Mathematical Analysis IB

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0 - Review on differentiation

Differentiability

Let f be a function on some open interval I containing x. The derivative of f at x, denoted by $f'(x)$, is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Differentiation rules

1. $\dfrac{d}{dx}(cf(x)) = cf'(x)$
2. $\dfrac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$
3. $\dfrac{d}{dx}(f(x)g(x)) = f(x)g'(x) + g(x)f'(x)$
4. $\dfrac{d}{dx}\Big(\dfrac{f(x)}{g(x)}\Big) = \dfrac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$
5. $\dfrac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$

Differentiation formulas I

1. $\dfrac{d}{dx}(c) = 0, c \in \mathbb{R}$
2. $\dfrac{d}{dx}(x^r) = rx^{r-1}, r \in \mathbb{R}$

3. $\dfrac{d}{dx}(\sin x) = \cos x$
4. $\dfrac{d}{dx}(\cos x) = \sin x$
5. $\dfrac{d}{dx}(\tan x) = \sec^2 x$
6. $\dfrac{d}{dx}(\cot x) = -\csc^2 x$
7. $\dfrac{d}{dx}(\sec x) = \sec x \tan x$
8. $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$

Differentiation formulas II

1. $\dfrac{d}{dx}(e^x) = e^x$
2. $\dfrac{d}{dx}(\ln |x|) = \dfrac{1}{x}$
3. $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$
4. $\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$
5. $\dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{x\sqrt{x^2-1}}$

Mean value theorem

Let f be a function that is continuous on $[a, b]$ and is differentiable on (a, b). Then there is a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Consequences of MVT

Zero derivative

If $f'(x) = 0 \; \forall x$ in interval I, then $f(x) = c \; \forall x \in I$ for some constant C.

Equal derivatives

If $f'(x) - g'(x) = 0 \; \forall x$ in an interval I, then $f(x) = g(x) + C$ for some constant C.

Example

Let $f(x) = \cos^{-1} x$ and $g(x) = -\sin^{-1} x$.

This implies that $x \in [-1, 1]$ and $f(x), g(x) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$f'(x) = -\frac{1}{\sqrt{x^2 + 1}}$$

$$g'(x) = -\frac{1}{\sqrt{x^2 + 1}}$$

Since $f'(x) - g'(x) = 0$ for $x \in [-1, 1]$, then $f(x) - g(x) = C$ for some constant C by a corollary.

$$\begin{aligned} \cos^{-1} x - (-\sin^{-1} x) &= C \\ \cos^{-1} x + \sin^{-1} x &= C \end{aligned}$$

Substituting $x \in [-1, 1]$, in this case, let's use $x = 0$,

$$\begin{aligned} \cos^{-1}(0) + \sin^{-1}(0) &= C \\ 0 + \frac{\pi}{2} &= C \\ C &= \frac{\pi}{2} \end{aligned}$$

$$\therefore \forall x \in [-1, 1], f(x) - g(x) = \frac{\pi}{2}$$

Differentials

$$\begin{aligned} f'(x) &= \frac{dy}{dx} \\ f'(x)dx &= dy \end{aligned}$$

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1 - Indefinite and definite integrals

Indefinite integral

The main interpretation of derivative is the slope of a tangent line of a curve.

Example

At any point (x, y) on a particular curve $y = F(x)$, the tangent line has a slope equal to $4x - 5$. If the curve contains the point $(3, 7)$, find $F(x)$.

Solution. Since the slope is equal to $4x - 5$ for any point (x, y), then the slope at $(3, 7)$ is $4(3) - 5 = 7$.

$4x - 5$ therefore represents the tangent slope for all values of x. So

$$F'(x) = 4x - 5$$

By intuition, we can conclude that $F(x) = 2x^2 - 5x$.

However given $F(x) = 2x^2 - 5x + 1$, $F'(x)$ remains the same. And so is $F(x) = 2x^2 - 5x - 3$, $F(x) = 2x^2 - 5x + \pi$, and infinitely more functions. We can arbitrarily assign a constant k, so that $F(x) = 2x^2 - 5x + k$.

Substituting $(x, y) = (3, 7)$,

$$\begin{aligned}
7 &= 2(3)^2 - 5(3) + k \\
7 &= 18 - 15 + k \\
k &= 4
\end{aligned}$$

So $F(x) = 2x^2 - 5x + 4$.

Definition of an antiderivative

A function F is called an antiderivative of the function f on an interval I if $F'(x) = f(x)\ \forall x \in I$.

$F(x) = 2x^2 - 5x$ is a **possible** antiderivative of $f(x) = 4x - 5$. $F(x) = 2x^2 - 5x + 4$ is also a **possible** antiderivative of $f(x) = 4x - 5$.

Equal derivatives

If $F'(x) = G'(x)\ \forall x$ in an interval I, then $F(x) = G(x) + C\ \forall x \in I$ for some constant C.

Integration notation

The collection of all antiderivatives of f is denoted by

$$\int f(x)dx$$

which is read as "the integral of $f(x)dx$."

This collection is also called the **indefinite integral** of f.

The reverse process if differentiation is called **antidifferentiation** or **integration**.

$\int (4x-5)dx = 2x^2 - 5x + C$ for some constant C.

C is the constant of integration.

$\int \sin x dx = -\cos x + C$

Integration rules

1. $\int kf(x)dx = k\int f(x)dx$, k constant
2. $\int f(x) \pm g(x)dx = \int f(x)dx \pm \int g(x)dx$

Integration formulas I

1. $\int kdx = kx + C, k \in \mathbb{R}$
2. $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C, n \in \mathbb{R}, n \neq -1$

Integration formulas II

1. $\int \sin x dx = -\cos x + C$
2. $\int \cos x dx = \sin x + C$
3. $\int \sec^2 x dx = \tan x + C$
4. $\int \csc^2 x dx = -\cot x + C$
5. $\int \sec x \tan x dx = \sec x + C$
6. $\int \csc x \cot x dx = -\csc x + C$

Integration formulas III

1. $\int e^x dx = e^x + C$
2. $\int \frac{1}{x} dx = \ln|x| + C$
3. $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$
4. $\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$
5. $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} + C$

Substitution rule

Chain rule for derivatives

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$$

If follows that

$$\int f'(g(x))g'(x)dx = f(g(x)) + C$$

Example

Evaluate $\int 2x \cos x^2 dx$.

Preliminary work. By intuition, we can get $f(x) = sinx$ and $g(x) = x^2$

$$\int 2x \cos x^2 dx = f(g(x)) = \sin x^2$$

Solution. Suppose that $f'(x) = \frac{dy}{dx}$

$$dy = f'(x)dx$$

Let $u = g(x)$, then $g'(x) = \frac{du}{dx}$

$$du = g'(x)dx$$

Let $u = x^2$

$$
\begin{aligned}
du &= 2xdx \\
\int 2x\cos x^2 dx &= \int \cos u du \\
&= \sin u + C \\
&= \sin x^2 + C
\end{aligned}
$$

Definition of the substitution rule

If $u = g(x)$ is a differentiable function whose range is interval I and f is continuous on I, then

$$\int f'(g(x))g'(x) = \int f(u)du$$

Definite integrals

The area problem

Let f be a continuous nonnegative function on $[a, b]$. Find the area of the regiom bounded by the curve $y = f(x)$, the lines $x = a$, $x = b$, and the x-axis.

> The area is often coined the **region under the curve**, which generally means the area in between the curve and the x-axis

Example

Consider $f(x) = x^2 + 1$ on $[0, 2]$.

Solution. Let A be the area under the curve

Using right endpoints (5 rectangles)

$$\Delta x = \frac{2-0}{5} = \frac{2}{5} = 0.4$$

Rectangle 1: $(\Delta x)(f(0.4)) = (0.4)(1.16)$

Rectangle 2: $(\Delta x)(f(0.8)) = (0.4)(1.64)$

Rectangle 3: $(\Delta x)(f(1.2)) = (0.4)(2.44)$

Rectangle 4: $(\Delta x)(f(1.6)) = (0.4)(3.56)$

Rectangle 5: $(\Delta x)(f(2.0)) = (0.4)(5)$

$A_5^+ = (0.4)(1.16)+(0.4)(1.64)+(0.4)(2.44)+(0.4)(3.56)+(0.4)(5) = 5.52$

A_5^+ is an overestimation of A.

Using left endpoints (5 rectangles):

Rectangle 1: $(\Delta x)(f(0)) = (0.4)(1)$

Rectangle 2: $(\Delta x)(f(0.4)) = (0.4)(1.16)$

Rectangle 3: $(\Delta x)(f(0.8)) = (0.4)(1.64)$

Rectangle 4: $(\Delta x)(f(1.2)) = (0.4)(2.44)$

Rectangle 5: $(\Delta x)(f(1.6)) = (0.4)(3.56)$

$A_5^- = (0.4)(1)+(0.4)(1.16)+(0.4)(1.64)+(0.4)(2.44)+(0.4)(3.56)+ = 3.92$

A_5^- is an underestimation of A.

We can increase the number of rectangles and compute the area A more **accurately** by computing the area as the number of rectangles approach infinity.

Let the number of rectangles be n

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

Let x_0 be the first point: $x_0 = 0$

$$\begin{array}{ccc} x_1 = \dfrac{2}{n} & x_2 = \dfrac{4}{n} & x_3 = \dfrac{6}{n} \\ x_4 = \dfrac{8}{n} & x_5 = \dfrac{10}{n} & x_6 = \dfrac{12}{n} \\ x_7 = \dfrac{14}{n} & \cdots & x_i = \dfrac{2i}{n} \end{array}$$

$$
\begin{aligned}
A_n &= R_1 + R_2 + R_3 + R_4 + \cdots + R_n \\
&= \sum_{i=1}^{n} \Delta x(f(x_i)) \\
&= \sum_{i=1}^{n} \frac{2}{n}\Big[\Big(\frac{2i}{n}\Big)^2 + 1\Big] \\
&= \frac{2}{n}\sum_{i=1}^{n}\Big(\frac{4i^2}{n^2} + 1\Big) \\
&= \frac{2}{n}\Big[\sum_{i=1}^{n}\Big(\frac{4i^2}{n^2}\Big) + \sum_{i=1}^{n}(1)\Big] \\
&= \frac{2}{n}\Big[\frac{4}{n^2}\sum_{i=1}^{n}(i^2) + \sum_{i=1}^{n}(1)\Big] \\
&= \frac{2}{n}\Big[\frac{4}{n^2}\Big(\frac{(n)(n+1)(2n+1)}{6}\Big) + n\Big] \\
&= \frac{8}{n^3}\frac{(n)(n+1)(2n+1)}{6} + 2 \\
A_n &= \frac{4}{3}\Big(1 + \frac{1}{n}\Big)\Big(2 + \frac{1}{n}\Big) + 2
\end{aligned}
$$

$$
\begin{aligned}
A &= \lim_{n\to\infty} A_n \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\Big[\Big(\frac{2i}{n}\Big)^2 + 1\Big] \\
&= \lim_{n\to\infty} \Big[\frac{4}{3}\Big(1 + \frac{1}{n}\Big)\Big(2 + \frac{1}{n}\Big) + 2\Big] \\
&= \frac{4}{3}(1)(2) + 2 \\
A &= \frac{14}{3}
\end{aligned}
$$

Riemann sum

Let f be a function defined on $[a, b]$.

Divide $[a, b]$ into n subintervals, each with width

$$\Delta x = \frac{b - a}{n}$$

Let $x_0 = a, x_1, x_2, \ldots, x_n = b$,

For each subinterval $[x_{i-1}, x_i]$, choose a sample point x_i^*

Compute the sum

$$A_n = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

This is also called the **Riemann sum**.

$$A_n = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Definite integral and integrability

The definite integral of f from a to b is

$$\int_a^b f(x)dx = \lim_{x\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided that such limit exists.

We say that f is integrable on $[a, b]$

Remarks on the definite integral

1. If a function is continuous on $[a, b]$, it is integrable on $[a, b]$.
2. If f is a nonnegative continuous function on $[a, b]$, then $\displaystyle\int_a^b f(x)dx$ is the area under the curve $y = f(x)$ from $x = a$ and $x = b$
3. $\displaystyle\int_a^b f(x)dx = \int_a^b f(y)dy$

Conventions on the definite integral

1. $\displaystyle\int_b^a f(x)dx = -\int_a^b f(x)dx$
2. $\displaystyle\int_a^a f(x)dx = 0$

Properties of the definite integral

1. $\displaystyle\int_a^b cf(x)dx = c\int_a^b f(x)dx$
2. $\displaystyle\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x) \pm \int_a^b g(x)$
3. $\displaystyle\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$
4. If $f(x) \geq 0 \ \forall x \in [a, b]$, then $\displaystyle\int_a^b f(x)dx \geq 0$
5. If $f(x) \geq g(x) \ \forall x \in [a, b]$, then $\displaystyle\int_a^b f(x)dx \geq \int_a^b g(x)dx$
6. If $m \leq f(x) \leq M \ \forall x \in [a, b]$, then $\displaystyle m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

The Fundamental Theorem of Calculus

Mean Value Theorem for integrals

Proof

Since f is continuous on $[a, b]$, then f is integrable on $[a, b]$ — i.e. $\int_a^b f(x)dx$ has a value.

Since f is continuous on [a,b], by the **Extreme Value Theorem**, $\exists m, M \in \mathbb{R}$ such that $f(x_m) = m, f(x_M) = M, m \leq f(x) \leq M \ \forall x \in [a, b]$ and for some $x_m, x_M \in [a, b]$.

By Property 6 of the definite integral, $\displaystyle m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

$$
\begin{aligned}
m &\leq \frac{\int_a^b f(x)dx}{b-a} \leq & M \\
f(x_m) &\leq \frac{\int_a^b f(x)dx}{b-a} \leq & f(x_M)
\end{aligned}
$$

By the IVT, $\exists c \in [a, b]$ such that

$$\frac{\int_a^b f(x)dx}{b-a} = f(c)$$
$$\int_a^b f(x)dx = f(c)(b-a)$$

If f is continuous on $[a, b]$, $\exists c \in [a, b]$ such that

$$\int_b^a f(x)dx = f(c)(b-a)$$

Average value of a function

Proof

Given a function continuous on $[a, b]$, we can get the average value of the function at $[a, b]$ by dividing the curve into n equal-width rectangles, getting the value of each sample points, and dividing by n.

Average area $= \frac{\sum_{i=1}^{n} f(x_i^*)i}{n}$

But then, $\Delta x = \frac{b-a}{n} \implies n = \frac{b-a}{\Delta x}$

$$\begin{aligned}\frac{\sum_{i=1}^{n} f(x_i^*)}{n} &= \frac{\sum_{i=1}^{n} f(x_i^*)}{\frac{b-a}{\Delta x}} \\ &= \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\Delta x\end{aligned}$$

We want to make n larger in order to make the average more accurate.

$$\begin{aligned}\lim_{n\to\infty} \frac{\sum_{i=1}^{n} f(x_i^*)}{n} &= \lim_{n\to\infty} \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)i\Delta x \\ &= \frac{1}{b-a}\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x \\ &= \frac{1}{b-a}\int_a^b f(x)dx\end{aligned}$$

Therefore, given function f that is continuous on $[a, b]$, there exists $c \in [a, b]$ such that

$$f_{avg} = f(c)$$

Let f be a continuous on $[a, b]$. The average value of f at $[a, b]$, denoted by f_{avg} is

$$f_{avg} = \frac{\int_a^b f(x)dx}{b - a}$$

First part of the Fundamental Theorem of Calculus

Let $y = f(t)$ that is continuous on $[a, b]$.

If $x \in [a, b]$, then the function is also continuous on $[a, b] \implies$ the function is also continuous on $[a, x]$.

$$\begin{aligned}
F(x) &= \int_a^x f(t)dt \\
F(a) &= \int_a^a f(t)dt = 0 \\
F(b) &= \int_a^b f(t)dt
\end{aligned}$$

Let f be continuous on $[a, b]$. If f is the function defined by

$$F(x) = \int_a^x f(t)dt$$

then $F'(x) = f(x)\ \forall x \in [a, b]$.

Proof

Let $x, x + h \in [a, b], h \neq 0$.

$$F(x + h) - F(x) = \int_a^{x+h} f(t)dt - \int_a^x f(t)dt$$

By the Property 3 of definite integrals,

$$\begin{aligned}
\int_a^{x+h} f(t)dt - \int_a^x f(t)dt &= \int_a^x f(t)dt + \int_x^{x+h} f(t)dt - \int_a^x f(t)dt \\
&= \int_x^{x+h} f(t)dt
\end{aligned}$$

By the Mean Value Theorem for integrals, $\exists c \in [x, x+h]$ such that

$$\begin{aligned}
\int_x^{x+h} f(t)dt &= f(c)(x+h-x) \\
&= hf(c) \\
\implies F(x+h) - F(x) &= hf(c) \\
\frac{F(x+h) - F(x)}{h} &= f(c) \\
\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} &= \lim_{h \to 0} f(c)
\end{aligned}$$

Note that $\lim_{h \to 0} x = x$ and $\lim_{h \to 0}(x+h) = x \implies \lim_{h \to 0} c = x$ by Squeeze Theorem.

Since f is continuous at x,

$$\lim_{h \to 0} f(c) = f(x)$$

$$\begin{aligned}
\implies F'(x) = f(x) \ \forall x \in [a, b] \\
\frac{d}{dx} \int_a^x f(t)dt = f(x)
\end{aligned}$$

Second part of the Fundamental Theorem of Calculus

Let's bring back $f(x) = x^2 + 1$ on $[0, 2]$.

f is continuous on $[0, 2] \implies f$ is integrable on $[0, 2]$.

$$\implies \int_0^2 (x^2 + 1)dx = \frac{14}{3}$$

Let $F(x) = \dfrac{x^3}{3} + x - 1$.

$$
\begin{aligned}
F(2) &= \frac{2^3}{3} + 2 - 1 = \frac{8}{3} + 1 = \frac{11}{3} \\
F(0) &= \frac{0^3}{3} + 0 - 1 = 0 - 1 = -1 \\
F(2) - F(0) &= \frac{11}{3} - (-1) = \frac{14}{3} \\
\int_0^2 (x^2 + 1)dx &= F(2) - F(0)
\end{aligned}
$$

Observe that $F'(x) = x^2 + 1 \implies F(x)$ is the an antiderivative of $x^2 + 1$.

If a function f is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is any antiderivative of f on $[a, b]$.

The following notations for $F(b) - F(a)$ are very useful in evaluating definite integrals:

$F(x)\Big]_a^b$ or $F(x)\Big|_a^b$

Proof

By FTC - Part 1, the function

$$\int_a^x f(t)dt$$

is an antiderivative of f on $[a, b]$.

By the Equal Derivatives Theorem,

$$\int_a^x f(t)dt = F(x) + C$$

where F is any antiderivative of f.

$$
\begin{aligned}
x = b, \int_a^b f(t)dt &= F(b) + C \\
x = a, \int_a^a f(t)dt &= F(a) + C = 0 \\
\int_a^b f(t)dt - \int_a^a f(t)dt &= [F(b) + C] - [F(a) + C] \\
\int_a^b f(t)dt &= F(b) - F(a)
\end{aligned}
$$

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2 - Application I

Areas between curves

Example 1

Find the area of the region under the curve $y = x^2 - 1$ from $x = -1$ to $x = 2$.

Solution. Area is simply not $\int_{-1}^{2}(x^2 - 1)dx$ because $\int_{-1}^{1}(x^2 - 1)dx$ is negative and cancels the positive area.

Therefore, we get $\int_{-1}^{1} -(x^2-1)dx$ to get the area of the curve between -1 and 1.

$$
\begin{aligned}
A &= \int_{-1}^{1} -(x^2 - 1)dx + \int_{1}^{2} (x^2 - 1)dx \\
&= \Big(-\frac{x^3}{3} + x\Big)\Big|_{-1}^{1} + \Big(\frac{x^3}{3} - x\Big)\Big|_{1}^{2} \\
&= \Big(\frac{1^3}{3} + 1\Big) - \Big[\frac{(-1)^3}{3} + (-1)\Big] + \Big(\frac{2^3}{3} - 2\Big) - \Big(\frac{1^3}{3} - 1\Big) \\
&= \frac{2}{3} + \frac{2}{3} + \frac{8}{3} - 2 + \frac{2}{3} \\
A &= \frac{8}{3}
\end{aligned}
$$

Example 2

Find the area of the region bounded by the curves of $y = x^2$ and $y = 4x - x^2$.

Solution. Note that both curves intersect at $(0, 0)$ and $(2, 4)$.

When we use Riemann sum, we only get the rectangles in between the region bounded by the area by subtracting the upper function ($y = 4x - x^2$) to the lower function ($y = x^2$)

$$\implies A_n = \sum [(4x - x^2) - x^2]\Delta x$$

$$\begin{aligned}
A &= \int_0^2 [(4x - x^2) - x^2]dx \\
&= \int_0^2 (4x - 2x^2)dx \\
&= \left(2x^2 - \frac{2x^3}{3}\right)\Bigg|_0^2 \\
&= \left[2(2)^2 - \frac{2(2)^3}{3}\right] - \left[2(0)^2 - \frac{2(0)^3}{3}\right] \\
&= \left[8 - \frac{16}{3}\right] - 0 \\
A &= \frac{8}{3}
\end{aligned}$$

Example 3

Find the area of the region bounded by the curve $y = \sqrt{x}$, the line $x + 2y = 3$, and the x-axis.

Solution. The graphs intersect at $(0, 0)$, $(1, 1)$, and $(3, 0)$.

$$x + 2y = 3 \implies y = -\frac{1}{2}x + \frac{3}{2}$$

$$
\begin{aligned}
A &= \int_0^1 (\sqrt{x})dx + \int_1^3 \Big(-\frac{1}{2}(3-x)\Big)dx \\
&= \Big(\frac{2x^{\frac{3}{2}}}{3}\Big)\Bigg|_0^1 + \Big(\frac{1}{2}(3x - \frac{x^2}{2})\Big)\Bigg|_1^3 \\
&= \frac{2(1)^{\frac{3}{2}}}{3} - \frac{2(0)^{\frac{3}{2}}}{+}\frac{1}{2}(3(3) - \frac{3^2}{2}) - \frac{1}{2}(3(1) - \frac{1^2}{2}) \\
&= \frac{2}{3} + \frac{9}{4} - \frac{5}{4} \\
&= \frac{8 - 27 + 15}{12} \\
&= \frac{20}{12} \\
A &= \frac{5}{3}
\end{aligned}
$$

Volumes and volumes of revolution using disks and washers

Volume of a right cylinder

$$V = ah$$

$$V_n = \sum_{i=1}^{n} A(x)\Delta x$$

Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x through x and perpendicular to the x-axis is $A(x)$, where A is a continuous function on $[a, b]$, then the volume V of S is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_b^a A(x)dx$$

Example 1

Let us find the volume of a sphere of radius r.

Solution.

radius of the cross-section circle at $x = \sqrt{r^2 - x^2}$

$$
\begin{aligned}
A(x) &= \pi(\sqrt{r^2 - x^2})^2 \\
&= \pi(r^2 - x^2)
\end{aligned}
$$

$$
\begin{aligned}
V_{\text{sphere}} &= \int_{-r}^{r} A(x)dx \\
&= \int_{-r}^{r} \pi(r^2 - x^2)dx \\
&= \pi(r^2x - \frac{x^3}{3})\Big|_{-r}^{r} \\
&= \pi\left[r^2(r) - \frac{r^3}{3}\right] - \pi\left[r^2(-r) - \frac{(-r)^3}{3}\right] \\
V_{\text{sphere}} &= \frac{4}{3}\pi r^3
\end{aligned}
$$

Example 2

The base of a solid is the region bounded by $y = x^2$ and $y = 4$. Its parallel cross-sections perpendicular to the base and the y-axis are squares. Find the volume of the solid.

Solution. side of the cross-section at $y = 2\sqrt{y}$

$$A(y) = (2\sqrt{y})^2 = 4y$$

$$
\begin{aligned}
V &= \int_{0}^{4} A(y)dy \\
&= \int_{0}^{4} 4ydy \\
&= 2y^2\Big|_{0}^{4} \\
&= 2(4)^2 - 2(0)^2 \\
V &= 32
\end{aligned}
$$

Volume of solids of revolution

If we revolve a region about a line, we obtain a **solid of revolution**.

Example 1

Consider the region under the curve $y = x^2 + 1$ from $x = -1$ to $x = 2$. We revolve this region about the x-axis.

Solution. radius of the cross-section at $x = f(x)$

$$A(x) = \pi[f(x)]^2$$

$$\begin{aligned}
V &= \int_{-1}^{2} \pi(x^2+1)^2 dx \\
&= \int_{-1}^{2} \pi(x^4+2x^2+1) dx \\
&= \pi\left(\frac{x^5}{5} + \frac{2x^3}{3} + x\right)\Big|_{-1}^{2} \\
&= \pi\left[\frac{2^5}{5} + \frac{2(2)^3}{3} + 2\right] - \pi\left[\frac{(-1)^5}{5} + \frac{2(-1)^3}{3} + (-1)\right] \\
V &= \frac{78\pi}{5}
\end{aligned}$$

The cross-section of a solid of revolution is always a circle.

Example 2

A solid is obtained by revolving about the x-axis the region bounded by $x = y^2$ and $2y = x$. Find the volume of the solid.

Solution.

$$\begin{aligned}
V &= \int_{0}^{4} \pi(\sqrt{x})^2 dx - \int_{0}^{4} \pi\left(\frac{x}{2}\right)^2 dx \\
&= \int_{0}^{4} \pi(x) dx - \int_{0}^{4} \pi\frac{x^4}{4} dx \\
V &= \frac{8\pi}{3}
\end{aligned}$$

Example 3

A solid is obtained by revolving about the y-axis the region bounded by $2x = y^2$, $y = 4$, and the y-axis. Find the volume of the solid.

Solution.

$$
\begin{aligned}
V &= \int_0^4 \pi(\frac{y^2}{2})^2 dy \\
&= \int_0^4 \pi(\frac{y^4}{4}) dy \\
&= \frac{\pi y^5}{20}\Bigg|_0^4 \\
V &= \frac{256\pi}{5}
\end{aligned}
$$

Volumes by cylindrical shells

There are times that disks-and-washers technique is not the best way to solve a volume problem – e.g. $y = 4x - x$ rotated about the y-axis.

Volume of a cylindrical shell

Let r_1 be the inner radius of the cylinder, r_2 be the outer (and larger) radius of the cylinder. r be the average of both

$$
\begin{aligned}
\Delta r &= r_2 - r_1 \\
r &= \frac{r_2 + r_1}{2}
\end{aligned}
$$

$$
\begin{aligned}
V_{\text{cylindrical shell}} &= \pi r_2^2 h - r_1^2 h \\
&= \pi(r_2^2 - r_1^2)h \\
&= \pi(r_2 + r_1)h(r_2 - r_1) \\
&= 2\pi\left(\frac{r_2 + r_1}{2}\right)h\Delta r \\
V_{\text{cylindrical shell}} &= 2\pi r h \Delta r
\end{aligned}
$$

Given a curve $y = f(x)$ in $[a, b]$ rotated about the y-axis, the Riemann sum of the volume is

$$V = \sum 2\pi r h \Delta r$$

In this context, $r = x$ (horizontal distance), $h = f(x)$ (vertical distance), and $\Delta r = \Delta x$

$$
\begin{aligned}
V &= \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*)\Delta x \\
&= \int_a^b 2\pi x f(x) dx
\end{aligned}
$$

The volume of a solid obtained by rotating about the y-axis the region under the curve $y = f(x)$ (continuous and nonnegative) from $x = a$ (nonnegative) to $x = b$ is

$$
V = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*)\Delta x = \int_a^b 2\pi x f(x) dx
$$

Example 1

A solid is obtained by revolving about the y-axis the region bounded by $y = 4x - x^2$ and the x-axis. Use cylindrical shells to find the volume of the solid.

Solution.

$$
\begin{aligned}
V &= \int_0^4 2\pi x(4x - x^2)dx \\
&= \int_0^4 2\pi(4x^2 - x^3)dx \\
&= 2\pi\Big(\frac{4}{3}x^3 - \frac{x^4}{4}\Big)\Big|_0^4 \\
&= 2\pi\Big[\frac{4}{3}(4)^3 - \frac{4^4}{4}\Big] - 0 \\
&= \frac{128\pi}{3}
\end{aligned}
$$

Example 2

A solid is obatined by revolving about the x-axis the region bounded by $y = \sqrt{x}$, $y = 2 - x$, and the x-axis. Find the volume of the solid.

Solution.

Disks:

$y = \sqrt{x}$ and $y = 2 - x$ intersect at $(1, 1)$

$\forall x \in [0, 1]$, $\sqrt{x} \leq 2 - x \implies$ we use $\sqrt{x}$ at this interval

$\forall x \in [1,2],\ 2-x \le \sqrt{x} \implies$ we use $2-x$ at this interval

$$
\begin{aligned}
V &= \int_0^1 \pi(\sqrt{x})^2 dx + \int_1^2 \pi(2-x)^2 dx \\
&= \int_0^1 \pi x dx + \int_1^2 \pi(x^2-4x+4)dx
\end{aligned}
$$

(This exercise is left to the reader lol don't wanna solve this myself)

cylindrical shell:

$y=\sqrt{x}$ and $y=2-x$ intersect at $(1,1)$.

$$
\begin{aligned}
y=\sqrt{x} &\implies x=y^2 \\
y=2-x &\implies x=2-y
\end{aligned}
$$

$$V = \int_0^1 2\pi y(2-y^2-y)dy$$

(Just solve this yourselves)

Example 3

A solid is obtained by the revolving about the line $x=-2$ the region bounded by $y=x^3$, $y=8$, and the y-axis. Find the volume of the solid.

Solution.

disks: $V = \displaystyle\int_0^8 \pi\big[(\sqrt[3]{y}+2)^2-2^2\big]dy = \frac{336\pi}{5}$

cylindrical shells: $V = \displaystyle\int_0^2 2\pi(x+2)(8-x^3)dx = \frac{336\pi}{5}$
