Graph Theory Fundamentals

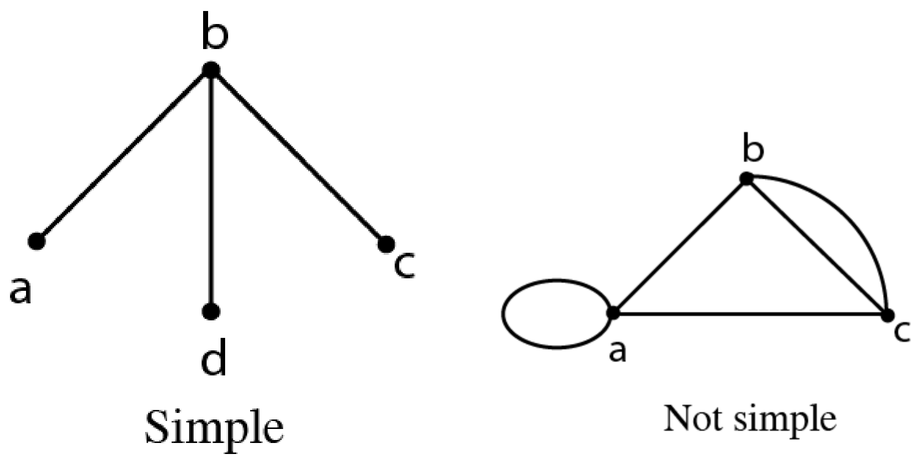
Graph

Notation: G = (V, E)

Where G is the graph, V represents the set of vertices and E represents the set of edges.

Simple Graph

A simple graph has no loops or multiple edges.



Neighborhood

A set of all vertices adjacent to a vertex.

Using the simple graph above:

N(b) = {a, d, c}

N(a) = {b}

Degree

Cardinality (count) of a neighborhood.

Using the simple graph shown earlier:

deg(b) = |N(b)| = 3

deg(a) = 1

Handshaking Theorem

The sum of the degrees of a graph is equal to twice the the number of edges.

Using the simple graph shown earlier:

$$2|E| = \sum_{v \in V} deg(v)$$

$$2 \times 3 = deg(a) + deg(b) + deg(c) + deg(d)$$

$$2 \times 3 = 1 + 3 + 1 + 1$$

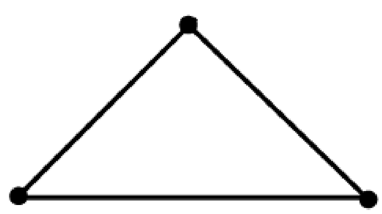
Complete Graph

Contains exactly one edge between every pair of different vertices.

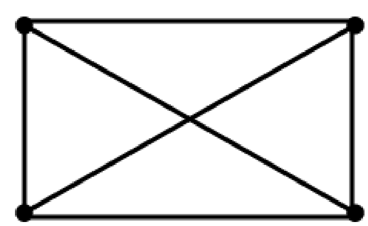
Notation: K_n where n is the number of vertices.

K_1

K_2



K_3

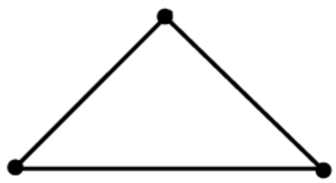


K_4

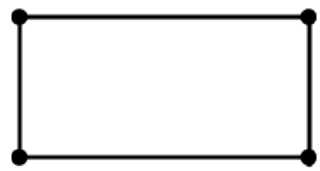
Cycles

Every vertex must have a degree of two, and the graph must be closed.

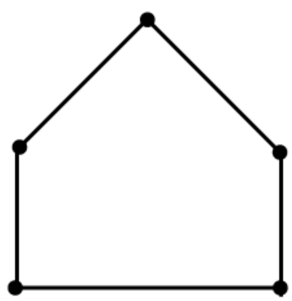
Spread vertices out to a circle for best image as done below.



C_3



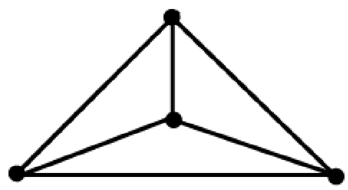
C_4



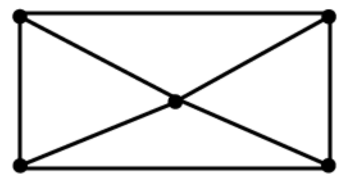
C_5

Wheels

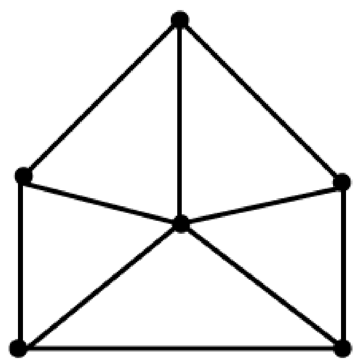
Like a cycle, but with a central vertex that connects to all other vertices.



W_3



W_4



W_5

Bipartite Graph

For sets of vertices v_1 and v_2, every edge connects a vertex in v_1 to v_2.

The following graph is bipartite:

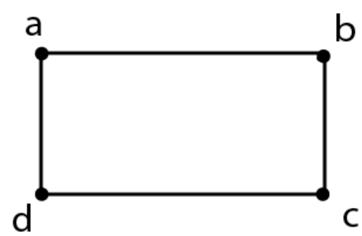


Figure 1:

It can be partitioned into the following sets:

$v_1 = \{a, c\}$

$v_2 = \{b, d\}$

Note: triangles and wheels are *not* bipartite.

Coloring/Color Theory

Coloring can be used to determine if a graph is bipartite. A graph's vertices can be colored with two alternating colors if it is bipartite:

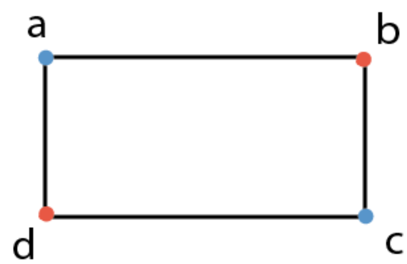


Figure 2:

The following graph can *not* be colored using two alternating colors:

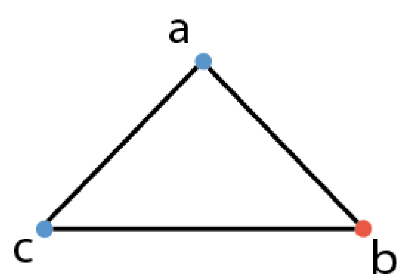


Figure 3:

Adjacency Matrix

A square n x n matrix for n vertices that counts number of edges in between vertices. It will always be symmetrical.

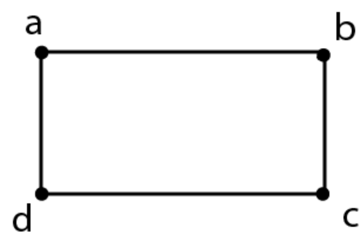


Figure 4:

The adjacency matrix for the graph above:

$$\begin{array}{c|cccc|} & a & b & c & d \\ a & 0 & 1 & 0 & 1 \\ b & 1 & 0 & 1 & 0 \\ c & 0 & 1 & 0 & 1 \\ d & 1 & 0 & 1 & 0 \end{array}$$

Figure 5: