Mathematical Analysis IB

Matt Alejo

Review on differentiation

Differentiability

Let f be a function on some open interval I containing x. The derivative of f at x, denoted by $f'(x)$, is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Differentiation rules

1. $\frac{d}{dx}(cf(x)) = cf'(x)$
2. $\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$
3. $\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + g(x)f'(x)$
4. $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$
5. $\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$

Differentiation formulas I

1. $\frac{d}{dx}(c) = 0, c \in \mathbb{R}$
2. $\frac{d}{dx}(x^r) = rx^{r-1}, r \in \mathbb{R}$
3. $\frac{d}{dx}(sinx) = cosx$
4. $\frac{d}{dx}(cosx) = -sinx$
5. $\frac{d}{dx}(tanx) = sec^2x$
6. $\frac{d}{dx}(cotx) = -csc^x$
7. $\frac{d}{dx}(secx) = secxtanx$
8. $\frac{d}{dx}(cscx) = -cscxcotx$

Differentiation formulas II

1. $\frac{d}{dx}(e^x) = e^x$
2. $\frac{d}{dx}(ln|x|) = \frac{1}{x}$
3. $\frac{d}{dx}(sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

4. $\frac{d}{dx}(tan^{-1}x) = \frac{1}{1+x^2}$
5. $\frac{d}{dx}(sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$

Mean value theorem

Let f be a function that is continuous on $[a, b]$ and is differentiable on (a, b). Then there is a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Consequences of MVT

Zero derivative

If $f'(x) = 0 \; \forall x$ in interval I, then $f(x) = c \; \forall x \in I$ for some constant C.

Equal derivatives

If $f'(x) - g'(x) = 0 \; \forall x$ in an interval I, then $f(x) = g(x) + C$ for some constant C.

Example Let $f(x) = cos^{-1}x$ and $g(x) = -sin^{-1}x$

This implies that $x \in [-1, 1]$ and $f(x), g(x) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$f'(x) = -\frac{1}{\sqrt{x^2 + 1}}$$

$$g'(x) = -\frac{1}{\sqrt{x^2 + 1}}$$

Since $f'(x) - g'(x) = 0$ for $x \in [-1, 1]$, then $f(x) - g(x) = C$ for some constant C by a corollary.

$$cos^{-1}x - (-sin^{-1}x) = C$$
$$cos^{-1}x + sin^{-1}x = C$$

Substituting $x \in [-1, 1]$, in this case, let's use $x = 0$,

$$cos^{-1}(0) + sin^{-1}(0) = C$$
$$0 + \frac{\pi}{2} = C$$

$$C = \frac{\pi}{2}$$

$$\therefore \forall x \in [-1, 1], f(x) - g(x) = \frac{\pi}{2}$$

Differentials

$$f'(x) = \frac{dy}{dx}$$
$$dy = f'(x)dx$$

Module 1: Indefinite and definite integrals

Indefinite integral

The main interpretation of derivative is the slope of a tangent line of a curve.

Example At any point (x, y) on a particular curve $y = F(x)$, the tangent line has a slope equal to $4x - 5$. If the curve contains the point $(3, 7)$, find $F(x)$.

Solution: Since the slope is equal to $4x - 5$ for any point (x, y), then the slope at $(3, 7)$ is $4(3) - 5 = 7$.

$4x - 5$ therefore represents the tangent slope for all values of x. So

$$F'(x) = 4x - 5$$

By intuition, we can conclude that $F(x) = 2x^2 - 5x$.

However given $F(x) = 2x^2 - 5x + 1$, $F'(x)$ remains the same. And so is $F(x) = 2x^2 - 5x - 3$, $F(x) = 2x^2 - 5x + \pi$, and infinitely more functions. We can arbitrarily assign a constant k, so that $F(x) = 2x^2 - 5x + k$.

Substituting $(x, y) = (3, 7)$,

$$7 = 2(3)^2 - 5(3) + k$$

$$7 = 18 - 15 + k$$

$$k = 4$$

So $F(x) = 2x^2 - 5x + 4$.

Definition of an antiderivative

A function F is called an antiderivative of the function f on an interval I if $F'(x) = f(x)\ \forall x \in I$.

$F(x) = 2x^2 - 5x$ is a *possible* antiderivative of $f(x) = 4x - 5$. $F(x) = 2x^2 - 5x + 4$ is also a *possible* antiderivative of $f(x) = 4x - 5$.

Equal derivatives

If $F'(x) = G'(x)\ \forall x$ in an interval I, then $F(x) = G(x) + C\ \forall x \in I$ for some constant C.

Integration notation

The collection of all antiderivatives of f is denoted by

$$\int f(x)dx$$

which is read as "the integral of $f(x)dx$."

This collection is also called the *indefinite integral* of f.

The reverse process if differentiation is called *antidifferentiation* or *integration*.

$\int(4x - 5)dx = 2x^2 - 5x + C$ for some constant C.

C is the constant of integration.

$\int sinxdx = -cosx + C$

Integration rules

1. $\int kf(x)dx = k\int f(x)dx$, k constant
2. $\int f(x) \pm g(x)dx = \int f(x)dx \pm \int g(x)dx$

Integration formulas I

1. $\int kdx = kx + C, k \in \mathbb{R}$
2. $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \in \mathbb{R}, n \neq -1$

Integration formulas II

1. $\int sinxdx = -cosx + C$
2. $\int cosxdx = sinx + C$
3. $\int sec^2xdx = tanx + C$
4. $\int csc^2xdx = -cotx + C$
5. $\int secxtanxdx = secx + C$
6. $\int cscxcotx = -cscx + C$

Integration formulas III

1. $\int e^x dx = e^x + C$
2. $\int \frac{1}{x}dx = ln|x| + C$
3. $\int \frac{1}{\sqrt{1-x^2}} = sin^{-1}x + C$
4. $\int \frac{1}{1+x^2} = tan^{-1}x + C$
5. $\int \frac{1}{x\sqrt{x^2-1}} = sec^{-1} + C$

Substitution rule

Chain rule for derivatives

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$$

If follows that

$$\int f'(g(x))g'(x)dx = f(g(x)) + C$$

Example Evaluate $\int 2xcosx^2dx$.

Preliminary work. By intuition, we can get $f(x) = sinx$ and $g(x) = x^2$

$$\int 2xcosx^2dx = f(g(x)) = sinx^2$$

Solution. Suppose that $f'(x) = \frac{dy}{dx}$

$$dy = f'(x)dx$$

Let $u = g(x)$, then $g'(x) = \frac{du}{dx}$

$$du = g'(x)dx$$

Let $u = x^2$

$$du = 2xdx$$

$$\int 2xcosx^2dx = \int cosudu$$
$$= sinu + C$$
$$= sinx^2 + C$$

The area problem

The definite Integrals

The Fundamental Theorem of Calculus

Proof of Fundamental Theorem of Calculus

Module 2: Application I

Areas between curves

Volumes and volumes of revolution using disks and washers

Volumes of solids of revolution using cylindrical shells

Module 3: Techniques of integration

Integration by parts

Trigonometric integrals

Trigonometric Substitution

Partial fractions

Module 4: Applications II

Arc length

Variable-separable differential equations and models for population growth