# Mathematical Analysis IB

Download the PDF copy of the notes here

## 0 - Review on differentiation

### Differentiability

Let $f$ be a function on some open interval $I$ containing $x$. The derivative of $f$ at $x$, denoted by $f'(x)$, is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

---

### Differentiation rules

1. $\dfrac{d}{dx}(cf(x)) = cf'(x)$
2. $\dfrac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$
3. $\dfrac{d}{dx}(f(x)g(x)) = f(x)g'(x) + g(x)f'(x)$
4. $\dfrac{d}{dx}\Big(\dfrac{f(x)}{g(x)}\Big) = \dfrac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$
5. $\dfrac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$

### Differentiation formulas I

1. $\dfrac{d}{dx}(c) = 0, c \in \mathbb{R}$
2. $\dfrac{d}{dx}(x^r) = rx^{r-1}, r \in \mathbb{R}$
3. $\dfrac{d}{dx}(\sin x) = \cos x$
4. $\dfrac{d}{dx}(\cos x) = \sin x$
5. $\dfrac{d}{dx}(\tan x) = \sec^2 x$
6. $\dfrac{d}{dx}(\cot x) = -\csc^2 x$
7. $\dfrac{d}{dx}(\sec x) = \sec x \tan x$
8. $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$

## Differentiation formulas II

1. $\dfrac{d}{dx}(e^x) = e^x$
2. $\dfrac{d}{dx}(\ln|x|) = \dfrac{1}{x}$
3. $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$
4. $\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$
5. $\dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{x\sqrt{x^2-1}}$

---

## Mean value theorem

Let $f$ be a function that is continuous on $[a, b]$ and is differentiable on $(a, b)$. Then there is a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

## Consequences of MVT

### Zero derivative

If $f'(x) = 0\ \forall x$ in interval $I$, then $f(x) = c\ \forall x \in I$ for some constant $C$.

### Equal derivatives

If $f'(x) - g'(x) = 0\ \forall x$ in an interval $I$, then $f(x) = g(x) + C$ for some constant $C$.

### Example

Let $f(x) = \cos^{-1} x$ and $g(x) = -\sin^{-1} x$

This implies that $x \in [-1, 1]$ and $f(x), g(x) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$f'(x) = -\frac{1}{\sqrt{x^2+1}}$$

$$g'(x) = -\frac{1}{\sqrt{x^2+1}}$$

Since $f'(x) - g'(x) = 0$ for $x \in [-1, 1]$, then $f(x) - g(x) = C$ for some constant $C$ by a corollary.

$$\begin{aligned}\cos^{-1} x - (-\sin^{-1} x) &= C \\ \cos^{-1} x + \sin^{-1} x &= C\end{aligned}$$

Substituting $x \in [-1, 1]$, in this case, let's use $x = 0$,

$$\begin{aligned}\cos^{-1}(0) + \sin^{-1}(0) &= C \\ 0 + \frac{\pi}{2} &= C \\ C &= \frac{\pi}{2}\end{aligned}$$

$$\therefore \forall x \in [-1, 1], f(x) - g(x) = \frac{\pi}{2}$$

---

### Differentials

$$\begin{aligned}f'(x) &= \frac{dy}{dx} \\ f'(x)dx &= dy\end{aligned}$$

---

# 1 - Indefinite and definite integrals

## Indefinite integral

The main interpretation of derivative is the slope of a tangent line of a curve.

#### Example

At any point $(x, y)$ on a particular curve $y = F(x)$, the tangent line has a slope equal to $4x - 5$. If the curve contains the point $(3, 7)$, find $F(x)$.

**Solution.** Since the slope is equal to $4x - 5$ for any point $(x, y)$, then the slope at $(3, 7)$ is $4(3) - 5 = 7$.

$4x - 5$ therefore represents the tangent slope for all values of $x$. So

$$F'(x) = 4x - 5$$

By intuition, we can conclude that $F(x) = 2x^2 - 5x$.

However given $F(x) = 2x^2 - 5x + 1$, $F'(x)$ remains the same. And so is $F(x) = 2x^2 - 5x - 3$, $F(x) = 2x^2 - 5x + \pi$, and infinitely more functions. We can arbitrarily assign a constant $k$, so that $F(x) = 2x^2 - 5x + k$.

Substituting $(x, y) = (3, 7)$,

$$\begin{aligned}&7 = 2(3)^2 - 5(3) + k \\ &7 = 18 - 15 + k \\ &k = 4\end{aligned}$$

So $F(x) = 2x^2 - 5x + 4$.

**Definition of an antiderivative**

A function $F$ is called an antiderivative of the function $f$ on an interval $I$ if $F'(x) = f(x)\ \forall x \in I$.

$F(x) = 2x^2 - 5x$ is a **possible** antiderivative of $f(x) = 4x - 5$. $F(x) = 2x^2 - 5x + 4$ is also a **possible** antiderivative of $f(x) = 4x - 5$.

**Equal derivatives**

If $F'(x) = G'(x)\ \forall x$ in an interval $I$, then $F(x) = G(x) + C\ \forall x \in I$ for some constant $C$.

**Integration notation**

The collection of all antiderivatives of $f$ is denoted by

$$\int f(x)dx$$

which is read as "the integral of $f(x)dx$."

This collection is also called the **indefinite integral** of $f$.

The reverse process if differentiation is called **antidifferentiation** or **integration**.

$\displaystyle\int (4x-5)dx = 2x^2 - 5x + C$ for some constant $C$.

$C$ is the constant of integration.

$\displaystyle\int \sin x dx = -\cos x + C$

---

**Integration rules**

1. $\displaystyle\int kf(x)dx = k\int f(x)dx$, $k$ constant
2. $\displaystyle\int f(x) \pm g(x)dx = \int f(x)dx \pm \int g(x)dx$

**Integration formulas I**

1. $\displaystyle\int kdx = kx + C, k \in \mathbb{R}$
2. $\displaystyle\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \in \mathbb{R}, n \neq -1$

**Integration formulas II**

1. $\displaystyle\int \sin x dx = -\cos x + C$
2. $\displaystyle\int \cos x dx = \sin x + C$
3. $\displaystyle\int \sec^2 x dx = \tan x + C$
4. $\displaystyle\int \csc^2 x dx = -\cot x + C$

5. $\int \sec x \tan x dx = \sec x + C$

6. $\int \csc x \cot x dx = -\csc x + C$

**Integration formulas III**

1. $\int e^x dx = e^x + C$

2. $\int \frac{1}{x} dx = \ln|x| + C$

3. $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$

4. $\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$

5. $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} + C$

---

## Substitution rule

**Chain rule for derivatives**

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$$

If follows that

$$\int f'(g(x))g'(x)dx = f(g(x)) + C$$

**Example**

Evaluate $\int 2xcosx^2 dx$.

**Preliminary work.** By intuition, we can get $f(x) = sinx$ and $g(x) = x^2$

$$\int 2x \cos x^2 dx = f(g(x)) = \sin x^2$$

**Solution.** Suppose that $f'(x) = \frac{dy}{dx}$

$$dy = f'(x)dx$$

Let $u = g(x)$, then $g'(x) = \frac{du}{dx}$

$$du = g'(x)dx$$

Let $u = x^2$

$$\begin{aligned} du &= 2xdx \\ \int 2x \cos x^2 dx &= \int \cos u du \\ &= \sin u + C \\ &= \sin x^2 + C \end{aligned}$$

**Definition of the substitution rule**

If $u = g(x)$ is a differentiable function whose range is interval $I$ and $f$ is continuous on $I$, then

$$\int f'(g(x))g'(x) = \int f(u)du$$

---

## Definite integrals

### The area problem

Let $f$ be a continuous nonnegative function on $[a, b]$. Find the area of the regiom bounded by the curve $y = f(x)$, the lines $x = a$, $x = b$, and the $x$-axis.

> The area is often coined the **region under the curve**, which generally means the area in between the curve and the $x$-axis

### Example

Consider $f(x) = x^2 + 1$ on $[0, 2]$.

**Solution.** Let $A$ be the area under the curve

Using right endpoints (5 rectangles)

$$\Delta x = \frac{2-0}{5} = \frac{2}{5} = 0.4$$

Rectangle 1: $(\Delta x)(f(0.4)) = (0.4)(1.16)$

Rectangle 2: $(\Delta x)(f(0.8)) = (0.4)(1.64)$

Rectangle 3: $(\Delta x)(f(1.2)) = (0.4)(2.44)$

Rectangle 4: $(\Delta x)(f(1.6)) = (0.4)(3.56)$

Rectangle 5: $(\Delta x)(f(2.0)) = (0.4)(5)$

$$A_5^+ = (0.4)(1.16) + (0.4)(1.64) + (0.4)(2.44) + (0.4)(3.56) + (0.4)(5) = 5.52$$

$A_5^+$ is an overestimation of $A$.

Using left endpoints (5 rectangles):

Rectangle 1: $(\Delta x)(f(0)) = (0.4)(1)$

Rectangle 2: $(\Delta x)(f(0.4)) = (0.4)(1.16)$

Rectangle 3: $(\Delta x)(f(0.8)) = (0.4)(1.64)$

Rectangle 4: $(\Delta x)(f(1.2)) = (0.4)(2.44)$

Rectangle 5: $(\Delta x)(f(1.6)) = (0.4)(3.56)$

$$A_5^- = (0.4)(1) + (0.4)(1.16) + (0.4)(1.64) + (0.4)(2.44) + (0.4)(3.56) + = 3.92$$

$A_5^-$ is an underestimation of $A$.

We can increase the number of rectangles and compute the area $A$ more **accurately** by computing the area as the number of rectangles approach infinity.

Let the number of rectangles be $n$

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

Let $x_0$ be the first point: $x_0 = 0$

$$\begin{aligned} x_1 &= \frac{2}{n} & x_2 &= \frac{4}{n} & x_3 &= \frac{6}{n} \\ x_4 &= \frac{8}{n} & x_5 &= \frac{10}{n} & x_6 &= \frac{12}{n} \\ x_7 &= \frac{14}{n} & & \cdots & x_i &= \frac{2i}{n} \end{aligned}$$

$$\begin{aligned} A_n &= R_1 + R_2 + R_3 + R_4 + \cdots + R_n \\ &= \sum_{i=1}^{n} \Delta x(f(x_i)) \\ &= \sum_{i=1}^{n} \frac{2}{n}\Big[\Big(\frac{2i}{n}\Big)^2 + 1\Big] \\ &= \frac{2}{n}\sum_{i=1}^{n}\Big(\frac{4i^2}{n^2} + 1\Big) \\ &= \frac{2}{n}\Big[\sum_{i=1}^{n}\Big(\frac{4i^2}{n^2}\Big) + \sum_{i=1}^{n}(1)\Big] \\ &= \frac{2}{n}\Big[\frac{4}{n^2}\sum_{i=1}^{n}(i^2) + \sum_{i=1}^{n}(1)\Big] \\ &= \frac{2}{n}\Big[\frac{4}{n^2}\Big(\frac{(n)(n+1)(2n+1)}{6}\Big) + n\Big] \\ &= \frac{8}{n^3}\frac{(n)(n+1)(2n+1)}{6} + 2 \\ A_n &= \frac{4}{3}\Big(1 + \frac{1}{n}\Big)\Big(2 + \frac{1}{n}\Big) + 2 \end{aligned}$$

$$
\begin{aligned}
A &= \lim_{n\to\infty} A_n \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left[\left(\frac{2i}{n}\right)^2 + 1\right] \\
&= \lim_{n\to\infty} \left[\frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + 2\right] \\
&= \frac{4}{3}(1)(2) + 2 \\
A &= \frac{14}{3}
\end{aligned}
$$

**Riemann sum**

Let $f$ be a function defined on $[a, b]$.

Divide $[a, b]$ into $n$ subintervales, each with width

$$\Delta x = \frac{b-a}{n}$$

Let $x_0 = a, x_1, x_2, \ldots, x_n = b$,

For each subinterval $[x_{i-1}, x_i]$, choose a sample point $x_i^*$

Compute the sum

$$A_n = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

This is also called the **Riemann sum**.

**Definite integral and integrability**

The definite integral of $f$ from $a$ to $b$ is

$$\int_a^b f(x)dx = \lim_{x\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided that such limit exists.

We say that $f$ is integrable on $[a, b]$

---

**Remarks on the definite integral**

1. If a function is continuous on $[a, b]$, it is integrable on $[a, b]$.
2. If $f$ is a nonnegative continuous function on $[a, b]$, then $\int_a^b f(x)dx$ is the area under the curve $y = f(x)$ from $x = a$ and $x = b$
3. $\int_a^b f(x)dx = \int_a^b f(y)dy$

**Conventions on the definite integral**

1. $\displaystyle\int_b^a f(x)dx = -\int_a^b f(x)dx$
2. $\displaystyle\int_a^a f(x)dx = 0$

**Properties of the definite integral**

1. $\displaystyle\int_a^b cf(x)dx = c\int_a^b f(x)dx$
2. $\displaystyle\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x) \pm \int_a^b g(x)$
3. $\displaystyle\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$
4. If $f(x) \geq 0 \ \forall x \in [a, b]$, then $\displaystyle\int_a^b f(x)dx \geq 0$
5. If $f(x) \geq g(x) \ \forall x \in [a, b]$, then $\displaystyle\int_a^b f(x)dx \geq \int_a^b g(x)dx$
6. If $m \leq f(x) \leq M \ \forall x \in [a, b]$, then $\displaystyle m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

## The Fundamental Theorem of Calculus

Let's bring back $f(x) = x^2 + 1$ on $[0, 2]$.

$f$ is continuous on $[0, 2] \implies f$ is integrable on $[0, 2]$.

$$\implies \int_0^2 (x^2+1)dx = \frac{14}{3}$$

Let $F(x) = \dfrac{x^3}{3} + x - 1$.

$$\begin{aligned}
F(2) &= \frac{2^3}{3} + 2 - 1 = \frac{8}{3} + 1 = \frac{11}{3} \\
F(0) &= \frac{0^3}{3} + 0 - 1 = 0 - 1 = -1 \\
F(2) - F(0) &= \frac{11}{3} - (-1) = \frac{14}{3} \\
\int_0^2 (x^2+1)dx &= F(2) - F(0)
\end{aligned}$$

Observe that $F'(x) = x^2 + 1 \implies F(x)$ is the an antiderivative of $x^2 + 1$.

**Second part of the Fundamental Theorem of Calculus**

If a function $f$ is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F$ is any antiderivative of $f$ on $[a, b]$

The following notations for $F(b) - F(a)$ are very useful in evaluating definite integrals:

$F(x)\Big]_a^b$ or $F(x)\Big|_a^b$

---

# 2 - Application I

## Areas between curves

**Example 1**

Find the area of the region under the curve $y = x^2 - 1$ from $x = -1$ to $x = 2$.

**Solution.** Area is simply not $\int_{-1}^{2}(x^2 - 1)dx$ because $\int_{-1}^{1}(x^2 - 1)dx$ is negative and cancels the positive area.

Therefore, we get $\int_{-1}^{1} -(x^2 - 1)dx$ to get the area of the curve between -1 and 1.

$$\begin{aligned}
A &= \int_{-1}^{1} -(x^2 - 1)dx + \int_{1}^{2} (x^2 - 1)dx \\
&= \Big(-\frac{x^3}{3} + x\Big)\Big|_{-1}^{1} + \Big(\frac{x^3}{3} - x\Big)\Big|_{1}^{2} \\
&= \Big(\frac{1^3}{3} + 1\Big) - \Big[\frac{(-1)^3}{3} + (-1)\Big] + \Big(\frac{2^3}{3} - 2\Big) - \Big(\frac{1^3}{3} - 1\Big) \\
&= \frac{2}{3} + \frac{2}{3} + \frac{8}{3} - 2 + \frac{2}{3} \\
A &= \frac{8}{3}
\end{aligned}$$

**Example 2**

Find the area of the region bounded by the curves of $y = x^2$ and $y = 4x - x^2$.

**Solution.** Note that both curves intersect at $(0, 0)$ and $(2, 4)$.

When we use Riemann sum, we only get the rectangles in between the region bounded by the area by subtracting the upper function ($y = 4x - x^2$) to the lower function ($y = x^2$)

$$\implies A_n = \sum[(4x - x^2) - x^2]\Delta x$$

$$
\begin{aligned}
A &= \int_0^2 [(4x - x^2) - x^2]dx \\
&= \int_0^2 (4x - 2x^2)dx \\
&= \Big(2x^2 - \frac{2x^3}{3}\Big)\Big|_0^2 \\
&= \Big[2(2)^2 - \frac{2(2)^3}{3}\Big] - \Big[2(0)^2 - \frac{2(0)^3}{3}\Big] \\
&= \Big[8 - \frac{16}{3}\Big] - 0 \\
A &= \frac{8}{3}
\end{aligned}
$$

**Example 3**

Find the area of the region bounded by the curve $y = \sqrt{x}$, the line $x + 2y = 3$, and the $x$-axis.

**Solution.** The graphs intersect at $(0, 0)$, $(1, 1)$, and $(3, 0)$.

$$
x + 2y = 3 \implies y = -\frac{1}{2}x + \frac{3}{2}
$$

$$
\begin{aligned}
A &= \int_0^1 (\sqrt{x})dx + \int_1^3 \Big(-\frac{1}{2}(3 - x)\Big)dx \\
&= \Big(\frac{2x^{\frac{3}{2}}}{3}\Big)\Big|_0^1 + \Big(\frac{1}{2}(3x - \frac{x^2}{2})\Big)\Big|_1^3 \\
&= \frac{2(1)^{\frac{3}{2}}}{3} - \frac{2(0)^{\frac{3}{2}}}{+}\frac{1}{2}(3(3) - \frac{3^2}{2}) - \frac{1}{2}(3(1) - \frac{1^2}{2}) \\
&= \frac{2}{3} + \frac{9}{4} - \frac{5}{4} \\
&= \frac{8 - 27 + 15}{12} \\
&= \frac{20}{12} \\
A &= \frac{5}{3}
\end{aligned}
$$

---

**Volumes and volumes of revolution using disks and washers**

**Volumes of solids of revolution using cylindrical shells**

## 3 - Techniques of integration

**Integration by parts**

**Trigonometric integrals**

**Trigonometric substitution**

**Partial fractions**

## 4 - Applications II

**Arc length**

**Variable-separable differential equations and models for population growth**