Mathematical Analysis II

Indeterminate forms and l'Hospital's rule

Recall:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

If both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, and $\lim_{x\to a} g(x) \neq 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists. It also holds if $x \to a$ is changed to $x \to a^+$ or $x \to \pm\infty$.

If $\lim_{x\to a} g(x) = 0$ but $\lim_{x\to a} f(x) \neq 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)}$ DNE.

If in $\lim_{x\to a} \frac{f(x)}{g(x)}$, $\lim_{x\to a} f(x) = 0$, $\lim_{x\to a} g(x) = 0$, then the limit may or may not exist, and we have what is called an indeterminate form of type $\frac{0}{0}$.

Similarly, if in $\lim_{x\to a} \frac{f(x)}{g(x)}$, $\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a} g(x) = \pm\infty$, then the limit may or may not exists, and we have what is called an indeterminate form of type $\frac{\infty}{\infty}$.

L'Hospital's rule (LR)

> Suppose f and g are diffrentiable and $g'(x) \neq 0$ on an open interval that contains a (except possibly at a),
>
> Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'x)}$
>
> > *Note:* L'Hospital's rule is also valid if $x \to a$ is changed to $x \to a^+$, $x \to a^-$, or $x \to \pm\infty$

Evaluate $\displaystyle\lim_{x\to 1} \frac{\ln x}{x-1}$

Solution. This has indeterminated form $\frac{0}{0}$, so we can apply LR:

$$\lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} \frac{1/x}{1} = 1$$

Indeterminate products

This kind of limit is also $0 \cdot \pm\infty$ indeterminate form, which happens if $\lim_{x\to a} f(x) = 0$, $\lim_{x\to a} g(x) = \pm\infty$. To solve, we must try to converty this to an indeterminate of either the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ before applying LR.

Evaluate $\displaystyle\lim_{x\to -\infty} (xe^x)$.

Solution. Note that the limit has the indeterminate form $0 \cdot -\infty$. We can write xe^x as a quotient

$$\lim_{x \to -\infty} (xe^x) = \lim_{x \to -\infty} \frac{x}{e^{-x}}$$

RHS is now a $\frac{-\infty}{\infty}$ indeterminate form. Hence we can apply LR.

$$\begin{aligned}\lim_{x \to -\infty} (xe^x) &= \lim_{x \to -\infty} \frac{x}{e^{-x}} \\ &= \lim_{x \to -\infty} \frac{1}{-e^{-x}} \\ &= 0\end{aligned}$$

Indeterminate differences

If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$, then the limit

$$\lim_{x \to a} [f(x) - g(x)]$$

is $\infty - \infty$ indeterminate form. Just like before, we need to convert this to an indeterminate limit of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ before applying LR.

Evaluate $\displaystyle\lim_{x \to 1^+} \left(\frac{1}{lnx} - \frac{1}{x-1} \right)$

Solution. Note that we have the indeterminate form $\infty - \infty$. Here, we can write the different as a quotient.

$$\lim_{x \to 1^+} \left(\frac{1}{lnx} - \frac{1}{x-1} \right) = \lim_{x \to 1^+} \frac{x - 1 - \ln x}{(x-1)\ln x}$$

Now, the new limit has indeterminate form $\frac{0}{0}$, so LR applies

$$\begin{aligned}\lim_{x \to 1^+} \left(\frac{x - 1 - \ln x}{(x-1)\ln x} \right) &= \lim_{x \to 1^+} \frac{1 - 1/x}{(x-1)(1/x) + \ln x} \\ &= \lim_{x \to 1^+} \frac{x-1}{x - 1 + x \ln x} \\ &= \lim_{x \to 1^+} \frac{1}{1 + x(1/x) + \ln x} \\ &= \lim_{x \to 1^+} \frac{1}{1 + 1 + \ln x} \\ &= \frac{1}{1 + 1 + \ln 1} \\ &= \frac{1}{2}\end{aligned}$$

Indeterminate powers

Improper integrals

The integral $\int_{-1}^{1}(1/x^2)dx$ looks like an ordinary integral that you encountered. But using FTC2,

$$\int_{-1}^{1}\frac{1}{x^2}dx = -\frac{1}{x}|_{-1}^{1} = -1-1 = -2$$

However, the computation is not valid because the integral is not continuous on $[-1,1]$.

Infinite intervals

1. If $\int_a^t f(x)dx$ exists for all $t \geq a$, then

$$\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx$$

   if this limit exists.

2. If $\int_t^b f(x)dx$ exists for all $t \leq a$, then

$$\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty}\int_t^b f(x)dx$$

   if this limit exists.

3. If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ exist (as finite numbers), then

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$$

   If either $\int_a^\infty f(x)dx$ or $\int_{-\infty}^a f(x)dx$ DNE, then $\int_{-\infty}^\infty f(x)dx$ DNE.

If an improper integral exists (as a finite number), then we say that it is convergent; otherwise, it is divergent.

Discontinuous integrands

1. If f is continuous on $[a,b)$ but discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{t\to b^-}\int_a^t f(x)dx$$

   if this limit exists (as a finite number)

2. If f is continuous on $(a,b]$ but discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{t\to a^+}\int_t^b f(x)dx$$

   if this limit exists.

3. If f is discontinuous at $c \in (a,b)$, but both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ exists, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

If either $\int_a^c f(x)dx$ or $\int_c^b f(x)dx$ DNE, then $\int_a^b f(x)dx$ DNE.

Like Type 1 improper integrals, a Type 2 improper integral is convergent if it exists, and divergent if it does not.

Recall $\displaystyle\int_{-1}^{1} \frac{1}{x^2}dx$

Solution. Note that $\frac{1}{x^2}$ is discontinuous at $x = 0$.

$$\begin{aligned}\int_{-1}^{1} \frac{1}{x^2}dx &= \int_{-1}^{0} \frac{1}{x^2} + \int_{0}^{1} \frac{1}{x^2} \\ &= \lim_{t\to 0^-}\int_{-1}^{t} \frac{1}{x^2} + \lim_{t\to 0^+}\int_{t}^{1} \frac{1}{x^2}\end{aligned}$$

We observe that both integrals are divergent becayse they both approach infinity.

$$\lim_{t\to 0^-}\int_{-1}^{t} \frac{1}{x^2} + \lim_{t\to 0^+}\int_{t}^{1} \frac{1}{x^2} = \infty + \infty = \infty$$

Comparison theorem

Suppose that f and g are continuous function with $f(x) \geq g(x) \geq 0 for x \geq a$.

1. If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is also convergent.
2. If $\int_a^\infty f(x)dx$ is divergent, then $\int_a^\infty g(x)dx$ is also divergent.