

The Calculus of Rainbows

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1 Introduction

A rainbow is a meteorological phenomenon that usually occurs after rain, characterized by a circular spectrum of light appearing in the sky. Because of its beauty and the difficulty of explaining the phenomenon, rainbows have a place in legends in different cultures. In Greco-Roman mythology, the rainbow was considered to be a path made by Iris, a messenger, between Earth and Heaven. In the Bible's Genesis flood narrative God put the rainbow in the sky after flooding the world to wash away sin as the sign of his promise that he would never again destroy the earth with a flood.

With scientific breakthroughs and discoveries throughout the history of humanity, the legends that attempt to explain the existence of rainbows are relegated to creative literature, while rainbows are explained to be optical illusions due to the reflection and refraction of light. This paper goes through the scientific explanation of rainbows and uses Calculus and Physics to derive the underlying mathematics from the phenomenon of rainbows.

2 Scientific explanation

2.1 Fermat's principle of optics

Fermat's principle states that the path taken by a ray between two given points is the path that can be traversed in the least time. In an ideal scenario – in a vacuum – the path of least time is the path least distance, the line, as described by the equation

$$d = vt \tag{1}$$

where d is the distance, v is the velocity, and t is the time.

However, when there are interferences to the path of the light, the path may take a different path, not a straight line, but the path of least time as suggested by Fermat.

2.2 Snell's law

The first case of Fermat's principle of optics is when light passes through two different objects. The phase velocity of light is computed as

$$v = \frac{c}{n} \quad (2)$$

where v is the phase velocity of light, c is the velocity of light in a vacuum, and n is the refractive index of the medium the light is passing through.

Now, we consider a light ray travelling from point A to point B with different refractive indices, n_1 and n_2 , respectively.

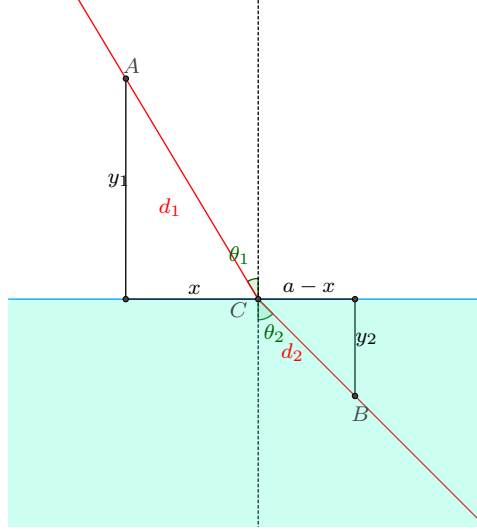


Figure 1: Refraction of light passing through different mediums.

Figure 1 visualizes how light behaves when there is a change in the medium it travels to. The calculation can be visualized using the figure below. Using the Pythagorean Theorem,

$$d_1 = \sqrt{x^2 + y_1^2}$$

$$d_2 = \sqrt{(a-x)^2 + y_2^2}$$

From Equation (1),

$$\begin{aligned} d_1 &= v_1 t_1 & d_2 &= v_2 t_2 \\ t_1 &= \frac{d_1}{v_1} & t_2 &= \frac{d_2}{v_2} \\ t_1 &= \frac{\sqrt{x^2 + y_1^2}}{v_1} & t_2 &= \frac{\sqrt{(a-x)^2 + y_2^2}}{v_2} \end{aligned} \quad (3)$$

We can then create a function $T(x)$ that measures the total time depending on the

value of x .

$$T(x) = t_1 + t_2 = \frac{\sqrt{(a-x)^2 + qy_2^2}}{v_1} + \frac{\sqrt{(a-x)^2 + qy_2^2}}{v_2} \quad (4)$$

Getting the derivative of T ,

$$\begin{aligned} T'(x) &= (2x) \left(\frac{1}{2} \right) \left(\frac{(x^2 + y_1^2)^{-1/2}}{v_1} \right) - 2(a-x) \left(\frac{1}{2} \right) \left(\frac{[(a-x)^2 + y_2^2]^{-1/2}}{v_2} \right) \\ T'(x) &= (x) \frac{(x^2 + y_1^2)^{-1/2}}{v_1} - (a-x) \frac{[(a-x)^2 + y_2^2]^{-1/2}}{v_2} \end{aligned} \quad (5)$$

To get the minimum T , we can equate $T'(x) = 0$.

$$\begin{aligned} (x) \frac{(x^2 + y_1^2)^{-1/2}}{v_1} - (a-x) \frac{[(a-x)^2 + y_2^2]^{-1/2}}{v_2} &= 0 \\ \frac{1}{v_1} \frac{x}{(x^2 + y_1^2)^{1/2}} - \frac{1}{v_2} \frac{a-x}{[(a-x)^2 + y_2^2]^{1/2}} &= 0 \end{aligned} \quad (6)$$

Observe from the figure that

$$\sin \theta_1 = \frac{x}{d_1} = \frac{x}{\sqrt{x^2 + y_1^2}} = \frac{x}{(x^2 + y_1^2)^{1/2}} \quad (7)$$

$$\sin \theta_2 = \frac{a-x}{d_2} = \frac{a-x}{\sqrt{(a-x)^2 + y_2^2}} = \frac{a-x}{[(a-x)^2 + y_2^2]^{1/2}} \quad (8)$$

giving us

$$\begin{aligned} \frac{1}{v_1} \frac{x}{(x^2 + y_1^2)^{1/2}} - \frac{1}{v_2} \frac{a-x}{[(a-x)^2 + y_2^2]^{1/2}} &= 0 \\ \sin(\theta_1) \frac{1}{v_1} - \sin(\theta_2) \frac{1}{v_2} &= 0 \end{aligned} \quad (9)$$

Using Equation (2),

$$\begin{aligned} v_1 &= \frac{c}{n_1} & v_2 &= \frac{c}{n_1} \\ \frac{1}{v_1} &= \frac{n_1}{c} & \frac{1}{v_2} &= \frac{n_2}{c} \end{aligned} \quad (10)$$

Substituting these to Equation (9),

$$\begin{aligned}
\sin(\theta_1) \frac{1}{v_1} - \sin(\theta_2) \frac{1}{v_2} &= 0 \\
\sin(\theta_1) \frac{n_1}{c} - \sin(\theta_2) \frac{n_2}{c} &= 0 \\
\sin(\theta_1) \frac{n_1}{c} &= \sin(\theta_2) \frac{n_2}{c} \\
n_1 \sin(\theta_1) &= n_2 \sin(\theta_2)
\end{aligned} \tag{11}$$

Equation (11) is what we call the law of refraction or *Snell's law*. Snell's Law is shows the relationship between the incident angle and the refracted angle in two different mediums.

2.3 Law of reflection

Another case of Fermat's principle of optics is if rather than passing through, the light ray bounces back.

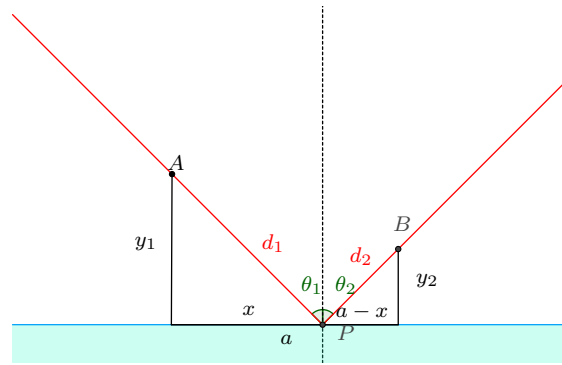


Figure 2: A light ray getting reflected at a flat surface

Figure 2 shows a ray of light from point A reflects off the surface at point P before arriving at point B , a horizontal distance a from point A .

Using the Pythagorean theorem,

$$\begin{aligned}
d_1 &= \sqrt{x^2 + y_1^2} \\
d_2 &= \sqrt{(a-x)^2 + y_2^2}
\end{aligned}$$

Using Equation (1),

$$\begin{aligned}
d_1 &= vt_1 & d_2 &= vt_2 & \text{note that the velocity of light at } d_1 \text{ and } d_2 \text{ are equal} \\
t_1 &= \frac{d_1}{v} & t_2 &= \frac{d_2}{v}
\end{aligned} \tag{12}$$

We can then create a function $T(x)$ that measures the total time depending on the value of x

$$T(x) = t_1 + t_2 = \frac{\sqrt{x^2 + y_1^2}}{v} + \frac{\sqrt{(a-x)^2 + y_2^2}}{v} \quad (13)$$

Getting the derivative of T ,

$$T'(x) = \frac{x}{v\sqrt{x^2 + y_1^2}} + \frac{-(a-x)}{v\sqrt{(a-x)^2 + y_2^2}} \quad (14)$$

Equating $T'(x) = 0$ to get the minimum value,

$$\begin{aligned} \frac{x}{v\sqrt{x^2 + y_1^2}} + \frac{-(a-x)}{v\sqrt{(a-x)^2 + y_2^2}} &= 0 \\ \frac{x}{v\sqrt{x^2 + y_1^2}} &= \frac{a-x}{v\sqrt{(a-x)^2 + y_2^2}} \\ \frac{x}{\sqrt{x^2 + y_1^2}} &= \frac{a-x}{\sqrt{(a-x)^2 + y_2^2}} \end{aligned} \quad (15)$$

Observe from Figure 2 that

$$\sin \theta_1 = \frac{x}{\sqrt{x^2 + y_1^2}} \quad (16)$$

$$\sin \theta_2 = \frac{a-x}{\sqrt{(a-x)^2 + y_2^2}} \quad (17)$$

Substituting these to Equation (15),

$$\begin{aligned} \sin \theta_1 &= \sin \theta_2 \\ \theta_1 &= \theta_2 \end{aligned} \quad (18)$$

Equation (18) is what we call the law of reflection, showing that the angle of incidence θ_1 is equal to the angle of reflection θ_2 .

Rough surfaces contain varying planes that diffuses light at different angles at the point where the light ray bounces. This is what we call *diffused reflection*. On the other hand, when light is directed in a smooth plane like polished metal or a glass mirror, the light ray is reflected at an angle equal to the incident angle with respect to the line perpendicular to the plane. This is what we call *specular reflection*.

2.4 Light wavelengths and their refractive interactions

Wavelength (nm)	Refractive Index	Color
400	1.3445	Violet
450	1.3406	Indigo
500	1.3377	Blue
550	1.3356	Green
600	1.3339	Yellow
650	1.3326	Orange
700	1.3314	Red

Table 1: Visible light wavelengths and their corresponding refractive indices on water

3 Mathematical derivation

3.1 The angle of a primary rainbow

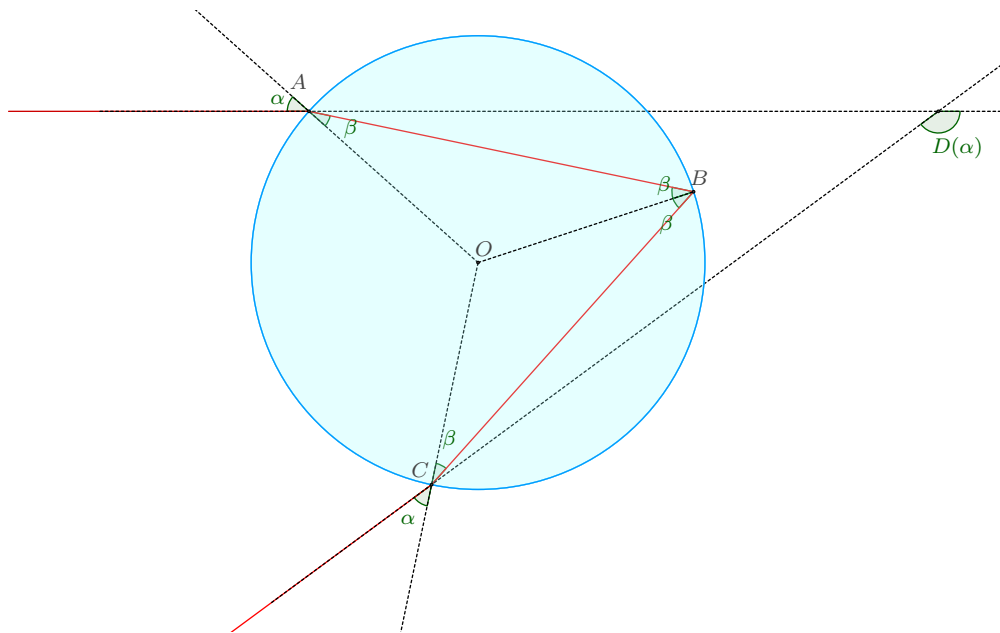


Figure 3: Formation of a primary rainbow, showing Snell's Law and the Law of Reflection at work.

Figure 3 shows a ray of sunlight entering a spherical raindrop at A . Some of the light is reflected, but the line AB shows the path of the part that enters the drop.

Notice that the light is refracted toward the normal line AO . Since there is a change in the medium of the passing light, Snell's law applies.

$$\sin \alpha = k \sin \beta \quad (19)$$

where α is the angle of incidence, β is the angle of refraction, and $k \approx 4/3$ is the approximate ratio between the index of refraction of water and the index of refraction of air. Note that the angle of refraction of air is very close to 1, so it can be assumed that k is the index of refraction of water.

At B some of the light passes through the drop and is refracted into the air, but some of the light is also internally reflected, as represented by BC . Note that the angle of incidence equals the angle of reflection. When the ray reaches C , part of it is reflected, but for the time being, we are more interested in the part that leaves the raindrop at C . (Notice that it is refracted away from the normal line.)

The angle of deviation $D(\alpha)$ is the amount of clockwise rotation that the ray has undergone during the stage of refraction, internal reflection, and another refraction. Using Geometry,

$$D(\alpha) = (\alpha - \beta) + (\pi - 2\beta) + (\alpha - \beta)$$

$$D(\alpha) = \pi + 2\alpha - 4\beta \quad (20)$$

$$(21)$$

Using Equation (19) to isolate β in terms of α ,

$$\begin{aligned} \sin \alpha &= k \sin \beta \\ \frac{\sin \alpha}{k} &= \sin \beta \\ \beta &= \arcsin \left(\frac{\sin \alpha}{k} \right) \end{aligned} \quad (22)$$

Substituting β on Equation (22),

$$D(\alpha) = \pi + 2\alpha - 4 \arcsin \left(\frac{\sin \alpha}{k} \right) \quad (23)$$

Getting the derivative of D ,

$$\begin{aligned} D'(\alpha) &= 0 + 2 - 4 \left[\frac{1}{\sqrt{1 - \left(\frac{\sin \alpha}{k} \right)^2}} \right] \left(\frac{\cos \alpha}{k} \right) \\ D(\alpha) &= 2 - \frac{4 \cos \alpha}{k \sqrt{1 - \frac{\sin^2 \alpha}{k^2}}} \end{aligned} \quad (24)$$

To get the minimum of $D'(\alpha)$, we shall equate it to 0.

$$\begin{aligned}
2 - \frac{4 \cos \alpha}{k \sqrt{1 - \frac{\sin^2 \alpha}{k^2}}} &= 0 \\
2 &= \frac{4 \cos \alpha}{k \sqrt{1 - \frac{\sin^2 \alpha}{k^2}}}
\end{aligned} \tag{25}$$

It is good to note that since $k > 1$, then $1 - \frac{\sin^2 \alpha}{k^2} > 0$ for all α .

$$\begin{aligned}
2k \sqrt{1 - \frac{\sin^2 \alpha}{k^2}} &= 4 \cos \alpha \\
k \sqrt{1 - \frac{\sin^2 \alpha}{k^2}} &= 2 \cos \alpha
\end{aligned} \tag{26}$$

Squaring both sides,

$$\begin{aligned}
k^2 \left(1 - \frac{\sin^2 \alpha}{k^2} \right) &= 4 \cos^2 \alpha \\
k^2 - \sin^2 \alpha &= 4 \cos^2 \alpha \\
k^2 &= \sin^2 \alpha + 4 \cos^2 \alpha \\
k^2 &= \sin^2 \alpha + \cos^2 \alpha + 3 \cos^2 \alpha \\
k^2 &= 1 + 3 \cos^2 \alpha \\
\sqrt{\frac{k^2 - 1}{3}} &= \cos \alpha \\
\alpha &= \arccos \sqrt{\frac{k^2 - 1}{3}}
\end{aligned} \tag{27}$$

$D(\alpha)$ is minimum when

$$\alpha = \arccos \sqrt{\frac{k^2 - 1}{3}} \tag{28}$$

Substituting $k \approx \frac{4}{3}$,

$$\begin{aligned}
\alpha &\approx \arccos \sqrt{\frac{\left(\frac{4}{3}\right)^2 - 1}{3}} \\
&\approx \arccos \sqrt{\frac{16/9 - 1}{3}} \\
&\approx \arccos \sqrt{\frac{7}{27}} \\
\alpha &\approx 1.0366 \\
\alpha &\approx 59.4^\circ
\end{aligned}$$

$D(\alpha)$ is minimum at $\alpha \approx 59.4^\circ$.

Substituting $\alpha \approx 59.4^\circ$,

$$\begin{aligned}
\min(D(\alpha)) &\approx D(1.0366) \approx \pi + 2\alpha - 4 \arcsin\left(\frac{\sin \alpha}{4/3}\right) \\
&\approx \pi + 2(1.0366) - 4 \arcsin\left(\frac{\sin 1.0366}{4/3}\right) \\
&\approx 138^\circ
\end{aligned}$$

The minimum value of the deviation is $D(\alpha) \approx 138^\circ$ which occurs when $\alpha \approx 59.4^\circ$.

The minimum deviation angle has huge significance since $D'(\alpha) \approx 0$ implies that $\Delta D/\Delta \alpha \approx 0$, which means that light rays deviate by approximately the same amount at $\alpha \approx 59.4^\circ$. This implies that there is a concentration of rays coming from near the direction of the minimum deviation, causing the formation of the rainbow from these concentrated light rays [1, p. 219].

The figure shows that the angle of elevation from the observer up to the highest point on the rainbow is $180^\circ - 138^\circ = 42^\circ$. This angle is called the *rainbow angle*.

3.2 Colors in a primary rainbow

Sunlight comprises a range of wavelengths, from the reds with the longest wavelengths and the violets with the shortest wavelengths. As Newton discovered in his prism experiments of 1666, the index of refraction is different for each color in a phenomenon called dispersion. For red light, the refractive index is $k \approx 1.3318$, whereas for violet light it is $k \approx 1.3435$.

As previously mentioned in Equation (28), the minimum of $D(\alpha)$ is when $\alpha = \arccos \sqrt{\frac{k^2 - 1}{3}}$

Substituting $k \approx 1.3318$ for red light,

$$\begin{aligned}\alpha &= \arccos \sqrt{\frac{1.3318^2 - 1}{3}} \\ &\approx 1.0381\end{aligned}$$

Substituting $\alpha \approx 1.0381$ and $k \approx 1.3318$ on $D(\alpha)$,

$$\begin{aligned}D(1.0381) &= \pi + 2(1.0381) - 4 \arcsin \left(\frac{\sin 1.0381}{1.3318} \right) \\ &\approx 2.4041 \\ &\approx 137.7^\circ\end{aligned}$$

The rainbow angle of the red light is $\approx 180^\circ - 137.7^\circ \approx 42.3^\circ$.

Substituting $k \approx 1.3435$ for violet light,

$$\begin{aligned}\alpha &= \arccos \sqrt{\frac{1.3435^2 - 1}{3}} \\ &\approx 1.0263\end{aligned}$$

Substituting $\alpha \approx 1.0263$ and $k \approx 1.3435$ on $D(\alpha)$,

$$\begin{aligned}D(1.0263) &= \pi + 2(1.0263) - 4 \arcsin \left(\frac{\sin 1.0263}{1.3435} \right) \\ &\approx 2.4334 \\ &\approx 139.4^\circ\end{aligned}$$

The rainbow angle of the violet light is $\approx 180^\circ - 139.4^\circ = 40.6^\circ$.

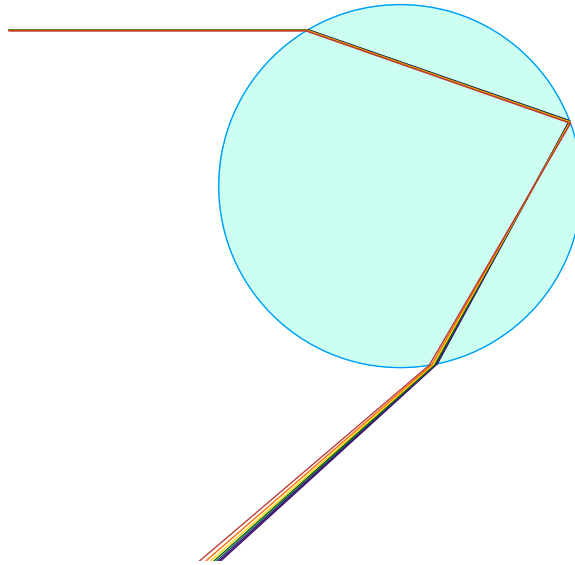


Figure 4: Colors of the rainbow based on their respective minimum α value and refractive indices.

As observed in the calculations and as visualized in Figure 4, the rainbow angle of the red light (42.3°) is larger than the rainbow angle of the violet light (40.6°). This implies that the rainbow's color appears red on top, violet on the bottom, and all other visible light wavelengths in between.

3.3 Double rainbows

There are instances where another fainter rainbow appears in addition of the stronger rainbow. This phenomenon is called *double rainbow*, and the fainter rainbow is colloquially known as the *secondary rainbow*.

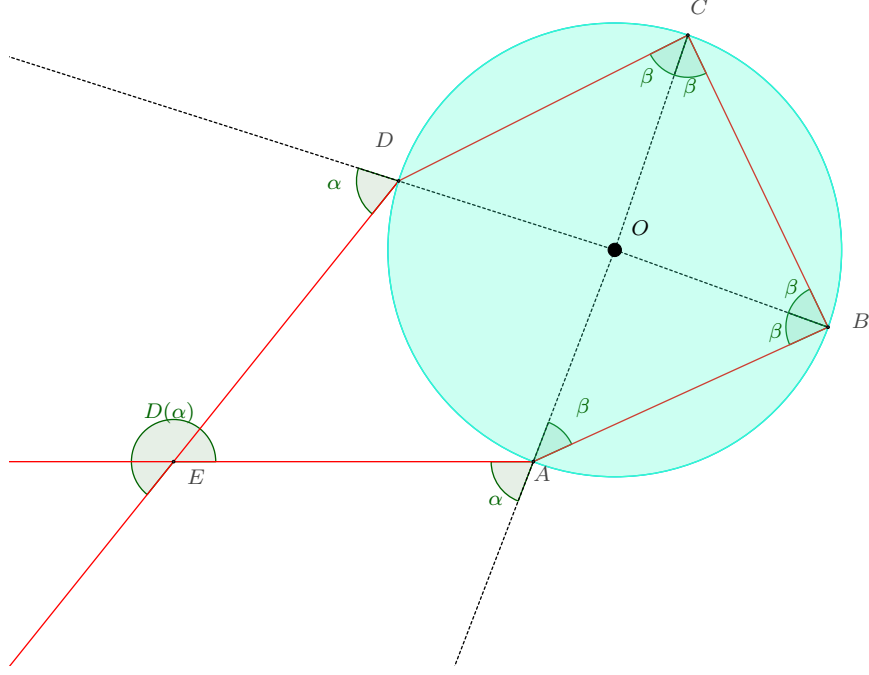


Figure 5: Formation of a secondary rainbow. Compared to the primary rainbow, the light ray is internally reflected twice.

As visualized in Figure 5, this phenomenon results from the part of a ray that enters a raindrop and is refracted at A , reflected twice (at B and C), and refracted as it leaves the drop at D . Whereas $D(\alpha)$ is the amount of *clockwise* deviation, in secondary rainbows, the deviation angle $D(\alpha)$ refers to the total amount of counterclockwise rotation that the ray undergoes in this four-stage process.

As seen in Figure 5, the light ray has undergone two refractions and two internal reflections. The refractions deviate the ray by $\alpha - \beta$ while the internal reflections deviate the ray by $\pi - 2\beta$. Therefore, the angle of deviation $D(\alpha)$ is

$$\begin{aligned} D(\alpha) &= (\alpha - \beta) + (\pi - 2\beta) + (\pi - 2\beta) + (\alpha - \beta) \\ D(\alpha) &= 2\pi + 2\alpha - 6\beta \end{aligned} \tag{29}$$

Using Equation (19) to get β in terms of α ,

$$\begin{aligned} \sin \alpha &= k \sin \beta \\ \beta &= \arcsin \left(\frac{\sin \alpha}{k} \right) \end{aligned}$$

Substuting β on Equation (29),

$$\begin{aligned}
D(\alpha) &= 2\pi + 2\alpha - 6\beta \\
D(\alpha) &= 2\pi + 2\alpha - 6 \arcsin \left(\frac{\sin \alpha}{k} \right)
\end{aligned} \tag{30}$$

In order to get the minimum value of D , we can equate its derivative, $D'(\alpha) = 0$.

$$\begin{aligned}
D'(\alpha) &= 0 + 2 - 6 \left[\frac{1}{\sqrt{1 - \left(\frac{\sin \alpha}{k} \right)^2}} \right] \left(\frac{\cos \alpha}{k} \right) \\
D'(\alpha) &= 2 - \frac{6 \cos \alpha}{k \sqrt{1 - \frac{\sin^2 \alpha}{k^2}}}
\end{aligned} \tag{31}$$

Equating $D'(\alpha) = 0$,

$$\begin{aligned}
0 &= 2 - \frac{6 \cos \alpha}{k \sqrt{1 - \frac{\sin^2 \alpha}{k^2}}} \\
\frac{6 \cos \alpha}{k \sqrt{1 - \frac{\sin^2 \alpha}{k^2}}} &= 2
\end{aligned} \tag{32}$$

As mentioned in Section 3.2, since $k > 1$, then $1 - \frac{\sin^2 \alpha}{k^2} > 0$ for all α .

$$\begin{aligned}
\frac{6 \cos \alpha}{k \sqrt{1 - \frac{\sin^2 \alpha}{k^2}}} &= 2 \\
6 \cos \alpha &= 2k \sqrt{1 - \frac{\sin^2 \alpha}{k^2}} \\
3 \cos \alpha &= k \sqrt{1 - \frac{\sin^2 \alpha}{k^2}}
\end{aligned} \tag{33}$$

Squaring both sides,

$$\begin{aligned}
9 \cos^2 \alpha &= k^2 \left(1 - \frac{\sin^2 \alpha}{k^2} \right) \\
9 \cos^2 \alpha &= k^2 - \sin^2 \alpha \\
k^2 &= \sin^2 \alpha + 9 \cos^2 \alpha \\
k^2 &= \sin^2 \alpha + \cos^2 \alpha + 8 \cos^2 \alpha \\
k^2 &= 1 + 8 \cos^2 \alpha \\
\cos^2 \alpha &= \frac{k^2 - 1}{8} \\
\cos \alpha &= \sqrt{\frac{k^2 - 1}{8}}
\end{aligned} \tag{34}$$

This shows that $D(\alpha)$ is minimum when

$$\cos \alpha = \sqrt{\frac{k^2 - 1}{8}} \tag{35}$$

Taking $k = 4/3$,

$$\cos \alpha = \sqrt{\frac{(4/3)^2 - 1}{8}} = \sqrt{\frac{16/9 - 1}{8}} = \sqrt{\frac{7/9}{8}} \cos \alpha = \sqrt{\frac{7}{72}} \alpha = \arccos \sqrt{\frac{7}{72}} \alpha \approx 1.2537 \alpha \approx 71.8^\circ$$

When $k = 4/3$, $D(\alpha)$ is minimum when $\alpha = 71.8^\circ$.

$$\min(D(\alpha)) \approx D(1.2537) = 2\pi + 2(1.2537) - 6 \arcsin \left(\frac{\sin 1.2537}{4/3} \right) \approx 4.031 \approx 231^\circ$$

As mentioned in Section 3.2, $dD/da \approx 0$ implies that light rays converge at that certain angle, meaning that it is the best angle to properly see the rainbow.

Let $D_2(\alpha)$ be the clockwise deviation angle of the light ray. This means that $D(\alpha) + D_2(\alpha) = 2\pi$

Since D is minimum $D(\alpha) = 231^\circ$, then $D_2(\alpha) = 360^\circ - 231^\circ = 129^\circ$.

The rainbow angle is $180^\circ - 129^\circ = 51^\circ$. Note that in Section ??, the maximum rainbow angle is on the red bow at $\approx 42.4^\circ$. This means that the secondary rainbow is *always* above the primary rainbow.

3.4 Colors in a secondary rainbow

4 Conclusion

Using the principles of light rays, laws of refraction and reflection, the rainbow angle deviations and locations were calculated. Applying the concept of Snell's Law, the minimum angle of deviation of the primary rainbow was shown, which is at 138 degrees. The angle of deviation of the primary rainbow is the amount of clockwise rotation the ray has undergone, and at minimum angle of deviation, the rainbow is at its brightest as the rays are deviated by approximately the same amount. It is elevated at 42 degrees from the observer to the highest point of the rainbow, which is called the rainbow angle. To explain the colors in the bow, the different refraction index of each color was used. Higher refraction index has the smaller rainbow angle there is, illustrating the order of the seven colors that can be seen from a rainbow.

Above the primary rainbow is the secondary rainbow that is fainter than the first. The minimum angle of deviation is at 231 degrees, but the angle of deviation in the secondary rainbow is in counterclockwise rotation and is equivalent to 129 degrees in a clockwise rotation. The bow is elevated higher than the primary one, which general rainbow angle is at 51 degrees elevation from the observer to the highest point of the bow. In the secondary rainbow, the rainbow angle gets larger as the index of refraction gets bigger. Consequently, the order of the colors in the secondary rainbow is in opposite of the primary rainbow.

References

- [1] James Stewart, Daniel Clegg, and Saleem Watson. *Calculus: International metric version*. 9th ed. Cengage Learning, 2021. ISBN: 9780357113462.