Mathematical Analysis IB

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Review on differentiation

Differentiability

Let f be a function on some open interval I containing x. The derivative of f at x, denoted by f'(x), is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Differentiation rules

- 1. $\frac{d}{dx}(cf(x)) = cf'(x)$
- 2. $\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$
- 3. $\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + g(x)f'(x)$
- 4. $\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{g(x)f'(x) f(x)g'(x)}{(g(x))^2}$
- 5. $\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$

Differentiation formulas I

- 1. $\frac{d}{dx}(c) = 0, c \in \mathbb{R}$
- $2. \ \frac{d}{dx}(x^r) = rx^{r-1}, r \in \mathbb{R}$
- 3. $\frac{d}{dx}(sinx) = cosx$
- 4. $\frac{d}{dx}(\cos x) = -\sin x$
- 5. $\frac{d}{dx}(tanx) = sec^2x$
- 6. $\frac{d}{dx}(cotx) = -csc^x$
- 7. $\frac{d}{dx}(secx) = secxtanx$
- 8. $\frac{d}{dx}(cscx) = -cscxcotx$

Differentiation formulas II

- $1. \ \frac{d}{dx}(e^x) = e^x$
- $2. \ \frac{d}{dx}(\ln|x|) = \frac{1}{x}$
- 3. $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$
- 4. $\frac{d}{dx}(tan^{-1}x) = \frac{1}{1+x^2}$
- 5. $\frac{d}{dx}(sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$

Mean value theorem

Let f be a function that is continuous on [a, b] and is differentiable on (a, b). Then there is a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Consequences of MVT

Zero derivative

If $f'(x) = 0 \ \forall x$ in interval I, then $f(x) = c \ \forall x \in I$ for some constant C.

Equal derivatives

If $f'(x) - g'(x) = 0 \ \forall x$ in an interval I, then f(x) = g(x) + C for some constant C.

Example Let $f(x) = cos^{-1}x$ and $g(x) = -sin^{-1}x$

This implies that $x \in [-1,1]$ and $f(x), g(x) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$f'(x) = -\frac{1}{\sqrt{x^2 + 1}}$$

$$g'(x) = -\frac{1}{\sqrt{x^2 + 1}}$$

Since f'(x) - g'(x) = 0 for $x \in [-1, 1]$, then f(x) - g(x) = C for some constant C by a corollary.

$$\cos^{-1}x - (-\sin^{-1}x) = C$$

$$\cos^{-1}x + \sin^{-1}x = C$$

Substituting $x \in [-1, 1]$, in this case, let's use x = 0,

$$\cos^{-1}(0) + \sin^{-1}(0) = C$$

$$0 + \frac{\pi}{2} = C$$

$$C=\frac{\pi}{2}$$

$$\therefore \forall x \in [-1, 1], f(x) - g(x) = \frac{\pi}{2}$$

Differentials

$$f'(x) = \frac{dy}{dx}$$

$$dy = f'(x)dx$$

Module 1: Indefinite and definite integrals

Indefinite integral

The main interpretation of derivative is the slope of a tangent line of a curve.

Example At any point (x, y) on a particular curve y = F(x), the tangent line has a slope equal to 4x - 5. If the curve contains the point (3, 7), find F(x).

Solution. Since the slope is equal to 4x - 5 for any point (x, y), then the slope at (3, 7) is 4(3) - 5 = 7. 4x - 5 therefore represents the tangent slope for all values of x. So

$$F'(x) = 4x - 5$$

By intuition, we can conclude that $F(x) = 2x^2 - 5x$.

However given $F(x) = 2x^2 - 5x + 1$, F'(x) remains the same. And so is $F(x) = 2x^2 - 5x - 3$, $F(x) = 2x^2 - 5x + \pi$, and infinitely more functions. We can arbitrarily assign a constant k, so that $F(x) = 2x^2 - 5x + k$.

Substituting (x, y) = (3, 7),

$$7 = 2(3)^2 - 5(3) + k$$

$$7 = 18 - 15 + k$$

$$k = 4$$

So
$$F(x) = 2x^2 - 5x + 4$$
.

Definition of an antiderivative

A function F is called an antiderivative of the function f on an interval I if $F'(x) = f(x) \ \forall x \in I$.

 $F(x) = 2x^2 - 5x$ is a possible antiderivative of f(x) = 4x - 5. $F(x) = 2x^2 - 5x + 4$ is also a possible antiderivative of f(x) = 4x - 5.

Equal derivatives

If $F'(x) = G'(x) \ \forall x$ in an interval I, then $F(x) = G(x) + C \ \forall x \in I$ for some constant C.

Integration notation

The collection of all antiderivatives of f is denoted by

$$\int f(x)dx$$

which is read as "the integral of f(x)dx."

This collection is also called the *indefinite integral* of f.

The reverse process if differentiation is called antidifferentiation or integration.

$$\int (4x-5)dx = 2x^2 - 5x + C$$
 for some constant C.

C is the constant of integration.

$$\int sinx dx = -cosx + C$$

Integration rules

- 1. $\int kf(x)dx = k \int f(x)dx$, k constant
- 2. $\int f(x) \pm g(x)dx = \int f(x)dx \pm \int g(x)dx$

Integration formulas I

- 1. $\int kdx = kx + C, k \in \mathbb{R}$
- 2. $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \in \mathbb{R}, n \neq -1$

Integration formulas II

- 1. $\int sinx dx = -cosx + C$
- 2. $\int cosxdx = sinx + C$
- 3. $\int sec^2xdx = tanx + C$
- 4. $\int csc^2x dx = -cotx + C$
- 5. $\int secxtanxdx = secx + C$
- 6. $\int cscxcotx = -cscx + C$

Integration formulas III

- 1. $\int e^x dx = e^x + C$
- $2. \int \frac{1}{x} dx = \ln|x| + C$
- 3. $\int \frac{1}{\sqrt{1-x^2}} = \sin^{-1}x + C$
- 4. $\int \frac{1}{1+x^2} = tan^{-1}x + C$
- 5. $\int \frac{1}{x\sqrt{x^2-1}} = sec^{-1} + C$

Substitution rule

Chain rule for derivatives

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$$

If follows that

$$\int f'(g(x))g'(x)dx = f(g(x)) + C$$

Example Evaluate $\int 2x \cos^2 dx$.

Preliminary work. By intuition, we can get f(x) = sinx and $g(x) = x^2$

$$\int 2x cos x^2 dx = f(g(x)) = sin x^2$$

Solution. Suppose that $f'(x) = \frac{dy}{dx}$

$$dy = f'(x)dx$$

Let u = g(x), then $g'(x) = \frac{du}{dx}$

$$du = g'(x)dx$$

Let $u = x^2$

$$du = 2xdx$$

$$\int 2x\cos^2 dx = \int \cos u du$$
$$= \sin u + C$$
$$= \sin x^2 + C$$

Definition of the substitution rule

If u = g(x) is a differentiable function whose range is interval I and f is continuous on I, then

$$\int f'(g(x))g'(x) = \int f(u)du$$

The area problem

The definite Integrals

The Fundamental Theorem of Calculus

Proof of Fundamental Theorem of Calculus

Module 2: Application I

Areas between curves

Volumes and volumes of revolution using disks and washers

Volumes of solids of revolution using cylindrical shells

Module 3: Techniques of integration

Integration by parts

Trigonometric integrals

Trigonometric Substitution

Partial fractions

Module 4: Applications II

Arc length

Variable-separable differential equations and models for population growth