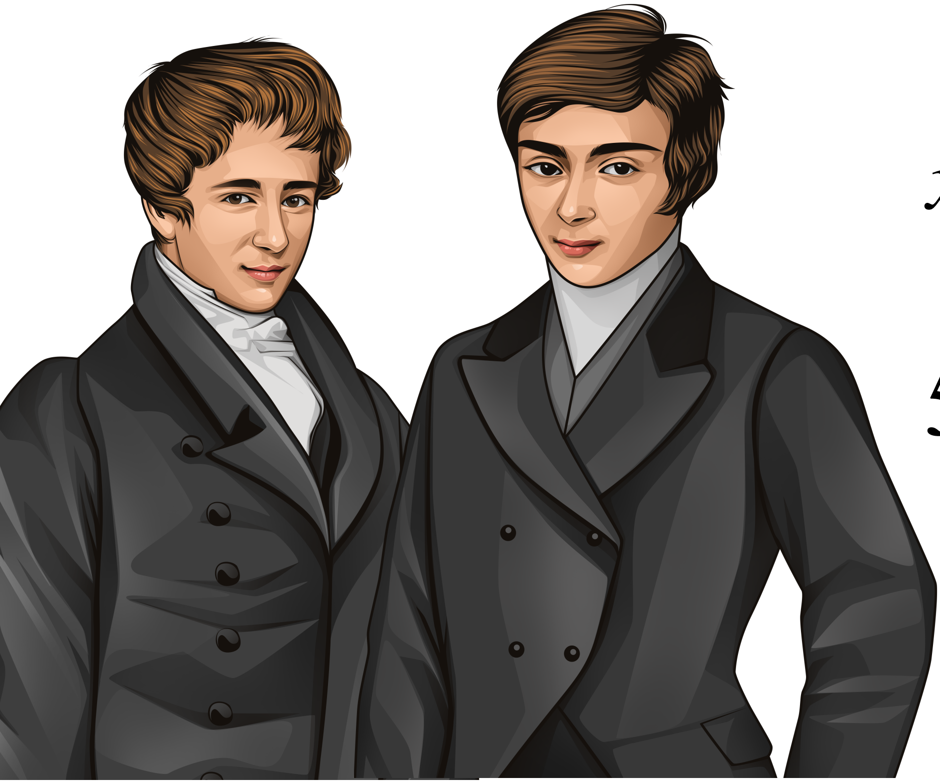
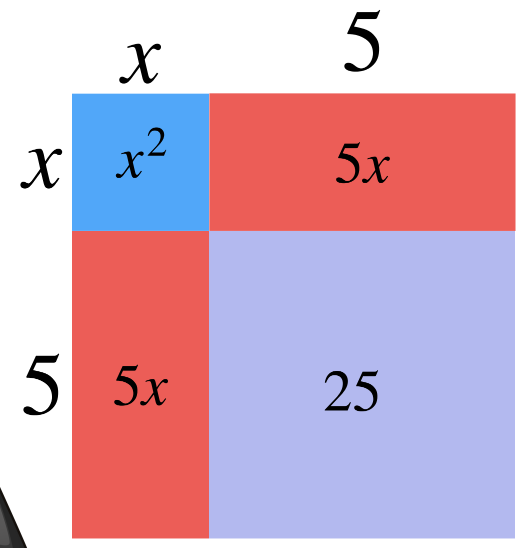
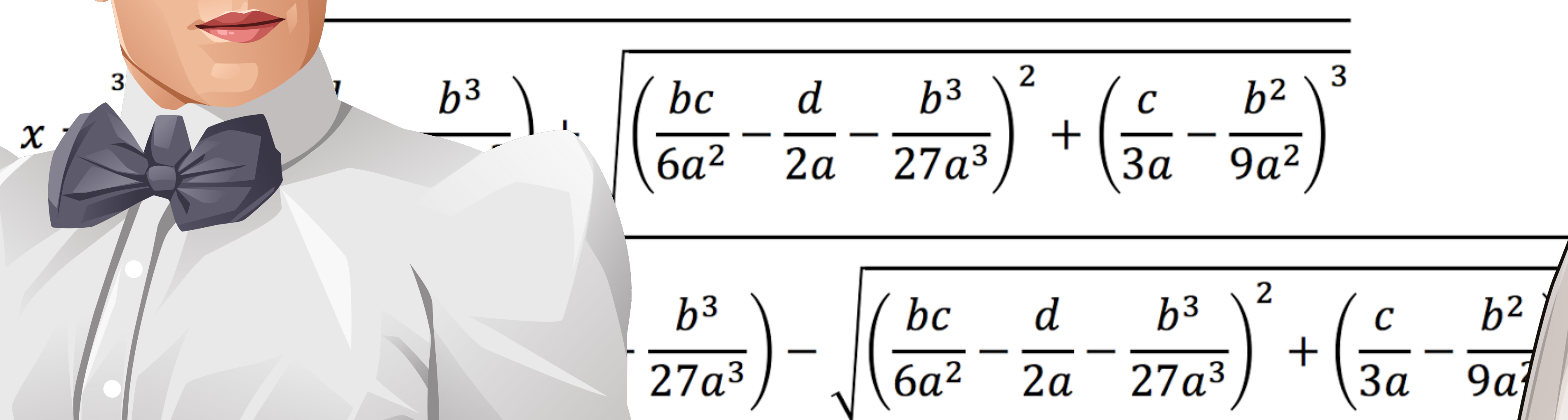
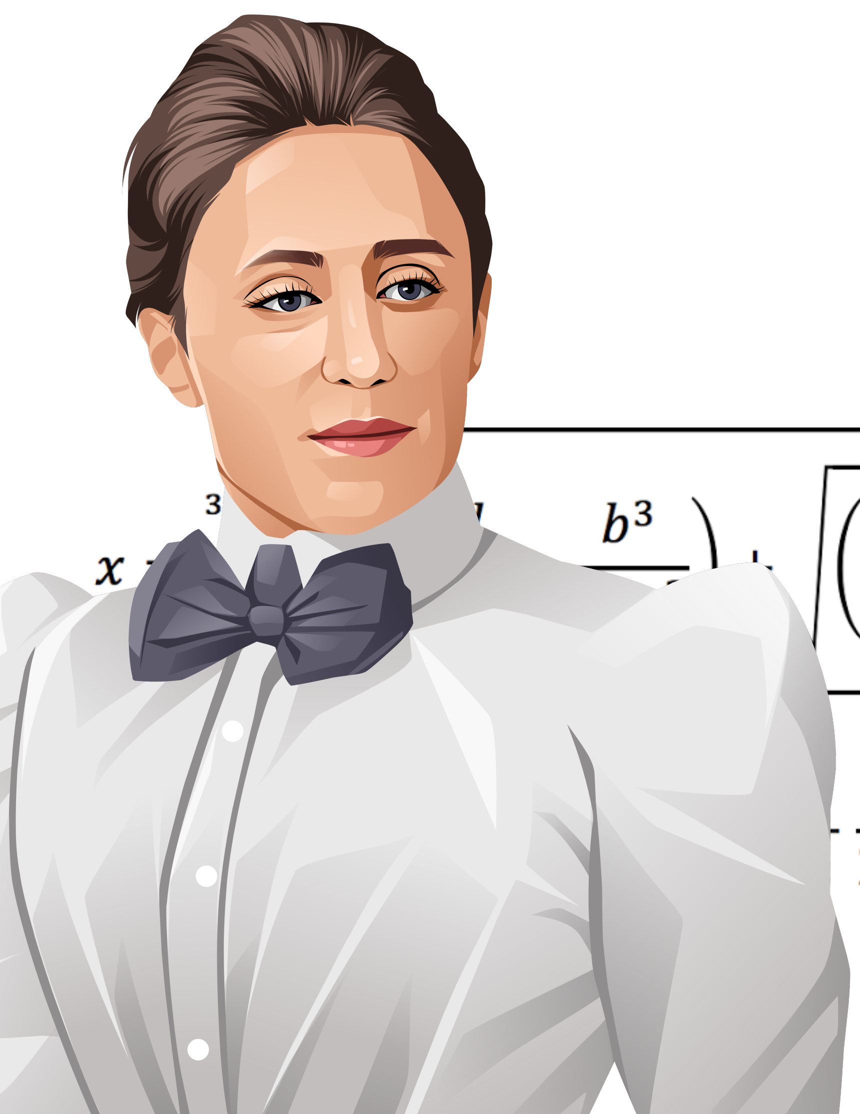
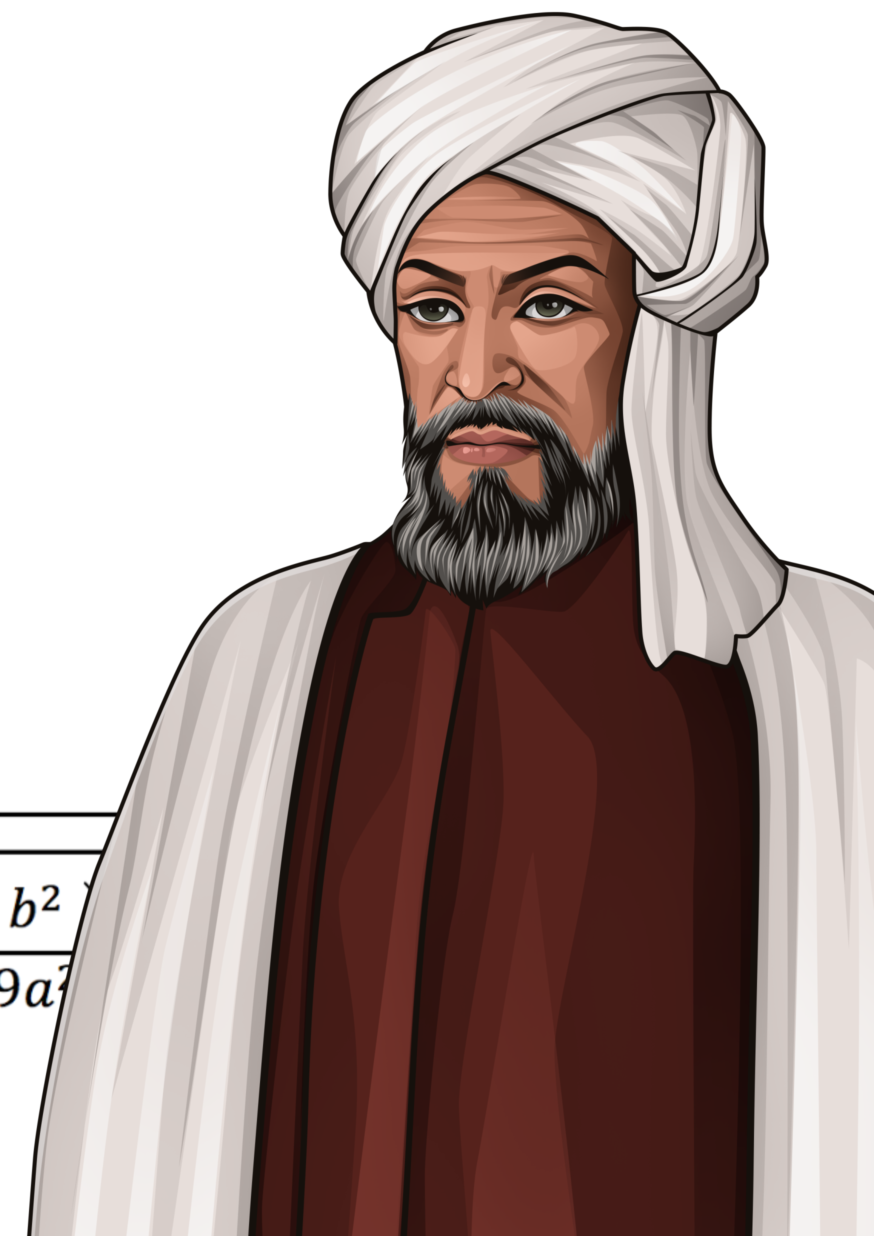




Chapter 6: Algebra





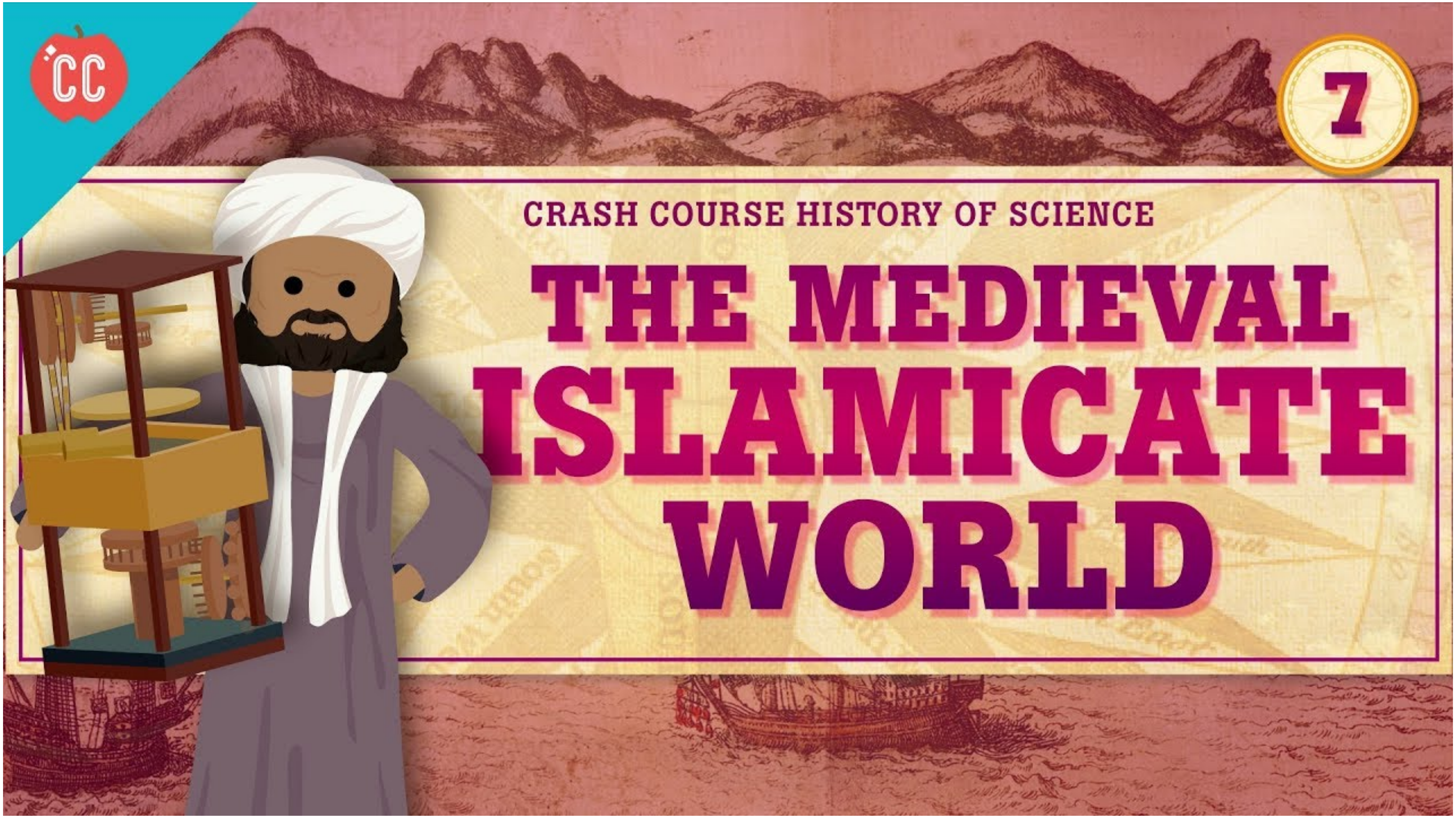


Arabic Mathematics

- ~800 AD algebra was developed by Arabic mathematicians.
- It was first used to solve practical problems.

- Example: Inheritance. Suppose a father leaves 1,000 dirhams to his 3 sons and 2 daughters. Rule: Each son gets twice as much as each daughter. How much does each get?

The Arab World

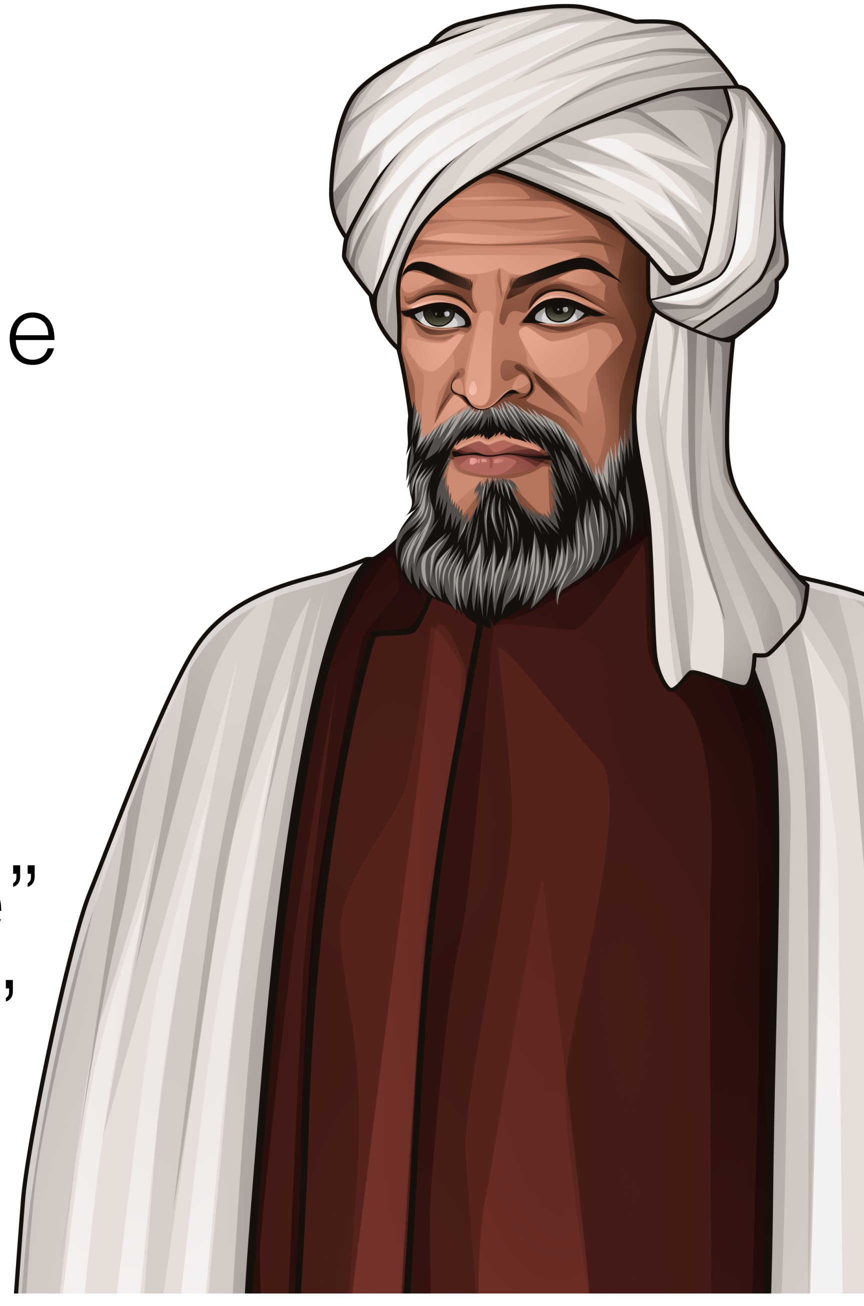


al-Khwārizmī

- Muḥammad ibn Mūsā al-Khwārizmī lived ~780-850AD.

- Born in modern-day Uzbekistan, but came to Baghdad which had become an intellectual center, attracting people from all over, including from India and China.

- Fun fact: the arabic word for “restore” is “al-jabr”, which became “algebra.”



al-Khwārizmī

- He called algebraic manipulation "restoring." Example:

$$2x - 5 = 11$$

$$2x - 5 + 5 = 11 + 5$$

$$2x = 16$$

$$\frac{2x}{2} = \frac{16}{2}$$

$$x = 8$$

- These symbols did not exist, though. It was all written out with words. I.e., "rhetorical algebra."

al-Khwārizmī

- Instead of writing $x^2 + 10x = 39$, he wrote

> "What must be the square which, when increased by ten of its own roots, amounts to thirty-nine?"

- He figured out how to solve quadratic equations like this.

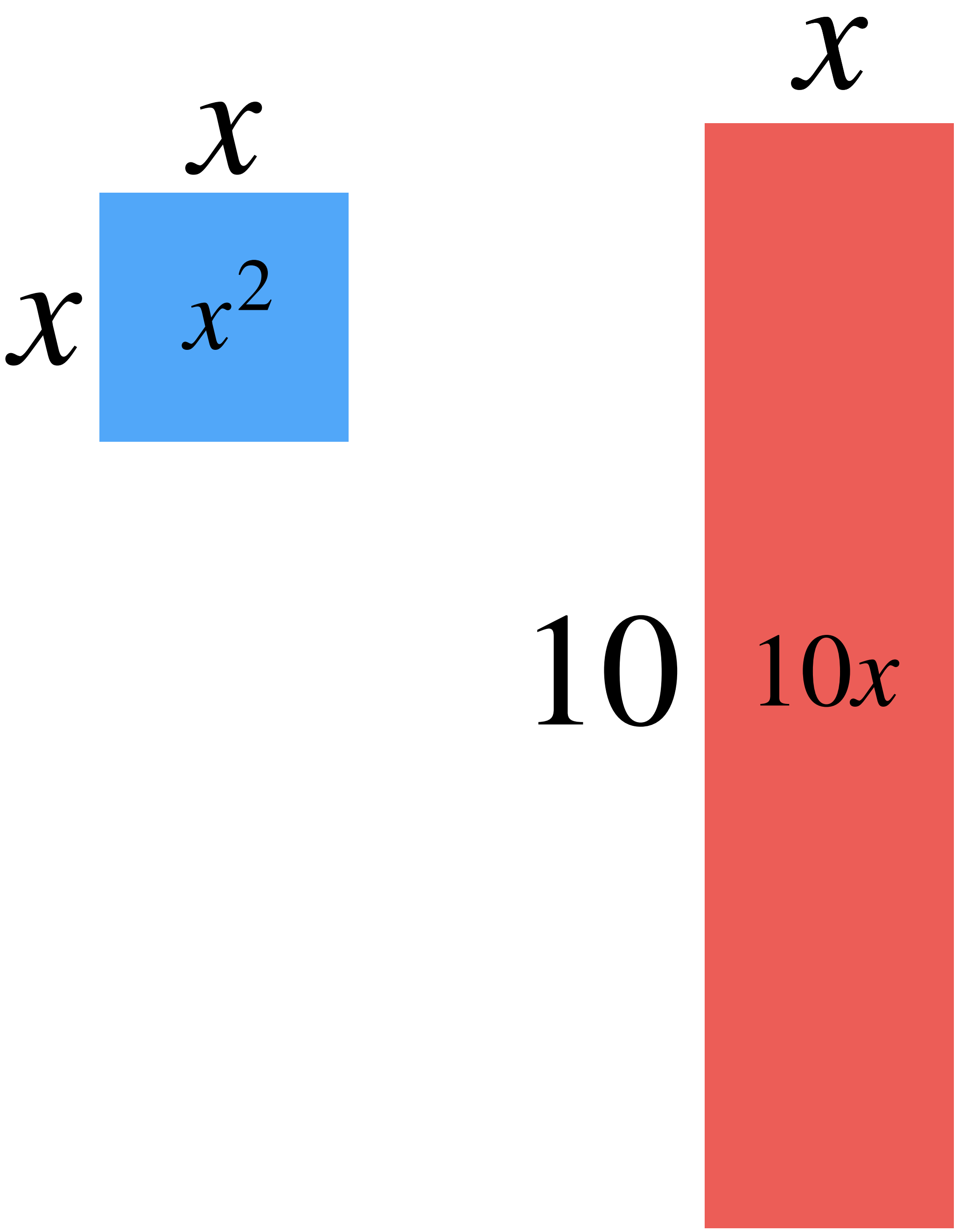
- It's tough because: when do you take the square root?

al-Khwārizmī

- How did he solve quadratic equations? Geometrically!

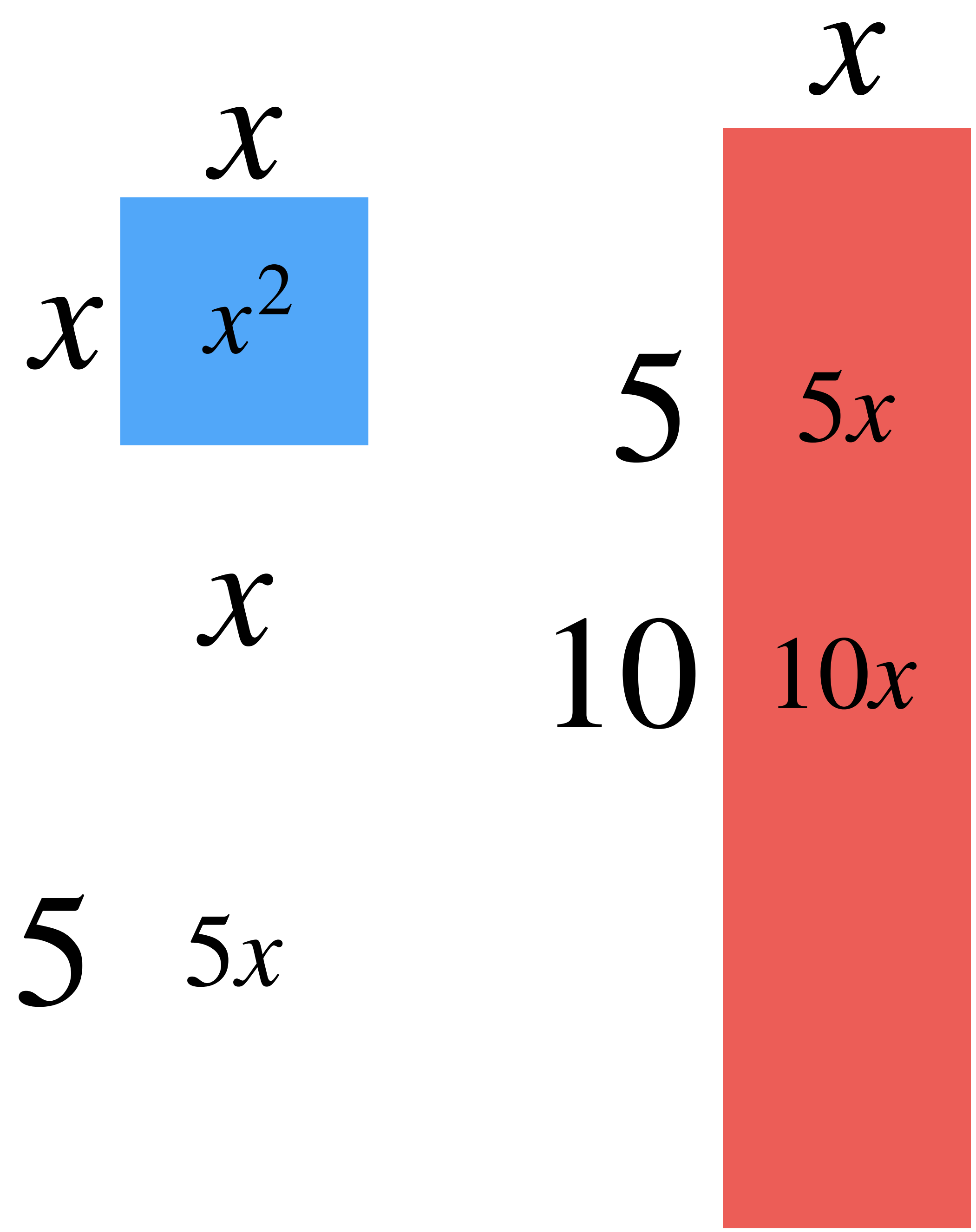
- Example: To solve $x^2 + 10x = 39$, start with a square of area x^2

- Then a rectangle of area $10x$



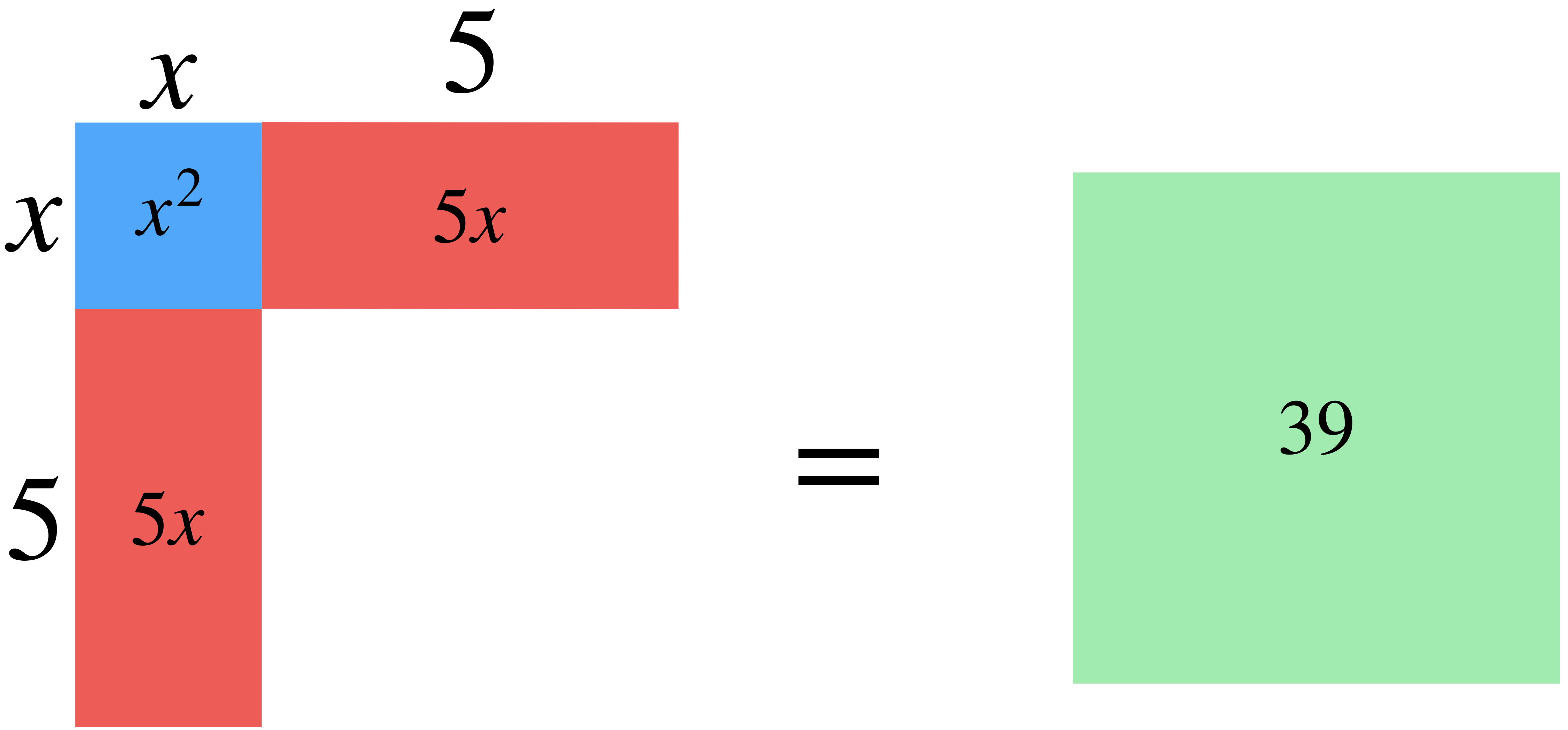
al-Khwārizmī

- Chop the rectangle in two.
- Attach the two halves to the square.
- The area in this picture represents $x^2 + 10x$.
- We want $x^2 + 10x = 39$.



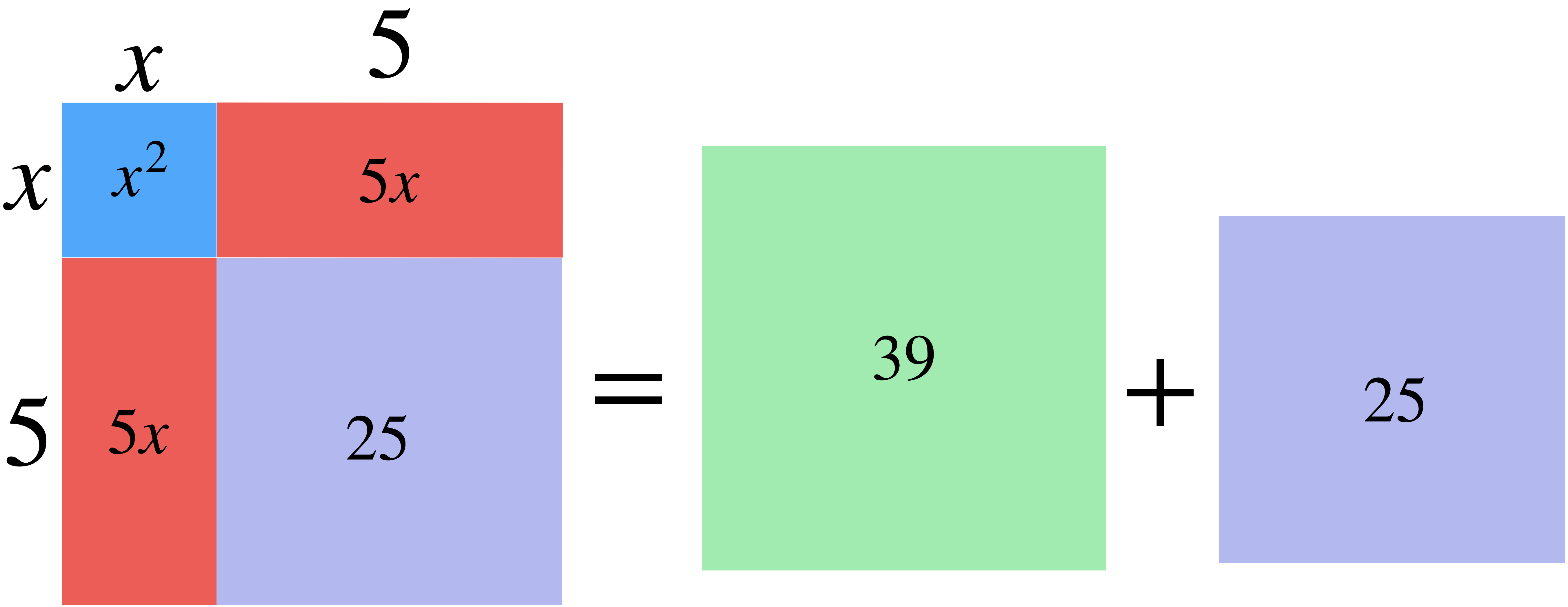
al-Khwārizmī

- Now represent $x^2 + 10x = 39$ with shapes.



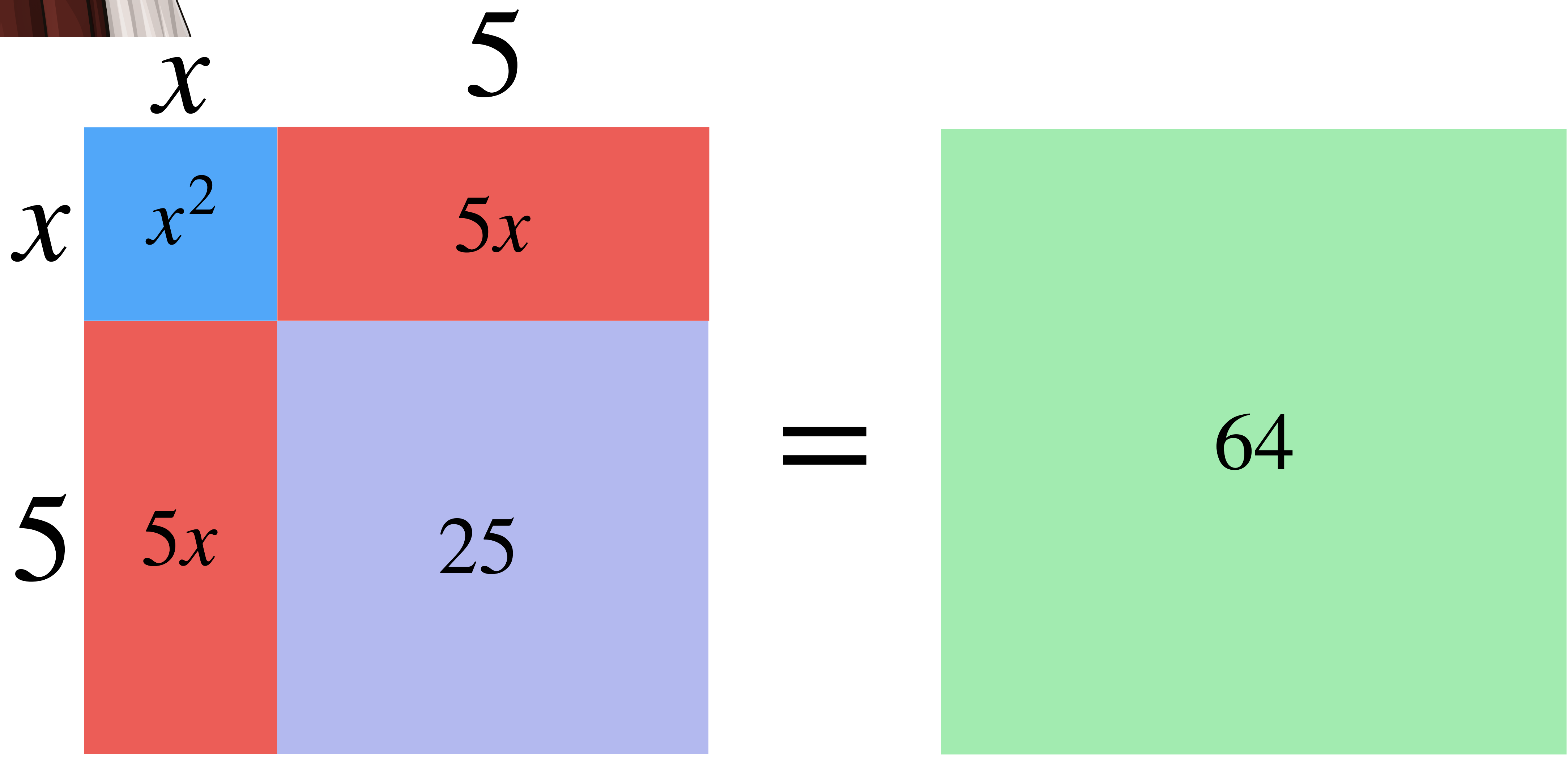
al-Khwārizmī

- Now represent $x^2 + 10x = 39$ with shapes.

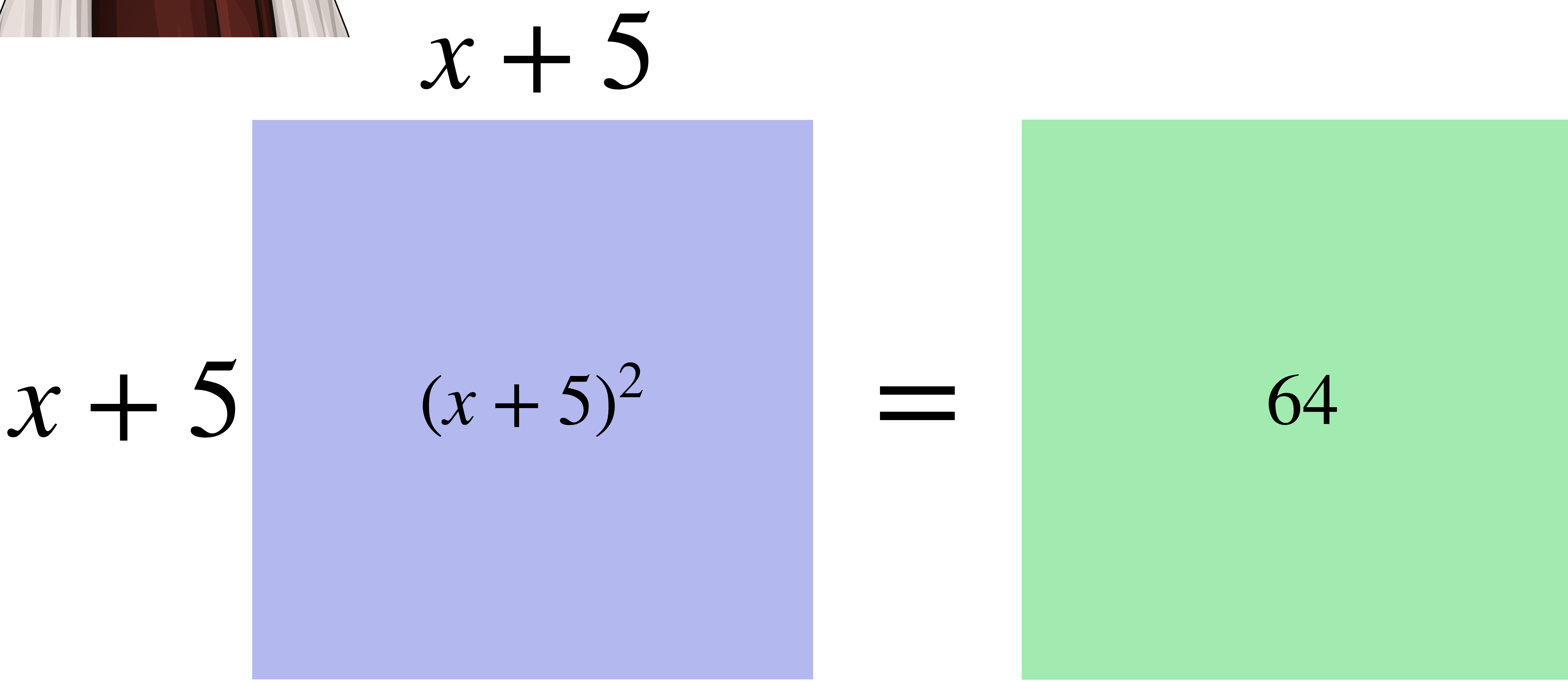


We *literally* completed the square!

al-Khwārizmī



al-Khwārizmī



$$(x + 5)^2 = 64$$
$$x + 5 = 8$$
$$x = 3$$

Now we can take the square root!

Notice: Negative solutions were not considered.

General Quadratic

- **Theorem:** If $a \neq 0$, the equation $ax^2 + bx + c = 0$ has the solutions $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

- **Proof:** Complete the square!

$$ax^2 + bx + c = 0$$

$$ax^2 + bx = -c$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

General Quadratic

- Next, chop linear term in two, complete the square, and solve.

$$x^2 + 2\frac{b}{2a}x = -\frac{c}{a}$$

$$x^2 + 2\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

General Quadratic

- Solve.

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{4ac}{4a^2}$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

General Quadratic

- Solve.

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

al-Khwārizmī

- With that, al-Khwārizmī solved the general quadratic equation.

- He wrote a very influential book on algebra called *The Compendious Book on Calculation by Completion and Balancing*.

- He also wrote books explaining the decimal position system, and with advancements in trigonometry.

- The next challenge for mathematicians: Solve the general cubic!

The Long Aftermath

*Including one of the most amazing stories in math history.

Italy

Ireland
London
Netherlands
Berlin
Poland
Warsaw
Belarus
Kyiv
Київ
Germany
Belgium
Paris
Czechia
Slovakia
Ukraine
Austria
Hungary
Moldova
France
Romania
Croatia
Serbia
Black Sea
Italy
Rome
Bulgaria
Portugal
Madrid
Spain
Lisbon
Tyrrhenian Sea
Istanbul
Ankara
Greece
Turkey
Syria
Lebanon
Tunisia
Mediterranean Sea
Israel
Morocco
Cairo
القاهرة
Algeria
Libya
Egypt
Jeddah

Cubic Equation

- As we saw, there is a formula for the roots of the general quadratic equation $ax^2 + bx + c = 0$, and the formula uses only the basic arithmetic operations: $+, -, \times, \div$ and roots.

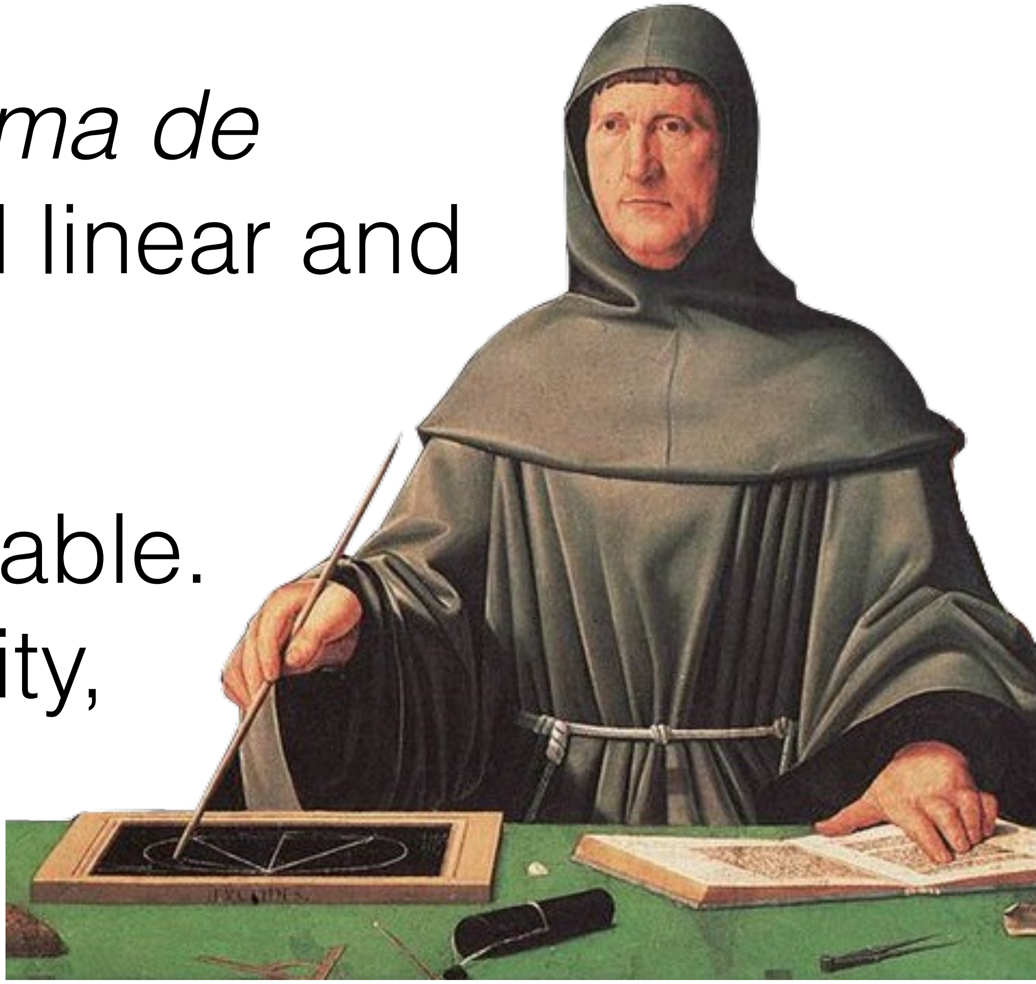
- What is the formula for the general cubic equation, $ax^3 + bx^2 + cx + d = 0$, using only the basic arithmetic operations?

- The story takes place in Italy, beginning in 1494.

Cubic Equation

- In 1494, Luca Pacioli wrote *Summa de Arithmetica*, in which he covered linear and quadratic equations thoroughly.

- Book basically used the first variable. It used *co* for an unknown quantity, short for *cosa*, which means "thing" in Italian.



- Pacioli challenged the Italian math community: Find a solution to the general cubic!

- Three main characters:

Cubic Equation

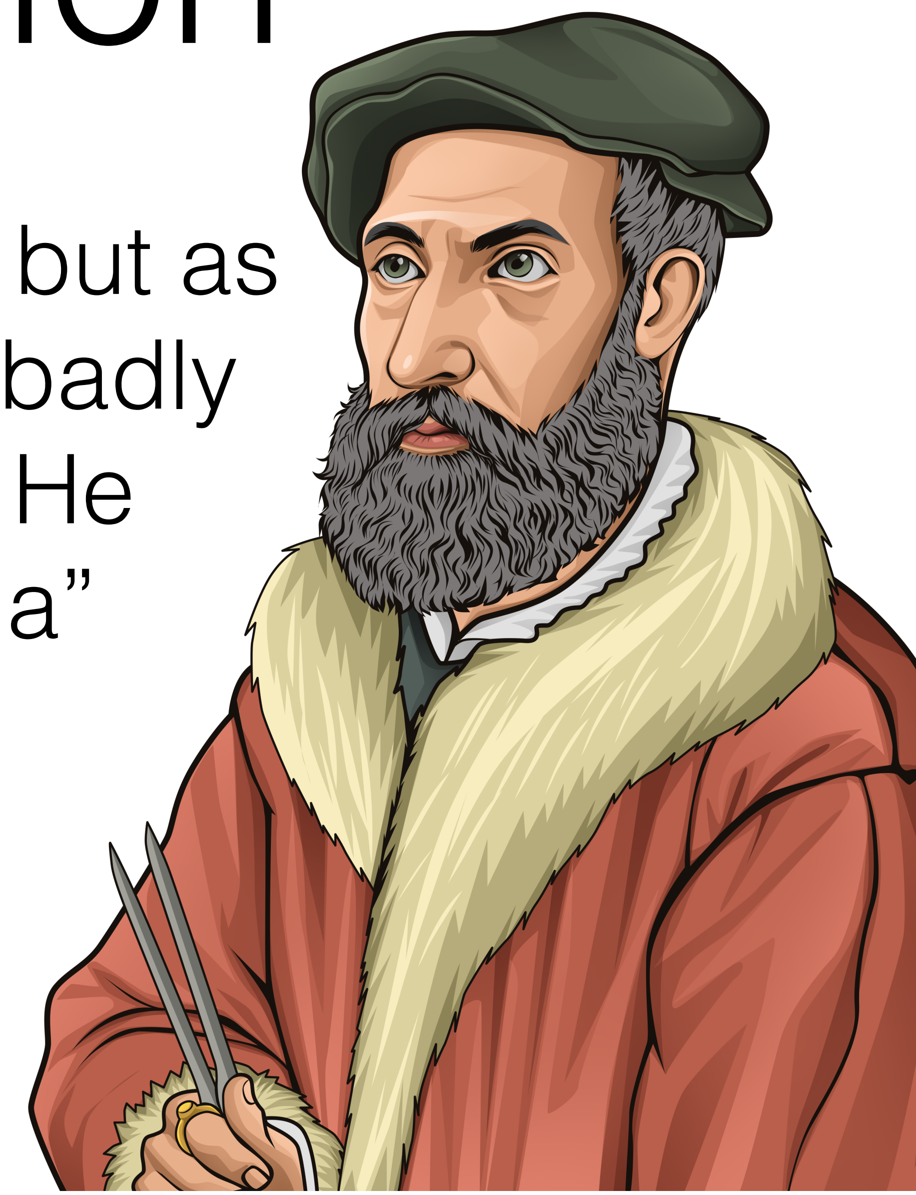
- Scipione del Ferro worked on a simpler problem: The *depressed cubic*, which is $ax^3 + cx + d = 0$.

- Equivalent to $x^3 + mx = n$.

- “cube and *cosa* equals number”

- He solved it! Important problem that would have given him notoriety. But…he kept his solution a secret…

Cubic Equation

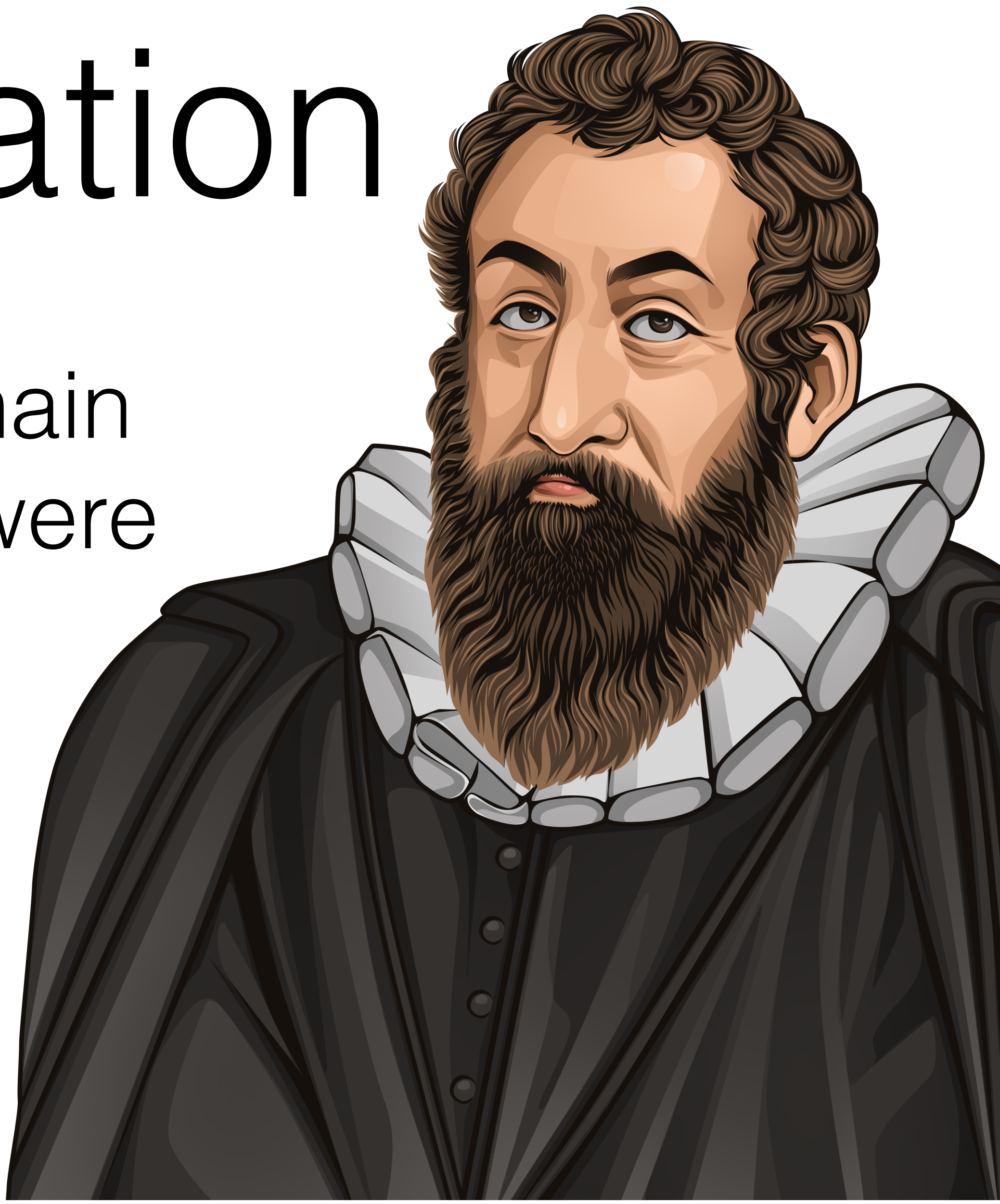


- Tartaglia, born as Niccolo Fontana, but as a boy a French soldier cut his face badly and his voice was forever affected. He was given cruel nickname: “Tartaglia” means “The Stutterer.”

- Boasted he could solve a different depressed cubic: $x^3 + mx^2 = n$.

- But Tartaglia did not publish his result, and del Ferro doubted whether he actually had a solution.

Cubic Equation



- Gerolamo Cardano is the third main character in this story. All three were Italian.
- Cardano was a man of contradictions, in his personal and professional life.
- Became deeply interested in solving the cubic but needed del Ferro's secret to succeed.

Cubic Equation

- Story time. Recall that Del Ferro solved the depressed cubic but kept it a secret. Why?

- Before tenure, professors had to maintain their professional standing. Losing a “math duel” with another math prof would damage their reputation.

- Math duel: Each gives the other ~30 problems to solve. The winner is the one who solves the most.

- Del Ferro kept his solution a secret to win duels.

Cubic Equation

Del Ferro solves the depressed cubic

Pacioli's challenge: Solve the cubic

Cubic Equation

- Del Ferro kept it a secret for his entire life. On his deathbed he gave it to his student, Antonio Fior.

Cubic Equation

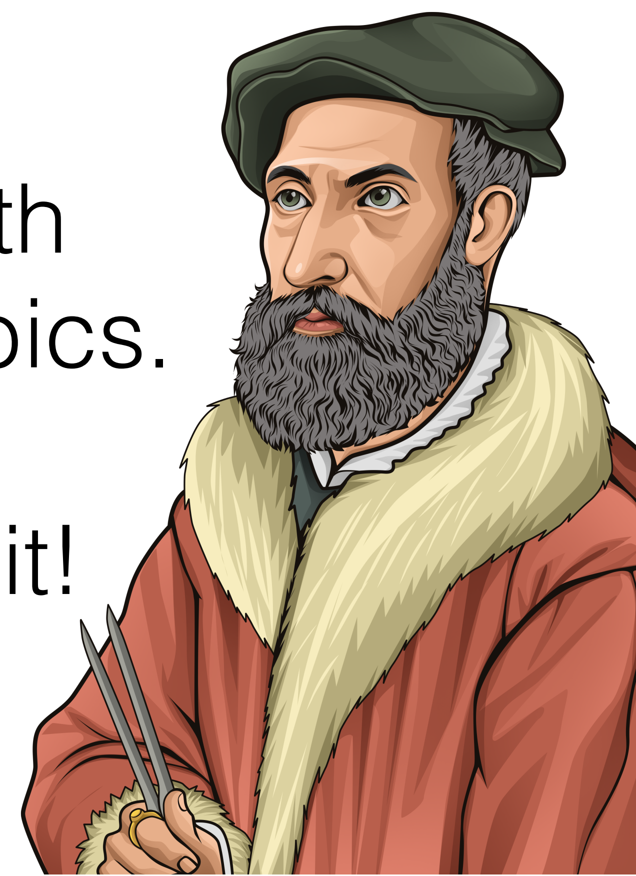
Del Ferro solves the depressed cubic

Pacioli's challenge: Solve the cubic

On death-bed, gives solution to Fior

Cubic Equation

- Del Ferro kept it a secret for his entire life. On his deathbed he gave it to his student, Antonio Fior.
- Fior was a mediocre mathematician with a big weapon.



- In 1535, Fior challenged *Tartaglia* to a math duel. Fior sent Tartaglia 30 depressed cubics.
- Just before the deadline, Tartaglia solved it!

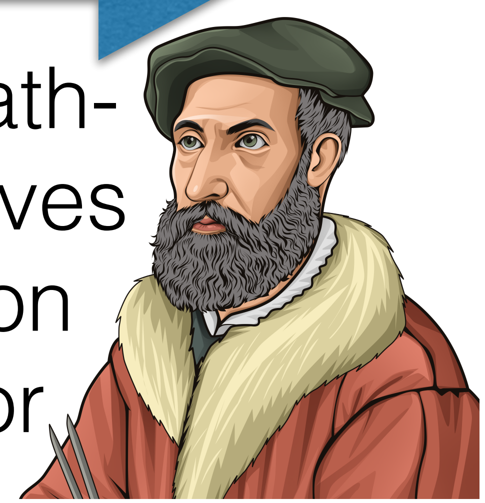
Cubic Equation

Del Ferro solves the depressed cubic

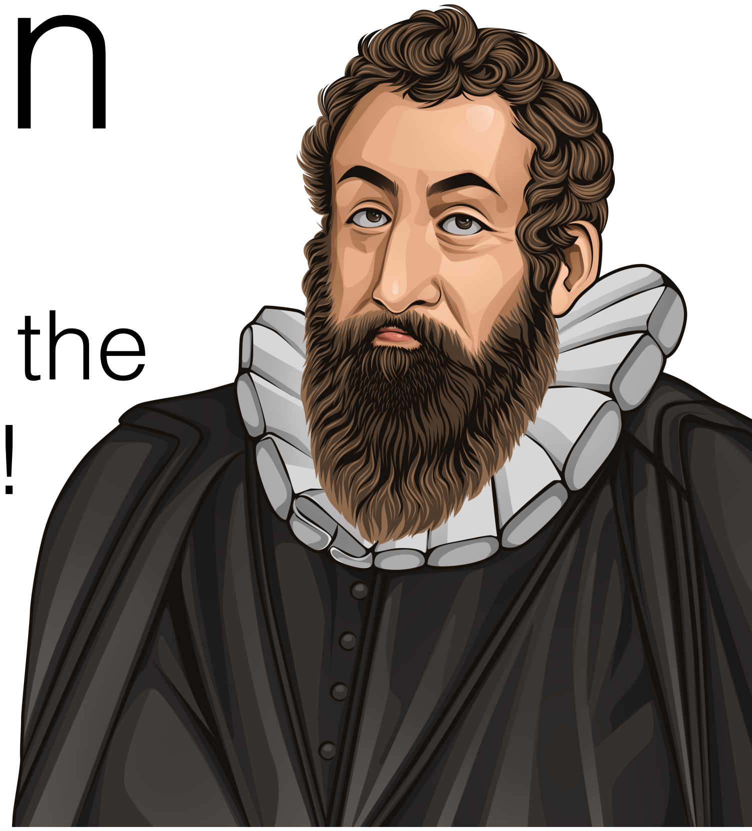
Fior duels Tartaglia, who solves depressed cubic

Pacioli's challenge: Solve the cubic

On death-bed, gives solution to Fior



Cubic Equation



- Tartaglia is the third person to discover the solution, and he…. also kept it a secret!

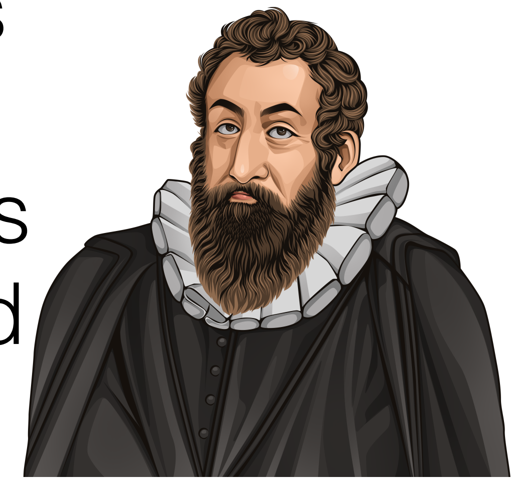
- This drove Cardano crazy. Begged Tartaglia for the solution.

- Visited him in Milan. Tartaglia at last divulged it, but only after Cardano swore an oath of secrecy.

Cubic Equation

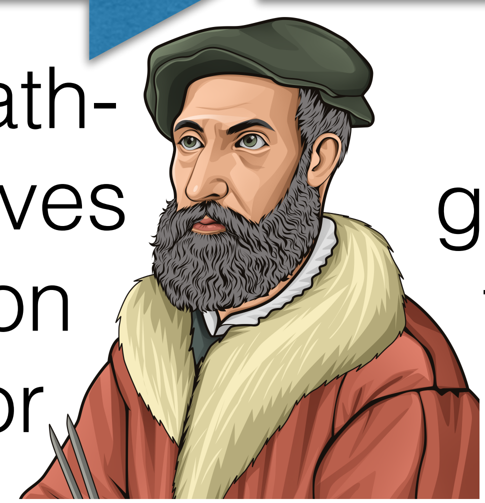
Del Ferro solves the depressed cubic

Fior duels Tartaglia, who solves depressed cubic



Pacioli's challenge: Solve the cubic

On death-bed, gives solution to Fior



Tartaglia gives solution to Cardano under vow of secrecy

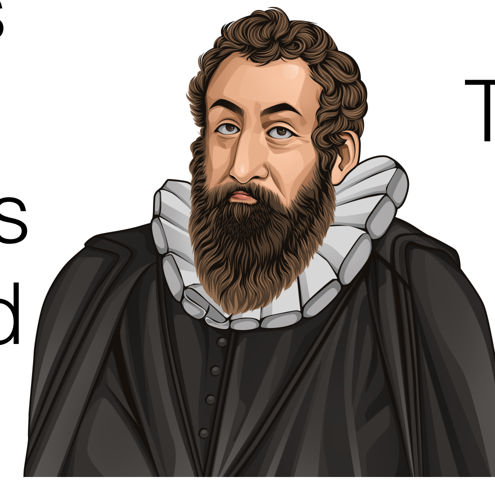
Cubic Equation

- Tartaglia is the third person to discover the solution, and he…. Also kept it a secret!
- This drove Cardano crazy. Begged Tartaglia for the solution.
- Visited him in Milan. Tartaglia at last divulged it, but only after Cardano swore an oath of secrecy.
- Cardano told it to his student Ludovico Ferrari. Together, *they solved the general cubic problem!*

Cubic Equation

Del Ferro solves the depressed cubic

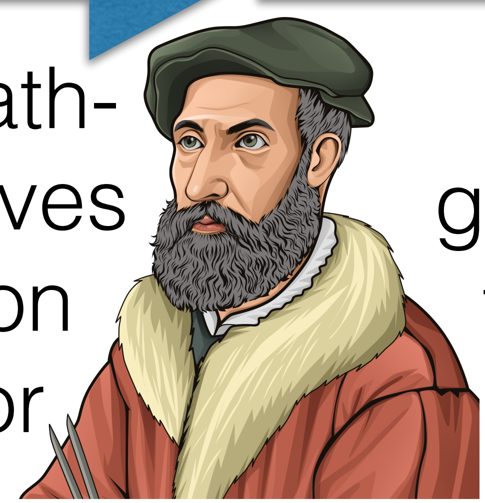
Fior duels Tartaglia, who solves depressed cubic



Tells Ferrari. Then they solve the general cubic.

Pacioli’s challenge: Solve the cubic

On death-bed, gives solution to Fior



Tartaglia gives solution to Cardano under vow of secrecy

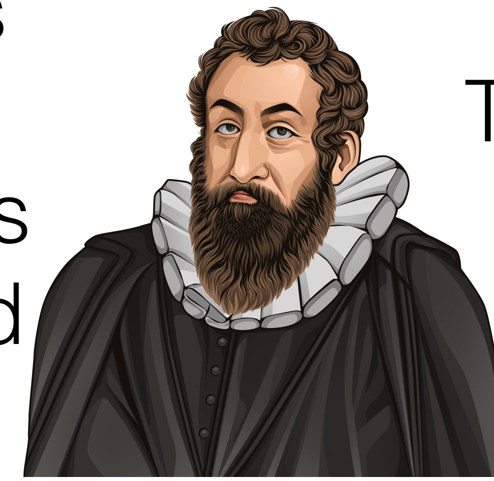
Cubic Equation

- But their solution worked like this: Take a general cubic, do these fancy things to it to reduce it to the depressed cubic, and then use Tartaglia’s secret solution to solve it.
- They solved an enormously important problem, but Cardano’s secrecy oath prevented them from sharing it with the world.
- Ferrari even found a way to solve *the general quartic polynomial*, relying on Tartaglia’s solution!

Cubic Equation

Del Ferro solves the depressed cubic

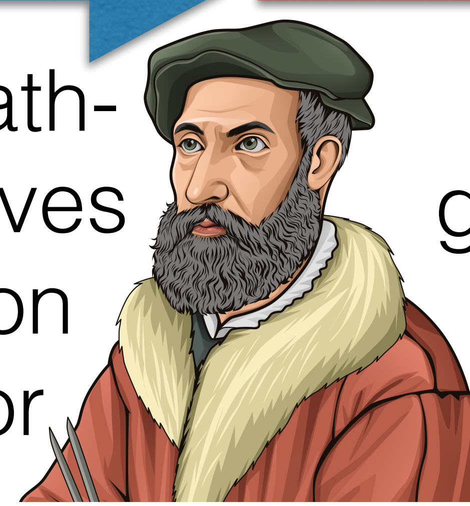
Fior duels Tartaglia, who solves depressed cubic



Tells Ferrari. Then they solve the general cubic.

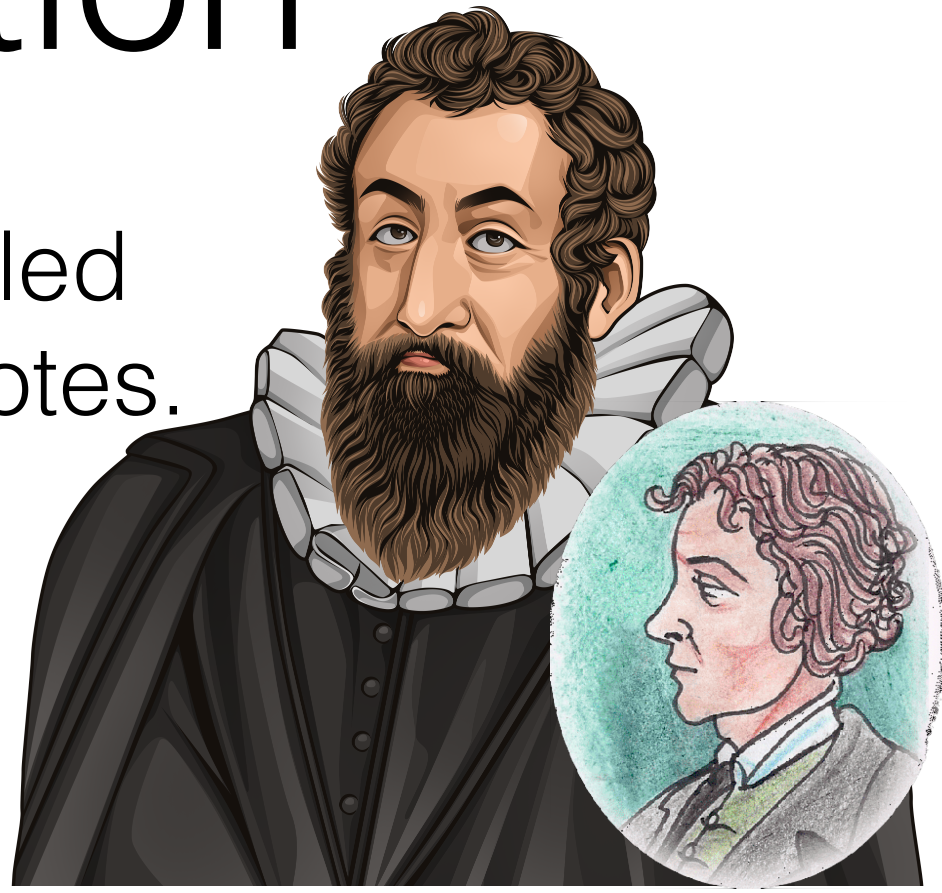
Pacioli's challenge: Solve the cubic

On death-bed, gives solution to Fior



Tartaglia gives solution to Cardano under vow of secrecy

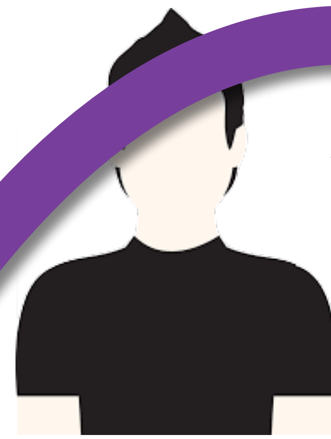
Cubic Equation



- In 1543, Cardano and Ferrari traveled to Bologna to inspect del Ferro’s notes.
- In his notes, they found del Ferro’s solution to the depressed cubic!
- Cardano: I no longer need *Tartaglia’s* solution, I can instead use del Ferro’s! So I can tell the world now!

Cubic Equation

Del Ferro solves the depressed cubic

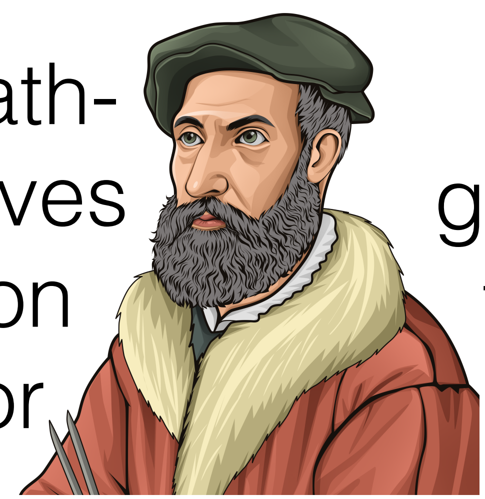


Fior duels Tartaglia, who solves depressed cubic

Tells Ferrari. Then they solve the general cubic.

Pacioli's challenge: Solve the cubic

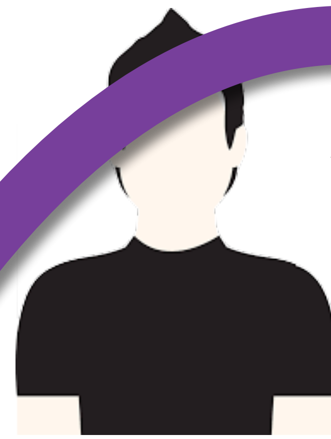
On death-bed, gives solution to Fior



Tartaglia gives solution to Cardano under vow of secrecy

Cubic Equation

Del Ferro solves the depressed cubic

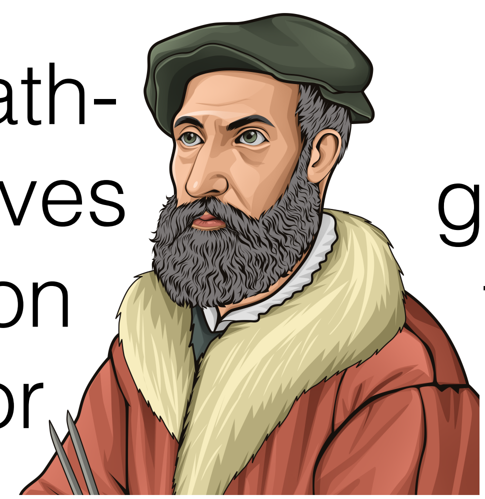


Fior duels Tartaglia, who solves depressed cubic

Tells Ferrari. Then they solve the general cubic and quartic and publish it.

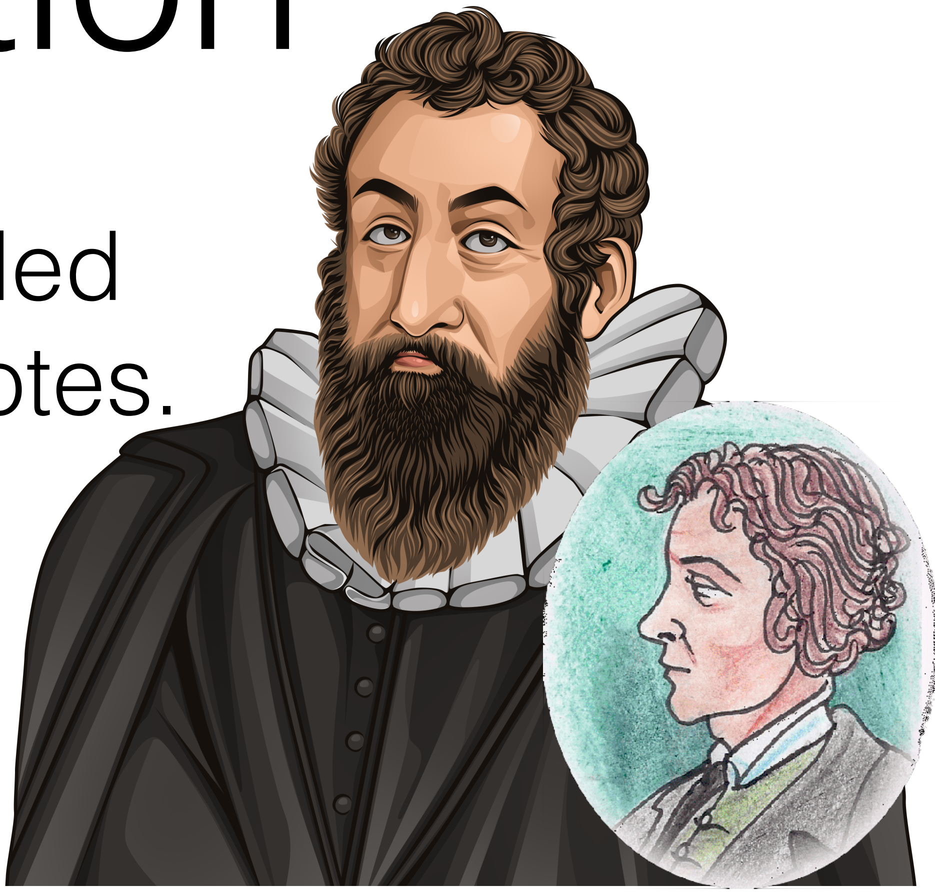
Pacioli's challenge: Solve the cubic

On death-bed, gives solution to Fior



Tartaglia gives solution to Cardano under vow of secrecy

Cubic Equation



- In 1543, Cardano and Ferrari traveled to Bologna to inspect del Ferro’s notes.
- In his notes, they found del Ferro’s solution to the depressed cubic!
- Cardano: I no longer need *Tartaglia’s* solution, I can instead use del Ferro’s! So I can tell the world now!
- Cardano published it in his book *Ars Magna*. Tartaglia felt betrayed.

Cubic Equation

- What was the answer?

Theorem.

Theorem 1.2 (*Root of the Depressed Cubic*)**.** A solution to $x^3 + mx = n$ is

$$x = \sqrt[3]{\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}} - \sqrt[3]{-\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}}.$$

The other solutions, if they exist, can be found by using long division and then the quadratic formula.

Cubic Equation

Theorem.

Theorem 1.3 (*Root of the Cubic*)**.** A solution to $ax^3 + bx^2 + cx + d = 0$ is

$$x = \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)}}$$
$$+ \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)}}$$
$$- \frac{b}{3a}.$$

The other solutions, if they exist, can be found by using long division and then the quadratic formula.

Quartic Equation

$$r_1 = \frac{-a}{4} - \frac{1}{2}\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \frac{2^{\frac{1}{3}}\left(b^2 - 3ac + 12d\right)}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}\right)^{\frac{1}{3}}} + \left(\frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}}{54}\right)^{\frac{1}{3}}} - \frac{1}{2}\sqrt{\frac{a^2}{2} - \frac{4b}{3} - \frac{2^{\frac{1}{3}}\left(b^2 - 3ac + 12d\right)}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}\right)^{\frac{1}{3}}} - \left(\frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}}{54}\right)^{\frac{1}{3}} - \frac{-a^3 + 4ab - 8c}{4\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \frac{2^{\frac{1}{3}}\left(b^2 - 3ac + 12d\right)}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}\right)^{\frac{1}{3}}} + \left(\frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}}{54}\right)^{\frac{1}{3}}}}}$$

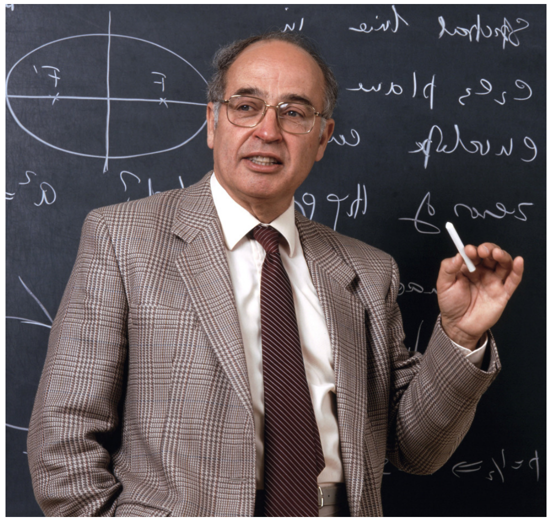
$$r_2 = \frac{-a}{4} - \frac{1}{2}\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \frac{2^{\frac{1}{3}}\left(b^2 - 3ac + 12d\right)}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}\right)^{\frac{1}{3}}} + \left(\frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}}{54}\right)^{\frac{1}{3}}} + \frac{1}{2}\sqrt{\frac{a^2}{2} - \frac{4b}{3} - \frac{2^{\frac{1}{3}}\left(b^2 - 3ac + 12d\right)}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}\right)^{\frac{1}{3}}} - \left(\frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}}{54}\right)^{\frac{1}{3}} - \frac{-a^3 + 4ab - 8c}{4\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \frac{2^{\frac{1}{3}}\left(b^2 - 3ac + 12d\right)}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}\right)^{\frac{1}{3}}} + \left(\frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}}{54}\right)^{\frac{1}{3}}}}}$$

$$r_3 = \frac{-a}{4} + \frac{1}{2}\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \frac{2^{\frac{1}{3}}\left(b^2 - 3ac + 12d\right)}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}\right)^{\frac{1}{3}}} + \left(\frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}}{54}\right)^{\frac{1}{3}}} - \frac{1}{2}\sqrt{\frac{a^2}{2} - \frac{4b}{3} - \frac{2^{\frac{1}{3}}\left(b^2 - 3ac + 12d\right)}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}\right)^{\frac{1}{3}}} - \left(\frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}}{54}\right)^{\frac{1}{3}} + \frac{-a^3 + 4ab - 8c}{4\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \frac{2^{\frac{1}{3}}\left(b^2 - 3ac + 12d\right)}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}\right)^{\frac{1}{3}}} + \left(\frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}}{54}\right)^{\frac{1}{3}}}}}$$

$$r_4 = \frac{-a}{4} + \frac{1}{2}\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \frac{2^{\frac{1}{3}}\left(b^2 - 3ac + 12d\right)}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}\right)^{\frac{1}{3}}} + \left(\frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}}{54}\right)^{\frac{1}{3}}} + \frac{1}{2}\sqrt{\frac{a^2}{2} - \frac{4b}{3} - \frac{2^{\frac{1}{3}}\left(b^2 - 3ac + 12d\right)}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}\right)^{\frac{1}{3}}} - \left(\frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}}{54}\right)^{\frac{1}{3}} + \frac{-a^3 + 4ab - 8c}{4\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \frac{2^{\frac{1}{3}}\left(b^2 - 3ac + 12d\right)}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}\right)^{\frac{1}{3}}} + \left(\frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}}{54}\right)^{\frac{1}{3}}}}}$$

Quartic Equation

$$\frac{-a}{4} - \frac{1}{2}\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \frac{2^{\frac{1}{3}}\left(b^2 - 3ac + 12d\right)}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4(b^2 - 3ac + 12d)^3 + (2b^3 - 9abc + 27c^2 +}\right.}}$$



Algebra: The Light or The Dark?

- Fields Medalist Michael Atiyah:

> Algebra is the offer made by the devil to the mathematician. The devil says: I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvelous machine.

- al-Khwārizmī:

> That fondness for science, … that affability and condescension which God shows to the learned, that promptitude with which he protects and supports them in the elucidation of obscurities and in the removal of difficulties, has encouraged me to compose a short work on calculating by al-jabr and al-muqabala, confining it to what is easiest and most useful in arithmetic.

Algebra Aftermath

Abstract Algebra

- Next challenge: Solve the general quintic. That is, using only basic arithmetic operations, find a formula for a root of

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0.$$

- The problem remained unsolved for 250 years.
- Then a 22 year old began work on it.
- His name was Niels Abel.

Niels Abel

- Born in 1802 in Norway under threat of war. Was sent to Oslo for school.
- Was poor and sick his whole life. Struggled to find work despite his obvious genius.
- Worked on the quintic problem. In 1824, at age 22, he proved *no such formula exists*. Huge result, but in his life he never got the credit he deserved.
- Died at age 26 of tuberculosis. Offered job 2 days later.

Évariste Galois

- Born in 1811. Was 18 when Abel died.
- Fell in love with math at boarding school. Absorbed the main textbooks, eventually read articles from Lagrange about the quintic problem.
- He worked on it, unaware of Abel's work.
- Galois' aim was broader than Abel's: He wanted to determine *which* polynomials can be solved with a formula.

Évariste Galois

- From Abel, we know that

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$$

cannot be solved with a formula.

- But $ax^5 + f = 0$ can. Formula: $x = \sqrt[5]{-f/a}$.

- Which can and which can't? Galois solved this problem by studying symmetries in the roots of equations. Today this is called *Galois theory*.

Évariste Galois

- In his life, Galois didn't fully get the credit his work deserved. He submitted his work for the Grand Prix for Mathematics but his manuscript was accidentally neglected.

- French politics were turbulent. Galois wanted revolution against King Charles X and joined a militant group.

- He died at the age of 20 in a duel. It was probably over a love interest, but the circumstances are unclear.

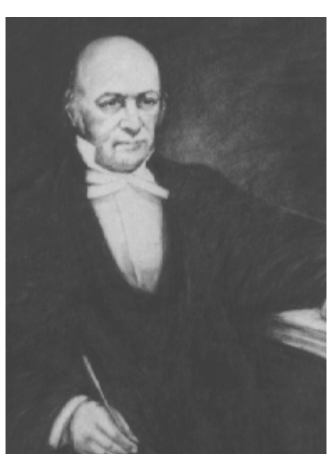
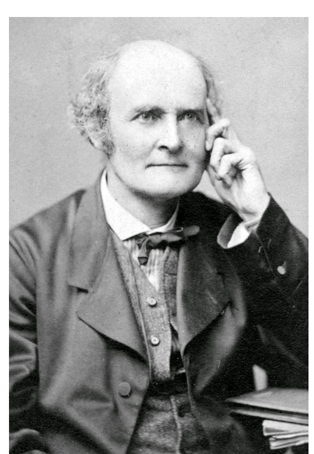
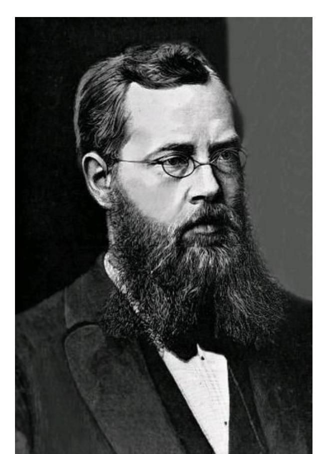
Galois' Final Letters

Group Theory

- Group theory was more fully developed by Cauchy, Hamilton, Cayley, Sylow, Lie, Frobenius, Klein, and Poincaré.

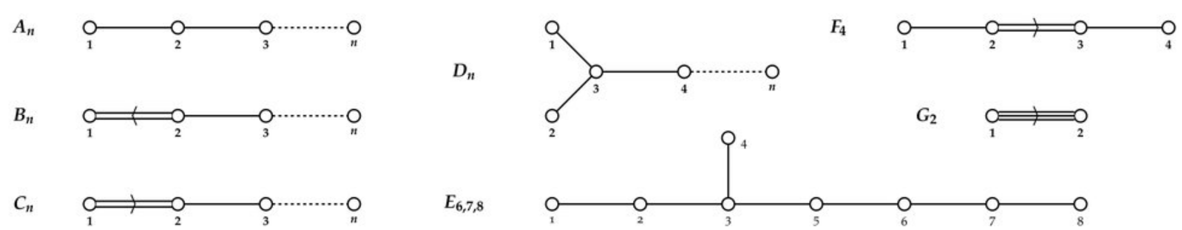
- It is still an active area of research today.

- A major problem was the classification of finite, simple groups. This was completed in 2008.

The Periodic Table Of Finite Simple Groups

Dynkin Diagrams of Simple Lie Algebras



$0, C_1, \mathbb{Z}_1$ **1** 1																	C_2 2
$A_1(4), A_1(5)$ A_5 60	$A_2(2)$ $A_1(7)$ 168											$^2A_3(4)$ $B_2(3)$ 25 920	$C_3(3)$ 4 585 351 680	$D_4(2)$ 174 182 400	$^2D_4(2^2)$ 197 406 720	$G_2(2)'$ $^2A_2(9)$ 6 048	C_3 3
$A_1(9), B_2(2)'$ A_6 360	$^2G_2(3)'$ $A_1(8)$ 504											$B_2(4)$ 979 200	$C_3(5)$ 228 501 000 000 000	$D_4(3)$ 4 952 179 814 400	$^2D_4(3^2)$ 10 151 968 619 520	$^2A_2(16)$ 62 400	C_5 5
A_7 2 520	$A_1(11)$ 660	$E_6(2)$ 214 841 575 522 005 575 270 400	$E_7(2)$ 7 997 476 042 075 799 759 100 487 262 680 802 918 400	$E_8(2)$ 337 804 753 143 634 806 261 388 190 614 085 595 079 991 692 242 467 651 576 160 959 909 068 800 000	$F_4(2)$ 3 311 126 603 366 400	$G_2(3)$ 4 245 696	$^3D_4(2^3)$ 211 341 312	$^2E_6(2^2)$ 76 532 479 683 774 853 939 200	$^2B_2(2^3)$ 29 120	Tits* $^2F_4(2)'$ 17 971 200	$^2G_2(3^3)$ 10 073 444 472	$B_3(2)$ 1 451 520	$C_4(3)$ 65 784 756 654 489 600	$D_5(2)$ 23 499 295 948 800	$^2D_5(2^2)$ 25 015 379 558 400	$^2A_2(25)$ 126 000	C_7 7
$A_3(2)$ A_8 20 160	$A_1(13)$ 1 092	$E_6(3)$ 7 257 703 347 541 463 210 028 258 395 214 643 200	$E_7(3)$ 1 271 375 236 818 136 742 240 479 751 139 021 644 554 379 203 770 766 254 617 395 200	$E_8(3)$ [illegible]	$F_4(3)$ 5 734 420 792 816 671 844 761 600	$G_2(4)$ 251 596 800	$^3D_4(3^3)$ 20 560 831 566 912	$^2E_6(3^2)$ 14 636 855 916 969 695 633 965 120 680 532 377 600	$^2B_2(2^5)$ 32 537 600	$^2F_4(2^3)$ 264 905 352 699 586 176 614 400	$^2G_2(3^5)$ 49 825 657 439 340 552	$B_2(5)$ 4 680 000	$C_3(7)$ 273 457 218 604 953 600	$D_4(5)$ 8 911 539 000 000 000 000	$^2D_4(4^2)$ 67 536 471 195 648 000	$^2A_3(9)$ 3 265 920	C_{11} 11
A_9 181 440	$A_1(17)$ 2 448	$E_6(4)$ 85 528 710 781 342 640 103 833 619 055 142 765 466 746 880 000	$E_7(4)$ 111 131 458 114 940 385 379 597 233 477 884 941 280 664 199 527 155 056 307 251 745 263 504 588 800 000 000	$E_8(4)$ [illegible]	$F_4(4)$ 19 009 825 523 840 945 451 297 669 120 000	$G_2(5)$ 5 859 000 000	$^3D_4(4^3)$ 67 802 350 642 790 400	$^2E_6(4^2)$ 85 696 576 147 617 709 485 896 772 387 584 983 695 360 000 000	$^2B_2(2^7)$ 34 093 383 680	$^2F_4(2^5)$ 1 318 633 155 799 591 447 702 161 609 782 722 560 000	$^2G_2(3^7)$ 239 189 910 264 352 349 332 632	$B_2(7)$ 138 297 600	$C_3(9)$ 54 025 731 402 499 584 000	$D_5(3)$ 1 289 512 799 941 305 139 200	$^2D_4(5^2)$ 17 880 203 250 000 000 000	$^2A_2(64)$ 5 515 776	C_{13} 13
A_n $\frac{n!}{2}$	$PSL_{n+1}(q), L_{n+1}(q)$ $A_n(q)$ $\frac{q^{n(n+1)/2}}{(n+1,q-1)}\prod_{i=1}^{n}(q^{i+1}-1)$	$E_6(q)$ $\frac{q^{36}(q^{12}-1)(q^9-1)(q^8-1)(q^6-1)(q^5-1)(q^2-1)}{(3,q-1)}$	$E_7(q)$ $\frac{q^{63}}{(2,q-1)}\prod_{i=1, i\neq 2,8}^{9}(q^{2i}-1)$	$E_8(q)$ $q^{120}(q^{30}-1)(q^{24}-1)(q^{20}-1)(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^8-1)(q^2-1)$	$F_4(q)$ $q^{24}(q^{12}-1)(q^8-1)(q^6-1)(q^2-1)$	$G_2(q)$ $q^6(q^6-1)(q^2-1)$	$^3D_4(q^3)$ $q^{12}(q^8+q^4+1)(q^6-1)(q^2-1)$	$^2E_6(q^2)$ $\frac{q^{36}(q^{12}-1)(q^9+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1)}{(3,q+1)}$	$^2B_2(2^{2n+1})$ $q^2(q^2+1)(q-1)$	$^2F_4(2^{2n+1})$ $q^{12}(q^6+1)(q^4-1)(q^3+1)(q-1)$	$^2G_2(3^{2n+1})$ $q^3(q^3+1)(q-1)$	$O_{2n+1}(q), \Omega_{2n+1}(q)$ $B_n(q)$ $\frac{q^{n^2}}{(2,q-1)}\prod_{i=1}^{n}(q^{2i}-1)$	$PSp_{2n}(q)$ $C_n(q)$ $\frac{q^{n^2}}{(2,q-1)}\prod_{i=1}^{n}(q^{2i}-1)$	$O_{2n}^{+}(q)$ $D_n(q)$ $\frac{q^{n(n-1)}(q^n-1)}{(4,q^n-1)}\prod_{i=1}^{n-1}(q^{2i}-1)$	$O_{2n}^{-}(q)$ $^2D_n(q^2)$ $\frac{q^{n(n-1)}(q^n+1)}{(4,q^n+1)}\prod_{i=1}^{n-1}(q^{2i}-1)$	$PSU_{n+1}(q)$ $^2A_n(q^2)$ $\frac{q^{n(n+1)/2}}{(n+1,q+1)}\prod_{i=2}^{n+1}(q^i-(-1)^i)$	$\mathbb{Z}_p$ C_p p

- Alternating Groups
- Classical Chevalley Groups
- Chevalley Groups
- Classical Steinberg Groups
- Steinberg Groups
- Suzuki Groups
- Ree Groups and Tits Group*
- Sporadic Groups
- Cyclic Groups

Alternates†
Symbol
Order‡

M_{11} 7 920	M_{12} 95 040	M_{22} 443 520	M_{23} 10 200 960	M_{24} 244 823 040	$J(1), J(11)$ J_1 175 560	HJ J_2 604 800	HJM J_3 50 232 960	J_4 86 775 571 046 077 562 880	HS 44 352 000	McL 898 128 000	F_7, HHM, HTH He 4 030 387 200	Ru 145 926 144 000
Sz Suz 448 345 497 600	$O'NS, O\text{–}S$ $O'N$ 460 815 505 920	$\cdot 3$ Co_3 495 766 656 000	$\cdot 2$ Co_2 42 305 421 312 000	$\cdot 1$ Co_1 4 157 776 806 543 360 000	F_5, D HN 273 030 912 000 000	LyS Ly 51 765 179 004 000 000	F_3, E Th 90 745 943 887 872 000	$M(22)$ Fi_{22} 64 561 751 654 400	$M(23)$ Fi_{23} 4 089 470 473 293 004 800	$F_{3+}, M(24)'$ Fi_{24}' 1 255 205 709 190 661 721 292 800	F_2 B 4 154 781 481 226 426 191 177 580 544 000 000	F_1, M_1 M 808 017 424 794 512 875 886 459 904 961 710 757 005 754 368 000 000 000

*The Tits group $^2F_4(2)'$ is not a group of Lie type, but is the (index 2) commutator subgroup of $^2F_4(2)$. It is usually given honorary Lie type status.

The groups starting on the second row are the classical groups. The sporadic suzuki group is unrelated to the families of Suzuki groups.

†For sporadic groups and families, alternate names in the upper left are other names by which they may be known. For specific non-sporadic groups these are used to indicate isomorphims. All such isomorphisms appear on the table except the family $B_n(2^m) \cong C_n(2^m)$.

‡Finite simple groups are determined by their order with the following exceptions:
$B_n(q)$ and $C_n(q)$ for q odd, $n > 2$;
$A_8 \cong A_3(2)$ and $A_2(4)$ of order 20160.

The Fundamental Theorem of Algebra

- **Theorem.** A non-constant polynomial of degree n, whether its coefficients are real numbers or complex numbers, has n complex roots, when counted with multiplicity.

- Many people contributed to this discovery, and many provided flawed proofs of it before it was at last rigorously proved by Jean-Robert Argand.

- Carl Friedrich Gauss proved it four times. His first proof was nearly correct and proceeded Argand's.

- This theorem helped complex numbers be accepted.

Linear Algebra

> The aim of the present work is to establish certain theorems valid in different functional domains, which I will specify in what follows. Nevertheless, in order not to have to prove them for each particular domain, which would be painful, I have chosen to take a different route … I consider sets of elements about which I postulate certain properties; I deduce from them certain theorems, and I then prove for each particular functional domain that the postulates adopted are true for it.

- Writing down axioms not only grounds a theory, but it *forms* a theory. It ties together many different problems, asserting they are all of the same type.

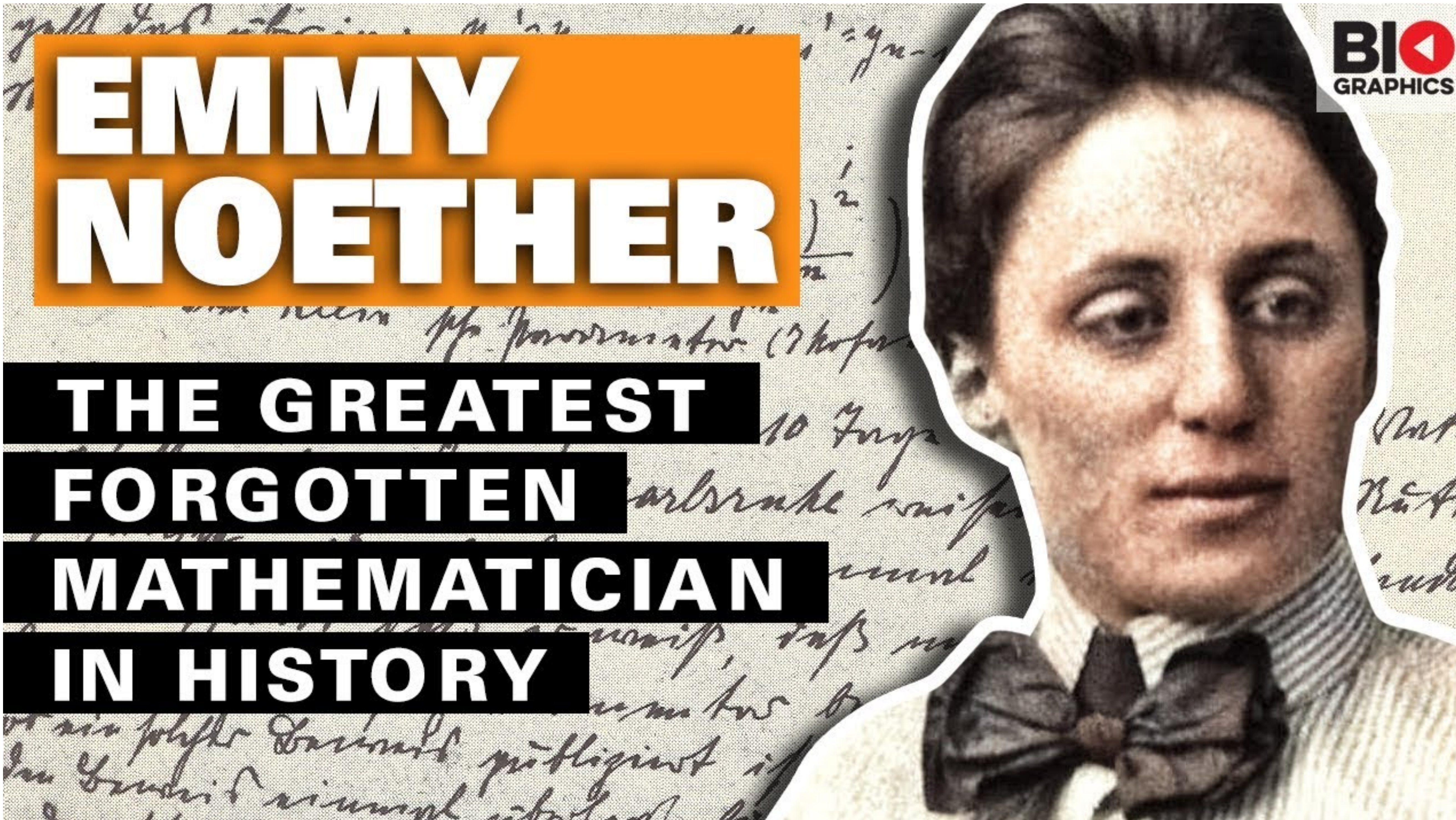
Ring Theory

- Nobody was more responsible for the growth of modern abstract algebra than German mathematician Emmy Noether.

- She developed ring theory in the 1920s, which united the well-studied theories of numbers and polynomials, and much else.

- She also played a major rule in cementing the abstract theory of groups.

Emmy Noether



Emmy Noether

- Throughout her life, she faced a mixture of small to moderate advantages, and enormous disadvantages.

- She faced repeated sexism as a woman in math and academia.

- She faced antisemitism as a Jew in 20th-century Germany.

- She mostly worked for no pay and let others publish much of her work.

Emmy Noether

- She also made seminal contributions to study of fields, algebras and mathematical physics.
- This included ground-breaking work related to Einstein's new theory of general relativity.
- When she fled Nazi Germany, Einstein personally arranged a position for her at Bryn Mawr College.
- She died as one of the greatest mathematicians in history.

Ultimate Abstraction

- Perhaps the most abstract form of math is called *category theory*. (More in the notes.)
- Category theory is the closest we come to Banach’s vision of the “ultimate mathematician”

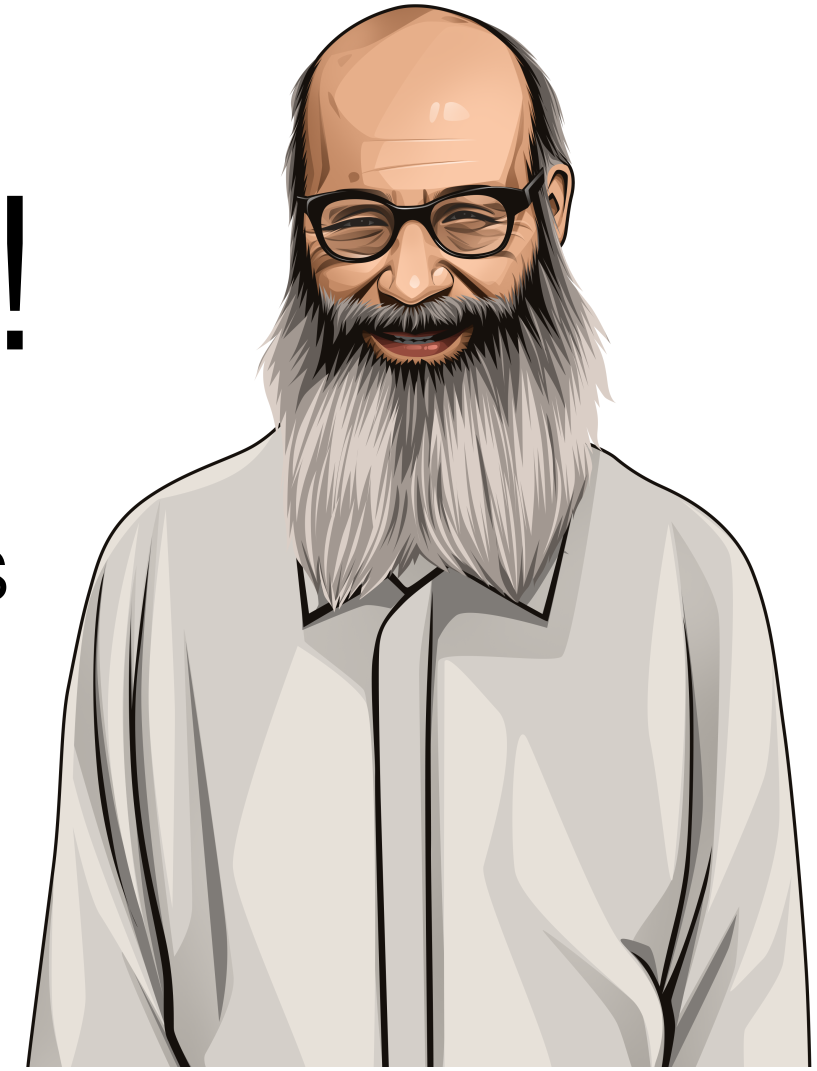
> A mathematician is a person who can find analogies between theorems; a better mathematician is one who can see analogies between proofs and the best mathematician can notice analogies between theories. One can imagine that the ultimate mathematician is one who can see analogies between analogies.

Shout-outs!

- A 4,000 year old clay tablet from Mesopotamia contains the values of $x^3 + x^2$ for $x = 1,2,3,\ldots,50$. Possibly they had a method to reduce certain cubic equations to the form $x^3 + x^2 = C$, and this tablet was used to solve or at least estimate the final answer from there.

- Diophantus of Alexandria lived in the third century AD. He wrote *Arithmetica*, which was the most prominant work on algebra by a Greek mathematician. *Diophantine equations* were a major study of research over the centuries following his life.

Shout-outs!



- A super deep area of math that uses a lot of algebra is called *algebraic geometry*. Alexander Grothendieck was a leader in its development.

- The Nine Chapter has a method to solve polynomial equations with a counting board.

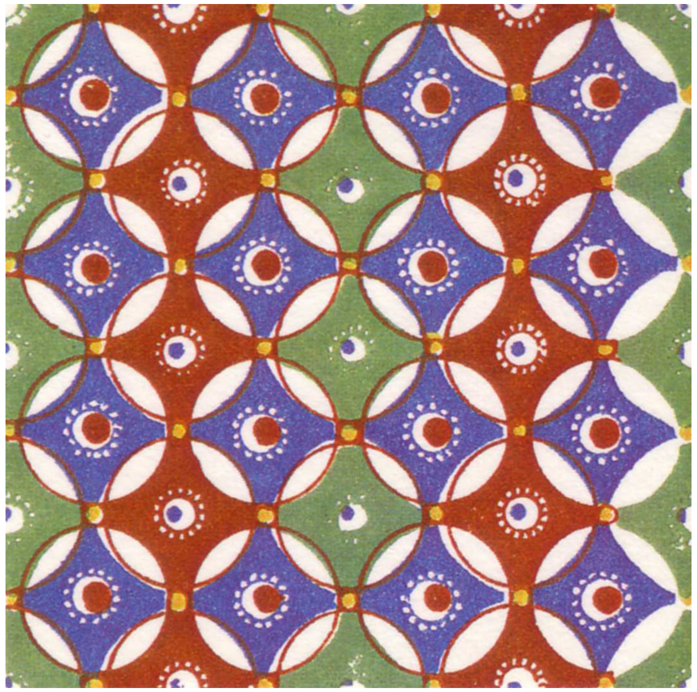
- Persian polymath Omar Khayyam came up with a neat approach to solve cubic equations by intersecting comics. More in the exercises if you want to explore it further.

A People's History of Symmetry

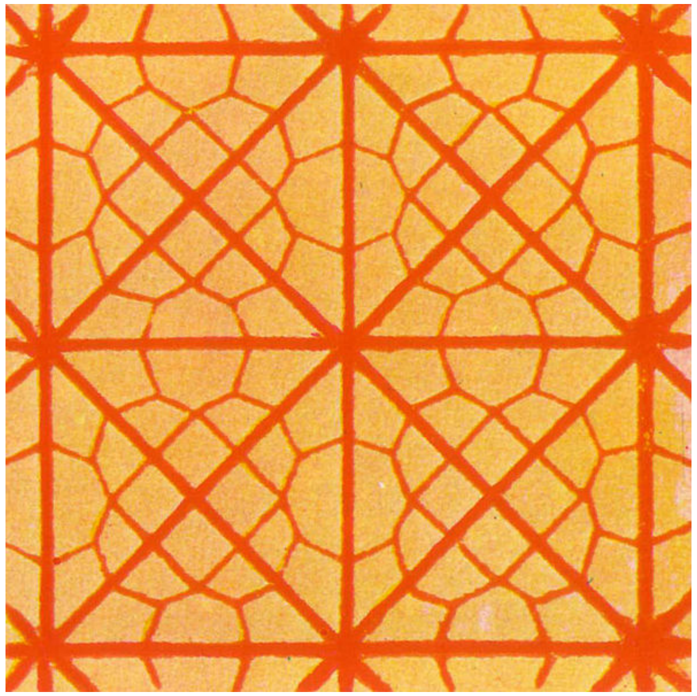
A People's History of Symmetry

- People have investigated symmetry for millennia. Notable example: Tessellations.

Metalwork, India



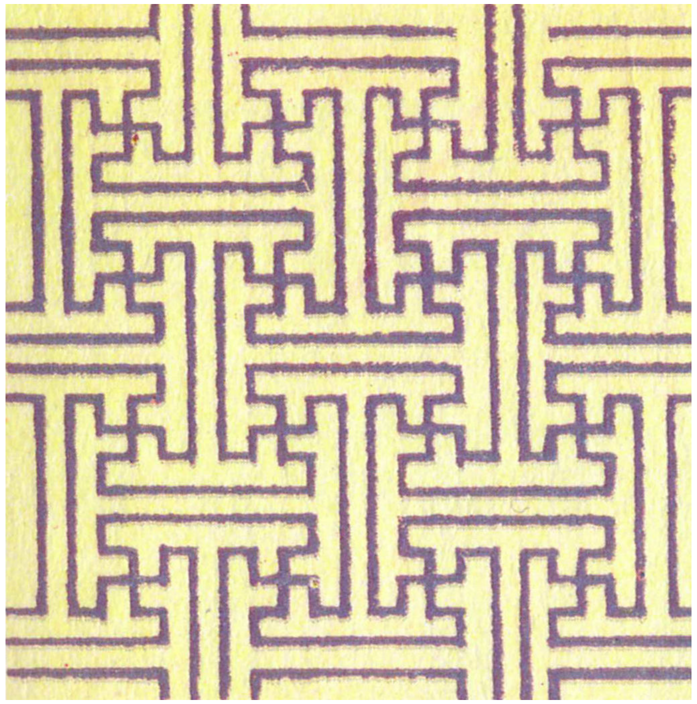
Mummy case, Egypt



Glazed tile, Persia

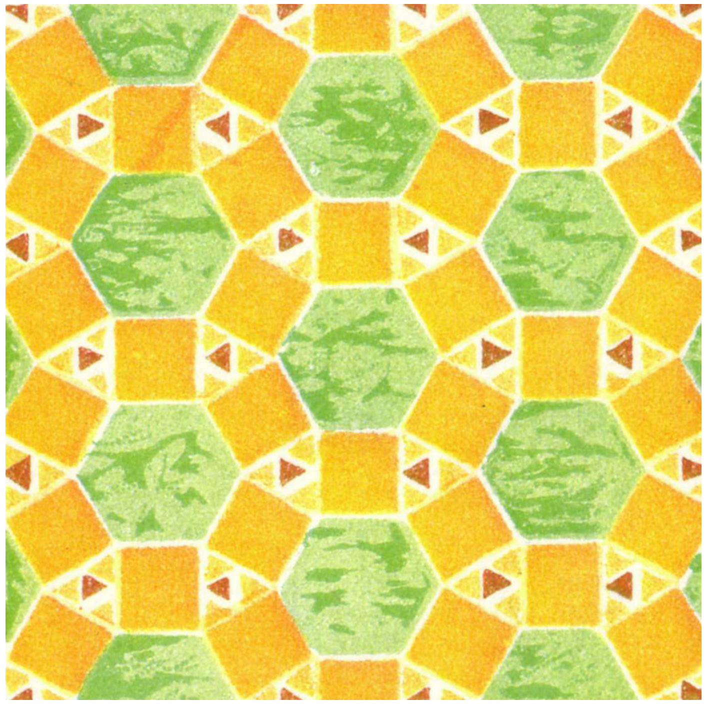
A People's History of Symmetry

- People have investigated symmetry for millennia. Notable example: Tessellations.



Painted porcelain, China

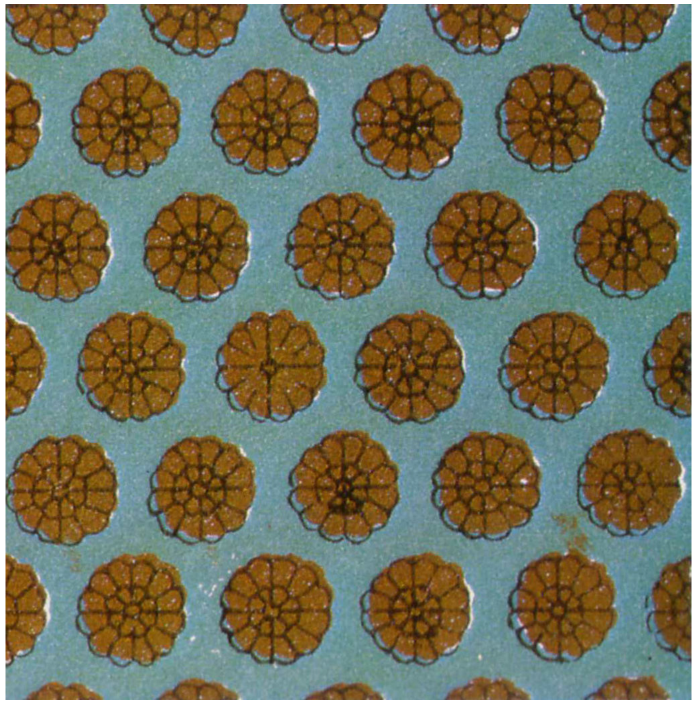
Wall painting, Spain



Painted marble, Persia

A People's History of Symmetry

- People have investigated symmetry for millennia. Notable example: Tessellations.



King's dress, Assyria

Cloth garment, Hawaii

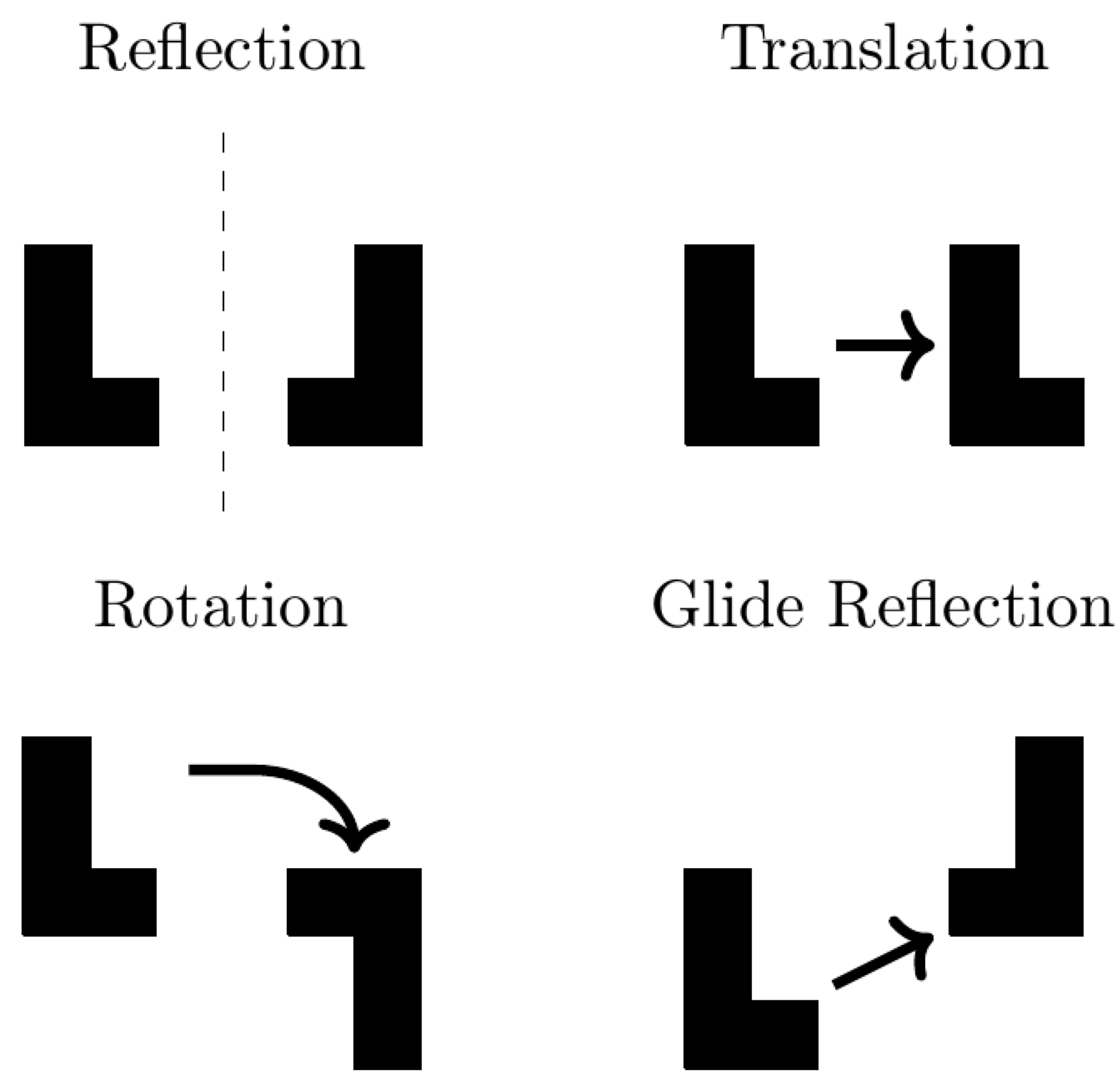
Jay's patio, Sacramento

A People’s History of Symmetry

- People have investigated symmetry for millennia. Notable example: Tessellations.

A People's History of Symmetry

- The symmetries of a tessellation are a combination of four types:



A People's History of Symmetry

- Deep theorem (1891): Every possible tessellation is of 17 symmetry types.

- Ancient artists found every single one. The Alhambra palace alone may contain all 17.

- Their understanding is "proven" via their art.

A People's History of Symmetry

- Hermann Weyl in 1938:

> The art of ornament contains in implicit form the oldest piece of high mathematics known to us. To be sure, the conceptual means for a complete abstract formulation of the underlying problem, namely the mathematical notion of a group of transformation, was not provided before the nineteenth century; and only on this basis is one able to prove that the seventeen symmetries already implicitly known to the Egyptian craftsman exhaust all possibilities.

A People's History of Symmetry

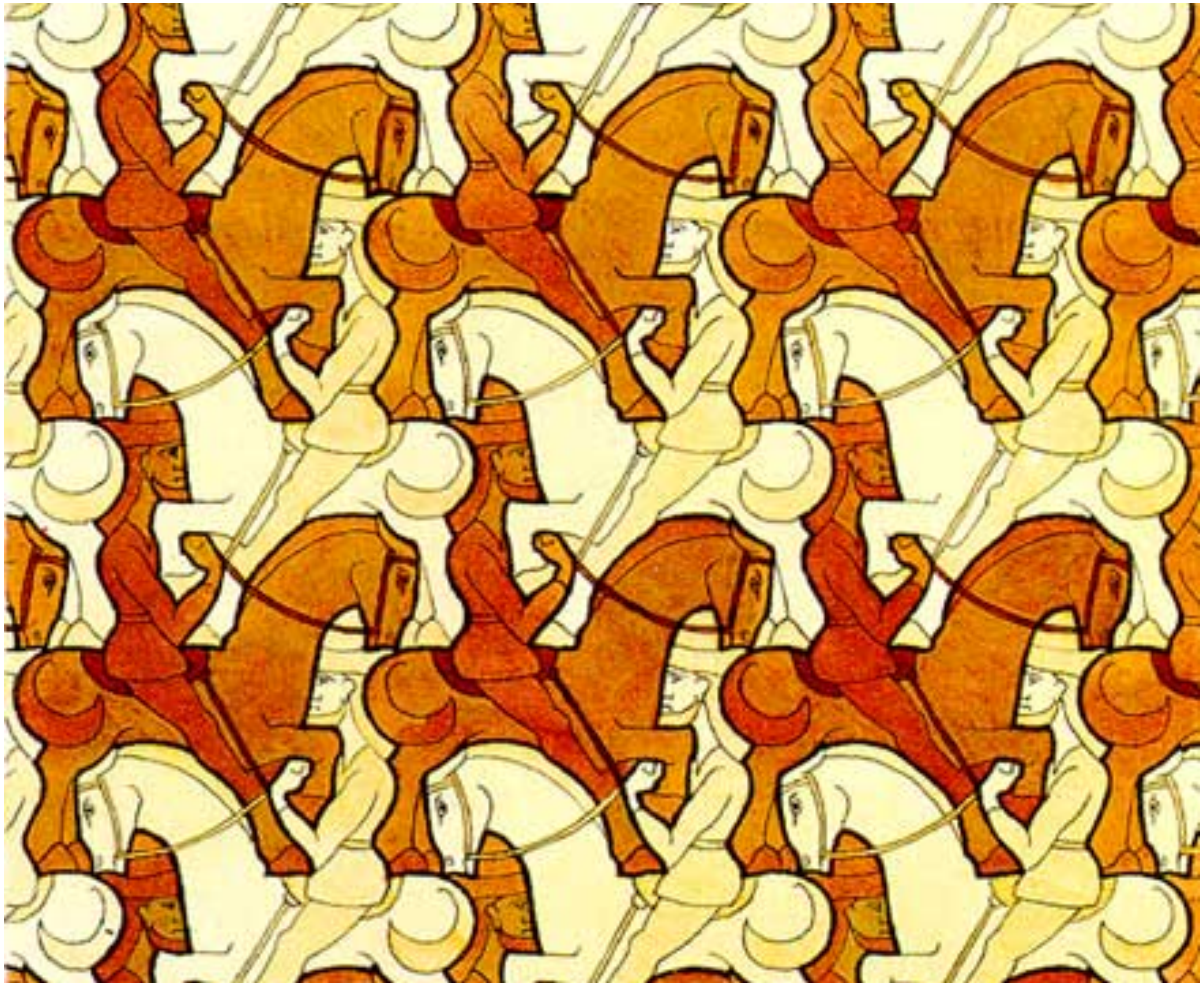
- *Tessellation groups* are well-studied.

- Andreas Speiser's 1923 book *The theory of groups of finite order* contained a chapter on this connection.

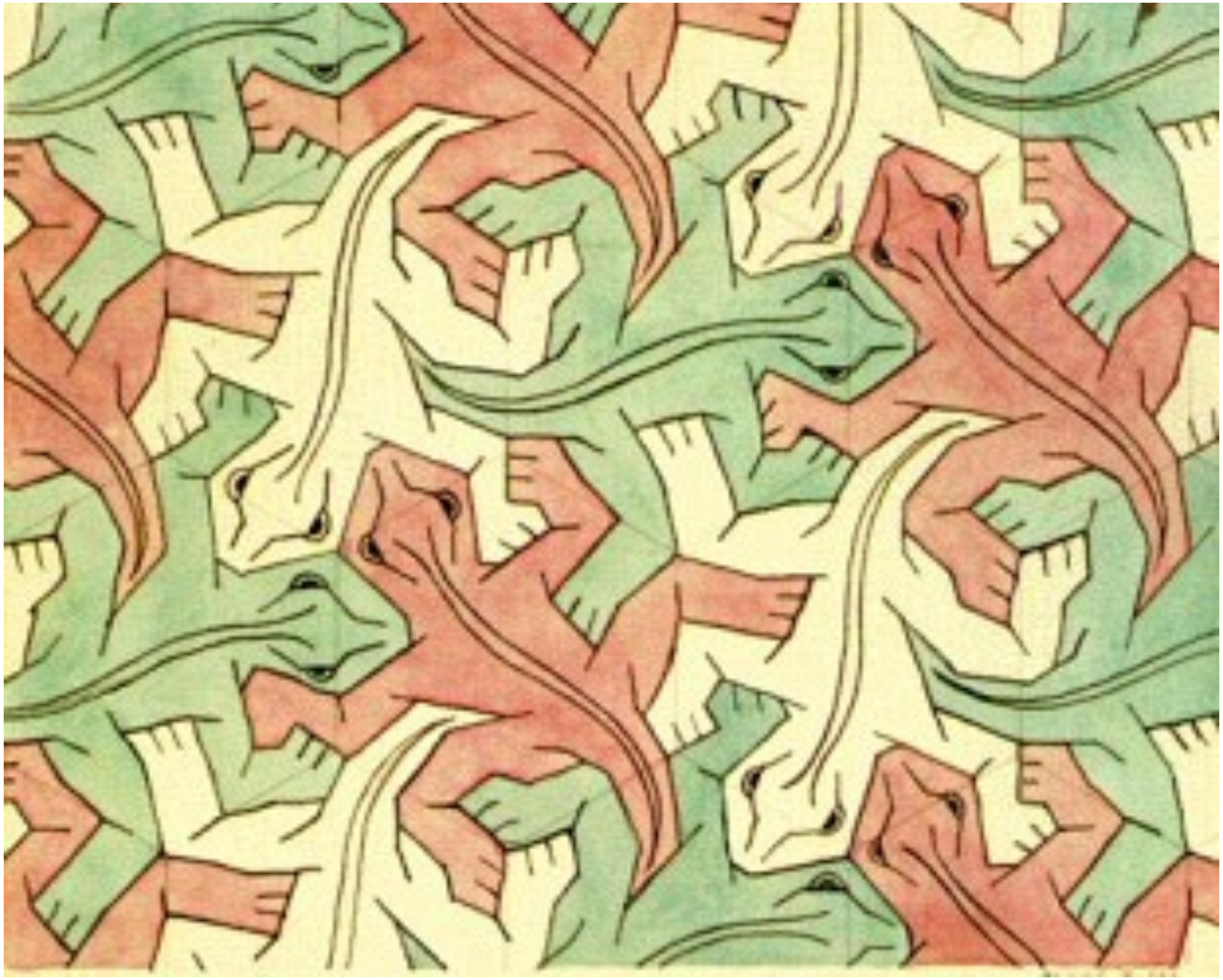
- Donald W. Crowe:

> [Speiser] persuaded many of us that designers of ornament from ancient times had at least a subconscious understanding of basic ideas of group theory. Speiser was, in the eyes of many, the first to make this connection.

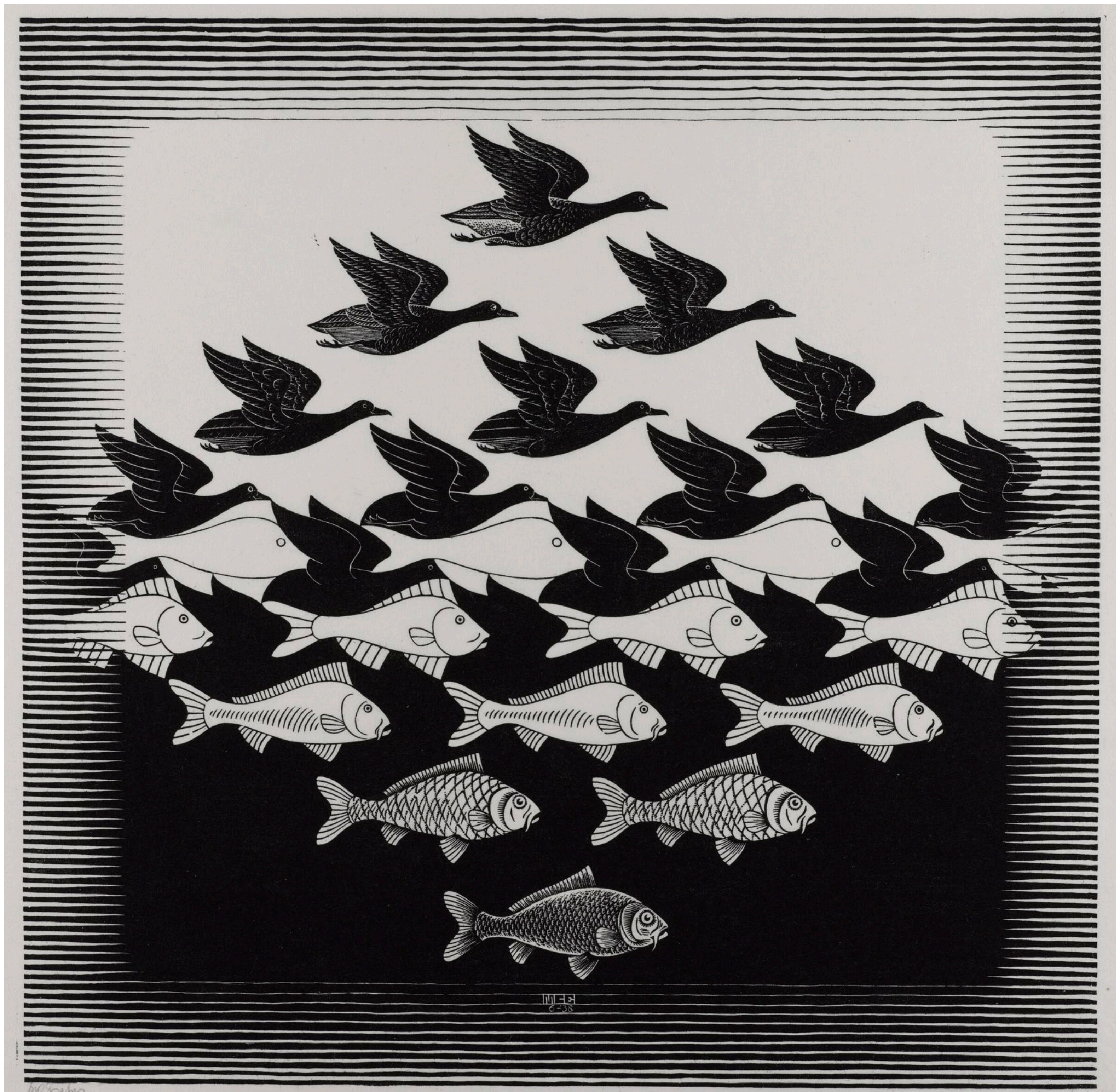
M.C. Escher



M.C. Escher

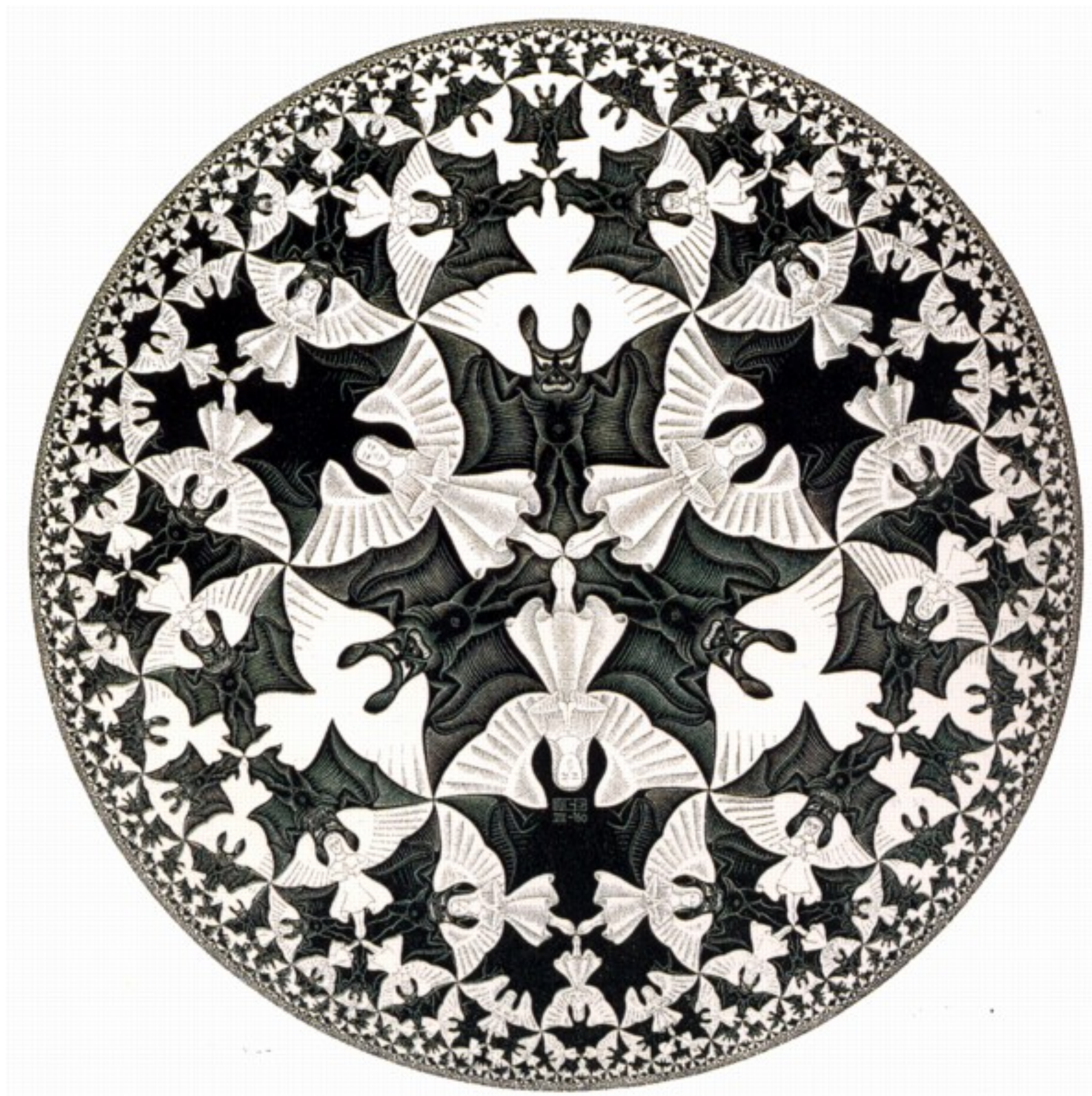


M.C. Escher



M.C. Escher

- His work inspired work in hyperbolic geometry.



A People’s History of Symmetry

- Often in math, a picture can capture an idea.
- Example: Proofs by picture, or a picture which summarizes a proof.
- The tessellations that adorn buildings, clothes, vases, etc., for thousands of years also exhibit deep understanding.
- It’s beautiful “people’s math.”

The History of Mathematical Symbols

Math Symbols

- In 1557, Robert Recorde was getting tired of writing “is equals to” over and over again. So he invented a symbol.

And to auoide the tediouse repetition of these woordes : is equalle to : I will sette as I doe often in woorke vse, a paire of paralleles, or Gemowe lines of one lengthe, thus: =====, bicause noe .2. thynges, can be moare equalle.

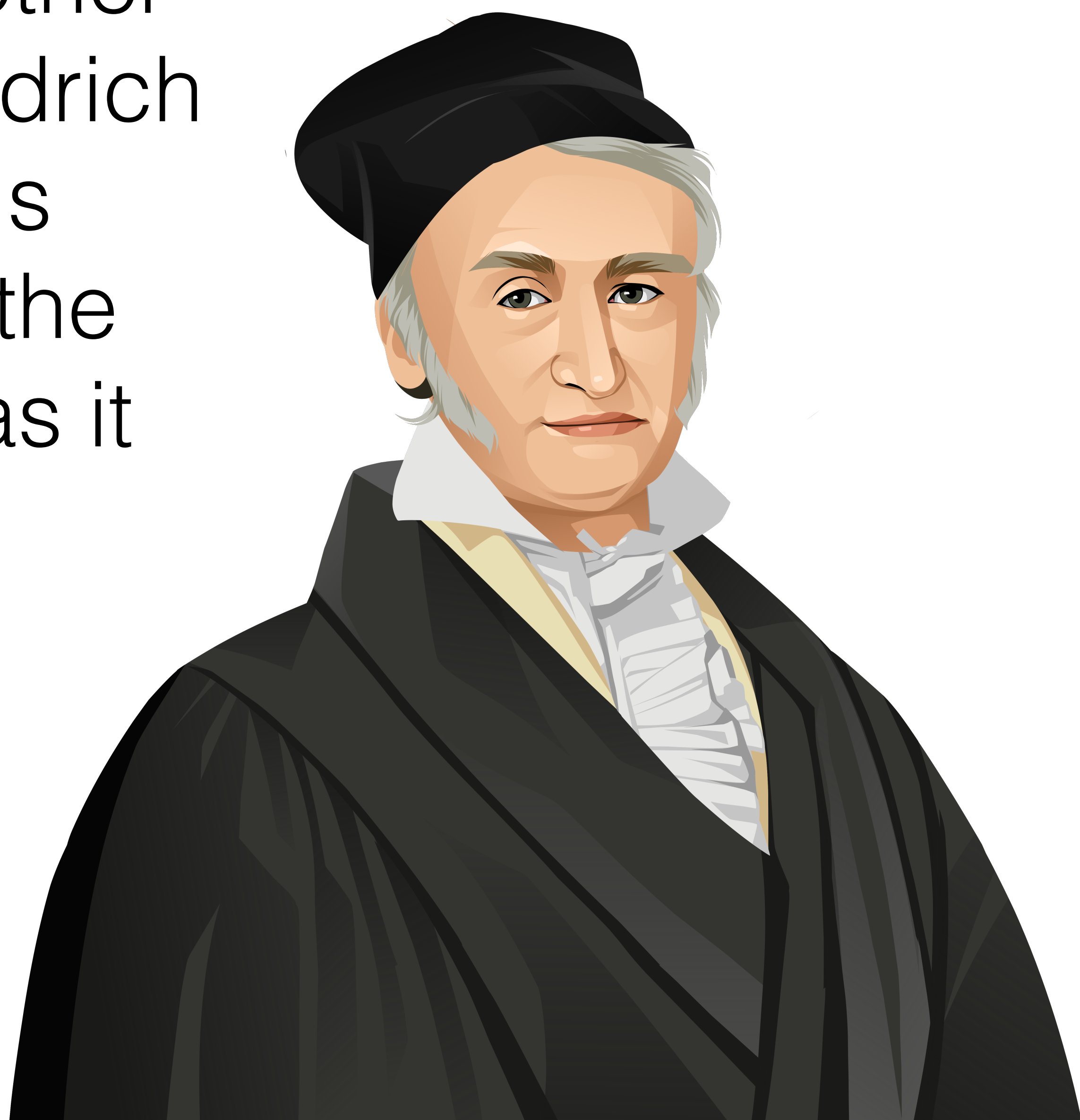
And to avoid the tedious repetition of these words: ``is equal to'' I will set as I do often in work use, a pair of parallels, or duplicate lines of one length, thus: =====, because no 2 things can be more equal.

- First equation:

$$14.\mathfrak{z}. + .15.\mathfrak{d} = 71.\mathfrak{d}.$$

Math Symbols

- Recorde's notation was very good.
- Good notation often leads to other good notation: When Carl Friedrich Gauss needed a symbol for his modular arithmetic, he chose the symbol $\equiv$. This is also good, as it shares many of the same properties as $=$.
- Other variants: $\neq$, $\approx$, $\cong$.



Math Symbols

- Robert Recorde also needed new language for powers higher than squares and cubes. He based his choice off the word *zenzic* which means "squared."

Because $x^4 = (x^2)^2$, his fourth power was *zenzizenzic*

Because $x^6 = (x^3)^2$, his sixth power was *zenzicubike*

Because $x^8 = ((x^2)^2)^2$, his eighth power was *zenzizenzizenzic*

Because $x^{16} = (((x^2)^2)^2)^2$, his sixteenth power was *zenzizenzizenzizenzike*

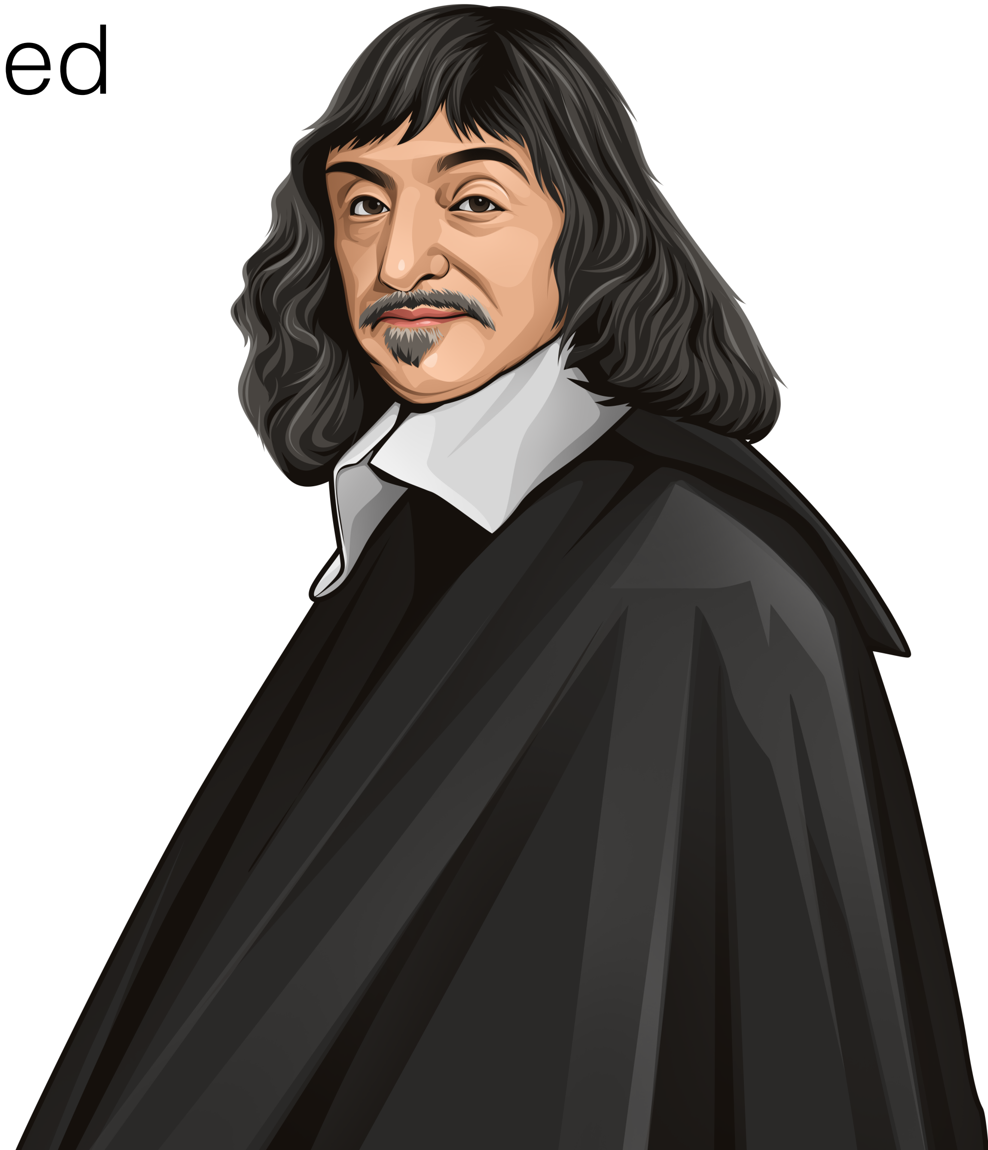
This is bad, and so it did not last.

Math Symbols

- In 1636, James Hune used

$$A^2 = A^{\text{ii}} \quad A^3 = A^{\text{iii}} \quad A^{16} = A^{\text{iiiiiiiiiiiiiiii}}$$

- In 1637, Rene Descartes introduced our modern notation:
  $A^2, A^3, A^{16}, \ \ldots$



- In *La Géométrie*, he wrote

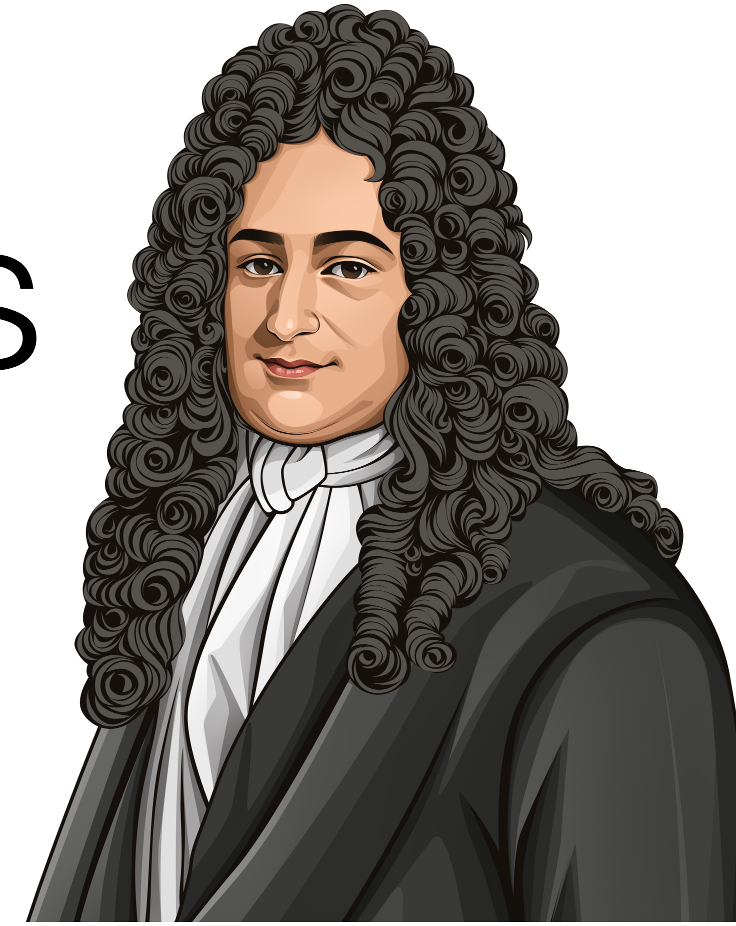
> I designate … aa, or a^2 in multiplying a by itself; and a^3 in multiplying it once more again by a, and thus to infinity.

Math Symbols

- This is great notation. It *encourages* new math. For example, if you can have a^n for a natural number n, what does would it mean for n to be negative? Rational? Irrational? Imaginary?

- Good notation has often inspired new mathematical ideas.

Math Symbols



Derivative Notation

For $y = f(x)$

Derivative	Newton	Leibniz
First	$\dot{y}$	$\dfrac{dy}{dx}$
Second	$\ddot{y}$	$\dfrac{d^2y}{dx^2}$
Eighteenth	$\overset{\vdots\vdots\vdots\vdots}{y}$	$\dfrac{d^{18}y}{dx^{18}}$
n^{th}	???	$\dfrac{d^ny}{dx^n}$

- Leibniz's notation also invites the idea of a *fractional derivative*.

Math Symbols

- Algebraic symbol progression:

    1. Rhetorical algebra: All words

    2. Syncopated algebra: Some abbreviations

    3. Symbolic algebra: Symbols everywhere

Math Symbols

- Rhetorical algebra examples:
- Problem 28 of the Rhind papyrus (≈1550 BC):

A quantity together with its two-thirds has one third its sum taken away to yield 10. What is the quantity?

- I.e.,

$$\left(x + \frac{2}{3}x\right) - \frac{x + \frac{2}{3}x}{3} = 10$$

Math Symbols

- Rhetorical algebra examples:
- In Al-Khwārizmī's book (9th-century):

$$\boxed{\text{Squares and Numbers are equal to Roots}}$$

- I.e.,

$$cx^2 + a = bx$$

Math Symbols

- Rhetorical algebra examples:
- In Bhaskara II's book (12th century):

> The eighth part of a group of monkeys, squared, were skipping in the forest and delighted with their sport. Another 12 were on the hill, screaming. How many monkeys were there in total?

- I.e.,

$$x = \left(\frac{x}{8}\right)^2 + 12$$

Math Symbols

- Syncopated algebra examples:

Diophantus wrote	$\kappa^{\Upsilon}\overline{\delta}$	⫛	$\delta^{\acute{\upsilon}}\overline{\beta}\varsigma\overline{\alpha}$	⫛	$\overline{\alpha}\mu^{\overline{o}}$	ι^{σ}	$\overline{\varepsilon}\mu^{\overline{o}}$
We would write	$4x^3$	$-$	$2x^2 + x$	$-$	1	$=$	5

Brahmagupta wrote	*ya ka 7 bha c12 ru* $\dot{8}$ *ya v 3 ya 10*
We would write	$7xy + \sqrt{12} - 8$ $= 3x^2 + 10x$

Cardano wrote	3*quadr. quad.*	$\overline{\text{p.}}$	6 pos.	aeq.	20
We would write	$3x^4$	$+$	$6x$	$=$	20

- Symbolic algebra examples:

Math Symbols

Descartes (1637)

Leibniz (1637)

Recorde (1557)

Euler (1748)

Euler (1727)

Jones (1706)

Oresme (1300s)

$$e^{i \cdot \pi} + 1 = 0$$

Indian (Brahmi) 1^{st} century		𑁒	𑁓	𑁔	𑁕	𑁖	𑁗	𑁘	𑁙	𑁚
Indian (Devanagari) 7^{th} century	०	१	२	३	४	५	६	७	८	९
East Arabic 11^{th} century	٠	١	٢	٣	۴	۵	۶	٧	٨	٩
Medieval European 15^{th} century	O	I	2	3	[illegible]	ϥ	6	Λ	8	9
Modern 21^{st} century	0	1	2	3	4	5	6	7	8	9

Math Symbols

- This identity is also due to Euler. He first wrote the more general form,

$$e^{ix} = \cos(x) + i\sin(x)$$

He first wrote it like this:

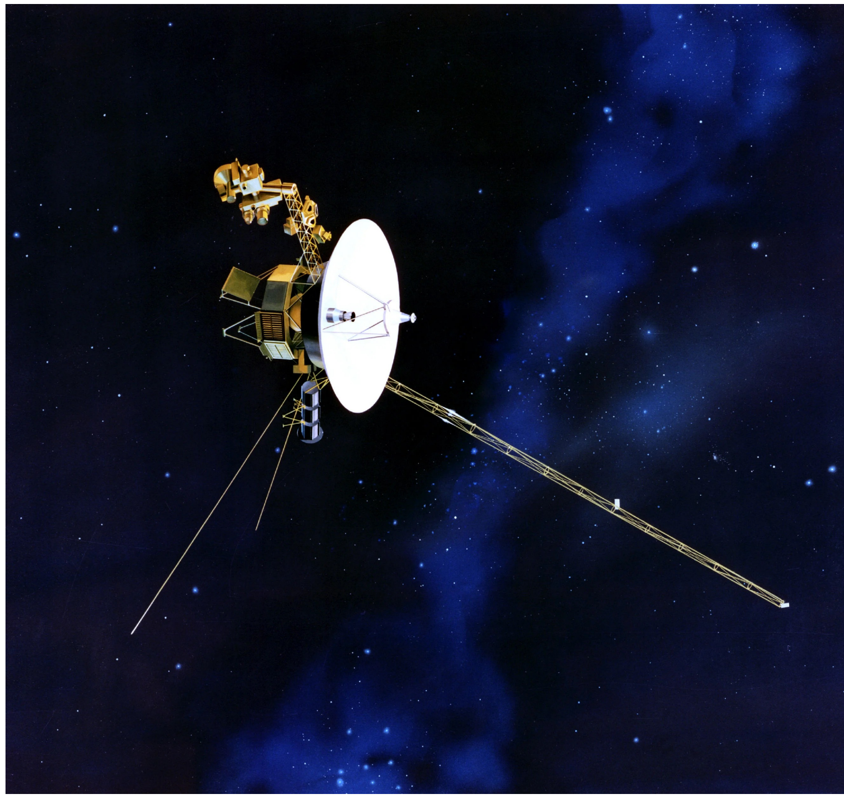
$$e^{+v\sqrt{-1}} = cos\,.\,v + \sqrt{-1}sin\,.\,v\,.$$

In his own hand:

$$e^{+v\sqrt{-1}} = co\text{ſ}.\,v + \sqrt{-1}.\,\text{ſ}in.\,v$$

Symbol(s)	Year	Person	Notes
$\surd$	1556	Niccolò Tartaglia	Eg., $\surd 5$. 100 years later: $\sqrt{5}$
$(\dots)$	1556	Niccolò Tartaglia	For precedence grouping
$.$	1593	Christopher Clavius	Decimal separator
$\sqrt[n]{\ }$	1629	Albert Girard	Still no line on top
dx, $\frac{dy}{dx}$, $\int$	1675	Gottfried Leubniz	Calculus symbols
∞	1655	John Wallis	Used $^1/_\infty$ for an infinitesimal
$f(x)$, e, i, $\sum$, sin, cos, tan	1736-94	Leonhard Euler	Introduced important symbols in nearly every area of math
$!$	1808	Christian Kramp	Factorial symbol
$\aleph$, ω, $\{\dots\}$	1893-95	Georg Cantor	All are from set theroy
$\mathbb{R}$	1872	Richard Dedekind	These began as **R,N,Q,Z,C**; $\mathbb{R}, \mathbb{N}, \mathbb{Q}, \mathbb{Z}, \mathbb{C}$ came later
$\cup$, $\cap$, $\mathbb{N}$, $\mathbb{Q}$, $\exists$	1888-97	Giuseppe Peano	0 was *not* a natural number
$\mathbb{Z}$	1930	Edmund Landau	From the German *Zahlen*
$\mathbb{C}$	1939	Nathan Jacobson	From the English *Complex*
$\emptyset$	1939	Nicolas Bourbaki	French group of mathematicians
$\blacksquare$	1950	Paul Halmos	End-of-proof "tombstone"

Math Symbols



- In 1977, the Voyager 1 space probe departed our atmosphere.
- In 2012 it left our atmosphere.
- Today it is 15 billion miles away
- It is more than a scientific instrument. It is also a time capsule, for whoever may one day discover it. It contains “The Golden Record” with images and sounds of who we are as people. It also discusses some of our math and science knowledge.

Math Symbols

- In
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Math Symbols

- How can one discuss our math and science with an intelligent but totally-alien life form?
- Step 1: teach them some of our symbols.
- Image 3 from The Golden Record: