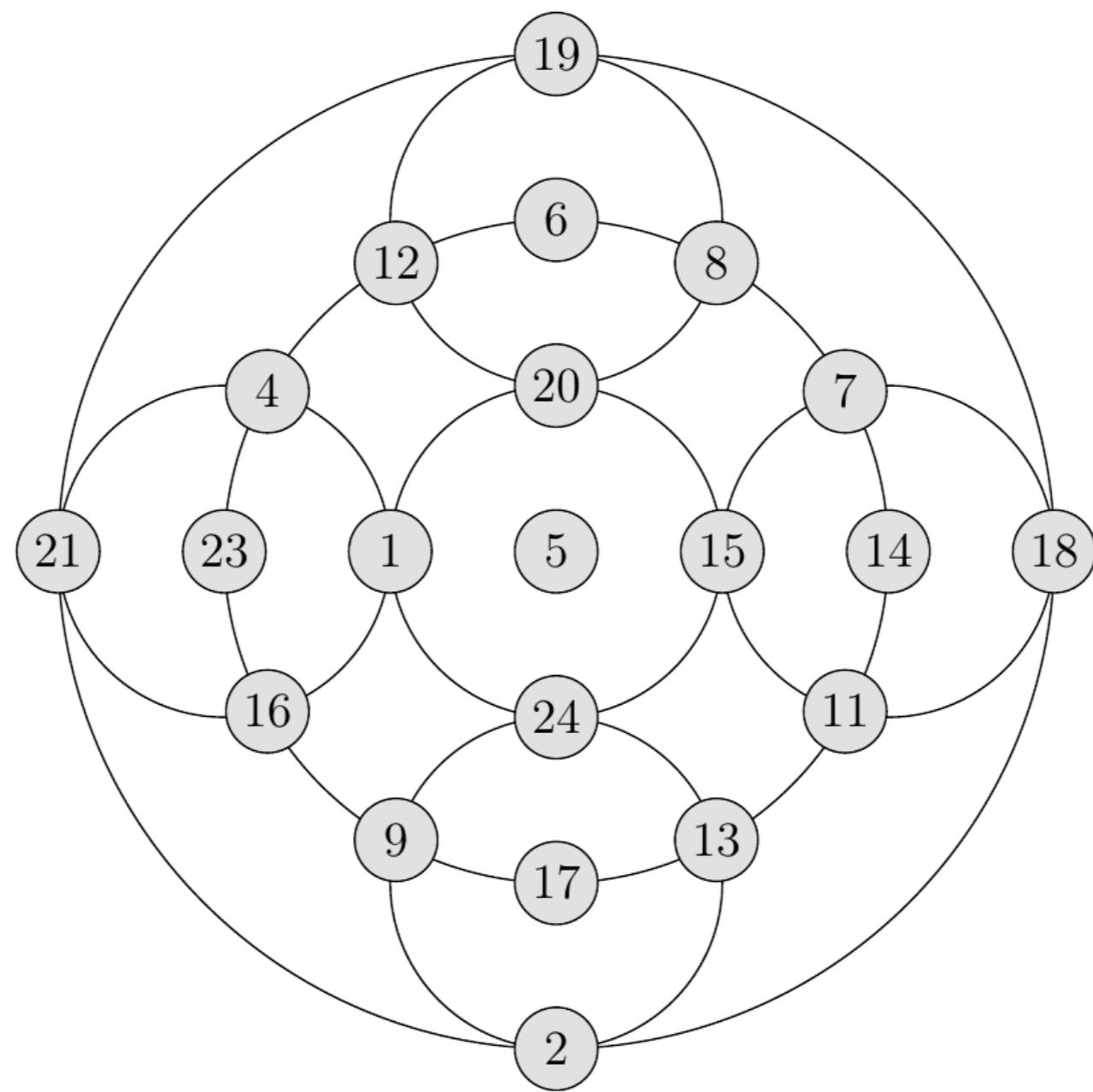
1	20	21	40	41	60	61	80	81	100
99	82	79	62	59	42	39	22	19	2
3	18	23	38	43	58	63	78	83	98
97	84	77	64	57	44	37	24	17	4
5	16	25	36	45	56	65	76	85	96
95	86	75	66	55	46	35	26	15	6
14	7	34	27	54	47	74	67	94	87
88	93	68	73	48	53	28	33	8	13
12	9	32	29	52	49	72	69	92	89
91	90	71	70	51	50	31	30	11	10

The Aftermath

- Yang Hui also constructed six magic circles.

The Aftermath

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Algorithm to Construct Magic Squares

Shout-Outs!

- Crows

Counting Crows

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- Other animals may count too, to some extent. For example, crows.

Counting Crows

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- Old story of a hunter hiding in a shelter.



Counting Crows



Counting Crows



Shout-Outs!

- Napier's Bones

Shout-Outs!

- Napier's Bones



Napier's Bones										
	0	1	2	3	4	5	6	7	8	9
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	1	0	1	2
3	0	0	0	0	0	1	2	1	2	2
4	0	0	0	0	1	1	2	2	2	3
5	0	0	0	1	1	2	2	3	3	4
6	0	0	0	1	1	2	3	3	4	5
7	0	0	0	1	2	2	3	4	4	5
8	0	0	0	1	2	3	4	4	5	6
9	0	0	0	1	2	3	4	5	6	7

Invented by John Napier 1550-1617 Made in USA by Creative Crafthouse

People's History

People's History of Numbers

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- A “people’s history” studies the history of non-elites.

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- Hindu-Arabic numerals “democratized” math.
- Some Roman elites fought back in an effort to keep math opaque.