Differentiation

Definition of derivatives

The derivative of f at x, denoted by $f'(x)$, is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Differentiation rules

1. $\frac{d}{dx}(cf(x)) = cf'(x)$
2. $\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$
3. $\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + g(x)f'(x)$
4. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$
5. $\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$

Differentiation formulas I

1. $\frac{d}{dx}(c) = 0, c \in \mathbb{R}$
2. $\frac{d}{dx}(x^r) = rx^{r-1}, r \in \mathbb{R}$
3. $\frac{d}{dx}(\sin x) = \cos x$
4. $\frac{d}{dx}(\cos x) = \sin x$
5. $\frac{d}{dx}(\tan x) = \sec^2 x$
6. $\frac{d}{dx}(\cot x) = -\csc^2 x$
7. $\frac{d}{dx}(\sec x) = \sec x \tan x$
8. $\frac{d}{dx}(\csc x) = -\csc x \cot x$

Differentiation formulas II

1. $\frac{d}{dx}(e^x) = e^x$
2. $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$
3. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
4. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
5. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

Mean value theorem

Let f be a function that is continuous on $[a,b]$ and is differentiable on (a,b). Then there is a number $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Consequences of MVT

Zero derivative

If $f'(x) = 0 \ \forall x$ in interval I, then $f(x) = c \ \forall x \in I$ for some constant C.

Equal derivatives

If $f'(x) - g'(x) = 0 \ \forall x$ in an interval I, then $f(x) = g(x) + C$ for some constant C.

Differentials

$$f'(x) = \frac{dy}{dx}$$
$$f'(x)dx = dy$$

Integration

Definition of an antiderivative

A function F is called an antiderivative of the function f on an interval I if $F'(x) = f(x) \ \forall x \in I$.

Equal derivatives (antiderivatives)

If $F'(x) = G'(x) \ \forall x$ in an interval I, then $F(x) = G(x) + C \ \forall x \in I$ for some constant C.

Integration rules

1. $\int kf(x)dx = k\int f(x)dx,\ k \in \mathbb{R}$
2. $\int f(x) \pm g(x)dx = \int f(x)dx \pm \int g(x)dx$

Integration formulas I

1. $\int kdx = kx + C, k \in \mathbb{R}$
2. $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \in \mathbb{R}, n \neq -1$

Integration formulas II

1. $\int \sin x dx = -\cos x + C$
2. $\int \cos x dx = \sin x + C$
3. $\int \sec^2 x dx = \tan x + C$
4. $\int \csc^2 x dx = -\cot x + C$
5. $\int \sec x \tan x dx = \sec x + C$
6. $\int \csc x \cot x dx = -\csc x + C$

Integration formulas III

1. $\int e^x dx = e^x + C$
2. $\int \frac{1}{x} dx = \ln |x| + C$
3. $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$
4. $\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$
5. $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} + C$

Substitution rule

If $u = g(x)$ is a differentiable function whose range is interval I and f is continuous on I, then

$$\int f'(g(x))g'(x)dx = \int f(u)du$$

Riemann Sum

$$A_n = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Definite integrals

The definite integral of f from a to b is

$$\int_a^b f(x)dx = \lim_{x\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided that such limit exists.

We say that f is integrable on $[a,b]$

Remarks on the definite integral

1. If a function is continuous on $[a,b]$, it is integrable on $[a,b]$.
2. If f is a nonnegative continuous function on $[a,b]$, then $\int_a^b f(x)dx$ is the area under the curve $y = f(x)$ from $x = a$ and $x = b$
3. $\int_a^b f(x)dx = \int_a^b f(y)dy$

Conventions on the definite integral

1. $\int_b^a f(x)dx = -\int_a^b f(x)dx$
2. $\int_a^a f(x)dx = 0$

Properties of the definite integral

1. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$
2. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x) \pm \int_a^b g(x)$
3. $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$
4. If $f(x) \geq 0 \ \forall x \in [a,b]$, then $\int_a^b f(x)dx \geq 0$
5. If $f(x) \geq g(x) \ \forall x \in [a,b]$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$
6. If $m \leq f(x) \leq M \ \forall x \in [a,b]$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

Mean value theorem (integrals)

If f is continuous on $[a,b]$, $\exists c \in [a,b]$ such that

$$\int_b^a f(x)dx = f(c)(b-a)$$

Average value of a function

Let f be a continuous on $[a,b]$. The average value of f at $[a,b]$, denoted by f_{avg} is

$$f_{avg} = \frac{\int_a^b f(x)dx}{b-a}$$

FTC 1

Let f be continuous on $[a,b]$. If f is the function defined by

$$F(x) = \int_a^x f(t)dt$$

then $F'(x) = f(x) \ \forall x \in [a,b]$.

FTC 2

If a function f is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

The following notations for $F(b) - F(a)$ are very useful in evaluating definite integrals: $F(x)\Big]_a^b$ or $F(x)\Big|_a^b$

Area between curves

Given two curves $y = f(x)$ and $y = g(x)$, where $f(x) > g(x)\ \forall x \in [a, b]$, then the area between both curves from $x = a$ and $x = b$ is

$$A = \int_a^b f(x) - g(x)dx$$

Volume of a solid

Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x through x and perpendicular to the x-axis is $A(x)$, where A is a continuous function on $[a, b]$, then the volume V of S is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_b^a A(x)dx$$

Volumes of revolution

Disk and washers technique

$$A(x) = \pi[f(x)]^2$$
$$\therefore V = \int_b^a \pi[f(x)]^2 dx$$

Cylindrical shells

The volume of a solid obtained by rotating about the y-axis the region under the curve $y = f(x)$ (continuous and nonnegative) from $x = a$ (nonnegative) to $x = b$ is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*)\Delta x = \int_a^b 2\pi x f(x)dx$$

Integration by parts

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

Letting $u = f(x)$, $v = g(x) \implies du = f'(x)dx$, $dv = g'(x)dx$,

$$\int udv = uv - \int vdu$$

Integration by parts and definite integrals

Combining the integration-by-parts formula and FTC2,

$$\int_a^b f(x)g'(x)dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)dx$$

Trigonometric identities

1. $\sin^2 x + \cos^2 x = 1$
2. $\tan^2 x + 1 = sec^2 x$
3. $\cot^2 x + 1 = csc^2 x$
4. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
5. $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
6. $\sin A \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)]$
7. $\sin A \sin B = \frac{1}{2}[\cos(A - B)] - \cos(A + B)]$
8. $\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$

Integrals of trigonometric functions

1. $\int \tan x dx = \ln|\sec x| + C$
2. $\int \sec x dx = \ln|\sec x + \tan x| + C$
3. $\int \cot x dx = \ln|\sin x| + C$
4. $\int \csc x dx = \ln|\csc x - \cot x| + C$

Trigonometric substitution

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta, 0 \leq \theta \leq \frac{\pi}{2}$ or $\pi \leq \theta \leq \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Partial fractions

Let $f(x) = \frac{P(x)}{Q(x)}$ where P,Q are polynomial functions.

If $\deg(P) \leq \deg(Q)$, then continue doing partial fractions.

If $\deg(P) \geq \deg(Q)$, then we need to do preliminary work:

$$f(x) = S(x) + \frac{R(x)}{Q(x)}$$

where S is a polynomial function and R is the remainder of the long division between P and Q.

Arc length

If f' is continuous on $[a, b]$, then the length L of the curve $y = f(x)$, $a \leq x \leq b$, is given by

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

Simple growth model

The solution of the initial-value problem

$$\frac{dP}{dt} = kP, P(0) = P_0$$

is

$$P(t) = P_0 e^{kt}$$

Logistic model

The solution fo the initial-value problem

$$\frac{\frac{dP}{dt}}{P} = k\left(1 - \frac{P}{M}\right)$$

is

$$P(t) = \frac{M}{1 - Ae^{kt}}, A = \frac{M - P_0}{P_0}$$