

# Calculus 3, Exam 2 Formulas

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## Gradients, Directional Derivatives

Chain Rule (Alternate Method and Multivariate Method):

Given  $y = f(x)$ ,  $x = g(t)$ , then

$$y = f(x) = f(g(t))$$

and

$$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}$$

The following is then true:

Given  $z = f(x, y)$ , where  $x = g(t)$ ,  $y = h(t)$ ,

$$z = f(x, y) = f(g(t), h(t))$$

and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

Gradient:

Given  $f(x, y)$ , the gradient of  $f$  is

$$\nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$$

Directional Derivative:

In two variables for  $f(x, y)$  in the direction of unit vector  $\vec{u} = \langle a, b \rangle$ :

$$D_{\vec{u}} f(x, y) = f_x(x, y) a + f_y(x, y) b$$

In three variables for  $f(x, y, z)$  in the direction of unit vector  $\vec{u} = \langle a, b, c \rangle$ :

$$D_{\vec{u}} f(x, y, z) = f_x(x, y, z) a + f_y(x, y, z) b + f_z(x, y, z) c$$

or in a simpler form:

Given  $f(x, y)$ , the directional derivative of  $f$  at  $p = (a, b)$  in the direction of unit vector  $\vec{u}$  is

$$D_{\vec{u}}f(p) = D_{\vec{u}}f(a, b) = \nabla f(a, b) \cdot \vec{u}$$

Rate of change of a function  $f$  in the direction of  $\nabla f(p)$ :

$$\|\nabla f(p)\|$$

Rate of change of a function  $f$  in the direction of a unit vector  $\vec{u}$  making an angle  $\theta$  with  $\nabla f(p)$ :

$$\nabla f(p) \cdot \vec{u} = \|\nabla f(p)\| \|\vec{u}\| \cos \theta$$

(This comes from the following identity):

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta$$

## Optimization

### Critical points

A point  $p = (a, b)$  in the domain of  $f$  is a *critical point* if:

$f_x(a, b) = 0$  or  $f_x(a, b)$  does not exist, and  
 $f_y(a, b) = 0$  or  $f_y(a, b)$  does not exist.

Solving the system of  $f_x = 0$ ,  $f_y = 0$  will find the critical point (if it exists).

### Second Derivative Test

The second derivative test finds local max., min., and saddle points.

A critical point of  $f(x, y)$  is needed, as is the discriminant of  $f(x, y)$  which is:

$$D(a, b) = \begin{bmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{bmatrix}$$

which yields

$$D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}^2(a, b)$$

Then, the second derivative test's rules are:

$p = (a, b)$  is a *critical point* of  $f(x, y)$ .

1. If  $D > 0$  and  $f_{xx}(a, b) > 0$ , then  $f(a, b)$  is a local minimum.
2. If  $D > 0$  and  $f_{xx}(a, b) < 0$ , then  $f(a, b)$  is a local maximum.
3. If  $D < 0$ , then  $f$  has a saddle point at  $(a, b)$ .
4. If  $D = 0$ , then the test is inconclusive.

### Global Extrema

Let  $f(x, y)$  be defined over a closed domain  $D$ . Then,

$f$ 's extreme values occur at either critical points in the interior of  $D$ , or at points on the boundary of  $D$ .

First, find and examine critical points. Then, evaluate  $f$  at the boundaries of  $D$ . Compare these points to find  $f_{max}$  and  $f_{min}$ .

### Lagrange Multipliers

Lagrange multipliers are used to find the min. and max. of a function under a given constraint. Typically, it will look like:

Given function  $f(x, y)$ , find the min. and max. under the constraint  $g(x, y)$ .

( $g$  might make a line or other geometric shape; this shape slices  $f$ . We then need to find the max./min. of  $f$  in this slice.)

The Lagrange condition is:

$$\nabla f = \lambda \nabla g$$

Then, solve for  $\lambda$  in terms of  $x$  and  $y$ . This might look like:

$$\lambda = m x, \quad \lambda = n y$$

(Taking  $m$  and  $n$  to represent some expressions).

Then, let these two  $\lambda$  expressions equal each other and solve for  $x$  and  $y$ :

$$m x = n y$$