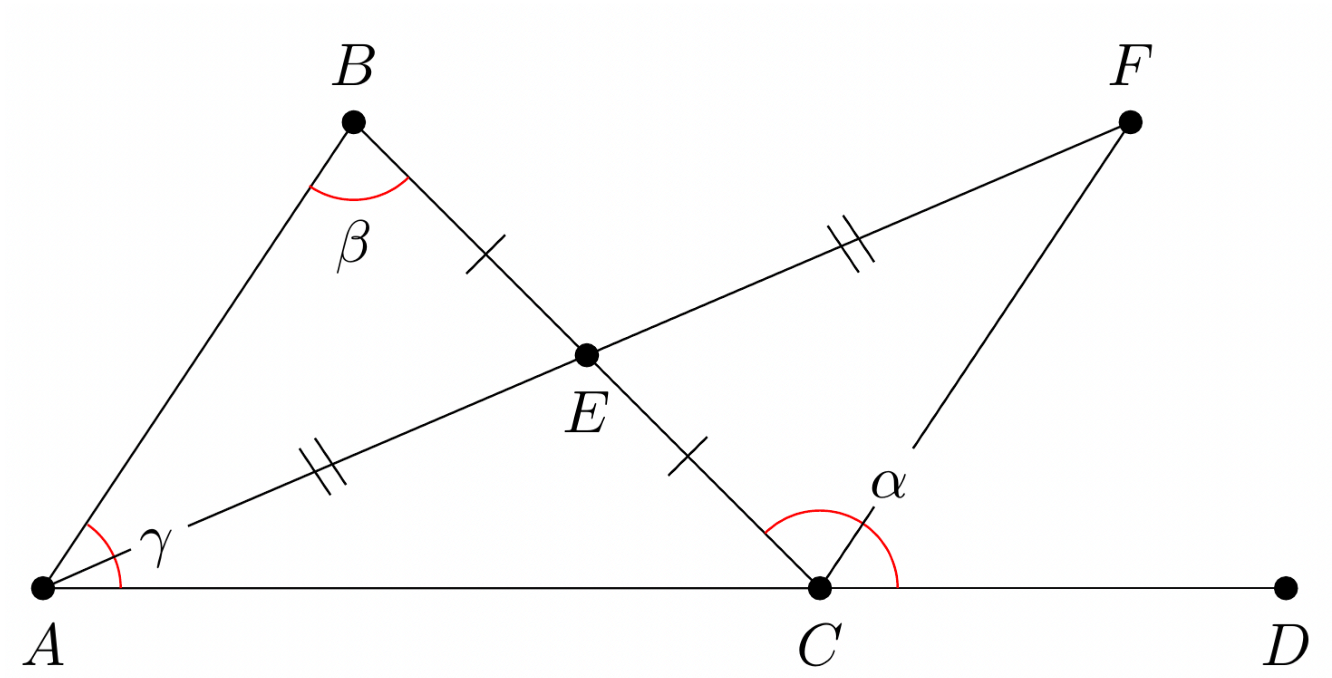
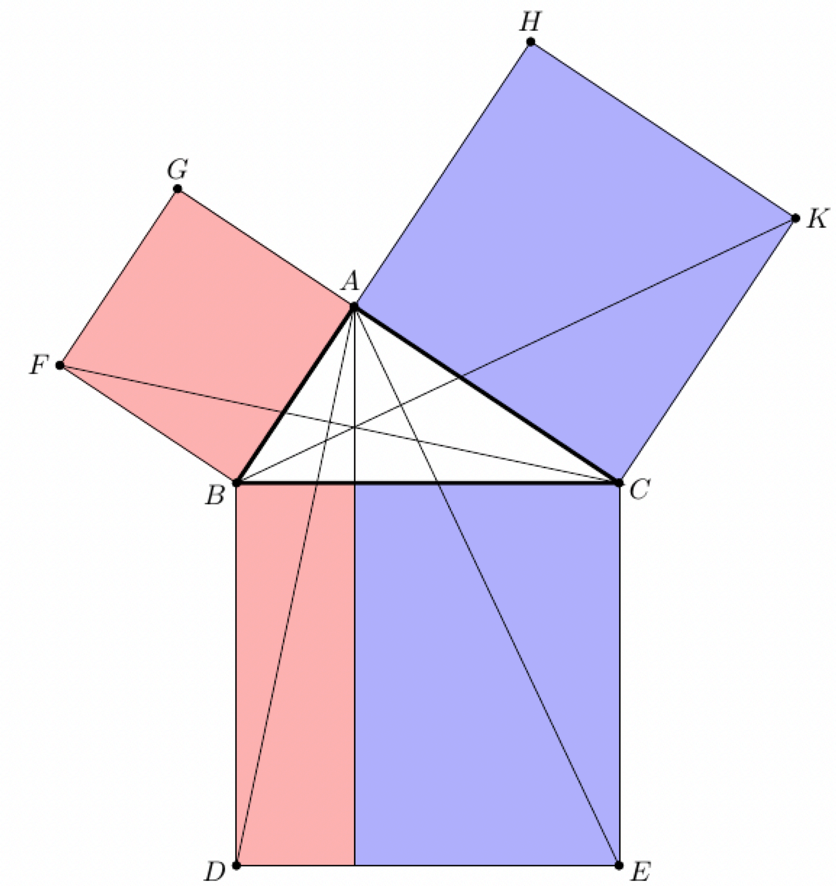
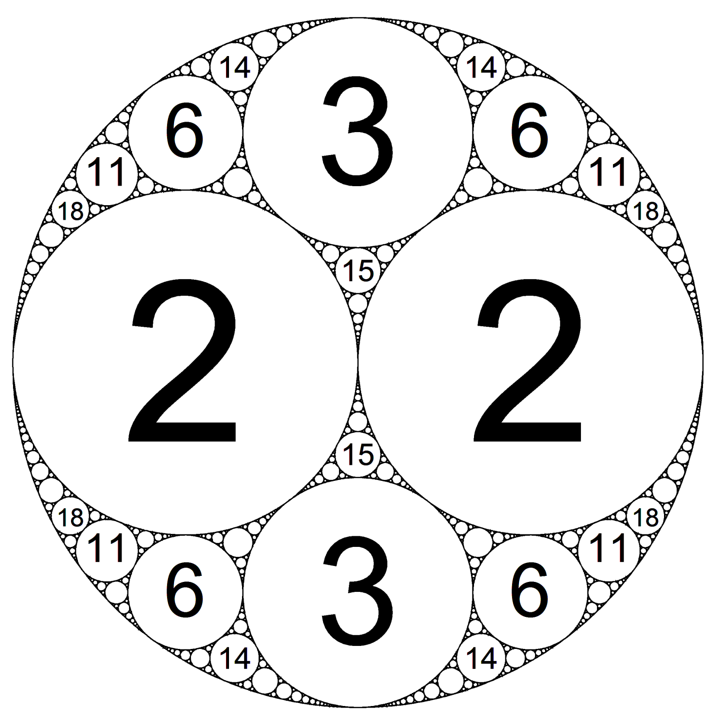




Chapter 4: Euclidean Geometry



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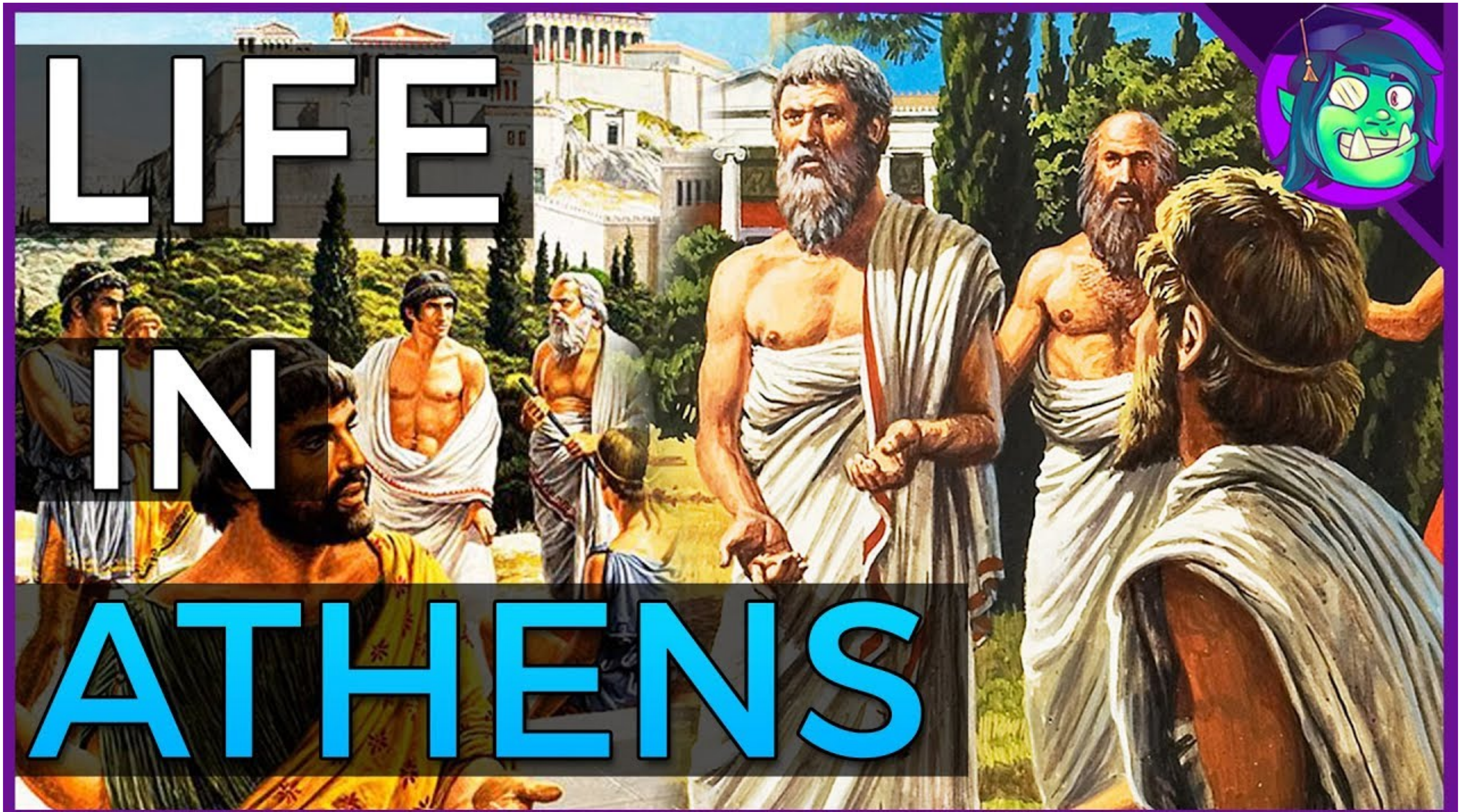
- Many great Greek mathematicians studied at the Academy. Greatest of them was Eudoxus of Cnidus.

Academy in Athens

Life in Ancient Athens



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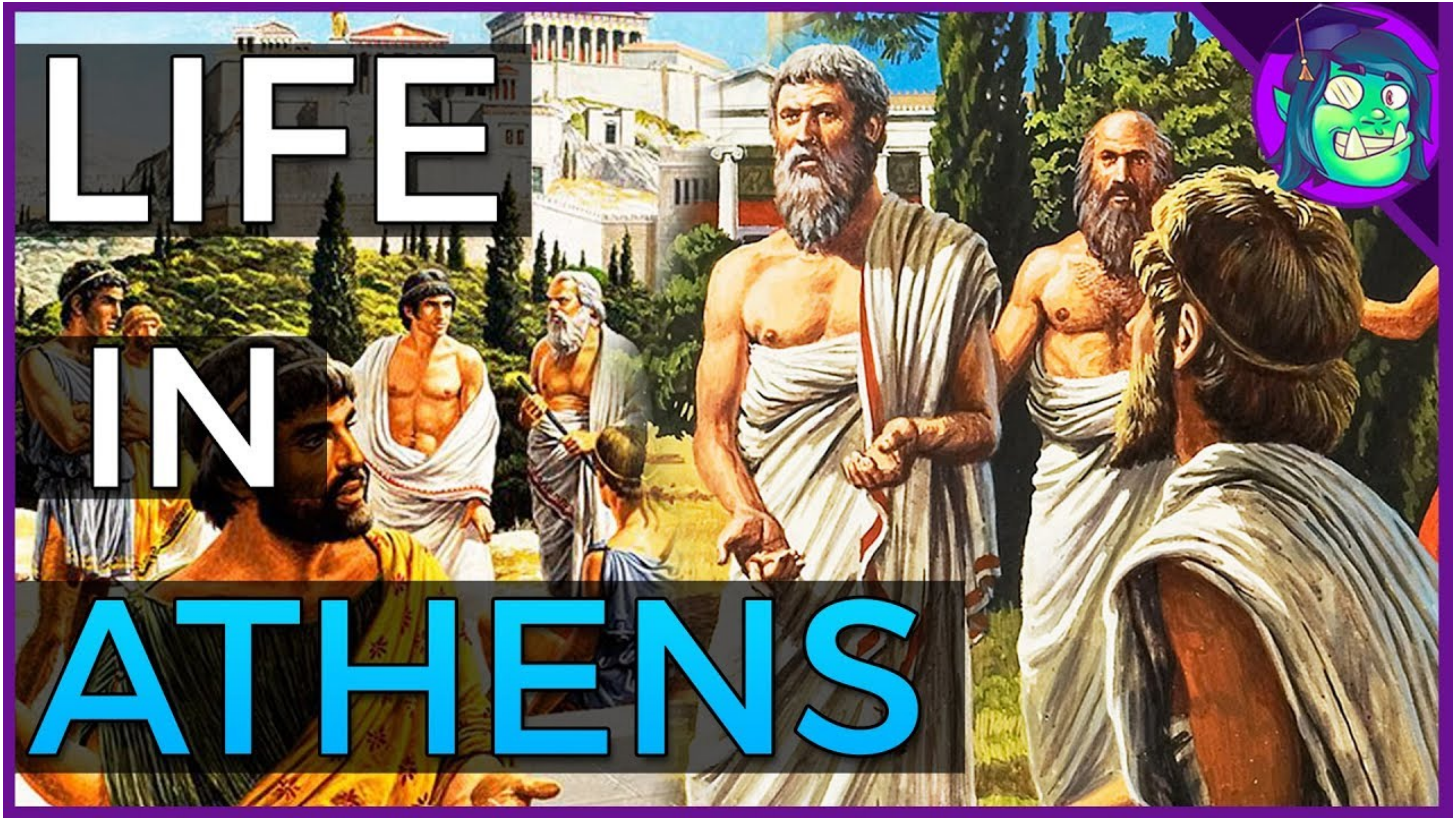

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- The major sources were likely the Pythagoreans, Hippocrates and Eudoxus. Many of the proofs are believed to be original to Euclid.
- Its success is due to its logical and axiomatic presentation. This deductive style is the central approach to mathematics today.

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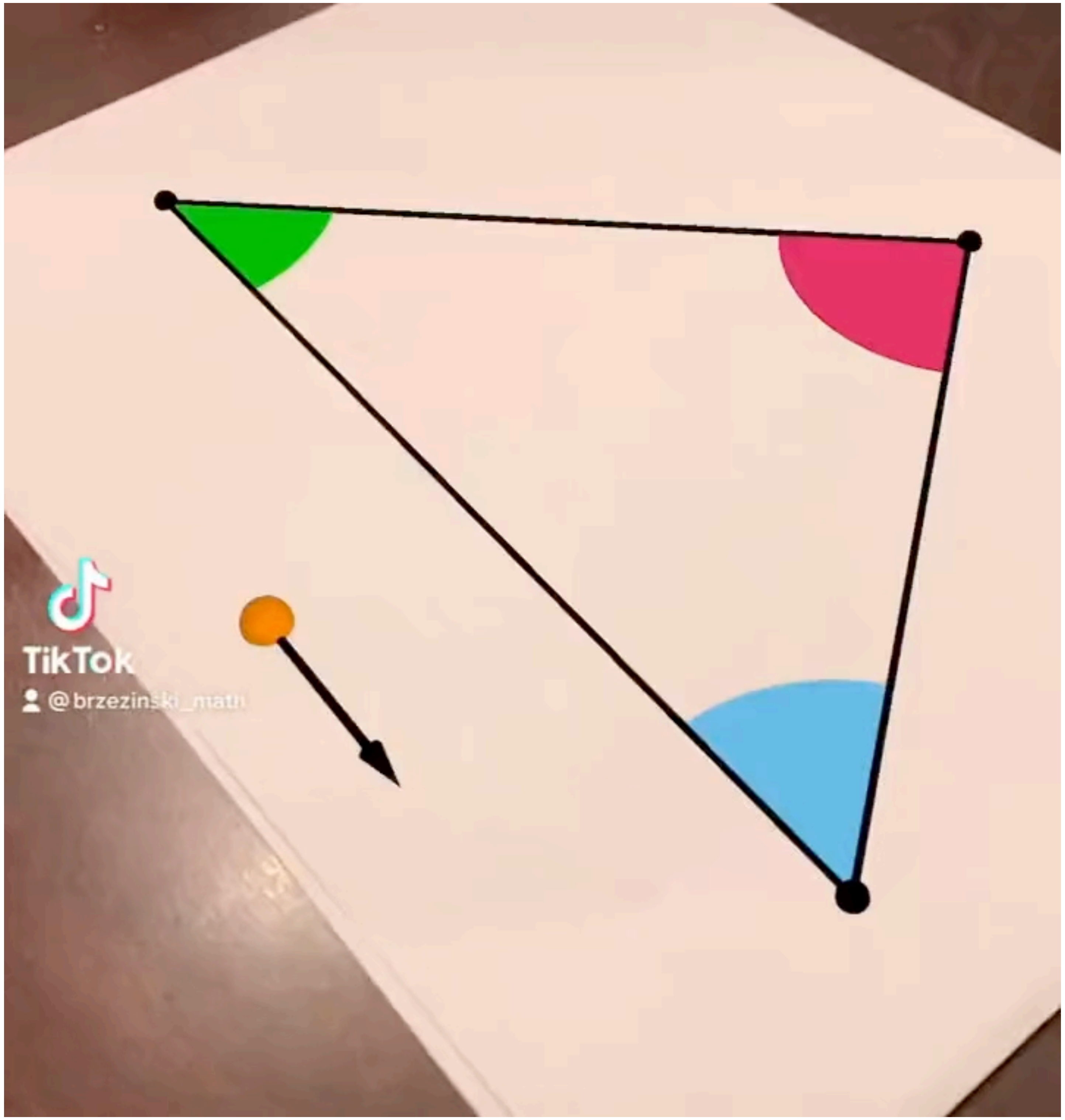
- Euclid’s postulates allow him to use a straightedge and compass, which play a central role.

Definitions, Common Notions and Postulates

Proving Propositions

Triangle Angle Sum



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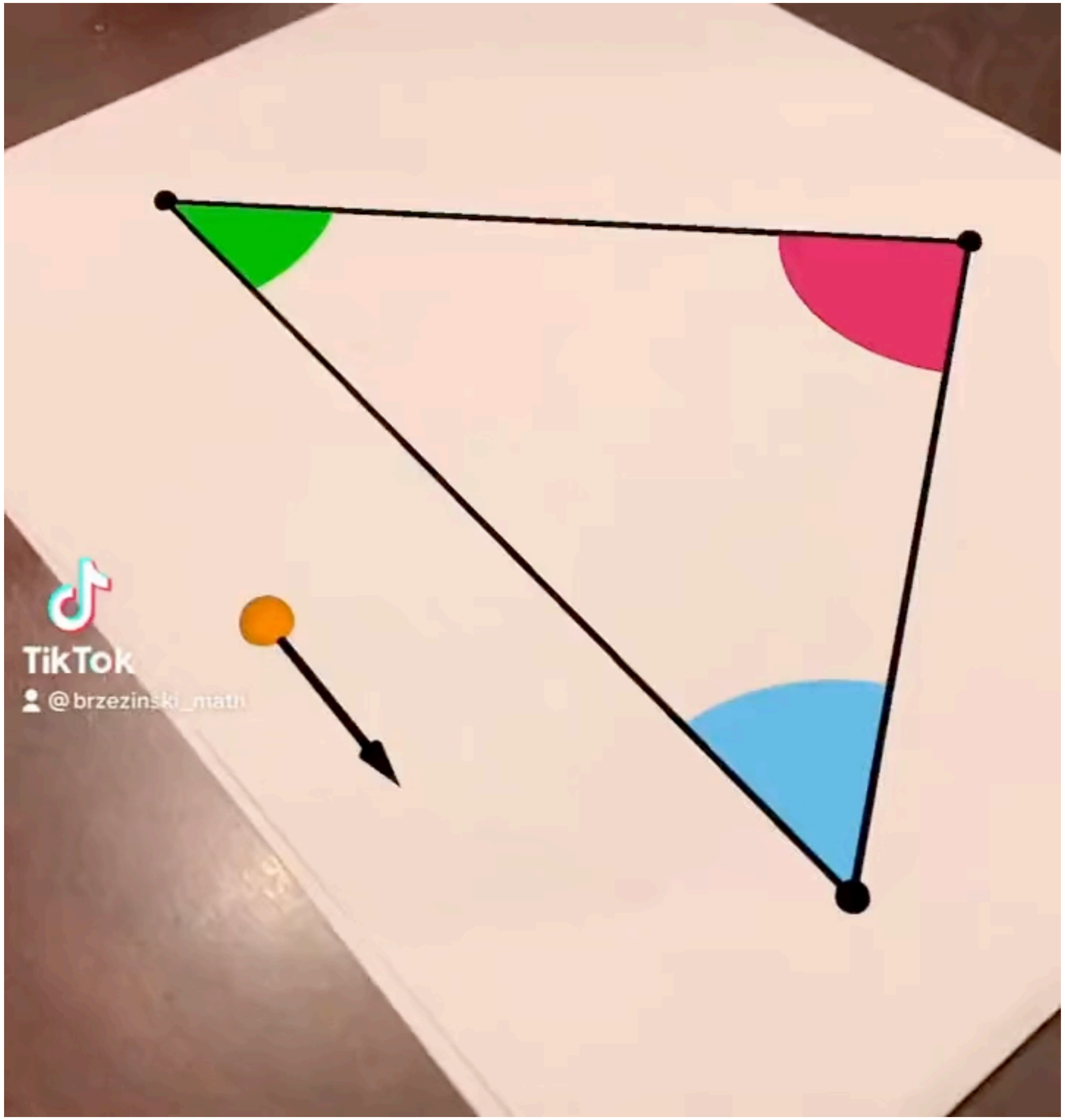

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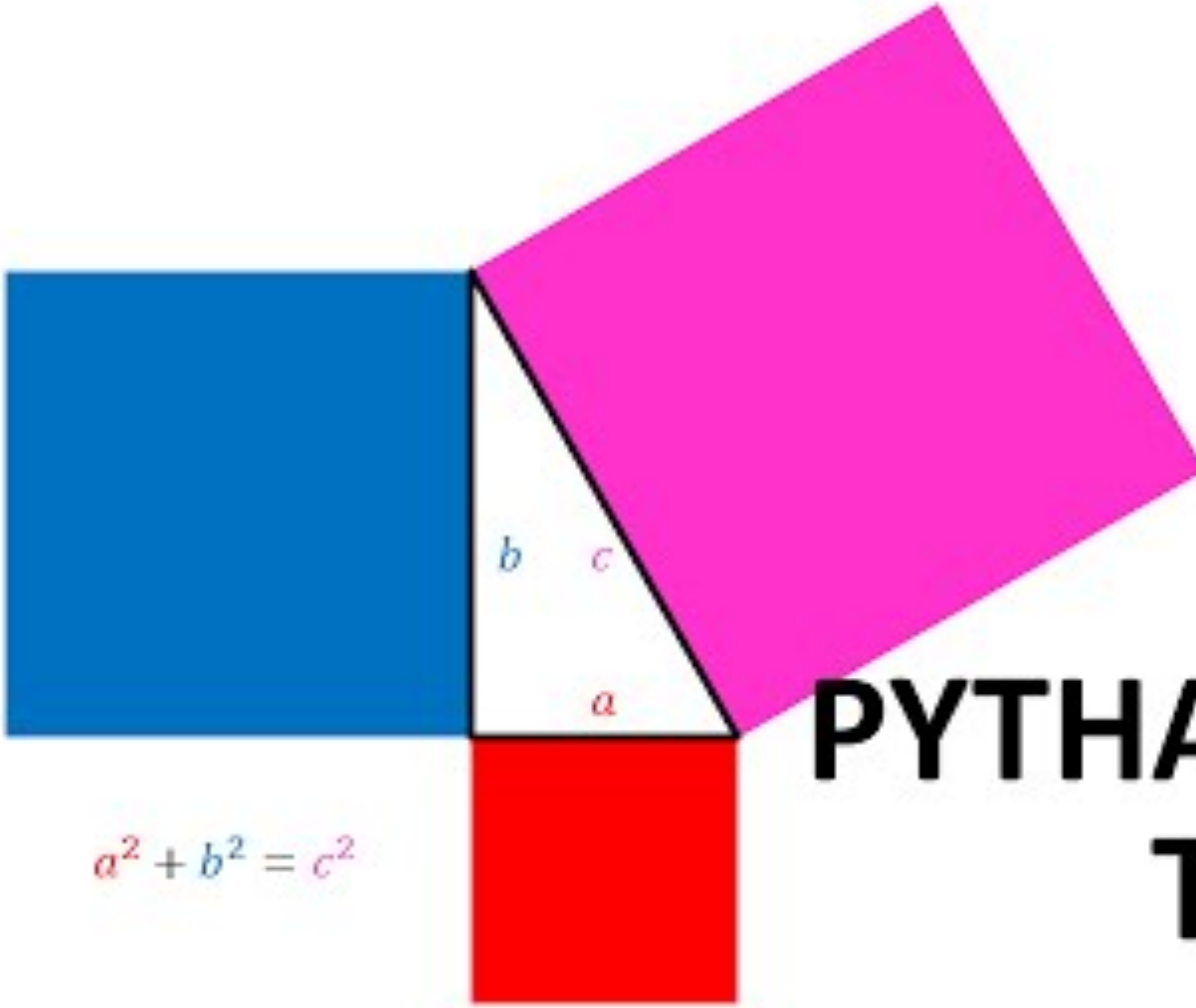
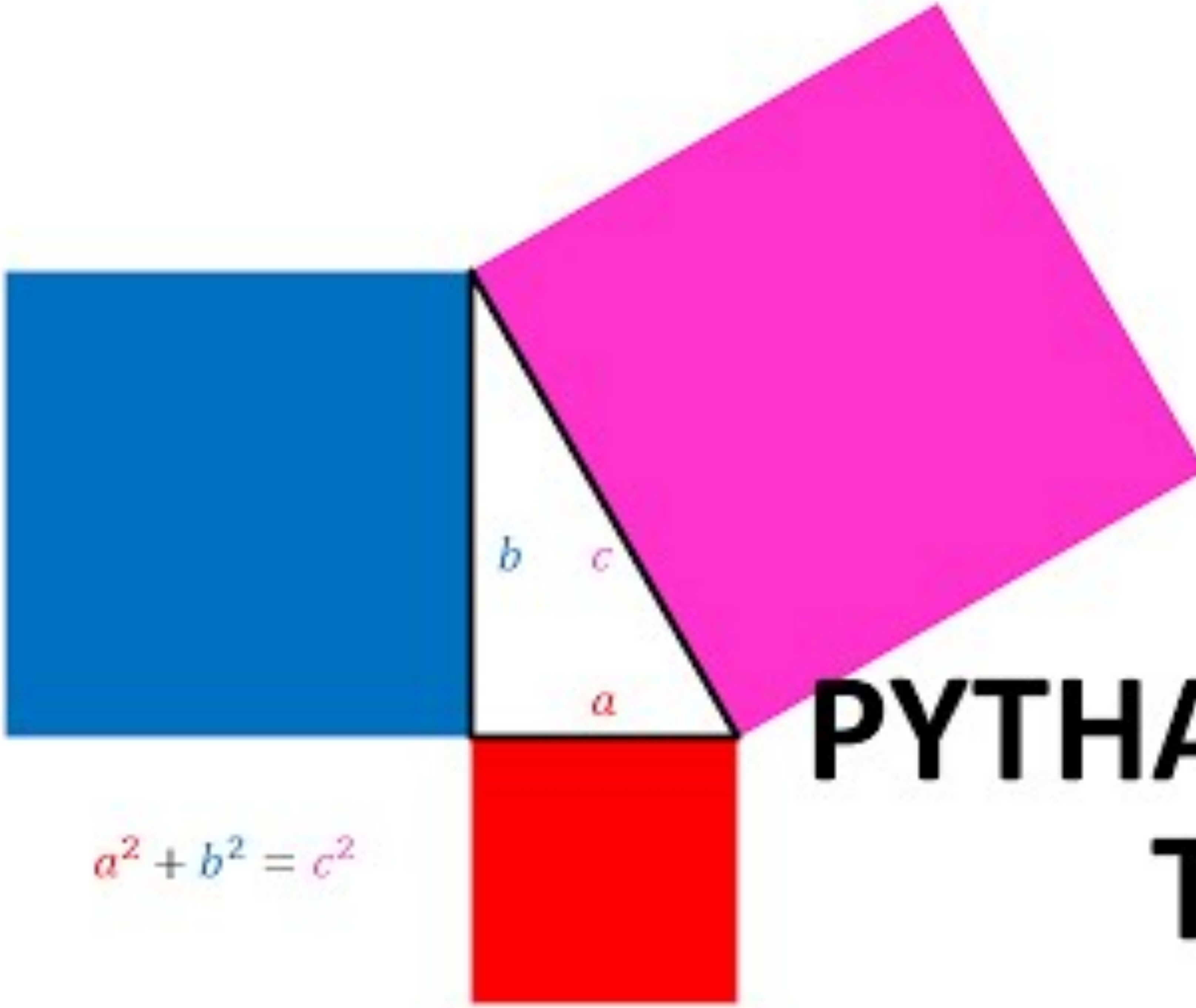


PYTHAGOREAN THEOREM



$$a^2 + b^2 = c^2$$

PYTHAGOREAN THEOREM

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- Ancient Greeks also know how to construct a regular pentagon, and given a regular polygon with m sides, they could construct a regular polygon with $2m$ sides.

- Major open question: For which values of n is the regular n-gon constructible with a straightedge and compass?

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- Then, at 24, he entirely classified which regular polygons are constructible.

- He did not prove one direction of his theorem, though. That was done by Pierre Wantzel.

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- A Fermat prime is a prime of the form $2^{2^k} + 1$.

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The Elements

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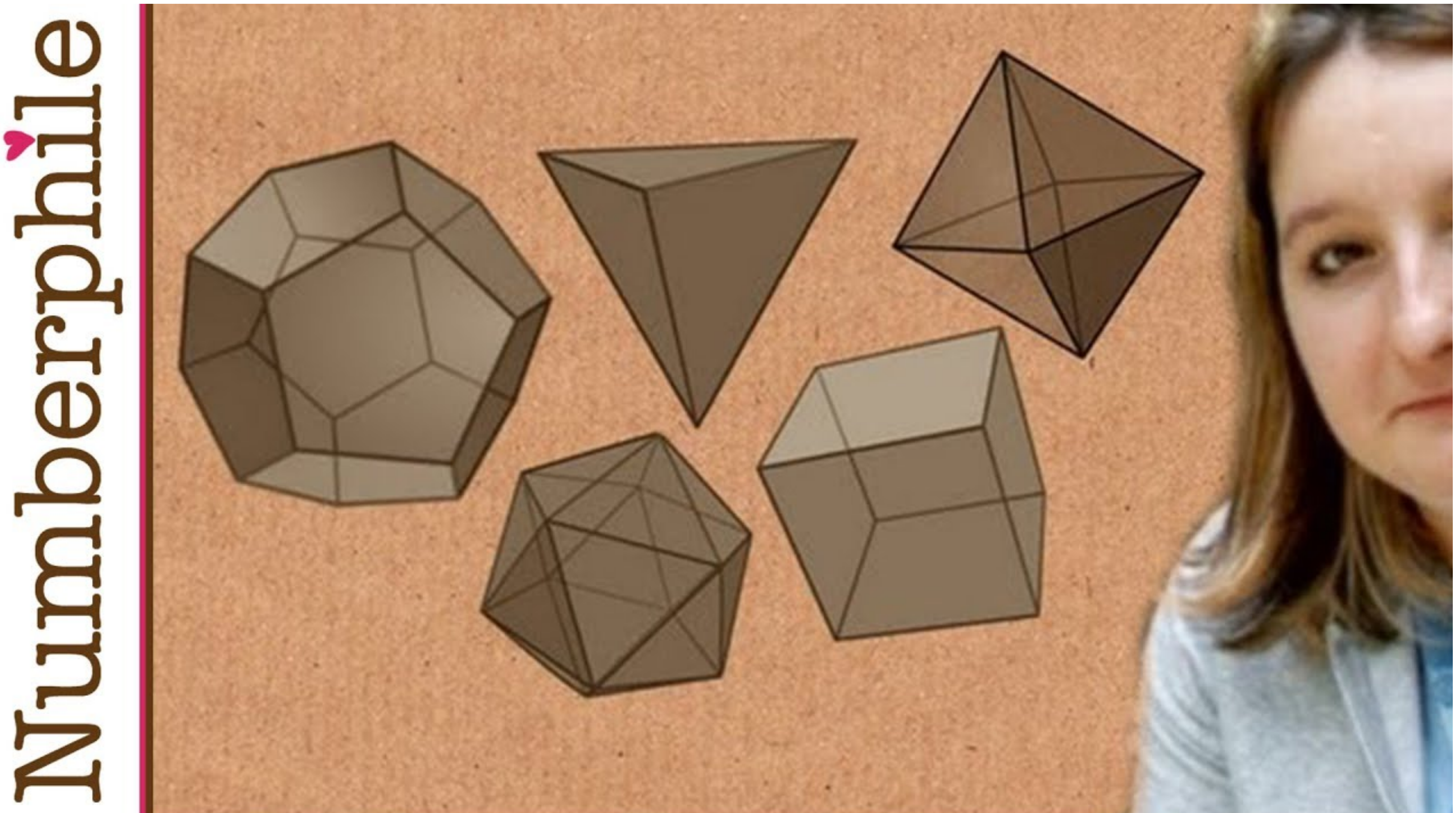
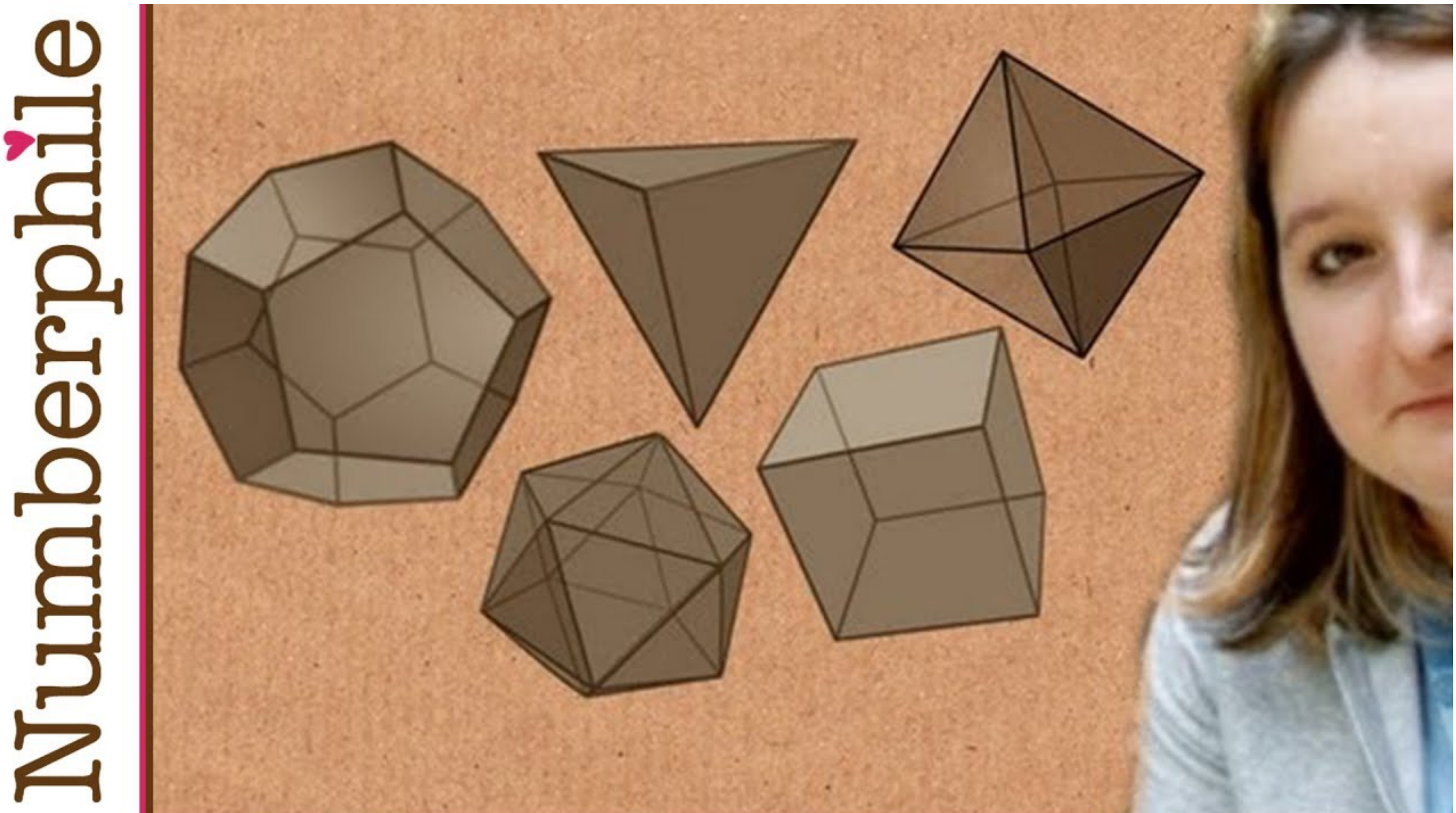
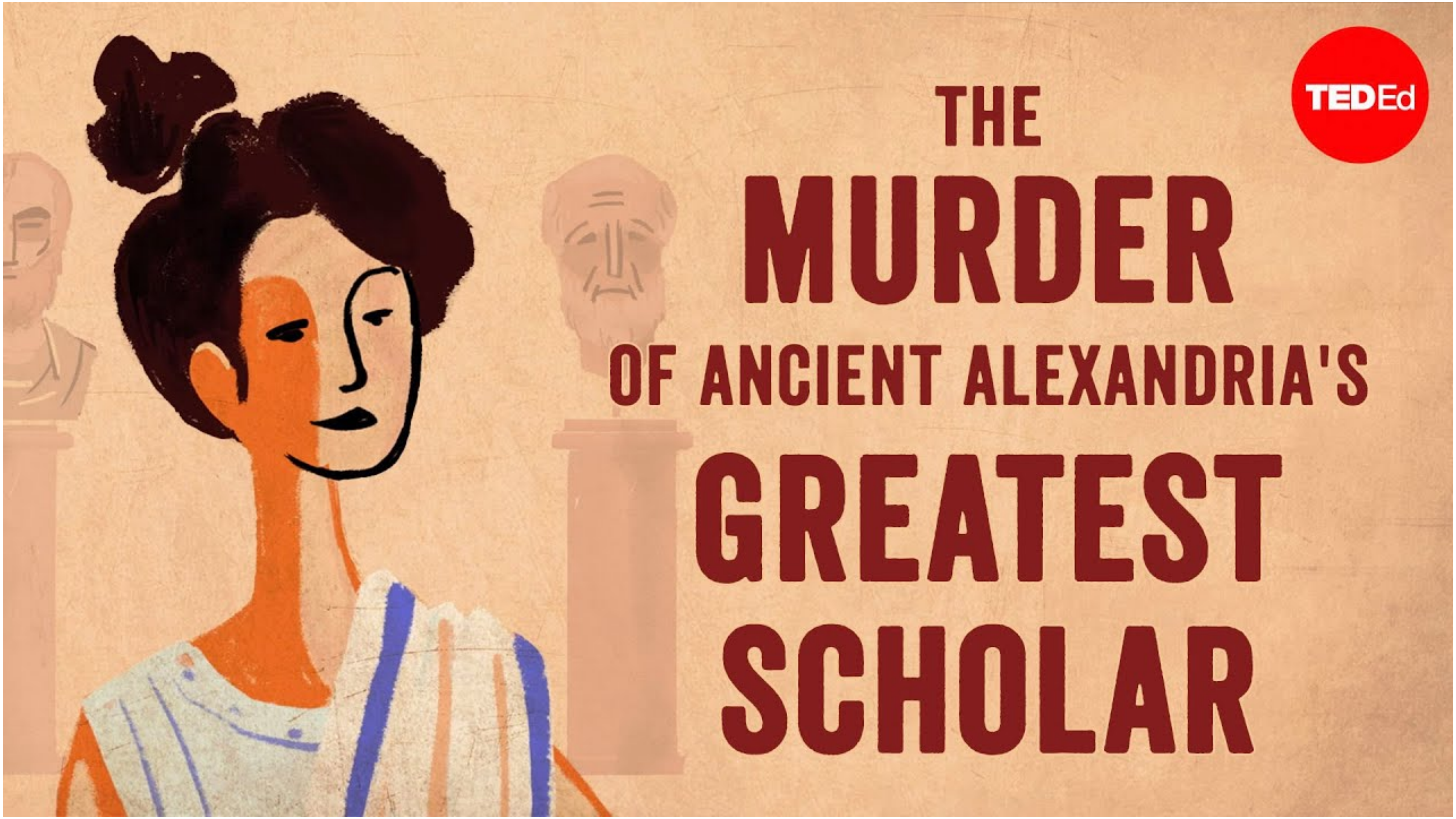
Platonic solids



Platonic solids



Hypatia



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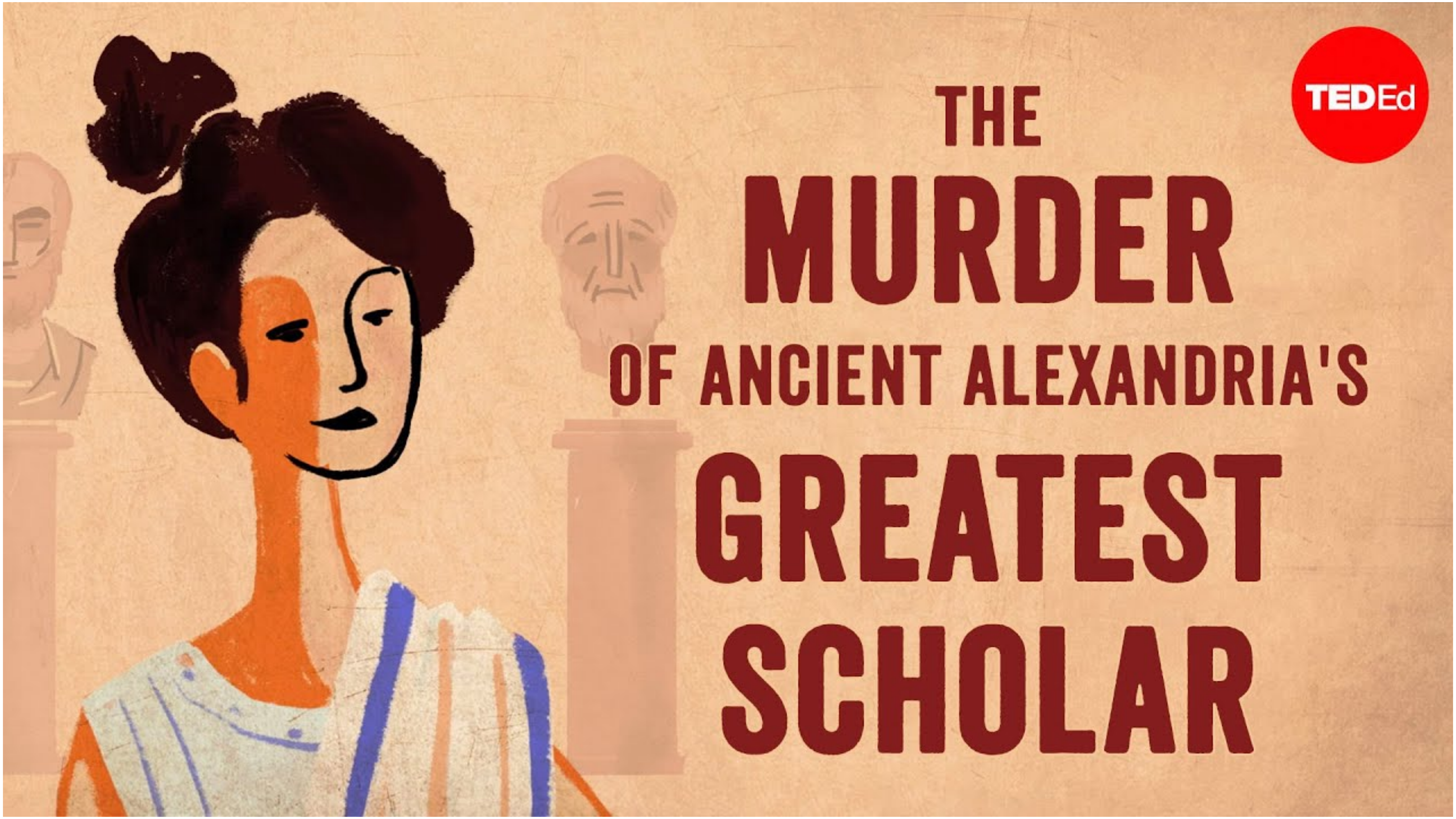

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- She was a first-class thinker and highly respected philosopher, mathematician and teacher.

- Her life ended in tragedy, as she was brutally murdered for defending religious freedom.

The Legacy of Axiomatic Thinking

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- Can that postulate be deduced from the other four? Unfortunately, no.

- In this way, the parallel postulate is *undecidable* if your axioms are only the first four.

The Aftermath

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- This implies that there exist non-Euclidean geometries. Instead of the parallel postulate, you can have an axiom saying that if ℓ is a line and P is a point off of ℓ, then there are 2+ lines through P that are parallel to ℓ. Or no lines that are parallel.

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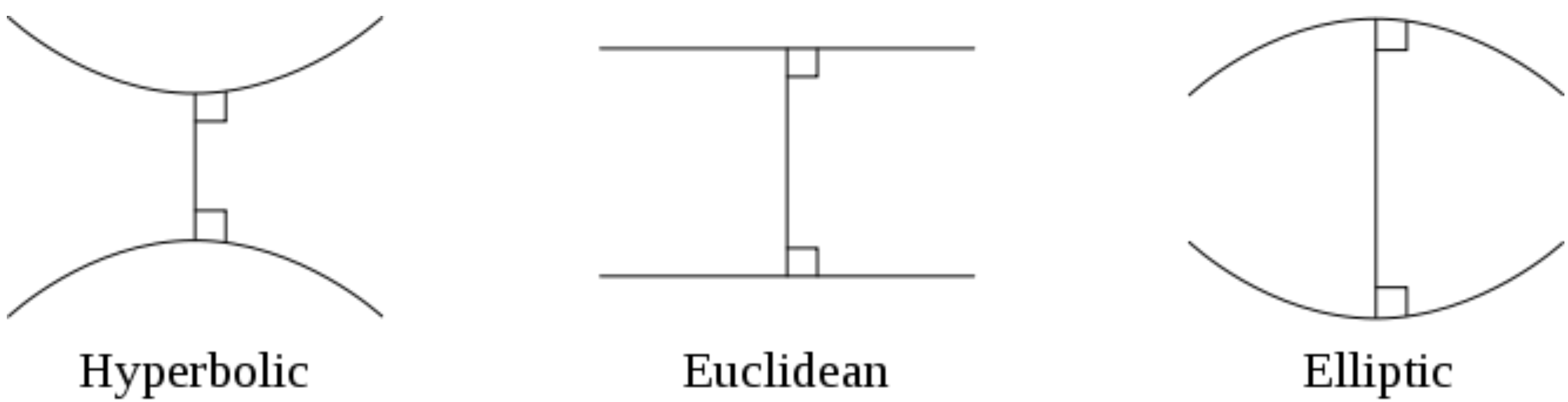

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- From these, not only is set theory built, but number theory, too.

- Kurt Gödel proved that every set of axioms that leads to number theory must have undecidable statements. There must be statements that can neither be proven nor disproven.

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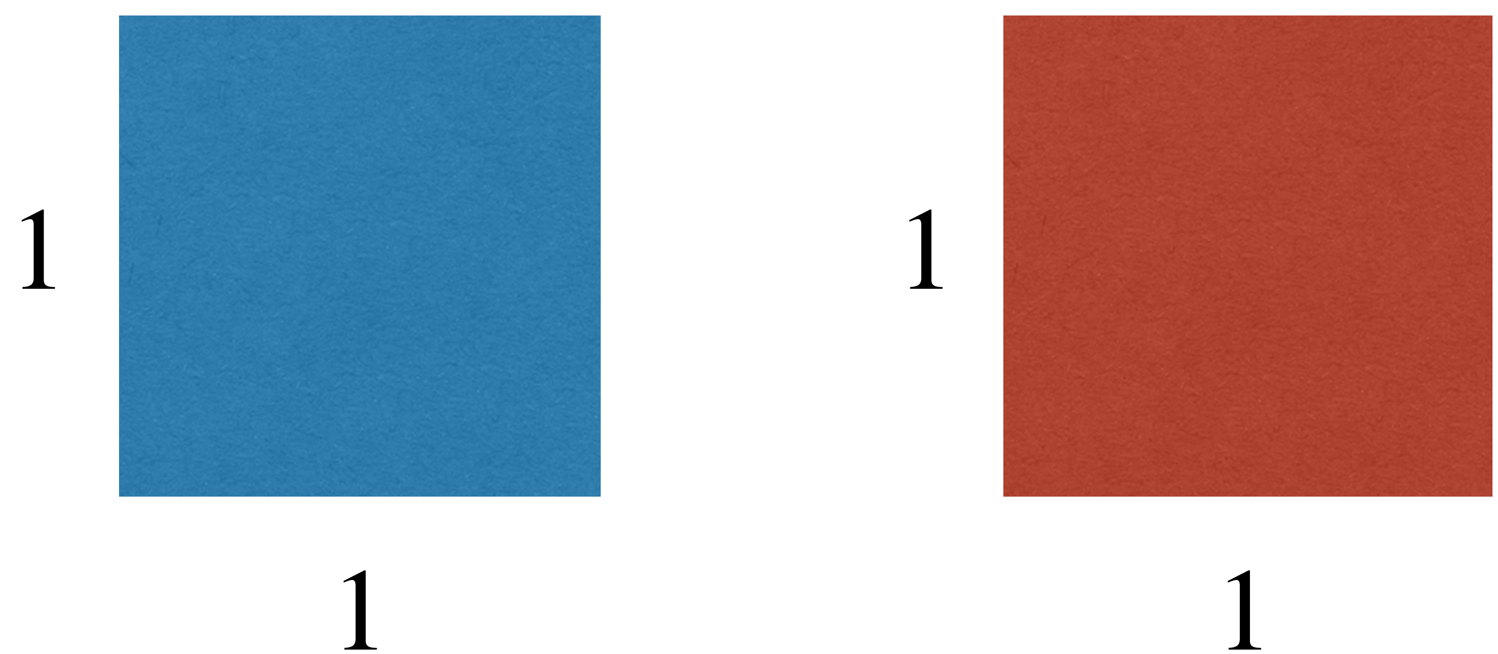
Approximating $\sqrt{2}$

Suppose you have two 1×1 squares



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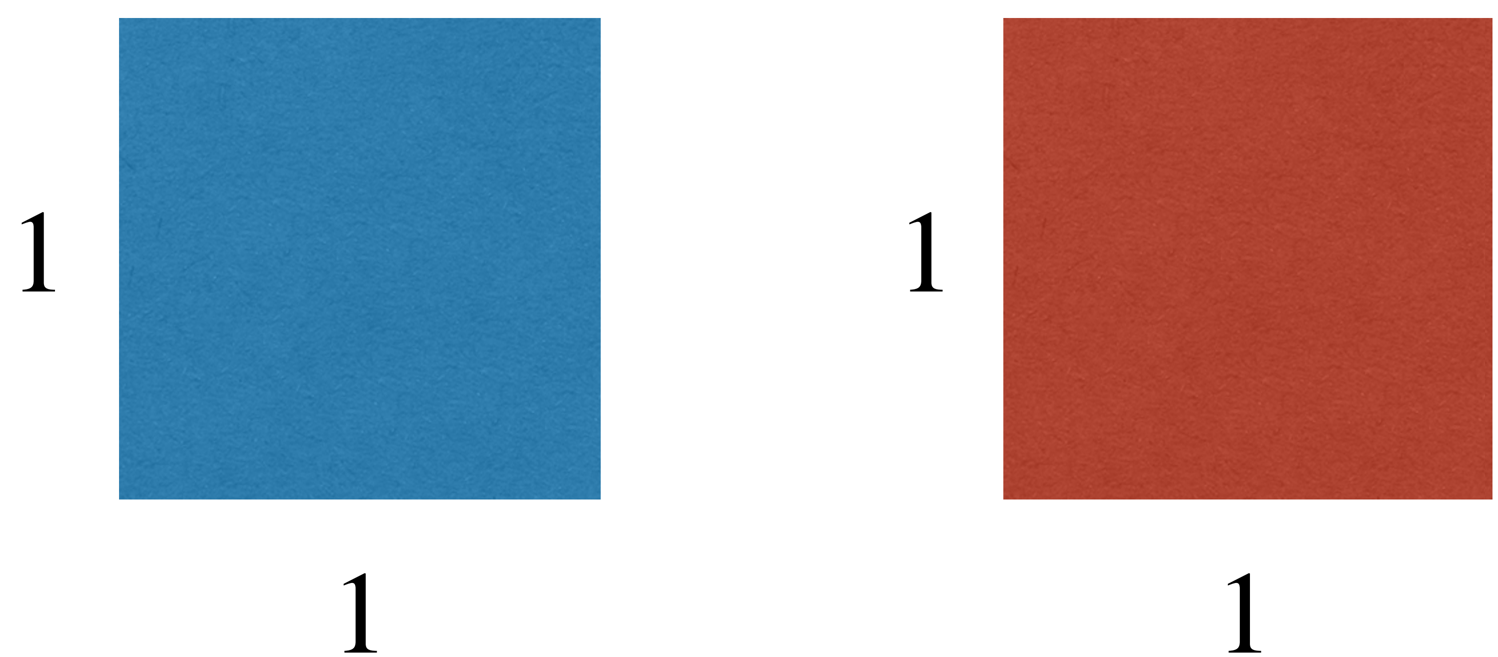

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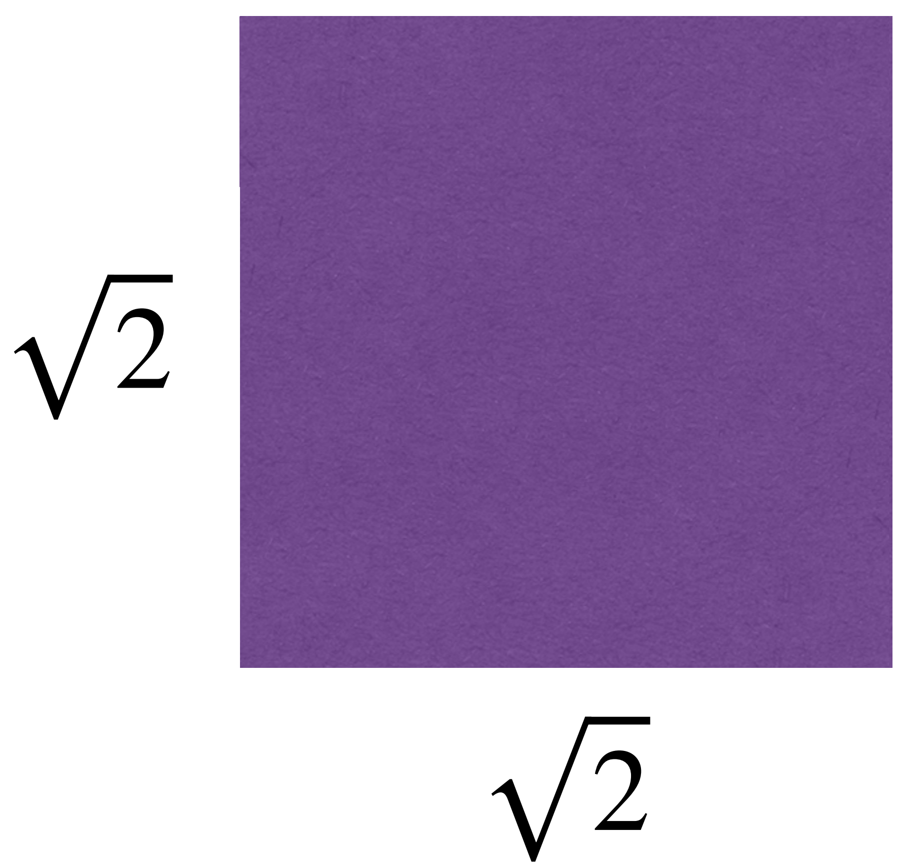
If you could combine them into a single square, it would be a $\sqrt{2} \times \sqrt{2}$ square



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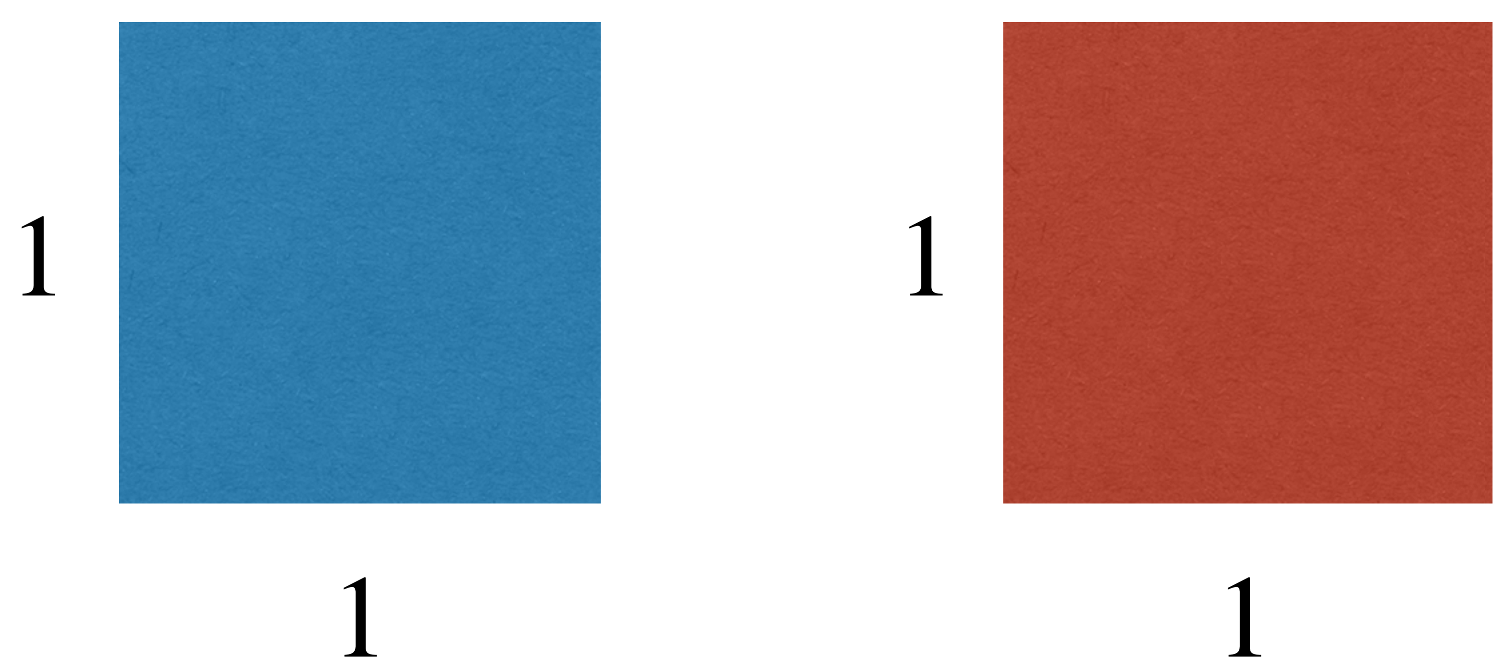

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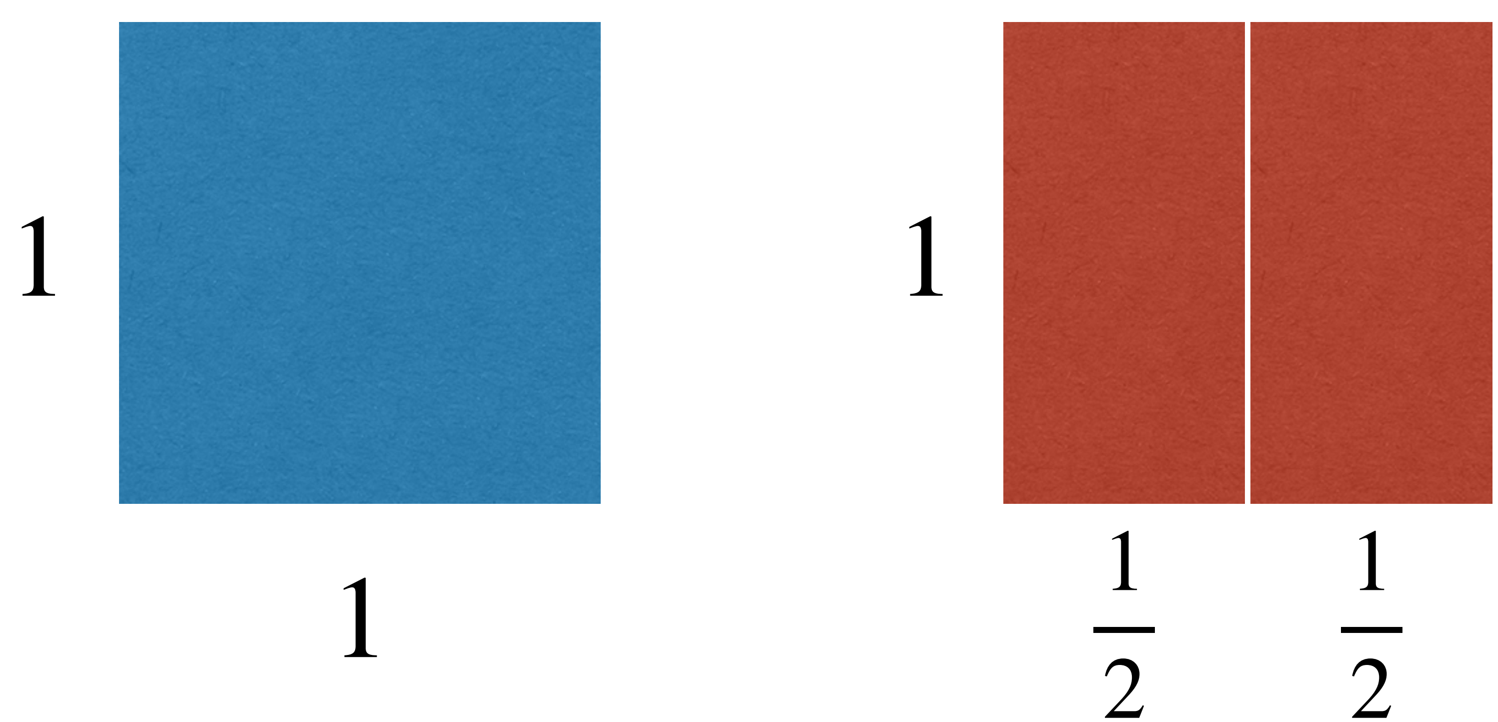

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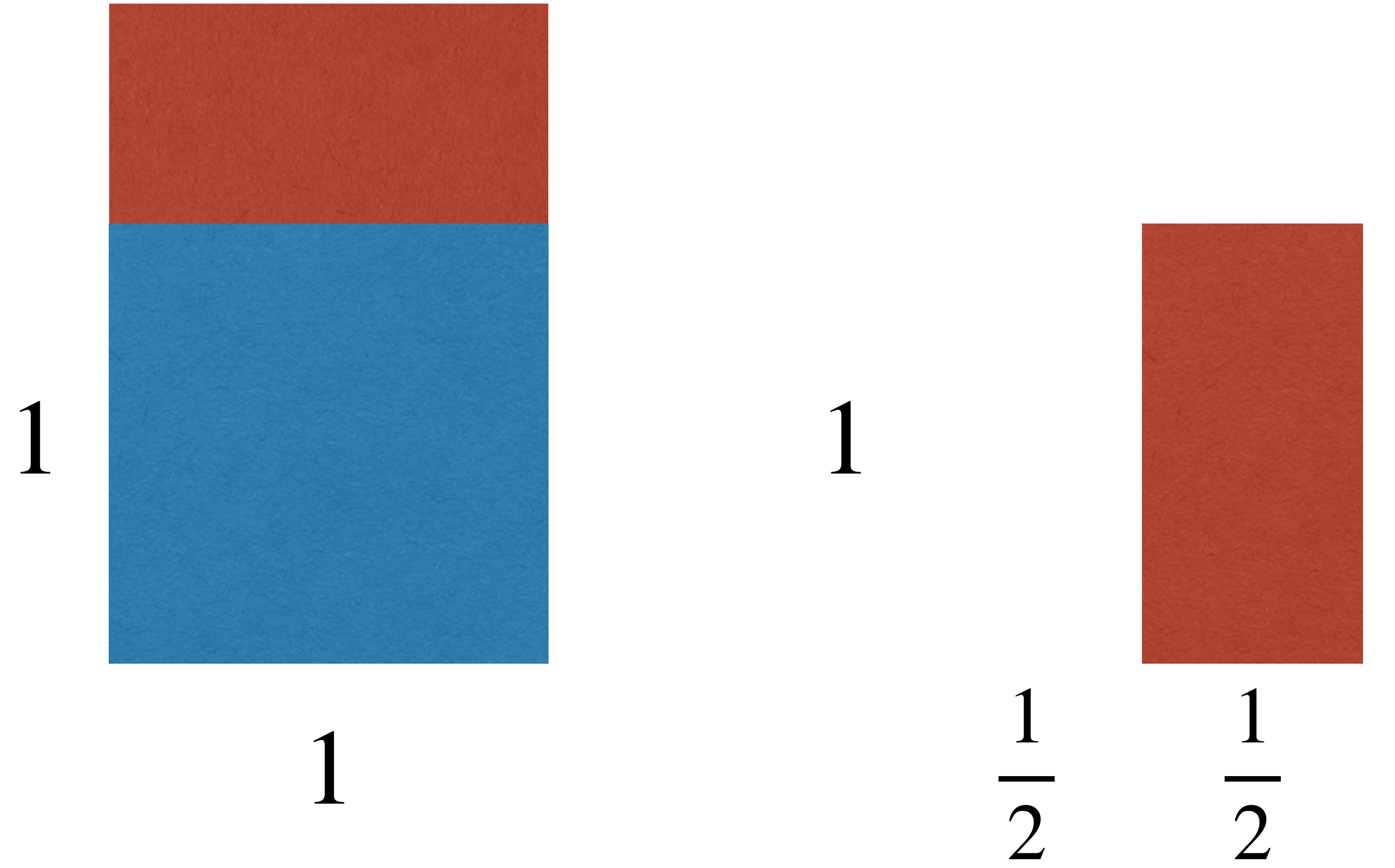

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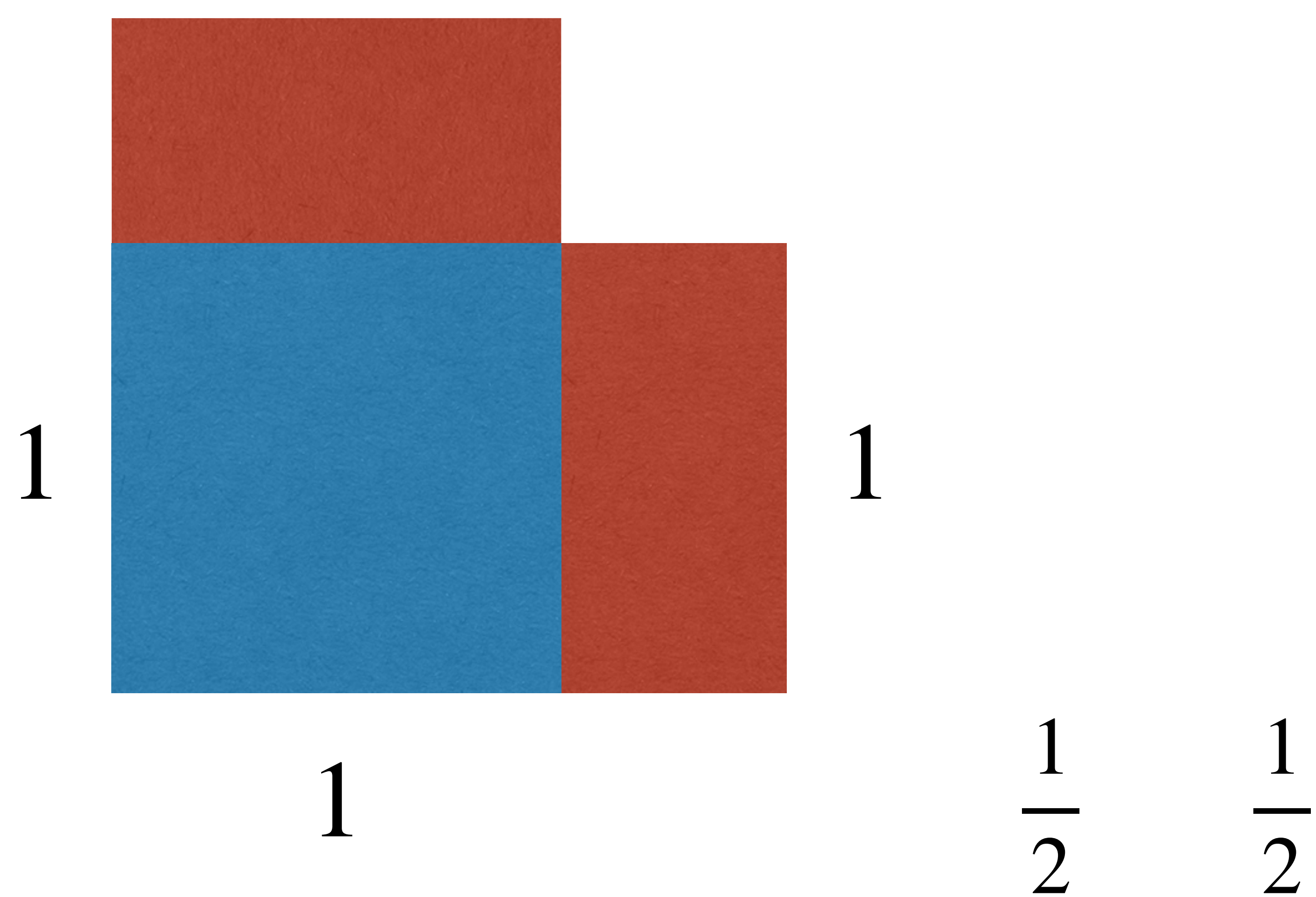

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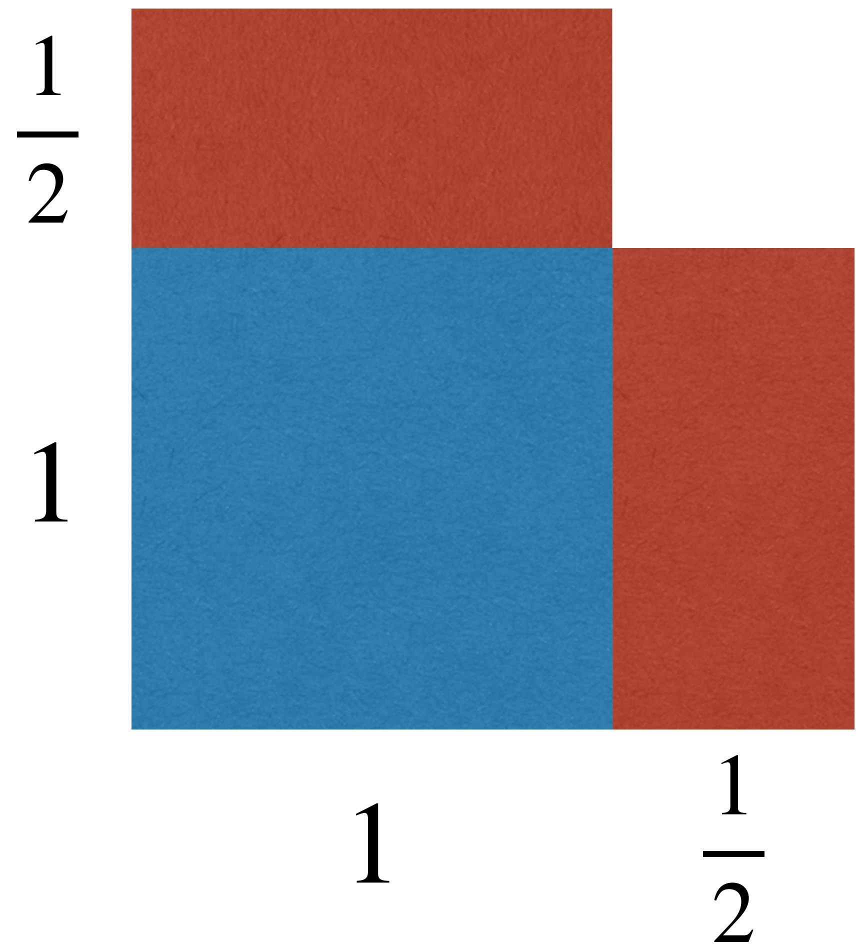

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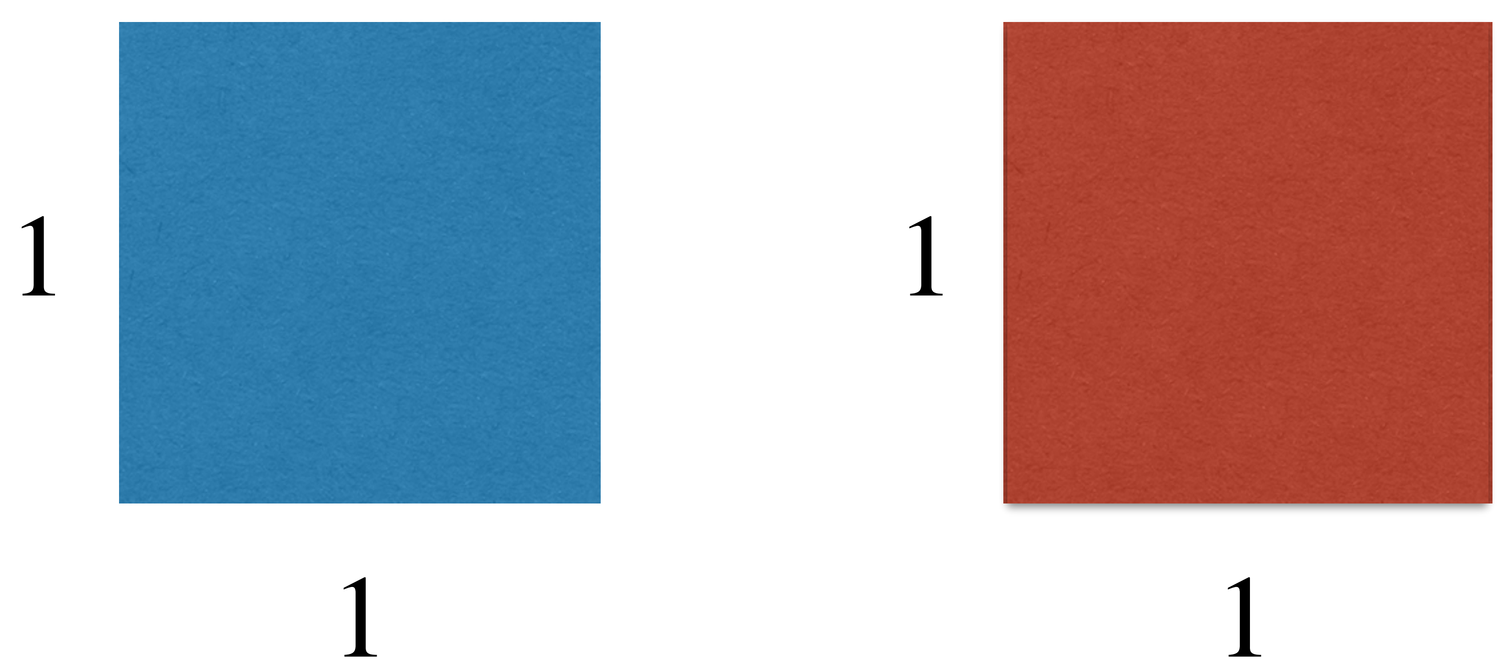

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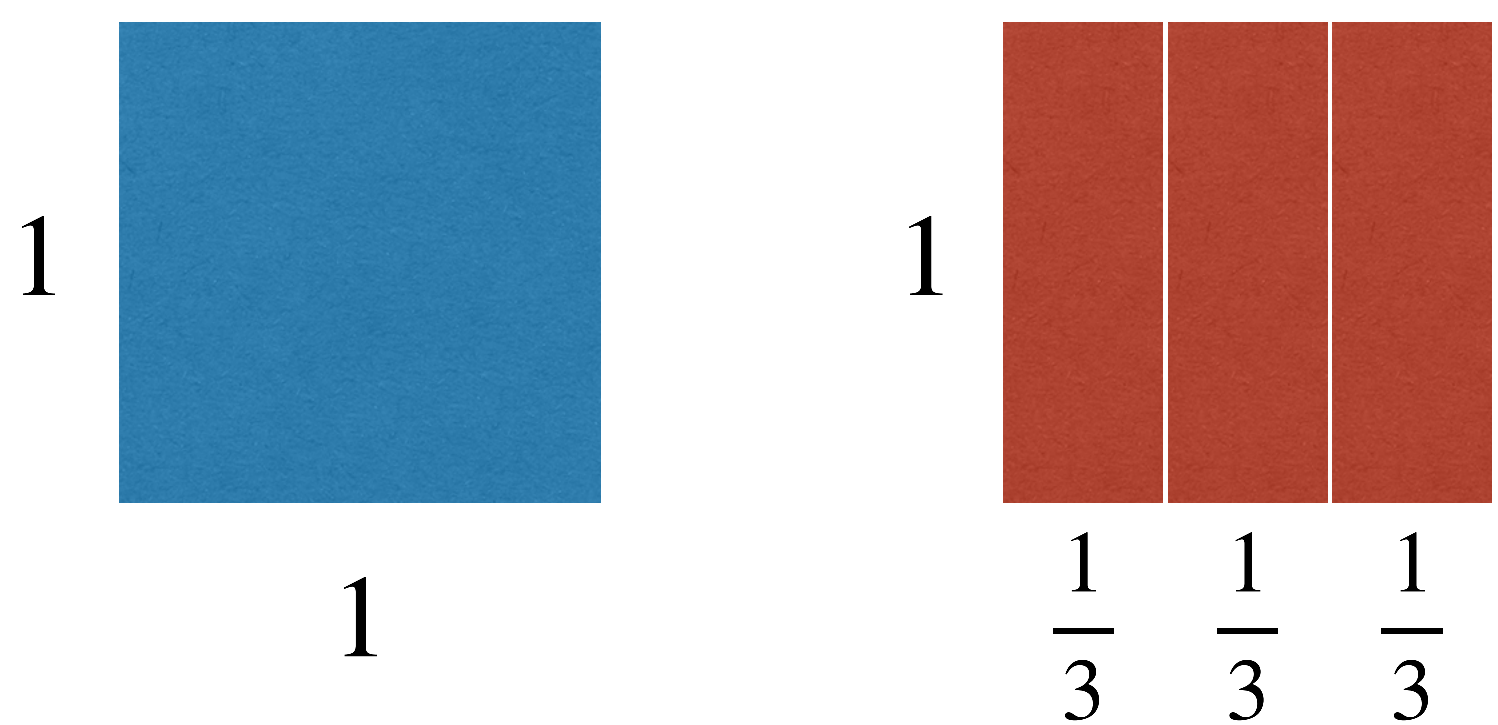

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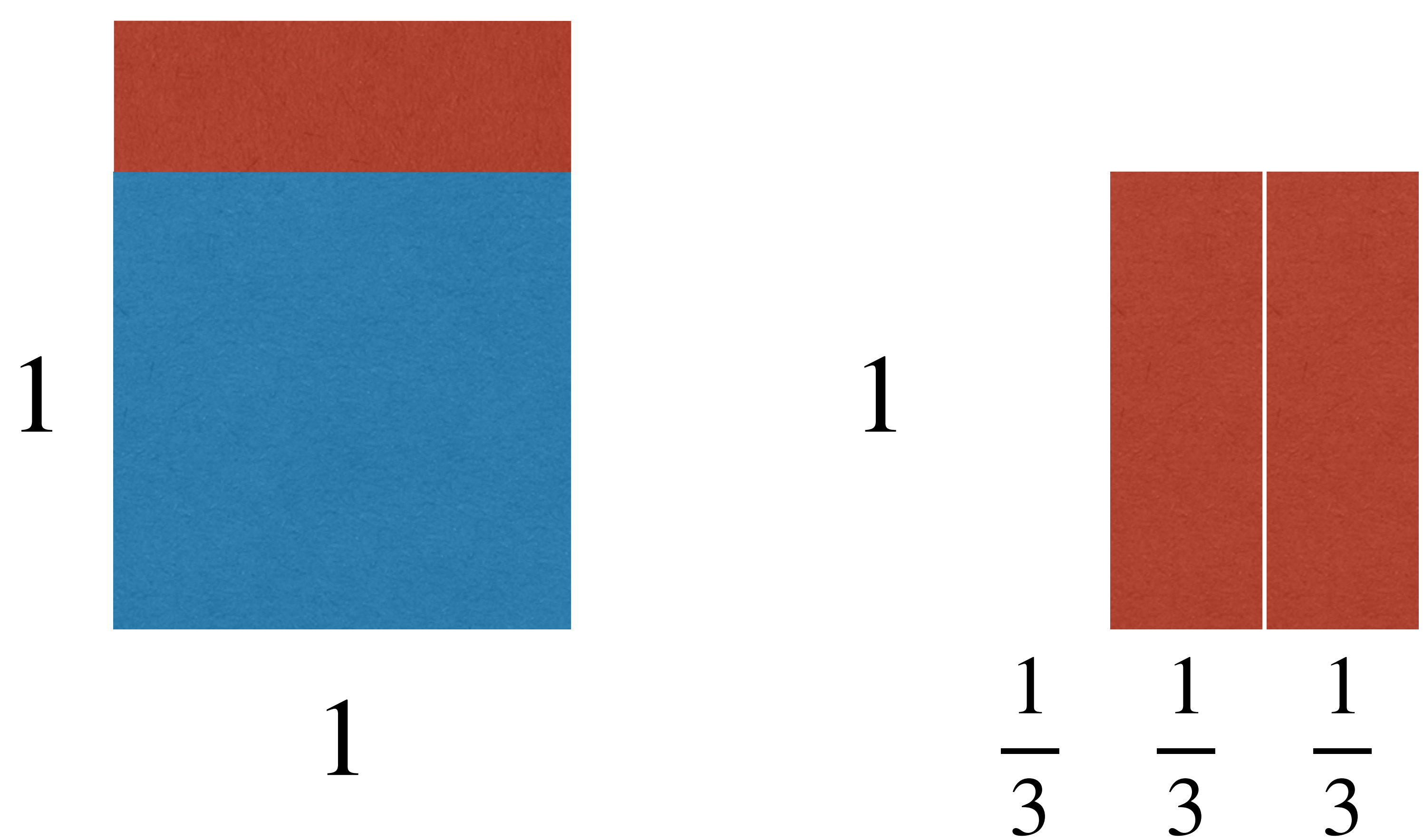

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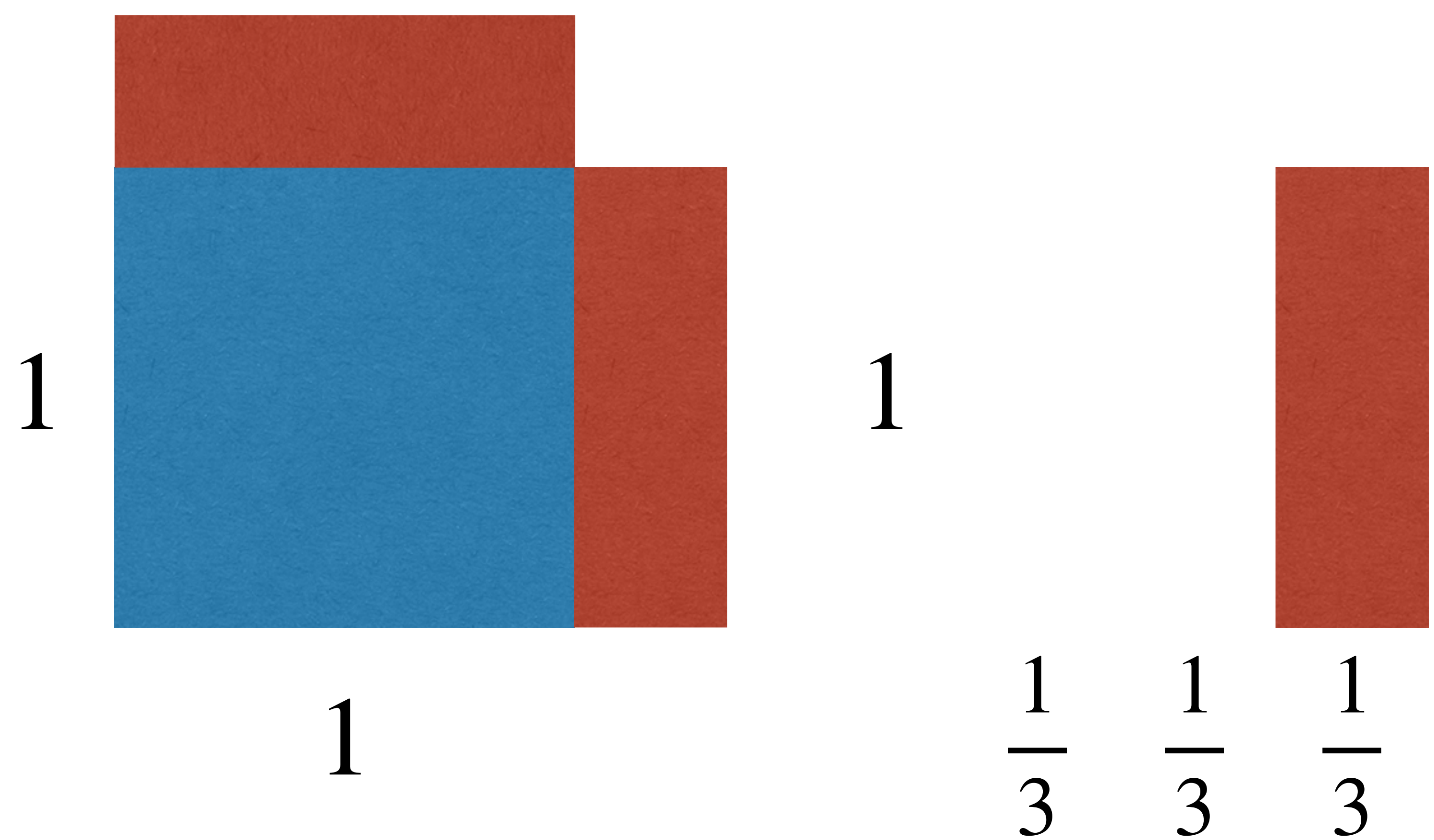

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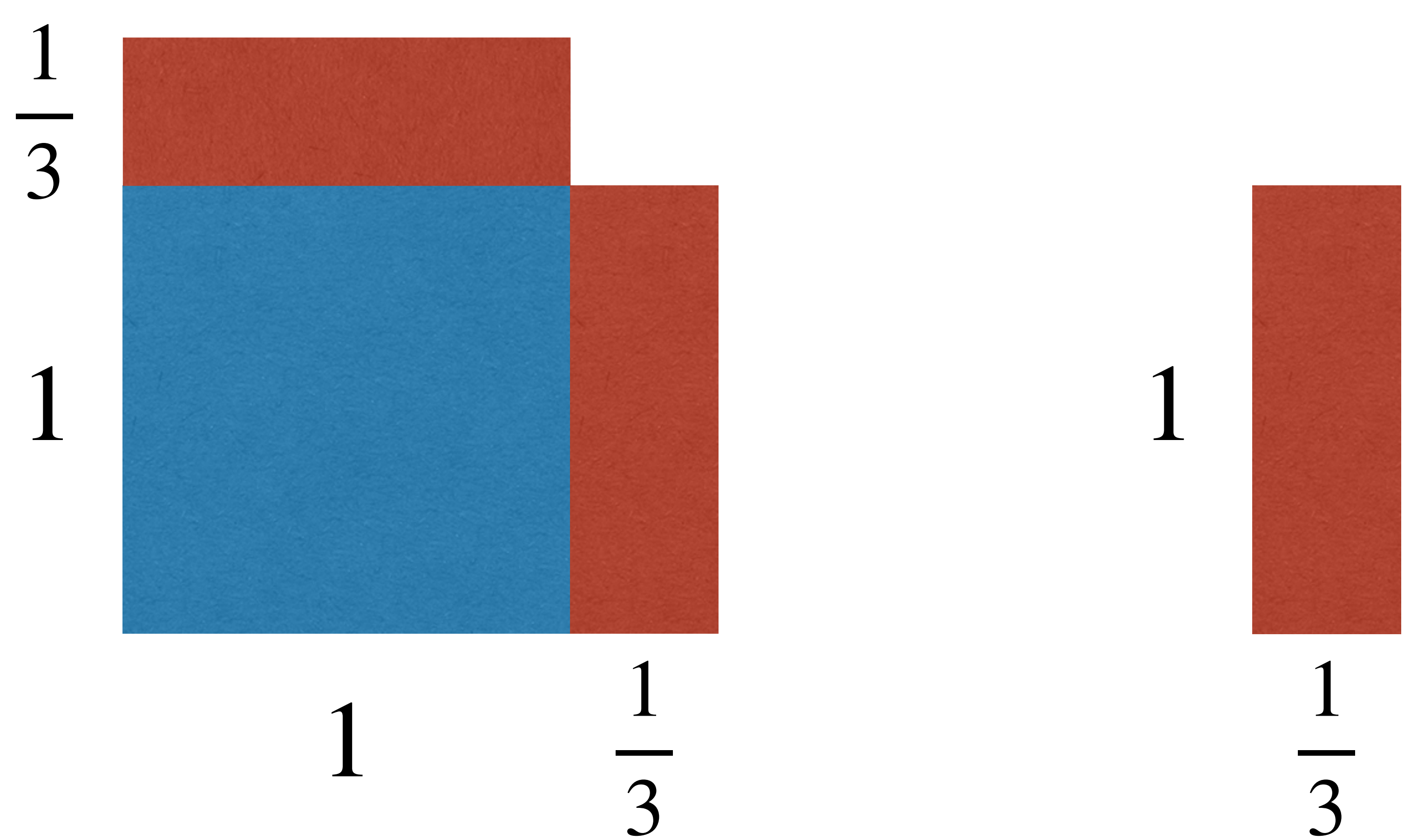

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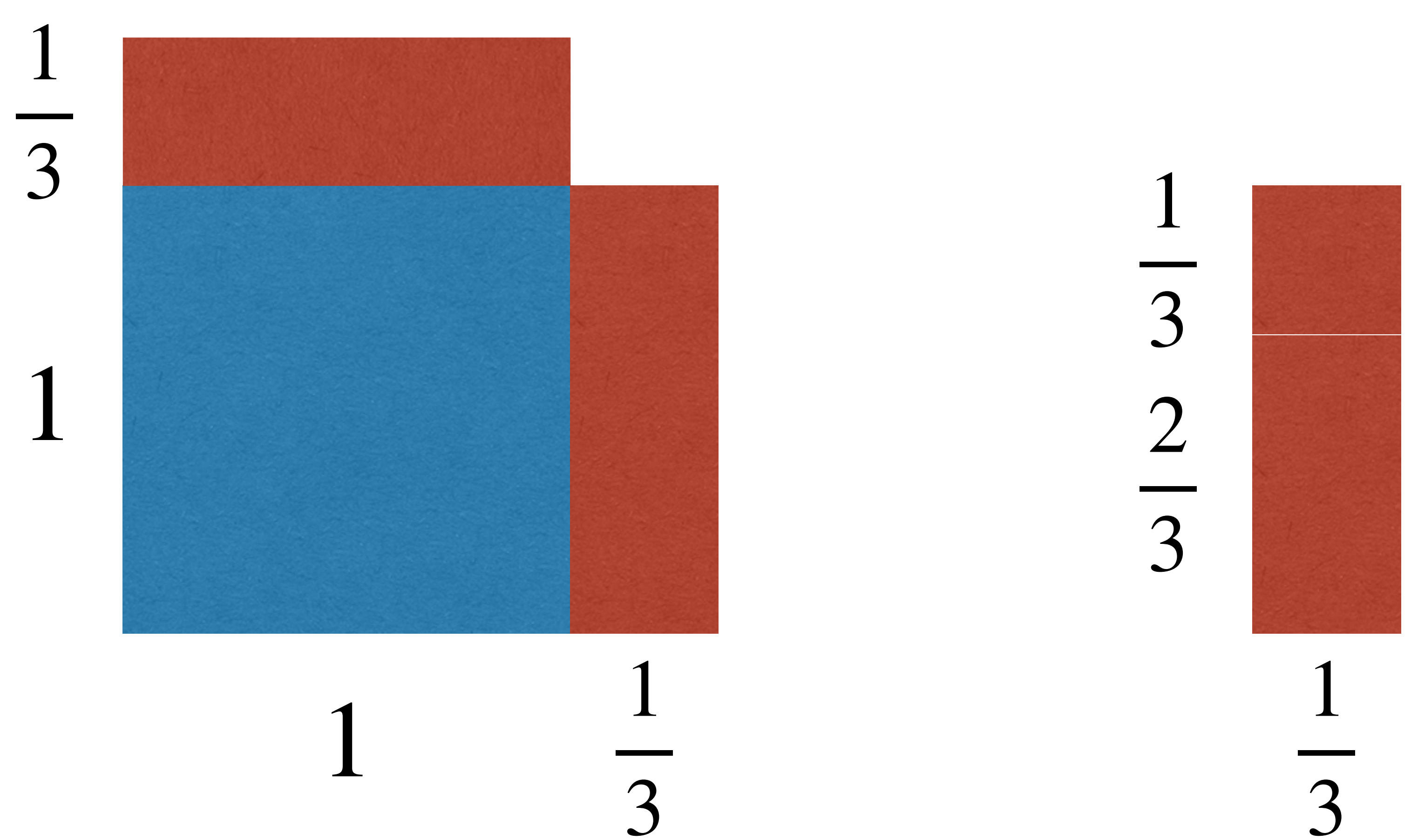

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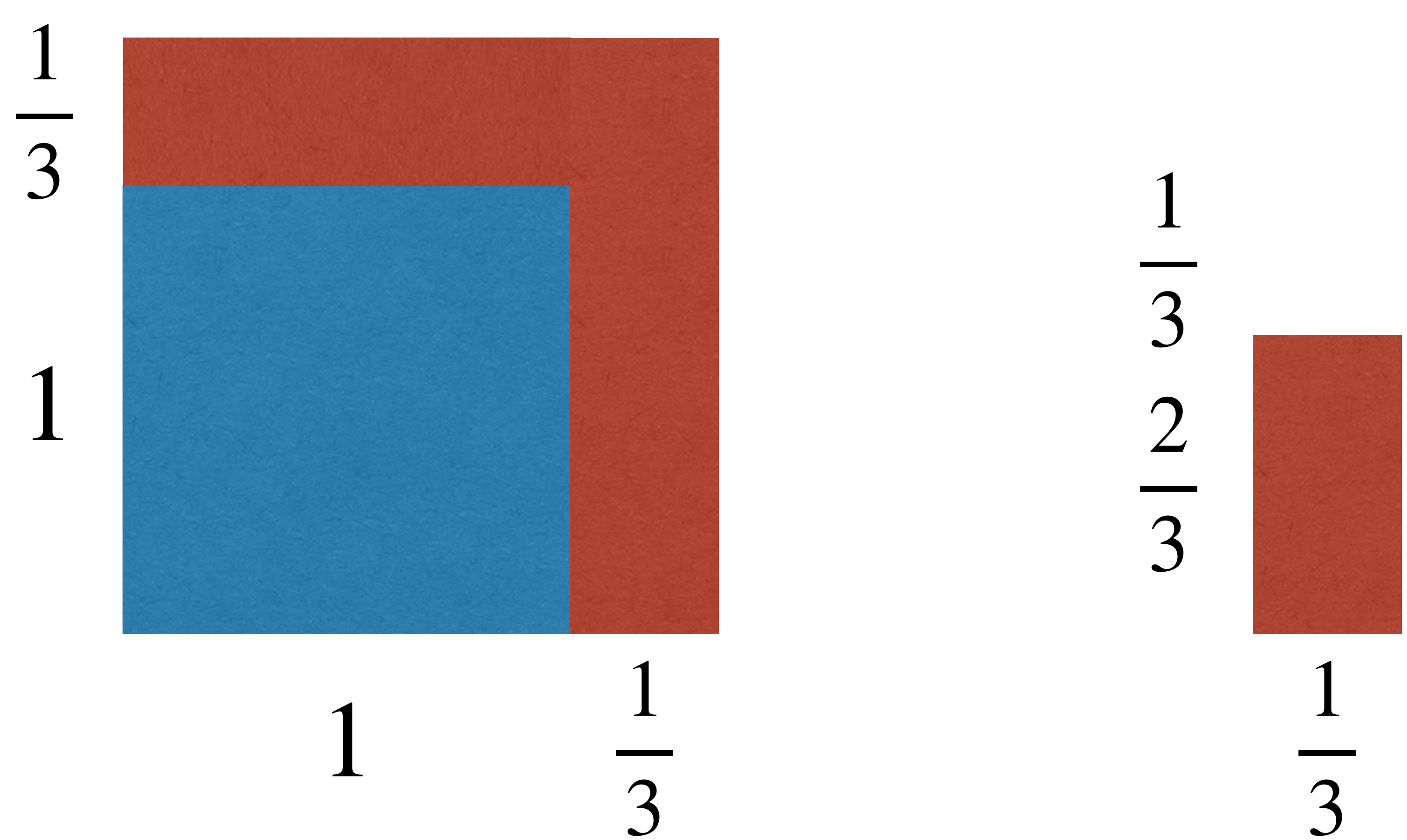

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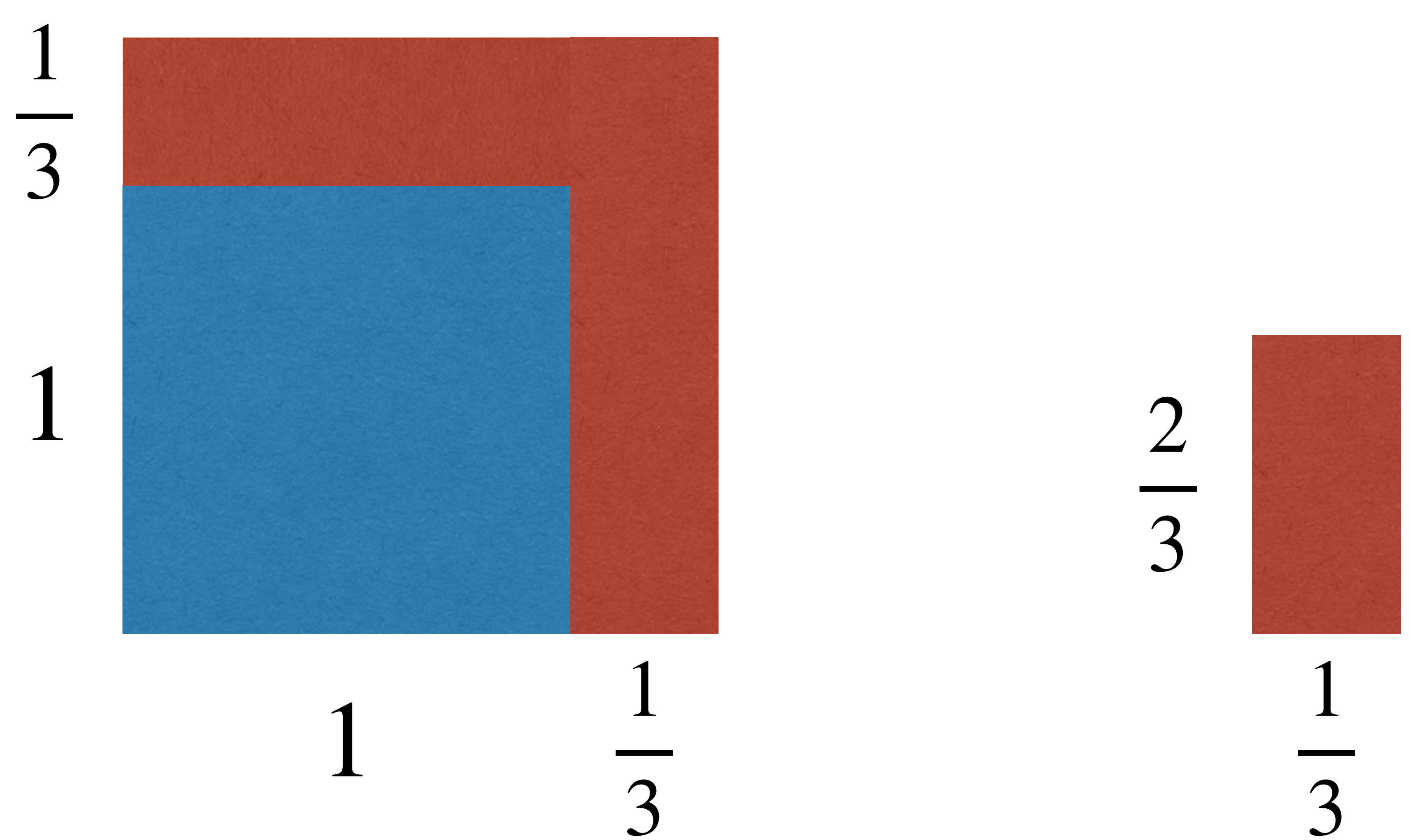

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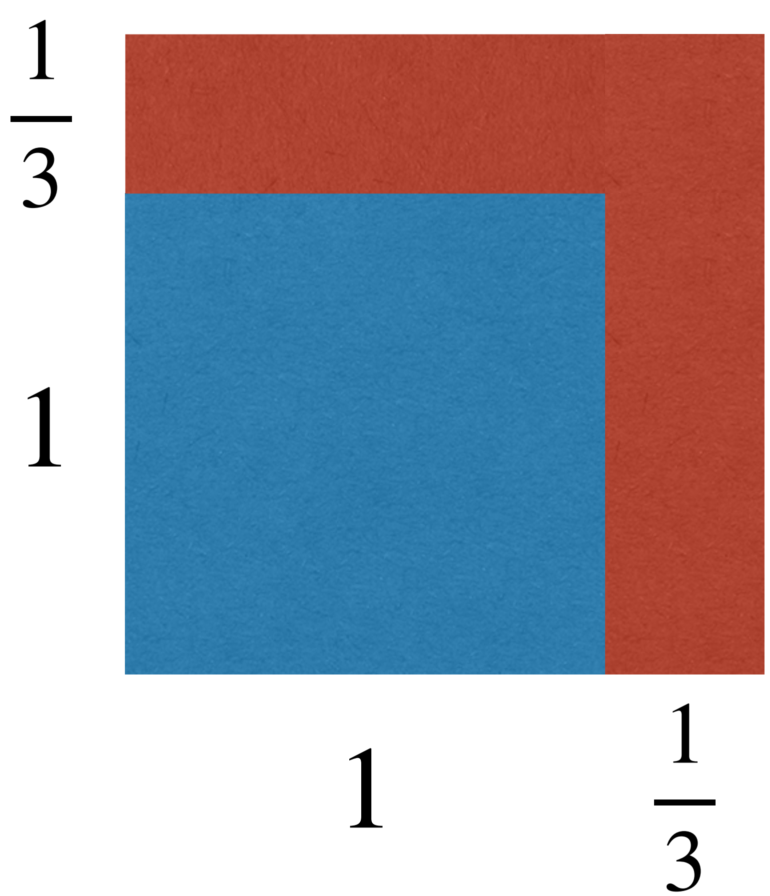
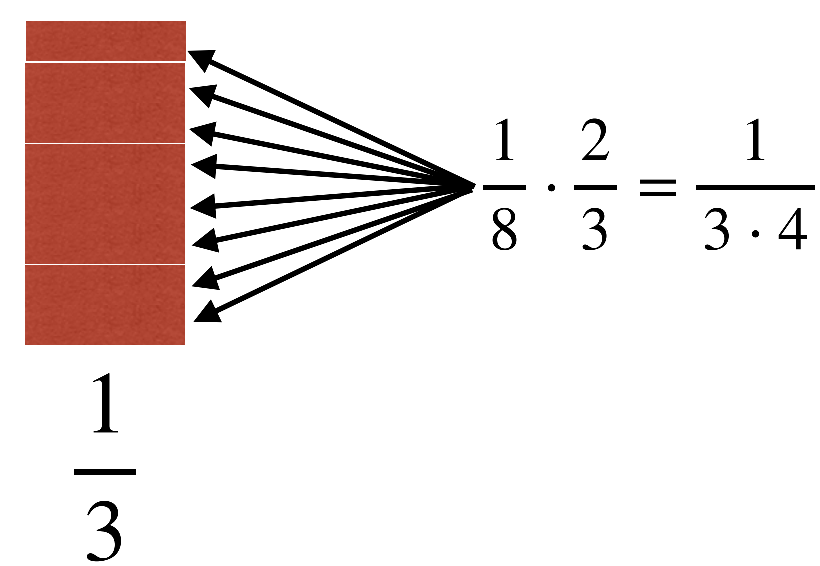
Divide into 8 pieces





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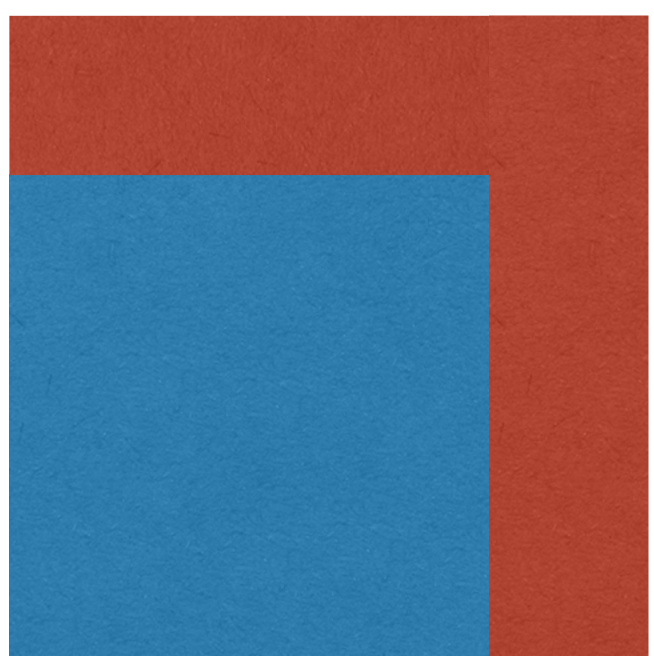
Place them on the diagram

$\frac{1}{3}$

1



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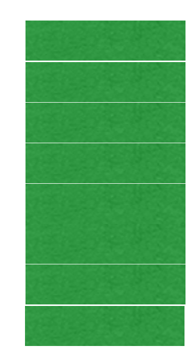

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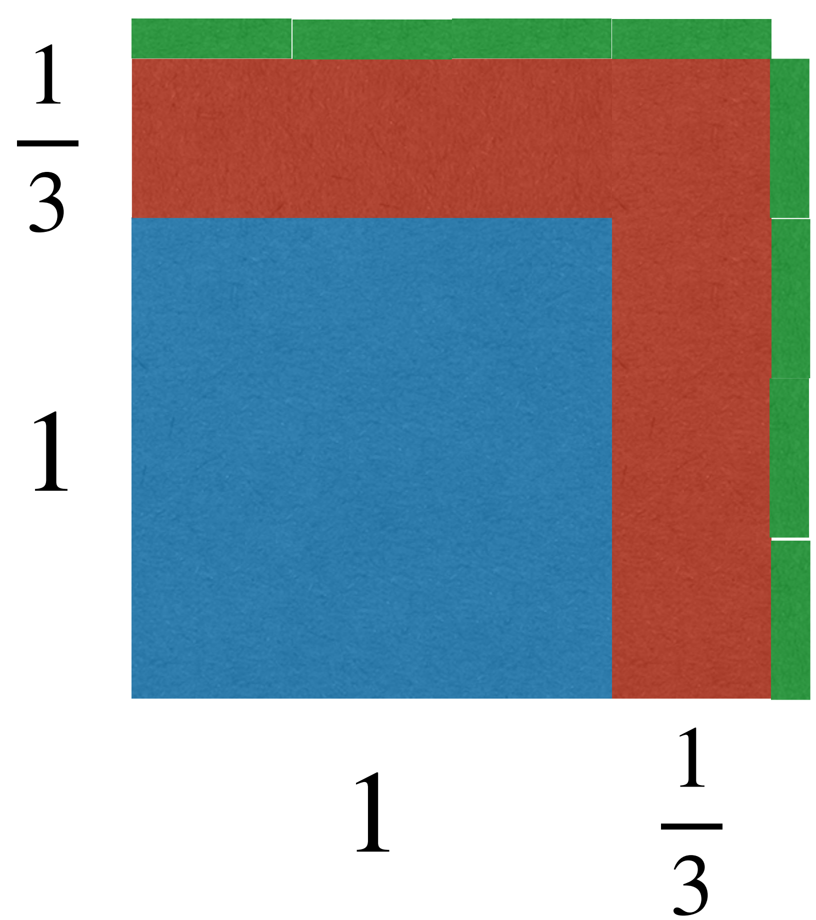

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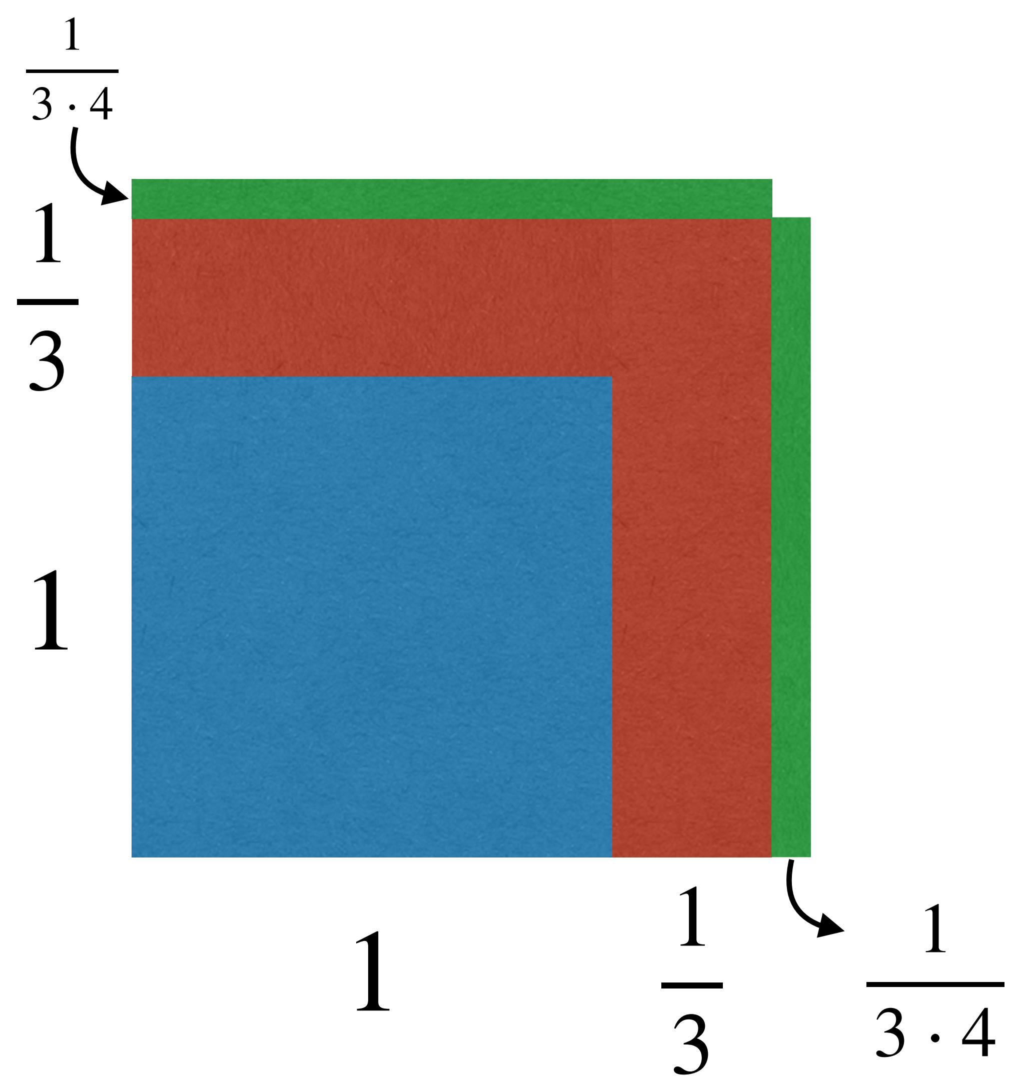

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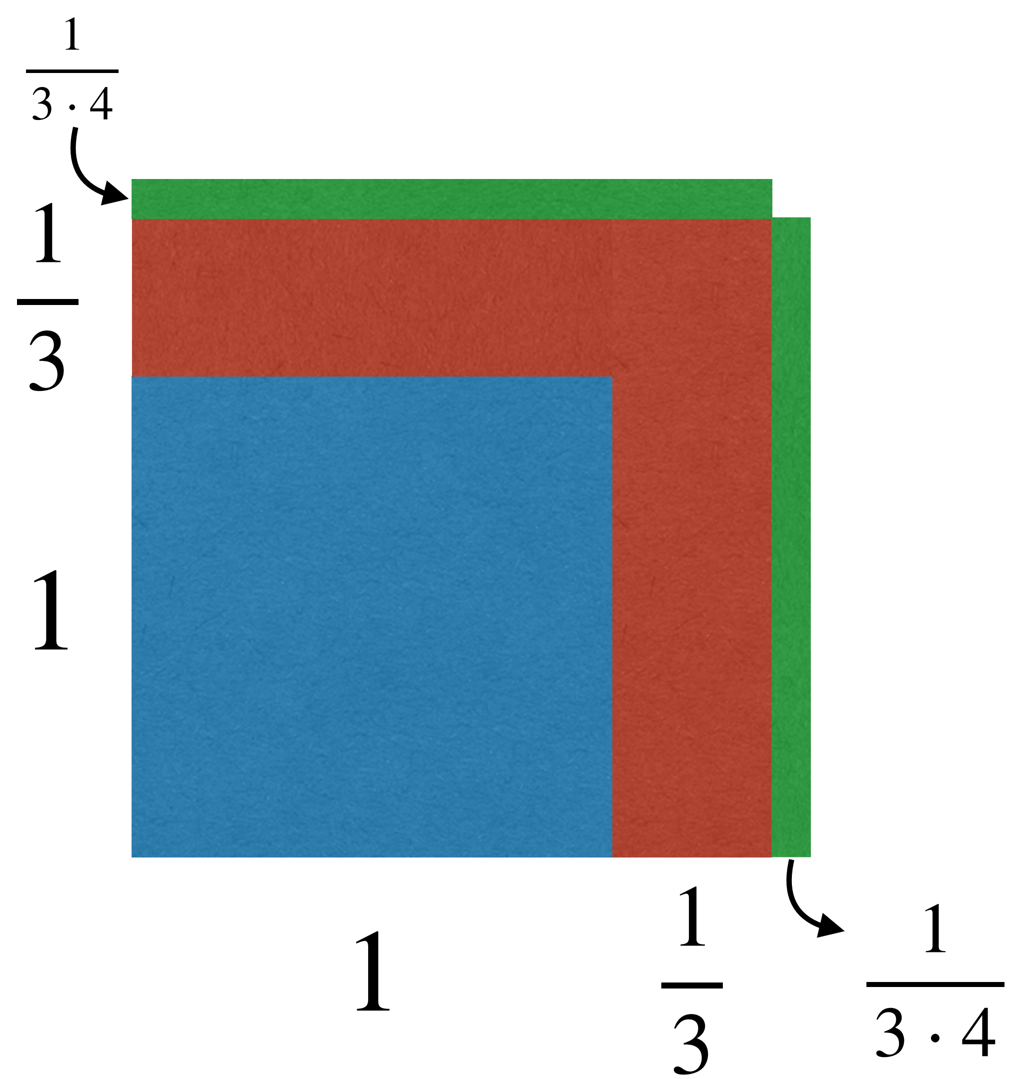

This is not a square, but it's close. And it does use all of the original two squares.

Approximating $\sqrt{2}$

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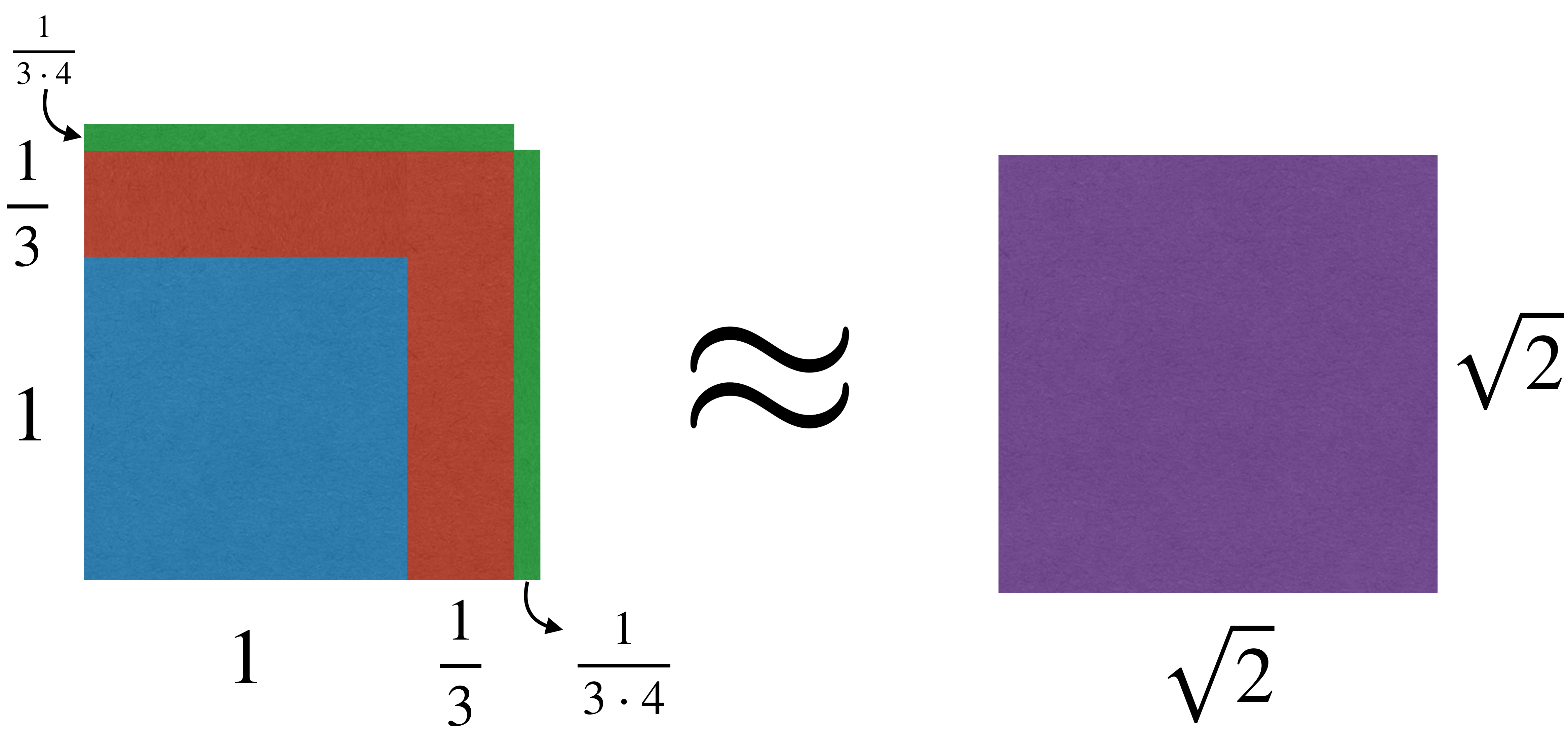
$$\sqrt{2} \approx 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34}.$$



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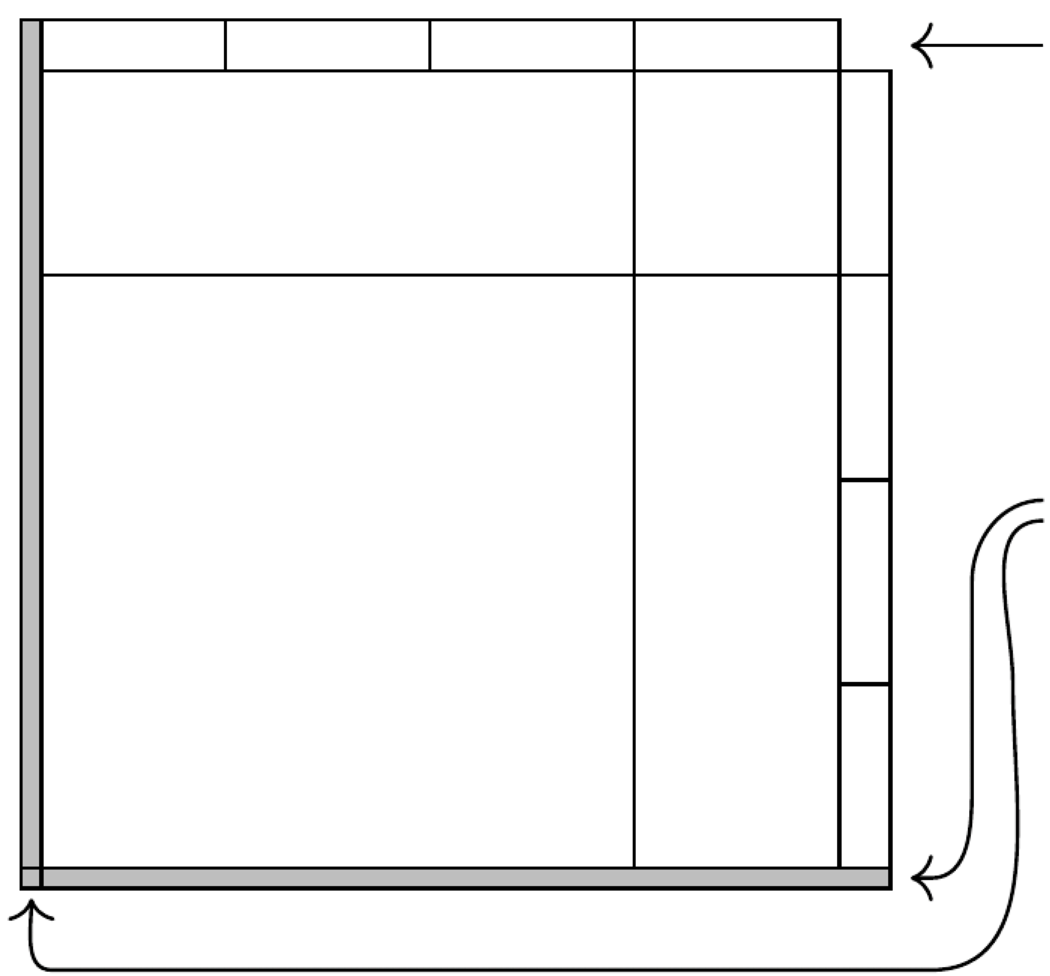

$\longleftarrow$ Missing $\frac{1}{3\cdot4} \times \frac{1}{3\cdot4}$ square

Two shaded strips to cut off, to equal the area of the missing square

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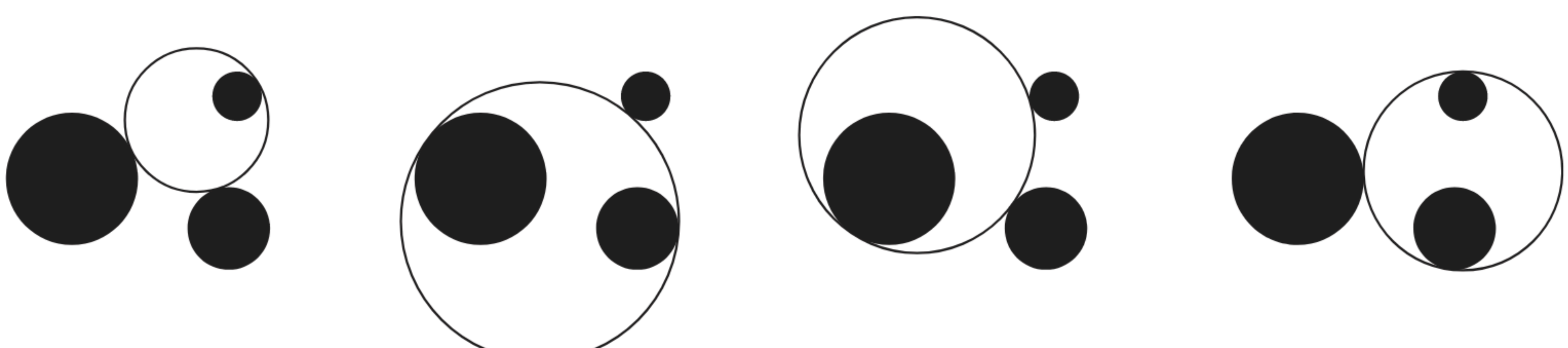

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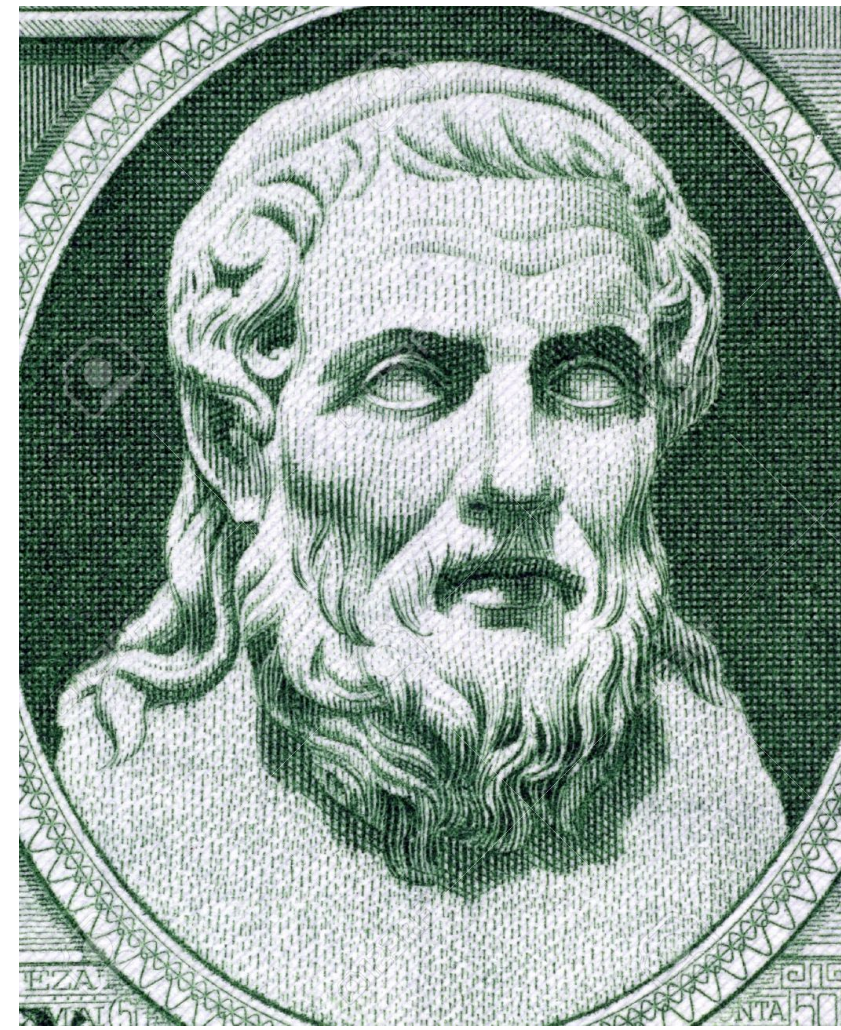

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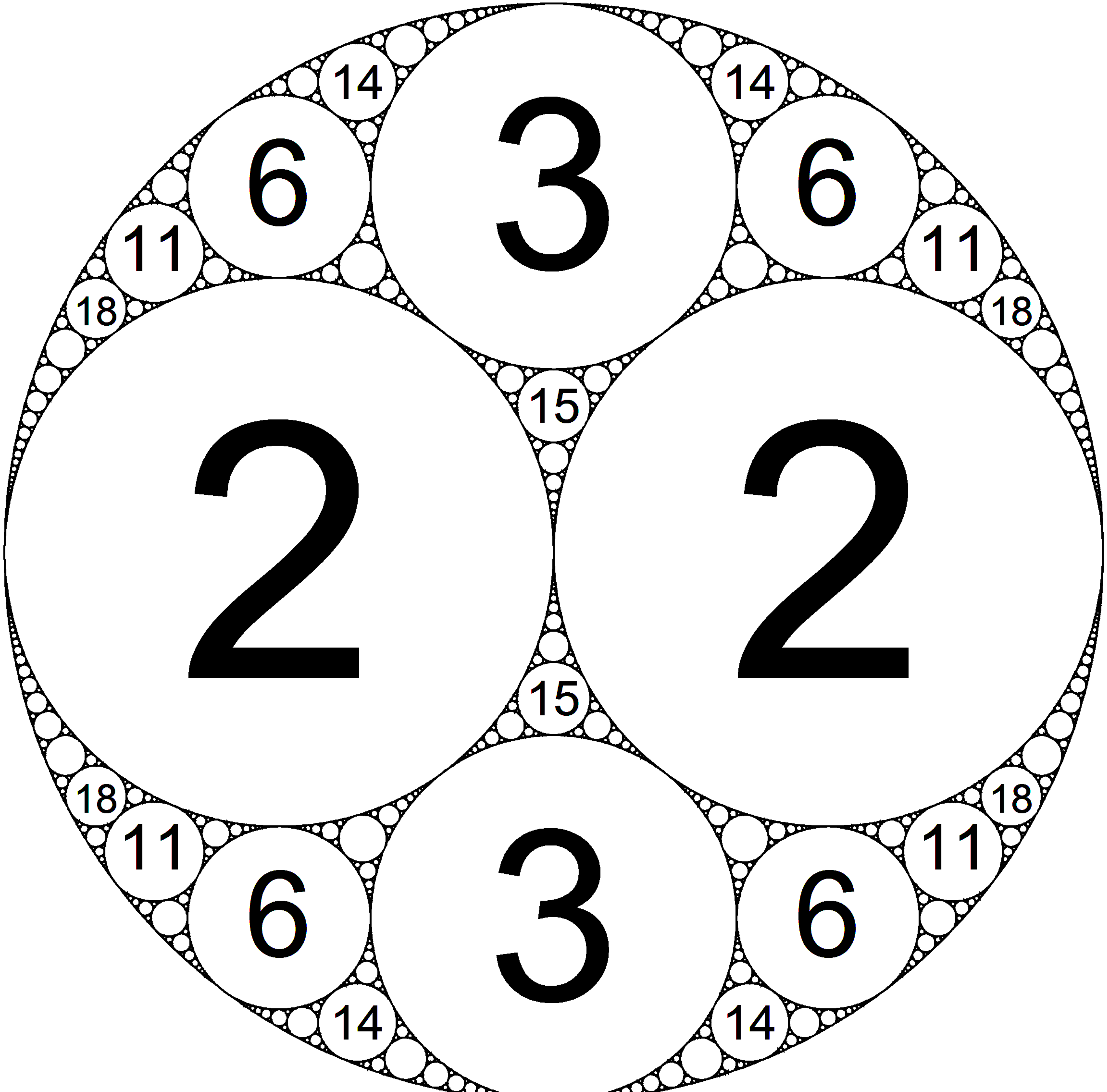

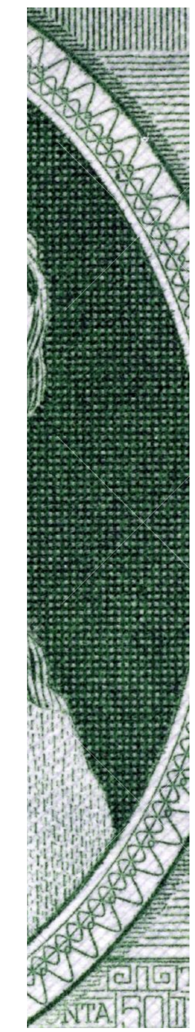

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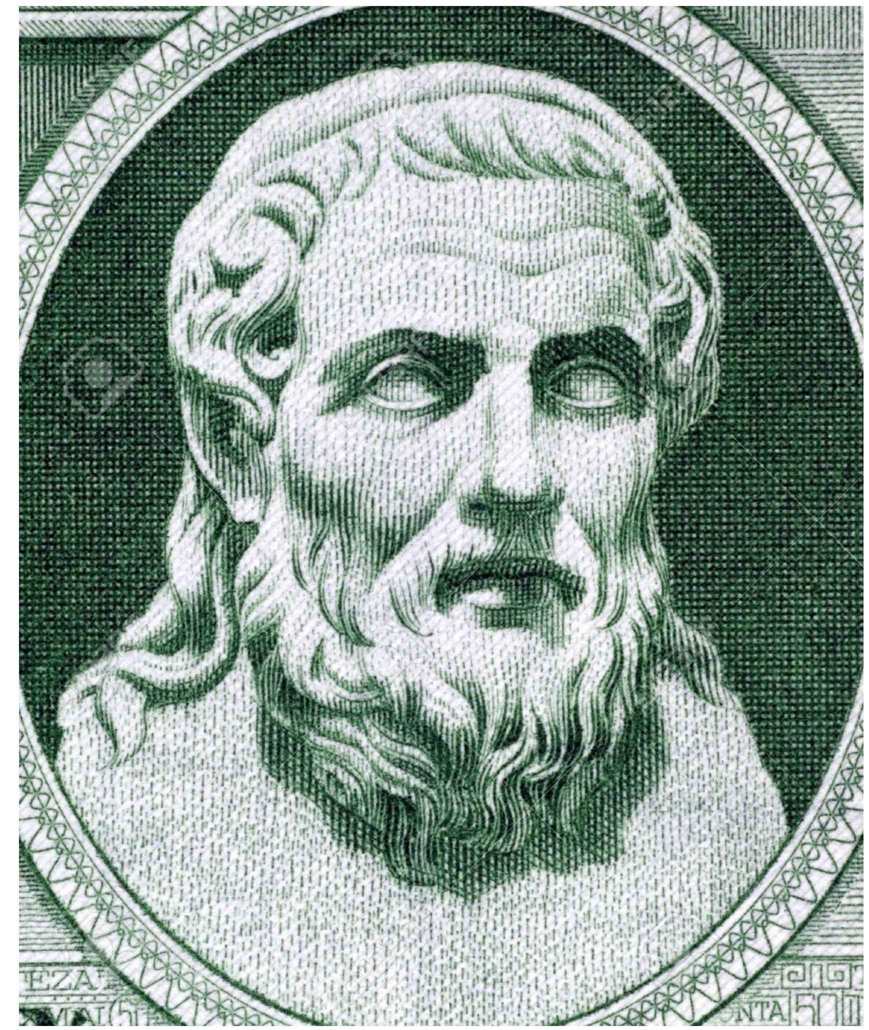

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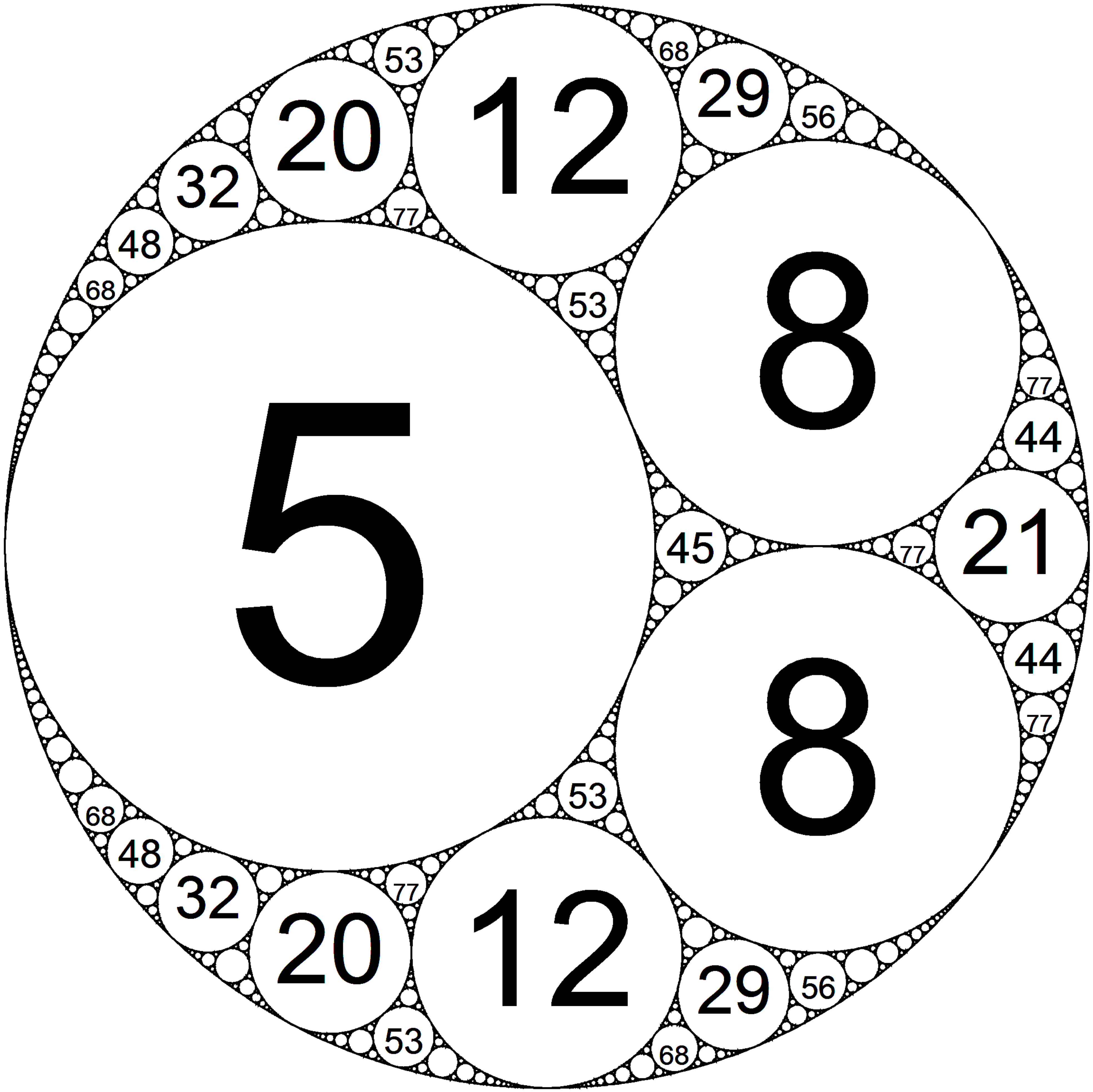
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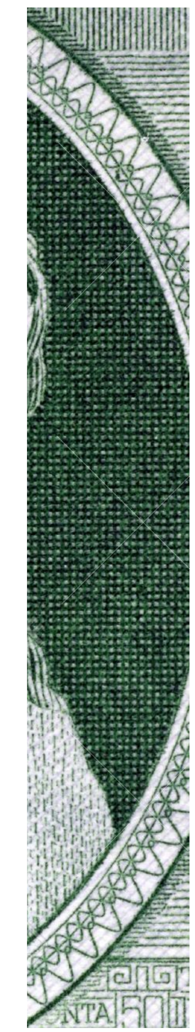

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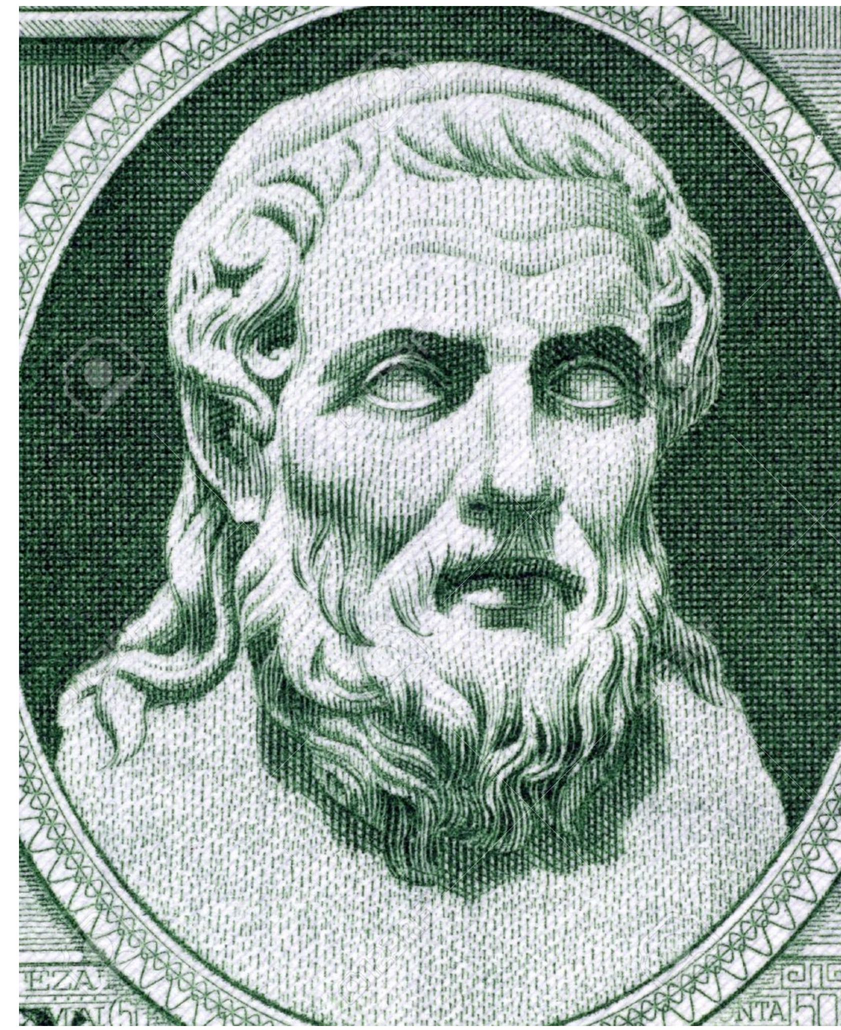

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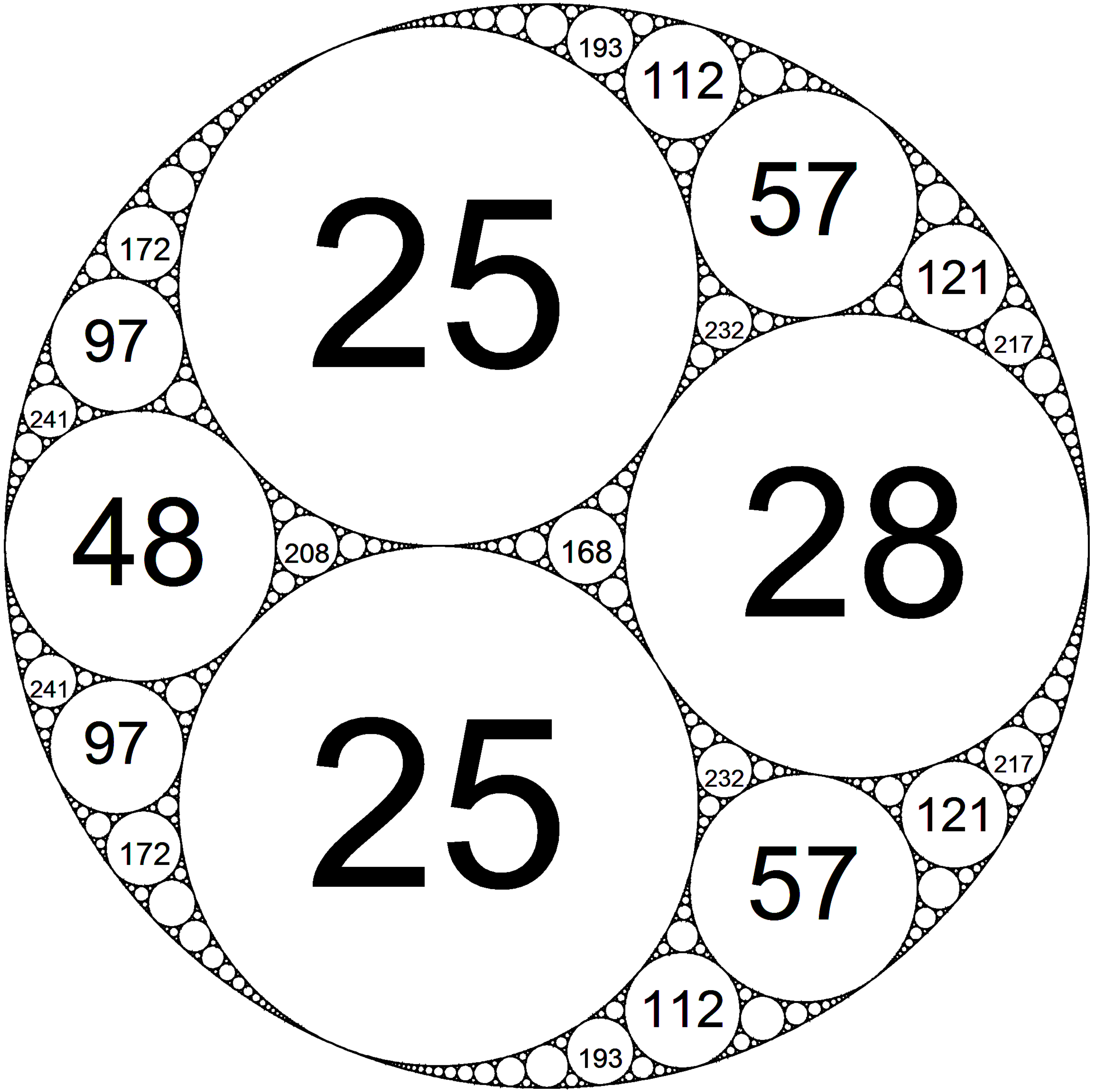
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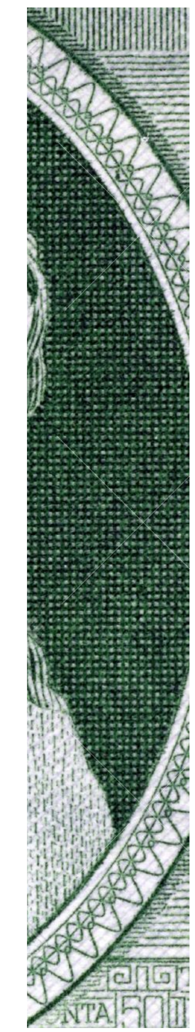
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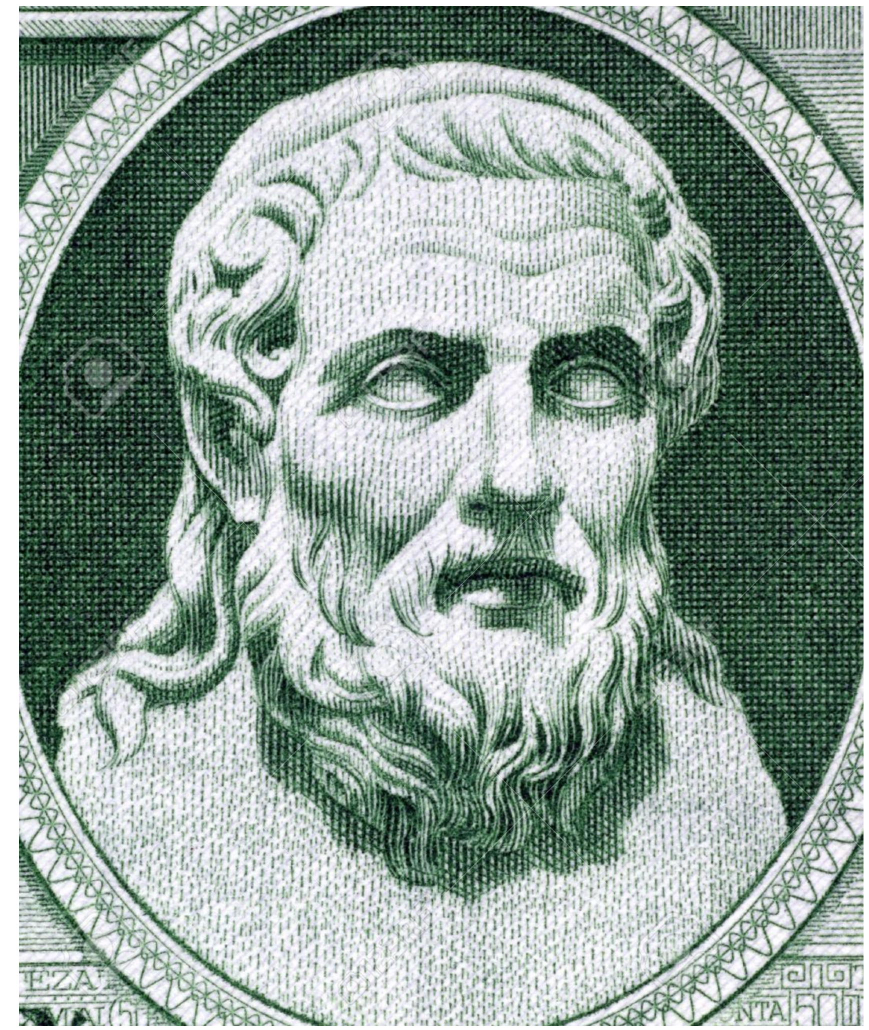

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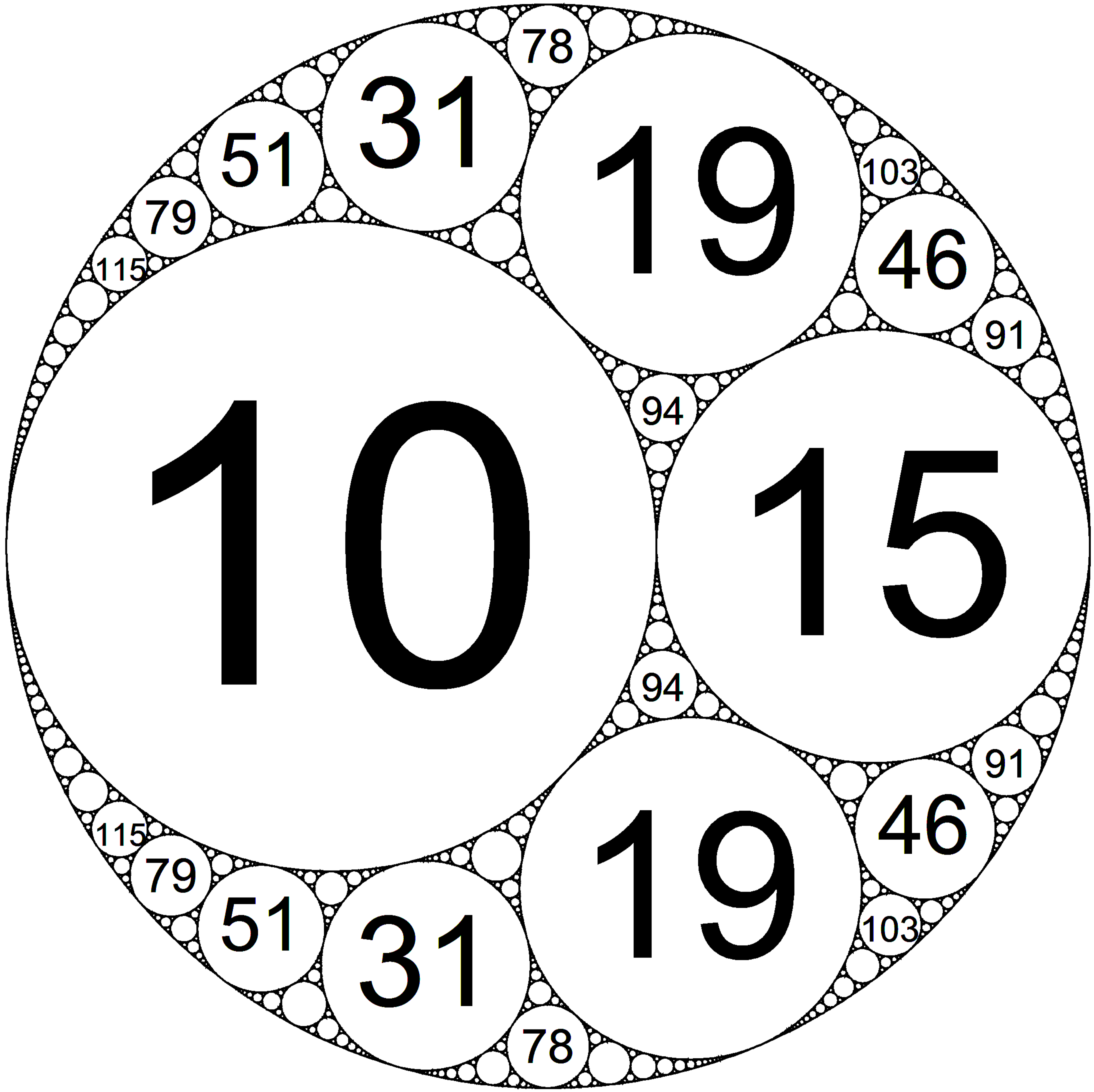
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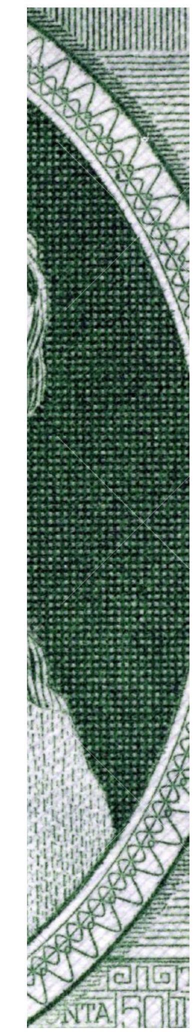
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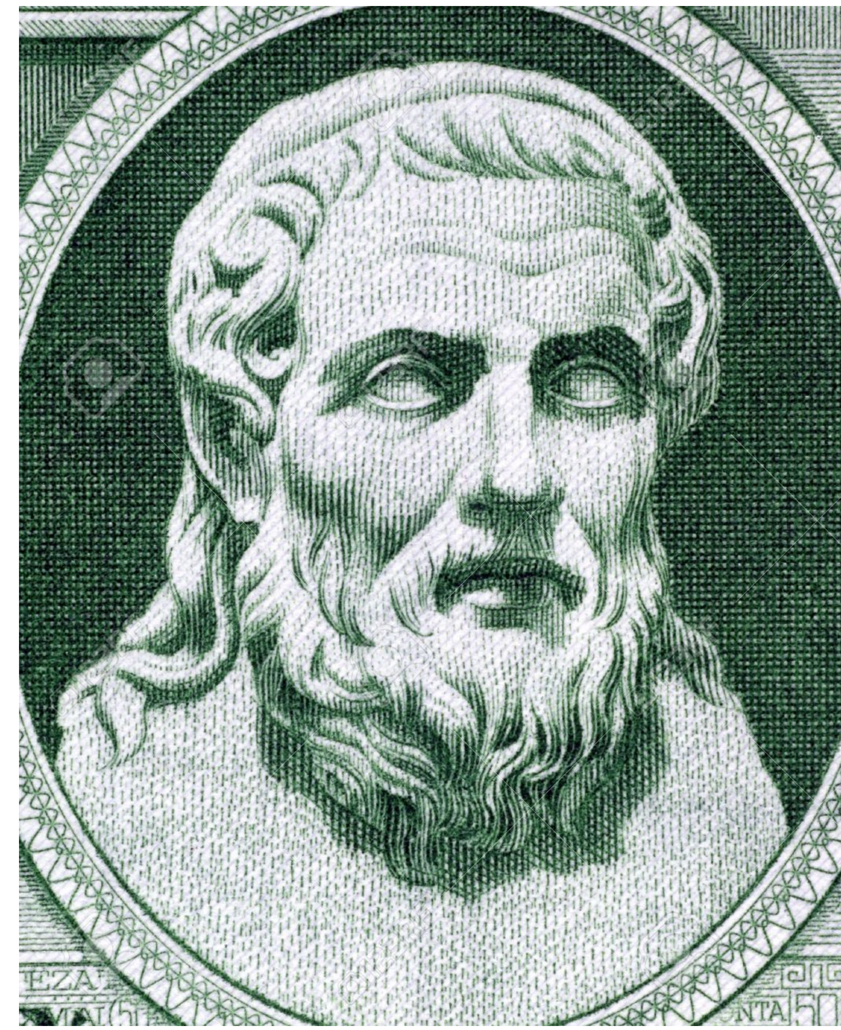

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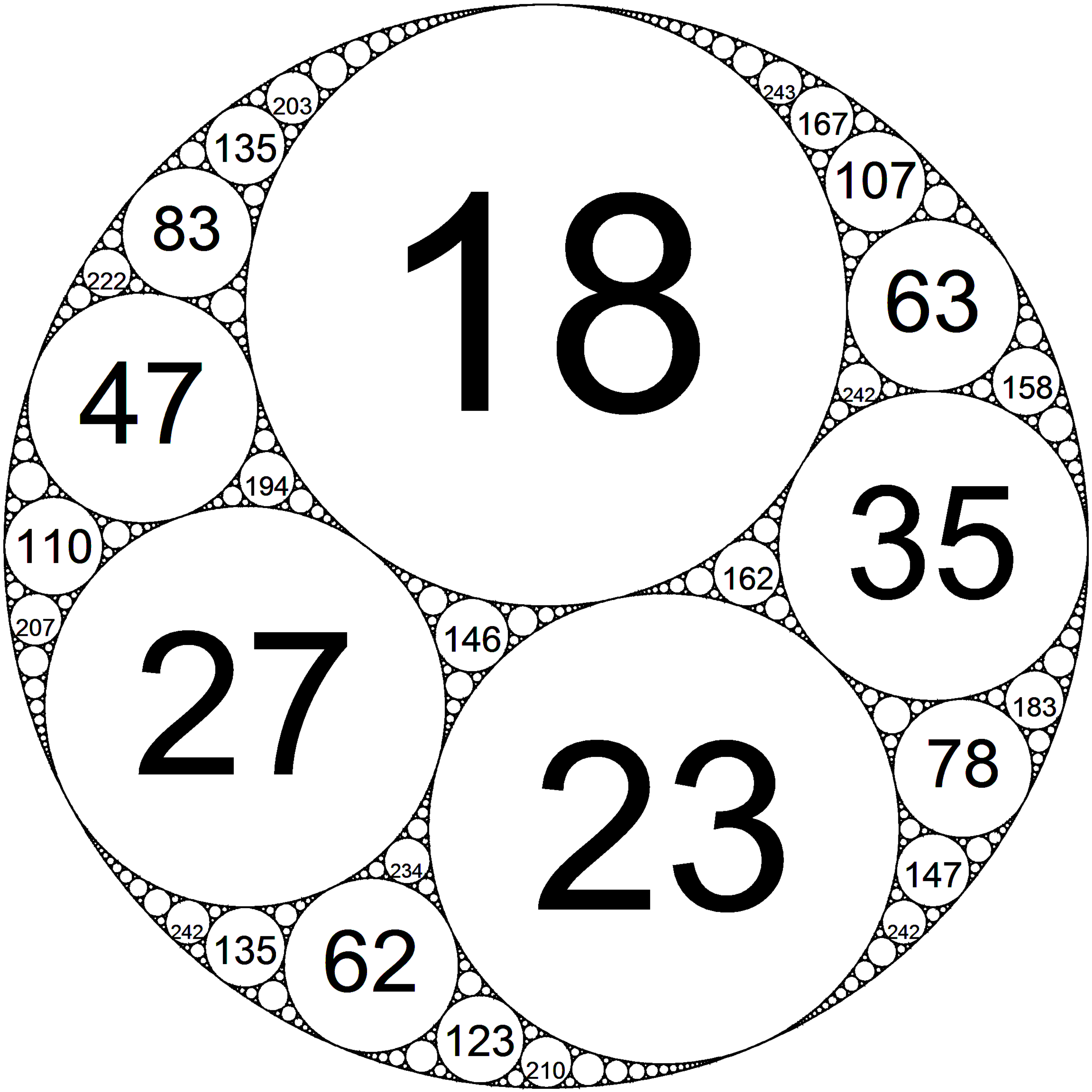
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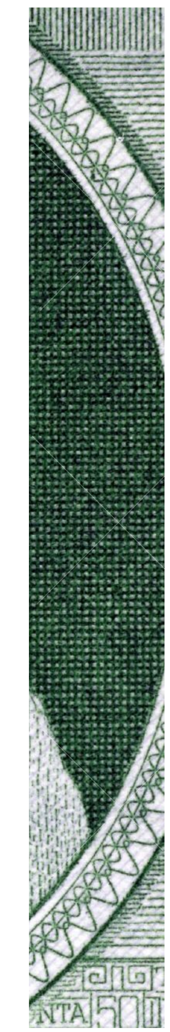
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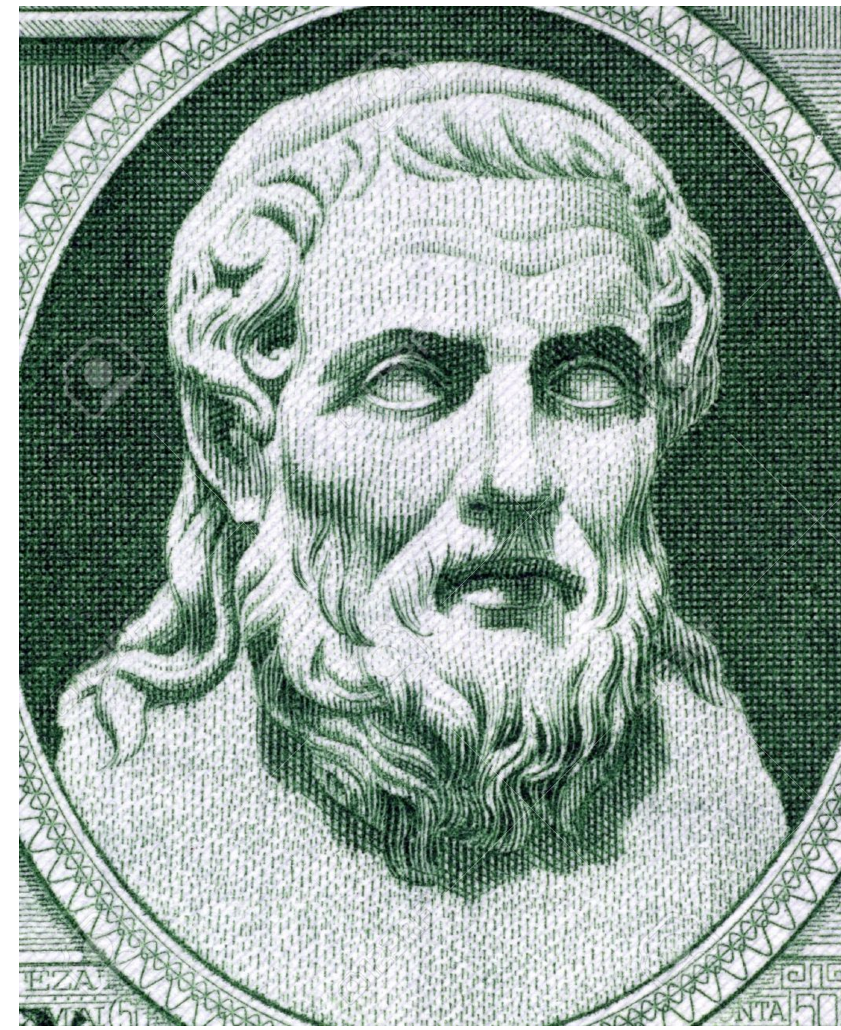

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- For her work, Viazovska won the 2022 Fields medal. She was the second woman in history to win it.

School of Athens

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People’s History

People’s History of Geometries