

# Differentiation

## Definition of derivatives

The derivative of  $f$  at  $x$ , denoted by  $f'(x)$ , is

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

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## Differentiation rules

1.  $\frac{d}{dx}(cf(x)) = cf'(x)$
  2.  $\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$
  3.  $\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + g(x)f'(x)$
  4.  $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$
  5.  $\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$
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## Differentiation formulas I

1.  $\frac{d}{dx}(c) = 0, c \in \mathbb{R}$
  2.  $\frac{d}{dx}(x^r) = rx^{r-1}, r \in \mathbb{R}$
  3.  $\frac{d}{dx}(\sin x) = \cos x$
  4.  $\frac{d}{dx}(\cos x) = -\sin x$
  5.  $\frac{d}{dx}(\tan x) = \sec^2 x$
  6.  $\frac{d}{dx}(\cot x) = -\csc^2 x$
  7.  $\frac{d}{dx}(\sec x) = \sec x \tan x$
  8.  $\frac{d}{dx}(\csc x) = -\csc x \cot x$
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## Differentiation formulas II

1.  $\frac{d}{dx}(e^x) = e^x$
  2.  $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$
  3.  $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
  4.  $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
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$$5. \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

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## Mean value theorem

Let  $f$  be a function that is continuous on  $[a, b]$  and is differentiable on  $(a, b)$ . Then there is a number  $c \in (a, b)$  such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

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## Consequences of MVT

### Zero derivative

If  $f'(x) = 0 \forall x$  in interval  $I$ , then  $f(x) = c \forall x \in I$  for some constant  $C$ .

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### Equal derivatives

If  $f'(x) - g'(x) = 0 \forall x$  in an interval  $I$ , then  $f(x) = g(x) + C$  for some constant  $C$ .

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## Differentials

$$f'(x) = \frac{dy}{dx}$$
$$f'(x)dx = dy$$

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## Integration

### Definition of an antiderivative

A function  $F$  is called an antiderivative of the function  $f$  on an interval  $I$  if  $F'(x) = f(x) \forall x \in I$ .

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### Equal derivatives (antiderivatives)

If  $F'(x) = G'(x) \forall x$  in an interval  $I$ , then  $F(x) = G(x) + C \forall x \in I$  for some constant  $C$ .

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## Integration rules

1.  $\int kf(x)dx = k \int f(x)dx, k \in \mathbb{R}$
  2.  $\int f(x) \pm g(x)dx = \int f(x)dx \pm \int g(x)dx$
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## Integration formulas I

1.  $\int k dx = kx + C, k \in \mathbb{R}$
  2.  $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \in \mathbb{R}, n \neq -1$
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## Integration formulas II

1.  $\int \sin x dx = -\cos x + C$
  2.  $\int \cos x dx = \sin x + C$
  3.  $\int \sec^2 x dx = \tan x + C$
  4.  $\int \csc^2 x dx = -\cot x + C$
  5.  $\int \sec x \tan x dx = \sec x + C$
  6.  $\int \csc x \cot x dx = -\csc x + C$
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## Integration formulas III

1.  $\int e^x dx = e^x + C$
  2.  $\int \frac{1}{x} dx = \ln |x| + C$
  3.  $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$
  4.  $\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$
  5.  $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + C$
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## Substitution rule

If  $u = g(x)$  is a differentiable function whose range is interval  $I$  and  $f$  is continuous on  $I$ , then

$$\int f'(g(x))g'(x)dx = \int f(u)du$$

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## Riemann Sum

$$A_n = \sum_{i=1}^n f(x_i^*)\Delta x$$

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## Definite integrals

The definite integral of  $f$  from  $a$  to  $b$  is

$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*)\Delta x$$

provided that such limit exists.

We say that  $f$  is integrable on  $[a, b]$

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## Remarks on the definite integral

1. If a function is continuous on  $[a, b]$ , it is integrable on  $[a, b]$ .
  2. If  $f$  is a nonnegative continuous function on  $[a, b]$ , then  $\int_a^b f(x)dx$  is the area under the curve  $y = f(x)$  from  $x = a$  and  $x = b$
  3.  $\int_a^b f(x)dx = \int_a^b f(y)dy$
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## Conventions on the definite integral

1.  $\int_b^a f(x)dx = -\int_a^b f(x)dx$
  2.  $\int_a^a f(x)dx = 0$
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## Properties of the definite integral

1.  $\int_a^b cf(x)dx = c \int_a^b f(x)dx$
  2.  $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
  3.  $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$
  4. If  $f(x) \geq 0 \forall x \in [a, b]$ , then  $\int_a^b f(x)dx \geq 0$
  5. If  $f(x) \geq g(x) \forall x \in [a, b]$ , then  $\int_a^b f(x)dx \geq \int_a^b g(x)dx$
  6. If  $m \leq f(x) \leq M \forall x \in [a, b]$ , then  $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$
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## Mean value theorem (integrals)

If  $f$  is continuous on  $[a, b]$ ,  $\exists c \in [a, b]$  such that

$$\int_a^b f(x)dx = f(c)(b-a)$$

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## Volumes of revolution

### Disk and washers technique

$$A(x) = \pi[f(x)]^2$$
$$\therefore V = \int_b^a \pi[f(x)]^2 dx$$

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### Cylindrical shells

The volume of a solid obtained by rotating about the  $y$ -axis the region under the curve  $y = f(x)$  (continuous and nonnegative) from  $x = a$  (nonnegative) to  $x = b$  is

$$V = \lim_{n \rightarrow \infty} \sum_{i=1}^n 2\pi x_i^* f(x_i^*) \Delta x = \int_a^b 2\pi x f(x) dx$$

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### Integration by parts

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

Letting  $u = f(x)$ ,  $v = g(x) \implies du = f'(x)dx$ ,  $dv = g'(x)dx$ ,

$$\int u dv = uv - \int v du$$

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### Integration by parts and definite integrals

Combining the integration-by-parts formula and FTC2,

$$\int_a^b f(x)g'(x)dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)dx$$

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### Trigonometric identities

1.  $\sin^2 x + \cos^2 x = 1$
  2.  $\tan^2 x + 1 = \sec^2 x$
  3.  $\cot^2 x + 1 = \csc^2 x$
  4.  $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
  5.  $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
  6.  $\sin A \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)]$
  7.  $\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$
  8.  $\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$
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## Average value of a function

Let  $f$  be a continuous on  $[a, b]$ . The average value of  $f$  at  $[a, b]$ , denoted by  $f_{avg}$  is

$$f_{avg} = \frac{\int_a^b f(x)dx}{b-a}$$

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## FTC 1

Let  $f$  be continuous on  $[a, b]$ . If  $F$  is the function defined by

$$F(x) = \int_a^x f(t)dt$$

then  $F'(x) = f(x) \forall x \in [a, b]$ .

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## FTC 2

If a function  $f$  is continuous on  $[a, b]$ , then

$$\int_a^b f(x)dx = F(b) - F(a)$$

The following notations for  $F(b) - F(a)$  are very useful in evaluating definite integrals:  $F(x)\Big|_a^b$  or  $F(x)\Big|_a^b$

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## Area between curves

Given two curves  $y = f(x)$  and  $y = g(x)$ , where  $f(x) > g(x) \forall x \in [a, b]$ , then the area between both curves from  $x = a$  and  $x = b$  is

$$A = \int_a^b f(x) - g(x)dx$$

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## Volume of a solid

Let  $S$  be a solid that lies between  $x = a$  and  $x = b$ . If the cross-sectional area of  $S$  in the plane  $P_x$  through  $x$  and perpendicular to the  $x$ -axis is  $A(x)$ , where  $A$  is a continuous function on  $[a, b]$ , then the volume  $V$  of  $S$  is

$$V = \lim_{n \rightarrow \infty} \sum_{i=1}^n A(x_i^*) \Delta x = \int_a^b A(x)dx$$

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## Integrals of trigonometric functions

$$1. \int \tan x dx = \ln |\sec x| + C$$

$$2. \int \sec x dx = \ln |\sec x + \tan x| + C$$

$$3. \int \cot x dx = \ln |\sin x| + C$$

$$4. \int \csc x dx = \ln |\csc x - \cot x| + C$$

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