

# Mathematical Analysis IB

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## 0 - Review on differentiation

### Differentiability

Let  $f$  be a function on some open interval  $I$  containing  $x$ . The derivative of  $f$  at  $x$ , denoted by  $f'(x)$ , is

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

### Differentiation rules

1.  $\frac{d}{dx}(cf(x)) = cf'(x)$
2.  $\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$
3.  $\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + g(x)f'(x)$
4.  $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$
5.  $\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$

### Differentiation formulas I

1.  $\frac{d}{dx}(c) = 0, c \in \mathbb{R}$
2.  $\frac{d}{dx}(x^r) = rx^{r-1}, r \in \mathbb{R}$
3.  $\frac{d}{dx}(\sin x) = \cos x$
4.  $\frac{d}{dx}(\cos x) = -\sin x$
5.  $\frac{d}{dx}(\tan x) = \sec^2 x$
6.  $\frac{d}{dx}(\cot x) = -\csc^2 x$
7.  $\frac{d}{dx}(\sec x) = \sec x \tan x$
8.  $\frac{d}{dx}(\csc x) = -\csc x \cot x$

### Differentiation formulas II

1.  $\frac{d}{dx}(e^x) = e^x$
2.  $\frac{d}{dx}(\ln |x|) = \frac{1}{x}$
3.  $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
4.  $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
5.  $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

## Mean value theorem

Let  $f$  be a function that is continuous on  $[a, b]$  and is differentiable on  $(a, b)$ . Then there is a number  $c \in (a, b)$  such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

## Consequences of MVT

### Zero derivative

If  $f'(x) = 0 \forall x$  in interval  $I$ , then  $f(x) = c \forall x \in I$  for some constant  $C$ .

### Equal derivatives

If  $f'(x) - g'(x) = 0 \forall x$  in an interval  $I$ , then  $f(x) = g(x) + C$  for some constant  $C$ .

### Example

Let  $f(x) = \cos^{-1} x$  and  $g(x) = -\sin^{-1} x$

This implies that  $x \in [-1, 1]$  and  $f(x), g(x) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$f'(x) = -\frac{1}{\sqrt{x^2 + 1}}$$

$$g'(x) = -\frac{1}{\sqrt{x^2 + 1}}$$

Since  $f'(x) - g'(x) = 0$  for  $x \in [-1, 1]$ , then  $f(x) - g(x) = C$  for some constant  $C$  by a corollary.

$$\begin{aligned}\cos^{-1} x - (-\sin^{-1} x) &= C \\ \cos^{-1} x + \sin^{-1} x &= C\end{aligned}$$

Substituting  $x \in [-1, 1]$ , in this case, let's use  $x = 0$ ,

$$\begin{aligned}\cos^{-1}(0) + \sin^{-1}(0) &= C \\ 0 + \frac{\pi}{2} &= C \\ C &= \frac{\pi}{2}\end{aligned}$$

$$\therefore \forall x \in [-1, 1], f(x) - g(x) = \frac{\pi}{2}$$

## Differentials

$$\begin{aligned}f'(x) &= \frac{dy}{dx} \\ f'(x)dx &= dy\end{aligned}$$

# 1 - Indefinite and definite integrals

## Indefinite integral

The main interpretation of derivative is the slope of a tangent line of a curve.

### Example

At any point  $(x, y)$  on a particular curve  $y = F(x)$ , the tangent line has a slope equal to  $4x - 5$ . If the curve contains the point  $(3, 7)$ , find  $F(x)$ .

**Solution.** Since the slope is equal to  $4x - 5$  for any point  $(x, y)$ , then the slope at  $(3, 7)$  is  $4(3) - 5 = 7$ .

$4x - 5$  therefore represents the tangent slope for all values of  $x$ . So

$$F'(x) = 4x - 5$$

By intuition, we can conclude that  $F(x) = 2x^2 - 5x$ .

However given  $F(x) = 2x^2 - 5x + 1$ ,  $F'(x)$  remains the same. And so is  $F(x) = 2x^2 - 5x - 3$ ,  $F(x) = 2x^2 - 5x + \pi$ , and infinitely more functions. We can arbitrarily assign a constant  $k$ , so that  $F(x) = 2x^2 - 5x + k$ .

Substituting  $(x, y) = (3, 7)$ ,

$$7 = 2(3)^2 - 5(3) + k$$

$$7 = 18 - 15 + k$$

$$k = 4$$

So  $F(x) = 2x^2 - 5x + 4$ .

### Definition of an antiderivative

A function  $F$  is called an antiderivative of the function  $f$  on an interval  $I$  if  $F'(x) = f(x) \forall x \in I$ .

$F(x) = 2x^2 - 5x$  is a **possible** antiderivative of  $f(x) = 4x - 5$ .  $F(x) = 2x^2 - 5x + 4$  is also a **possible** antiderivative of  $f(x) = 4x - 5$ .

### Equal derivatives

If  $F'(x) = G'(x) \forall x$  in an interval  $I$ , then  $F(x) = G(x) + C \forall x \in I$  for some constant  $C$ .

### Integration notation

The collection of all antiderivatives of  $f$  is denoted by

$$\int f(x)dx$$

which is read as “the integral of  $f(x)dx$ .”

This collection is also called the **indefinite integral** of  $f$ .

The reverse process of differentiation is called **antidifferentiation** or **integration**.

$\int (4x - 5)dx = 2x^2 - 5x + C$  for some constant  $C$ .

$C$  is the constant of integration.

$$\int \sin x dx = -\cos x + C$$

### Integration rules

1.  $\int kf(x)dx = k \int f(x)dx$ ,  $k$  constant
2.  $\int f(x) \pm g(x)dx = \int f(x)dx \pm \int g(x)dx$

### Integration formulas I

1.  $\int kdx = kx + C$ ,  $k \in \mathbb{R}$
2.  $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ ,  $n \in \mathbb{R}$ ,  $n \neq -1$

### Integration formulas II

1.  $\int \sin x dx = -\cos x + C$
2.  $\int \cos x dx = \sin x + C$
3.  $\int \sec^2 x dx = \tan x + C$
4.  $\int \csc^2 x dx = -\cot x + C$
5.  $\int \sec x \tan x dx = \sec x + C$
6.  $\int \csc x \cot x dx = -\csc x + C$

### Integration formulas III

1.  $\int e^x dx = e^x + C$
2.  $\int \frac{1}{x} dx = \ln |x| + C$
3.  $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$
4.  $\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$
5.  $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + C$

### Substitution rule

#### Chain rule for derivatives

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$$

It follows that

$$\int f'(g(x))g'(x)dx = f(g(x)) + C$$

#### Example

Evaluate  $\int 2x \cos x^2 dx$ .

**Preliminary work.** By intuition, we can get  $f(x) = \sin x$  and  $g(x) = x^2$

$$\int 2x \cos x^2 dx = f(g(x)) = \sin x^2$$

**Solution.** Suppose that  $f'(x) = \frac{dy}{dx}$

$$dy = f'(x)dx$$

Let  $u = g(x)$ , then  $g'(x) = \frac{du}{dx}$

$$du = g'(x)dx$$

Let  $u = x^2$

$$\begin{aligned} du &= 2x dx \\ \int 2x \cos x^2 dx &= \int \cos u du \\ &= \sin u + C \\ &= \sin x^2 + C \end{aligned}$$

### Definition of the substitution rule

If  $u = g(x)$  is a differentiable function whose range is interval  $I$  and  $f$  is continuous on  $I$ , then

$$\int f'(g(x))g'(x) = \int f(u)du$$

The area problem

The definite Integrals

The Fundamental Theorem of Calculus

Proof of Fundamental Theorem of Calculus

## 2 - Application I

Areas between curves

Volumes and volumes of revolution using disks and washers

Volumes of solids of revolution using cylindrical shells

## 3 - Techniques of integration

Integration by parts

Trigonometric integrals

Trigonometric substitution

Partial fractions

## 4 - Applications II

Arc length

Variable-separable differential equations and models for population growth