Mathematical Analysis II

Indeterminate forms and l'Hospital's rule

Recall:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

If both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, and $\lim_{x\to a} g(x) \neq 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists. It also holds if $x \to a$ is changed to $x \to a^+$ or $x \to \pm\infty$.

If $\lim_{x\to a} g(x) = 0$ but $\lim_{x\to a} f(x) \neq 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)}$ DNE.

If in $\lim_{x\to a} \frac{f(x)}{g(x)}$, $\lim_{x\to a} f(x) = 0$, $\lim_{x\to a} g(x) = 0$, then the limit may or may not exist, and we have what is called an indeterminate form of type $\frac{0}{0}$.

Similarly, if in $\lim_{x\to a} \frac{f(x)}{g(x)}$, $\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a} g(x) = \pm\infty$, then the limit may or may not exists, and we have what is called an indeterminate form of type $\frac{\infty}{\infty}$.

L'Hospital Rule (LR)

Supposed f and g are diffrentiable and $g'(x) \neq 0$ on an open interval that contains a (except possibly at a),

Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'x)}$

Note: L'Hospital's rule is also valid if $x \to a$ is changed to $x \to a^+$, $x \to a^-$, or $x \to \pm\infty$

Example Evaluate $\displaystyle\lim_{x\to 1}\frac{\ln x}{x-1}$

Solution. This has indeterminated form $\frac{0}{0}$, so we can apply LR:

$$\lim_{x\to 1}\frac{\ln x}{x-1} = \lim_{x\to 1}\frac{1/x}{1} = 1$$

$0 \cdot \pm\infty$ **indeterminate form**

Example Evaluate $\displaystyle\lim_{x\to -\infty}(xe^x)$.

Solution. Note that the limit has the indeterminate form $0 \cdot -\infty$. We can write xe^x as a quotient

$$\lim_{x\to -\infty}(xe^x) = \lim_{x\to -\infty}\frac{x}{e^{-x}}$$

RHS is now a $\frac{-\infty}{\infty}$ indeterminate form. Hence we can apply LR.

$$
\begin{aligned}
\lim_{x\to-\infty}(xe^x) &= \lim_{x\to-\infty}\frac{x}{e^{-x}}\\
&= \lim_{x\to-\infty}\frac{1}{-e^{-x}}\\
&= 0
\end{aligned}
$$

$\infty - \infty$ indeterminate form

Example Evaluate $\lim\limits_{x\to 1^+}\left(\dfrac{1}{lnx} - \dfrac{1}{x-1}\right)$

Solution: Note that we have the indeterminate form $\infty - \infty$. Here, we can write the different as a quotient.

$$
\lim_{x\to 1^+}\left(\frac{1}{lnx} - \frac{1}{x-1}\right) = \lim_{x\to 1^+}\frac{x-1-\ln x}{(x-1)\ln x}
$$

Now, the new limit has indeterminate form $\frac{0}{0}$, so LR applies

$$
\begin{aligned}
\lim_{x\to 1^+}\left(\frac{x-1-\ln x}{(x-1)\ln x}\right) &= \lim_{x\to 1^+}\frac{1-1/x}{(x-1)(1/x)+\ln x}\\
&= \lim_{x\to 1^+}\frac{x-1}{x-1+x\ln x}\\
&= \lim_{x\to 1^+}\frac{1}{1+x(1/x)+\ln x}\\
&= \lim_{x\to 1^+}\frac{1}{1+1+\ln x}\\
&= \frac{1}{1+1+\ln 1}\\
&= \frac{1}{2}
\end{aligned}
$$