Mathematical Analysis IB

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0 - Review on differentiation

Differentiability

Let f be a function on some open interval I containing x. The derivative of f at x, denoted by $f'(x)$, is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Differentiation rules

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Mean value theorem

Let f be a function that is continuous on $[a, b]$ and is differentiable on (a, b). Then there is a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Consequences of MVT

Zero derivative

If $f'(x) = 0 \; \forall x$ in interval I, then $f(x) = c \; \forall x \in I$ for some constant C.

Equal derivatives

If $f'(x) - g'(x) = 0 \; \forall x$ in an interval I, then $f(x) = g(x) + C$ for some constant C.

Example

Let $f(x) = \cos^{-1} x$ and $g(x) = -\sin^{-1} x$.
This implies that $x \in [-1, 1]$ and $f(x), g(x) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$f'(x) = -\frac{1}{\sqrt{x^2+1}}$$

$$g'(x) = -\frac{1}{\sqrt{x^2+1}}$$

Since $f'(x) - g'(x) = 0$ for $x \in [-1, 1]$, then $f(x) - g(x) = C$ for some constant C by a corollary.

Differentials

$$\begin{aligned} f'(x) &= \frac{dy}{dx} \\ f'(x)dx &= dy \end{aligned}$$

1 - Indefinite and definite integrals

Indefinite integral

The main interpretation of derivative is the slope of a tangent line of a curve.

Example

At any point (x, y) on a particular curve $y = F(x)$, the tangent line has a slope equal to $4x - 5$. If the curve contains the point $(3, 7)$, find $F(x)$.

Solution. *Since the slope is equal to $4x - 5$ for any point (x, y), then the slope at $(3, 7)$ is $4(3) - 5 = 7$.*
$4x - 5$ therefore represents the tangent slope for all values of x. So

$$F'(x) = 4x - 5$$

By intuition, we can conclude that $F(x) = 2x^2 - 5x$.
However given $F(x) = 2x^2 - 5x + 1$, $F'(x)$ remains the same. And so is $F(x) = 2x^2 - 5x - 3$, $F(x) = 2x^2 - 5x + \pi$, and infinitely more functions. We can arbitrarily assign a constant k, so that $F(x) = 2x^2 - 5x + k$.
Substituting $(x, y) = (3, 7)$

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Substitution rule

Chain rule for derivatives

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$$

If follows that

$$\int f'(g(x))g'(x)dx = f(g(x)) + C$$

Example

Evaluate $\int 2x \cos x^2 dx$.

Preliminary work. *By intuition, we can get $f(x) = sinx$ and $g(x) = x^2$*

$$\int 2x \cos x^2 dx = f(g(x)) = \sin x^2$$

Solution. *Suppose that $f'(x) = \frac{dy}{dx}$*

$$dy = f'(x)dx$$

Definite integrals

The area problem

Let f be a continuous nonnegative function on $[a, b]$. Find the area of the regiom bounded by the curve $y = f(x)$, the lines $x = a$, $x = b$, and the x-axis.

The area is often coined the ***region under the curve****, which generally means the area in between the curve and the x-axis*

Example

Consider $f(x) = x^2 + 1$ on $[0, 2]$.

Solution. *Let A be the area under the curve*

Using right endpoints (5 rectangles)

$$\Delta x = \frac{2-0}{5} = \frac{2}{5} = 0.4$$

Rectangle 1: $(\Delta x)(f(0.4)) = (0.4)(1.16)$

Rectangle 2: $(\Delta x)(f(0.8)) = (0.4)(1.64)$

Rectangle 3: $(\Delta x)(f(1.2)) = (0.4)(2.44)$

Rectangle 4: $(\Delta x)(f(1.6)) = (0.4)(3.56)$

Rectangle 5: $(\Delta x)(f(2.0)) = (0.4)(5)$

Remarks on the definite integral

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Conventions on the definite integral

Properties of the definite integral

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The Fundamental Theorem of Calculus

Mean Value Theorem for integrals

Proof

Since f is continuous on $[a, b]$, then f is integrable on $[a, b]$ — i.e. $\int_a^b f(x)dx$ has a value.

Since f is continuous on [a,b], by the ***Extreme Value Theorem****, $\exists m, M \in \mathbb{R}$ such that $f(x_m) = m, f(x_M) = M, m \leq f(x) \leq M \; \forall x \in [a, b]$ and for some $x_m, x_M \in [a, b]$.*

By Property 6 of the definite integral, $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

$$
\begin{aligned}
m \leq \frac{\int_a^b f(x)dx}{b-a} \leq & \quad M \\
f(x_m) \leq \frac{\int_a^b f(x)dx}{b-a} \leq & \quad f(x_M)
\end{aligned}
$$

First part of the Fundamental Theorem of Calculus

Let $y = f(t)$ that is continuous on $[a, b]$.
If $x \in [a, b]$, then the function is also continuous on $[a, b] \implies$ the function is also continuous on $[a, x]$.

$$F(x) = \int_a^x f(t)dt$$
$$F(a) = \int_a^a f(t)dt = 0$$
$$F(b) = \int_a^b f(t)dt$$

Let f be continuous on $[a, b]$. If f is the function defined by

$$F(x) = \int_a^x f(t)dt$$

then $F'(x) = f(x)\ \forall x \in [a, b]$.

Proof

Let $x, x + h \in [a, b], h \neq 0.$

Second part of the Fundamental Theorem of Calculus

Let's bring back $f(x) = x^2 + 1$ *on* $[0, 2]$.
f *is continuous on* $[0, 2] \implies f$ *is integrable on* $[0, 2]$.

$$\implies \int_0^2 (x^2 + 1)dx = \frac{14}{3}$$

Let $F(x) = \frac{x^3}{3} + x - 1$.

$$\begin{aligned} F(2) &= \frac{2^3}{3} + 2 - 1 = \frac{8}{3} + 1 = \frac{11}{3} \\ F(0) &= \frac{0^3}{3} + 0 - 1 = 0 - 1 = -1 \\ F(2) - F(0) &= \frac{11}{3} - (-1) = \frac{14}{3} \\ \int_0^2 (x^2 + 1)dx &= F(2) - F(0) \end{aligned}$$

Observe that $F'(x) = x^2 + 1 \implies F(x)$ *is the an antiderivative of* $x^2 + 1$.

2 - Application I

Areas between curves

Example 1

Find the area of the region under the curve $y = x^2 - 1$ from $x = -1$ to $x = 2$.

Solution. *Area is simply not $\int_{-1}^{2}(x^2 - 1)dx$ because $\int_{-1}^{1}(x^2 - 1)dx$ is negative and cancels the positive area. Therefore, we get $\int_{-1}^{1} -(x^2 - 1)dx$ to get the area of the curve between -1 and 1.*

$$\begin{aligned}
A &= \int_{-1}^{1} -(x^2 - 1)dx + \int_{1}^{2}(x^2 - 1)dx \\
&= \Big(-\frac{x^3}{3} + x\Big)\Big|_{-1}^{1} + \Big(\frac{x^3}{3} - x\Big)\Big|_{1}^{2} \\
&= \Big(\frac{1^3}{3} + 1\Big) - \Big[\frac{(-1)^3}{3} + (-1)\Big] + \Big(\frac{2^3}{3} - 2\Big) - \Big(\frac{1^3}{3} - 1\Big) \\
&= \frac{2}{3} + \frac{2}{3} + \frac{8}{3} - 2 + \frac{2}{3}
\end{aligned}$$

$$A = \frac{8}{3}$$

Volumes and volumes of revolution using disks and washers

Volume of a right cylinder

$$V = ah$$

$$V_n = \sum_{i=1}^{n} A(x)\Delta x$$

Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x through x and perpendicular to the x-axis is $A(x)$, where A is a continuous function on $[a, b]$, then the volume V of S is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_b^a A(x)dx$$

Example 1

Let us find the volume of a sphere of radius r.

Solution.

radius of the cross-section circle at $x = \sqrt{r^2 - x^2}$

Volume of solids of revolution

If we revolve a region about a line, we obtain a **solid of revolution**.

Example 1

Consider the region under the curve $y = x^2 + 1$ from $x = -1$ to $x = 2$. We revolve this region about the x-axis.

Solution. *radius of the cross-section at $x = f(x)$*

$$A(x) = \pi[f(x)]^2$$

$$\begin{aligned} V &= \int_{-1}^{2} \pi(x^2+1)^2 dx \\ &= \int_{-1}^{2} \pi(x^4+2x^2+1) dx \\ &= \pi\left(\frac{x^5}{5} + \frac{2x^3}{3} + x\right)\Big|_{-1}^{2} \\ &= \pi\left[\frac{2^5}{5} + \frac{2(2)^3}{3} + 2\right] - \pi\left[\frac{(-1)^5}{5} + \frac{2(-1)^3}{3} + (-1)\right] \end{aligned}$$

Volumes by cylindrical shells

There are times that disks-and-washers technique is not the best way to solve a volume problem – e.g. $y = 4x - x$ rotated about the y-axis.

Volume of a cylindrical shell

Let r_1 be the inner radius of the cylinder, r_2 be the outer (and larger) radius of the cylinder. r be the average of both

$$\begin{aligned} \Delta r &= r_2 - r_1 \\ r &= \frac{r_2 + r_1}{2} \end{aligned}$$

$$\begin{aligned} V_{cylindrical\ shell} &= \pi r_2^2 h - r_1^2 h \\ &= \pi(r_2^2 - r_1^2)h \\ &= \pi(r_2 + r_1)h(r_2 - r_1) \\ &= 2\pi\left(r_2 + r_1 \right) h\Delta r \end{aligned}$$