

Mathematical Analysis IB

Download the PDF copy of the notes here

Download the formula sheet here

Review on differentiation

Differentiability

Let f be a function on some open interval I containing x . The derivative of f at x , denoted by $f'(x)$, is

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Differentiation rules

1. $\frac{d}{dx}(cf(x)) = cf'(x)$
2. $\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$
3. $\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + g(x)f'(x)$
4. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$
5. $\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$

Differentiation formulas I

1. $\frac{d}{dx}(c) = 0, c \in \mathbb{R}$
2. $\frac{d}{dx}(x^r) = rx^{r-1}, r \in \mathbb{R}$
3. $\frac{d}{dx}(\sin x) = \cos x$
4. $\frac{d}{dx}(\cos x) = -\sin x$
5. $\frac{d}{dx}(\tan x) = \sec^2 x$
6. $\frac{d}{dx}(\cot x) = -\csc^2 x$
7. $\frac{d}{dx}(\sec x) = \sec x \tan x$
8. $\frac{d}{dx}(\csc x) = -\csc x \cot x$

Differentiation formulas II

1. $\frac{d}{dx}(e^x) = e^x$
 2. $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$
 3. $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$
 4. $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$
 5. $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$
-

Mean value theorem

Let f be a function that is continuous on $[a, b]$ and is differentiable on (a, b) . Then there is a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Consequences of MVT

Zero derivative

If $f'(x) = 0 \forall x$ in interval I , then $f(x) = c \forall x \in I$ for some constant C .

Equal derivatives

If $f'(x) - g'(x) = 0 \forall x$ in an interval I , then $f(x) = g(x) + C$ for some constant C .

Example

Let $f(x) = \cos^{-1}x$ and $g(x) = -\sin^{-1}x$.

This implies that $x \in [-1, 1]$ and $f(x), g(x) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$f'(x) = -\frac{1}{\sqrt{x^2+1}}$$

$$g'(x) = -\frac{1}{\sqrt{x^2+1}}$$

Since $f'(x) - g'(x) = 0$ for $x \in [-1, 1]$, then $f(x) - g(x) = C$ for some constant C by a corollary.

$$\begin{aligned}\cos^{-1}x - (-\sin^{-1}x) &= C \\ \cos^{-1}x + \sin^{-1}x &= C\end{aligned}$$

Substituting $x \in [-1, 1]$, in this case, let's use $x = 0$,

$$\cos^{-1}(0) + \sin^{-1}(0) = C$$

$$0 + \frac{\pi}{2} = C$$

$$C = \frac{\pi}{2}$$

$$\therefore \forall x \in [-1, 1], f(x) - g(x) = \frac{\pi}{2}$$

Differentials

$$f'(x) = \frac{dy}{dx}$$

$$f'(x)dx = dy$$

Indefinite and definite integrals

Indefinite integral

The main interpretation of derivative is the slope of a tangent line of a curve.

Example

At any point (x, y) on a particular curve $y = F(x)$, the tangent line has a slope equal to $4x - 5$. If the curve contains the point $(3, 7)$, find $F(x)$.

Solution. Since the slope is equal to $4x - 5$ for any point (x, y) , then the slope at $(3, 7)$ is $4(3) - 5 = 7$.

$4x - 5$ therefore represents the tangent slope for all values of x . So

$$F'(x) = 4x - 5$$

By intuition, we can conclude that $F(x) = 2x^2 - 5x$.

However given $F(x) = 2x^2 - 5x + 1$, $F'(x)$ remains the same. And so is $F(x) = 2x^2 - 5x - 3$, $F(x) = 2x^2 - 5x + \pi$, and infinitely more functions. We can arbitrarily assign a constant k , so that $F(x) = 2x^2 - 5x + k$.

Substituting $(x, y) = (3, 7)$,

$$7 = 2(3)^2 - 5(3) + k$$

$$7 = 18 - 15 + k$$

$$k = 4$$

So $F(x) = 2x^2 - 5x + 4$.

Definition of an antiderivative

A function F is called an antiderivative of the function f on an interval I if $F'(x) = f(x) \forall x \in I$.

$F(x) = 2x^2 - 5x$ is a **possible** antiderivative of $f(x) = 4x - 5$. $F(x) = 2x^2 - 5x + 4$ is also a **possible** antiderivative of $f(x) = 4x - 5$.

Equal derivatives

If $F'(x) = G'(x) \forall x$ in an interval I , then $F(x) = G(x) + C \forall x \in I$ for some constant C .

Integration notation

The collection of all antiderivatives of f is denoted by

$$\int f(x)dx$$

which is read as “the integral of $f(x)dx$.”

This collection is also called the **indefinite integral** of f .

The reverse process of differentiation is called **antidifferentiation** or **integration**.

$$\int (4x - 5)dx = 2x^2 - 5x + C \text{ for some constant } C.$$

C is the constant of integration.

$$\int \sin x dx = -\cos x + C$$

Integration rules

1. $\int kf(x)dx = k \int f(x)dx, k \in \mathbb{R}$
2. $\int f(x) \pm g(x)dx = \int f(x)dx \pm \int g(x)dx$

Integration formulas I

1. $\int kdx = kx + C, k \in \mathbb{R}$
2. $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \in \mathbb{R}, n \neq -1$

Integration formulas II

1. $\int \sin x dx = -\cos x + C$
2. $\int \cos x dx = \sin x + C$
3. $\int \sec^2 x dx = \tan x + C$
4. $\int \csc^2 x dx = -\cot x + C$

$$5. \int \sec x \tan x dx = \sec x + C$$

$$6. \int \csc x \cot x dx = -\csc x + C$$

Integration formulas III

$$1. \int e^x dx = e^x + C$$

$$2. \int \frac{1}{x} dx = \ln |x| + C$$

$$3. \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$$

$$4. \int \frac{1}{1+x^2} dx = \tan^{-1} x + C$$

$$5. \int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + C$$

Substitution rule

Chain rule for derivatives

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$$

If follows that

$$\int f'(g(x))g'(x)dx = f(g(x)) + C$$

Example

Evaluate $\int 2x \cos x^2 dx$.

Preliminary work. By intuition, we can get $f(x) = \sin x$ and $g(x) = x^2$

$$\int 2x \cos x^2 dx = f(g(x)) = \sin x^2$$

Solution. Suppose that $f'(x) = \frac{dy}{dx}$

$$dy = f'(x)dx$$

Let $u = g(x)$, then $g'(x) = \frac{du}{dx}$

$$du = g'(x)dx$$

Let $u = x^2$

$$\begin{aligned}
du &= 2x dx \\
\int 2x \cos x^2 dx &= \int \cos u du \\
&= \sin u + C \\
&= \sin x^2 + C
\end{aligned}$$

Definition of the substitution rule

If $u = g(x)$ is a differentiable function whose range is interval I and f is continuous on I , then

$$\int f'(g(x))g'(x)dx = \int f(u)du$$

Definite integrals

The area problem

Let f be a continuous nonnegative function on $[a, b]$. Find the area of the region bounded by the curve $y = f(x)$, the lines $x = a$, $x = b$, and the x -axis.

The area is often coined the **region under the curve**, which generally means the area in between the curve and the x -axis

Example

Consider $f(x) = x^2 + 1$ on $[0, 2]$.

Solution. Let A be the area under the curve

Using right endpoints (5 rectangles)

$$\Delta x = \frac{2 - 0}{5} = \frac{2}{5} = 0.4$$

Rectangle 1: $(\Delta x)(f(0.4)) = (0.4)(1.16)$

Rectangle 2: $(\Delta x)(f(0.8)) = (0.4)(1.64)$

Rectangle 3: $(\Delta x)(f(1.2)) = (0.4)(2.44)$

Rectangle 4: $(\Delta x)(f(1.6)) = (0.4)(3.56)$

Rectangle 5: $(\Delta x)(f(2.0)) = (0.4)(5)$

$$A_5^+ = (0.4)(1.16) + (0.4)(1.64) + (0.4)(2.44) + (0.4)(3.56) + (0.4)(5) = 5.52$$

A_5^+ is an overestimation of A .

Using left endpoints (5 rectangles):

Rectangle 1: $(\Delta x)(f(0)) = (0.4)(1)$

Rectangle 2: $(\Delta x)(f(0.4)) = (0.4)(1.16)$

Rectangle 3: $(\Delta x)(f(0.8)) = (0.4)(1.64)$

Rectangle 4: $(\Delta x)(f(1.2)) = (0.4)(2.44)$

Rectangle 5: $(\Delta x)(f(1.6)) = (0.4)(3.56)$

$$A_5^- = (0.4)(1) + (0.4)(1.16) + (0.4)(1.64) + (0.4)(2.44) + (0.4)(3.56) + = 3.92$$

A_5^- is an underestimation of A .

We can increase the number of rectangles and compute the area A more **accurately** by computing the area as the number of rectangles approach infinity.

Let the number of rectangles be n

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

Let x_0 be the first point: $x_0 = 0$

$$\begin{array}{lll} x_1 = \frac{2}{n} & x_2 = \frac{4}{n} & x_3 = \frac{6}{n} \\ x_4 = \frac{8}{n} & x_5 = \frac{10}{n} & x_6 = \frac{12}{n} \\ x_7 = \frac{14}{n} & \dots & x_i = \frac{2i}{n} \end{array}$$

$$\begin{aligned} A_n &= R_1 + R_2 + R_3 + R_4 + \dots + R_n \\ &= \sum_{i=1}^n \Delta x(f(x_i)) \\ &= \sum_{i=1}^n \frac{2}{n} \left[\left(\frac{2i}{n} \right)^2 + 1 \right] \\ &= \frac{2}{n} \sum_{i=1}^n \left(\frac{4i^2}{n^2} + 1 \right) \\ &= \frac{2}{n} \left[\sum_{i=1}^n \left(\frac{4i^2}{n^2} \right) + \sum_{i=1}^n (1) \right] \\ &= \frac{2}{n} \left[\frac{4}{n^2} \sum_{i=1}^n (i^2) + \sum_{i=1}^n (1) \right] \\ &= \frac{2}{n} \left[\frac{4}{n^2} \left(\frac{(n)(n+1)(2n+1)}{6} \right) + n \right] \\ &= \frac{8}{n^3} \frac{(n)(n+1)(2n+1)}{6} + 2 \\ A_n &= \frac{4}{3} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) + 2 \end{aligned}$$

$$\begin{aligned}
A &= \lim_{n \rightarrow \infty} A_n \\
&= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{2}{n} \left[\left(\frac{2i}{n} \right)^2 + 1 \right] \\
&= \lim_{n \rightarrow \infty} \left[\frac{4}{3} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) + 2 \right] \\
&= \frac{4}{3} (1)(2) + 2 \\
A &= \frac{14}{3}
\end{aligned}$$

Riemann sum

Let f be a function defined on $[a, b]$.

Divide $[a, b]$ into n subintervals, each with width

$$\Delta x = \frac{b - a}{n}$$

Let $x_0 = a, x_1, x_2, \dots, x_n = b$,

For each subinterval $[x_{i-1}, x_i]$, choose a sample point x_i^*

Compute the sum

$$A_n = \sum_{i=1}^n f(x_i^*) \Delta x$$

This is also called the **Riemann sum**.

$$A_n = \sum_{i=1}^n f(x_i^*) \Delta x$$

Definite integral and integrability

The definite integral of f from a to b is

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$$

provided that such limit exists.

We say that f is integrable on $[a, b]$

Remarks on the definite integral

1. If a function is continuous on $[a, b]$, it is integrable on $[a, b]$.
2. If f is a nonnegative continuous function on $[a, b]$, then $\int_a^b f(x) dx$ is the area under the curve $y = f(x)$ from $x = a$ and $x = b$

$$3. \int_a^b f(x)dx = \int_a^b f(y)dy$$

Conventions on the definite integral

$$1. \int_b^a f(x)dx = - \int_a^b f(x)dx$$

$$2. \int_a^a f(x)dx = 0$$

Properties of the definite integral

$$1. \int_a^b cf(x)dx = c \int_a^b f(x)dx$$

$$2. \int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x) \pm \int_a^b g(x)$$

$$3. \int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$$

$$4. \text{ If } f(x) \geq 0 \ \forall x \in [a, b], \text{ then } \int_a^b f(x)dx \geq 0$$

$$5. \text{ If } f(x) \geq g(x) \ \forall x \in [a, b], \text{ then } \int_a^b f(x)dx \geq \int_a^b g(x)dx$$

$$6. \text{ If } m \leq f(x) \leq M \ \forall x \in [a, b], \text{ then } m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

The Fundamental Theorem of Calculus

Mean Value Theorem for integrals

Proof

Since f is continuous on $[a, b]$, then f is integrable on $[a, b]$ — i.e. $\int_a^b f(x)dx$ has a value.

Since f is continuous on $[a, b]$, by the **Extreme Value Theorem**, $\exists m, M \in \mathbb{R}$ such that $f(x_m) = m, f(x_M) = M, m \leq f(x) \leq M \ \forall x \in [a, b]$ and for some $x_m, x_M \in [a, b]$.

By Property 6 of the definite integral, $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

$$\begin{aligned} m &\leq \frac{\int_a^b f(x)dx}{b-a} \leq M \\ f(x_m) &\leq \frac{\int_a^b f(x)dx}{b-a} \leq f(x_M) \end{aligned}$$

By the IVT, $\exists c \in [a, b]$ such that

$$\frac{\int_a^b f(x)dx}{b-a} = f(c)$$

$$\int_a^b f(x)dx = f(c)(b-a)$$

If f is continuous on $[a, b]$, $\exists c \in [a, b]$ such that

$$\int_b^a f(x)dx = f(c)(b-a)$$

Average value of a function

Proof

Given a function continuous on $[a, b]$, we can get the average value of the function at $[a, b]$ by dividing the curve into n equal-width rectangles, getting the value of each sample points, and dividing by n .

$$\text{Average area} = \frac{\sum_{i=1}^n f(x_i^*) \Delta x}{n}$$

$$\text{But then, } \Delta x = \frac{b-a}{n} \implies n = \frac{b-a}{\Delta x}$$

$$\begin{aligned} \frac{\sum_{i=1}^n f(x_i^*)}{n} &= \frac{\sum_{i=1}^n f(x_i^*)}{\frac{b-a}{\Delta x}} \\ &= \frac{1}{b-a} \sum_{i=1}^n f(x_i^*) \Delta x \end{aligned}$$

We want to make n larger in order to make the average more accurate.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n f(x_i^*)}{n} &= \lim_{n \rightarrow \infty} \frac{1}{b-a} \sum_{i=1}^n f(x_i^*) \Delta x \\ &= \frac{1}{b-a} \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x \\ &= \frac{1}{b-a} \int_a^b f(x)dx \end{aligned}$$

Therefore, given function f that is continuous on $[a, b]$, there exists $c \in [a, b]$ such that

$$f_{avg} = f(c)$$

Let f be a continuous on $[a, b]$. The average value of f at $[a, b]$, denoted by f_{avg} is

$$f_{avg} = \frac{\int_a^b f(x)dx}{b-a}$$

First part of the Fundamental Theorem of Calculus

Let $y = f(t)$ that is continuous on $[a, b]$.

If $x \in [a, b]$, then the function is also continuous on $[a, b] \implies$ the function is also continuous on $[a, x]$.

$$\begin{aligned}F(x) &= \int_a^x f(t)dt \\F(a) &= \int_a^a f(t)dt = 0 \\F(b) &= \int_a^b f(t)dt\end{aligned}$$

Let f be continuous on $[a, b]$. If F is the function defined by

$$F(x) = \int_a^x f(t)dt$$

then $F'(x) = f(x) \forall x \in [a, b]$.

Proof

Let $x, x+h \in [a, b], h \neq 0$.

$$F(x+h) - F(x) = \int_a^{x+h} f(t)dt - \int_a^x f(t)dt$$

By the Property 3 of definite integrals,

$$\begin{aligned}\int_a^{x+h} f(t)dt - \int_a^x f(t)dt &= \int_a^x f(t)dt + \int_x^{x+h} f(t)dt - \int_a^x f(t)dt \\&= \int_x^{x+h} f(t)dt\end{aligned}$$

By the Mean Value Theorem for integrals, $\exists c \in [x, x+h]$ such that

$$\begin{aligned}\int_x^{x+h} f(t)dt &= f(c)(x+h-x) \\&= hf(c) \\\implies F(x+h) - F(x) &= hf(c) \\\frac{F(x+h) - F(x)}{h} &= f(c) \\\lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} &= \lim_{h \rightarrow 0} f(c)\end{aligned}$$

Note that $\lim_{h \rightarrow 0} x = x$ and $\lim_{h \rightarrow 0} (x+h) = x \implies \lim_{h \rightarrow 0} c = x$ by Squeeze Theorem.

Since f is continuous at x ,

$$\lim_{h \rightarrow 0} f(c) = f(x)$$

$$\begin{aligned} \implies F'(x) &= f(x) \quad \forall x \in [a, b] \\ \frac{d}{dx} \int_a^x f(t) dt &= f(x) \end{aligned}$$

Second part of the Fundamental Theorem of Calculus

Let's bring back $f(x) = x^2 + 1$ on $[0, 2]$.

f is continuous on $[0, 2] \implies f$ is integrable on $[0, 2]$.

$$\implies \int_0^2 (x^2 + 1) dx = \frac{14}{3}$$

Let $F(x) = \frac{x^3}{3} + x - 1$.

$$\begin{aligned} F(2) &= \frac{2^3}{3} + 2 - 1 = \frac{8}{3} + 1 = \frac{11}{3} \\ F(0) &= \frac{0^3}{3} + 0 - 1 = 0 - 1 = -1 \\ F(2) - F(0) &= \frac{11}{3} - (-1) = \frac{14}{3} \\ \int_0^2 (x^2 + 1) dx &= F(2) - F(0) \end{aligned}$$

Observe that $F'(x) = x^2 + 1 \implies F(x)$ is the an antiderivative of $x^2 + 1$.

If a function f is continuous on $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a)$$

where F is any antiderivative of f on $[a, b]$.

The following notations for $F(b) - F(a)$ are very useful in evaluating definite integrals:

$$F(x) \Big|_a^b \text{ or } F(x) \Big|_a^b$$

Proof

By FTC - Part 1, the function

$$\int_a^x f(t) dt$$

is an antiderivative of f on $[a, b]$.

By the Equal Derivatives Theorem,

$$\int_a^x f(t) dt = F(x) + C$$

where F is any antiderivative of f .

$$\begin{aligned}x = b, \int_a^b f(t)dt &= F(b) + C \\x = a, \int_a^a f(t)dt &= F(a) + C = 0 \\ \int_a^b f(t)dt - \int_a^a f(t)dt &= [F(b) + C] - [F(a) + C] \\ \int_a^b f(t)dt &= F(b) - F(a)\end{aligned}$$

Application I

Areas between curves

Example 1

Find the area of the region under the curve $y = x^2 - 1$ from $x = -1$ to $x = 2$.

Solution. Area is simply not $\int_{-1}^2 (x^2 - 1)dx$ because $\int_{-1}^1 (x^2 - 1)dx$ is negative and cancels the positive area.

Therefore, we get $\int_{-1}^1 -(x^2 - 1)dx$ to get the area of the curve between -1 and 1.

$$\begin{aligned}A &= \int_{-1}^1 -(x^2 - 1)dx + \int_1^2 (x^2 - 1)dx \\ &= \left(-\frac{x^3}{3} + x\right)\Big|_{-1}^1 + \left(\frac{x^3}{3} - x\right)\Big|_1^2 \\ &= \left(\frac{1^3}{3} + 1\right) - \left[\frac{(-1)^3}{3} + (-1)\right] + \left(\frac{2^3}{3} - 2\right) - \left(\frac{1^3}{3} - 1\right) \\ &= \frac{2}{3} + \frac{2}{3} + \frac{8}{3} - 2 + \frac{2}{3} \\ A &= \frac{8}{3}\end{aligned}$$

Example 2

Find the area of the region bounded by the curves of $y = x^2$ and $y = 4x - x^2$.

Solution. Note that both curves intersect at $(0, 0)$ and $(2, 4)$.

When we use Riemann sum, we only get the rectangles in between the region bounded by the area by subtracting the upper function ($y = 4x - x^2$) to the lower function ($y = x^2$)

$$\implies A_n = \sum [(4x - x^2) - x^2] \Delta x$$

$$\begin{aligned}
A &= \int_0^2 [(4x - x^2) - x^2] dx \\
&= \int_0^2 (4x - 2x^2) dx \\
&= \left(2x^2 - \frac{2x^3}{3} \right) \Big|_0^2 \\
&= \left[2(2)^2 - \frac{2(2)^3}{3} \right] - \left[2(0)^2 - \frac{2(0)^3}{3} \right] \\
&= \left[8 - \frac{16}{3} \right] - 0 \\
A &= \frac{8}{3}
\end{aligned}$$

Example 3

Find the area of the region bounded by the curve $y = \sqrt{x}$, the line $x + 2y = 3$, and the x -axis.

Solution. The graphs intersect at $(0, 0)$, $(1, 1)$, and $(3, 0)$.

$$x + 2y = 3 \implies y = -\frac{1}{2}x + \frac{3}{2}$$

$$\begin{aligned}
A &= \int_0^1 (\sqrt{x}) dx + \int_1^3 \left(-\frac{1}{2}(3 - x) \right) dx \\
&= \left(\frac{2x^{\frac{3}{2}}}{3} \right) \Big|_0^1 + \left(\frac{1}{2} \left(3x - \frac{x^2}{2} \right) \right) \Big|_1^3 \\
&= \frac{2(1)^{\frac{3}{2}}}{3} - \frac{2(0)^{\frac{3}{2}}}{3} + \frac{1}{2} \left(3(3) - \frac{3^2}{2} \right) - \frac{1}{2} \left(3(1) - \frac{1^2}{2} \right) \\
&= \frac{2}{3} + \frac{9}{4} - \frac{5}{4} \\
&= \frac{8 - 27 + 15}{12} \\
&= \frac{20}{12} \\
A &= \frac{5}{3}
\end{aligned}$$

Volumes and volumes of revolution using disks and washers

Volume of a right cylinder

$$V = ah$$

$$V_n = \sum_{i=1}^n A(x) \Delta x$$

Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x through x and perpendicular to the x -axis is $A(x)$, where A is a continuous function on $[a, b]$, then the volume V of S is

$$V = \lim_{n \rightarrow \infty} \sum_{i=1}^n A(x_i^*) \Delta x = \int_b^a A(x) dx$$

Example 1

Let us find the volume of a sphere of radius r .

Solution.

radius of the cross-section circle at $x = \sqrt{r^2 - x^2}$

$$\begin{aligned} A(x) &= \pi(\sqrt{r^2 - x^2})^2 \\ &= \pi(r^2 - x^2) \end{aligned}$$

$$\begin{aligned} V_{\text{sphere}} &= \int_{-r}^r A(x) dx \\ &= \int_{-r}^r \pi(r^2 - x^2) dx \\ &= \pi \left(r^2 x - \frac{x^3}{3} \right) \Big|_{-r}^r \\ &= \pi \left[r^2(r) - \frac{r^3}{3} \right] - \pi \left[r^2(-r) - \frac{(-r)^3}{3} \right] \\ V_{\text{sphere}} &= \frac{4}{3} \pi r^3 \end{aligned}$$

Example 2

The base of a solid is the region bounded by $y = x^2$ and $y = 4$. Its parallel cross-sections perpendicular to the base and the y -axis are squares. Find the volume of the solid.

Solution. side of the cross-section at $y = 2\sqrt{y}$

$$A(y) = (2\sqrt{y})^2 = 4y$$

$$\begin{aligned} V &= \int_0^4 A(y) dy \\ &= \int_0^4 4y dy \\ &= 2y^2 \Big|_0^4 \\ &= 2(4)^2 - 2(0)^2 \\ V &= 32 \end{aligned}$$

Volume of solids of revolution

If we revolve a region about a line, we obtain a **solid of revolution**.

Example 1

Consider the region under the curve $y = x^2 + 1$ from $x = -1$ to $x = 2$. We revolve this region about the x -axis.

Solution. radius of the cross-section at $x = f(x)$

$$A(x) = \pi[f(x)]^2$$

$$\begin{aligned} V &= \int_{-1}^2 \pi(x^2 + 1)^2 dx \\ &= \int_{-1}^2 \pi(x^4 + 2x^2 + 1) dx \\ &= \pi \left(\frac{x^5}{5} + \frac{2x^3}{3} + x \right) \Big|_{-1}^2 \\ &= \pi \left[\frac{2^5}{5} + \frac{2(2)^3}{3} + 2 \right] - \pi \left[\frac{(-1)^5}{5} + \frac{2(-1)^3}{3} + (-1) \right] \\ V &= \frac{78\pi}{5} \end{aligned}$$

The cross-section of a solid of revolution is always a circle.

Example 2

A solid is obtained by revolving about the x -axis the region bounded by $x = y^2$ and $2y = x$. Find the volume of the solid.

Solution.

$$\begin{aligned} V &= \int_0^4 \pi(\sqrt{x})^2 dx - \int_0^4 \pi\left(\frac{x}{2}\right)^2 dx \\ &= \int_0^4 \pi(x) dx - \int_0^4 \pi \frac{x^2}{4} dx \\ V &= \frac{8\pi}{3} \end{aligned}$$

Example 3

A solid is obtained by revolving about the y -axis the region bounded by $2x = y^2$, $y = 4$, and the y -axis. Find the volume of the solid.

Solution.

$$\begin{aligned}
V &= \int_0^4 \pi \left(\frac{y^2}{2}\right)^2 dy \\
&= \int_0^4 \pi \left(\frac{y^4}{4}\right) dy \\
&= \frac{\pi y^5}{20} \Big|_0^4 \\
V &= \frac{256\pi}{5}
\end{aligned}$$

Volumes by cylindrical shells

There are times that disks-and-washers technique is not the best way to solve a volume problem – e.g. $y = 4x - x^2$ rotated about the y -axis.

Volume of a cylindrical shell

Let r_1 be the inner radius of the cylinder, r_2 be the outer (and larger) radius of the cylinder. r be the average of both

$$\begin{aligned}
\Delta r &= r_2 - r_1 \\
r &= \frac{r_2 + r_1}{2}
\end{aligned}$$

$$\begin{aligned}
V_{\text{cylindrical shell}} &= \pi r_2^2 h - \pi r_1^2 h \\
&= \pi (r_2^2 - r_1^2) h \\
&= \pi (r_2 + r_1) h (r_2 - r_1) \\
&= 2\pi \left(\frac{r_2 + r_1}{2} \right) h \Delta r \\
V_{\text{cylindrical shell}} &= 2\pi r h \Delta r
\end{aligned}$$

Given a curve $y = f(x)$ in $[a, b]$ rotated about the y -axis, the Riemann sum of the volume is

$$V = \sum 2\pi r h \Delta r$$

In this context, $r = x$ (horizontal distance), $h = f(x)$ (vertical distance), and $\Delta r = \Delta x$

$$\begin{aligned}
V &= \lim_{n \rightarrow \infty} \sum_{i=1}^n 2\pi x_i^* f(x_i^*) \Delta x \\
&= \int_a^b 2\pi x f(x) dx
\end{aligned}$$

The volume of a solid obtained by rotating about the y -axis the region under the curve $y = f(x)$ (continuous and nonnegative) from $x = a$ (nonnegative) to $x = b$ is

$$V = \lim_{n \rightarrow \infty} \sum_{i=1}^n 2\pi x_i^* f(x_i^*) \Delta x = \int_a^b 2\pi x f(x) dx$$

Example 1

A solid is obtained by revolving about the y -axis the region bounded by $y = 4x - x^2$ and the x -axis. Use cylindrical shells to find the volume of the solid.

Solution.

$$\begin{aligned}
 V &= \int_0^4 2\pi x(4x - x^2)dx \\
 &= \int_0^4 2\pi(4x^2 - x^3)dx \\
 &= 2\pi \left(\frac{4}{3}x^3 - \frac{x^4}{4} \right) \Big|_0^4 \\
 &= 2\pi \left[\frac{4}{3}(4)^3 - \frac{4^4}{4} \right] - 0 \\
 &= \frac{128\pi}{3}
 \end{aligned}$$

Example 2

A solid is obtained by revolving about the x -axis the region bounded by $y = \sqrt{x}$, $y = 2 - x$, and the x -axis. Find the volume of the solid.

Solution.

Disks:

$y = \sqrt{x}$ and $y = 2 - x$ intersect at $(1, 1)$

$\forall x \in [0, 1], \sqrt{x} \leq 2 - x \implies$ we use \sqrt{x} at this interval

$\forall x \in [1, 2], 2 - x \leq \sqrt{x} \implies$ we use $2 - x$ at this interval

$$\begin{aligned}
 V &= \int_0^1 \pi(\sqrt{x})^2 dx + \int_1^2 \pi(2 - x)^2 dx \\
 &= \int_0^1 \pi x dx + \int_1^2 \pi(x^2 - 4x + 4) dx
 \end{aligned}$$

(This exercise is left to the reader lol don't wanna solve this myself)

cylindrical shell:

$y = \sqrt{x}$ and $y = 2 - x$ intersect at $(1, 1)$.

$$\begin{aligned}
 y = \sqrt{x} &\implies x = y^2 \\
 y = 2 - x &\implies x = 2 - y
 \end{aligned}$$

$$V = \int_0^1 2\pi y(2 - y^2 - y) dy$$

(Just solve this yourselves)

Example 3

A solid is obtained by the revolving about the line $x = -2$ the region bounded by $y = x^3$, $y = 8$, and the y -axis. Find the volume of the solid.

Solution.

$$\text{disks: } V = \int_0^8 \pi [(\sqrt[3]{y} + 2)^2 - 2^2] dy = \frac{336\pi}{5}$$

$$\text{cylindrical shells: } V = \int_0^2 2\pi(x+2)(8-x^3)dx = \frac{336\pi}{5}$$

Techniques of integration

Integration by parts

Product rule and the differentials

Recalling the product rule in derivatives,

$$\begin{aligned}\frac{d}{dx}(f(x)g(x)) &= f(x)g'(x) + g(x)f'(x) \\ d(f(x)g(x)) &= f(x)g'(x)dx + g(x)f'(x)dx\end{aligned}$$

Let $u = f(x)$, $v = g(x)$

$du = f'(x)dx$, $dv = g'(x)dx$

$$\begin{aligned}d(f(x)g(x)) &= f(x)g'(x)dx + g(x)f'(x)dx \implies d(uv) = u dv + v du \\ u dv &= d(uv) - v du \\ \int u dv &= \int d(uv) - \int v du \\ \int u dv &= uv - \int v du\end{aligned}$$

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

Letting $u = f(x)$, $v = g(x) \implies du = f'(x)dx$, $dv = g'(x)dx$,

$$\int u dv = uv - \int v du$$

This is also called the **integration-by-parts formula**.

Integration by parts and definite integrals

Combining the integration-by-parts formula and FTC2,

$$\int_a^b f(x)g'(x)dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)dx$$

Example 1

Evaluate $\int \ln x dx$

Preliminary work.

$$\int \ln x dx \implies x > 0$$

Let $u = 1$, $dv = \ln x dx$

$$du = 0, v = \int \ln x dx$$

Note that this solution is not correct because we just went back to our original statement, leading us nowhere.

Solution. Let $u = \ln x$, $dv = dx$

$$du = \frac{1}{x} dx, v = \int dx = x + C$$

Following the integration-by-parts formula,

$$\begin{aligned} \int \ln x dx &= (\ln x)(x + c) - \int (x + C) \frac{1}{x} dx \\ &= x \ln x + C \ln x - \int \left(1 + \frac{C}{x}\right) dx \\ &= x \ln x + C \ln x - (x + C \ln |x| + C_1) \\ &= x \ln x + C \ln x - x - C \ln |x| - C_1; \text{ note that } x > 0, \text{ thus } \ln |x| = \ln x \\ &= x \ln x - x + C \end{aligned}$$

Note that the contribution of $+C$ in v just cancels at $uv - \int v du$, so it is **not necessary** to put $+C$ when using integration by parts.

Example 2

Evaluate $\int t^2 \sin \beta t dt$

Solution. Let $u = t^2$, $dv = \sin \beta t dt$

$$du = 2t dt, v = \int \sin \beta t dt = -\frac{\cos \beta t}{\beta}$$

$$\int t^2 \sin \beta t dt = -\frac{t^2 \cos \beta t}{\beta} - \int -\frac{2t \cos \beta t}{\beta} dt = -\frac{t^2 \cos \beta t}{\beta} - \int \frac{2}{\beta} t \cos \beta t dt$$

Let $u = t$, $dv = \cos \beta t dt$

$$du = dt, v = \frac{\sin \beta t}{\beta}$$

$$\begin{aligned} -\frac{t^2 \cos \beta t}{\beta} - \int \frac{2}{\beta} t \cos \beta t dt &= -\frac{t^2 \cos \beta t}{\beta} - \frac{2}{\beta} \left(\frac{t \sin \beta t}{\beta} - \int \frac{\sin \beta t}{\beta} dt \right) \\ &= -\frac{t^2 \cos \beta t}{\beta} - \frac{2t \sin \beta t}{\beta^2} - \frac{2}{\beta} \left(\int \frac{\sin \beta t}{\beta} dt \right) \\ &= -\frac{t^2 \cos \beta t}{\beta} - \frac{2t \sin \beta t}{\beta^2} + \frac{2 \cos \beta t}{\beta^3} + C \end{aligned}$$

Example 3

Evaluate $\int e^x \sin \pi x dx$

Solution. Let $u = \sin \pi x$, $dv = e^x dx$

$$du = \pi \cos \pi x dx, v = e^x$$

By integration by parts,

$$\int e^x \sin \pi x dx = e^x \sin \pi x - \pi \int e^x \cos \pi x dx$$

Let $u = \cos \pi x$, $dv = e^x dx$

$$du = -\pi \sin \pi x, v = e^x$$

By integration by parts,

$$\begin{aligned} \int e^x \sin \pi x dx &= e^x \sin \pi x - \pi \int e^x \cos \pi x dx = e^x \sin \pi x - \pi \left(e^x \cos \pi x - \pi \int -e^x \sin \pi x dx \right) \\ &= e^x \sin \pi x - \pi e^x \cos \pi x - \pi^2 \int e^x \sin \pi x dx \\ \int e^x \sin \pi x dx + \pi^2 \int e^x \sin \pi x dx &= e^x \sin \pi x - \pi e^x \cos \pi x \\ (1 + \pi^2) \int e^x \sin \pi x dx &= e^x \sin \pi x - \pi e^x \cos \pi x \\ \int e^x \sin \pi x dx &= \frac{e^x \sin \pi x - \pi e^x \cos \pi x}{1 + \pi^2} + C \end{aligned}$$

Example 4 (definite integral)

Evaluate $\int_0^1 \tan^{-1} x dx$.

Solution. We shall first get the antiderivative of the function before evaluating the definite integral.

Let $u = \arctan x$, $dv = dx$

$$du = \frac{1}{x^2 + 1} dx, v = x$$

By integration by parts,

$$\int \tan^{-1} x dx = x \arctan x - \int \frac{x}{x^2 + 1} dx$$

Let $u = x^2 + 1 \implies du = 2x dx$

Using the substitution rule,

$$\begin{aligned}
x \arctan x - \int \frac{x}{x^2 + 1} dx &= x \arctan x - \frac{1}{2} \int \frac{1}{u} du \\
&= x \arctan x - \frac{\ln |u|}{2} \\
&= x \arctan x - \frac{\ln |x^2 + 1|}{2}, \text{ note that } \forall x \in \mathbb{R}, x^2 + 1 > 0 \\
&= x \arctan x - \frac{\ln(x^2 + 1)}{2}
\end{aligned}$$

By FTC2,

$$\begin{aligned}
\int_0^1 \tan^{-1} x dx &= \left(x \arctan x - \frac{\ln(x^2 + 1)}{2} \right) \Big|_0^1 \\
&= \left[\arctan 1 - \frac{\ln(1^2 + 1)}{2} \right] - \left[-\frac{\ln(0^2 + 1)}{2} \right] \\
&= \frac{\pi - 2 \ln(2)}{4}
\end{aligned}$$

Trigonometric integrals

Trigonometric identities

1. $\sin^2 x + \cos^2 x = 1$
 2. $\tan^2 x + 1 = \sec^2 x$
 3. $\cot^2 x + 1 = \csc^2 x$
 4. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
 5. $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
 6. $\sin A \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)]$
 7. $\sin A \sin B = \frac{1}{2}[\cos(A - B)] - \cos(A + B)]$
 8. $\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$
-

Integrals of trigonometric functions

$$1. \int \tan x dx = \ln |\sec x| + C$$

$$\begin{aligned}
\int \tan x dx &= \int \frac{\sin x}{\cos x} dx = -\ln |\cos x| + C \\
&= \ln |\sec x| + C
\end{aligned}$$

$$2. \int \sec x dx = \ln |\sec x + \tan x| + C$$

$$\begin{aligned}
\int \sec x dx &= \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} dx \\
&= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx \\
&= \ln |\sec x + \tan x| + C
\end{aligned}$$

$$3. \int \cot x dx = \ln |\sin x| + C$$

$$4. \int \csc x dx = \ln |\csc x - \cot x| + C$$

Example 1

Evaluate $\int \sqrt{\cos \theta} \sin^3 \theta d\theta$.

Solution.

$$\begin{aligned}
\int \sqrt{\cos \theta} \sin^3 \theta d\theta &= \int \sqrt{\cos \theta} \sin^2 \theta \sin \theta d\theta \\
&= \int \sqrt{\cos \theta} (1 - \cos^2 \theta) \sin \theta d\theta
\end{aligned}$$

Let $u = \sqrt{\cos \theta}$

$$\implies u^2 = \cos \theta$$

$$\implies 2u du = -\sin \theta d\theta$$

$$\implies -2u du = \sin \theta d\theta$$

$$\begin{aligned}
\int \sqrt{\cos \theta} (1 - \cos^2 \theta) \sin \theta d\theta &= \int u(1 - u^4)(-2u) du \\
&= \int (-2u^2 + 2u^6) du \\
&= -\frac{2}{3}u^3 + \frac{2}{7}u^7 + C \\
&= -\frac{2}{3}(\cos \theta)^{3/2} + \frac{2}{7}(\cos \theta)^{7/2} + C
\end{aligned}$$

Example 2

Evaluate $\int \tan^2 t \sec^4 t dt$.

Solution.

$$\begin{aligned}
\int \tan^2 t \sec^4 t dt &= \int \tan^2 t \sec^2 t \sec^2 t dt \\
&= \int \tan^2 t (1 + \tan^2 t) \sec^2 t dt
\end{aligned}$$

Let $u = \tan t \implies du = \sec^2 t dt$.

$$\begin{aligned}\int \tan^2 t (1 + \tan^2 t) \sec^2 t dt &= \int u^2 (1 + u^2) du \\ &= \int (u^2 + u^4) du \\ &= \frac{u^3}{3} + \frac{u^5}{5} \\ &= \frac{\tan^3 t}{3} + \frac{\tan^5 t}{5} + C\end{aligned}$$

Example 3 (definite integral)

Evaluate $\int_{\pi/4}^{\pi/2} \csc^5 x \cot^3 x dx$.

$$\begin{aligned}\int_{\pi/4}^{\pi/2} \csc^5 x \cot^3 x dx &= \int_{\pi/4}^{\pi/2} \csc^4 x \cot^2 x \csc x \cot x dx \\ &= \int_{\pi/4}^{\pi/2} \csc^4 x (\csc^2 x - 1) \csc x \cot x dx\end{aligned}$$

Let $u = \csc x \implies du = -\csc x \cot x dx \implies -du = \csc x \cot x dx$

$$x = \frac{\pi}{4} \implies u = \sqrt{2}$$

$$x = \frac{\pi}{2} \implies u = 1$$

$$\begin{aligned}\int_{\pi/4}^{\pi/2} \csc^4 x (\csc^2 x - 1) \csc x \cot x dx &= \int_{\sqrt{2}}^1 u^4 (u^2 - 1) (-du) \\ &= \int_{\sqrt{2}}^1 -u^6 + u^4 du \\ &= \int_1^{\sqrt{2}} u^6 - u^4 du \\ &= \left. \frac{u^7}{7} - \frac{u^5}{5} \right|_1^{\sqrt{2}} \\ &= \frac{8\sqrt{2}}{7} - \frac{4\sqrt{2}}{5} - \frac{1}{7} + \frac{1}{5} \\ &= \frac{40\sqrt{2} - 28\sqrt{2} - 5 + 7}{35} \\ &= \frac{12\sqrt{2} + 2}{35}\end{aligned}$$

Example 4 (definite integral)

Evaluate $\int_0^{\pi/2} \cos 5t \cos 10t dt$.

$$\begin{aligned}
\int_0^{\pi/2} \cos 5t \cos 10t dt &= \int_0^{\pi/2} \frac{1}{2} [\cos(5t - 10t) + \cos(5t + 10t)] dt \\
&= \int_0^{\pi/2} \frac{1}{2} [\cos(-5t) + \cos 15t] dt; \text{ note that } \cos(-5t) = \cos 5t \forall t \\
&= \int_0^{\pi/2} \frac{1}{2} \cos 5t + \frac{1}{2} \cos 15t \\
&= \frac{1}{10} \sin 5t + \frac{1}{30} \sin 15t \Big|_0^{\pi/2} \\
&= \frac{1}{10} \sin \frac{5\pi}{2} + \frac{1}{30} \sin \frac{15\pi}{2} - 0 \\
&= \frac{1}{15}
\end{aligned}$$

Example 5

Evaluate $\int \frac{\sin^2(1/t)}{t^2} dt$.

$$\text{Let } u = \frac{1}{t} \implies du = -\frac{1}{t^2} \implies -du = \frac{1}{t^2}$$

$$\int \frac{\sin^2(1/t)}{t^2} dt = \int -\sin^2 u du$$

Note that $\sin^2 u = \frac{1}{2}(1 - \cos 2u)$

$$\begin{aligned}
\int -\sin^2 u du &= \int -\frac{1}{2}(1 - \cos 2u) du \\
&= -\frac{1}{2} \left(u - \frac{\sin 2u}{2} \right) \\
&= -\frac{1}{2t} + \frac{\sin(2/t)}{4} + C
\end{aligned}$$

Trigonometric substitution

Circular functions

$$x^2 + y^2 = r^2$$

which proves that trigonometric functions are also circular functions

$$\sin \theta = \frac{y}{r} \implies y = r \sin \theta$$

$$\cos \theta = \frac{x}{r} \implies x = r \cos \theta$$

We can therefore rename (x, y) to $(r \cos \theta, r \sin \theta)$. It implies that for every (x, y) coordinates, there corresponds (r, θ) coordinates.

Recall

Evaluate $\int_{-2}^2 \sqrt{4-x^2} dx$.

Intuitive solution. Observe that the function is a semicircle with $r=2$, thus the integral looks for the area of the semicircle.

$$\int_{-2}^2 \sqrt{4-x^2} dx = \frac{1}{2} \pi 2^2 = 2\pi$$

Solution. Let $y = \sqrt{4-x^2}$.

$$y^2 = 4 - x^2$$

$$x^2 + y^2 = 4$$

$$r^2 = 4 \implies r = 2$$

$$(x, y) = (x, \sqrt{4-x^2}) = (2 \cos \theta, 2 \sin \theta)$$

$$x = 2 \cos \theta \implies dx = 2(-\sin \theta) d\theta$$

$$\sqrt{4-x^2} = 2 \sin \theta$$

$$\begin{aligned} \int_{-2}^2 \sqrt{4-x^2} dx &= \int_{\pi}^0 (2 \sin \theta)(-2 \sin \theta d\theta); \text{ note that we converted the } x\text{-value to } \theta\text{-value} \\ &= \int_{\pi}^0 -4 \sin^2 \theta d\theta \\ &= \int_0^{\pi} -4 \sin^2 \theta d\theta \\ &= \int_0^{\pi} \frac{4}{2} (1 - \cos 2\theta) d\theta \\ &= \int_0^{\pi} 2(1 - \cos 2\theta) d\theta \\ &= 2x - \frac{\sin 2\theta}{2} \Big|_0^{\pi} \\ &= 2\pi - 0 - 0 \\ &= 2\pi \end{aligned}$$

Trigonometric substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin \theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2 \theta = \cos^2 \theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 + \tan^2 \theta = \sec^2 \theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta, 0 \leq \theta \leq \frac{\pi}{2} \text{ or } \pi \leq \theta \leq \frac{3\pi}{2}$	$\sec^2 \theta - 1 = \tan^2 \theta$

Example 1

Evaluate $\int_0^1 \frac{dx}{(x^2+1)^2}$

$$\int_0^1 \frac{dx}{(x^2+1)^2} = \int_0^1 \frac{dx}{(\sqrt{x^2+1})^4}$$

Visualizing a right triangle with legs x and 1 ,

$$x = \tan \theta \implies dx = \sec^2 \theta d\theta$$

$$\cos \theta = \frac{1}{\sqrt{x^2+1}} \implies \sqrt{x^2+1} = \sec \theta$$

$$x = 0 \implies 0 = \tan \theta \implies \theta = 0$$

$$x = 1 \implies \theta = \frac{\pi}{4}$$

$$\begin{aligned} \int_0^1 \frac{dx}{(\sqrt{x^2+1})^4} &= \int_0^{\pi/4} \frac{\sec^2 \theta}{\sec^4 \theta} d\theta \\ &= \int_0^{\pi/4} \frac{1}{\sec^2 \theta} d\theta \\ &= \int_0^{\pi/4} \cos^2 \theta d\theta \\ &= \int_0^{\pi/4} \frac{1}{2}(1 + \cos 2\theta) d\theta \\ &= \frac{1}{2} \left(\theta + \frac{\sin 2\theta}{2} \right) \Big|_0^{\pi/4} \end{aligned}$$

(solution to be continued)

Example 2

Evaluate $\int \frac{\sqrt{x^2-1}}{x^4} dx$.

Solution.

By trigonometric substitution,

$$x = \sec \theta \implies dx = \sec \theta \tan \theta d\theta$$

$$\sqrt{x^2-1} = \tan \theta$$

$$\begin{aligned}
\int \frac{\sqrt{x^2-1}}{x^4} dx &= \int \frac{\tan \theta}{\sec^4 \theta} (\sec \theta \tan \theta d\theta) \\
&= \int \frac{\tan^2 \theta}{\sec^3 \theta} d\theta \\
&= \int \sin^2 \theta \cos \theta d\theta \\
&= \frac{\sin^3 \theta}{3} + C \\
&= \frac{1}{3} \left(\frac{\sqrt{x^2-1}}{x} \right)^3 + C \\
&= \frac{(x^2-1)^{3/2}}{3x^3} + C
\end{aligned}$$

Example 3

Evaluate $\int \frac{x^2}{(3+4x-4x^2)^{3/2}} dx$.

Solution.

$$\int \frac{x^2}{(3+4x-4x^2)^{3/2}} dx = \int \frac{x^2}{[4-(2x-1)^2]^{3/2}} dx$$

Let there be a triangle with legs $2x-1$ and $\sqrt{4-(2x-1)^2}$

$$2x-1 = 2 \sin \theta \implies x = \frac{1+2 \sin \theta}{2}$$

$$2dx = 2 \cos \theta d\theta$$

$$dx = \cos \theta d\theta$$

$$\sqrt{4-(2x-1)^2} = 2 \cos \theta$$

$$\begin{aligned}
\int \frac{x^2}{[4-(2x-1)^2]^{3/2}} dx &= \int \frac{\left(\frac{1+2 \sin \theta}{2}\right)^2}{2 \cos^3 \theta} \cos \theta d\theta \\
&= \int \frac{1+4 \sin \theta + 4 \sin^2 \theta}{32 \cos \theta} d\theta \\
&= \frac{1}{32} \int (\sec^2 \theta + 4 \tan \theta \sec \theta + 4 \tan^2 \theta) d\theta \\
&= \frac{1}{32} \tan \theta + \frac{1}{8} \sec \theta + \frac{1}{8} \tan \theta - \frac{1}{8} \theta + C \\
&= \frac{5}{32} \tan \theta + \frac{1}{8} \sec \theta - \frac{1}{8} \theta + C \\
&= \frac{5}{32} \frac{2x-1}{\sqrt{3+4x-4x^2}} + \frac{1}{8} \frac{2}{\sqrt{3+4x-4x^2}} - \frac{1}{8} \arcsin \left(\frac{2x-1}{2} \right) + C
\end{aligned}$$