Calculus 3, Exam 2 Formulas

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Gradients, Directional Derivatives

Chain Rule (Alternate Method and Multivariate Method):

Given $y = f(x)$, $x = g(t)$, then

$$y = f(x) = f(\ g(t)\ )$$

and

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

The following is then true:

Given $z = f(x, y)$, where $x = g(t)$, $y = h(t)$,

$$z = f(x, y) = f(\ g(t), h(t)\ )$$

and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} \ + \ \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Gradient:

Given $f(x, y)$, the gradient of f is

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

Directional Derivative:

In two variables for $f(x, y)$ in the direction of unit vector $\vec{u} =< a, b >$:

$$D_{\vec{u}}f\,(x, y) = f_x\,(x, y)\,a + f_y\,(x, y)\,b$$

In three variables for $f(x, y, z)$ in the direction of unit vector $\vec{u} =< a, b, c >$:

$$D_{\vec{u}}f\,(x, y, z) = f_x\,(x, y, z)\,a + f_y\,(x, y, z)\,b + f_z\,(x, y, z)\,c$$

or in a simpler form:
Given $f(x, y)$, the directional derivative of f at $p = (a, b)$ in the direction of unit vector $\vec{u}$ is

$$D_{\vec{u}}f\ (p) = D_{\vec{u}}f\ (a,b) = \nabla f(a,b) \cdot \vec{u}$$

Rate of change of a function f in the direction of $\nabla f(p)$:

$$||\nabla f(p)||$$

Rate of change of a function f in the direction of a unit vector $\vec{u}$ making an angle θ with $\nabla f(p)$:

$$\nabla f(p) \cdot \vec{u} = ||\nabla f(p)||\ ||\vec{u}||\ \cos\theta$$

(This comes from the following identity):

$$\vec{u} \cdot \vec{v} = ||u||\ ||v||\ \cos\theta$$

Optimization

<u>Critical points</u>
A point $p = (a,b)$ in the domain of f is a *critical point* if:

$f_x(a,b) = 0$ or $f_x(a,b)$ does not exist, and
$f_y(a,b) = 0$ or $f_y(a,b)$ does not exist.

Solving the system of $f_x = 0,\ f_y = 0$ will find the critical point (if it exists).

<u>Second Derivative Test</u>
The second derivative test finds local max., min., and saddle points.
A critical point of $f(x,y)$ is needed, as is the discriminant of $f(x,y)$ which is:

$$D(a,b) = \begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{bmatrix}$$

which yields

$$D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b)$$

Then, the second derivative test's rules are:

$p = (a,b)$ is a *critical point* of $f(x,y)$.
1. If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.
2. If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.
3. If $D < 0$, then f has a saddle point at (a,b).
4. If $D = 0$, then the test is inconclusive.

<u>Global Extrema</u>
Let $f(x,y)$ be defined over a closed domain D. Then,

f's extreme values occur at either critical points in the interior of D, or
at points on the boundary of D.

First, find and examine critical points. Then, evaluate f at the boundaries of D.
Compare these points to find f_{max} and f_{min}.

Lagrange Multipliers

Lagrange multipliers are used to find the min. and max. of a function under a given constraint. Typically, it will look like:

Given function $f(x, y)$, find the min. and max. under the constraint $g(x, y)$.

(g might make a line or other geometric shape; this shape slices f. We then need to find the max./min. of f in this slice.)

The Lagrange condition is:

$$\nabla f = \lambda \nabla g$$

Then, solve for λ in terms of x and y. This might look like:

$$\lambda = m\ x,\ \ \lambda = n\ y$$

(Taking m and n to represent some expressions).

Then, let these two λ expressions equal each other and solve for x and y:

$$m\ x = n\ y$$

This new x and y is the crit. point.
Sub. the newly found x and y into constraint g to find max. and min. of f.

Double Integrals