Graph theory is the study of graphs which are composed of mathematical structures used to model relations between objects. A graph is a combination of nodes (points or vertices) that are connected to one another creating edges. Graph theory was originated in the Eighteenth century by Leonhard Euler in the German city of Königsberg, when he posed the question of whether it was possible to traverse 4 bodies of land connected by seven bridges, where each bridge would be crossed exactly once. From the question he posed in image A, the first representation of a modern graph was generated, consisting of four vertices, and seven edges. As a result, a famous Euler path theorem was born, which states that “A connected multigraph has a Euler path if and only if it has exactly two vertices of odd degrees”.

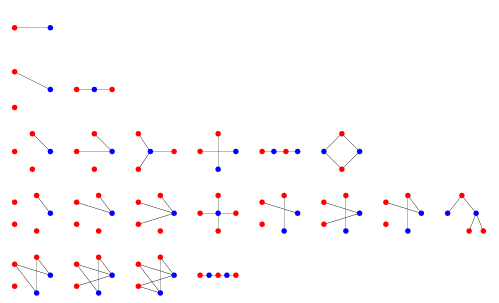
A close up of a map

Description automatically generated

In the modern day, graph theory is utilized as a tool to quantify and simplify the moving parts of dynamic systems (city layouts, computer systems, etc.). Although graph theory is a powerful tool that we use, there are some problems that have been encountered throughout its history, one of the most famous being the graph coloring problem. One of the most interesting problems in graph theory is the graph coloring theory which works closely with whether a certain graph is **bipartite**. A graph is bipartite when the vertices can be partitioned into two separate sets V1 and V2 such that all the edges connect a vertex within one set to a vertex in the other set. There are no edges comprised of vertices in a single set within a bipartite graph. We can consider bipartite graphs to be equivalent to two-colorable graphs, in which the vertices are colored in a manner where the color does repeat along an edge. Cycles are bipartite, while wheels, and complete graphs are not. An application of bipartite graphs is to find matchings within a graph. Matching of a graph is a set of edges in the graph, but no two edges share the same vertex.

Graph coloring is used as a method of labeling graph components such as vertices and edges with some certain constraints. The purpose of graph coloring is to assign labels (colors) to vertices of a graph in a manner such that no two adjacent vertices, edges or regions have the same color. Within a graph there exists a chromatic number, which is the representation of minimal number of colors that will need to be used for graph coloring to be possible. Certain types of graphs, such as complete (Kn) or bipartite (Km,n) only have a few choices possible for how they could be colored.

Bipartite graphs would then have the chromatic number 2, and could also be referred to as a 2-chromatic graph.



Depicted above are some 2-chromatic (bipartite) graphs.

When a graph’s chromatic number is less than or equal to some *k,* it can be considered to be *k*-colorable. Disconnected graphs have the special distinction of being 1-colorable.

There is currently no way to calculate the chromatic number of an arbitrary graph, however, and this problem is classified as NP-Complete, meaning it is NP and NP-Hard (computable in polynomial time). There is however an algorithm by Mehrotra and Trick used for vertex coloring and finding chromatic numbers that works for reasonably-sized graphs.

For certain types of graphs, the chromatic number is easily found based on its classification. A complete graph *kn*, for example, will have a chromatic number *n*.



An example of this is complete graph K4.

A cycle graph will have a chromatic number of either 3 or 2, depending on the parity of its order; meaning all even-ordered cycles will be bipartite. Below is cycle C6, which has a chromatic number of 2 and is bipartite.



All star graphs (similar to a wheel, but only the “spokes” are present) will have a chromatic number of 2. Below is star graph S6, which is also bipartite.



Wheel graphs, similar to cycle graphs, will have a chromatic number of either 3 or 4, depending on the parity of the graph’s order as well; a chromatic number of 3 for odd-ordered graphs and 4 for even-ordered graphs. Pictured below is wheel W6.

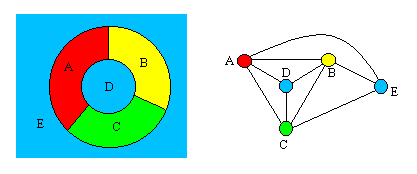


One algorithm used to find a graph’s chromatic number is the greedy coloring algorithm. Such an algorithm can return a solution in linear time, meaning as the size of the graph increases, so will the time it takes to compute a graph’s chromatic number.

A greedy algorithm in general will make the locally optimal choice at each stage of the algorithm, with the end goal of finding the globally optimal choice. An example for this is the traveling salesman problem, where choosing the next city to visit would involve only the available choices for the next step, not the future n steps possible.

The greedy coloring algorithm is also referred to as sequential coloring, for the reason stated above; nodes are visited in a given sequence. Each node is assigned a number (which can be easily used to represent a color in a program), from 0 to n. Each node will be given the smallest number that is not already used by one of the node’s neighbors; this will be the local optimum.

Another very famous theory stemming from color theory is the four-color theorem, which states that any planar graph that is formed from a map is 4-coloarable. A graph is said to be planar if it can be represented by a drawing in the plane where no edges cross. The vertices of this planar graph can then be colored with at most 4-colors, so that no 2 adjacent vertices receive the same color. Represented in the following image, is a map transformed to be represented as a planar graph:



The above graph is a complete graph, in the sense that if another connection is made that will cause lines to cross, deeming it a non-planar graph. The first origination of the four-color theorem dates to 1852 when Francis Guthrie colored the map of countries of England with 4 colors. From this moment on there have been several failed proofs of this theorem, until in 1977 it was proven by mathematicians Kenneth Appel and Wolfgang Haken who had constructed a computer assisted proof. This theory can be proved by contradiction, if you start by assuming there is a network that is unable to be colored using 4 colors. If this network exists, the there will be at least one that has the fewest number of nodes, that is the network that should be looked at. Then the idea is to show that a node can be removed from this network without altering the number of colors that are needed to color the network. Now since the new network has one fewer node than the original network you started with, and that network was chosen as the smallest that could not be colored with four colors, the new network can be colored using 4 colors. But as we use contradiction, because of the way the node to remove was chosen, that means the original map can also be colored with 4 colors. To prove this, Appel and Haken used a computer program to examine an unavoidable set of reducible configurations, proving a minimal counterexample could not exist. They looked at a particular set of 1,936 maps, and determined that each of these maps could not be part of a smallest-sized counter example to the four-color theorem. This theory proof was the first of its kind, as it was the first to be proven with the help of machines, as checking all these maps by hand would being extremely time consuming. They concluded that no smallest counterexample exists, because it must contain, yet cannot contain of the specific maps, thus through contradiction proves the four-color theorem. Since this proof, more efficient algorithms have been discovered, and the unavoidable set of configurations has been reduced to 633. Some applications of this theory include of course coloring maps, scheduling tasks, designing seating charts, cell phone tower coverage, etc.

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