Mathematical Analysis IB

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Review on differentiation

Differentiability

Let f be a function on some open interval I containing x. The derivative of f at x, denoted by $f'(x)$, is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Differentiation rules

1. $\dfrac{d}{dx}(cf(x)) = cf'(x)$
2. $\dfrac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$
3. $\dfrac{d}{dx}(f(x)g(x)) = f(x)g'(x) + g(x)f'(x)$
4. $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$
5. $\dfrac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$

Differentiation formulas I

1. $\dfrac{d}{dx}(c) = 0, c \in \mathbb{R}$
2. $\dfrac{d}{dx}(x^r) = rx^{r-1}, r \in \mathbb{R}$
3. $\dfrac{d}{dx}(\sin x) = \cos x$
4. $\dfrac{d}{dx}(\cos x) = \sin x$
5. $\dfrac{d}{dx}(\tan x) = \sec^2 x$
6. $\dfrac{d}{dx}(\cot x) = -\csc^2 x$
7. $\dfrac{d}{dx}(\sec x) = \sec x \tan x$
8. $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$

Differentiation formulas II

1. $\dfrac{d}{dx}(e^x) = e^x$
2. $\dfrac{d}{dx}(\ln|x|) = \dfrac{1}{x}$
3. $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$
4. $\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$
5. $\dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{x\sqrt{x^2-1}}$

Mean value theorem

Let f be a function that is continuous on $[a, b]$ and is differentiable on (a, b). Then there is a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Consequences of MVT

Zero derivative

If $f'(x) = 0 \ \forall x$ in interval I, then $f(x) = c \ \forall x \in I$ for some constant C.

Equal derivatives

If $f'(x) - g'(x) = 0 \ \forall x$ in an interval I, then $f(x) = g(x) + C$ for some constant C.

Example

Let $f(x) = \cos^{-1} x$ and $g(x) = -\sin^{-1} x$.

This implies that $x \in [-1, 1]$ and $f(x), g(x) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$f'(x) = -\frac{1}{\sqrt{x^2+1}}$$

$$g'(x) = -\frac{1}{\sqrt{x^2+1}}$$

Since $f'(x) - g'(x) = 0$ for $x \in [-1, 1]$, then $f(x) - g(x) = C$ for some constant C by a corollary.

$$\begin{aligned} \cos^{-1} x - (-\sin^{-1} x) &= C \\ \cos^{-1} x + \sin^{-1} x &= C \end{aligned}$$

Substituting $x \in [-1, 1]$, in this case, let's use $x = 0$,

$$\begin{aligned}\cos^{-1}(0) + \sin^{-1}(0) &= C\\ 0 + \frac{\pi}{2} &= C\\ C &= \frac{\pi}{2}\end{aligned}$$

$$\therefore \forall x \in [-1, 1], f(x) - g(x) = \frac{\pi}{2}$$

Differentials

$$\begin{aligned}f'(x) &= \frac{dy}{dx}\\ f'(x)dx &= dy\end{aligned}$$

Indefinite and definite integrals

Indefinite integral

The main interpretation of derivative is the slope of a tangent line of a curve.

Example

At any point (x, y) on a particular curve $y = F(x)$, the tangent line has a slope equal to $4x - 5$. If the curve contains the point $(3, 7)$, find $F(x)$.

Solution. Since the slope is equal to $4x - 5$ for any point (x, y), then the slope at $(3, 7)$ is $4(3) - 5 = 7$.

$4x - 5$ therefore represents the tangent slope for all values of x. So

$$F'(x) = 4x - 5$$

By intuition, we can conclude that $F(x) = 2x^2 - 5x$.

However given $F(x) = 2x^2 - 5x + 1$, $F'(x)$ remains the same. And so is $F(x) = 2x^2 - 5x - 3$, $F(x) = 2x^2 - 5x + \pi$, and infinitely more functions. We can arbitrarily assign a constant k, so that $F(x) = 2x^2 - 5x + k$.

Substituting $(x, y) = (3, 7)$,

$$\begin{aligned}&7 = 2(3)^2 - 5(3) + k\\ &7 = 18 - 15 + k\\ &k = 4\end{aligned}$$

So $F(x) = 2x^2 - 5x + 4$.

Definition of an antiderivative

A function F is called an antiderivative of the function f on an interval I if $F'(x) = f(x)\ \forall x \in I$.

$F(x) = 2x^2 - 5x$ is a **possible** antiderivative of $f(x) = 4x - 5$. $F(x) = 2x^2 - 5x + 4$ is also a **possible** antiderivative of $f(x) = 4x - 5$.

Equal derivatives

If $F'(x) = G'(x)\ \forall x$ in an interval I, then $F(x) = G(x) + C\ \forall x \in I$ for some constant C.

Integration notation

The collection of all antiderivatives of f is denoted by

$$\int f(x)dx$$

which is read as "the integral of $f(x)dx$."

This collection is also called the **indefinite integral** of f.

The reverse process if differentiation is called **antidifferentiation** or **integration**.

$\int (4x-5)dx = 2x^2 - 5x + C$ for some constant C.

C is the constant of integration.

$\int \sin x dx = -\cos x + C$

Integration rules

1. $\int kf(x)dx = k \int f(x)dx,\ k \in \mathbb{R}$
2. $\int f(x) \pm g(x)dx = \int f(x)dx \pm \int g(x)dx$

Integration formulas I

1. $\int kdx = kx + C, k \in \mathbb{R}$
2. $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C, n \in \mathbb{R}, n \neq -1$

Integration formulas II

1. $\int \sin x dx = -\cos x + C$
2. $\int \cos x dx = \sin x + C$
3. $\int \sec^2 x dx = \tan x + C$
4. $\int \csc^2 x dx = -\cot x + C$

5. $\int \sec x \tan x dx = \sec x + C$

6. $\int \csc x \cot x dx = -\csc x + C$

Integration formulas III

1. $\int e^x dx = e^x + C$

2. $\int \frac{1}{x} dx = \ln |x| + C$

3. $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$

4. $\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$

5. $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} + C$

Substitution rule

Chain rule for derivatives

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$$

If follows that

$$\int f'(g(x))g'(x)dx = f(g(x)) + C$$

Example

Evaluate $\int 2x \cos x^2 dx$.

Preliminary work. By intuition, we can get $f(x) = sinx$ and $g(x) = x^2$

$$\int 2x \cos x^2 dx = f(g(x)) = \sin x^2$$

Solution. Suppose that $f'(x) = \frac{dy}{dx}$

$$dy = f'(x)dx$$

Let $u = g(x)$, then $g'(x) = \frac{du}{dx}$

$$du = g'(x)dx$$

Let $u = x^2$

$$\begin{aligned} du &= 2xdx \\ \int 2x\cos x^2 dx &= \int \cos u du \\ &= \sin u + C \\ &= \sin x^2 + C \end{aligned}$$

Definition of the substitution rule

If $u = g(x)$ is a differentiable function whose range is interval I and f is continuous on I, then

$$\int f'(g(x))g'(x)dx = \int f(u)du$$

Definite integrals

The area problem

Let f be a continuous nonnegative function on $[a, b]$. Find the area of the regiom bounded by the curve $y = f(x)$, the lines $x = a$, $x = b$, and the x-axis.

> The area is often coined the **region under the curve**, which generally means the area in between the curve and the x-axis

Example

Consider $f(x) = x^2 + 1$ on $[0, 2]$.

Solution. Let A be the area under the curve

Using right endpoints (5 rectangles)

$$\Delta x = \frac{2-0}{5} = \frac{2}{5} = 0.4$$

Rectangle 1: $(\Delta x)(f(0.4)) = (0.4)(1.16)$

Rectangle 2: $(\Delta x)(f(0.8)) = (0.4)(1.64)$

Rectangle 3: $(\Delta x)(f(1.2)) = (0.4)(2.44)$

Rectangle 4: $(\Delta x)(f(1.6)) = (0.4)(3.56)$

Rectangle 5: $(\Delta x)(f(2.0)) = (0.4)(5)$

$$A_5^+ = (0.4)(1.16) + (0.4)(1.64) + (0.4)(2.44) + (0.4)(3.56) + (0.4)(5) = 5.52$$

A_5^+ is an overestimation of A.

Using left endpoints (5 rectangles):

Rectangle 1: $(\Delta x)(f(0)) = (0.4)(1)$

Rectangle 2: $(\Delta x)(f(0.4)) = (0.4)(1.16)$

Rectangle 3: $(\Delta x)(f(0.8)) = (0.4)(1.64)$

Rectangle 4: $(\Delta x)(f(1.2)) = (0.4)(2.44)$

Rectangle 5: $(\Delta x)(f(1.6)) = (0.4)(3.56)$

$$A_5^- = (0.4)(1) + (0.4)(1.16) + (0.4)(1.64) + (0.4)(2.44) + (0.4)(3.56) + = 3.92$$

A_5^- is an underestimation of A.

We can increase the number of rectangles and compute the area A more **accurately** by computing the area as the number of rectangles approach infinity.

Let the number of rectangles be n

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

Let x_0 be the first point: $x_0 = 0$

$$\begin{array}{ccc} x_1 = \dfrac{2}{n} & x_2 = \dfrac{4}{n} & x_3 = \dfrac{6}{n} \\ x_4 = \dfrac{8}{n} & x_5 = \dfrac{10}{n} & x_6 = \dfrac{12}{n} \\ x_7 = \dfrac{14}{n} & \cdots & x_i = \dfrac{2i}{n} \end{array}$$

$$\begin{aligned} A_n &= R_1 + R_2 + R_3 + R_4 + \cdots + R_n \\ &= \sum_{i=1}^{n} \Delta x(f(x_i)) \\ &= \sum_{i=1}^{n} \frac{2}{n}\Big[\Big(\frac{2i}{n}\Big)^2 + 1\Big] \\ &= \frac{2}{n}\sum_{i=1}^{n}\Big(\frac{4i^2}{n^2} + 1\Big) \\ &= \frac{2}{n}\Big[\sum_{i=1}^{n}\Big(\frac{4i^2}{n^2}\Big) + \sum_{i=1}^{n}(1)\Big] \\ &= \frac{2}{n}\Big[\frac{4}{n^2}\sum_{i=1}^{n}(i^2) + \sum_{i=1}^{n}(1)\Big] \\ &= \frac{2}{n}\Big[\frac{4}{n^2}\Big(\frac{(n)(n+1)(2n+1)}{6}\Big) + n\Big] \\ &= \frac{8}{n^3}\frac{(n)(n+1)(2n+1)}{6} + 2 \\ A_n &= \frac{4}{3}\Big(1 + \frac{1}{n}\Big)\Big(2 + \frac{1}{n}\Big) + 2 \end{aligned}$$

$$
\begin{aligned}
A &= \lim_{n\to\infty} A_n \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left[\left(\frac{2i}{n}\right)^2 + 1\right] \\
&= \lim_{n\to\infty} \left[\frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + 2\right] \\
&= \frac{4}{3}(1)(2) + 2 \\
A &= \frac{14}{3}
\end{aligned}
$$

Riemann sum

Let f be a function defined on $[a, b]$.

Divide $[a, b]$ into n subintervals, each with width

$$\Delta x = \frac{b-a}{n}$$

Let $x_0 = a, x_1, x_2, \ldots, x_n = b$,

For each subinterval $[x_{i-1}, x_i]$, choose a sample point x_i^*

Compute the sum

$$A_n = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

This is also called the **Riemann sum**.

$$A_n = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Definite integral and integrability

The definite integral of f from a to b is

$$\int_a^b f(x)dx = \lim_{x\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided that such limit exists.

We say that f is integrable on $[a, b]$

Remarks on the definite integral

1. If a function is continuous on $[a, b]$, it is integrable on $[a, b]$.
2. If f is a nonnegative continuous function on $[a, b]$, then $\int_a^b f(x)dx$ is the area under the curve $y = f(x)$ from $x = a$ and $x = b$

3. $\displaystyle\int_a^b f(x)dx = \int_a^b f(y)dy$

Conventions on the definite integral

1. $\displaystyle\int_b^a f(x)dx = -\int_a^b f(x)dx$
2. $\displaystyle\int_a^a f(x)dx = 0$

Properties of the definite integral

1. $\displaystyle\int_a^b cf(x)dx = c\int_a^b f(x)dx$
2. $\displaystyle\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x) \pm \int_a^b g(x)$
3. $\displaystyle\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$
4. If $f(x) \geq 0 \ \forall x \in [a,b]$, then $\displaystyle\int_a^b f(x)dx \geq 0$
5. If $f(x) \geq g(x) \ \forall x \in [a,b]$, then $\displaystyle\int_a^b f(x)dx \geq \int_a^b g(x)dx$
6. If $m \leq f(x) \leq M \ \forall x \in [a,b]$, then $m(b-a) \leq \displaystyle\int_a^b f(x)dx \leq M(b-a)$

The Fundamental Theorem of Calculus

Mean Value Theorem for integrals

Proof

Since f is continuous on $[a,b]$, then f is integrable on $[a,b]$ — i.e. $\int_a^b f(x)dx$ has a value.

Since f is continuous on [a,b], by the **Extreme Value Theorem**, $\exists m, M \in \mathbb{R}$ such that $f(x_m) = m, f(x_M) = M, m \leq f(x) \leq M \ \forall x \in [a,b]$ and for some $x_m, x_M \in [a,b]$.

By Property 6 of the definite integral, $m(b-a) \leq \displaystyle\int_a^b f(x)dx \leq M(b-a)$

$$\begin{aligned} m \leq \frac{\int_a^b f(x)dx}{b-a} \leq & \qquad M \\ f(x_m) \leq \frac{\int_a^b f(x)dx}{b-a} \leq & \qquad f(x_M) \end{aligned}$$

By the IVT, $\exists c \in [a,b]$ such that

$$\frac{\int_a^b f(x)dx}{b-a} = f(c)$$
$$\int_a^b f(x)dx = f(c)(b-a)$$

If f is continuous on $[a,b]$, $\exists c \in [a,b]$ such that

$$\int_b^a f(x)dx = f(c)(b-a)$$

Average value of a function

Proof

Given a function continuous on $[a,b]$, we can get the average value of the function at $[a,b]$ by dividing the curve into n equal-width rectangles, getting the value of each sample points, and dividing by n.

Average area $= \frac{\sum_{i=1}^n f(x_i^*)i}{n}$

But then, $\Delta x = \frac{b-a}{n} \implies n = \frac{b-a}{\Delta x}$

$$\begin{aligned}\frac{\sum_{i=1}^n f(x_i^*)}{n} &= \frac{\sum_{i=1}^n f(x_i^*)}{\frac{b-a}{\Delta x}} \\ &= \frac{1}{b-a}\sum_{i=1}^n f(x_i^*)\Delta x\end{aligned}$$

We want to make n larger in order to make the average more accurate.

$$\begin{aligned}\lim_{n\to\infty} \frac{\sum_{i=1}^n f(x_i^*)}{n} &= \lim_{n\to\infty} \frac{1}{b-a}\sum_{i=1}^n f(x_i^*)i\Delta x \\ &= \frac{1}{b-a}\lim_{n\to\infty}\sum_{i=1}^n f(x_i^*)\Delta x \\ &= \frac{1}{b-a}\int_a^b f(x)dx\end{aligned}$$

Therefore, given function f that is continuous on $[a,b]$, there exists $c \in [a,b]$ such that

$$f_{avg} = f(c)$$

Let f be a continuous on $[a,b]$. The average value of f at $[a,b]$, denoted by f_{avg} is

$$f_{avg} = \frac{\int_a^b f(x)dx}{b-a}$$

First part of the Fundamental Theorem of Calculus

Let $y = f(t)$ that is continuous on $[a, b]$.

If $x \in [a, b]$, then the function is also continuous on $[a, b] \implies$ the function is also continuous on $[a, x]$.

$$\begin{aligned} F(x) &= \int_a^x f(t)dt \\ F(a) &= \int_a^a f(t)dt = 0 \\ F(b) &= \int_a^b f(t)dt \end{aligned}$$

Let f be continuous on $[a, b]$. If f is the function defined by

$$F(x) = \int_a^x f(t)dt$$

then $F'(x) = f(x)\ \forall x \in [a, b]$.

Proof

Let $x, x + h \in [a, b], h \neq 0$.

$$F(x + h) - F(x) = \int_a^{x+h} f(t)dt - \int_a^x f(t)dt$$

By the Property 3 of definite integrals,

$$\begin{aligned} \int_a^{x+h} f(t)dt - \int_a^x f(t)dt &= \int_a^x f(t)dt + \int_x^{x+h} f(t)dt - \int_a^x f(t)dt \\ &= \int_x^{x+h} f(t)dt \end{aligned}$$

By the Mean Value Theorem for integrals, $\exists c \in [x, x + h]$ such that

$$\begin{aligned} \int_x^{x+h} f(t)dt &= f(c)(x + h - x) \\ &= hf(c) \\ \implies F(x + h) - F(x) &= hf(c) \\ \frac{F(x + h) - F(x)}{h} &= f(c) \\ \lim_{h\to 0} \frac{F(x + h) - F(x)}{h} &= \lim_{h\to 0} f(c) \end{aligned}$$

Note that $\lim_{h\to 0} x = x$ and $\lim_{h\to 0}(x + h) = x \implies \lim_{h\to 0} c = x$ by Squeeze Theorem.

Since f is continuous at x,

$$\lim_{h\to 0} f(c) = f(x)$$

$$\implies F'(x) = f(x) \; \forall x \in [a, b]$$

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Second part of the Fundamental Theorem of Calculus

Let's bring back $f(x) = x^2 + 1$ on $[0, 2]$.

f is continuous on $[0, 2] \implies f$ is integrable on $[0, 2]$.

$$\implies \int_0^2 (x^2 + 1)dx = \frac{14}{3}$$

Let $F(x) = \dfrac{x^3}{3} + x - 1$.

$$\begin{aligned} F(2) &= \frac{2^3}{3} + 2 - 1 = \frac{8}{3} + 1 = \frac{11}{3} \\ F(0) &= \frac{0^3}{3} + 0 - 1 = 0 - 1 = -1 \\ F(2) - F(0) &= \frac{11}{3} - (-1) = \frac{14}{3} \\ \int_0^2 (x^2 + 1)dx &= F(2) - F(0) \end{aligned}$$

Observe that $F'(x) = x^2 + 1 \implies F(x)$ is the an antiderivative of $x^2 + 1$.

If a function f is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is any antiderivative of f on $[a, b]$.

The following notations for $F(b) - F(a)$ are very useful in evaluating definite integrals:

$F(x)\Big]_a^b$ or $F(x)\Big|_a^b$

Proof

By FTC - Part 1, the function

$$\int_a^x f(t)dt$$

is an antiderivative of f on $[a, b]$.

By the Equal Derivatives Theorem,

$$\int_a^x f(t)dt = F(x) + C$$

where F is any antiderivative of f.

$$\begin{aligned} x = b, \int_a^b f(t)dt &= F(b) + C \\ x = a, \int_a^a f(t)dt &= F(a) + C = 0 \\ \int_a^b f(t)dt - \int_a^a f(t)dt &= [F(b) + C] - [F(a) + C] \\ \int_a^b f(t)dt &= F(b) - F(a) \end{aligned}$$

Application I

Areas between curves

Example 1

Find the area of the region under the curve $y = x^2 - 1$ from $x = -1$ to $x = 2$.

Solution. Area is simply not $\int_{-1}^{2}(x^2-1)dx$ because $\int_{-1}^{1}(x^2-1)dx$ is negative and cancels the positive area.

Therefore, we get $\int_{-1}^{1} -(x^2-1)dx$ to get the area of the curve between -1 and 1.

$$\begin{aligned} A &= \int_{-1}^{1} -(x^2-1)dx + \int_{1}^{2}(x^2-1)dx \\ &= \Big(-\frac{x^3}{3} + x\Big)\Big|_{-1}^{1} + \Big(\frac{x^3}{3} - x\Big)\Big|_{1}^{2} \\ &= \Big(\frac{1^3}{3} + 1\Big) - \Big[\frac{(-1)^3}{3} + (-1)\Big] + \Big(\frac{2^3}{3} - 2\Big) - \Big(\frac{1^3}{3} - 1\Big) \\ &= \frac{2}{3} + \frac{2}{3} + \frac{8}{3} - 2 + \frac{2}{3} \\ A &= \frac{8}{3} \end{aligned}$$

Example 2

Find the area of the region bounded by the curves of $y = x^2$ and $y = 4x - x^2$.

Solution. Note that both curves intersect at $(0,0)$ and $(2,4)$.

When we use Riemann sum, we only get the rectangles in between the region bounded by the area by subtracting the upper function ($y = 4x - x^2$) to the lower function ($y = x^2$)

$$\implies A_n = \sum[(4x - x^2) - x^2]\Delta x$$

$$
\begin{aligned}
A &= \int_0^2 [(4x - x^2) - x^2]dx \\
&= \int_0^2 (4x - 2x^2)dx \\
&= \Big(2x^2 - \frac{2x^3}{3}\Big)\Big|_0^2 \\
&= \Big[2(2)^2 - \frac{2(2)^3}{3}\Big] - \Big[2(0)^2 - \frac{2(0)^3}{3}\Big] \\
&= \Big[8 - \frac{16}{3}\Big] - 0 \\
A &= \frac{8}{3}
\end{aligned}
$$

Example 3

Find the area of the region bounded by the curve $y = \sqrt{x}$, the line $x + 2y = 3$, and the x-axis.

Solution. The graphs intersect at $(0, 0)$, $(1, 1)$, and $(3, 0)$.

$$x + 2y = 3 \implies y = -\frac{1}{2}x + \frac{3}{2}$$

$$
\begin{aligned}
A &= \int_0^1 (\sqrt{x})dx + \int_1^3 \Big(-\frac{1}{2}(3 - x)\Big)dx \\
&= \Big(\frac{2x^{\frac{3}{2}}}{3}\Big)\Big|_0^1 + \Big(\frac{1}{2}(3x - \frac{x^2}{2})\Big)\Big|_1^3 \\
&= \frac{2(1)^{\frac{3}{2}}}{3} - \frac{2(0)^{\frac{3}{2}}}{+} \frac{1}{2}(3(3) - \frac{3^2}{2}) - \frac{1}{2}(3(1) - \frac{1^2}{2}) \\
&= \frac{2}{3} + \frac{9}{4} - \frac{5}{4} \\
&= \frac{8 - 27 + 15}{12} \\
&= \frac{20}{12} \\
A &= \frac{5}{3}
\end{aligned}
$$

Volumes and volumes of revolution using disks and washers

Volume of a right cylinder

$$V = ah$$

$$V_n = \sum_{i=1}^{n} A(x)\Delta x$$

Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x through x and perpendicular to the x-axis is $A(x)$, where A is a continuous function on $[a, b]$, then the volume V of S is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_b^a A(x)dx$$

Example 1

Let us find the volume of a sphere of radius r.

Solution.

radius of the cross-section circle at $x = \sqrt{r^2 - x^2}$

$$\begin{aligned} A(x) &= \pi(\sqrt{r^2 - x^2})^2 \\ &= \pi(r^2 - x^2) \end{aligned}$$

$$\begin{aligned} V_{\text{sphere}} &= \int_{-r}^{r} A(x)dx \\ &= \int_{-r}^{r} \pi(r^2 - x^2)dx \\ &= \pi(r^2 x - \frac{x^3}{3})\Big|_{-r}^{r} \\ &= \pi\left[r^2(r) - \frac{r^3}{3}\right] - \pi\left[r^2(-r) - \frac{(-r)^3}{3}\right] \\ V_{\text{sphere}} &= \frac{4}{3}\pi r^3 \end{aligned}$$

Example 2

The base of a solid is the region bounded by $y = x^2$ and $y = 4$. Its parallel cross-sections perpendicular to the base and the y-axis are squares. Find the volume of the solid.

Solution. side of the cross-section at $y = 2\sqrt{y}$

$$A(y) = (2\sqrt{y})^2 = 4y$$

$$\begin{aligned} V &= \int_0^4 A(y)dy \\ &= \int_0^4 4ydy \\ &= 2y^2\Big|_0^4 \\ &= 2(4)^2 - 2(0)^2 \\ V &= 32 \end{aligned}$$

Volume of solids of revolution

If we revolve a region about a line, we obtain a **solid of revolution**.

Example 1

Consider the region under the curve $y = x^2 + 1$ from $x = -1$ to $x = 2$. We revolve this region about the x-axis.

Solution. radius of the cross-section at $x = f(x)$

$$A(x) = \pi[f(x)]^2$$

$$\begin{aligned}
V &= \int_{-1}^{2} \pi(x^2+1)^2 dx \\
&= \int_{-1}^{2} \pi(x^4+2x^2+1) dx \\
&= \pi\left(\frac{x^5}{5} + \frac{2x^3}{3} + x\right)|_{-1}^{2} \\
&= \pi\left[\frac{2^5}{5} + \frac{2(2)^3}{3} + 2\right] - \pi\left[\frac{(-1)^5}{5} + \frac{2(-1)^3}{3} + (-1)\right] \\
V &= \frac{78\pi}{5}
\end{aligned}$$

The cross-section of a solid of revolution is always a circle.

Example 2

A solid is obtained by revolving about the x-axis the region bounded by $x = y^2$ and $2y = x$. Find the volume of the solid.

Solution.

$$\begin{aligned}
V &= \int_{0}^{4} \pi(\sqrt{x})^2 dx - \int_{0}^{4} \pi\left(\frac{x}{2}\right)^2 dx \\
&= \int_{0}^{4} \pi(x) dx - \int_{0}^{4} \pi\frac{x^4}{4} dx \\
V &= \frac{8\pi}{3}
\end{aligned}$$

Example 3

A solid is obtained by revolving about the y-axis the region bounded by $2x = y^2$, $y = 4$, and the y-axis. Find the volume of the solid.

Solution.

$$\begin{aligned} V &= \int_0^4 \pi(\frac{y^2}{2})^2 dy \\ &= \int_0^4 \pi(\frac{y^4}{4}) dy \\ &= \frac{\pi y^5}{20}\Bigg|_0^4 \\ V &= \frac{256\pi}{5} \end{aligned}$$

Volumes by cylindrical shells

There are times that disks-and-washers technique is not the best way to solve a volume problem – e.g. $y = 4x - x$ rotated about the y-axis.

Volume of a cylindrical shell

Let r_1 be the inner radius of the cylinder, r_2 be the outer (and larger) radius of the cylinder. r be the average of both

$$\begin{aligned} \Delta r &= r_2 - r_1 \\ r &= \frac{r_2 + r_1}{2} \end{aligned}$$

$$\begin{aligned} V_{\text{cylindrical shell}} &= \pi r_2^2 h - r_1^2 h \\ &= \pi(r_2^2 - r_1^2)h \\ &= \pi(r_2 + r_1)h(r_2 - r_1) \\ &= 2\pi\left(\frac{r_2 + r_1}{2}\right)h\Delta r \\ V_{\text{cylindrical shell}} &= 2\pi r h \Delta r \end{aligned}$$

Given a curve $y = f(x)$ in $[a, b]$ rotated about the y-axis, the Riemann sum of the volume is

$$V = \sum 2\pi r h \Delta r$$

In this context, $r = x$ (horizontal distance), $h = f(x)$ (vertical distance), and $\Delta r = \Delta x$

$$\begin{aligned} V &= \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*) \Delta x \\ &= \int_a^b 2\pi x f(x) dx \end{aligned}$$

The volume of a solid obtained by rotating about the y-axis the region under the curve $y = f(x)$ (continuous and nonnegative) from $x = a$ (nonnegative) to $x = b$ is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*) \Delta x = \int_a^b 2\pi x f(x) dx$$

Example 1

A solid is obtained by revolving about the y-axis the region bounded by $y = 4x - x^2$ and the x-axis. Use cylindrical shells to find the volume of the solid.

Solution.

$$\begin{aligned}
V &= \int_0^4 2\pi x(4x - x^2)dx \\
&= \int_0^4 2\pi(4x^2 - x^3)dx \\
&= 2\pi\left(\frac{4}{3}x^3 - \frac{x^4}{4}\right)\Bigg|_0^4 \\
&= 2\pi\left[\frac{4}{3}(4)^3 - \frac{4^4}{4}\right] - 0 \\
&= \frac{128\pi}{3}
\end{aligned}$$

Example 2

A solid is obatined by revolving about the x-axis the region bounded by $y = \sqrt{x}$, $y = 2 - x$, and the x-axis. Find the volume of the solid.

Solution.

Disks:

$y = \sqrt{x}$ and $y = 2 - x$ intersect at $(1, 1)$

$\forall x \in [0, 1]$, $\sqrt{x} \leq 2 - x \implies$ we use $\sqrt{x}$ at this interval

$\forall x \in [1, 2]$, $2 - x \leq \sqrt{x} \implies$ we use $2 - x$ at this interval

$$\begin{aligned}
V &= \int_0^1 \pi(\sqrt{x})^2 dx + \int_1^2 \pi(2 - x)^2 dx \\
&= \int_0^1 \pi x dx + \int_1^2 \pi(x^2 - 4x + 4)dx
\end{aligned}$$

(This exercise is left to the reader lol don't wanna solve this myself)

cylindrical shell:

$y = \sqrt{x}$ and $y = 2 - x$ intersect at $(1, 1)$.

$$y = \sqrt{x} \implies x = y^2$$
$$y = 2 - x \implies x = 2 - y$$

$$V = \int_0^1 2\pi y(2 - y^2 - y)dy$$

(Just solve this yourselves)

Example 3

A solid is obtained by the revolving about the line $x = -2$ the region bounded by $y = x^3$, $y = 8$, and the y-axis. Find the volume of the solid.

Solution.

disks: $V = \int_0^8 \pi\left[(\sqrt[3]{y}+2)^2 - 2^2\right]dy = \frac{336\pi}{5}$

cylindrical shells: $V = \int_0^2 2\pi(x+2)(8-x^3)dx = \frac{336\pi}{5}$

Techniques of integration

Integration by parts

Product rule and the differentials

Recalling the product rule in derivatives,

$$\begin{aligned}\frac{d}{dx}(f(x)g(x)) &= f(x)g'(x) + g(x)f'(x)\\ d(f(x)g(x)) &= f(x)g'(x)dx + g(x)f'(x)dx\end{aligned}$$

Let $u = f(x)$, $v = g(x)$

$du = f'(x)dx$, $dv = g'(x)dx$

$$\begin{aligned}d(f(x)g(x)) = f(x)g'(x)dx + g(x)f'(x)dx \implies d(uv) &= udv + vdu\\ udv &= d(uv) - vdu\\ \int udv &= \int d(uv) - \int vdu\\ \int udv &= uv - \int vdu\end{aligned}$$

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

Letting $u = f(x)$, $v = g(x) \implies du = f'(x)dx$, $dv = g'(x)dx$,

$$\int udv = uv - \int vdu$$

This is also called the **integration-by-parts formula**.

Integration by parts and definite integrals

Combining the integration-by-parts formula and FTC2,

$$\int_a^b f(x)g'(x)dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)dx$$

Example 1

Evaluate $\int \ln x dx$

Preliminary work.

$$\int \ln x dx \implies x > 0$$

Let $u = 1$, $dv = \ln x dx$

$du = 0$, $v = \int \ln x dx$

Note that this solution is not correct because we just went back to our original statement, leading us nowhere.

Solution. Let $u = \ln x$, $dv = dx$

$du = \frac{1}{x}dx$, $v = \int dx = x + C$

Following the integration-by-parts formula,

$$\begin{aligned}\int \ln x dx &= (\ln x)(x + c) - \int (x + C)\frac{1}{x}dx \\ &= x \ln x + C \ln x - \int (1 + \frac{C}{x})dx \\ &= x \ln x + C \ln x - (x + C \ln |x| + C_1) \\ &= x \ln x + C \ln x - x - C \ln |x| - C_1; \text{ note that } x > 0, \text{ thus } \ln |x| = \ln x \\ &= x \ln x - x + C\end{aligned}$$

Note that the contribution of $+C$ in v just cancels at $uv - \int vdu$, so it is **not necessary** to put $+C$ when using integration by parts.

Example 2

Evaluate $\int t^2 \sin \beta t dt$

Solution. Let $u = t^2$, $dv = \sin \beta t dt$

$du = 2tdt$, $v = \int \sin \beta t dt = -\dfrac{\cos \beta t}{\beta}$

$$\int t^2 \sin \beta t dt = -\frac{t^2 \cos \beta t}{\beta} - \int -\frac{2t \cos \beta t}{\beta}dt = -\frac{t^2 \cos \beta t}{\beta} - \int \frac{2}{\beta} t \cos \beta t dt$$

Let $u = t$, $dv = \cos \beta t dt$

$du = dt$, $v = \dfrac{\sin \beta t}{\beta}$

$$\begin{aligned}-\frac{t^2 \cos \beta t}{\beta} - \int \frac{2}{\beta} t \cos \beta t dt &= -\frac{t^2 \cos \beta t}{\beta} - \frac{2}{\beta}\Big(\frac{t \sin \beta t}{\beta} - \int \frac{\sin \beta t}{\beta}dt\Big) \\ &= -\frac{t^2 \cos \beta t}{\beta} - \frac{2t \sin \beta t}{\beta^2} - \frac{2}{\beta}\Big(\int \frac{\sin \beta t}{\beta}dt\Big) \\ &= -\frac{t^2 \cos \beta t}{\beta} - \frac{2t \sin \beta t}{\beta^2} + \frac{2 \cos \beta t}{\beta^3} + C\end{aligned}$$

Example 3

Evaluate $\int e^x \sin \pi x dx$

Solution. Let $u = \sin \pi x$, $dv = e^x dx$

$du = \pi \cos \pi x dx$, $v = e^x$

By integration by parts,

$$\int e^x \sin \pi x dx = e^x \sin \pi x - \pi \int e^x \cos \pi x dx$$

Let $u = \cos \pi x$, $dv = e^x dx$

$du = -\pi \sin \pi x$, $v = e^x$

By integration by parts,

$$\begin{aligned}
\int e^x \sin \pi x dx = e^x \sin \pi x - \pi \int e^x \cos \pi x dx &= e^x \sin \pi x - \pi \Big(e^x \cos \pi x - \pi \int -e^x \sin \pi x dx \Big) \\
&= e^x \sin \pi x - \pi e^x \cos \pi x - \pi^2 \int e^x \sin \pi x dx \\
\int e^x \sin \pi x dx + \pi^2 \int e^x \sin \pi x dx &= e^x \sin \pi x - \pi e^x \cos \pi x \\
(1+\pi^2) \int e^x \sin \pi x dx &= e^x \sin \pi x - \pi e^x \cos \pi x \\
\int e^x \sin \pi x dx &= \frac{e^x \sin \pi x - \pi e^x \cos \pi x}{1+\pi^2} + C
\end{aligned}$$

Example 4 (definite integral)

Evaluate $\int_0^1 \tan^{-1} x dx$.

Solution. We shall first get the antiderivative of the function before evaluating the definite integral.

Let $u = \arctan x$, $dv = dx$

$du = \dfrac{1}{x^2+1} dx$, $v = x$

By integration by parts,

$$\int \tan^{-1} x dx = x \arctan x - \int \frac{x}{x^2+1} dx$$

Let $u = x^2 + 1 \implies du = 2x dx$

Using the substitution rule,

$$
\begin{aligned}
x\arctan x - \int \frac{x}{x^2+1}dx &= x\arctan x - \frac{1}{2}\int \frac{1}{u}du \\
&= x\arctan x - \frac{\ln|u|}{2} \\
&= x\arctan x - \frac{\ln|x^2+1|}{2}, \text{ note that } \forall x \in \mathbb{R}, x^2+1>0 \\
&= x\arctan x - \frac{\ln(x^2+1)}{2}
\end{aligned}
$$

By FTC2,

$$
\begin{aligned}
\int_0^1 \tan^{-1} x dx &= (x\arctan x - \frac{\ln(x^2+1)}{2})|_0^1 \\
&= [\arctan 1 - \frac{\ln(1^2+1)}{2}] - [-\frac{\ln(0^2+1)}{2}] \\
&= \frac{\pi - 2\ln(2)}{4}
\end{aligned}
$$

Trigonometric integrals

Trigonometric identities

1. $\sin^2 x + \cos^2 x = 1$
2. $\tan^2 x + 1 = sec^2 x$
3. $\cot^2 x + 1 = csc^2 x$
4. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
5. $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
6. $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
7. $\sin A \sin B = \frac{1}{2}[\cos(A-B)] - \cos(A+B)]$
8. $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

Integrals of trigonometric functions

1. $\int \tan x dx = \ln|\sec x| + C$

$$
\begin{aligned}
\int \tan x dx = \int \frac{\sin x}{\cos x}dx &= -\ln|\cos x| + C \\
&= \ln|\sec x| + C
\end{aligned}
$$

2. $\int \sec x dx = \ln|\sec x + \tan x| + C$

$$\begin{aligned}\int \sec x dx &= \int secx \frac{\sec x + \tan x}{\sec x + \tan x} dx \\ &= \int \frac{\sec^2 + \sec x \tan x}{\sec x + \tan x} dx \\ &= \ln|\sec x + \tan x| + C\end{aligned}$$

3. $\int \cot x dx = \ln|\sin x| + C$
4. $\int \csc x dx = \ln|\csc x - \cot x| + C$

Example 1

Evaluate $\int \sqrt{cos\theta} sin^3 \theta d\theta$.

Solution.

$$\begin{aligned}\int \sqrt{\cos\theta} \sin^3\theta d\theta &= \int \sqrt{\cos\theta} \sin^2\theta \sin\theta d\theta \\ &= \int \sqrt{\cos\theta}(1 - \cos^2\theta) \sin\theta d\theta\end{aligned}$$

Let $u = \sqrt{\cos\theta}$

$\implies u^2 = \cos\theta$

$\implies 2udu = -\sin\theta d\theta$

$\implies -2udu = \sin\theta d\theta$

$$\begin{aligned}\int \sqrt{\cos\theta}(1 - \cos^2\theta) \sin\theta d\theta &= \int u(1 - u^4)(-2u) du \\ &= \int (-2u^2 + 2u^6) du \\ &= -\frac{2}{3}u^3 + \frac{2}{7}u^7 + C \\ &= -\frac{2}{3}(\cos\theta)^{3/2} + \frac{2}{7}(\cos\theta)^{7/2} + C\end{aligned}$$

Example 2

Evaluate $\int \tan^2 t \sec^4 t dt$.

Solution.

$$\begin{aligned}\int \tan^2 t \sec^4 t dt &= \int \tan^2 t \sec^2 t \sec^2 t dt \\ &= \int \tan^2 t(1 + \tan^2 t) \sec^2 t dt\end{aligned}$$

Let $u = \tan t \implies du = \sec^2 t dt$.

$$\begin{aligned}
\int \tan^2 t(1+\tan^2 t)\sec^2 t dt &= \int u^2(1+u^2)du \\
&= \int (u^2+u^4)du \\
&= \frac{u^3}{3} + \frac{u^5}{5} \\
&= \frac{\tan^5 t}{5} + \frac{\tan^3 t}{3} + C
\end{aligned}$$

Example 3 (definite integral)

Evaluate $\int_{\pi/4}^{\pi/2} \csc^5 x \cot^3 x dx$.

$$\begin{aligned}
\int_{\pi/4}^{\pi/2} \csc^5 x \cot^3 x dx &= \int_{\pi/4}^{\pi/2} \csc^4 x \cot^2 x \csc x \cot x dx \\
&= \int_{\pi/4}^{\pi/2} \csc^4 x(\csc^2 x - 1)\csc x \cot x dx
\end{aligned}$$

Let $u = \csc x \implies du = -\csc x \cot x dx \implies -du = \csc x \cot x dx$

$x = \frac{\pi}{4} \implies u = \sqrt{2}$

$x = \frac{\pi}{2} \implies u = 1$

$$\begin{aligned}
\int_{\pi/4}^{\pi/2} \csc^4 x(\csc^2 x - 1)\csc x \cot x dx &= \int_{\sqrt{2}}^{1} u^4(u^2-1)(-du) \\
&= \int_{\sqrt{2}}^{1} -u^6 + u^4 du \\
&= \int_{1}^{\sqrt{2}} u^6 - u^4 du \\
&= \left. \frac{u^7}{7} - \frac{u^5}{5} \right|_1^{\sqrt{2}} \\
&= \frac{8\sqrt{2}}{7} - \frac{4\sqrt{2}}{5} - \frac{1}{7} + \frac{1}{5} \\
&= \frac{40\sqrt{2} - 28\sqrt{2} - 5 + 7}{35} \\
&= \frac{12\sqrt{2}+2}{35}
\end{aligned}$$

Example 4 (definite integral)

Evaluate $\int_{0}^{\pi/2} \cos 5t \cos 10t dt$.

$$
\begin{aligned}
\int_0^{\pi/2} \cos 5t \cos 10t dt &= \int_0^{\pi/2} \frac{1}{2}[\cos(5t-10t) + cos(5t+10t)]dt \\
&= \int_0^{\pi/2} \frac{1}{2}[\cos(-5t) + \cos 15t]dt; \text{ note that } \cos(-5t) = \cos 5t \ \forall t \\
&= \int_0^{\pi/2} \frac{1}{2}\cos 5t + \frac{1}{2}\cos 15t \\
&= \frac{1}{10}\sin 5t + \frac{1}{30}\sin 15t \Big|_0^{\pi/2} \\
&= \frac{1}{10}\sin\frac{5\pi}{2} + \frac{1}{30}\sin\frac{15\pi}{2} - 0 \\
&= \frac{1}{15}
\end{aligned}
$$

Example 5

Evaluate $\displaystyle\int \frac{\sin^2(1/t)}{t^2}dt$.

Let $u = \dfrac{1}{t} \implies du = -\dfrac{1}{t^2} \implies -du = \dfrac{1}{t^2}$

$$\int \frac{\sin^2(1/t)}{t^2}dt = \int -\sin^2 u du$$

Note that $\sin^2 u = \frac{1}{2}(1-\cos 2u)$

$$
\begin{aligned}
\int -\sin^2 u du &= \int -\frac{1}{2}(1 - cos2u)du \\
&= -\frac{1}{2}(u - \frac{sin2u}{2}) \\
&= -\frac{1}{2t} + \frac{\sin(2/t)}{4} + C
\end{aligned}
$$

Trigonometric substitution

Circular functions

$$x^2 + y^2 = r^2$$

which proves that trigonometric functions are also circular functions

$$
\begin{aligned}
\sin\theta = \frac{y}{r} &\implies y = r\sin\theta \\
\cos\theta = \frac{x}{r} &\implies x = r\cos\theta
\end{aligned}
$$

We can therefore rename (x, y) to $(r\cos\theta, r\sin\theta)$. It implies that for every (x, y) coordinates, there corresponds (r, θ) coordinates.

Recall

Evaluate $\int_{-2}^{2} \sqrt{4-x^2}dx$.

Intuitive solution. Observe that the function is a semicircle with r=2, thus the integral looks for the area of the semicircle.

$$\int_{-2}^{2} \sqrt{4-x^2}dx = \frac{1}{2}\pi 2^2 = 2\pi$$

Solution. Let $y = \sqrt{4-x^2}$.

$y^2 = 4 - x^2$

$x^2 + y^2 = 4$

$r^2 = 4 \implies r = 2$

$(x, y) = (x, \sqrt{4-x^2}) = (2\cos\theta, 2\sin\theta)$

$x = 2\cos\theta \implies dx = 2(-\sin\theta)d\theta$

$\sqrt{4-x^2} = 2\sin\theta$

$$\begin{aligned}
\int_{-2}^{2} \sqrt{4-x^2}dx &= \int_{\pi}^{0} (2\sin\theta)(-2\sin\theta d\theta); \text{ note that we converted the } x\text{-value to}\theta\text{-value} \\
&= \int_{\pi}^{0} -4\sin^2\theta d\theta \\
&= \int_{0}^{\pi} -4\sin^2\theta d\theta \\
&= \int_{0}^{\pi} \frac{4}{2}(1 - cos2\theta)d\theta \\
&= \int_{0}^{\pi} 2(1 - cos2\theta)d\theta \\
&= 2x - \frac{sin2\theta}{2}\Bigg|_{0}^{\pi} \\
&= 2\pi - 0 - 0 \\
&= 2\pi
\end{aligned}$$

Trigonometric substitutions

Expression	Substitution	Identity
$\sqrt{a^2-x^2}$	$x = a\sin\theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta, 0 \leq \theta \leq \frac{\pi}{2}$ or $\pi \leq \theta \leq \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Example 1

Evaluate $\displaystyle\int_0^1 \frac{dx}{(x^2+1)^2}$

$$\int_0^1 \frac{dx}{(x^2+1)^2} = \int_0^1 \frac{dx}{(\sqrt{x^2+1})^4}$$

Visualizing a right triangle with legs x and 1,

$x = \tan\theta \implies dx = \sec^2\theta d\theta$

$\cos\theta = \dfrac{1}{\sqrt{x^2+1}} \implies \sqrt{x^2+1} = \sec\theta$

$x = 0 \implies 0 = \tan\theta \implies \theta = 0$

$x = 1 \implies \theta = \dfrac{\pi}{4}$

$$\begin{aligned}
\int_0^1 \frac{dx}{(\sqrt{x^2+1})^4} &= \int_0^{\pi/4} \frac{\sec^2\theta}{\sec^4\theta} d\theta \\
&= \int_0^{\pi/4} \frac{1}{\sec^2\theta} d\theta \\
&= \int_0^{\pi/4} \cos^2\theta d\theta \\
&= \int_0^{\pi/4} \frac{1}{2}(1+\cos 2\theta) d\theta \\
&= \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\Bigg|_0^{\pi/4}
\end{aligned}$$

(solution to be continued)

Example 2

Evaluate $\displaystyle\int \frac{\sqrt{x^2-1}}{x^4} dx.$

Solution.

By trigonometric substitution,

$x = \sec\theta \implies dx = \sec\theta\tan\theta d\theta$

$\sqrt{x^2-1} = \tan\theta$

$$
\begin{aligned}
\int \frac{\sqrt{x^2-1}}{x^4}dx &= \int \frac{\tan\theta}{\sec^4\theta}(\sec\theta\tan\theta d\theta) \\
&= \int \frac{\tan^2\theta}{\sec^3\theta}d\theta \\
&= \int \sin^2\theta\cos\theta d\theta \\
&= \frac{\sin^3\theta}{3}+C \\
&= \frac{1}{3}(\frac{\sqrt{x^2-1}}{x})^3+C \\
&= \frac{(x^2-1)^{3/2}}{3x^3}+C
\end{aligned}
$$

Example 3

Evaluate $\displaystyle\int \frac{x^2}{(3+4x-4x^2)^{3/2}}dx$.

Solution.

$$
\int \frac{x^2}{(3+4x-4x^2)^{3/2}}dx = \int \frac{x^2}{[4-(2x-1)^2]^{3/2}}dx
$$

Let there be a triangle with legs $2x-1$ and $\sqrt{4-(2x-1)^2}$

$2x-1 = 2\sin\theta \implies x = \dfrac{1+2\sin\theta}{2}$

$2dx = 2\cos\theta d\theta$

$dx = \cos\theta d\theta$

$\sqrt{4-(2x-1)^2} = 2\cos\theta$

$$
\begin{aligned}
\int \frac{x^2}{[4-(2x-1)^2]^{3/2}}dx &= \int \frac{\left(\frac{1+2\sin\theta}{2}\right)^2}{2\cos^3\theta}\cos\theta d\theta \\
&= \int \frac{1+4\sin\theta+4\sin^2\theta}{32\cos\theta}d\theta \\
&= \frac{1}{32}\int(\sec^2\theta+4\tan\theta\sec\theta+4\tan^2\theta)d\theta \\
&= \frac{1}{32}\tan\theta+\frac{1}{8}\sec\theta+\frac{1}{8}\tan\theta-\frac{1}{8}\theta+C \\
&= \frac{5}{32}\tan\theta+\frac{1}{8}\sec\theta-\frac{1}{8}\theta+C \\
&= \frac{5}{32}\frac{2x-1}{\sqrt{3+4x-4x^2}}+\frac{1}{8}\frac{2}{\sqrt{3+4x-4x^2}}-\frac{1}{8}\arcsin\left(\frac{2x-1}{2}\right)+C
\end{aligned}
$$

Integration of rational functions by partial fractions

Quick review

$$\int \frac{2}{3-2x}dx = \ln|3-2x| + C$$
$$\int \frac{2}{(3-2x)^2}dx = \frac{1}{3-2x} + C$$
$$\int \frac{x}{x^2+4}dx = \frac{1}{2}\ln(x^2+4) + C;\text{ note that } x^2+4 \geq 0\ \forall x \in \mathbb{R}$$
$$\int \frac{x}{(x^2+4)^2}dx = -\frac{1}{2(x^2+4)} + C$$
$$\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

Partial fractions

Example

$$\frac{1}{x+2} - \frac{1}{x-3} = \frac{x-2-(x+2)}{(x+2)(x-3)} = \frac{-5}{(x+2)(x-3)}$$

This is an identity because it is true for all x.

$$\implies \frac{-5}{(x+2)(x-3)} = \frac{A}{x+2} + \frac{B}{x-3},\ A, B \in \mathbb{R}$$

$$\begin{aligned}(x+2)(x-3)\left[\frac{-5}{(x+2)(x-3)} = \frac{A}{x+2} + \frac{B}{x-3}\right](x+2)(x-3)&\\ -5 &= A(x-3) + B(x+2)\\ -5 &= Ax - 3A + Bx + 2B\\ -5 &= (A+B)x + (-3A+2B)\end{aligned}$$

By systems of equations,

$$\begin{cases} A + B = 0 \implies A = -B \\ -3A + 2B = -5 \implies = -3(-B) + 2B = -5 \implies B = -1, A = 1 \end{cases}$$

By substitution,

$$\begin{aligned} x = 3:\ -5 &= 0 + B(3+2) \\ B &= -1 \end{aligned}$$

$$\begin{aligned} x = -2:\ -5 &= A(-2-3) + 0 \\ A &= 1 \end{aligned}$$

Applications II

Arc length

Suppose a continuous function $f(x)$ continuous on $[a, b]$. We want to get the length of the curve from $[a, b]$.

Let Δx be the spacing of the values of x when divided in to n equal intervals.

$$\Delta x = x_i - x_{i-1} = \frac{b-a}{n}$$

We have two adjacent points, $P_{i-1}(x, f(x))$ and $P_i(x+\Delta x, f(x+\Delta x))$. Rewriting the coordinates, we get $P_{i-1}(x_{i-1}, f(x_{i-1}))$ and $P_i(x_i, f(x_i))$. Getting the distance between these two points,

$$|P_{i-1}P_i| = \sqrt{(x_i - x_{i-1})^2 + [f(x_i) - f(x_{i-1})]^2}$$

Recalling MVT for derivatives,

- f is continuous on $[x_{i-1}, x_i]$
- f is differentiable on (x_{i-1}, x_i)

$\implies \exists x_i^* \in (x_{i-1}, x_i)$ such that

$$f'(x_i^*) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

Going back to $|P_{i-1}P_i|$,

$$\begin{aligned} |P_{i-1}P_i| &= \sqrt{(x_i - x_{i-1})^2 + [f(x_i) - f(x_{i-1})]^2} \\ &= \sqrt{(x_i - x_{i-1})^2 + [f'(x_i^*)(x_i - x_{i-1})]^2} \\ &= \sqrt{(\Delta x)^2 + [f'(x_i^*)\Delta x]^2} \\ &= \sqrt{1 + [f'(x_i^*)]^2}\Delta x \end{aligned}$$

Let the length of the curve be L.

$$\begin{aligned} L &= \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i| \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\Delta x \\ L &= \int_a^b \sqrt{1 + [f'(x)]^2} dx \end{aligned}$$

The arc length formula

If f' is continuous on $[a, b]$, then the length L of the curve $y = f(x)$, $a \leq x \leq b$, is given by

$$L = \int_a^b \sqrt{1 + [f(x)]^2} dx$$

Differential equations

incomplete

Models for population growth

Malthus (c. 1798) said that in an ideal population, the rate of growth of a population is proportional to the size of the population.

Let P be a function of time stating the number of individuals in a population at time t.

$$\begin{aligned} &\frac{dP}{dt} \propto P \\ \implies &\frac{dP}{dt} = kP; k \text{ is the proportionality constant} \\ &\frac{\frac{dP}{dt}}{P} = k \end{aligned}$$

This equations is known as the **simple growth model**.

Verholst (c. 1838) mentioned that Malthus's population model is only correct for small populations, and that population growth is bounded by environmental restrictions.

Let M be the carrying capacity of the environment.

Then when the population surpasses M, the population growth pattern changes.

$$\frac{\frac{dP}{dt}}{P} = k\left(1 - \frac{P}{M}\right)$$

Note that when $P > M$, then the proportion becomes negative.

This is knows as the **logistic model**.

Differential equation

A **differential equation** is an equation that contains an unknown function and one or more of its derivatives.

Differential equation terminologies

- **order** - highest order of the derivative that appears
- **solution** - the equation that satisfies the differential equation
- **solve** - refers to finding **ALL** equations that satisfies the differential equation
- **initial-value problem** - finding the solution to the differential equation with an initial condition given – e.g. a coordinate

Simple growth model

The solution of the initial-value problem

$$\frac{dP}{dt} = kP, P(0) = P_0$$

is

$$P(t) = P_0 e^{kt}$$

Logistic model

The solution fo the initial-value problem

$$\frac{\frac{dP}{dt}}{P} = k\left(1 - \frac{P}{M}\right)$$

is

$$P(t) = \frac{M}{1 - Ae^{kt}}, A = \frac{M - P_0}{P_0}$$