Calculus 3, Final Formulas

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Vector Operations/Identities

$v \cdot w = ||v||\ ||w||\ cos\theta$
$v \cdot v = ||v||^2$
$v \perp w \Leftrightarrow v \cdot w = 0$
Unit vector: $e_u = \frac{u}{||u||}$
Projection of u along v: $u_{||v} = \left(\frac{u \cdot v}{v \cdot v}\right) v = \left(\frac{u \cdot v}{||v||^2}\right) v = \left(\frac{u \cdot v}{||v||}\right) e_v$
Component of u along v: $\frac{u \cdot v}{||v||} = ||u||\ cos\theta$
u perpendicular to v: $u_{\perp v} = u - u_{||v}$
Decomposition of vector u with respect to v (derived from above): $u = u_{\perp v} + u_{||v}$

Cross product of v and w:

$$v \times w = \begin{pmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} = \begin{pmatrix} v_2 & v_3 \\ w_2 & w_3 \end{pmatrix} i - \begin{pmatrix} v_1 & v_3 \\ w_1 & w_3 \end{pmatrix} j + \begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix} k$$

$(v \times w) \perp v$ and $(v \times w) \perp w$
$||v \times w|| = ||v||\ ||w||\ sin\theta$
Area of parallelogram spanned by v and $w = ||v \times w||$

Lines and Planes in Space

Equation of a line:

A line through $p_0 = (x_0, y_0, z_0)$ in the direction of $v =< a, b, c >$:
$r(t) = r_0 + t\ v\ \ = < x_0, y_0, z_0 > + t < a, b, c >$

Parametric equations for line (derived from above):
$x(t) = x_0 + at$
$y(t) = y_0 + bt$
$z(t) = z_0 + ct$

Equation of a plane:
Vector form:
$n \cdot < x, y, z > = \ d$
Scalar form:
$a(x - x_0) + b(y - y_0) + c(z - z_0)\ \ =\ \ 0$
or $ax + by + cz = d$ where $d = ax_0 + by_0 + cz_0$.
Arc length:
$s(t) = \int_0^T ||r'(t)||\ dt$
Speed:
$\frac{d}{dt} s(t) = ||r'(t)||$

Tangent plane:
For $z = f(x, y)$:
tangent plane at (x_0, y_0): $f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

Gradients, Directional Derivatives

Chain Rule (Alternate Method and Multivariate Method):
Given $y = f(x)$, $x = g(t)$, then $y = f(x) = f(\ g(t)\ )$ and $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$. The following is then true:
Given $z = f(x, y)$, where $x = g(t)$, $y = h(t)$, $z = f(x, y) = f(\ g(t), h(t)\ )$
and $\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt}\ +\ \frac{\partial f}{\partial y}\frac{dy}{dt}$.

Gradient: Given $f(x, y)$, the gradient of f is $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$.

Directional Derivative:
In the direction of unit vector $\vec{u} =< a, b >$: $D_{\vec{u}}f\,(x, y) = f_x\,(x, y)\,a + f_y\,(x, y)\,b$.

In three variables: $D_{\vec{u}}f\,(x, y, z) = f_x\,(x, y, z)\,a + f_y\,(x, y, z)\,b + f_z\,(x, y, z)\,c$.
or in a simpler form:
Given $f(x, y)$, the directional derivative of f at $p = (a, b)$ in the direction of unit vector $\vec{u}$ is

$$D_{\vec{u}}f\,(p) = D_{\vec{u}}f\,(a, b) = \nabla f(a, b) \cdot \vec{u}$$

Rate of change of a function f in the direction of $\nabla f(p)$:
$||\nabla f(p)||$.
Rate of change of a function f in the direction of a unit vector $\vec{u}$ making an angle θ with $\nabla f(p)$:

$$\nabla f(p) \cdot \vec{u} = ||\nabla f(p)||\ ||\vec{u}||\ \cos\theta$$

(This comes from the following identity):$\vec{u} \cdot \vec{v} = ||u||\ ||v||\ \cos\theta$.

Optimization

Critical points
A point $p = (a, b)$ in the domain of f is a *critical point* if:
$f_x(a, b) = 0$ or $f_x(a, b)$ does not exist, and
$f_y(a, b) = 0$ or $f_y(a, b)$ does not exist.

Solving the system of $f_x = 0$, $f_y = 0$ will find the critical point (if it exists).

Second Derivative Test
The second derivative test finds local max., min., and saddle points.
A critical point of $f(x, y)$ is needed, as is the discriminant of $f(x, y)$ which is:

$$D(a, b) = \begin{bmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{bmatrix}$$

which yields

$$D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}^2(a, b)$$

Then, the second derivative test's rules are:

$p = (a, b)$ is a *critical point* of $f(x, y)$.
1. If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
2. If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
3. If $D < 0$, then f has a saddle point at (a, b).
4. If $D = 0$, then the test is inconclusive.

Global Extrema
Let $f(x, y)$ be defined over a closed domain D. Then, f's extreme values occur at either critical points in

the interior of D, or at points on the boundary of D.
First, find and examine critical points. Then, evaluate f at the boundaries of D. Compare these points to find f_{max} and f_{min}.

Lagrange Multipliers

The Lagrange condition is: $\nabla f = \lambda \nabla g$.
Then, solve for λ in terms of x and y: $\lambda = m\ x,\ \lambda = n\ y$. (Taking m and n to represent some expressions).
Then, let these two λ expressions equal each other and solve for x and y: $m\ x = n\ y$.
This new x and y is the crit. point. Sub. the newly found x and y into constraint g to find max. and min. of f.