

# Mathematical Analysis IB

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## Review on differentiation

### Differentiability

Let  $f$  be a function on some open interval  $I$  containing  $x$ . The derivative of  $f$  at  $x$ , denoted by  $f'(x)$ , is

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

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### Differentiation rules

1.  $\frac{d}{dx}(cf(x)) = cf'(x)$
2.  $\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$
3.  $\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + g(x)f'(x)$
4.  $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$
5.  $\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$

### Differentiation formulas I

1.  $\frac{d}{dx}(c) = 0, c \in \mathbb{R}$
2.  $\frac{d}{dx}(x^r) = rx^{r-1}, r \in \mathbb{R}$
3.  $\frac{d}{dx}(\sin x) = \cos x$
4.  $\frac{d}{dx}(\cos x) = -\sin x$
5.  $\frac{d}{dx}(\tan x) = \sec^2 x$
6.  $\frac{d}{dx}(\cot x) = -\csc^2 x$
7.  $\frac{d}{dx}(\sec x) = \sec x \tan x$
8.  $\frac{d}{dx}(\csc x) = -\csc x \cot x$

## Differentiation formulas II

1.  $\frac{d}{dx}(e^x) = e^x$
  2.  $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$
  3.  $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$
  4.  $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$
  5.  $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$
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## Mean value theorem

Let  $f$  be a function that is continuous on  $[a, b]$  and is differentiable on  $(a, b)$ . Then there is a number  $c \in (a, b)$  such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

## Consequences of MVT

### Zero derivative

If  $f'(x) = 0 \forall x$  in interval  $I$ , then  $f(x) = c \forall x \in I$  for some constant  $C$ .

### Equal derivatives

If  $f'(x) - g'(x) = 0 \forall x$  in an interval  $I$ , then  $f(x) = g(x) + C$  for some constant  $C$ .

### Example

Let  $f(x) = \cos^{-1}x$  and  $g(x) = -\sin^{-1}x$ .

This implies that  $x \in [-1, 1]$  and  $f(x), g(x) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$f'(x) = -\frac{1}{\sqrt{x^2+1}}$$

$$g'(x) = -\frac{1}{\sqrt{x^2+1}}$$

Since  $f'(x) - g'(x) = 0$  for  $x \in [-1, 1]$ , then  $f(x) - g(x) = C$  for some constant  $C$  by a corollary.

$$\begin{aligned}\cos^{-1}x - (-\sin^{-1}x) &= C \\ \cos^{-1}x + \sin^{-1}x &= C\end{aligned}$$

Substituting  $x \in [-1, 1]$ , in this case, let's use  $x = 0$ ,

$$\cos^{-1}(0) + \sin^{-1}(0) = C$$

$$0 + \frac{\pi}{2} = C$$

$$C = \frac{\pi}{2}$$

$$\therefore \forall x \in [-1, 1], f(x) - g(x) = \frac{\pi}{2}$$


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## Differentials

$$f'(x) = \frac{dy}{dx}$$

$$f'(x)dx = dy$$


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## Indefinite and definite integrals

### Indefinite integral

The main interpretation of derivative is the slope of a tangent line of a curve.

#### Example

At any point  $(x, y)$  on a particular curve  $y = F(x)$ , the tangent line has a slope equal to  $4x - 5$ . If the curve contains the point  $(3, 7)$ , find  $F(x)$ .

**Solution.** Since the slope is equal to  $4x - 5$  for any point  $(x, y)$ , then the slope at  $(3, 7)$  is  $4(3) - 5 = 7$ .

$4x - 5$  therefore represents the tangent slope for all values of  $x$ . So

$$F'(x) = 4x - 5$$

By intuition, we can conclude that  $F(x) = 2x^2 - 5x$ .

However given  $F(x) = 2x^2 - 5x + 1$ ,  $F'(x)$  remains the same. And so is  $F(x) = 2x^2 - 5x - 3$ ,  $F(x) = 2x^2 - 5x + \pi$ , and infinitely more functions. We can arbitrarily assign a constant  $k$ , so that  $F(x) = 2x^2 - 5x + k$ .

Substituting  $(x, y) = (3, 7)$ ,

$$7 = 2(3)^2 - 5(3) + k$$

$$7 = 18 - 15 + k$$

$$k = 4$$

So  $F(x) = 2x^2 - 5x + 4$ .

### Definition of an antiderivative

A function  $F$  is called an antiderivative of the function  $f$  on an interval  $I$  if  $F'(x) = f(x) \forall x \in I$ .

$F(x) = 2x^2 - 5x$  is a **possible** antiderivative of  $f(x) = 4x - 5$ .  $F(x) = 2x^2 - 5x + 4$  is also a **possible** antiderivative of  $f(x) = 4x - 5$ .

### Equal derivatives

If  $F'(x) = G'(x) \forall x$  in an interval  $I$ , then  $F(x) = G(x) + C \forall x \in I$  for some constant  $C$ .

### Integration notation

The collection of all antiderivatives of  $f$  is denoted by

$$\int f(x)dx$$

which is read as “the integral of  $f(x)dx$ .”

This collection is also called the **indefinite integral** of  $f$ .

The reverse process of differentiation is called **antidifferentiation** or **integration**.

$$\int (4x - 5)dx = 2x^2 - 5x + C \text{ for some constant } C.$$

$C$  is the constant of integration.

$$\int \sin x dx = -\cos x + C$$

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### Integration rules

1.  $\int kf(x)dx = k \int f(x)dx, k \in \mathbb{R}$
2.  $\int f(x) \pm g(x)dx = \int f(x)dx \pm \int g(x)dx$

### Integration formulas I

1.  $\int kdx = kx + C, k \in \mathbb{R}$
2.  $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \in \mathbb{R}, n \neq -1$

### Integration formulas II

1.  $\int \sin x dx = -\cos x + C$
2.  $\int \cos x dx = \sin x + C$
3.  $\int \sec^2 x dx = \tan x + C$
4.  $\int \csc^2 x dx = -\cot x + C$

$$5. \int \sec x \tan x dx = \sec x + C$$

$$6. \int \csc x \cot x dx = -\csc x + C$$

### Integration formulas III

$$1. \int e^x dx = e^x + C$$

$$2. \int \frac{1}{x} dx = \ln |x| + C$$

$$3. \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$$

$$4. \int \frac{1}{1+x^2} dx = \tan^{-1} x + C$$

$$5. \int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + C$$


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### Substitution rule

#### Chain rule for derivatives

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$$

If follows that

$$\int f'(g(x))g'(x)dx = f(g(x)) + C$$

#### Example

Evaluate  $\int 2x \cos x^2 dx$ .

**Preliminary work.** By intuition, we can get  $f(x) = \sin x$  and  $g(x) = x^2$

$$\int 2x \cos x^2 dx = f(g(x)) = \sin x^2$$

**Solution.** Suppose that  $f'(x) = \frac{dy}{dx}$

$$dy = f'(x)dx$$

Let  $u = g(x)$ , then  $g'(x) = \frac{du}{dx}$

$$du = g'(x)dx$$

Let  $u = x^2$

$$\begin{aligned}
 du &= 2x dx \\
 \int 2x \cos x^2 dx &= \int \cos u du \\
 &= \sin u + C \\
 &= \sin x^2 + C
 \end{aligned}$$

### Definition of the substitution rule

If  $u = g(x)$  is a differentiable function whose range is interval  $I$  and  $f$  is continuous on  $I$ , then

$$\int f'(g(x))g'(x)dx = \int f(u)du$$


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## Definite integrals

### The area problem

Let  $f$  be a continuous nonnegative function on  $[a, b]$ . Find the area of the region bounded by the curve  $y = f(x)$ , the lines  $x = a$ ,  $x = b$ , and the  $x$ -axis.

The area is often coined the **region under the curve**, which generally means the area in between the curve and the  $x$ -axis

### Example

Consider  $f(x) = x^2 + 1$  on  $[0, 2]$ .

**Solution.** Let  $A$  be the area under the curve

Using right endpoints (5 rectangles)

$$\Delta x = \frac{2 - 0}{5} = \frac{2}{5} = 0.4$$

Rectangle 1:  $(\Delta x)(f(0.4)) = (0.4)(1.16)$

Rectangle 2:  $(\Delta x)(f(0.8)) = (0.4)(1.64)$

Rectangle 3:  $(\Delta x)(f(1.2)) = (0.4)(2.44)$

Rectangle 4:  $(\Delta x)(f(1.6)) = (0.4)(3.56)$

Rectangle 5:  $(\Delta x)(f(2.0)) = (0.4)(5)$

$$A_5^+ = (0.4)(1.16) + (0.4)(1.64) + (0.4)(2.44) + (0.4)(3.56) + (0.4)(5) = 5.52$$

$A_5^+$  is an overestimation of  $A$ .

Using left endpoints (5 rectangles):

Rectangle 1:  $(\Delta x)(f(0)) = (0.4)(1)$

Rectangle 2:  $(\Delta x)(f(0.4)) = (0.4)(1.16)$

Rectangle 3:  $(\Delta x)(f(0.8)) = (0.4)(1.64)$

Rectangle 4:  $(\Delta x)(f(1.2)) = (0.4)(2.44)$

Rectangle 5:  $(\Delta x)(f(1.6)) = (0.4)(3.56)$

$$A_5^- = (0.4)(1) + (0.4)(1.16) + (0.4)(1.64) + (0.4)(2.44) + (0.4)(3.56) + = 3.92$$

$A_5^-$  is an underestimation of  $A$ .

We can increase the number of rectangles and compute the area  $A$  more **accurately** by computing the area as the number of rectangles approach infinity.

Let the number of rectangles be  $n$

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

Let  $x_0$  be the first point:  $x_0 = 0$

$$\begin{array}{lll} x_1 = \frac{2}{n} & x_2 = \frac{4}{n} & x_3 = \frac{6}{n} \\ x_4 = \frac{8}{n} & x_5 = \frac{10}{n} & x_6 = \frac{12}{n} \\ x_7 = \frac{14}{n} & \dots & x_i = \frac{2i}{n} \end{array}$$

$$\begin{aligned} A_n &= R_1 + R_2 + R_3 + R_4 + \dots + R_n \\ &= \sum_{i=1}^n \Delta x(f(x_i)) \\ &= \sum_{i=1}^n \frac{2}{n} \left[ \left( \frac{2i}{n} \right)^2 + 1 \right] \\ &= \frac{2}{n} \sum_{i=1}^n \left( \frac{4i^2}{n^2} + 1 \right) \\ &= \frac{2}{n} \left[ \sum_{i=1}^n \left( \frac{4i^2}{n^2} \right) + \sum_{i=1}^n (1) \right] \\ &= \frac{2}{n} \left[ \frac{4}{n^2} \sum_{i=1}^n (i^2) + \sum_{i=1}^n (1) \right] \\ &= \frac{2}{n} \left[ \frac{4}{n^2} \left( \frac{(n)(n+1)(2n+1)}{6} \right) + n \right] \\ &= \frac{8}{n^3} \frac{(n)(n+1)(2n+1)}{6} + 2 \\ A_n &= \frac{4}{3} \left( 1 + \frac{1}{n} \right) \left( 2 + \frac{1}{n} \right) + 2 \end{aligned}$$

$$\begin{aligned}
A &= \lim_{n \rightarrow \infty} A_n \\
&= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{2}{n} \left[ \left( \frac{2i}{n} \right)^2 + 1 \right] \\
&= \lim_{n \rightarrow \infty} \left[ \frac{4}{3} \left( 1 + \frac{1}{n} \right) \left( 2 + \frac{1}{n} \right) + 2 \right] \\
&= \frac{4}{3} (1)(2) + 2 \\
A &= \frac{14}{3}
\end{aligned}$$

### Riemann sum

Let  $f$  be a function defined on  $[a, b]$ .

Divide  $[a, b]$  into  $n$  subintervals, each with width

$$\Delta x = \frac{b - a}{n}$$

Let  $x_0 = a, x_1, x_2, \dots, x_n = b$ ,

For each subinterval  $[x_{i-1}, x_i]$ , choose a sample point  $x_i^*$

Compute the sum

$$A_n = \sum_{i=1}^n f(x_i^*) \Delta x$$

This is also called the **Riemann sum**.

$$A_n = \sum_{i=1}^n f(x_i^*) \Delta x$$

### Definite integral and integrability

The definite integral of  $f$  from  $a$  to  $b$  is

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$$

provided that such limit exists.

We say that  $f$  is integrable on  $[a, b]$

### Remarks on the definite integral

1. If a function is continuous on  $[a, b]$ , it is integrable on  $[a, b]$ .
2. If  $f$  is a nonnegative continuous function on  $[a, b]$ , then  $\int_a^b f(x) dx$  is the area under the curve  $y = f(x)$  from  $x = a$  and  $x = b$



$$3. \int_a^b f(x)dx = \int_a^b f(y)dy$$

### Conventions on the definite integral

$$1. \int_b^a f(x)dx = - \int_a^b f(x)dx$$

$$2. \int_a^a f(x)dx = 0$$

### Properties of the definite integral

$$1. \int_a^b cf(x)dx = c \int_a^b f(x)dx$$

$$2. \int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x) \pm \int_a^b g(x)$$

$$3. \int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$$

$$4. \text{ If } f(x) \geq 0 \forall x \in [a, b], \text{ then } \int_a^b f(x)dx \geq 0$$

$$5. \text{ If } f(x) \geq g(x) \forall x \in [a, b], \text{ then } \int_a^b f(x)dx \geq \int_a^b g(x)dx$$

$$6. \text{ If } m \leq f(x) \leq M \forall x \in [a, b], \text{ then } m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$


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## The Fundamental Theorem of Calculus

### Mean Value Theorem for integrals

Proof

Since  $f$  is continuous on  $[a, b]$ , then  $f$  is integrable on  $[a, b]$  — i.e.  $\int_a^b f(x)dx$  has a value.

Since  $f$  is continuous on  $[a, b]$ , by the **Extreme Value Theorem**,  $\exists m, M \in \mathbb{R}$  such that  $f(x_m) = m, f(x_M) = M, m \leq f(x) \leq M \forall x \in [a, b]$  and for some  $x_m, x_M \in [a, b]$ .

By Property 6 of the definite integral,  $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

$$\begin{aligned} m &\leq \frac{\int_a^b f(x)dx}{b-a} \leq M \\ f(x_m) &\leq \frac{\int_a^b f(x)dx}{b-a} \leq f(x_M) \end{aligned}$$

By the IVT,  $\exists c \in [a, b]$  such that

$$\frac{\int_a^b f(x)dx}{b-a} = f(c)$$

$$\int_a^b f(x)dx = f(c)(b-a)$$

If  $f$  is continuous on  $[a, b]$ ,  $\exists c \in [a, b]$  such that

$$\int_b^a f(x)dx = f(c)(b-a)$$

### Average value of a function

Proof

Given a function continuous on  $[a, b]$ , we can get the average value of the function at  $[a, b]$  by dividing the curve into  $n$  equal-width rectangles, getting the value of each sample points, and dividing by  $n$ .

$$\text{Average area} = \frac{\sum_{i=1}^n f(x_i^*) \Delta x}{n}$$

$$\text{But then, } \Delta x = \frac{b-a}{n} \implies n = \frac{b-a}{\Delta x}$$

$$\begin{aligned} \frac{\sum_{i=1}^n f(x_i^*)}{n} &= \frac{\sum_{i=1}^n f(x_i^*)}{\frac{b-a}{\Delta x}} \\ &= \frac{1}{b-a} \sum_{i=1}^n f(x_i^*) \Delta x \end{aligned}$$

We want to make  $n$  larger in order to make the average more accurate.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n f(x_i^*)}{n} &= \lim_{n \rightarrow \infty} \frac{1}{b-a} \sum_{i=1}^n f(x_i^*) \Delta x \\ &= \frac{1}{b-a} \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x \\ &= \frac{1}{b-a} \int_a^b f(x)dx \end{aligned}$$

Therefore, given function  $f$  that is continuous on  $[a, b]$ , there exists  $c \in [a, b]$  such that

$$f_{avg} = f(c)$$

Let  $f$  be a continuous on  $[a, b]$ . The average value of  $f$  at  $[a, b]$ , denoted by  $f_{avg}$  is

$$f_{avg} = \frac{\int_a^b f(x)dx}{b-a}$$


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### First part of the Fundamental Theorem of Calculus

Let  $y = f(t)$  that is continuous on  $[a, b]$ .

If  $x \in [a, b]$ , then the function is also continuous on  $[a, b] \implies$  the function is also continuous on  $[a, x]$ .

$$\begin{aligned} F(x) &= \int_a^x f(t) dt \\ F(a) &= \int_a^a f(t) dt = 0 \\ F(b) &= \int_a^b f(t) dt \end{aligned}$$

Let  $f$  be continuous on  $[a, b]$ . If  $F$  is the function defined by

$$F(x) = \int_a^x f(t) dt$$

then  $F'(x) = f(x) \forall x \in [a, b]$ .

Proof

Let  $x, x+h \in [a, b], h \neq 0$ .

$$F(x+h) - F(x) = \int_a^{x+h} f(t) dt - \int_a^x f(t) dt$$

By the Property 3 of definite integrals,

$$\begin{aligned} \int_a^{x+h} f(t) dt - \int_a^x f(t) dt &= \int_a^x f(t) dt + \int_x^{x+h} f(t) dt - \int_a^x f(t) dt \\ &= \int_x^{x+h} f(t) dt \end{aligned}$$

By the Mean Value Theorem for integrals,  $\exists c \in [x, x+h]$  such that

$$\begin{aligned} \int_x^{x+h} f(t) dt &= f(c)(x+h-x) \\ &= hf(c) \\ \implies F(x+h) - F(x) &= hf(c) \\ \frac{F(x+h) - F(x)}{h} &= f(c) \\ \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} &= \lim_{h \rightarrow 0} f(c) \end{aligned}$$

Note that  $\lim_{h \rightarrow 0} x = x$  and  $\lim_{h \rightarrow 0} (x+h) = x \implies \lim_{h \rightarrow 0} c = x$  by Squeeze Theorem.

Since  $f$  is continuous at  $x$ ,

$$\lim_{h \rightarrow 0} f(c) = f(x)$$

$$\begin{aligned} \implies F'(x) &= f(x) \quad \forall x \in [a, b] \\ \frac{d}{dx} \int_a^x f(t) dt &= f(x) \end{aligned}$$


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## Second part of the Fundamental Theorem of Calculus

Let's bring back  $f(x) = x^2 + 1$  on  $[0, 2]$ .

$f$  is continuous on  $[0, 2] \implies f$  is integrable on  $[0, 2]$ .

$$\implies \int_0^2 (x^2 + 1) dx = \frac{14}{3}$$

Let  $F(x) = \frac{x^3}{3} + x - 1$ .

$$\begin{aligned} F(2) &= \frac{2^3}{3} + 2 - 1 = \frac{8}{3} + 1 = \frac{11}{3} \\ F(0) &= \frac{0^3}{3} + 0 - 1 = 0 - 1 = -1 \\ F(2) - F(0) &= \frac{11}{3} - (-1) = \frac{14}{3} \\ \int_0^2 (x^2 + 1) dx &= F(2) - F(0) \end{aligned}$$

Observe that  $F'(x) = x^2 + 1 \implies F(x)$  is the an antiderivative of  $x^2 + 1$ .

If a function  $f$  is continuous on  $[a, b]$ , then

$$\int_a^b f(x) dx = F(b) - F(a)$$

where  $F$  is any antiderivative of  $f$  on  $[a, b]$ .

The following notations for  $F(b) - F(a)$  are very useful in evaluating definite integrals:

$$F(x) \Big|_a^b \text{ or } F(x) \Big|_a^b$$

Proof

By FTC - Part 1, the function

$$\int_a^x f(t) dt$$

is an antiderivative of  $f$  on  $[a, b]$ .

By the Equal Derivatives Theorem,

$$\int_a^x f(t) dt = F(x) + C$$

where  $F$  is any antiderivative of  $f$ .

$$\begin{aligned}
 x = b, \int_a^b f(t)dt &= F(b) + C \\
 x = a, \int_a^a f(t)dt &= F(a) + C = 0 \\
 \int_a^b f(t)dt - \int_a^a f(t)dt &= [F(b) + C] - [F(a) + C] \\
 \int_a^b f(t)dt &= F(b) - F(a)
 \end{aligned}$$


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## Application I

### Areas between curves

#### Example 1

Find the area of the region under the curve  $y = x^2 - 1$  from  $x = -1$  to  $x = 2$ .

**Solution.** Area is simply not  $\int_{-1}^2 (x^2 - 1)dx$  because  $\int_{-1}^1 (x^2 - 1)dx$  is negative and cancels the positive area.

Therefore, we get  $\int_{-1}^1 -(x^2 - 1)dx$  to get the area of the curve between -1 and 1.

$$\begin{aligned}
 A &= \int_{-1}^1 -(x^2 - 1)dx + \int_1^2 (x^2 - 1)dx \\
 &= \left( -\frac{x^3}{3} + x \right) \Big|_{-1}^1 + \left( \frac{x^3}{3} - x \right) \Big|_1^2 \\
 &= \left( \frac{1^3}{3} + 1 \right) - \left[ \frac{(-1)^3}{3} + (-1) \right] + \left( \frac{2^3}{3} - 2 \right) - \left( \frac{1^3}{3} - 1 \right) \\
 &= \frac{2}{3} + \frac{2}{3} + \frac{8}{3} - 2 + \frac{2}{3} \\
 A &= \frac{8}{3}
 \end{aligned}$$

#### Example 2

Find the area of the region bounded by the curves of  $y = x^2$  and  $y = 4x - x^2$ .

**Solution.** Note that both curves intersect at  $(0, 0)$  and  $(2, 4)$ .

When we use Riemann sum, we only get the rectangles in between the region bounded by the area by subtracting the upper function ( $y = 4x - x^2$ ) to the lower function ( $y = x^2$ )

$$\implies A_n = \sum [(4x - x^2) - x^2] \Delta x$$

$$\begin{aligned}
A &= \int_0^2 [(4x - x^2) - x^2] dx \\
&= \int_0^2 (4x - 2x^2) dx \\
&= \left( 2x^2 - \frac{2x^3}{3} \right) \Big|_0^2 \\
&= \left[ 2(2)^2 - \frac{2(2)^3}{3} \right] - \left[ 2(0)^2 - \frac{2(0)^3}{3} \right] \\
&= \left[ 8 - \frac{16}{3} \right] - 0 \\
A &= \frac{8}{3}
\end{aligned}$$

### Example 3

Find the area of the region bounded by the curve  $y = \sqrt{x}$ , the line  $x + 2y = 3$ , and the  $x$ -axis.

**Solution.** The graphs intersect at  $(0, 0)$ ,  $(1, 1)$ , and  $(3, 0)$ .

$$x + 2y = 3 \implies y = -\frac{1}{2}x + \frac{3}{2}$$

$$\begin{aligned}
A &= \int_0^1 (\sqrt{x}) dx + \int_1^3 \left( -\frac{1}{2}(3 - x) \right) dx \\
&= \left( \frac{2x^{\frac{3}{2}}}{3} \right) \Big|_0^1 + \left( \frac{1}{2} \left( 3x - \frac{x^2}{2} \right) \right) \Big|_1^3 \\
&= \frac{2(1)^{\frac{3}{2}}}{3} - \frac{2(0)^{\frac{3}{2}}}{3} + \frac{1}{2} \left( 3(3) - \frac{3^2}{2} \right) - \frac{1}{2} \left( 3(1) - \frac{1^2}{2} \right) \\
&= \frac{2}{3} + \frac{9}{4} - \frac{5}{4} \\
&= \frac{8 - 27 + 15}{12} \\
&= \frac{20}{12} \\
A &= \frac{5}{3}
\end{aligned}$$


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## Volumes and volumes of revolution using disks and washers

### Volume of a right cylinder

$$V = ah$$

$$V_n = \sum_{i=1}^n A(x) \Delta x$$

Let  $S$  be a solid that lies between  $x = a$  and  $x = b$ . If the cross-sectional area of  $S$  in the plane  $P_x$  through  $x$  and perpendicular to the  $x$ -axis is  $A(x)$ , where  $A$  is a continuous function on  $[a, b]$ , then the volume  $V$  of  $S$  is

$$V = \lim_{n \rightarrow \infty} \sum_{i=1}^n A(x_i^*) \Delta x = \int_b^a A(x) dx$$

### Example 1

Let us find the volume of a sphere of radius  $r$ .

**Solution.**

radius of the cross-section circle at  $x = \sqrt{r^2 - x^2}$

$$\begin{aligned} A(x) &= \pi(\sqrt{r^2 - x^2})^2 \\ &= \pi(r^2 - x^2) \end{aligned}$$

$$\begin{aligned} V_{\text{sphere}} &= \int_{-r}^r A(x) dx \\ &= \int_{-r}^r \pi(r^2 - x^2) dx \\ &= \pi \left( r^2 x - \frac{x^3}{3} \right) \Big|_{-r}^r \\ &= \pi \left[ r^2(r) - \frac{r^3}{3} \right] - \pi \left[ r^2(-r) - \frac{(-r)^3}{3} \right] \\ V_{\text{sphere}} &= \frac{4}{3} \pi r^3 \end{aligned}$$

### Example 2

The base of a solid is the region bounded by  $y = x^2$  and  $y = 4$ . Its parallel cross-sections perpendicular to the base and the  $y$ -axis are squares. Find the volume of the solid.

**Solution.** side of the cross-section at  $y = 2\sqrt{y}$

$$A(y) = (2\sqrt{y})^2 = 4y$$

$$\begin{aligned} V &= \int_0^4 A(y) dy \\ &= \int_0^4 4y dy \\ &= 2y^2 \Big|_0^4 \\ &= 2(4)^2 - 2(0)^2 \\ V &= 32 \end{aligned}$$

### Volume of solids of revolution

If we revolve a region about a line, we obtain a **solid of revolution**.

**Example 1**

Consider the region under the curve  $y = x^2 + 1$  from  $x = -1$  to  $x = 2$ . We revolve this region about the  $x$ -axis.

**Solution.** radius of the cross-section at  $x = f(x)$

$$A(x) = \pi[f(x)]^2$$

$$\begin{aligned} V &= \int_{-1}^2 \pi(x^2 + 1)^2 dx \\ &= \int_{-1}^2 \pi(x^4 + 2x^2 + 1) dx \\ &= \pi \left( \frac{x^5}{5} + \frac{2x^3}{3} + x \right) \Big|_{-1}^2 \\ &= \pi \left[ \frac{2^5}{5} + \frac{2(2)^3}{3} + 2 \right] - \pi \left[ \frac{(-1)^5}{5} + \frac{2(-1)^3}{3} + (-1) \right] \\ V &= \frac{78\pi}{5} \end{aligned}$$

The cross-section of a solid of revolution is always a circle.

**Example 2**

A solid is obtained by revolving about the  $x$ -axis the region bounded by  $x = y^2$  and  $2y = x$ . Find the volume of the solid.

**Solution.**

$$\begin{aligned} V &= \int_0^4 \pi(\sqrt{x})^2 dx - \int_0^4 \pi \left( \frac{x}{2} \right)^2 dx \\ &= \int_0^4 \pi(x) dx - \int_0^4 \pi \frac{x^2}{4} dx \\ V &= \frac{8\pi}{3} \end{aligned}$$

**Example 3**

A solid is obtained by revolving about the  $y$ -axis the region bounded by  $2x = y^2$ ,  $y = 4$ , and the  $y$ -axis. Find the volume of the solid.

**Solution.**



$$\begin{aligned}
V &= \int_0^4 \pi \left(\frac{y^2}{2}\right)^2 dy \\
&= \int_0^4 \pi \left(\frac{y^4}{4}\right) dy \\
&= \frac{\pi y^5}{20} \Big|_0^4 \\
V &= \frac{256\pi}{5}
\end{aligned}$$

## Volumes by cylindrical shells

There are times that disks-and-washers technique is not the best way to solve a volume problem – e.g.  $y = 4x - x^2$  rotated about the  $y$ -axis.

### Volume of a cylindrical shell

Let  $r_1$  be the inner radius of the cylinder,  $r_2$  be the outer (and larger) radius of the cylinder.  $r$  be the average of both

$$\begin{aligned}
\Delta r &= r_2 - r_1 \\
r &= \frac{r_2 + r_1}{2}
\end{aligned}$$

$$\begin{aligned}
V_{\text{cylindrical shell}} &= \pi r_2^2 h - \pi r_1^2 h \\
&= \pi (r_2^2 - r_1^2) h \\
&= \pi (r_2 + r_1) h (r_2 - r_1) \\
&= 2\pi \left( \frac{r_2 + r_1}{2} \right) h \Delta r \\
V_{\text{cylindrical shell}} &= 2\pi r h \Delta r
\end{aligned}$$

Given a curve  $y = f(x)$  in  $[a, b]$  rotated about the  $y$ -axis, the Riemann sum of the volume is

$$V = \sum 2\pi r h \Delta r$$

In this context,  $r = x$  (horizontal distance),  $h = f(x)$  (vertical distance), and  $\Delta r = \Delta x$

$$\begin{aligned}
V &= \lim_{n \rightarrow \infty} \sum_{i=1}^n 2\pi x_i^* f(x_i^*) \Delta x \\
&= \int_a^b 2\pi x f(x) dx
\end{aligned}$$

The volume of a solid obtained by rotating about the  $y$ -axis the region under the curve  $y = f(x)$  (continuous and nonnegative) from  $x = a$  (nonnegative) to  $x = b$  is

$$V = \lim_{n \rightarrow \infty} \sum_{i=1}^n 2\pi x_i^* f(x_i^*) \Delta x = \int_a^b 2\pi x f(x) dx$$

**Example 1**

A solid is obtained by revolving about the  $y$ -axis the region bounded by  $y = 4x - x^2$  and the  $x$ -axis. Use cylindrical shells to find the volume of the solid.

**Solution.**

$$\begin{aligned}
 V &= \int_0^4 2\pi x(4x - x^2)dx \\
 &= \int_0^4 2\pi(4x^2 - x^3)dx \\
 &= 2\pi \left( \frac{4}{3}x^3 - \frac{x^4}{4} \right) \Big|_0^4 \\
 &= 2\pi \left[ \frac{4}{3}(4)^3 - \frac{4^4}{4} \right] - 0 \\
 &= \frac{128\pi}{3}
 \end{aligned}$$

**Example 2**

A solid is obtained by revolving about the  $x$ -axis the region bounded by  $y = \sqrt{x}$ ,  $y = 2 - x$ , and the  $x$ -axis. Find the volume of the solid.

**Solution.**

*Disks:*

$y = \sqrt{x}$  and  $y = 2 - x$  intersect at  $(1, 1)$

$\forall x \in [0, 1], \sqrt{x} \leq 2 - x \implies$  we use  $\sqrt{x}$  at this interval

$\forall x \in [1, 2], 2 - x \leq \sqrt{x} \implies$  we use  $2 - x$  at this interval

$$\begin{aligned}
 V &= \int_0^1 \pi(\sqrt{x})^2 dx + \int_1^2 \pi(2 - x)^2 dx \\
 &= \int_0^1 \pi x dx + \int_1^2 \pi(x^2 - 4x + 4) dx
 \end{aligned}$$

(This exercise is left to the reader lol don't wanna solve this myself)

*cylindrical shell:*

$y = \sqrt{x}$  and  $y = 2 - x$  intersect at  $(1, 1)$ .

$$\begin{aligned}
 y = \sqrt{x} &\implies x = y^2 \\
 y = 2 - x &\implies x = 2 - y
 \end{aligned}$$

$$V = \int_0^1 2\pi y(2 - y^2 - y) dy$$

(Just solve this yourselves)

### Example 3

A solid is obtained by the revolving about the line  $x = -2$  the region bounded by  $y = x^3$ ,  $y = 8$ , and the  $y$ -axis. Find the volume of the solid.

**Solution.**

$$\text{disks: } V = \int_0^8 \pi [(\sqrt[3]{y} + 2)^2 - 2^2] dy = \frac{336\pi}{5}$$

$$\text{cylindrical shells: } V = \int_0^2 2\pi(x+2)(8-x^3)dx = \frac{336\pi}{5}$$

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## Techniques of integration

### Integration by parts

#### Product rule and the differentials

Recalling the product rule in derivatives,

$$\begin{aligned}\frac{d}{dx}(f(x)g(x)) &= f(x)g'(x) + g(x)f'(x) \\ d(f(x)g(x)) &= f(x)g'(x)dx + g(x)f'(x)dx\end{aligned}$$

Let  $u = f(x)$ ,  $v = g(x)$

$du = f'(x)dx$ ,  $dv = g'(x)dx$

$$\begin{aligned}d(f(x)g(x)) &= f(x)g'(x)dx + g(x)f'(x)dx \implies d(uv) = u dv + v du \\ u dv &= d(uv) - v du \\ \int u dv &= \int d(uv) - \int v du \\ \int u dv &= uv - \int v du\end{aligned}$$

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

Letting  $u = f(x)$ ,  $v = g(x) \implies du = f'(x)dx$ ,  $dv = g'(x)dx$ ,

$$\int u dv = uv - \int v du$$

This is also called the **integration-by-parts formula**.

### Integration by parts and definite integrals

Combining the integration-by-parts formula and FTC2,

$$\int_a^b f(x)g'(x)dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)dx$$

**Example 1**

Evaluate  $\int \ln x dx$

**Preliminary work.**

$$\int \ln x dx \implies x > 0$$

Let  $u = 1$ ,  $dv = \ln x dx$

$$du = 0, v = \int \ln x dx$$

Note that this solution is not correct because we just went back to our original statement, leading us nowhere.

**Solution.** Let  $u = \ln x$ ,  $dv = dx$

$$du = \frac{1}{x} dx, v = \int dx = x + C$$

Following the integration-by-parts formula,

$$\begin{aligned} \int \ln x dx &= (\ln x)(x + c) - \int (x + C) \frac{1}{x} dx \\ &= x \ln x + C \ln x - \int \left(1 + \frac{C}{x}\right) dx \\ &= x \ln x + C \ln x - (x + C \ln |x| + C_1) \\ &= x \ln x + C \ln x - x - C \ln |x| - C_1; \text{ note that } x > 0, \text{ thus } \ln |x| = \ln x \\ &= x \ln x - x + C \end{aligned}$$

Note that the contribution of  $+C$  in  $v$  just cancels at  $uv - \int v du$ , so it is **not necessary** to put  $+C$  when using integration by parts.

**Example 2**

Evaluate  $\int t^2 \sin \beta t dt$

**Solution.** Let  $u = t^2$ ,  $dv = \sin \beta t dt$

$$du = 2t dt, v = \int \sin \beta t dt = -\frac{\cos \beta t}{\beta}$$

$$\int t^2 \sin \beta t dt = -\frac{t^2 \cos \beta t}{\beta} - \int -\frac{2t \cos \beta t}{\beta} dt = -\frac{t^2 \cos \beta t}{\beta} - \int \frac{2}{\beta} t \cos \beta t dt$$

Let  $u = t$ ,  $dv = \cos \beta t dt$

$$du = dt, v = \frac{\sin \beta t}{\beta}$$

$$\begin{aligned} -\frac{t^2 \cos \beta t}{\beta} - \int \frac{2}{\beta} t \cos \beta t dt &= -\frac{t^2 \cos \beta t}{\beta} - \frac{2}{\beta} \left( \frac{t \sin \beta t}{\beta} - \int \frac{\sin \beta t}{\beta} dt \right) \\ &= -\frac{t^2 \cos \beta t}{\beta} - \frac{2t \sin \beta t}{\beta^2} - \frac{2}{\beta} \left( \int \frac{\sin \beta t}{\beta} dt \right) \\ &= -\frac{t^2 \cos \beta t}{\beta} - \frac{2t \sin \beta t}{\beta^2} + \frac{2 \cos \beta t}{\beta^3} + C \end{aligned}$$

**Example 3**

Evaluate  $\int e^x \sin \pi x dx$

**Solution.** Let  $u = \sin \pi x$ ,  $dv = e^x dx$

$$du = \pi \cos \pi x dx, v = e^x$$

By integration by parts,

$$\int e^x \sin \pi x dx = e^x \sin \pi x - \pi \int e^x \cos \pi x dx$$

Let  $u = \cos \pi x$ ,  $dv = e^x dx$

$$du = -\pi \sin \pi x, v = e^x$$

By integration by parts,

$$\begin{aligned} \int e^x \sin \pi x dx &= e^x \sin \pi x - \pi \int e^x \cos \pi x dx = e^x \sin \pi x - \pi \left( e^x \cos \pi x - \pi \int -e^x \sin \pi x dx \right) \\ &= e^x \sin \pi x - \pi e^x \cos \pi x - \pi^2 \int e^x \sin \pi x dx \\ \int e^x \sin \pi x dx + \pi^2 \int e^x \sin \pi x dx &= e^x \sin \pi x - \pi e^x \cos \pi x \\ (1 + \pi^2) \int e^x \sin \pi x dx &= e^x \sin \pi x - \pi e^x \cos \pi x \\ \int e^x \sin \pi x dx &= \frac{e^x \sin \pi x - \pi e^x \cos \pi x}{1 + \pi^2} + C \end{aligned}$$

**Example 4 (definite integral)**

Evaluate  $\int_0^1 \tan^{-1} x dx$ .

**Solution.** We shall first get the antiderivative of the function before evaluating the definite integral.

Let  $u = \arctan x$ ,  $dv = dx$

$$du = \frac{1}{x^2 + 1} dx, v = x$$

By integration by parts,

$$\int \tan^{-1} x dx = x \arctan x - \int \frac{x}{x^2 + 1} dx$$

Let  $u = x^2 + 1 \implies du = 2x dx$

Using the substitution rule,

$$\begin{aligned}
x \arctan x - \int \frac{x}{x^2 + 1} dx &= x \arctan x - \frac{1}{2} \int \frac{1}{u} du \\
&= x \arctan x - \frac{\ln |u|}{2} \\
&= x \arctan x - \frac{\ln |x^2 + 1|}{2}, \text{ note that } \forall x \in \mathbb{R}, x^2 + 1 > 0 \\
&= x \arctan x - \frac{\ln(x^2 + 1)}{2}
\end{aligned}$$

By FTC2,

$$\begin{aligned}
\int_0^1 \tan^{-1} x dx &= \left( x \arctan x - \frac{\ln(x^2 + 1)}{2} \right) \Big|_0^1 \\
&= \left[ \arctan 1 - \frac{\ln(1^2 + 1)}{2} \right] - \left[ -\frac{\ln(0^2 + 1)}{2} \right] \\
&= \frac{\pi - 2 \ln(2)}{4}
\end{aligned}$$


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## Trigonometric integrals

### Trigonometric identities

1.  $\sin^2 x + \cos^2 x = 1$
  2.  $\tan^2 x + 1 = \sec^2 x$
  3.  $\cot^2 x + 1 = \csc^2 x$
  4.  $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
  5.  $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
  6.  $\sin A \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)]$
  7.  $\sin A \sin B = \frac{1}{2}[\cos(A - B)] - \cos(A + B)]$
  8.  $\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$
- 

### Integrals of trigonometric functions

$$1. \int \tan x dx = \ln |\sec x| + C$$

$$\begin{aligned}
\int \tan x dx &= \int \frac{\sin x}{\cos x} dx = -\ln |\cos x| + C \\
&= \ln |\sec x| + C
\end{aligned}$$

$$2. \int \sec x dx = \ln |\sec x + \tan x| + C$$

$$\begin{aligned}
\int \sec x dx &= \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} dx \\
&= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx \\
&= \ln |\sec x + \tan x| + C
\end{aligned}$$

$$3. \int \cot x dx = \ln |\sin x| + C$$

$$4. \int \csc x dx = \ln |\csc x - \cot x| + C$$


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### Example 1

Evaluate  $\int \sqrt{\cos \theta} \sin^3 \theta d\theta$ .

**Solution.**

$$\begin{aligned}
\int \sqrt{\cos \theta} \sin^3 \theta d\theta &= \int \sqrt{\cos \theta} \sin^2 \theta \sin \theta d\theta \\
&= \int \sqrt{\cos \theta} (1 - \cos^2 \theta) \sin \theta d\theta
\end{aligned}$$

Let  $u = \sqrt{\cos \theta}$

$$\implies u^2 = \cos \theta$$

$$\implies 2u du = -\sin \theta d\theta$$

$$\implies -2u du = \sin \theta d\theta$$

$$\begin{aligned}
\int \sqrt{\cos \theta} (1 - \cos^2 \theta) \sin \theta d\theta &= \int u(1 - u^4)(-2u) du \\
&= \int (-2u^2 + 2u^6) du \\
&= -\frac{2}{3}u^3 + \frac{2}{7}u^7 + C \\
&= -\frac{2}{3}(\cos \theta)^{3/2} + \frac{2}{7}(\cos \theta)^{7/2} + C
\end{aligned}$$

### Example 2

Evaluate  $\int \tan^2 t \sec^4 t dt$ .

**Solution.**

$$\begin{aligned}
\int \tan^2 t \sec^4 t dt &= \int \tan^2 t \sec^2 t \sec^2 t dt \\
&= \int \tan^2 t (1 + \tan^2 t) \sec^2 t dt
\end{aligned}$$

Let  $u = \tan t \implies du = \sec^2 t dt$ .

$$\begin{aligned}\int \tan^2 t (1 + \tan^2 t) \sec^2 t dt &= \int u^2 (1 + u^2) du \\ &= \int (u^2 + u^4) du \\ &= \frac{u^3}{3} + \frac{u^5}{5} \\ &= \frac{\tan^3 t}{3} + \frac{\tan^5 t}{5} + C\end{aligned}$$

**Example 3 (definite integral)**

Evaluate  $\int_{\pi/4}^{\pi/2} \csc^5 x \cot^3 x dx$ .

$$\begin{aligned}\int_{\pi/4}^{\pi/2} \csc^5 x \cot^3 x dx &= \int_{\pi/4}^{\pi/2} \csc^4 x \cot^2 x \csc x \cot x dx \\ &= \int_{\pi/4}^{\pi/2} \csc^4 x (\csc^2 x - 1) \csc x \cot x dx\end{aligned}$$

Let  $u = \csc x \implies du = -\csc x \cot x dx \implies -du = \csc x \cot x dx$

$$x = \frac{\pi}{4} \implies u = \sqrt{2}$$

$$x = \frac{\pi}{2} \implies u = 1$$

$$\begin{aligned}\int_{\pi/4}^{\pi/2} \csc^4 x (\csc^2 x - 1) \csc x \cot x dx &= \int_{\sqrt{2}}^1 u^4 (u^2 - 1) (-du) \\ &= \int_{\sqrt{2}}^1 -u^6 + u^4 du \\ &= \int_1^{\sqrt{2}} u^6 - u^4 du \\ &= \left. \frac{u^7}{7} - \frac{u^5}{5} \right|_1^{\sqrt{2}} \\ &= \frac{8\sqrt{2}}{7} - \frac{4\sqrt{2}}{5} - \frac{1}{7} + \frac{1}{5} \\ &= \frac{40\sqrt{2} - 28\sqrt{2} - 5 + 7}{35} \\ &= \frac{12\sqrt{2} + 2}{35}\end{aligned}$$

**Example 4 (definite integral)**

Evaluate  $\int_0^{\pi/2} \cos 5t \cos 10t dt$ .



$$\begin{aligned}
\int_0^{\pi/2} \cos 5t \cos 10t dt &= \int_0^{\pi/2} \frac{1}{2} [\cos(5t - 10t) + \cos(5t + 10t)] dt \\
&= \int_0^{\pi/2} \frac{1}{2} [\cos(-5t) + \cos 15t] dt; \text{ note that } \cos(-5t) = \cos 5t \forall t \\
&= \int_0^{\pi/2} \frac{1}{2} \cos 5t + \frac{1}{2} \cos 15t \\
&= \frac{1}{10} \sin 5t + \frac{1}{30} \sin 15t \Big|_0^{\pi/2} \\
&= \frac{1}{10} \sin \frac{5\pi}{2} + \frac{1}{30} \sin \frac{15\pi}{2} - 0 \\
&= \frac{1}{15}
\end{aligned}$$

### Example 5

Evaluate  $\int \frac{\sin^2(1/t)}{t^2} dt$ .

$$\text{Let } u = \frac{1}{t} \implies du = -\frac{1}{t^2} \implies -du = \frac{1}{t^2}$$

$$\int \frac{\sin^2(1/t)}{t^2} dt = \int -\sin^2 u du$$

Note that  $\sin^2 u = \frac{1}{2}(1 - \cos 2u)$

$$\begin{aligned}
\int -\sin^2 u du &= \int -\frac{1}{2}(1 - \cos 2u) du \\
&= -\frac{1}{2} \left( u - \frac{\sin 2u}{2} \right) \\
&= -\frac{1}{2t} + \frac{\sin(2/t)}{4} + C
\end{aligned}$$


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